A

Interner Bericht
DESY T-71/1
February 1971

Lectures on
Lagrangian Quantum Field Theory

by

K. Symanzik

Eigentum der DESY Bibliothek
Property of DESY library
Zugang: 13. März 2006
Accessions:
Keine Ausleihe
Not for loan

Lectures on

Lagrangian Quantum Field Theory

by

K. Symanzik

delivered in January 1968 at the
University of Islamabad, Pakistan,
under the auspices of the Ford Foundation.
Revised and edited by H.-J. Thun

Preface

These lecture notes give an almost self-contained exposition of relativistic quantum field theory from a relatively modern but still pragmatic standpoint. They lead from the principles of canonical quantization over some stages to the renormalization of quantum electrodynamics and of neutral vector meson theory. However, many topics of practical interest are omitted, e.g. discrete symmetry operations, calculation of cross sections, and in particular discussion and computation of Feynman integrals. The reader will find the material of these lectures more easily comprehensible if he has some pre-knowledge of QFT such as transmitted in introductory courses. He is referred to the standard textbooks, e.g. those given on page 182.

Acknowledgement

The lecturer thanks Professor Riazuddin and the Ford Foundation for the invitation to lecture at the University of Islamabad, and is in particular indebted to the Staff of the Ford Foundation in Pakistan for their cordial hospitality on that occasion. He thanks Dipl.Phys. H.-J. Thun who revised and edited the notes and to whom sect. 8.1 - 8.4 in their present form are entirely due.

TABLE OF CONTENTS

1. Canonical Formalism

In this lecture we derive the canonical commutation relations from their analogy to Poisson brackets. Hereby we shall find that the elementary rule

$$[q, \frac{\partial}{\partial \dot{q}} L]_{\mp} = i\hbar \tag{1.1}$$

does not always hold. However, in any case the Peierls- and the closely related Schwinger principle, which we shall explain, allow to circumvent the complexities of the elementary quantization approach.

1.1 Commutation Relations

We shall only consider Lagrangians which are linear in time-derivatives and have the simple form (cf. e.g. [1])

$$L(\dot{q}, q, t) = q\, G\, \dot{q} - H(q, t) \tag{1.2}$$

where all variables are combined into a real N-component variable q, G is a constant real $N \times N$ matrix, and H is a real function of q and t. (For the time being we consider (1.2) classically.) We call the Lagrangian in (1.2) a first order Lagrangian and speak of a first order formulation in contrast to a second (or higher) order one where Lagrangians quadratic (or higher non-linear) in time-derivatives are used. The reasons for the above choice are

1. theories with second order Lagrangians can be formulated in terms of such first order Lagrangians and
2. this form appears in quantum field theory (QFT), e.g. for the Dirac field, and the usual Bose field Lagrangians can be brought into this form (cf. Lecture 3).

With the help of the antisymmetric, imaginary and hence Hermitean matrix

$$A := -\, i(G - G^T) \tag{1.3}$$

the Lagrangian (1.2) can be written in a more symmetric and manifestly Hermitean form

$$L = \frac{i}{4}(q\, A\, \dot{q} - \dot{q}\, A\, q) - H(q, t) \tag{1.4}$$

where only a time derivative part has been omitted which, however, does not affect the equations of motion. From the variational principle we obtain the Lagrange equations

$$i\,A\,\dot{q} - H_q(q,t) = 0 \tag{1.5}$$

where

$$H_q = \frac{\partial H}{\partial q} \quad .$$

(We are suppressing indices everywhere.)

In the following we try to solve the Lagrange equations in various cases.

1.1.1 Poisson Brackets and Commutators

Assume first that A is non-singular. Then

$$i\,\dot{q} = A^{-1}\,H_q \quad . \tag{1.6}$$

Since from (1.3) N in this case must be even, i.e. $N = 2n$, A can be written as

$$A = -\,i\,U^T\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U \tag{1.7}$$

with U a real non-singular matrix. Each block in $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is an $n \times n$ matrix.

Exercise:

Prove this from more familiar matrix theorems (cf. e.g. [2]).

Denoting the column vector Uq by $\begin{pmatrix} P \\ Q \end{pmatrix}$ where P and Q are both n-component column vectors, (1.4) becomes

$$L = \tfrac{1}{2}(P\dot{Q} - Q\dot{P}) - H(P,Q,t) \quad . \tag{1.8}$$

Exercise:

Under what conditions can this L be considered to be derived in the familiar manner from a higher order $L(\dot{Q},Q,t)$? (Answer: The $n \times n$ matrix H_{PP} must be nonsingular. More general H are allowed, if they are reducible to this case by change of variables.)

We now define the Poisson bracket of two functions X and Y to be

$$(X,Y) := -\,i\,X\,\overleftarrow{\partial}_q\,A^{-1}\,\overrightarrow{\partial}_q\,Y \; . \tag{1.9}$$

It has the following properties:

i. $(X,Y) = -(Y,X)$ (anti-symmetry) (1.10)

ii. $((X,Y),Z) + ((Y,Z),X) + ((Z,X),Y)$ (Jacobi identity) (1.11)

iii. invariance under canonical transformations, i.e. an infinitesimal change $q' = q + \varepsilon(q,W)$ implies

$$(X,Y)' = -i X \overleftarrow{\partial}_{q'} A^{-1} \overrightarrow{\partial}_{q'} Y = (X,Y) \text{ up to } O(\varepsilon^2) \tag{1.12}$$

<u>Remark</u>: For (1.10...12) to hold A need not be q-independent.

In terms of the above Poisson brackets (1.6) becomes

$$\dot{q} = (q,H) \quad .$$

The listed properties allow, as discussed e.g. in Dirac's book, [4] to elevate these Poisson brackets to the status of quantum-mechanical commutators

$$(X,Y) \to -\frac{i}{h}[X,Y] \ . \tag{1.13}$$

(Throughout these lectures we will always use the Heisenberg representation. The commutator $[X,Y]$ is an equal-time commutator (ETC) of Heisenberg operators.) In particular we have

$$[q,q] = \hbar A^{-1} \ . \tag{1.14}$$

<u>Exercise</u>:
Choose G in (1.2) in the form $\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}$, g nonsingular, and all submatrices square matrices. Calculate A and A^{-1} and show that this quantization is identical with the elementary rule (1.1) which here applies only when L is in this particular form at least after relabelling of variables.

<u>Remark</u>: Contrary to superficial appearence, there is no isomorphism between the CM Poisson brackets and the QM commutators whenever these are considered between polynomials of order higher than two (otherwise QM would not differ essentially from CM). However, in special cases this correspondence may hold beyond. An important example hereto are the structure relations of generators of groups

under which the variables transform linearly. This case will be discussed in connection with field theory in Lecture 3.

1.1.2 Solvable Constraints

In QFT N is countably infinite. However this makes no difference at this formal level. A is, however, in QFT almost always singular. This means that for some q-components (or more generally, for certain linear combinations of them) there are no equations of motion. In such cases A can be written in the form

$$A = - i\, U^T \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U \tag{1.15}$$

where again U is a real nonsingular matrix (cf. e.g. [3]). With $Uq =: \begin{pmatrix} P \\ Q \\ R \end{pmatrix}$ we call the R-variables "dependent" ones. The Lagrangian (1.2) then takes the form

$$L = P\dot{Q} - H(P,Q,R,t) \quad . \tag{1.16}$$

The quations of motion are

$$\dot{Q} = H_P \quad , \tag{1.17}$$

$$\dot{P} = - H_Q \quad , \tag{1.18}$$

and

$$H_R = 0 \tag{1.19}$$

are the constraint equations, or to be more specific, primary constraints.

Suppose H_{RR} to be non-singular. Then we can immediately solve the primary constraints (1.19) in the form

$$R = R(P,Q,t) \tag{1.20}$$

and inserting it in (1.16) leads due to (1.19) again to the correct equations of motion. The commutators become, therefore,

$$\begin{aligned} &[Q,Q] = [P,P] = 0 \quad , \\ &[Q,P] = i\hbar\, 1 \quad . \end{aligned} \tag{1.21}$$

Commutators involving R are to be obtained from these and (1.20).
We now formulate this result without going over to the normal form (1.16). Let P be the real symmetric projection operator on the zero space of A and $\bar{P} = 1 - P$. Then

$$PA = AP = 0 \ , \quad P^2 = P \quad . \tag{1.22}$$

From (1.5) one now obtains the equations of motion

$$i \, \bar{P} \, A\bar{P} \, \bar{P}\dot{q} - \bar{P}H_q = 0 \tag{1.23}$$

and the primary constraints

$$PH_q = 0 \quad . \tag{1.24}$$

If now $PH_{qq}P$ is non-singular in the subspace on which P projects, then (1.24) can be solved for Pq in terms of $\bar{P}q$. The commutator (1.21) is found to be

$$[\bar{P}q, q\bar{P}] = \hbar \, (\bar{P}A\bar{P})^{-1} \tag{1.25}$$

or equivalently

$$[Aq, qA] = \hbar \, A \quad . \tag{1.26}$$

Indeed due to (1.24) we have

$$\begin{aligned} \bar{P} \, \dot{q} &= - \frac{i}{\hbar}[\bar{P}q, H] = - \frac{i}{\hbar}[\bar{P}q, q]H_q \\ &= - \frac{i}{\hbar}[\bar{P}q, q\bar{P}] \, \bar{P} \, H_q - \frac{i}{\hbar}[\bar{P}q, qP] \, PH_q = - \, i(\bar{P}A\bar{P})^{-1} \, \bar{P}H_q \end{aligned} \tag{1.27}$$

which coincides with (1.23).

The above argument as it stands is correct in QM only if $[\bar{P}q, qP]$ is a c-number since otherwise we are confronted with the problem of ordering the operators in this calculation. To my knowledge there does not exist a really completely general treatment of this problem. It turns out, however, that the Lagrangian QFT systems that can be handled at least in perturbation theory are necessarily so simple that they do not offer trouble from this source.

Exercise:

Derive the commutator of the dependent variables with the Hamiltonian:

$$[\mathcal{P}q,H] = -\hbar(\mathcal{P}\, H_{qq}\, \mathcal{P})^{-1}\, \mathcal{P}H_{qq}\bar{\mathcal{P}}(\bar{\mathcal{P}}\, A\, \bar{\mathcal{P}})^{-1}\, \bar{\mathcal{P}}H_{q} \quad . \tag{1.28}$$

Differentiating (1.24) w.r.t. time gives

$$\mathcal{P}H_{qq}\mathcal{P}\,\mathcal{P}\dot{q} + \mathcal{P}H_{qq}\bar{\mathcal{P}}\,\bar{\mathcal{P}}\dot{q} + \mathcal{P}H_{qt} = 0 \tag{1.29}$$

which can be solved for $\mathcal{P}\dot{q}$ giving

$$\mathcal{P}\dot{q} = -(\mathcal{P}H_{qq}\mathcal{P})^{-1}[-i\ \mathcal{P}H_{qq}\bar{\mathcal{P}}(\bar{\mathcal{P}}\, A\, \bar{\mathcal{P}})^{-1}\, \bar{\mathcal{P}}H_{q} + \mathcal{P}H_{qt}] \quad . \tag{1.30}$$

Using (1.28) this can also be written as

$$\mathcal{P}\dot{q} + \frac{i}{\hbar}[\mathcal{P}q,H] + (\mathcal{P}H_{qq}\mathcal{P})^{-1}\, \mathcal{P}H_{qt} = 0 \quad . \tag{1.31}$$

The term $-(\mathcal{P}H_{qq}\mathcal{P})^{-1}\ \mathcal{P}H_{qt}$ is equal to $\frac{\partial}{\partial t}(\mathcal{P}q)$ if $\mathcal{P}q$ is written as (in general t-dependent) function of $\bar{\mathcal{P}}q$, since from (1.24) follows

$$\mathcal{P}H_{qq}\mathcal{P}\ \frac{\partial q}{\partial t} + \mathcal{P}H_{qt} = 0 \quad . \tag{1.32}$$

In a theory invariant under time translation, or what is the same, with conserved H , $H_{qt} = 0$ and the Heisenberg equation

$$\dot{q} = -\frac{i}{\hbar}[q,H] \tag{1.33}$$

holds for all variables (and for their time-independent functions). There is however good reason to consider more general theories as we shall see.

1.1.3 Higher Order Constraints

Suppose now that $\mathcal{P}H_{qq}\mathcal{P}$ from (1.25) is singular, i.e. (1.24) cannot be solved for the dependent variables $\mathcal{P}q$. Then there exists another projection operator $\mathcal{Q}$ which projects on the zero space of $\mathcal{P}H_{qq}\mathcal{P}$:

$$\mathcal{Q}\mathcal{P}H_{qq}\mathcal{P} = \mathcal{P}H_{qq}\mathcal{P}\mathcal{Q} = 0, \quad \mathcal{Q}^2 = \mathcal{Q}, \quad \bar{\mathcal{Q}} = 1 - \mathcal{Q} \quad ,$$

such that from (1.29) and (1.23) further relations can be produced which do not contain time-derivatives of the variables

$$- i\, \mathcal{Q}\, \mathcal{P}H_{qq}\bar{\mathcal{P}}(\bar{\mathcal{P}}\, A\, \bar{\mathcal{P}})^{-1}\, \bar{\mathcal{P}}H_q + \mathcal{Q}\mathcal{P}H_{qt} = 0 \tag{1.34}$$

called secondary constraints.

For shortcutting the discussion let us assume $\bar{\mathcal{P}}H_q$ to be linear in the variables $\mathcal{Q}\mathcal{P}q$, e.g. H to be of the form

$$H = \frac{1}{2}\, q\, B\, q + H'(\bar{\mathcal{P}}q, \bar{\mathcal{Q}}\mathcal{P}q) + H''(\bar{\mathcal{P}}q, \bar{\mathcal{Q}}\mathcal{P}q)\mathcal{Q}\mathcal{P}q \tag{1.35}$$

where the real Hermitean matrix B is time-independent. Then the secondary constraints (1.34) can be solved for the variables $\mathcal{Q}\mathcal{P}q$ provided $V := \mathcal{Q}\mathcal{P}\, H_{qq}\, \bar{\mathcal{P}}(\bar{\mathcal{P}}\, A\, \bar{\mathcal{P}})^{-1}\bar{\mathcal{P}}H_{qq}\mathcal{P}\mathcal{Q}$ is nonsingular. However, if secondary or higher order constraints exist, then the commutation relations are no longer of the simple form (1.26). In the case of (1.35) and nonsingular V one finds (most easily from Peierls' principle which will be discussed later)

$$\frac{1}{\hbar}[Aq, qA] = A - \bar{\mathcal{P}}H_{qq}\mathcal{P}\mathcal{Q}V^{-1}\mathcal{Q}\mathcal{P}H_{qq}\bar{\mathcal{P}} \quad , \tag{1.36}$$

i.e. there is a modification of (1.26) which generally will not be a c-number term.

If the secondary constraints are not solvable for $\mathcal{Q}\mathcal{P}q$ then by further differentiation w.r.t. time tertiary constraints can be found and so on. We will not spend more time on this problem, however, since we shall not need it in QFT as far as we discuss it.

There are interesting Lagrangians such that the development in time of some variables cannot be determined from the Lagrange equations. This occurs if the Lagrangian is invariant under a group of transformations (e.g. gauge transformations of the second kind) with space-time arbitrariness. Such "gauge cases" are most conveniently dealt with by giving up the dynamically redundant invariance, as we shall show for QED in Lectures 5 and 6.

1.2 Anticommuting Variables

So far we have been discussing commuting variables only. Equally important are anticommuting ones which we shall treat now briefly. Classically, we consider

N anticommuting variables q_i such that $\{q_i,q_j\} = 0$ for all i,j. Their polynomials form a "Grassmann algebra" with N generators and degree (number of independent elements) 2^N. There exist simple $2^N \times 2^N$ matrix representations of such algebras [4]. We call an element of this algebra "even" or "odd" according to whether it commutes or anticommutes with all q_i. In other words it can be expressed as a polynomial with all terms containing either an even or an odd number of q's.

Now the whole discussion up to (1.36) can be repeated for the case of anticommuting variables, but we will write down only the main formulas. Because of the anticommutivity of the variables the antisymmetric matrix A of (1.4) has to be replaced by the real symmetric and hence Hermitean matrix

$$S := -i(G + G^T) \tag{1.37}$$

(G in (1.2) has now to be taken imaginary)

and the equations of motion are instead of (1.6) in the case of nonsingular S

$$i\,\dot{q} = S^{-1}\,\partial_q^L H = -\,\partial_q^R H\,S^{-1} \tag{1.38}$$

where ∂_q^L and ∂_q^R are "left (right) derivatives" which means that we have to reorder all terms such that the q w.r.t. which we differentiate appears always on the left (right) and is taken away from there. For the variation of an "even" quantity, e.g. the Hamiltonian H, therefore the relation

$$\delta H = \delta q\,\partial_q^L H = +\,\partial_q^R H\,\delta q = -\,\delta q\,\partial_q^R H$$

holds.

The matrix S can be written in the form

$$S = U^T \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} U$$

where U is real and nonsingular and the matrix $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ has n 1's and $N-n$ (-1)'s along its diagonal. Writing $Uq =: \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$ we obtain the normal form of the Lagrangian

$$L = \frac{i}{2}(Q_1\dot{Q}_1 - Q_2\dot{Q}_2) - H(Q_1,Q_2,t) \quad . \tag{1.39}$$

The Poisson bracket now reads

$$(X,Y) := -i\, X \overleftarrow{\partial}^{R}_{q} S^{-1} \overrightarrow{\partial}^{L}_{q}\, Y$$

and has the property of being antisymmetric if at least one of X,Y is even, whereas it is symmetric if both X and Y are odd. For a quantum mechanical system we have the correspondence

$$(X,Y) \to -\frac{i}{\hbar}[X,Y]$$

if at least one of the factors X,Y is even, but

$$(X,Y) \to -\frac{i}{\hbar}\,\{X,Y\}$$

if both are odd. In particular for Fermi variables we have

$$\{q,q\} = \hbar\, S^{-1}$$

instead of (1.14) which becomes in "normal form" (1.39)

$$\{Q_1,Q_1\} = -\{Q_2,Q_2\} = \hbar\, 1\ ,$$

$$\{Q_1,Q_2\} = 0$$

such that, if the metric in the QM state space is to be positive, i.e. we have a Hilbert space, there must be no Q_2-variables, such that the matrix S must be positive definite, since we supposed the q and therefore the Q to be Hermitean.

Since the case of dependent anticommuting variables offers little new compared to commuting ones and, moreoever, does not occur in our applications, we leave its discussion as an exercise.

Sometimes it is convenient to use non-real (i.e. in QM non-Hermitean) variables. Then one starts from

$$L = \frac{i}{2}\,(a^{+}B\dot{a} - \dot{a}^{+}Ba) - H(a^{+},a,t) \tag{1.40}$$

and must for reality (hermiticity) of L have $B = B^{+}$. There are then two ways to proceed. Either we take

$$a_1 = \frac{a + a^+}{\sqrt{2}} \quad . \qquad a_2 = \frac{a - a^+}{i\sqrt{2}} \tag{1.41}$$

as the Hermitean combinations and the problem reduces by substitution to the previous case. More economically, we consider a and a^+ as independent for variational purposes.

One finds now, when B is non-singular,

$$\begin{aligned} &[a,a^+]_{\mp} = \hbar\, B^{-1} \\ &[a\, ,\, a]_{\mp} = [a^+,a^+]_{\mp} = 0 \end{aligned} \tag{1.42}$$

($[\ ,\ ]_-$ = commutator, $[\ ,\ ]_+$ = anticommutator, upper sign corresponding to Bose variables, lower to Fermi variables.

