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BETRIFFT: Potential of a three-dimensional Gaussian bunch.

The influence of one bunch on another depends on its charge distribution. Investigation of the beam-beam interaction comes into a question of finding a force or potential of a bunch. Although the expression for a potential of three-dimensional Gaussian bunch is known [1], I was not able to find it myself. So I believe that a simple way to obtain this expression is of some interest.

Let us start from the charge density of three-dimensional Gaussian bunch with center of gravity at the origin (N is number of particles in the bunch):

$$\rho(x,y,z) = \frac{N}{(2\pi)^{3/2} abc} \exp\left(-\frac{x^2}{2a^2} - \frac{y^2}{2b^2} - \frac{z^2}{2c^2}\right).$$

Here a, b, c are standard deviations along the corresponding axes.

The potential created by such a charge distribution in the point (x, y, z) is the solution of the Laplace equation with the boundary conditions:

$$V(x, y, z)\Big|_{\substack{x \to \infty, \\ \text{or } y \to \infty, \\ \text{or } z \to \infty}} \to 0$$

Instead of solving the Laplace equation let us consider diffusion equation

$$\Delta V_A - A^2 \frac{\partial V_A}{\partial t} = -4\pi\rho . \quad (t \geq 0)$$

The potential $V(x, y, z)$ we are looking for could be obtained from $V_A(x, y, z)$ simply by going to the limit $A \to 0$:

$$V = \lim_{A \to 0} V_A .$$

Although the diffusion equation seems to be more complicated then the Laplace one, the Green function of it appears to be much simpler[2]:

$$G(x,y,z\,|\,x_0,y_0,z_0\,|\,t) = \frac{A\, e^{-\frac{1}{4t}[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]}}{2\pi^{1/2} t^{3/2}} \qquad (t \geq 0).$$

The solution of diffusion equation can now be written for any time $t > 0$ in the following form:

$$V_t(x,y,z) = \int_0^t d\tau \int_{-\infty}^{\infty} G(x,y,z\,|\,x_0,y_0,z_0\,|\,\tau)\,\rho(x_0,y_0,z_0)\,dx_0\,dy_0\,dz_0$$

Performing the integration over $dx_0\,dy_0\,dz_0$ and slightly changing the independent variable ($4\tau/A^2 = q$) we get:

$$V_t(x,y,z) = \frac{Ne}{\sqrt{\pi}} \int_0^{4\pi/t} dq\, \frac{\exp\left\{-\frac{x^2}{2a^2+q} - \frac{y^2}{2b^2+q} - \frac{z^2}{2c^2+q}\right\}}{\left[(2a^2+q)(2b^2+q)(2c^2+q)\right]^{1/2}}.$$

Going now to the limit $A \to 0$ in this expression we easily get the searched potential of the three-dimensional Gaussian bunch:

$$V(x,y,z) = \frac{Ne}{\sqrt{\pi}} \int_0^{\infty} dq\, \frac{\exp\left\{-\frac{x^2}{2a^2+q} - \frac{y^2}{2b^2+q} - \frac{z^2}{2c^2+q}\right\}}{\left[(2a^2+q)(2b^2+q)(2c^2+q)\right]^{1/2}}.$$

Literature:

1. Houssais
2. Morse & Feshbach, "Methods of Theoretical Physics", Part 1, p. 793 (1953)