Internal Report
DESY H2-77/01
January 1977

THE INFLUENCE OF A RADIALLY INHOMOGENEOUS RF FIELD ON PHASE OSCILLATIONS

by

R. D. Kohaupt

Introduction

In investigating beam-loading effects one usually assumes that the rf-accelerating field is radially homogeneous. This is approximately satisfied, if the diameter of the cavities is sufficiently large.

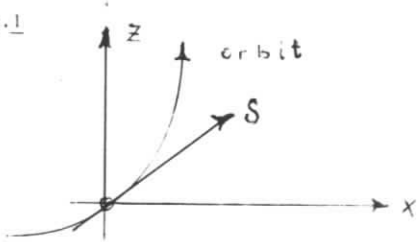
In the case of a machine with superconducting cavities there are mainly two reasons making an investigation of inhomogeneities necessary:

i: The radius of superconducting cavities should be smaller as usual
i.i. The circumferential voltage for such machines exceeds 1 GV (PETRA)
i.i.i. The shunt impedance is in the 1000 GΩ-range.

In this note single particle effects on the phase oscillation are discussed and the excitation of coherent phase-instabilities is studied.

I. Single particle effects

First we introduce a right handed coordinate system {x, s, z} as shown in fig.1



Fields:

The particles are moving in the positive s direction. We assume that the electric rf-field in a cavity is of the form:

$$E = \left\{0,\ \bar{E}_s,\ 0\right\} \sin \omega t, \tag{1}$$

where $\frac{\omega}{2\pi}$ is the radio frequency. The phase angle (above transition) is then defined by

$$\omega t = \pi - \phi \tag{2}$$

Now we make use of the fact, that $\bar{E}_s$ is not radially homogeneous. If for simplicity we consider the horizontal direction only, the field $\bar{E}_s$ can be written as:

$$\bar{E}_s = \bar{E}_{s0} + k_s x \tag{3}$$

From (1), (3) and Maxwell's equations we obtain for the magnetic field:

$$\vec{B} = \frac{1}{\omega}\left\{0,\ 0,\ \bar{k}_s\right\} \cos \omega t \tag{4}$$

and the horizontal force yields

$$F_x = \frac{ec}{\omega} \bar{k}_s \cos \omega t \tag{5}$$

This force causes a kick with an angle

$$\Theta_x = \frac{eC\ \bar{k}_s}{\omega} \frac{d}{E} \cos \omega t \tag{6}$$

where d is the length of the cavity and E is the beam energy. If there is a horizontal dispersion D_x in the cavity, the closed orbit gets lengthened by a horizontal force according

$$\frac{\Delta L_x}{L} = \Theta_x \frac{D_x}{L} \tag{7}$$

Thus we arrive at

$$\boxed{\begin{aligned} E_s &= \bar{E}_s \sin(\pi - \phi) \\ \frac{\Delta L_x}{L} &= \frac{eC\ \bar{k}_s \cdot d}{\omega E} \frac{D_x}{L} \cos(\pi - \phi) \end{aligned}} \tag{8}$$

Equations of motion:

For the description of phase motion we introduce $\psi = \phi - \phi_0$ the deviation from the equilibrium phase angle and $\varepsilon = \Delta E/E$ the relative energy deviation. Using the fields (8) the equations of motion are:

$$\dot{\eta} = A \cdot \phi + \frac{e \bar{k}_s d \cdot f_o}{E} D_x \eta \sin(\pi - \phi)$$
$$\dot{\phi} = -B\eta - \frac{B}{\alpha} \frac{ec \bar{k}_s d}{\omega E} \frac{D_x}{L} \cos(\pi - \phi) \quad (9)$$

Here α denotes the momentum compaction factor and B is defined as

$$B = \omega \cdot \alpha \quad (10)$$

The synchrotron frequency f_s is given by

$$f_s = \frac{\Omega_s}{2\pi}, \quad \Omega_s^2 = A \cdot B \quad (11)$$

and f_o is the revolution frequency, A is defined by (11). If we keep only terms linear in ϕ and η we obtain from (9)

$$\dot{\eta} = A\phi + \frac{e \bar{k}_s d f_o}{E} D_x \cdot \eta \sin \phi_o$$
$$\dot{\phi} = -B\eta - \frac{B}{\alpha} \frac{ec \bar{k}_s d}{\omega E} \frac{D_x}{L} \phi \sin \phi_o \quad (12)$$