In the case of solvable primary constraints we obtain

$$\begin{aligned} &[Ba,a^+B]_{\mp} = \hbar\, B \\ &[Ba,aB^T]_{\mp} = [B^Ta^+,\ a^+B]_{\mp} = 0 \quad . \end{aligned} \tag{1.43}$$

1.3 Variational Formulas

Next we recast the canonical quantization prescription in a more flexible form, as is essential for a (on this level) easy treatment of QFT .

1.3.1 Schwinger's Action Principle

Our starting point are the Heisenberg equations (cf. (1.31,33))

$$\bar{P}\,\dot{q} = -\frac{i}{\hbar}[\bar{P}q,H] \tag{1.44}$$

$$P\,\dot{q} = -\frac{i}{\hbar}[Pq,H] + \frac{\partial Pq}{\partial t} \tag{1.45}$$

for independent and dependent variables, respectively. We will now express H in terms of only the independent variables as

$$H(\bar{P}q,Pq(\bar{P}q,t),t) =: \hat{H}(\bar{P}q,t) \tag{1.46}$$

and (1.44) will be abbreviated as

$$\frac{dq(t)}{dt} = -\frac{i}{\hbar}[q(t),\hat{H}(q(t),t)] \tag{1.47}$$

by suppressing $\bar{P}$ everywhere. Now (1.47) is solved by

$$q(t) = U(t,t_o)q(t_o)U(t,t_o)^{-1} \tag{1.48}$$

where the unitary operator $U(t,t_o)$ satisfies

$$\frac{d}{dt}U(t,t_o) = \frac{i}{\hbar}\hat{H}(q(t),t)U(t,t_o) = \frac{i}{\hbar}U(t,t_o)\hat{H}(q(t_o),t) \tag{1.49}$$

and the initial condition

$$U(t_o,t_o) = 1 \quad . \tag{1.50}$$

Exercise:

Prove the unitarity of U from (1.49,50).

We are interested in the scalar product of a pair of states

$$|\phi_1 \, , \, t_1\rangle \quad , \quad |\phi_2 \, , \, t_2\rangle$$

which are defined by measurement or preparation operations at times t_1 and t_2, respectively, e.g. as eigenstates to a complete commuting subset of the operator family $q(t_1)$ or $q(t_2)$, the eigenvalues being represented by ϕ_1, ϕ_2 . Using the relation

$$|\phi_i,t_i\rangle = U(t_i,t_o)|\phi_i,t_i\rangle_o \tag{1.51}$$

where $|\phi_i,t_i\rangle_o$ means that we have replaced t_i in the argument of q implicit in the specification of $|\phi_i,t_i\rangle$ by t_o but have otherwise not changed the specification prescription, we have

$$\langle\phi_1,t_1|\phi_2,t_2\rangle = {}_o\langle\phi_1,t_1|U(t_1,t_o)^{-1}U(t_2,t_o)|\phi_2,t_2\rangle_o \quad . \tag{1.52}$$

We consider a family of different dynamics by taking $H = H(q,t,\lambda)$ to depend on parameters λ . For variations $\delta_\lambda\hat{H}(q,t,\lambda) = \hat{H}(q,t,\lambda+\delta\lambda) - \hat{H}(q,t,\lambda)$ i.e. taking into account only the explicit dependence of $\hat{H}$ while keeping the (independent) q fixed, we find from (1.49,50) to first order in infinitesimals

$$\frac{d}{dt}[\delta_\lambda U(t,t_o)U(t,t_o)^{-1}] = \frac{i}{\hbar} U(t,t_o)\delta_\lambda \hat{H}(q(t_o),t,\lambda)\Big|_{q(t_o)\ \text{fixed}} U(t,t_o)^{-1}$$

which gives upon integration

$$\delta_\lambda U(t,t_o) = \frac{i}{\hbar} \int_{t_o}^{t} \delta_\lambda \hat{H}(q(\tau),\tau,\lambda)\Big|_{q(\tau)\ \text{fixed}} d\tau\ U(t,t_o) \quad . \tag{1.53}$$

Using the identity $U(t,t_o)U(t,t_o)^{-1} = 1$ we therefore have

$$\delta_\lambda (U(t_1,t_o)^{-1}\, U(t_2,t_o)) = U(t_1,t_o)^{-1}(-\frac{i}{\hbar}) \int_{t_2}^{t_1} \delta_\lambda \hat{H}(q(\tau),\tau,\lambda)\Big|_{q(\tau)\ \text{fixed}} d\tau\ U(t_2,t_o)$$

such that the variation of (1.52) is given by

$$\delta_\lambda \langle\phi_1,t_1|\phi_2,t_2\rangle = -\frac{i}{\hbar} \langle\phi_1,t_1| \int_{t_2}^{t_1} \delta_\lambda \hat{H}(q(\tau),\tau,\lambda)\Big|_{q(\tau)\ \text{fixed}} d\tau\ |\phi_2,t_2\rangle \tag{1.54}$$

provided the specification of the states in terms of the independent operators at times t_1 , t_2 is not changed. All reference to t_o has disappeared. One only must keep in mind that at some time t_o the independent operators were kept fixed and the change of all operators at other times (and, if it applies, of the dependent operators at t_o) is defined through changed Lagrange equations. The arbitrariness of t_o reflects the freedom of similarity transformations in the representation of states and operators in QM in the calculation of amplitudes and probabilities.

We now calculate $\delta_\lambda \hat{H}(q(\tau),\tau,\lambda)\Big|_{q(\tau)\ \text{fixed}}$. This is, by (1.46),

$$\delta_\lambda H(\bar{P}q(\tau),\ Pq(\bar{P}q(\tau),\tau,\lambda),\tau,\lambda)\Big|_{\bar{P}q(\tau)\ \text{fixed}} =$$

$$= \delta_\lambda H(\bar{P}q(\tau),Pq(\bar{P}q(\tau),\tau,\lambda),\tau,\lambda)\Big|_{\bar{P}q,\ Pq\ \text{fixed}} + \tag{1.55}$$

$$+ H_q(\bar{P}q(\tau),Pq(\bar{P}q(\tau),\tau,\lambda)P\ \delta_\lambda Pq(\bar{P}q(\tau),\tau,\lambda)\Big|_{\bar{P}q\ \text{fixed}} .$$

The first term arises on account of the parameter dependence of H, the second since the solution of the constraint equation gives parameter dependence in $Pq(\bar{P}q(\tau),\tau,\lambda)$. However, due to (1.24) it vanishes. Because of our form (1.2) of the Lagrangian, with G taken as not varied, we have from (1.54,55) the simple result

$$\delta_\lambda\langle\phi_1,t_1|\phi_2,t_2\rangle = \frac{i}{\hbar}\langle\phi_1,t_2|\int_{t_2}^{t_1}\delta_\lambda L(\dot{q}(\tau),\ q(\tau),\tau,\lambda)\Big|_{\dot{q},q\ \text{fixed}} d\tau|\phi_2,t_2\rangle \tag{1.56}$$

which formula is known as the "Schwinger action principle" [5]. We have derived it only for our Lagrangian and the case of solvable constraints and up to possibly an ordering problem in QM (in the treatment of the second term in (1.55)) if dependent variables occur. But this formula actually is more generally valid. The reason why the generalization is not so easy to prove is that the higher order constraint equations mostly are very involved. The above derivation is adapted from C.S. Lam [6].

There is good reason why one should not take the Schwinger action principle as fundamental or primary and deduce the canonical formulas from it. We just showed it to be a consequence of the canonical quantization prescription. Schwinger himself in a series of papers [7] has expressed the view that one should not postulate the quantum action principle but derive it. In these papers he derived it from a sort of Feynman path or history formulation of QM or of QFT (cf. e.g. Dyson [5]). This derivation is, however, essentially void of mathematical meaning and succeeds even formally only in certain simple cases. For our applications of (1.56) it is adequate to consider it as an elegant and (as we shall see) most useful rewriting of canonical quantization at least for those cases where the latter is sufficiently established. As we have seen above, the formula does actually not necessarily take care of specifically QM fine points - in short, it is a "classical" formula.

Exercise:

Write out all steps that led to (1.56).

The Schwinger action principle can easily be generalized to the variation of matrix elements of arbitrary operators $F(q(t),t,\lambda)$ depending on parameters λ. Admitting the variation of the parameters in F as well as in H one finds

$$\begin{aligned} \delta_\lambda \langle\phi_1,t_1|F(q(t),t,\lambda)|\phi_2,t_2\rangle &= \langle\phi_1,t_1|\delta_\lambda F(q(t),t,\lambda)|\phi_2,t_2\rangle + \\ &+ \langle\phi_1,t_1|\frac{i}{\hbar}\int_t^{t_1} \delta_\lambda L(\dot q(\tau),q(\tau),\tau,\lambda)\Big|_{\dot q(\tau),q(\tau)\ \text{fixed}} d\tau\, F(q(t),t,\lambda)|\phi_2,t_2\rangle + \\ &+ \langle\phi_1,t_1|F(q(t),t,\lambda)\,\frac{i}{\hbar}\int_{t_2}^{t} \delta_\lambda L(\dot q(\tau),q(\tau),\tau,\lambda)\Big|_{\dot q(\tau),q(\tau)\ \text{fixed}} d\tau|\phi_2,t_2\rangle \end{aligned} \tag{1.57}$$

which is obtained by introducing two complete sets of intermediate states ψ,ψ' specified at time t such that the term

$$\langle\psi,t|F(q(t),t,\lambda)|\psi',t\rangle = \langle\psi,t_o|F(q(t_o),t,\lambda)|\psi',t_o\rangle$$

does not depend on the dynamics.
The generalization of formula (1.57) to a product of operator expression at different times is obvious.

1.3.2 Peierls' Formula

Specializing (1.57) to $t_1 = t_2 = \mp\infty$ gives (suppressing λ from now on)

$$\delta^{\substack{\text{ret}\\ \text{adv}}} F(q(t),t) = \delta F(q(t),t)\Big|_{q\ \text{fixed}} + \frac{i}{\hbar}\int_{\mp\infty}^{t} [F(q(t),t),\delta L(\dot q(\tau),q(\tau),\tau)\Big|_{\dot q,q\ \text{fixed}}]d\tau$$

where "retarded" or "advanced" means that we integrate the infinitesimally varied equation of motion from zero variation at $t = \mp \infty$ forward resp. backward in time. By subtraction of the two terms we obtain the important Peierls' formula [8],[5]

$$(\delta^{ret} - \delta^{adv})F(q(t),t) = \frac{i}{\hbar}\int_{-\infty}^{\infty} [F(q(t),t), \delta L(\dot{q}(\tau),q(\tau),\tau)\Big|_{\dot{q},q \text{ fixed}}]d\tau \qquad (1.58)$$

for determining commutators. In the case of free fields, i.e. when the Lagrangian is only bilinear, (1.58) gives immediately the commutators at all times in a convenient form without need of going over the clumsy p,q-method.

Exercise:

Derive again the commutation relations given before, and the commutation relations at all times, if in (1.4)

$$H(q,t,\lambda) = \frac{1}{2}\, q\, B\, q - q\lambda(t)$$

where B is a real symmetric matrix in the Bose case, resp. imaginary and antisymmetric in the Fermi case, and $\lambda(t)$ a commuting respectively anticommuting "c-number" function of time which is subjected to variation.

2. Relativistic Invariance

2.1 Notation

Latin indices take the values 1,2,3 Greek indices take the values 0,1,2,3. Summation convention is used throughout.

$$x^\mu = (x^o, \vec{x}) = (x^o, x^i) ,$$

i.e. in transition to non-covariant notation take as space components the 1,2,3 components of a contravariant vector. The metric tensor is

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} .$$

Throughout we shall take $\hbar = c = 1$,

$$\partial_\mu = \left(\frac{\partial}{\partial x^o}, \vec{\nabla}\right) = \left(\frac{\partial}{\partial x^o}, \frac{\partial}{\partial x^i}\right) \quad ,$$

$$\vec{x} \cdot \vec{p} = x^i p^i = - x_i p^i \quad ,$$

$$(\vec{x} \times \vec{y})^i = \varepsilon_{ijk} x^j y^k = - \varepsilon^i{}_{jk} x^j y^k = \varepsilon_{oijk} x^j y^k$$

where

$$\varepsilon_{0123} = - \varepsilon^{0123} = 1 \; , \qquad \varepsilon_{123} = 1 \; ,$$

both tensors being totally antisymmetric.

$$\partial_\mu x^\nu = g_\mu{}^\nu = \delta_{\mu\nu} \quad . \qquad \text{(Kronecker symbol)}$$

2.2 Lorentz Transformations

A Lorentz transformation is a real, linear, homogeneous transformation of x^μ

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{2.1}$$

abbreviated as $x' = \Lambda x$ that leaves $x^\mu x_\mu = x^\mu g_{\mu\nu} x^\nu$ invariant. Therefrom follows that the matrix Λ must satisfy

$$\Lambda^T g \Lambda = g \; . \tag{2.2}$$

The matrix Λ^{-1} (for brevity we identify matrices and transformations) is the inverse transformation which is also a Lorentz transformation. Further Λ, Λ' being Lorentz transformations implies that $\Lambda \Lambda'$ is such a transformation, too. Thus the Lorentz transformations form a group L . From (2.2) follows

$$(\det \Lambda)^2 = 1 \; , \quad \text{i.e.} \quad \det \Lambda = \pm 1 \tag{2.3}$$

and

$$(\Lambda^o{}_o)^2 - \sum_1^3{}_i \, (\Lambda^i{}_o)^2 = 1 \tag{2.4}$$

implying $|\Lambda^o{}_o| \geqslant 1$.

We will concern ourselves only with the matrices with $\det\Lambda = +1$, $\Lambda^0{}_0 \geq 1$ which form an invariant subgroup $L_+^\uparrow$ of the Lorentz group. The factor group $L/L_+^\uparrow$ has four elements and is isomorphic to the group of space and time reflections.

Due to $g^2 = 1$ also $\Lambda g \Lambda^T = g$ holds. The metric tensor g also is used to lower or raise indices according to

$$\Lambda_{\mu\nu} = g_{\mu\varkappa}\Lambda^{\varkappa}{}_{\nu} \quad , \Lambda^{\mu\nu} = \Lambda^{\mu}{}_{\varkappa} g^{\varkappa\nu} \quad , \Lambda_{\mu}{}^{\nu} = g_{\mu\varkappa} g^{\nu\lambda} \Lambda^{\varkappa}{}_{\lambda}$$

such that e.g.

$$(\Lambda^{-1})^{\mu}{}_{\nu} = (g\Lambda^T g)^{\mu}{}_{\nu} = g^{\mu\rho}\Lambda^{\sigma}{}_{\rho}\, g_{\sigma\nu} = \Lambda_{\nu}{}^{\mu}.$$

2.3 Poincarè Transformations

An inhomogeneous Lorentz transformation, also called Poincaré transformation, is a linear inhomogeneous transformation that leaves $(x-y)^{\mu}(x-y)_{\mu}$ invariant. Thus

$$x^{\mu} \rightarrow x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu} \tag{2.5}$$

under an inhomogeneous Lorentz transformation denoted by (Λ, a) where $\Lambda \in L$. Sequences of such transformations are computed by

$$(\Lambda'', a'')(\Lambda', a') = (\Lambda''\Lambda', \Lambda'' a' + a'') . \tag{2.6}$$

Finally we have

$$(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1} a). \tag{2.7}$$

Thus these transformations form a group called the Poincaré group. In terms of matrices they are represented by

$$(\Lambda, a) \longrightarrow \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix} . \tag{2.8}$$

For the Poincaré transformations on states in quantum mechanics we take the active point of view: Let ϕ be a Heisenberg state characterized e.g. by a set of measurements at some spacetime points x_i with (generally merely statistical) outcomes M_i. Then we call ϕ' the state that is characterized by analogous measurements with the same outcomes M_i at the points $x_i' = \Lambda x_i + a$ and write it after Wigner $\phi' = U(\Lambda, a)\phi$ with U unitary corresponding to the fact that

Λ is an orthochronous Lorentz transformation and in particular a proper one (we did restrict ourselves to $L_{+}^{\uparrow}$).
By "analogous" above is meant that any tensorial quantity is to be measured with respect to a reference system as moved along unter the Lorentz transformation. It is only then that we can expect numerically unchanged results.

From the theory of the rotation group we know that the group SO(3) is doubly connected. One can construct its universal covering group SU(2) and to each rotation O in SO(3) there are associated two matrices in SU(2), U and -U. If we consider the covering group then the multiplication law is

$$U(U')U(U'') = U(U'U'') \tag{2.9}$$

against merely

$$U(O')U(O'') = \pm U(O'O'') \tag{2.10}$$

for the double-valued i.e. spinorial representations of SO(3). The same holds for the inhomogeneization iSO(3) of the rotation group, the Euclidean group. Likewise the Lorentz group and the Poincaré group are doubly connected and one goes from $SO^{\uparrow}(1,3)$ to its (simply connected) universal covering group SL(2,C), the elements of which are denoted by A. Again to each $\Lambda \in SO^{\uparrow}(1,3)$ there are associated two elements A and -A of SL(2,C). The matrices A which form the representation of the group SL(2,C) by these matrices themselves are the starting point of spinor calculus.
For a sequence of Poincaré transformations we have according to (2.6)

$$U(A',a)U(A'',a'') = U(A'A'',a' + \Lambda(A')a''). \tag{2.11}$$

2.3.1 Generators

The generators of the Poincaré group are

$$P^{\mu} = - i \frac{\partial}{\partial a_{\mu}} U(1,a)\Big|_{a=0} \tag{2.12}$$

and

$$M^{\mu\nu} = i \frac{\partial}{\partial \omega_{\mu\nu}} U(\Lambda,0)\Big|_{\Lambda=1} \quad . \tag{2.13}$$

From (2.2) we deduce for a Λ close to identity

$$\Lambda_{\mu\nu} = g_{\mu\nu} + \omega_{\mu\nu} \tag{2.14}$$

the relations

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \quad . \tag{2.15}$$

Thus there are ten infinitesimals, four a_μ and six $\omega_{\mu\nu}$, and we have

$$U(\Lambda,a) = 1 + i a_\mu P^\mu - \frac{i}{2}\,\omega_{\mu\nu}M^{\mu\nu} + \ldots \quad . \tag{2.16}$$

Structure relations (Lie brackets) can be computed from any faithful representation e.g. from the formerly given one (2.8), and using (2.12,13) we find

$$[P^\mu, P^\nu] = 0 \tag{2.17}$$

$$[M^{\mu\nu}, P^\alpha] = -\,i(g^{\mu\alpha}P^\nu - g^{\nu\alpha}P^\mu) \tag{2.18}$$

$$[M^{\mu\nu}, M^{\varkappa\lambda}] = i(g^{\nu\varkappa}M^{\mu\lambda} - g^{\mu\varkappa}M^{\nu\lambda} - g^{\nu\lambda}M^{\mu\varkappa} + g^{\mu\lambda}M^{\nu\varkappa}) \quad . \tag{2.19}$$

The Poincaré group is the semi-direct product of the translation group and the Lorentz group. The translations form an Abelian subgroup.

Evidently $P^\mu P_\mu$ is an invariant, called the rest mass squared. From physical considerations we require $P^{0'} > 0$, $(P^\mu P_\mu)' > 0$ (the prime indicates eigenvalues) except for the vacuum state which is to be a non-degenerate simultaneous eigenstate of all the P^μ and $M^{\mu\nu}$ with eigenvalue zero.