The second terms on the right hand side in the first and second equation of (12) cause a change of the single particle damping rates. However, these two terms cancel. This is due to a general proof given by K.W. Robinson[(2)] and is established here in a special case. The "damping" term in the first equation is due to the fact that the energy gain of the particles is different for different horizontal (radial) positions.

Differentiating the second equation of (12) with respect to time one obtains from (12) a single differential equation:

$$\ddot{\phi} + (\Omega_s^2 - \Delta^2)\, \phi = o$$
$$\Delta = f_o \frac{e \bar{k}_s D_x d}{E} \sin \phi_o \quad (13)$$

Although there is no effect on the damping rates of a single particle a defocussing frequency shift remains present.

Estimate of the single particle frequency shift:

The electric field of the fundamental mode of the cavity can roughly be written as:

$$\bar{E}_s = \bar{E}_{so} \left(1 - \frac{\xi^2}{r_c^2}\right) \quad (14)$$

where ξ is the radial (horizontal) displacement and r_c is the cavity radius, approximately.

If there is a closed orbit deviation x_o with respect to the symmetry axis of the fundamental mode we obtain from (14)

$$\bar{E}_s = \bar{E}_{so} \left(1 - \frac{x_o^2}{r_c^2} - \frac{2x_o}{r_c} \frac{x}{r_c} + \ldots\right) \quad (15)$$

and

$$\bar{E}_s d = \bar{E}_{so} d \left(1 - \frac{x_o^2}{r_c^2} - g\, x\right) \quad (16)$$

$$g = \frac{2x_o}{r_c^2} \quad (17)$$

Since $\bar{E}_{so} \cdot d$ is the ordinary cavity voltage U_c one finds

$$\bar{k}_s d = g\, U_c \quad (18)$$

and (13) leads to

$$\Delta = \bar{\Delta} \sin \phi_o$$
$$\bar{\Delta} = 2 f_o \frac{U_c}{U_b} \frac{x_o D_x}{r_c^2} \quad (19)$$

$U_b \hat{=}$ beam energy in Volts.

Numerically one obtains:

DORIS:	PETRA:	PETRA (superconducting cavities):
$f_o = 10^6$ Hz	10^5 Hz	10^5 Hz
U_c = 2 MV	100 MV	1.6 GV
E_b = 2 GV	15 GV	40 GV
r_c = 20 cm	20 cm	20 cm
D_x = 2 m	10 cm	10 cm
X_o = 1 cm	1 cm	1 cm
$\bar{\Delta} \approx 1$ kHz	$\bar{\Delta} \approx 30$ Hz	$\bar{\Delta} \approx 0.2$ kHz

From these numbers one concludes that the single particle effects can be negleted even for smaller cavity radii.

II. Coherent instabilities, equations of motion

In this section Φ and η denote the coherent phase- and relative energy deviation of the bunch. The equations of coherent motion are then:

$$\boxed{\begin{aligned} \dot{\eta} &= A\,\Phi + e\,f_o\,U\,(I_b;\Phi,\eta) \\ \dot{\Phi} &= -B\,\eta - \frac{B}{\alpha}\,\frac{\Delta L_x}{L}\,(I_b;\Phi,\eta) \end{aligned}} \qquad (20)$$

Here I_b denotes the beam current, $U(I_b;\Phi,\eta)$ describes the induced voltage while $\frac{\Delta L_x}{L}(I_b;\Phi,\eta)$ describes the orbit lengthening due to the induced fields.

In order to solve (20) we keep only terms linear in Φ and η and the perturbations. Then the solutions is of the form

$$\Phi = \Phi_o\,e^{i\Omega t}\;; \qquad \eta = \eta_o\,e^{i\Omega t} \qquad (21)$$

with Ω near Ω_s.