2.3.2 Representations

We consider an n-component field $\chi_i(x)$ $i = 1,2,..,n$. Then we have

$$\sum_j S_{ij}(A^{-1})(\phi', \chi_j(\Lambda x + a)\phi') = (\phi, \chi_i(x)\phi) \tag{2.20}$$

where the n x n matrix is to account for the need to relate nonscalar (e.g. tensorial) measurements to new "axis systems" as mentioned before. Since with $\phi' = U(\Lambda,a)\phi$ this equation is to hold for all ϕ, we have

$$S_{ij}(A^{-1})\,\chi_j(\Lambda \mathbf{x} + a) = U(\Lambda,a)\chi_i(x)U(\Lambda,a)^{-1} \quad . \tag{2.21}$$

This equation shows that

$$S_{ij}(A^{-1})\ S_{jk}(A'^{-1})\,\chi_k(\Lambda'\Lambda x + \Lambda' a + a')$$

$$= U(\Lambda'\Lambda,\ \Lambda' a + a')\ \chi_i(x) U(\Lambda'\Lambda, \Lambda' a + a')^{-1}$$

$$= S_{ik}(A^{-1}\ A'^{-1})\chi_k(\Lambda'\Lambda x + \Lambda' a + a')$$

and consequently the $S_{ij}(A)$ form a representation of SL(2,C). Some of the simplest representations are (we list the matrices, their conventional designations, and examples of fields that transform under these matrices):

1	$D(0,0)$	scalar field
A	$D(1/2\ ,\ 0)$	neutrino-antineutrino field
$\bar{A}$	$D(0\ ,\ 1/2)$	
$\begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}$	$D(1/2\ ,\ 0) \oplus D(0,\ 1/2)$	Dirac field
Λ	$D(1/2\ ,\ 1/2)$	vector field (contravariant)
$(\Lambda^{-1})^T$	$D(1/2\ ,\ 1/2)$	vector field (covariant)
$\Lambda \otimes \Lambda$	$D(1/2\ ,\ 1/2) \otimes D(1/2\ ,\ 1/2) =$	
	$= D(1,1) \oplus$	graviton field
	(symmetric traceless 2nd rank tensor)	
	$\oplus\ D(1.0) \oplus D(0,1) \oplus$	**electromagnetic field**
	(self-dual antisymmetric tensors)	
	$\oplus\ D(0,0)$	scalar field
	(scalar)	

The decomposition in the last example follows from the general formula for the reduction of direct products

$$D\left(\frac{m}{2}, \frac{n}{2}\right) \otimes D\left(\frac{k}{2}, \frac{l}{2}\right) = \sum_{a=0}^{\min(m,k)} \sum_{b=0}^{\min(n,r)} \oplus\ D\left(\frac{m+k-2a}{2}, \frac{n+r-2b}{2}\right) \ - .$$

If we work with hermitian fields the S_{ij} must be real. The Dirac field representation is equivalent to the real one

$$\begin{pmatrix} \mathrm{Re}\,A & \mathrm{Im}\,A \\ -\mathrm{Im}\,A & \mathrm{Re}\,A \end{pmatrix}$$ called the Majorana representation.

By differentiation of (2.21) we obtain the commutation relations

$$[P^\mu, \chi_i(x)] = -i\,\partial^\mu \chi_i(x) \tag{2.22}$$

$$[M^{\mu\nu}, \chi_i(x)] = -i\,(x^\mu\partial^\nu - x^\nu\partial^\mu)\chi_i(x) - i\,S^{\mu\nu}_{ij}\,\chi_j(x) \tag{2.23}$$

where the

$$S^{\mu\nu}_{ij} = \left.\frac{\partial S_{ij}(A)}{\partial \omega_{\mu\nu}}\right|_{A=1} \tag{2.24}$$

are up to a factor i the infinitesimal generators in the representation $S(A)$ and therefore obey (indices are suppressed)

$$[S^{\mu\nu}, S^{\varkappa\lambda}] = g^{\nu\varkappa} S^{\mu\lambda} - g^{\mu\varkappa} S^{\nu\lambda} - g^{\nu\lambda} S^{\mu\varkappa} + g^{\mu\lambda} S^{\nu\varkappa}\,. \tag{2.25}$$

Herefrom it easily follows that all $S^{\mu\nu}$ are traceless i.e. the $S(A)$ are unimodular. This can be deduced also from the fact that the proper Lorentz group is simple.

For more thorough treatments of the topic of this lecture the reader is referred to [7],[11],[9],[10].

3. Relativistic Lagrangians

3.1 First Order Lagrangian Densities

In Lecture I we discussed the classical and quantized first order Lagrangian

$$L = \frac{i}{2}\, q\, A\, \dot{q} - H(q,t)$$

with A imaginary and antisymmetric (i.e. Hermitean) for Bose variables and real and symmetric (i.e. Hermitean) for Fermi variables, respectively. As relativistic analog we set for the Lagrangian density

$$\mathcal{L} = \frac{i}{2}\,\chi\alpha^\mu\partial_\mu\chi - \frac{m}{2}\,\chi\beta\chi - \mathcal{H}(\chi,x) \tag{3.1}$$

or in an explicit Hermitean form

$$\mathcal{L} = \frac{i}{4}\,(\chi\alpha^\mu\overrightarrow{\partial}_\mu\chi - \chi\overleftarrow{\partial}_\mu\alpha^\mu\chi) - \frac{m}{2}\,\chi\beta\chi - \mathcal{H}(\chi,x) \tag{3.2}$$

where

$$\alpha^\mu = \alpha^{\mu +}\ ,\quad \beta = \beta^+\ ,\qquad \mathcal{H} = \mathcal{H}^+\ ,$$

$$\alpha^\mu = \mp\,\alpha^{\mu T},\quad \beta = \pm\,\beta^T \qquad \text{in the} \begin{Bmatrix}\text{Bose}\\ \text{Fermi}\end{Bmatrix} \text{case}$$

for Hermitean fields.

The analog of

$$L = i\ a^+B\ \dot{a}\ \ -\ \ H(a^+,a,t)$$

is, correspondingly,

$$\mathcal{L} = i\ \psi^+\alpha^\mu\partial_\mu\psi\ -\ m\,\psi^+\beta\,\psi\ -\ \mathcal{H}(\psi^+,\psi,x) \tag{3.3}$$

where $\alpha^\mu = \alpha^{\mu +}\ ,\quad \beta = \beta^+\ ,\quad \mathcal{H} = \mathcal{H}^+$

for non-Hermitean (Bose resp. Fermi) fields.

We have separated out from the non-derivative part a bilinear term because of the prominent occurrence of such terms in all cases, and we have also included a quantity m of the dimension of mass for dimensional reasons. If we choose to have all components of χ to have the same dimension, then α^μ and β can now be taken dimensionless. The analogy is complete in the sense that our new Lagrangian density leads upon integration over space to a Lagrangian of the ordinary type with a countable infinite number of variables. Namely, we introduce a complete orthonormal system of real functions $f_k(\vec{x})$ on R^3 such that

$$\sum_k f_k(\vec{x})\ f_k(\vec{y})\ =\ \delta\,(\vec{x}-\vec{y}) \tag{3.4}$$

and

$$\int f_k(\vec{x})\ f_l(\vec{x})\ dx = \delta_{kl}. \tag{3.5}$$

We define

$$\chi_{i,\kappa}(x^0) = \int \chi_i(x^0,\vec{x})\ f_\kappa(\vec{x})\ d\vec{x}\ . \tag{3.6}$$

Then

$$\chi_i(x^0,\vec{x}) = \sum_\kappa f_\kappa(\vec{x})\ \chi_{i,\kappa}(x^0) \tag{3.7}$$

inserted into the Lagrangian density (3.2) gives upon integrating over $\vec{x}$ a Lagrangian of the form (1.4) with

$$A = \begin{pmatrix} \alpha^0 & & & \\ & \alpha^0 & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} \tag{3.8}$$

countably many often.
Then the projection operator on the zero-space of A is

$$\mathcal{P} = \begin{pmatrix} P & & & \\ & P & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} \tag{3.9}$$

where P is the projection operator on the zero-space of α^0, which satisfies ($\overline{P} = 1 - P$)

$$P\alpha^0 = \alpha^0 P = 0 \tag{3.10}$$

and $\overline{P}\alpha^0\overline{P}$ nonsingular in the subspace $\overline{P}$ projects on.

3.2 Restrictions by relativistic invariance

Now we look at the restrictions imposed on the matrices α^μ and β by relativistic invariance. These are obtained e.g. as follows: think $\mathcal{L}$ written in terms of the field operators, integrate over spacetime to obtain the action and apply $U(\Lambda, a) \ldots U(\Lambda, a)^{-1}$ to it, and require that the action goes over into itself.

Now

$$\begin{aligned} \mathcal{U}(\Lambda,a)\, \chi(x)\, \alpha^\mu \partial_\mu \chi(x)\, \mathcal{U}(\Lambda,a)^{-1} &= \\ &= \chi(\Lambda x+a)\; S^T(A^{-1})\, \alpha^\mu \partial_\mu\, S(A^{-1})\, \chi(\Lambda x+a) \\ &= \chi(\Lambda x+a)\; S^T(A^{-1})\, \alpha^\mu\, S(A^{-1})\, \Lambda^\nu{}_\mu \frac{\partial}{\partial(\Lambda x+a)^\nu}\, \chi(\Lambda x+a)\ . \end{aligned}$$

Since $d(\Lambda x) = (\det \Lambda)\, dx = dx$, the condition for the derivative term is fulfilled if and only if

$$S^T(A^{-1})\, \alpha^\mu S(A^{-1})\, \Lambda^\nu{}_\mu = \alpha^\nu \tag{3.11}$$

or in more usual form

$$S^T(A)\, \alpha^\mu S(A) = \Lambda^\mu{}_\nu\, \alpha^\nu\ . \tag{3.12}$$

Thus if α^μ is transformed on all indices simultanously with the appropriate transformation it must go into itself i.e. it must be an invariant matrix, or, it is invariant under the representation $S \otimes S \otimes \Lambda$. Such an α therefore exists if and only if in the direct product representation the trivial representation is contained. Clearly S cannot be the scalar (trivial) representation. It can also not be the vector representation. The simplest is a five component sum of a scalar and the vector representations. Also the Majorana representation discussed before, which is $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$, is allowed. In fact the product $[(\frac{1}{2},0) \oplus (0,\frac{1}{2})] \otimes [(\frac{1}{2},0) \oplus (0,\frac{1}{2})] \otimes (\frac{1}{2},\frac{1}{2})$ contains the trivial representation twice.

The case of complex fields is similar. In this case we require (instead of (3.11))

$$S^+(A^{-1})\, \alpha^\mu S(A^{-1})\, \Lambda^\nu{}_\mu = \alpha^\nu \tag{3.13}$$

or more usual

$$S^{+}(A)\,\alpha^{\mu}\,S(A) \;=\; \Lambda^{\mu}{}_{\nu}\,\alpha^{\nu}\ . \tag{3.14}$$

Now the representation $S(A) = A$ suffices, for
$[(\frac{1}{2},0) \otimes (0,\frac{1}{2})] \otimes (\frac{1}{2},\frac{1}{2}) = (0,0) \oplus (1,0) \oplus (0,1) \oplus (1,1)$.

In fact

$$\alpha^{\mu} = \left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, -\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, -\begin{pmatrix} & -i \\ i & \end{pmatrix}, -\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right) \tag{3.15}$$

is the set of α^{μ} in question, and the transformation formula is just the one with the help of which to a Lorentz transformation an A is associated (cf. the literature cited at the end of lecture 2). Since $\alpha^{0} = 1$, all fields are independent. In the first order formalism this spin $\frac{1}{2}$ case is the only one where this happens.

The same reasoning for the mass term in (3.1) resp. (3.3) yields

$$S^{T}(A)\,\beta\,S(A) = \beta \tag{3.16}$$

and

$$S^{+}(A)\,\beta\,S(A) = \beta \tag{3.17}$$

for hermitean and non-hermitean variables respectively.

Thus for such a β to exist, we must have $S \otimes S$ (or respectively $S \otimes \bar{S}$) contain the trivial representation. This is always so for real representations, e.g. $(0,0)$, $(\frac{1}{2},\frac{1}{2})$ etc. But $(\frac{1}{2},0)$, i.e. $S(A) = A$, is ruled out. This is the reason why we cannot write a two component equation with mass term in the form given in (3.3) (omitting the β-term and $\varkappa$ we have for $S(A) = A$ the Weyl neutrino Lagrangian). But $S \otimes S$ always contains the trivial representation whether S is real or complex, so we can have with the α^{μ} of (3.15)

$$\mathcal{L} = i\,\psi^{+}\alpha^{\mu}\partial_{\mu}\psi - \frac{m}{2}\,\psi\,\varepsilon\,\psi + \frac{m}{2}\,\psi^{+}\varepsilon\,\psi^{+} \tag{3.18}$$

for a non-hermitean two-component ψ , with the "spinor metric" matrix $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This Lagrangian is equivalent to the Majorana Lagrangian for a four-component Hermitean field.

The "Dirac form" for non-Hermitean (or also for Hermitean) fields we obtain for non-singular β by writing

$$\psi^{+}\beta = \overline{\psi} \quad , \quad \beta^{-1}\alpha^{\mu} = \gamma^{\mu} \tag{3.19}$$

which upon omitting $\mathcal{H}$ gives from (3.3) the "Dirac Lagrangian"

$$\mathcal{L} = i\,\overline{\psi}\,\gamma^{\mu}\partial_{\mu}\psi - m\,\overline{\psi}\psi\ . \tag{3.20}$$

This form is convenient since the γ^{μ} have the simple transformation property

$$S^{-1}(A)\ \gamma^{\mu}\,S(A) = \Lambda^{\mu}{}_{\nu}\,\gamma^{\nu} \tag{3.21}$$

such that products of the γ's transform like the corresponding tensors with respect to the Minkowski indices, i.e.

$$S^{-1}(A)\gamma^{\mu_1}\cdots\gamma^{\mu_r}\,S(A) = \Lambda^{\mu_1}{}_{\nu_1}\cdots\Lambda^{\mu_r}{}_{\nu_r}\,\gamma^{\nu_1}\cdots\gamma^{\nu_r}\ . \tag{3.22}$$

Finally for the $\mathcal{H}$-term in (3.1) resp. (3.3) we must for relativistic invariance have no explicit x-dependence and

$$\mathcal{H}(S\chi) = \mathcal{H}(\chi) \tag{3.23}$$

for Hermitean fields and

$$\mathcal{H}(\psi^{+}S^{+}, S\psi) = \mathcal{H}(\psi^{+}, \psi) \tag{3.24}$$

for non-Hermitean fields.

For Fermi fields, where we are only interested in spin $\frac{1}{2}$ fields, the Dirac γ^{μ}, particularly conveniently in the imaginary Majorana form, provide us with the solution of our problem. For Bose fields, fields of spin o and 1 are of interest, but here the S contain Λ (and their direct products) only and it is convenient to separate the various tensors from each other as follows: Consider the Klein-Gordon Lagrangian

$$\mathcal{L} = \frac{1}{2}\, \partial^\mu A\, \partial_\mu A - \frac{m^2}{2} A^2 \tag{3.25}$$

which is quadratic in time derivatives. In the usual transition to the Hamiltonian formalism we can linearize it in derivatives, in fact covariantly, by introducing an auxiliary vector field

$$\partial^\mu A = B^\mu \quad . \tag{3.26}$$

Then

$$\mathcal{L} = -\frac{1}{2} B^\mu B_\mu + B^\mu \partial_\mu A - \frac{m^2}{2} A^2 \tag{3.27}$$

giving

$$\partial_\mu B^\mu - m^2 A = o \tag{3.28}$$

and (3.26) as field equations. The Lagrangian (3.27) is equivalent to the Klein-Gordon Lagrangian. Written in terms of α^μ and β for the five-component field (A, B^μ) or for dimensional equality $(m^{1/2} A,\ m^{-1/2} B^\mu)$ provided $m \neq o$ gives the "Duffin-Kemmer-Pétiau" form of the Lagrangian for a scalar field. (If m = o it cannot be used since the scale change is not possible.)

<u>Exercise:</u> Write the α^μ , β , and γ^μ explicitly in matrix form.

For further discussions of general covariant first order Lagrangians and some of their properties the reader is referred to [1] (see especially § 20) and to [1] .
Some considerations on the basis of Schwinger's action principle can be found in [11].

3.3 Quantization

The Peierls' formula (1.58) now reads

$$(\delta^{ret} - \delta^{adv}) F(\chi(x),x) = i \int dy \, [F(\chi(x),x), \delta\mathcal{L}(\partial^\mu \chi(y), \chi(y), y)]_{\partial^\mu\chi, \chi \text{ fixed}} \tag{3.29}$$

and thus, since reference to a coordinate frame has disappeared, the covariance of the quantization procedure is assured if $\mathcal{L}$ satisfies the conditions we gave before. We will consider separately the cases:

1. $\mathcal{H} \equiv 0$, i.e. the Lagrangian density in (3.1) or (3.3) is bilinear (free fields),
2. $\mathcal{H} \not\equiv 0$, or more precisely contains trilinear or higher terms (interacting fields).

In case 2 we can essentially only obtain the canonical formulas back as they correspond to the elementary ones of lecture 1, via the relations (3.6...8) with the knowledge however, that the quantization is covariant. The case 1, however, will show the power and elegance of the Peierls' principle.

First we have to introduce parameters into the Lagrangian, which we do most conveniently via a term linear in χ . Thus

$$\mathcal{L} = \frac{i}{2} \chi \alpha^\mu \partial_\mu \chi - \frac{m}{2} \chi \beta \chi - \mathcal{H}(\chi) + \chi J \, . \tag{3.30}$$

Here J is a "source", and is an arbitrary space-time function in the Bose case, whereas it is an arbitrary space-time dependent algebraic element with the properties

$$\{J(x), \chi(y)\} = 0, \quad \{J(x), J(y)\} = 0 \text{ for all } x,y \tag{3.31}$$

in the Fermi case. The first property we have to require is to have the new term "even" like the other terms in the Lagrangian, the second to have it also commute rather than anticommute with itself (in the sense of classical variables). Now the field equations are

$$-i\alpha^\mu \partial_\mu \chi + m\beta\chi + \partial^L \mathcal{H} = J \tag{3.32}$$

and

$$i \chi \overleftarrow{\partial}_\mu \alpha^\mu + m \chi \beta + \partial^R \mathcal{H} = \pm J \tag{3.33}$$

where $\pm$ refers to the commutation resp. anticommutation case. By variation as a consequence of J-change follows

$$-i \alpha^\mu \partial_\mu \delta \chi + m \beta \delta \chi + \partial^L \partial^R \mathcal{H} \delta \chi = \delta J \tag{3.34}$$

and

$$i \delta \chi \overleftarrow{\partial}_\mu \alpha^\mu + m \delta \chi \beta + \delta \chi \partial^L \partial^R \mathcal{H} = \pm \delta J . \tag{3.35}$$

Let us now treat the two cases separately.

3.3.1 Free fields

Here we have $\mathcal{H} \equiv 0$. Define $S_{\substack{ret\\adv}}$ by

$$(-i \alpha^\mu \partial_\mu^x + m\beta) S_{\substack{ret\\adv}} (x-y) = \delta (x-y) I \tag{3.36}$$

with

$$S_{\substack{ret\\adv}} (x-y) = 0 \quad \text{if } x^0 \lessgtr y^0. \tag{3.37}$$

If the Lagrange equations determine the motion (cf. section 1.1.3), $S_{\substack{ret\\adv}}$ are hereby uniquely determined, since the homogeneous equations have no solution by assumption. The solution is directly obtained in Fourier representation:

$$S_{\substack{ret\\adv}} (x-y) = \frac{1}{(2\pi)^4} \int e^{-ik(x-y)} \frac{dk}{m\beta - \alpha^\mu k_\mu \mp i \alpha^0 \varepsilon} \tag{3.38}$$

where ε is positive infinitesimal.

Exercise: Derive the reality, hermiticity, and $x \longleftrightarrow y$ symmetry properties of $S_{\substack{ret \\ adv}}$.

Now $(m\beta - \alpha^\mu k_\mu \mp i\alpha^\circ \varepsilon)^{-1}$ as a matrix inverse has the form

$$\frac{\text{matrix with elements polynomial in } k}{\text{determinant}} .$$

Here the determinant is, on the basis of the transformation properties of α^μ and β and the unimodularity of $S(A)$, easily seen to be a polynomial in $(k_\mu \pm i\varepsilon\,\delta_{\mu o})(k^\mu \pm i\varepsilon\,\delta^{\mu o}) = k^2 \pm i\varepsilon k^\circ = k^2 + i\varepsilon \text{ sign } k^\circ$ with real coefficients. As such it can be written as $\text{const.}\prod_{\nu=1}^{s}(k^2 + i\,\varepsilon \text{ sign } k^o - m_\nu^2)$ if it has degree s in this variable with const. $\neq$ o and the m_ν^2 real or in complex conjugate pairs. (The determinant vanishes identically only in the gauge case which we excluded). After cancelling factors against possibly the same ones in the numerator (as occurs in the Dirac case) we make a partial-fraction decomposition of the remaining denominator. By a suitable choice of the α^μ and β one can always (i.e. for any spin) achieve that only one single term with mass m, say, remains in the denominator. How to choose α^μ and β or equivalently γ^μ and β, for this to occur is the subject of a paper by Pauli and Fierz [12] and in particular of the books of Naimark and of Gelfand, Minlos, and Shapiro on the representations of the Lorentz group [13,14]. The case of spin o we did just before, for the case of spin $\frac{1}{2}$ it is done in the familiar Dirac way, and the spin 1 case will concern us later.

Thus, considering for simplicity only this case of one real mass, we obtain

$$S_{\substack{ret \\ adv}}(x-y) = \text{Pol.}(i\,\partial^x)\,\Delta_{\substack{ret \\ adv}}(x-y) \qquad (3.39)$$

where Pol.$(i\,\partial^x)$ is a matrix, whose elements are polynomials of derivatives, with certain covariance properties. It is called "Klein-Gordon divisor" in the book by Takahashi on field quantization [15] .

Exercise: Derive those covariance properties.

Now

$$\delta_{\substack{ret\\adv}} \chi(x) = \int S_{\substack{ret\\adv}}(x-y)\ \delta J(y)\ dy \quad . \tag{3.40}$$

Thus

$$(\delta^{ret} - \delta^{adv})\chi(x) = \int (S_{ret}(x-y) - S_{adv}(x-y))\ \delta J(y)\ dy$$

$$= i\int [\chi(x), \chi(y)\ \delta J(y)]\ dy \tag{3.41}$$

or

$$(\delta^{ret} - \delta^{adv})\ \chi(x) = i\int [\chi(x), \chi(y)]_{\mp}\ \delta J(y)\ dy \ . \tag{3.42}$$

Defining

$$S_{adv}(x-y) - S_{ret}(x-y) = S(x-y) \tag{3.43}$$

which has the explicit form

$$S(x-y) = \mathrm{Pol.}(i\,\partial^x)\ \Delta(x-y) \tag{3.44}$$

we find due to the arbitrariness of δJ,

$$[\chi(x), \chi(y)]_{\mp} = i\ S(x-y), \tag{3.45}$$

i.e. the Peierls' method gives immediately the commutator (respectively anticommutator) for all times, which vanishes for spacelike distance and is a c-number. If we define

$$\chi_{\substack{in\\out}}(x) = \chi(x) - \int S_{\substack{ret\\adv}}(x-y)\ J(y)\ dy \tag{3.46}$$

then also

$$[\chi_{\substack{in\\out}}(x), \chi_{\substack{in\\out}}(y)]_{\mp} = i\ S(x-y) \quad . \tag{3.47}$$

In addition

$$S\ \chi_{out}(x) = \chi_{in}(x)\ S \tag{3.48}$$

where

$$S = e^{i\int \chi_{out}(x)\ J(x)\ dx} = e^{i\int \chi_{in}(x)\ J(x)\ dx} \tag{3.49}$$

is the (up to a c-number factor) unique solution of the operator equation (3.48).

3.3.2 Interacting fields

In this case we have $\mathcal{H} \not\equiv 0$. Now integration of the equation of motion over arbitrary time intervals, as we did before, is no longer possible. We therefore integrate only infinitesimally, and set

$$\delta J(x) = \varepsilon\ \delta(x-y) \tag{3.50}$$

where ε is an infinitesimal c-number (respectively an anticommuting algebraic element) with as many components as χ has. The Lagrange equations (3.32) give for the variation

$$\begin{aligned} \alpha^0 \partial_0\, \delta\chi(x) = & - \alpha^i \partial_i\, \delta\chi(x) - im\beta\, \delta\chi(x) \\ & - i\, \partial^L \partial^R \mathcal{H}(\chi(x))\, \delta\chi(x) + i\varepsilon\, \delta(x-y) \end{aligned} \tag{3.51}$$

Let $\mathcal{P}$ be the projection operator on the zero-subspace of the Hermitean matrix α^0 . Then with $\overline{\mathcal{P}} = 1 - \mathcal{P}$ the variation of the constraint equations gives

$$\begin{aligned} & - \mathcal{P}\alpha^i\mathcal{P}\ \partial_i \mathcal{P}\delta\chi(x) - \mathcal{P}\alpha^i\overline{\mathcal{P}}\ \partial_i\overline{\mathcal{P}}\delta\chi(x) - im\mathcal{P}\beta\mathcal{P}\ \mathcal{P}\delta\chi(x) - \\ & - im\mathcal{P}\beta\overline{\mathcal{P}}\ \overline{\mathcal{P}}\delta\chi(x) - i\,\mathcal{P}\partial^L\partial^R\mathcal{H}(\chi(x))\mathcal{P}\ \mathcal{P}\delta\chi(x) - \\ & - i\,\mathcal{P}\partial^L\partial^R\mathcal{H}(\chi(x))\overline{\mathcal{P}}\ \overline{\mathcal{P}}\delta\chi(x) + i\,\mathcal{P}\varepsilon\,\delta(x-y) = 0 \end{aligned} \tag{3.52}$$

We now use the fact that

$$\mathcal{P}\alpha^i\mathcal{P} = 0 \tag{3.53}$$

(This can be seen as follows: The infinitesimal form of (3.12) is

$$S^{\varkappa\lambda\,T}\alpha^{\mu} + \alpha^{\mu} S^{\varkappa\lambda} = \frac{\partial \Lambda^{\mu}{}_{\nu}}{\partial \omega_{\varkappa\lambda}}\alpha^{\nu} = (g^{\mu\varkappa} g^{\lambda}{}_{\nu} - g^{\mu\lambda} g^{\varkappa}{}_{\nu})\,\alpha^{\nu}$$

$$= (g^{\mu\varkappa}\alpha^{\lambda} - g^{\mu\lambda}\alpha^{\varkappa})$$

where (2.14) and (2.24) are used. Setting $\mu = o$, $\lambda = i$, $\varkappa = o$, and multiplying from both sides with $\mathcal{P}$ gives (3.53) due to (3.10).)
Due to (3.53) equation (3.52) can be solved for $\mathcal{P}\delta\chi(x)$ ($m \neq o$ is understood) if

i) $\mathcal{P}\beta\mathcal{P}$ is non-singular in the subspace $\mathcal{P}$ projects on and $\mathcal{H}$ contains dependent variables at most linearly (i.e. $\mathcal{P}\partial^L\partial^R\mathcal{H}\mathcal{P} = o$)

ii) $\mathcal{H}$ contains $\mathcal{P}\chi$ bilinearly but $m\mathcal{P}\beta\mathcal{P} + \mathcal{P}\partial^L\partial^R\mathcal{H}\mathcal{P}$ is non-singular in the subspace $\mathcal{P}$ projects on.

In the latter case $\mathcal{P}\chi(x)$, expressed in terms of $\mathcal{P}\varepsilon$ and $\overline{\mathcal{P}}\delta\chi(x)$ may involve an operator denominator which can give rise to complications. We will come to this point in lecture 4. In the cases i) and ii) the variational equations of motion are

$$\begin{aligned}\overline{\mathcal{P}}\alpha^{o}\overline{\mathcal{P}}\,\partial_{o}\overline{\mathcal{P}}\,\delta\chi(x) &= -\overline{\mathcal{P}}\alpha^{i}\overline{\mathcal{P}}\,\partial_{i}\overline{\mathcal{P}}\,\delta\chi(x) - \overline{\mathcal{P}}\alpha^{i}\mathcal{P}\,\partial_{i}\mathcal{P}\,\delta\chi(x) - \\ &\quad - im\,\overline{\mathcal{P}}\beta\overline{\mathcal{P}}\;\overline{\mathcal{P}}\delta\chi(x) - im\,\overline{\mathcal{P}}\beta\mathcal{P}\;\mathcal{P}\delta\chi(x) - \\ &\quad - i\,\overline{\mathcal{P}}\partial^L\partial^R\mathcal{H}\overline{\mathcal{P}}\;\overline{\mathcal{P}}\delta\chi(x) - i\,\overline{\mathcal{P}}\partial^L\partial^R\mathcal{H}\mathcal{P}\;\mathcal{P}\delta\chi(x) + \\ &\quad + i\,\overline{\mathcal{P}}\,\varepsilon\,\delta(x-y)\ .\end{aligned} \tag{3.54}$$

If we now set, for simplicity, $\mathcal{P}\varepsilon = o$ i.e. vary the sources only for the independent components, or, what is the same, replace ε by $\overline{\mathcal{P}}\varepsilon$ then $\mathcal{P}\delta\chi$ is, as we established, of the same order as $\overline{\mathcal{P}}\delta\chi$ (since linearely expressed by it) and (3.54) gives upon infinitesimal integration, using (3.10),

$$\begin{aligned}\overline{\mathcal{P}}\,\delta^{\substack{\mathrm{ret}\\ \mathrm{adv}}}\chi(x) &= \pm i\,(\overline{\mathcal{P}}\alpha^{o}\overline{\mathcal{P}})^{-1}\,\overline{\mathcal{P}}\varepsilon\,\delta(\vec{x}-\vec{y})\;\Theta(\pm(x^{o}-y^{o})) + \\ &\quad + O(|x^{o}-y^{o}|)\;O(\varepsilon)\ ,\end{aligned} \tag{3.55}$$

i.e.

$$(\delta^{ret} - \delta^{adv})\, \overline{\mathcal{P}}\, \chi(x) = i\, (\overline{\mathcal{P}} \alpha^{\circ} \overline{\mathcal{P}})^{-1}\; \overline{\mathcal{P}}\, \varepsilon\, \delta(\vec{x}-\vec{y}) \;+\; O(|x^{\circ}-y^{\circ}|)\, O(\varepsilon)$$

$$= i\, [\, \overline{\mathcal{P}}\, \chi(x),\, \chi(y)]_{\mp}\; \overline{\mathcal{P}}\varepsilon \;+ O(\varepsilon^{2}) \qquad (3.56)$$

according to (3.29). Setting $x^{o} = y^{o}$, it follows due to the arbitrariness of $\overline{\mathcal{P}}\varepsilon$,

$$[\overline{\mathcal{P}}\chi(x),\, \chi(y)\overline{\mathcal{P}}]_{\mp}\;{}_{x^{\circ}=y^{\circ}} = (\overline{\mathcal{P}}\alpha^{\circ}\overline{\mathcal{P}})^{-1}\; \delta(\vec{x}-\vec{y}) \qquad (3.57)$$

which is written more simply as

$$[\alpha^{\circ}\chi(x),\, \chi(y)\alpha^{\circ}]_{\mp}\, \delta(x^{\circ}-y^{\circ}) = \alpha^{\circ}\; \delta(x-y) \qquad (3.58)$$

or in general form, with $\alpha^{\mu} n_{\mu} = \alpha n$,

$$\delta(n(x-y))\, [(\alpha n)\, \chi(x),\, \chi(y)\, (\alpha n)]_{\mp} = (\alpha n)\; \delta(x-y) \qquad (3.59)$$

where n is an arbitrary positive timelike four-vector with

$$n^{\mu}\; n_{\mu} = 1. \qquad (3.60)$$

These commutation relations in a fixed reference system we could have derived immediately from the comparison with the elementary formula (1.14) resp. the corresponding one for anticommuting variables since by assumption the constraints are solvable. What we have shown in addition is that also the dependent components, namely $\mathcal{P}\chi$, have as a consequence of (3.53) local commutation relations since we obtained at most operators but not derivatives in the denominator arising from (3.52) in the case ii). Of course this locality had to arise since manifest relativistic covariance was assumed. We don't discuss the question of relativistic covariance i.e. the n-independence of (3.59). It is treated for instance in [16] .

It is to be kept in mind that for relativistic invariance of a QFT as a whole the conditions are, however, less stringent than manifest covariance. Namely it is only necessary that observable quantities (which are always of tensorial rather than spinorial character, i.e. transform under the one valued representations of the Lorentz group) transform covariantly, and that there exist P^μ and $M^{\mu\nu}$ that transform them so and obey, of course, the structure relations of the Poincaré group. The fields themselves need not transform covariantly insofar as they are not observables. The Lagrangian should for causality reasons (cf. our comparison of nonrelativistic systems and quantum field theory) be expressible by fields on a spacelike, e.g. a plane, surface but not necessarily as a space integral of a local Lagrangian density as we have assumed so far. Operators corresponding to localizable observables (field strength, charge and current density, and energy-momentum density) should commute for spacelike distances (they will, due to the required covariant transformation laws, do so if they commute at a fixed time in some frame) but those corresponding to unobservable fields, being objects introduced from the physical point of view for mathematical reasons only, need not. The Coulomb gauge of quantum electrodynamics, to be briefly described later, has just these properties that the fields (precisely: the vector potential and the spinor fields) do not commute in spacelike distances and do not transform covariantly (namely, they transform covariantly only up to a gauge transformation). Actually in QED one can on the demonstratibly harmless expense of enlarging the Hilbert space and letting the metric be indefinite achieve covariance of all fields.

We will now return to our manifestly covariant case and give the P^μ and $M^{\mu\nu}$ here. They are obtained by the familiar Noether method we need not describe here:

$$P^\mu = \int T^{\mu\nu} \, d\sigma_\nu \, , \tag{3.61}$$

$$M^{\mu\nu} = \int M^{\mu\nu\lambda} \, d\sigma_\lambda \, . \tag{3.62}$$

For (3.1) we have

$$T^{\mu\nu} = \frac{i}{4}(\chi\alpha^\nu\partial^\mu\chi - \partial^\mu\chi\alpha^\nu\chi) - g^{\mu\nu}\mathcal{L} \quad , \tag{3.63}$$

$$M^{\mu\nu\lambda} = x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda} + \frac{i}{4}(\chi\alpha^\lambda S^{\mu\nu}\chi - \chi S^{\mu\nu T}\alpha^\lambda\chi) \quad , \tag{3.64}$$

where in (3.63) $\mathcal{L}$ must be used in its explicitly Hermitean form.

<u>Exercise:</u> Show that $T^{\mu\nu}$ and $M^{\mu\nu\lambda}$ are conserved, i.e.

$$\partial_\nu T^{\mu\nu} = 0 \quad , \tag{3.65}$$

$$\partial_\lambda M^{\mu\nu\lambda} = 0 \quad . \tag{3.66}$$

For (3.65) it is needed that $\mathcal{H}(\chi)$ does not depend on x explicitly (if it did, one would get $\partial_\nu T^{\mu\nu} = \partial^\mu \mathcal{H}(\chi(x),x)\big|_{\chi=\text{const.}}$)
and for (3.66) that

$$\mathcal{H}(S\chi) = \mathcal{H}(\chi)$$

or in infinitesimal form

$$\partial^R \mathcal{H} S^{\mu\nu}\chi + \chi S^{\mu\nu T}\partial^L\mathcal{H} = 0 \quad .$$

Writing

$$M^{\mu\nu\lambda} = x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda} + f^{\mu\nu\lambda} \tag{3.67}$$

one easily can show that the "symmetric energy-momentum tensor"

$$\Theta^{\mu\nu} = \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu}) + \frac{1}{2}\partial_\lambda(f^{\nu\lambda\mu} + f^{\mu\lambda\nu}) \tag{3.68}$$

gives

$$P^\mu = \int \Theta^{\mu\nu} d\sigma_\nu \tag{3.69}$$

and

$$M^{\mu\nu} = \int (x^\mu \Theta^{\nu\lambda} - x^\nu \Theta^{\mu\lambda})\, d\sigma_\lambda \,. \qquad (3.70)$$

Exercise: Prove equations (3.69,70).

(Hereby one uses repeatedly Stokes theorem which is

$$\int \partial_\mu R \, d\sigma_\nu = \int \partial_\nu R \, d\sigma_\mu \qquad (3.71)$$

for any spacelike surface and an integrand R that vanishes more strongly than $\frac{1}{r^2}$ in space infinity.)

That P^μ and $M^{\mu\nu}$ have the correct commutation relations (2.22,23) with χ can be shown by direct, though in the case of singular α° somewhat lengthy, calculation. Since they are conserved, they can be commuted according to these formulas also under differentation signs. Using the Jacobi identity we therefore find that they must obey the structure relations of the Poincaré group (2.17,18,19) up to terms that commute with χ at all times. Briefly, the argument is for any Lie group generators G_i that transform the fields linearly,

$$[G_i, \chi] = Op_i \, \chi \qquad (3.72)$$

where Op_i is some differential and matrix operation, the following: From (3.72) we have

$$[G_j, [G_i, \chi]] = Op_i Op_j \, \chi$$

such that

$$[G_j, [G_i, \chi]] - [G_i, [G_j, \chi]] = - [[G_i, G_j], \chi] = (Op_i Op_j - Op_j Op_i)\, \chi \,.$$

If now $[Op_i, Op_j] = -c_{ij}{}^k Op_k$ with $c_{ij}{}^k$ as structure constants, which is easily verified in the Poincaré case, with the help of (2.24) then

$$[([G_i, G_j] - c_{ij}{}^k G_k), \chi] = o \quad .$$

Such quantities that commute with all operators at all times (we suppose the χ to be irreducible such that Schur's lemma applies) are, however, c-numbers and thus equal to their vacuum expectation values. If we define "truncated" operators P^μ and $M^{\mu\nu}$ from the naive forms (3.61) and (3.62) by subtracting the vacuum expectation values (VEV) under the integral sign (otherwise the integrals would not converge, nor any of the manipulations that rely on the vanishing of boundary terms at infinity be justified), then for the truncated operators the structure relations will be satisfied provided $\langle [P^\mu, P^\nu] \rangle$, etc. vanish. One can give plausibility arguments that this last condition is satisfied. Clearly, one can give no mathematical proofs here since one cannot show even the existence of any one physically nontrivial QFT, let alone that of e.g. QED. In this theory, all one has so far is perturbation theory, which is manifestly covariant such that the general invariance argumentation is superfluous.