From (20) then follows:

$$\boxed{\begin{aligned} i\Omega\eta_o &= A\,\Phi_o + f_o\,\frac{e\,U_\Phi\,(I_b)}{E}\,\Phi_o + f_o\,\frac{e\,U_\eta\,(I_b)}{E}\,\eta_o \\ i\Omega\Phi_o &= -B\,\eta_o - \frac{B}{\alpha}\,\frac{\Delta L_x}{L}{}_\Phi\,(I_b)\Phi_o - \frac{B}{\alpha}\,\frac{\Delta L_x}{L}{}_\eta\,(I_b)\eta_o \end{aligned}} \qquad (22)$$

Induced fields

The current component at radio frequency of a beam performing coherent phase oscillations is given by

$$I = I_b\left\{e^{i\,\omega t\,+\,i\,\Phi} + e^{-i\,\omega t\,-\,i\,\Phi}\right\} \qquad (23)$$

According to the linear approximation we have confined to we can write

$$I = I_b\left\{(e^{i\,\omega t} + e^{-i\,\omega t}) + i\,\Phi_o(e^{i(\omega\,+\,\Omega)t} - e^{-i(\omega\,-\,\Omega)t})\right\} \qquad (24)$$

The induced electric field at least linear in Φ_o and η_o follows from (24) and the field impedances $\mathfrak{Z}$:

$$\begin{aligned} \vec{E} = -\,I_b\Big\{&i\,\Phi_o\,\mathfrak{Z}_+\,e^{i(\omega\,+\,\Omega)t} - \mathfrak{Z}_-^*\,e^{-i(\omega\,-\,\Omega)t}) + gD_x\,\eta_o \wedge \\ &\times\,(\mathfrak{Z}_+\,e^{i(\omega\,+\,\Omega)t} + \mathfrak{Z}_-^*\,e^{-i(\omega\,-\,\Omega)t})\Big\}\left\{0,\ \bar{e}_{os} + \overset{\circ}{k}_s\,x,\ 0\right\} \end{aligned} \qquad (25)$$

where $\mathfrak{Z}_\pm = \mathfrak{Z}(\omega \pm \Omega)$.

The impedances $\mathfrak{Z}$ are related to the shunt impedances Z by

$$\begin{aligned} Z &= \mathfrak{Z}\,\bar{e}_{so}\cdot d \\ Zg &= \mathfrak{Z}\cdot\overset{\circ}{k}_s\cdot d \end{aligned} \qquad (26)$$

and from (25) we obtain:

$$U_\Phi \quad (I_b) = - i \; I_b (Z_+ - Z_-^*)$$
$$U_\eta \quad (I_b) = - I_b \; (Z_+ + Z_-^*) \; gD_x \qquad (27)$$

Using Maxwell's equation we derive an expression for the magnetic field:

$$\vec{B} = \frac{I_b}{\omega} \left\{ \Phi_o \left(\frac{\mathfrak{Z}_+}{1 + \frac{\Omega}{\omega}} e^{i(\omega + \Omega)t} + \frac{\mathfrak{Z}_-^*}{1 - \frac{\Omega}{\omega}} e^{-i(\omega - \Omega)t} \right) + gD_x \eta_o \left(\frac{\mathfrak{Z}_+}{1 + \frac{\Omega}{\omega}} e^{i(\omega + \Omega t)} - \frac{\mathfrak{Z}_-^*}{1 - \frac{\Omega}{\omega}} e^{-i(\omega - \Omega)t} \right) \right\} \left\{ 0, 0, \overset{o}{k}_s \right\} \qquad (28)$$

From this equation one derives:

$$- \frac{B}{\alpha} \frac{\Delta L_x}{L} \; \Phi \; (I_b) = -f_o \frac{I_b (Z_+ + Z_-^*)}{U_b} \; g \; D_x = B_1$$
$$- \frac{B}{\alpha} \frac{\Delta L_x}{L} \; \eta \; (I_b) = if_o \frac{I_b (Z_+ - Z_-^*)}{U_b} \; g^2 D_x^2 = B_2 \qquad (29)$$

where use has been made of $|\Omega| \ll \omega$.