Remark: It is well known that the Lagrange equations remain unchanged if one adds to the Lagrangian density a four-divergence (the local quantum field theory analog of adding a total time derivative to the nonrelativistic Lagrangian):

$$\mathcal{L} \to \mathcal{L} + \partial_\mu V^\mu(\chi) \quad .$$

From Peierls' formula (3.29) we see that thereby the commutation relations aren't changed either.

We have so far assumed in writing our formulas that all fields that enter χ are either of the Bose or of the Fermi type (commuting or anticommuting fields). In practice, e.g. in QED, one has both types of fields in the same Lagrangian. We can take over all our formulas with α^μ, β interpreted as reducible, i.e.

$$\alpha^\mu = \begin{pmatrix} \alpha^\mu_B & \\ & \alpha^\mu_F \end{pmatrix} \quad , \quad \beta = \begin{pmatrix} \beta_B & \\ & \beta_F \end{pmatrix} \tag{3.73}$$

with

$$\begin{aligned} \alpha^\mu_B &= - \alpha^{\mu T}_B \quad , & \alpha^\mu_F &= + \alpha^{\mu T}_F , \\ \beta_B &= + \beta^T_B \quad , & \beta_F &= - \beta^T_F \\ \alpha^\mu_{\substack{B\\F}} &= \alpha^{\mu +}_{\substack{B\\F}} \quad , & \beta_{\substack{B\\F}} &= \beta^+_{\substack{B\\F}} \end{aligned}$$

and wherever in some formulas a difference in signs, $\pm$ in our writing, for Bose and Fermi case should appear, the upper sign is to be taken for the Bose the lower for the Fermi part. That this reducibility holds and in the bilinear terms in $\mathcal{L}$ Bose and Fermi fields do not mix comes of course from the requirement that $\mathcal{L}$ be altogether an even operator since it is almost observable and closely related to the energy which is an observable.
As reference to this lecture see e.g. [51] .

4. GREEN'S FUNCTIONS, FEYNMAN AND NON-FEYNMAN RULES

We now investigate further the effect of having the manifestly co-variant Lagrangian amended by a source term

$$\mathcal{L} = \frac{i}{2} \chi \alpha^\mu \partial_\mu \chi - \frac{m}{2} \chi \beta \chi - \mathcal{H}(\chi) + \chi J \quad .$$

J is a multicomponent function of space-time. Therefore any number coming from this theory will depend on this function, i.e. be a functional. Of particular interest is the scalar product

$$\hat{G}_{disc} \{J\} := {}_{out}\langle \;|\; \rangle_{in} \tag{4.1}$$

of in- and out-vacuum states already mentioned before (1.51). Namely, we imagine $J = 0$ for very early and for very late times such that $H(x^0) = \int T^{00}(x,x^0)\, d\vec{x}$ is x^0-independent and $\rangle_{\text{in/out}}$ is defined as the lowest-lying eigenstate to $H(\mp\infty)$. If $J \not\equiv 0$, $\rangle_{in} \neq \rangle_{out}$ in general. (We always consider Heisenberg states, and for a system with explicitly time dependent external perturbation the state that was the ground state at $x^0 = -\infty$ is not the ground state at $x^0 = +\infty$.) However, $\hat{G}_{disc}\{0\} = 1$ as a phase convention, while $|\hat{G}_{disc}\{J\}| \leq 1$.

4.1 Functional derivatives

For a functional $F\{J\}$ we define a functional derivative by

$$\frac{\delta F\{J\}}{\delta J(x)} = \lim_{\varepsilon \to 0} \frac{F\{J'\} - F\{J\}}{\varepsilon} \tag{4.2}$$

with $J'(\cdot) = J(\cdot) + \varepsilon\, \delta(\cdot - x)$ or, somewhat more safely,

$$\int f(x) \frac{\delta F\{J\}}{\delta J(x)}\, dx = \lim_{\varepsilon \to 0} \frac{F\{J+\varepsilon f\} - F\{J\}}{\varepsilon} \tag{4.3}$$

for suitable smooth test functions. (Note that functional differentiability is a special property of a functional just as ordinary differentiability is a special property of ordinary functions. E.g. the functional $F\{J\} = ($nondifferentiable function of $\operatorname{Max}_x |J(x)|)$, well defined for continous $J(x)$, does not possess a functional derivative.)

A functional $F\{J + \lambda J'\}$ is an ordinary function of λ and, if it is sufficiently differentiable, allows the Taylor expansion

$$\begin{aligned} F\{J+\lambda J'\} = F\{J\} &+ \\ &+ \lambda\left(\frac{\partial}{\partial\lambda} F\{J+\lambda J'\}\right)_{\lambda=0} + \\ &+ \frac{\lambda^2}{2!}\left(\frac{\partial^2}{\partial\lambda^2} F\{J+\lambda J'\}\right)_{\lambda=0} + \\ &+ \ldots \end{aligned} \tag{4.4}$$

with, if desired, breaking the expansion off with remainder term. From the definition of functional derivative one easily finds by recursion that

$$\frac{\partial^n}{\partial \lambda^n} F\{J+\lambda J'\} = \tag{4.5}$$

$$= \int \cdots \int dx_1 \cdots dx_n \; J'(x_1) \cdots J'(x_n) \; \frac{\delta^n F\{J+\lambda J'\}}{\delta J(x_1) \cdots \delta J(x_n)}$$

such that, with $\lambda = 1$,

$$F\{J+J'\} = F\{J\} + \int dx \; J'(x) \frac{\delta F\{J\}}{\delta J(x)} + \tag{4.6}$$

$$+ \frac{1}{2!} \iint dx_1 \, dx_2 \; J'(x_1) \, J'(x_2) \frac{\delta^2 F\{J\}}{\delta J(x_1) \; \delta J(x_2)} + \ldots$$

again with remainder term if desired. This is the Volterra series. Using it for $J = o$ and writing J for J', we can say that knowledge of $F\{J\}$ is tantamount to knowledge of the infinite set of ordinary functions

$$F\{o\} = \text{a number} ,$$

$$\left.\frac{\delta F\{J\}}{\delta J(x)}\right|_{J=o} \equiv F_x\{o\} ,$$

$$\left.\frac{\delta^2 F\{J\}}{\delta J(x_1)\,\delta J(x_2)}\right|_{J=o} \equiv F_{x_1 x_2}\{o\} ,$$

etc. From these functions we can recover $F\{J\}$ by the Volterra series. Sometimes one uses $F\{J\}$ only as a shorthand for that series not caring whether the series converges or not, similary as in mathematics one sometimes considers the properties of "formal power series" not caring about convergence; in that case as in ours the "functional" is merely

a convenient tool to manipulate the infinite set of coefficients, which
symmetric
are functions of an increasing number of variables, and learning about their properties. This "algebraic" use of J need not be the end, however, as also in mathematics, formal power series often converge to an interesting object or are the asymptotic expansion of one. In fact, in our case $\hat{G}_{disc}\{J\}$ itself is a transition amplitude that can in principle be measured as such and need not be considered as symbol for its expansion only, indeed the property $|\hat{G}_{disc}\{J\}| \leq 1$ makes no sense otherwise.

4.2 Green's functions

The expansion coefficients of the Volterra series for $G_{disc}\{J\}$ are called (disconnected) Green's functions, since they are analogous to Green's functions in more elementary situations where they can be introduced in the same manner.

Example: Consider the potential energy $E\{\varrho\}$ for an elastic string as a functional of the weight distribution $\varrho(x)$.
Then

$$\frac{\delta E\{\varrho\}}{\delta \varrho(x)} = a\{\varrho, x\}$$

is the elongation at point x, and

$$\left.\frac{\delta^2 E\{\varrho\}}{\delta\varrho(x)\, \delta\varrho(y)}\right|_{\varrho=0} = G(x,y)$$

is the Green's function corresponding to the elasticity equation

$$-\frac{d^2 a(x)}{d x^2} = \varrho(x) \times \text{constant}$$

with appropriate boundary conditions, which is easily solved in this simple linearized case.

We now define

$$\hat{G}_{disc}(x_1\cdots x_n;J\} \equiv {}_{out}\langle(\chi(x_1)\cdots\chi(x_n))_+\rangle_{in} \equiv \left(\pm\frac{1}{i}\right)^n \frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)}\,\hat{G}_{disc}\{J\}$$

(in the Fermi case all derivatives are to be left ones) which is according to (1.57)

$$= {}_{out}\langle T(\chi(x_1)\cdots\chi(x_n))\rangle_{in} \tag{4.7}$$

if all times are distinct. T(...) indicates (naive) time ordering: increasing times from right to left, with change of sign according to each transposition of Fermi operators. $\hat{G}_{disc}(x_1\cdots x_n;0\} \equiv G_{disc}(x_1\cdots x_n)$ with sources switched off, $G_{disc}(x_1,\ldots,x_n)$ being the covariant Green's function. For possibly coinciding times we have

$$\begin{aligned}\hat{G}_{disc}(x_1x_2;J\} &= {}_{out}\langle T(\chi(x_1)\chi(x_2))\rangle_{in} + \left(\pm\frac{1}{i}\right){}_{out}\langle\frac{\delta}{\delta J(x_1)}\chi(x_2)\rangle_{in} \\ &= \pm\,{}_{out}\langle T(\chi(x_2)\chi(x_1))\rangle_{in} \pm\left(\pm\frac{1}{i}\right){}_{out}\langle\frac{\delta}{\delta J(x_2)}\chi(x_1)\rangle_{in}\end{aligned} \tag{4.8}$$

etc. The explicit dependence of χ on J arises if the component of χ in question is a dependent variable, since from lecture 1 we know that any dependent variable has to be expressed in terms of the independent ones through solving the constraint equations, and parameter variation, i.e. in our case functional derivation with respect to J has to be carried out on that expression. The property $\bar{\mathcal{P}}\alpha^i\mathcal{P} = 0$ which was essential to have local commutation relations also for dependent variables, again insures here that the correction terms added to the time ordered products to obtain the Green's functions are local, i.e. are different from zero only if not only the time but also the space components of x_1 and x_2 coincide such that these terms are proportional to δ^4-functions or derivatives thereof.

Since amendment of the Lagrangian by a source term and functional differentiation of $\hat{G}_{disc}$ with respect to J do not make reference to any particular coordinate system, the Green's functions must be covariant if J = o. The functions $\hat{G}_{disc}(x_1 \cdots x_n; J\}$ with $J \neq o$ are of course not covariant, but they have simple transformation properties only if J is also transformed correspondingly, such that the correction terms above must remedy the possible noncovariance of the T-products. Indeed, the separation of the, by themselves covariant, Lagrange equations into constraint equations and equations of motion does require to fix the coordinate system, and as a rule also the correction terms are non-covariant, though, of course, space rotation invariant. A sufficient condition for a T-product to require a correction term for covariance is that the equal time commutator of the quantities in question is proportional to some derivative of a δ-function:

$$
\begin{aligned}
T(A(x)B(y)) &= \Theta(x^o-y^o)A(x)B(y) \pm \Theta(y^o-x^o)B(y)A(x) \\
&= \Theta(n(x-y))A(x)B(y) \pm \Theta(n(y-x))B(y)A(x)
\end{aligned}
$$

with $n^\mu = (1,o,o,o)$ as in lecture 3. From the constraint $n^\mu n_\mu = 1$ we see that $\delta n^\mu n_\mu = o$, i.e. only the space components of n^μ can be varied independently. We therefore have

$$
\begin{aligned}
\frac{\partial}{\partial n_i} T(A(x)B(y)) &= (x^i-y^i)\left(A(x)B(y) \mp B(y)A(x)\right)\delta(x^o-y^o) \\
&= \delta(x^o-y^o)(x^i-y^i)\left[A(x), B(y)\right]_\mp \quad .
\end{aligned} \tag{4.9}
$$

This is zero if $[\,,\,]_\mp \sim \delta(\vec{x}-\vec{y})$ but $\neq o$ if it is $\sim \partial^i \delta(\vec{x}-\vec{y})$ since $a\,\delta(a) = o$ but $a\,\delta'(a) = -\delta(a) \neq o$. By the method used here one can also easily see that if the equal time commutator of two quantities is singular like the n^{th} derivative (with respect to some space coordinate) of a δ^3-function then the correction term must contain a δ-function in time up to $(n-1)^{th}$ derivatives at least. Since $\frac{\delta\chi}{\delta J}$ can be as singular as a derivative of a δ-function in time only if in order to express χ by independent variables one has to differentiate the primary constraints with respect to time, it follows that the equal time commutator of any two components of χ can be as singular as a second derivative of a space-δ-function only if there are secondary constraints, e.g. in the case of spin $\geqslant \frac{3}{2}$. First space derivatives are, however, quite common and occur in the spin o and spin 1 case already (in the first order treatment we are following, where with the scalar (respectively vector) fields also their time derivatives appear as canonical variables).

4.3 Generating functional

We now develop the consequences of the equations of motion for the Green's functions generating functional $\hat{G}_{disc}\{J\}$. From (1.56) we have

$$\frac{\delta^L}{\delta J(x)} \hat{G}_{disc} = \pm i \;{}_{out}\langle \chi(x) \rangle_{in} \tag{4.10}$$

and

$$\frac{\delta^R}{\delta J(x)} \hat{G}_{disc} = i \;{}_{out}\langle \chi(x) \rangle_{in} \quad . \tag{4.11}$$

Then (cp. (3.34,35))

$$i \alpha^\mu \partial_\mu \chi - m \beta \chi - \partial^L \mathcal{H} + J = 0 \tag{4.12}$$

and

$$-i \chi \overleftarrow{\partial}_\mu \alpha^\mu - m \chi \beta - \partial^R \mathcal{H} \pm J = 0 \tag{4.13}$$

give

$$(-i\alpha^\mu \partial_\mu + m\beta) \frac{\delta^L}{\delta J(x)} \hat{G}_{disc} = \pm i J(x) \hat{G}_{disc} \mp i \;{}_{out}\langle \partial^L \mathcal{H}(\chi(x)) \rangle_{in} \tag{4.14}$$

and

$$\frac{\delta^R}{\delta J(x)} \hat{G}_{disc} (i\alpha^\mu \overleftarrow{\partial}_\mu + \beta m) = \pm i J(x) \hat{G}_{disc} - i \;{}_{out}\langle \partial^R \mathcal{H}(\chi(x)) \rangle_{in} \quad . \tag{4.15}$$

We have already defined

$$S_{\substack{ret\\adv}}(x-y) = \mathrm{Pol.}(i\,\partial^x)\ \Delta_{\substack{ret\\adv}}(x-y) \quad ,$$

$$S(x-y) = \mathrm{Pol.}(i\,\partial^x)\ \Delta(x-y) \ .$$

We need now also

$$S^{(\pm)}(x-y) = \mathrm{Pol.}(i\,\partial^x)\ \Delta^{(\pm)}(x-y)$$

and

$$S_F(x-y) = \mathrm{Pol.}(i\,\partial^x)\ \Delta_F(x-y),$$

whereof the latter is the unique solution of

$$(m\beta - i\,\alpha^\mu \partial_\mu)\ S_F(x-y) = -\,i\ \ \delta(x-y) \tag{4.16}$$

that has no positive-frequency part for $x^o < y^o$ and no negative-frequency part for $x^o > y^o$. Uniqueness is shown as follows: Every solution of

$$(m\beta - i\,\alpha^\mu \partial_\mu)\ \ f(x) = o \tag{4.17}$$

can be represented as

$$f(x) = i\int S(x-y)\ \alpha^\mu f(y)\ d\,\sigma_\mu \quad . \tag{4.18}$$

Exercise: Prove this in three steps:

1. Show that the expression on the r.h.s. of (4.18) is independent of the surface σ .
2. For y earlier than x replace S by $-S_{ret}$ and close the surface at points y later than x which does not change the integral.
3. Use Gauss' theorem and the property of retarded functions

$$S_{ret}(x-y)\ (m\beta + i\,\alpha^\mu \overleftarrow{\partial}{}^y_\mu) = \delta(x-y)$$

which follows easily from its definition.

We define the $(\pm)$-frequency parts of $f(x)$ on the basis of (4.18) by

$$f^{(\pm)}(x) = i \int S^{(\pm)}(x-y)\, \alpha^{\mu} f(y)\, d\sigma_{\mu} \quad . \tag{4.19}$$

All solutions of

$$(m \beta - i \alpha^{\mu} \partial_{\mu})\, S_{\dots}(x-y) = \delta\,(x-y) \tag{4.20}$$

can therefore at $x^o > y^o$ or $x^o < y^o$ be uniquely decomposed into positive- and negative-frequency solutions. The solution of the homogeneous equation to (4.20) that has no positive-frequency parts in the past of y and no negative-frequency parts in the future of y has neither positive- nor negative-frequency parts at any time and thus is identically zero.

We note that

$$S_F(x-y) \neq \Theta(x^o-y^o)\, iS^{(+)}(x-y) - \Theta(y^o-x^o)\, iS^{(-)}(x-y)$$

in general, since

$$\begin{aligned} S_F(x-y) &= \mathrm{Pol.}(i\,\partial^x)\, \Theta(x^o-y^o)\, i\, \Delta^{(+)}(x-y) - \\ &\quad - \mathrm{Pol.}(i\,\partial^x)\, \Theta(y^o-x^o)\, i\, \Delta^{(-)}(x-y) \\ &= \Theta(x^o-y^o)\, iS^{(+)}(x-y) - \Theta(y^o-x^o)\, iS^{(-)}(y-x) \\ &\quad + [\mathrm{Pol.}(i\,\partial^x), \Theta(x^o-y^o)]\; i\, \Delta(x-y) \\ &= S_F(x-y)_{reg} - i\, S(x-y)_{sing} \quad . \end{aligned} \tag{4.21}$$

The two terms on the right hand side are not separately covariant unless the second vanishes. Here one notes that ∂^x_i commutes with $\Theta(x^o-y^o)$ but

$$[(\partial^x_o)^n, \Theta(x^o-y^o)]\, \Delta(x-y) = \sum_{\nu=0}^{\left[\frac{n-2}{2}\right]} (\partial^x_o)^{n-2\nu-2} (\Delta - m^2)^{\nu}\, \delta^4(x-y) \tag{4.22}$$

after an easy calculation, using $\Delta(x-y)\,\big|_{x^o=y^o} = 0$,

$$\partial^x_o\, \Delta(x-y)\big|_{x^o=y^o} = -\,\delta(\vec{x}-\vec{y}) \quad \text{and} \quad (\Box + m^2)\, \Delta(x-y) = 0$$

The role of the singular term in (4.21) in connection with the dependent components of χ is seen as follows:
From (4.12) we obtain

$$\chi = Z^{\frac{1}{2}} \chi_{\substack{in \\ out}} + S_{\substack{ret \\ adv}} J - S_{\substack{ret \\ adv}} \partial^L \mathcal{H} \tag{4.23}$$

$Z^{\frac{1}{2}}$ is the "amplitude renormalisation factor" to be talked more about later. We may consider $\chi_{\substack{in \\ out}}$ as defined by this formula.

Now the Peierls' formula (1.58) becomes

$$\delta^{\substack{ret \\ adv}} \chi(x) = \delta\chi(x)\Big|_{\substack{\text{independent} \\ \text{variables fixed}}} + \int_{\mp\infty}^{x^0} [\chi(x), \chi(y)]_{\mp}\, \delta J(y)\, dy \tag{4.24}$$

while variation of (4.23) gives

$$\begin{aligned} \delta^{\substack{ret \\ adv}} \chi &= S_{\substack{ret \\ adv}} \delta J - S_{\substack{ret \\ adv}} \partial^L \partial^R \mathcal{H}\, \delta^{\substack{ret \\ adv}} \chi = \\ &= S_{sing}\, \delta J - S_{sing} \partial^L \partial^R \mathcal{H}\, \delta^{\substack{ret \\ adv}} \chi + \\ &\quad + S_{\substack{ret \\ adv}\, reg}\, \delta J - S_{\substack{ret \\ adv}\, reg}\, \partial^L \partial^R \mathcal{H}\, \delta^{\substack{ret \\ adv}} \chi \end{aligned} \tag{4.25}$$

where the last terms stem from

$$S_{\substack{ret \\ adv}}(x-y) = S_{sing}(x-y) \mp \Theta(x^0-y^0)\, S\,(x-y) \tag{4.26}$$

arrived at as in (4.21), with for (4.25)

$$S_{\substack{ret \\ adv}\, reg}(x-y) = \mp \Theta(x^0-y^0)\, S(x-y)\,. \tag{4.27}$$

The Θ-function which arises in writing out the integral in (4.24) leads (at least in the case of no secondary constraints) to

$$\begin{aligned} \delta\chi\Big|_{\substack{\text{independent} \\ \text{variables fixed}}} &= S_{sing}\, \delta J - S_{sing}\, \partial^L \partial^R \mathcal{H}\, \delta\chi\Big|_{\substack{\text{independent} \\ \text{variables fixed}}} \\ &= (1 + S_{sing} \partial^L \partial^R \mathcal{H})^{-1} S_{sing}\, \delta J \end{aligned} \tag{4.28}$$

where $\partial^L \partial^R \mathcal{H}$ contributes only if $\mathcal{H}$ is at least bilinear in dependent variables since $S_{sing.} \neq o$ only for right- and left-index belonging to a dependent component.

Exercise: Show that in the case of solvable constraints

$$S_{sing} = \frac{1}{m} (P \beta P)^{-1} = \left. \frac{\partial \chi}{\partial J} \right|_{\text{ind. variables fixed}} \qquad \text{(if } \mathcal{H} \equiv o\text{)} \tag{4.29}$$

and that for $\mathcal{H} \not\equiv o$ (3.52) leads to (4.28).

We now use S_F (which is, if we treat several fields simultanously, actually a collection of simple S_F-functions combined in a matrix) to convert the differential equations (4.14,15) into the integral equations:

$$\frac{\delta^L}{\delta J(x)} \hat{G}_{disc} = \mp \int S_F(x-y)\, J(y)\, dy \;\; \hat{G}_{disc} \;\pm$$
$$\pm \int S_F(x-y)\, dy \;\; {}_{out}\!\langle \partial^L \mathcal{H}(\chi(y)) \rangle_{in} \tag{4.30}$$

$$\frac{\delta^R}{\delta J(x)} \hat{G}_{disc} = \mp \int J(y)\, S_F(y-x)\, dy \;\; \hat{G}_{disc} \;+$$
$$+ \int {}_{out}\!\langle \partial^R \mathcal{H}(\chi(y)) \rangle_{in}\; S_F(y-x)\, dy \tag{4.31}$$

where no boundary terms appear for the following reason: The operators (cp. (4.23))

$$\chi_{\substack{in\\out}}(x) = Z^{-\frac{1}{2}} \left(\chi(x) - \int S_{\substack{ret\\adv}}(x-y) \left[J(y) - \partial^L \mathcal{H}(\chi(y)) \right] dy \right) \tag{4.32}$$

satisfy the equations

$$(m\beta - i\alpha^{\mu}\partial_{\mu})\ \chi_{\substack{in\\out}}(x) = 0 \ .$$

This does not prove that these are free fields, for which it would also be necessary that

$$[\chi_{\substack{in\\out}}(x),\ \chi_{\substack{in\\out}}(y)]_{\mp} = i\,S(x-y)\,. \qquad (4.33)$$

Actually, this is so provided $\mathcal{H}$ has been properly adjusted. The proof is not easily given in the Lagrangian formalism but supposes that the theory has axiomatic properties [26] , i.e. among other things that the mass square operator has a discrete and isolated eigenvalue m^2, and furthermore requires $\chi_{\substack{in\\out}}$ to be defined more carefully, e.g.

$$\chi^{f}_{\substack{in\\out}} = \lim_{\sigma \to \mp\infty} \chi^{f}(\sigma) \ , \qquad (4.34)$$

$$\chi^{f}(\sigma) = \int \bar{f}(x)\, \alpha^{\mu}\, \chi(x)\, d\sigma_{\mu} \ , \qquad (4.35)$$

$$\chi^{f}_{\substack{in\\out}} = \int \bar{f}(x)\, \alpha^{\mu}\, \chi_{\substack{in\\out}}(x)\ d\sigma_{\mu}\ , \qquad (4.36)$$

where f runs over a complete set of positive and negative frequency solutions of (4.17). These $\chi_{\substack{in\\out}}$ then, again frequency decomposed, are used as creation and annihilation operators in the Fock spaces of in- and out-particle states: One first defines by

$$\chi^{(+)}_{\substack{in\\out}}(x) >_{\substack{in\\out}} = 0 \quad \text{for all } x \qquad (4.37)$$

or more properly by

$$\chi^{f}_{\substack{in\\out}} >_{\substack{in\\out}} = 0 \qquad (4.38)$$

for all positive-frequency solutions f of (4.17) the in- and out-vacua, and by

$$(\chi^{f_n}_{\substack{in\\out}})^+ \cdots (\chi^{f_1}_{\substack{in\\out}})^+ >_{\substack{in\\out}}$$

with $f_1,\ldots,f_n$ positive-frequency solutions of (4.17) the n-in— respectively n-out-particle states. Since the in- and out-operators are defined through operations at arbitrarily early resp. late times, when all particles the theory is thought to describe in a fixed Heisenberg state are travelling practically freely, the order in which one performs operations on such particles is immaterial since they do not interact, unless one happens to perform two operations on the same free particle, and the contribution therefrom to the r.h.s. of (4.33) is the same as in a free field theory, i.e. is a c-number.

If $J \equiv o$, then $>_{in} = >_{out}$, but χ_{in} is not equal to χ_{out} if $\mathcal{H}$ is not merely quadratic and time-independent, such that the scattering problem is nontrivial. The S matrix elements are the scalar products

$$\begin{aligned} &\langle n \text{ out-particles} \mid n' \text{ in-particles} \rangle \\ = &\langle n \text{ in-particles} | S | n' \text{ in-particles} \rangle \\ = &\langle n \text{ out-particles} | S | n' \text{ out-particles} \rangle \end{aligned} \tag{4.39}$$

where S is defined by

$$S\ \chi_{out}(x) = \chi_{in}(x)\ S\ ,$$

$$S\,S^+ = S^+S = 1, \qquad S >_{out} = >_{in} .$$

For a thorough treatment of the problems of scattering theory see e.g. [27] .

Now the boundary term arising in the integration that leads to (4.30,31) has an in-annihilation operator acting on the in-vacuum which gives zero by definition of that state, and an out-annihilation operator action on the out-vacuum, for which also zero is obtained. One-, two- etc. particle states are characterized by the corresponding nonvanishing boundary

terms arising in the integration. However, the "reduction technique", which will be recapitulated at the end of this lecture, allows to obtain matrix elements involving non-vacuum states directly from the Green's functions with vacuum states on the sides.
Now

$$_{out}\langle\, \partial^L \mathcal{H}(\chi(x)) \,\rangle_{in} = \partial^L \mathcal{H}\left(\pm \frac{1}{i} \frac{\delta^L}{\delta J(x)}\right) \hat{G}_{disc}\{J\}$$

if all components in $\partial^L \mathcal{H}(\chi(x))$ are independent ones, or if at most one is a dependent one, since in that case

$$_{out}\langle\, \frac{\delta}{\delta J(x)} (\chi(x) \cdots \chi(x))\Big|_{indep.\ var.\ fixed} \,\rangle_{in} = 0$$

even if $\frac{\delta}{\delta J(x)}$ refers to a dependent component provided all the independent components have been produced first. If $\mathcal{H}$ contains dependent components quadratically, the above formula has nonzero terms in the right hand side. Let us treat these cases separately:

4.3.1 Feynman rules

In the simple case when $\mathcal{H}$ is at most linear in dependent variables, from (4.30) there remains to solve

$$\frac{\delta^L}{\delta J(x)} \hat{G}_{disc}\{J\} = \mp \int S_F(x-y)\, J(y)\, dy\ \hat{G}_{disc}\{J\} \pm \pm \int S_F(x-y)\, dy\ \partial^L \mathcal{H}\left(\pm \frac{1}{i} \frac{\delta^L}{\delta J(y)}\right) \hat{G}_{disc}\{J\} \,. \tag{4.40}$$

If furthermore $\mathcal{H} \equiv 0$, the solution is easily to be seen

$$\hat{G}_{disc}\{J\} = \text{const.}\ e^{\mp \frac{1}{2} \iint J(x)\, S_F(x-y)\, J(y)\, dx\, dy} \tag{4.41}$$

where, due to $\hat{G}_{disc}\{o\} = 1$ (see the beginning of this lecture), the constant must be equal to one. It is then not difficult to show that (with $S^{(1)} = i\,S^{(+)} - i\,S^{(-)}$)

$$|\hat{G}_{disc}\{J\}| = e^{\mp \frac{1}{4} \iint J(x)\, S^{(1)}(x-y)\, J(y)\, dx\, dy} \tag{4.42}$$

and that this satisfies $|G_{disc}\{J\}| \leq 1$ in the spin o, spin $\frac{1}{2}$, and spin 1 case, and the Volterra series expansion of $\hat{G}_{disc}$ clearly converges (if the exponent in (4.41) is finite) as it is an exponential series. If $\mathcal{H} \not\equiv o$, one has the unique formal solution

$$\hat{G}_{disc}\{J\} = \text{const.}\; e^{-i\int \mathcal{H}\left(\pm \frac{1}{i} \frac{\delta^L}{\delta J(x)}\right) dx} \times$$

$$\times\; e^{\mp \frac{1}{2} \iint J(x)\, S_F(x-y)\, J(y)\, dx\, dy} \tag{4.43}$$

which follows from the fact that on account of (4.40)

$$e^{i\int \mathcal{H}\left(\pm \frac{1}{i} \frac{\delta^L}{\delta J(x)}\right) dx} \cdot \hat{G}_{disc}\{J\}$$

satisfies (4.40) with $\mathcal{H} = o$. (4.43) is the final formula with the constant to be chosen such that $\hat{G}_{disc}\{o\} = 1$. If one evaluates this formula by expanding the first exponential (and to the extent that one is interested in Green's functions only, and not in $\hat{G}_{disc}\{J\}$ as a functional, expanding, also, the second exponential), one finds that a perturbation theoretical contribution to a Green's function is given by (in our case) "Feynman rules":

> Write vertices $-i\mathcal{H}(\ldots\ldots.)$ any number of times, connect them in all possible ways with each other by lines S_F and let a line S_F go to each external argument of the Green's function; integrate the vertices over all space time and sum over all such contributions, with, in case a graph admits a nontrivial mapping onto itself, the corresponding contribution to be divided by the symmetry number, i.e. the number of such possible nontrivial mappings. (Here, if $\mathcal{H}$ has L equivalent legs, it is understood that it has a factor $\frac{1}{L!}$ multiplying the coupling constant.

2 6 4

Symmetry numbers

(provided the lines are equivalent)

The constant in (4.43) has the effect that vacuum graphs, i.e. isolated graph parts with no external lines, are to be omitted. The easiest way to see that is to go to the generating functional $\hat{G} := \ln \hat{G}_{disc}$ of "truncated Green's function" also called "connected" ones. One finds by differentiating

$$\hat{G}_{disc} = e^{\hat{G}} \tag{4.44}$$

$$\begin{aligned} G_{disc}(x_1 \ldots\ldots x_n) &= G(x_1 \ldots\ldots x_n) + \\ &+ \sum_{\text{partitions}} G(x_{i_1} \ldots x_{i_r}) \quad G(x'_{j_1} \ldots x'_{j_{n-r}}) + \\ &+ \sum_{\text{partitions}} G(\ldots) \quad G(\ldots) \quad G(\ldots) \quad + \ldots \end{aligned} \tag{4.45}$$

with the sum going over all different partitions of arguments as long as each G(...) has at least two arguments that allow it to be different from zero, while inversely

$$\hat{G} = \ln \hat{G}_{disc} \tag{4.46}$$

gives

$$\begin{aligned} G(x_1 \ldots\ldots x_n) &= G_{disc}(x_1 \ldots\ldots x_n) - \\ &- 1! \sum_{\text{partitions}} G_{disc}(\ldots) \quad G_{disc}(\ldots) \quad + \\ &+ 2! \sum_{\text{partitions}} G_{disc}(\ldots) \quad G_{disc}(\ldots) \quad G_{disc}(\ldots) \quad - + \ldots \end{aligned} \tag{4.47}$$

(Whenever the order or arguments in the product terms on the r.h.s. of (4.45,47) differs from the one on the l.h.s., a factor (-1) is to be provided for each necessary transposition of Fermi variables.) which shows incidentally that $G(x_1 \cdots x_n)$ has the same covariance properties that $G_{disc}(x_1 \cdots x_n)$ has. Now for $\hat{G}\{J\}$ we have

$$\frac{\delta^L}{\delta J(x)} \hat{G}\{J\} = \mp \int S_F(x-y)\, J(y)\, dy \pm \int S_F(x-y)\, dy\; \partial^L \mathcal{H}\left(\pm \frac{1}{i} \frac{\delta^L \hat{G}}{\delta J(x)} \pm \frac{1}{i} \frac{\delta^L}{\delta J(x)}\right)$$

$$= \mp \int S_F(x-y)\, J(y)\, dy \pm \qquad (4.48)$$

$$\pm \int S_F(x-y)\, dy \Big[\partial^L \mathcal{H}\left(\pm \frac{1}{i} \frac{\delta^L}{\delta J(x)}\right) \hat{G} +$$

$$+ \sum_{\text{partitions}} \partial^L \mathcal{H}\left(\pm \frac{1}{i} \frac{\delta^L}{\delta J}, \ldots, \pm \frac{1}{i} \frac{\delta^L}{\delta J}\right) \hat{G} \cdots \hat{G} +$$

$$+ \ldots \Big] .$$

This means: if we "look into" a Green's function from one argument we see along, in any case, a S_F function, at the end of which is either another Green's function argument (the first term on the r.h.s. of (4.48)) or a vertex. All other legs starting from that vertex may either go into a connected Green's function, or into two connected Green's functions that hang together only at that vertex, or into three etc., as many as the legs of the vertex (and the other arguments of the original Green's function) permit. Iteration of (4.48) obviously leads only to connected perturbation theoretical contributions to $\hat{G}\{J\}$, and $\hat{G}\{0\} = 0$ from $\hat{G}_{disc}\{0\} = 1$.

4.3.2 Non-Feynman rules

In the more complicated case when $\mathcal{H}$ contains the dependent variables also quadratically, the present method works quite straightforwardly in giving from (1.57) and (4.28) the necessary corrections to the "Feynman rules" from

$${}_{out}\langle \partial^L \mathcal{H}(\chi(x)) \rangle_{in} = \partial^L \mathcal{H}\left(\pm \frac{1}{i} \frac{\delta^L}{\delta J(x)}\right) \hat{G}_{disc}\{J\} + \text{correction terms} \qquad (4.49)$$

which leads to "non-Feynman vertices". A particular simple way to see how they would look like is to write $g\mathcal{H}$ in place of $\mathcal{H}$ and to consider

$$\frac{\partial}{\partial g}\hat{G}_{disc}\{J\} = -i\int {}_{out}\!\!<\mathcal{H}(\chi(x))>_{in}\,dx \qquad + \text{"boundary terms"} \tag{4.50}$$

where the "boundary terms" come from the explicit dependence of ${}_{out}\!\!<$ and $>_{in}$ themselves on g. (Cf. the assumptions we had to make to obtain the formula (1.54) without such terms.) Disregarding the boundary terms for a moment gives

$$\frac{\partial}{\partial g}\hat{G}_{disc}\{J\} = -i\int dx\ \mathcal{H}\left(\pm\frac{1}{i}\frac{\delta^L}{\delta J(x)}\right)\ \hat{G}_{disc}\{J\}$$

$$+ \text{ terms from } \left(\frac{\delta\chi}{\delta J}\Big|_{\text{indep. var. fixed}} \neq 0\right) \tag{4.51}$$

where the last term can always be written

$$-i\frac{\partial}{\partial g}\int dx\ \hat{\mathcal{H}}\left(\pm\frac{1}{i}\frac{\delta^L}{\delta J(x)}, g\right)\hat{G}_{disc}\{J\}$$

and then

$$\hat{G}_{disc}\{J\} = e^{-i\int\left[g\mathcal{H}\left(\pm\frac{1}{i}\frac{\delta^L}{\delta J(x)}\right) + \hat{\mathcal{H}}\left(\pm\frac{1}{i}\frac{\delta^L}{\delta J(x)}, g\right)\right]dx}$$

$$\times\ e^{\mp\frac{1}{2}\iint J(x)\,S_F(x-y)\,J(y)\,dy} \tag{4.52}$$

The boundary terms neglected above have, as comparision with the differential-equation method would show, only the effect of supplying the constant that insures $\hat{G}_{disc}\{0\} = 1$. The present method, however, shows that the correction terms simply take the form of occurrence of extra vertices, which are, however, an infinite number since $\hat{\mathcal{H}}$ turns out as a rule to be some non-polynomial function of g (usually of the form ln(1+...g) related to the formula (4.28)) that must be expanded.

An example of this need of "non-Feynman vertices" is the theory of charged vector bosons (the W-bosons that mediate weak interaction, say) with anomalous magnetic momentum of Lee and Yang [17] in second order formulation. C.S. Lam showed in [6] that it can elegantly be treated with the Green's functions method described above. On the other hand, to avoid the non-Feynman vertices Lee and Yang introduce a unitarity and positive metric violating ξ -formalism, effectively a regularisation. Such step seemed desirable because the non-Feynman vertices are actually non-covariant and quite pathological (having a factor $\delta^4(o)$). Since ultimately the Green's functions must be covariant, the plainly untolerable non-Feynman terms must compensate similar undesirable terms the faithful evaluation [52] of "ordinary-Feynman-rules" terms give, but the way of avoiding them altogether by the first order method would seem preferable. What the first order method of course cannot do is to make the non-renormalisable theory one is considering here renormalisable; the ξ - formalism does it but it is doubtful whether this is of any significance.
Another example with non-Feynman rules in first as well as second order formalisms is that of "phenomenological Lagrangians" first obtained by Gell-Mann and Lévy from the σ-model, and later considered by Stech and Gürsey, Weinberg, Schwinger, Cronin, Wess and Zumino, L.S. Brown and others.*) Let us draw some conclusions: The method we have shown here to obtain the perturbation theoretical expansion of Green's functions (and therefore of scattering amplitudes, see later) has the following advantages over the Dyson $Te^{-i\int H}$ - formula:
i) It is manifestly covariant in every step. The Dyson formula is so only in QED and other such simple cases; it is not manifestly covariant already in e.g. QED of scalar charged particles, not to speak of charged particles of higher spins. In the more complicated cases the interaction Hamiltonian that appears in the Dyson formula contains non-covariant ("normal-dependent") terms, and the contraction functions

*) **Non-Feynman rules play a role in recent discussions of nonpolynomial chiral invariant Lagrangian models, too, [59],[60].**

$\langle T\chi\chi\rangle$ are not S_F but its non-covariant "regular part" $S_{F\ reg.}$. When one completes the $S_{F\ reg.}$ to the full covariant S_F one finds that the changes in the vertices one must introduce to keep the whole expression unchanged is just the removal of the non-covariant parts of the vertices.

ii) As we have seen above the Green's functions method gives quite straightforwardly the "non-Feynman vertices" in more complicated cases, too.

iii) Finally, the Green's function method is most suited to perform renormalization. This we will show in some more detail when we discuss QED.