If in addition to the definition of the B_i in (29) we define

$$- if_o \frac{I_b (Z_+ - Z_-^*)}{U_b} = A_1$$
$$- f_o \frac{I_b (Z_+ + Z_-^*)}{U_b} \; gD_x = A_2 \qquad (30)$$

the system (22) becomes

$$\begin{pmatrix} B_1 & B_2 - B \\ A_1 + A_1 & A_2 \end{pmatrix} \begin{pmatrix} \Phi_o \\ \eta_o \end{pmatrix} = i\Omega \begin{pmatrix} \Phi_o \\ \eta_o \end{pmatrix} \qquad (31)$$

With

$$D = \begin{vmatrix} B_1 - i\Omega & B_\alpha - B \\ A_1 + A & A_2 - i\Omega \end{vmatrix} \qquad (32)$$

we have to look for solutions of $D = o$ in Ω.

Putting $\Omega = \Omega_s + \Delta\Omega$ and keeping only linear terms in $\Delta\Omega$, A_i, B_i; $i = 1,2$ one obtains finally:

$$\Delta\Omega = \Omega_s \frac{A_1}{2A} - \Omega_s \frac{B_2}{2B} - i \frac{A_2 + B_1}{2} \qquad (33)$$

The first term of equ. (33) is well known and describes Robinson-damping (antidamping).

The second term describes an instability due to the magnetic field and corresponds to the longitudinal instabilities excited by TM_{1nm} deflecting modes[3].

The third term corresponds to a change of the coherent damping rates. In the case of coherent instabilities there is no cancellation as in the single particle case. This can be seen by the fact that the "damping terms" are of different origin in the case of coherent oscillations:
in the single particle case the lengthening term proportional to Φ comes about because of the phase dependence of the magnetic field.

In the case of coherent oscillations the lengthening term proportional to Φ comes about since the magnetic field at Φ_o is dependent on Φ due to the excitation of the cavity according to phase oscillations Φ (t).

The damping rates for the effects of an inhomogenious rf field can be derived from the corresponding imaginary parts of (33).

Instability due to magnetic field:

$$1/\tau_M = \Omega_s \frac{I_b(R_- - R_+)}{U_b} \frac{g^2 D_x^2}{4\pi\alpha q} \qquad (34)$$

Change of coherent damping:

$$1/\tau_D = f_o \frac{I_b(R_- + R_+)}{U_b} g D_x \qquad (35)$$

$R \hat{=}$ real part of shunt impedance

$q \hat{=}$ Hormonic number

$\tau \hat{=}$ rise time

From (34) follows that the beam is stable if the cavity is tuned above the radio frequency ($R_+ > R_-$), in contrast to the Robinson-Instability.

From (35) follows that the beam is stable or unstable depending on the sign of $D_x \cdot X_o$ irrespective of the tune around the radio frequency. Numerically one obtains:

	DORIS:	PETRA:
I_b	= 200 mA	160 mA
α	= 0,018	0,002
$\Omega_s/2\pi$	= 40 kHz	10 kHz
q	= 480	3840
R	= 24 MΩ	1 GΩ
U_b	= 2 GV	7,5 GV
τ_M	= 200 Msec	29 sec
τ_D	= 0.4 msec	10 msec

Calculating τ_D from (35) for a superconducting structure of 10^{12} Ω one has to emphasize that the impedance Z is a rapidly varying function of the frequency because of the high quality factor ($Q \approx 4 \cdot 10^7$). So one has to account for the fact that Z depends on the frequency shift $\Delta\Omega$:

PETRA (superconducting structure):

I_b = 40 mA

α = 0.002

$\Omega_s/2\pi$ = 10 kHz

q = 3840

R = 1000 GΩ

U_b = 7.5 GV

$\tau_M \approx$ 120 msec

$\tau_D \approx$ 1 msec

Thus also from the instability point of view there is no major danger connected with the inhomogeneities of the rf field.

The author is grateful to discussions with K.G. Steffen.

References:

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3. R.D. Kohaupt, H2-6/75, Techn. Notiz, 1975