Since what one desires to have are covariant Green's functions, if one is satisfied with a perturbation expansion only, it appears most economic to formulate that expansion directly in terms of covariant Feynman rules and associated renormalization prescriptions, as done by Stueckelberg and most completely by Bogoliubov and Shirkov [3] , and reformulated by Weinberg [53] .

However, perturbation theory appears not always to be sufficiently sharp a tool. E.g., for a particle of spin $\frac{3}{2}$ in interaction with an external electromagnetic field, Johnson and Sudarshan [51] and Velo and Zwanziger [54] have pointed out difficulties that do not show up in perturbation theory. Also when a theory is to possess certain symmetries (expressed e.g. in the validity of Ward identities, see lectures 7,8), it is apparently useful to formulate it as limit of a correspondingly symmetric but regularized theory such that the Lagrangian formalism, Noether's theorem, etc., do apply. It is for these reasons that the Lagrangian formalism as presented in these lectures is not outdated.

4.4 Reduction technique

We have now to go from Green's functions to observables, i.e. scattering amplitudes. This is done by the "reduction technique". After (4.38) we have already constructed the non-vacuum in- and out-states:

$$\chi_{in}^{f_1+} \cdots \chi_{in}^{f_r+} >_{in}$$

$$\chi_{out}^{g_1+} \cdots \chi_{out}^{g_m+} >_{out}$$

with $f_1 \ldots f_r$ and $g_1 \ldots g_m$ as before positive frequency solutions of the equation

$$(m\beta - i\alpha^\mu \partial_\mu) f = 0 \quad . \tag{4.17}$$

(If we take α^μ and β not only Hermitean but also to satisfy $\beta = \pm\beta^T$, $\alpha^\mu = \mp\alpha^{\mu T}$ we have in (4.17) a real equation which allows the use of real solutions. The positive- and negative-frequency parts of such solutions are complex conjugates to each other. For the spin $\frac{1}{2}$ -case this treatment means using a Majorana representation for the γ's, which can be made imaginary actually for any spin).
Now the S-matrix element (with sources)

$${}_{out}< \chi_{out}^{g_1} \cdots \chi_{out}^{g_m} \; \chi_{in}^{f_1+} \cdots \chi_{in}^{f_r+} >_{in}$$

is to be converted into an integral over a Green's function. The asymptotic condition (4.34) allows us to write this as

$$\begin{aligned} {}_{out}< \cdots >_{in} = \lim_{\sigma_m \to \infty} \cdots \lim_{\sigma_1 \to \infty} \lim_{\sigma_1' \to -\infty} \cdots\cdot \lim_{\sigma_r' \to -\infty} \; . \\ \cdot \, {}_{out}< \chi^{g_1}(\sigma_1) \cdots \chi^{g_m}(\sigma_m) \left(\chi^{f_1}(\sigma_1')\right)^+ \cdots \left(\chi^{f_r}(\sigma_r')\right)^+ >_{in} \end{aligned} \tag{4.53}$$

where we can keep all spacelike surfaces well separated in the limiting process. Then, however, since times do not coincide, we can replace this by

$$\lim_{\sigma_m\to\infty} \cdots \lim_{\sigma_1\to\infty} \lim_{\sigma_1'\to-\infty} \cdots \lim_{\sigma_r'\to-\infty} \cdot$$

$$\cdot \int\cdots\int \bar{g}_1(x_1)\,\alpha^{\mu_1}\,d\sigma_{\mu_1} \cdots \bar{g}_m\,\alpha^{\mu_m}\,d\sigma_{\mu_m} \cdot \tag{4.54}$$

$$\cdot {}_{out}\!\!<(\chi(x_1)\cdots\chi(x_m)\,\chi(x_1')\cdots\chi(x_r'))_+>_{in} \cdot$$

$$\cdot\ \alpha^{\mu_1'} f_1(x_1') \cdots \alpha^{\mu_r'} f_r(x_r')\, d\sigma_{\mu_1'} \cdots d\sigma_{\mu_r'} \quad .$$

Assume that we are interested only in scattering in the theory with no sources. Then $>_{out} = \ >_{in} \ = \ >$, and we have

$$G_{disc}(x_1\ldots x_m x_1'\ldots x_r') =$$

$$= G(x_1\ldots x_m x_1'\ldots x_r') \quad + \sum_{\text{partitions}} G(\ldots)G(\quad) + \ldots$$

for the function in (4.54). Any G(...) of two arguments hereby gives zero when both arguments are at $+\infty$ or $-\infty$ but gives a "straight through" term if one argument is at $+\infty$ and the other at $-\infty$, since in the expression

$$\lim_{\sigma\to+\infty} \lim_{\sigma\to-\infty} \iint \bar{g}(x)\,\alpha^{\mu}\,d\sigma_{\mu} <(\chi(x)\,\chi(x'))_+>\alpha^{\mu'} d\sigma_{\mu'}\; f(x') \tag{4.55}$$

the VEV $<(\chi(x)\,\chi(x'))_+>$, which is of the form "S_F + correction", is replaceable by "$iS^{(+)}$ + correction" where the correction term does not contribute in view of (4.21). (This will later be made more explicit in terms of spectral representations, and is what is achieved by mass- and amplitude renormalisation.) But because of (4.19)

$$\iint \bar{g}(x)\, \alpha^{\mu} d\sigma_{\mu}\ \ i\, S^{(+)}(x-x')\, \alpha^{\mu'} d\sigma'_{\mu'}\, f(x') =$$

$$= \int \bar{g}(x)\, \alpha^{\mu}\, f(x)\, d\sigma_{\mu} \ = \ (\bar{g}, f) \tag{4.56}$$

is the "scalar product" of the two positive-frequency solutions, i.e. a Kronecker- δ if we have an orthonormalized set, and $2k^0\, \delta(\vec{k}-\vec{k}')\, \delta(k^2-m^2)$ multiplying a covariant polarisation vector product when we use plane waves. The G with three arguments do not contribute; G with four or more arguments represent "subscattering" except for the G of all arguments, which is the "connected" scattering amplitude for all m + r particles here involved. For any such a connected term of at least four arguments Gauss' theorem gives

$$\lim_{\sigma\to\infty} \cdots \lim_{\sigma'\to-\infty} \int\cdots\int \bar{g}(x)\, \alpha^{\mu} d\sigma_{\mu} \cdots\ G(\cdots x \cdots x' \cdots)\ \ \alpha^{\mu'} d\sigma'_{\mu'}\, f(x') \ldots =$$

$$= \int i\, \bar{g}(x) (m\beta - i\alpha^{\mu}\vec{\partial}_{\mu}) \cdots\ G(\cdots x \cdots x' \cdots)\, (m\beta + i\alpha^{\mu'}\overleftarrow{\partial}_{\mu'})\, i\, f(x') \ldots dx \ldots dx' \ldots \tag{4.57}$$

Namely

$$\partial_{\mu}\, \bar{g}(x)\, \alpha^{\mu} \ldots\ G(\ldots x \ldots) \ =$$

$$= \ -i\, \bar{g}(x)\, (m\beta + i\alpha^{\mu}\overleftarrow{\partial}_{\mu}) \cdots G(\ldots x \ldots) \ + \tag{4.58}$$

$$+\ i\, \bar{g}(x)\ (m\beta - i\alpha^{\mu}\vec{\partial}_{\mu}) \cdots\ G(\ldots x \ldots)$$

whereof the first term vanishes, and similarly for the $-\infty$-terms, and a boundary term could only be a scalar product term of the type (4.56) which, however, would be disconnected from the remaining part of the

Green's function which is not possible for a connected function.

In recent literature the reduction formula is used very often in connection with current-algebra calculations for spin 1 particles where (if one uses positive metric) the T-product vacuum expectation value differs from the Green's function since the zero-components of the vector fields are then dependent variables. Then one usually goes first to the more familiar T-product and then argues that it is actually for the scattering amplitude equivalent to the covariant Green's product since the difference term is, as we have seen, local, i.e. proportional to $\delta(x-x')$ or some derivative of it, and such terms do not give singularities on the "mass shell". This is entirely correct. For some very special purposes, like use of current algebra the T-product is somewhat more convenient in spite of its non-covariance because it allows a more direct use of e.g. PCAC. That for currents (instead of, above, fields) the T-product is in general not covariant is seen without reference to a Lagrangian theory from our earlier argument that whenever an equal-time commutator of two operators contains a derivative of the δ-function, the T-product of those operators cannot be covariant, and the presence of such derivative terms, the socalled "Schwinger terms", is well known from the spectral representation of the vacuum expectation value of the product of two currents. Electromagnetic current commutators and correlation functions (defined by functional derivatives of $_{out}\langle \mid \rangle_{in}$ with respect to A^{μ}_{ext} after adding a term $j_{\mu}A^{\mu}_{ext}$ to the Lagrangian) have been investigated e.g. in [18] , [19] , [20] .

5. BASIC EQUATIONS OF QUANTUM ELECTRODYNAMICS

5.1 Maxwell equations

The Maxwell equations are

$$\dot{\vec{E}} = \vec{\nabla} \times \vec{B} - \vec{j} \quad (5.1a) \quad , \quad \dot{\vec{B}} = -\vec{\nabla} \times \vec{E} \quad (5.1b)$$

$$\vec{\nabla}\cdot\vec{E} = \varrho \quad (5.1c) \quad , \quad \vec{\nabla}\cdot\vec{B} = 0 \quad (5.1d)$$

and imply

$$\dot{\varrho} = - \vec{\nabla} \cdot \vec{j} \quad . \tag{5.2}$$

Classically, ϱ and $\vec{j}$ satisfying these equations are formed from matter variables in the particle approximation, or from matter field variables, while in QFT they are formed from the quantized matter field. We need consider their explicit form only later. (The formulas of lecture 3 naively would give, if we use, as is more familiar, a complex field instead of a Hermitean field of twice the number of components to describe charged particles, in the first-order formalism

$$j^o = e\, \psi^+ \alpha^o \psi \quad , \vec{j} = e\, \psi^+ \vec{\alpha}\, \psi$$

for "minimal electromagnetic coupling".) Passing to relativistic notation (with conventions as e.g. in Bjorken and Drell [2] , we introduce the second rank tensor of field strengths

$$F^{\mu\nu} = - F^{\nu\mu} \quad , \tag{5.3}$$

where

$$F^{oi} = E^i$$

$$F^{ij} = \varepsilon_{ijk}\, B^k = F_{ij} \quad \text{such that } B^i = \tfrac{1}{2} \varepsilon_{ijk}\, F^{jk} \quad ,$$

and the current four-vector

$$j^\mu = (\varrho , \vec{j}) \quad . \tag{5.4}$$

Thereupon (5.1a,c) and (5.1b,d) take the form

$$\partial_\nu F^{\mu\nu} = j^\mu \tag{5.5a}$$

and

$$\partial^\mu F^{\nu\varkappa} + \partial^\varkappa F^{\mu\nu} + \partial^\nu F^{\varkappa\mu} = o \tag{5.5b}$$

respectively, while (5.2) becomes

$$\partial_\mu j^\mu = 0 \quad . \tag{5.6}$$

(Another way of writing (5.5b) is by introducing the dual tensor

$$\hat{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\varkappa\lambda} \quad F^{\varkappa\lambda}$$

such that

$$\hat{F}_{oi} = B^i, \quad \hat{F}_{ij} = - \varepsilon_{ijk} E^k$$

and

$$\frac{1}{2} \varepsilon^{\mu\nu\varkappa\lambda} \hat{F}_{\varkappa\lambda} = - F^{\mu\nu} \quad .$$

Then (5.5b) becomes

$$\partial^\nu \hat{F}_{\mu\nu} = 0.$$

Together with (5.5a), this symmetric way of writing the Maxwell equations is the starting point of introducing "magnetic charge" which has received some renewed discussion recently, but we will not concern ourselves with it.)

Remark: If we integrate (5.5a) over a spacelike surface and use Stokes' theorem (3.71) we obtain

$$\int \partial_\nu F^{\mu\nu} d\sigma_\mu = \int \partial_\mu F^{\mu\nu} d\sigma_\nu = \frac{1}{2} \int [\partial_\nu F^{\mu\nu} d\sigma_\mu + \partial_\mu F^{\mu\nu} d\sigma_\nu] = 0$$

because of (5.3), and thus $\int j^\mu d\sigma_\mu = 0$ i.e. the total charge must be zero which is generally not the case. The error is in the incorrect use of Stokes' theorem, which requires the integrand R to vanish faster than $\frac{1}{r^2}$ in space-like infinity. We therefore conclude that $F^{\mu\nu}$ (more precisely: F^{oi}) cannot vanish that fast in spacelike infinity if the total charge is not zero. (The same conclusion can also be drawn by integrating (5.1c) over space.) One has to keep that long range in mind, and we will later always have to be careful when writing integrals over all space, and in particular when manipulating them, and convince ourselves that we are not committing errors. We could be more careless if the photon mass were finite and the Coulomb potential replaced by the exponentially decreasing Yukawa potential.

5.2 Vector potentials and gauge freedom

Equations (5.5a,b) cannot be derived from an invariant Lagrangian density since their l.h.s. transform like a vector or a third rank tensor, and we have available as variable only a second rank tensor. However, as in classical theory, this is no obstacle for the solution of the quantum problem, if we can take j^μ as given operator field that commutes with $F^{\mu\nu}$, in particular, as a c-number field. Namely, (5.5a,b) give

$$\Box F^{\mu\nu} = \partial^\nu j^\mu - \partial^\mu j^\nu \tag{5.7}$$

with the solution

$$F^{\mu\nu}(x) = F^{\mu\nu}_{\substack{in\\out}}(x) + \partial^\nu \int D_{\substack{ret\\adv}}(x-y) j^\mu(y) dy - \partial^\mu \int D_{\substack{ret\\adv}}(x-y) j^\nu(y) dy \tag{5.8}$$

where $F^{\mu\nu}_{\substack{in\\out}}$ satisfy (5.5a,b) with $j^\mu \equiv o$, and

$$D_{\substack{ret\\adv}}(x-y) = \frac{1}{(2\pi)^4} \int \frac{e^{-ik(x-y)}}{-k^2 \mp i\varepsilon k^o} dk \quad , \quad \varepsilon \to +o \quad .$$

The Maxwell energy-momentum tensor is symmetric

$$\Theta^{\mu\nu}_{Max} = - F^{\mu\lambda} F^\nu{}_\lambda + \frac{1}{4} g^{\mu\nu} F^{\varkappa\lambda} F_{\varkappa\lambda} \tag{5.9}$$

such that

$$M^{\mu\nu\lambda}_{Max} = x^{\mu}\,\Theta^{\nu\lambda}_{Max} - x^{\nu}\,\Theta^{\mu\lambda}_{Max}$$

and we have

$$\partial_{\nu}\,\Theta^{\mu\nu}_{Max} = j_{\lambda}\,F^{\mu\lambda} \tag{5.10}$$

where the r.h.s. represents the density of energy-momentum transfer from matter to the electromagnetic field. The commutation relations that give for $j^{\mu} = o$ (i.e. for the in- and out-fields) the desired relations (2.22,23)

$$[P^{\mu}, F^{\varkappa\lambda}] = -i\,\partial^{\mu}F^{\varkappa\lambda} \tag{5.11}$$

$$[M^{\mu\nu}, F^{\varkappa\lambda}] = -i\,(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})\,F^{\varkappa\lambda} \tag{5.12}$$

$$-i\,(g^{\nu\varkappa}F^{\mu\lambda} - g^{\mu\varkappa}F^{\nu\lambda} + g^{\mu\lambda}F^{\nu\varkappa} - g^{\nu\lambda}F^{\mu\varkappa})$$

are

$$[F^{\mu\nu}(x), F^{\varkappa\lambda}(y)] = i\,(g^{\mu\varkappa}\partial^{\nu}\partial^{\lambda} - g^{\nu\varkappa}\partial^{\mu}\partial^{\lambda} +$$

$$+ g^{\nu\lambda}\partial^{\mu}\partial^{\varkappa} - g^{\mu\lambda}\partial^{\nu}\partial^{\varkappa})\;D(x-y) \tag{5.13}$$

where

$$D(x-y) = D_{adv}(x-y) - D_{ret}(x-y)\;.$$

If j^{μ} commutes at all times with $F^{\mu\nu}$ and j^{ν} , (5.13) also gives, as follows from (5.8), the commutation relations of the "interacting" $F^{\varkappa\lambda}$ itself. The solution of (5.13) that fulfills (5.5a,b) is easily obtained by analysing the Fourier transforms in terms of creation and annihilation operators. One finds photons with helicities +1 and -1. For details, see e.g. [1] , [2] , [3] , [6] . These photons are emitted and absorbed by the given current j^μ , and if j^μ can be taken to commute with $F^{\mu\nu}$ at all times (or physically speaking, in the approximation where we can neglect the reaction of the electromagnetic field on the relatively heavy charged bodies), one obtains the S-matrix and all transition amplitudes for the electromagnetic field easily in explicit form in terms of matrix elements of j^μ or products thereof with respect to matter states. For details about this we recommend particularly [5] and [12] .

This approach does not suffice if the reaction of the electromagnetic field on the charges is to be taken into account, i.e. if the $F^{\mu\nu}$ cannot be taken to commute with the j^μ . We know the equation of motion for point-like particles

$$\frac{d}{d\tau} m \frac{d\vec{x}(\tau)}{d\tau} = e\left(\vec{E}(x(\tau))\frac{dx^0(\tau)}{d\tau} + \frac{d\vec{x}(\tau)}{d\tau} \times \vec{B}(x(\tau))\right) \tag{5.14}$$

with τ being the proper time

$$d\tau^2 = dx_\mu dx^\mu = (dx^0)^2 - (d\vec{x})^2$$

and j^μ is related to the particle trajectory $x(\tau)$ by

$$\rho(x) = e\int_{-\infty}^{\infty} dx^0(\tau)\,\delta(x - x(\tau)) ,$$

$$\vec{j}(x) = e\int_{-\infty}^{\infty} d\vec{x}(\tau)\,\delta(x - x(\tau)) .$$

(5.14) is in covariant form

$$\frac{d}{d\tau} m \frac{dx^\mu(\tau)}{d\tau} = -e\, F^{\mu\nu}(x(\tau)) \frac{dx_\nu(\tau)}{d\tau} =$$

$$= -\int F^{\mu\nu}(\vec{x}, t(\tau))\, j_\nu(\vec{x}, t(\tau)) \frac{dx^0(\tau)}{d\tau}\, d\vec{x}$$

where the r.h.s., which is essentially the negative of r.h.s. of (5.10), represents the energy-momentum transferred from the electromagnetic field to the charged particle (one finds an analysis of these equations in [12]). However, for a consistent QFT treatment of matter and electromagnetic field in interaction one has to have quantum field equations and commutation relations consistent with them, and the only way known to obtain these is to start from a Lagrangian for the whole system and to apply canonical quantization.

Obtaining (5.5a,b) from a Lagrangian density clearly demands the use of a vector field, the field whose necessity is also known from CED, where (5.14) is put in Lagrangian form as derived from

$$\begin{aligned} L(t)\, dt &= -m\, d\tau + e\, A^\mu(x(\tau))\, dx_\mu(\tau) = \\ &= -m\, d\tau + dt \int A^\mu(\vec{x}, t)\, j_\mu(\vec{x}, t)\, d\vec{x} \end{aligned} \qquad (5.15)$$

where the space components of the vector potential A^μ are necessitated as "generalized potential" by the velocity-dependent force in (5.14), the Lorentz force (Cf. e.g. [21]). Here we have

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) \qquad (5.16)$$

which is the general solution of (5.5b). (Our A^μ differs in sign from the one used by Bjorken and Drell [2] .) Namely, it is immediate that (5.16) satisfies (5.5b), and if (5.16) holds with some A^μ then, since the most general solution of

$$\partial^\nu \Delta A^\mu(x) - \partial^\mu \Delta A^\nu(x) = 0$$

is

$$\Delta A^\mu(x) = \partial^\mu \Lambda(x), \tag{5.17}$$

the general A^μ differs from any particular one by a "gauge transformation" term $\partial^\mu \Lambda$. Thus it suffices to exhibit some A^μ that satisfies (5.16):
Consider

$$A^\mu(x) = \int_x^\infty F^{\mu\nu}(x')\, dx'_\nu = \int_0^\infty F^{\mu\nu}(x+z)\, dz_\nu \tag{5.18}$$

where we integrate along an x-independent spacelike z-path to obtain for A^μ a simple transformation law under translations analogous to (5.11). (In (5.18) we need not fear a convergence problem since $F^{\mu\nu}$ needs, for (5.18) to converge, only to go to zero more strongly than $\frac{1}{r}$ in infinity which decrease is consistent also with nonzero charge.)
Now

$$\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) = \int_0^\infty [\partial^\mu F^{\nu\varkappa}(x+z) - \partial^\nu F^{\mu\varkappa}(x+z)]\, dz_\varkappa =$$

$$= \int_0^\infty [\partial^\varkappa F^{\nu\mu}(x+z)]\, dz_\varkappa =$$

$$= F^{\mu\nu}(x) - F^{\mu\nu}(\infty) =$$

$$= F^{\mu\nu}(x)$$

as required. Here we have used (5.3), (5.5b), and the fact that is safe to set $F^{\mu\nu}(\infty) = 0$. (The differentiation under the integral sign is

permissible here, since in general the new integrals will converge even better than that in (5.18).)
Different choices of z-paths will give different gauges; in fact,

$$\left(\int_0^{\infty^{(1)}} - \int_0^{\infty^{(2)}} \right) F^{\mu\nu}(x+z)\, dz_\nu =$$

$$= \frac{1}{2} \int_V dO_{\nu\lambda}(z) \left[\partial^\lambda F^{\mu\nu} - \partial^\nu F^{\mu\lambda} \right](x+z) = \tag{5.19a}$$

$$= \partial^\mu \Lambda(x)$$

the two dimensional surface integral going over a sector with vertex at x (respectively 0 for z), and with

$$\Lambda(x) = -\frac{1}{2} \int_V dO_{\nu\lambda}(z)\; F^{\nu\lambda}(x+z) \tag{5.19b}$$

by use of (5.5b). (For this Λ -expression to exist $F^{\nu\lambda}$ should go to zero more strongly than $\frac{1}{r^2}$ in infinity, if it does not go so fast but more strongly than $\frac{1}{r}$, $\partial_\mu \Lambda(x)$ still exists with the property $\partial_\nu(\partial_\mu \Lambda(x)) - \partial_\mu(\partial_\nu \Lambda(x)) = 0$). The gauges we have so obtained belong to a narrow class since they lead due to (5.5a) to

$$\partial_\mu A^\mu(x) = -\int_0^\infty j_\nu(x+z)\, dz^\nu$$

which vanishes in the absence of charges, i.e. A^μ then satisfies the Lorentz condition which is clearly a special property.
One might think to obtain a covariantly transforming $A^\mu(x)$ by averaging (5.18) over paths covariantly, which would formally yield

$$A^\mu(x) = \int \partial^x_\nu \bar{D}(x-x')\, F^{\mu\nu}(x')\, dx' \tag{5.20}$$

with

$$\bar{D}(x) = -\frac{1}{2}\,\mathrm{sign}(x^0)\, D(x) \; .$$

(<u>Exercise:</u> Check this.)
Unfortunately, this integral does not converge since (5.7) indicates that the Fourier transform $F^{\mu\nu}(k)$ of $F^{\mu\nu}(x)$ is singular on $k^2 = o$, and this is also the singularity of $\bar{D}(k) = -P\,\frac{1}{k^2}$. In fact, we shall later prove that there cannot exist a covariant vector potential that could be expressed by $F^{\mu\nu}$ in any manner. (For a theory of massive photons the coincidence of singularities just mentioned does not occur, and the expression corresponding to (5.20) does give a covariant vector potential). The best one can do is to obtain a vector potential that transforms covariantly under space rotations only, by averaging (5.18) spherically symmetrically over straight line paths at a fixed time. To the resulting Coulomb or radiation gauge we shall come back later. From what we found about (5.18) it is clear that when we perform a Lorentz transformation that involves an acceleration, the radiation-gauge A^μ will transform like a four-vector only up to a gauge transformation whose explicit form in terms of $F^{\mu\nu}$ can be deduced from (5.19).

Equations (5.5a,b) in the solved form (5.16) are obtainable from the scalar Lagrangian density

$$L = -\frac{1}{2} F_{\mu\nu}(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j_\mu A^\mu + \frac{m^2}{2} A_\mu A^\mu \qquad (5.21)$$

where we have also added a mass term for the purpose of later comparison, since the Lagrange equations to (5.21) are (5.16) and (5.5a), the latter in the mass-term amended form

$$\partial_\nu F^{\mu\nu} - m^2 A^\mu = j^\mu \qquad (5.22)$$

Putting (5.21) into the first order standard form would require to introduce a 1o-component $\chi = (A^{\cdot}, F^{\cdot\cdot})$ and 10 x 10 matrices α^{μ} and β. By a change of scale, if $m^2 \neq o$, one then obtains the Duffin-Kemmer - Pétiau form of the Lagrangian for a spin 1 field. If $m^2 = o$, the scale change is not possible, and the β of the normal form turns out to be so singular, that the Lagrange equations do not determine the motion. We see this by separating equations of motion

$$\partial^o F^{oi} = -j^i + \partial_j F^{ij} - m^2 A^i \tag{5.23a}$$

$$\partial^o A^i = \partial^i A^o + F^{oi} \tag{5.23b}$$

from equations of constraint

$$F^{ij} = \partial^i A^j - \partial^j A^i \tag{5.24a}$$

$$\partial_i F^{oi} = j^o + m^2 A^o \quad . \tag{5.24b}$$

Equation (5.24a) gives the F^{ij} occuring in (5.23a) in terms of A^i, but (5.24b) can be solved for A^o needed in (5.23b) only if $m^2 \neq o$ in which case we would have the usual canonical scheme. With $m^2 = o$, A^o occuring only in (5.23b) is completely arbitrary while (5.24b) rather becomes one constraint too many, and is compatible with (5.23a) if and only if (5.6) holds. Thus, a retarded or advanced Green's function does not exist in the $m^2 = o$ case since the motion of the variables is not determined by any Cauchy data, and moreover the classical action integral has a stationary point (or rather, manifold) only if the current is conserved.

In more detail, for $m^2 = o$ we define of any three-vector fields B^i the longitudinal part B^{iL} by

$$B^{iL}(x) := \partial_i \left[(-\Delta)^{-1} \partial_j B^j \right](x)$$

where $\Delta = -\partial_i \partial^i$ is the Laplacian and

$$(-\Delta)^{-1}\Big|_{xx'} = \delta(x^o - x'^o) \frac{1}{4\pi\,|\vec{x} - \vec{x}'|}$$

the Coulomb potential as an integral operator.

$$B^{iT} = B^i - B^{iL} = (g^i{}_j + \partial^i(-\Delta)^{-1}\partial_j)\, B^j =$$

$$= T^i{}_j\, B^j$$

is the transverse part. Here $T^i{}_j$ is a projection operator that commutes with derivatives (for fields that go to zero sufficiently strongly in infinity such that the space integrals involved here converge). Now (5.24b) gives the longitudinal part of F^{oi} as

$$F^{oiL} = \partial_i (-\Delta)^{-1} j^o \tag{5.25}$$

and the longitudinal part of (5.23a) becomes the consistency condition (5.6). There remains the transverse part of (5.23a),

$$\partial^o F^{oiT} = -j^{iT} + \partial_j F^{ij} \tag{5.26}$$

We also decompose A^i and have from (5.24a)

$$F^{ij} = \partial^i A^{jT} - \partial^j A^{iT} \tag{5.27}$$

and from (5.23b)

$$\partial^o A^{iT} = F^{oiT} \tag{5.28}$$

There remains from (5.23b)

$$\partial^o A^{iL} = \partial^i A^o + F^{oiL} ,$$

i.e. with (5.25)

$$\partial^0 A^{iL} - \partial^i A^0 = \partial_i (-\Delta)^{-1} j^0 \tag{5.29}$$

which yields A^{iL} in terms of A^0, or somewhat more naturally vice versa, the general solution being obtained from any particular one by

$$A^0 , A^{iL} \longrightarrow A^0 + \partial^0 \Lambda , \quad A^{iL} + \partial^i \Lambda$$

with Λ an arbitrary space-time function, which transition is the gauge transformation (5.17). $A^{iL} \equiv 0$ gives the Coulomb gauge

$$A^0 = (-\Delta)^{-1} j^0 \tag{5.30}$$

while e.g. setting $A^0 = 0$ and determining A^{iL} from (5.29) would be less acceptable since it would amount to integration in (5.18) over a straight line parallel to the time axis which sometimes does not give a convergent integral.

If one inserts $F^{\mu\nu}$ and A^μ in the decompositions given above into (5.21), adopts for definiteness the Coulomb gauge and discards terms that vanish by partial integration due to vanishing boundary terms, one obtains a non-covariant Lagrangian and action integral whose Lagrange equations are the equations (5.26) - (5.28). This Lagrangian is one possible starting point to treat QED along the canonical scheme and is followed by e.g. Bjorken and Drell [2] .

5.3 Canonical quantization of quantum electrodynamics

It is extremely difficult, however, to make non-lowest-order calculations in the Coulomb gauge because for the unambiguous cancellation of infinite expressions that arise in higher orders manifest covariance is almost mandatory. In fact, Bjorken and Drell [2] proceed by first manipulating the Coulomb gauge S-matrix, in which they are ultimately only interested, formally into covariant form (Cf. also [22]). We shall go the other way round and obtain for all quantities, Green's funtions and not only S-matrix elements, covariant expressions, and will show that our theory, the Gupta-Bleuler gauge approach, is equivalent to the Coulomb gauge one, i.e. does describe the physics of QED correctly.

We now turn back to the Lagrangian (5.21) with $m^2 = o$ understood unless said otherwise. As it stands it is gauge invariant only up to a four-divergence which is, however, cancelled by a term coming from the gauge transformation of the matter field variable in the matter Lagrangian not written here. We now add to (5.21) terms that do break gauge invariance but allow to quantize canonically. In the next lecture these terms will be shown to be innocuous as just remarked, essentially because they only restrict the physically unnecessary freedom in (5.23,24). We choose

$$L = -\frac{1}{2} F_{\mu\nu}(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + j^\mu A_\mu + \tag{5.31}$$

$$- \underbrace{B\,\partial^\mu A_\mu + \frac{s}{2} B^2}_{\sim\, -\frac{1}{2s}(\partial A)^2} + J^\mu A_\mu + K B$$

where we have added also source terms KB and $J^\mu A_\mu$ for application of the Peierls' variational method, with the c-number function K later set equal to zero but possibly retaining $J^\mu \neq o$ representing an externally given c-number current distribution subject to $\partial_\mu J^\mu = o$. B is to be a Hermitean scalar field, and s a real number to be fixed later. The Lagrange equations are now

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu , \tag{5.32}$$

$$\partial_\nu F^{\mu\nu} = j^\mu + J^\mu + \partial^\mu B , \tag{5.33}$$

$$\partial_\mu A^\mu = s B + K , \tag{5.34}$$

where we have assumed that the functional expression for j^μ does not contain $F^{\mu\nu}$, A^μ, B or their derivatives. (Although the form of j^μ given after (5.2) satisfies this requirement, this is not quite so trivial a point as it may seem. We will discuss this question some time later in detail.) To depart from the original situation as little as possible, we will keep

$$\partial_\mu j^\mu = o \tag{5.35}$$

which is here to be taken as a consequence of the matter field equations, since (5.33) no longer demands it by itself as a consistency condition. The fact that (5.33) is not the Maxwell equation (5.5a) will later be a central point.

To explain (5.31), we note that due to (5.34) for s = o the B-term in (5.31) plays simply the role of a Lagrange multiplier that enforces the Lorentz condition $\partial_\mu A^\mu = o$ (respectively, with source, $\partial_\mu A^\mu = K$) to hold. In CM a constraint imposed by a Lagrange multiplier term contributes a constraining force to the equations of motion, the precise value of which is to be calculated with the help of the constraint itself. In QM the Lagrangian multiplier must become, for consistency, a q-variable, and in particular in QFT a quantized field; in (5.33) we have the amended equation of motion and in (5.34) with s = K = o the constraint. Once we have recognized, however, that we must admit an extrafield, we can change the dynamics of that field a little by the quadratic B-term in (5.31), which turns out to be useful since in QM we also have to keep an eye on the state space and this happens to be simplest with s having a particular nonzero value. The "transition to suitable variables" that in CM often relieves the need for a constraint (think of the Lagrange equations of motion in generalized coordinates) is here an operator gauge transformation. We will in the next lecture see how such a noncovariant transformation leads us from (5.31) and its variables to the Coulomb gauge. In lecture 7 we will treat for comparison also the case of nonzero photon mass where, in addition to the noncovariant operator gauge transformation to the Yukawa gauge, there exists a covariant one to the Proca or Lorentz gauge, the one that is most natural in that theory anyway; however, the way to it via a Lagrangian with constraint built in by extra variables does have advantages as we shall see.

The Lagrangian (5.31) is of the canonical form $L = p\dot{q} - H(p,q,r)$ discussed in section 1.1.2, with the constraint equations $H_r(p,q,r) = o$ solvable for r. The independent field variables are $q = (A^o, A^i)$ and $p = (-B, F^{oi})$, with the dependent variables $r = (F^{ij})$ to be obtained from (5.32). The canonical equal-time commutators (1.21) then read

$$[F^{oi}(x), A^j(y)]_{x^o = y^o} = i\, g^{ij}\, \delta^3(\vec{x}-\vec{y}) \tag{5.36}$$

$$[B(x), A^o(y)]_{x^o = y^o} = i\, \delta^3(\vec{x}-\vec{y}) \tag{5.37}$$

$$[F^{oi}(x), A^o(y)]_{x^o = y^o} = o \tag{5.38a}$$

$$[B(x), A^i(y)]_{x^o = y^o} = o \tag{5.38b}$$

$$[F^{oi}(x), F^{oj}(y)]_{x^o = y^o} = o \tag{5.38c}$$

$$[F^{oi}(x), B(y)]_{x^o = y^o} = o \tag{5.38d}$$

$$[B(x), B(y)]_{x^o = y^o} = o \tag{5.38e}$$

$$[A^\mu(x), A^\nu(y)]_{x^o = y^o} = o \tag{5.38f}$$

Note that we have here assumed that j^μ does not contain hidden time derivatives of F^{oi}, A^μ, or B. Although this seems already implied in our remark about (5.32) - (5.34), it is here a somewhat stronger assumption since ETC are more singular objects than single terms in the field equations. In fact, we shall later see that j^i does contain the fields A^k in a hidden manner, such that, however, the equations (5.32) - (5.34) require only a somewhat elaborate writing of j^μ in (5.33), but no change of (5.36) - (5.38) takes place since no time derivatives, even hidden ones, occur in j^μ.

We emphasize that we are not discussing the most general theory with a formal appearance similar to QED, but QED ultimately defined by its famous renormalized perturbation series solution itself, and that we want to formulate in a reasonably logically consistent and intuitively satisfactory way a theory for which that expansion is a formal solution. With more elaborate mathematics the points glossed over here can be cleared up [55] (except for, at present, the state space problem of the infrared catastrophy, cf. sect. 6.2)

First we will derive some immediate consequences from the field equations and the canonical commutation relations (CCR). We will take these as source of information about the so far only vaguely defined term j^μ in the equations of motion, for which we will use its vector character and (5.35). We therefore separate the equations of motion

$$\partial^o A^i = \partial^i A^o + F^{oi} \quad , \tag{5.32a}$$

$$\partial^o F^{oi} = \partial_j F^{ij} - j^i - J^i - \partial^i B \; , \tag{5.33a}$$

$$\partial^o B = + \partial_i F^{oi} - j^o - J^o \quad , \tag{5.33b}$$

$$\partial^o A^o = - \partial_i A^i + sB + K \; . \tag{5.34}$$

from the constraint equations

$$F^{ij} = \partial^i A^j - \partial^j A^i \tag{5.32b}$$

and rewrite the imposed property of current conservation

$$\partial^o j^o = - \partial_i j^i \; . \tag{5.35}$$

We now differentiate in turn all commutators (5.36) - (5.38) with respect to time and obtain in that order (from (5.39) till (5.48) all commutators are to be taken at equal times):

$$\left[j^i,\ A^j\right] = 0 \tag{5.39}$$

$$\left[j^0,\ A^0\right] = 0 \tag{5.40}$$

$$\left[j^i,\ A^0\right] = 0 \tag{5.41a}$$

$$\left[j^0,\ A^i\right] = 0 \tag{5.41b}$$

$$\left[j^i,\ F^{0j}\right] + \left[F^{0i},\ j^j\right] = 0 \tag{5.41c}$$

$$\left[j^i,\ B\right] + \left[F^{0i},\ j^0\right] = 0 \tag{5.41d}$$

$$\left[j^0,\ B\right] + \left[B\ ,\ j^0\right] = 0 \tag{5.41e}$$

(5.41f) is an identity

Here (5.35) has not yet been used. (5.39,40,41a,b) can be combined to

$$\left[j^\mu,\ A^\nu\right] = 0 \tag{5.42}$$

which means in view of (5.36,37) that the current operator does not have a hidden dependence on F^{0i} or B.
The equation (5.34,35,40) give

$$\left[j^0,\ B\right] = 0 \quad \text{if} \quad s \neq 0 \tag{5.43}$$

while (5.32b,35,41b) give

$$\left[j^0, F^{0i}\right] = 0 \ . \tag{5.44}$$

Thus (5.41d) becomes

$$\left[j^i,\ B\right] = 0 \tag{5.45}$$

and time differentiation of (5.41f) gives with (5.33b,35)

$$[j^o, j^o] = o \tag{5.46}$$

whereas (5.44) gives upon time differentiation

$$[j^o, j^i] = - [\partial_k j^k, F^{oi}] - [j^o, \partial^i B] . \tag{5.47}$$

From (5.43) we see that the last term in this equation vanishes if $s \neq o$ but later on we shall see (cf. (5.53)) that this is also generally true. Therefore the charge-current-density commutator is a derivative one.

Commuting (5.38c) with M^{ok} and using (5.41c) formally gives (cf. [23])

$$(x^k - y^k) [j^i, F^{oj}] = o \quad \text{for all } i,j,k$$

which means that

$$[j^i(x), F^{oj}(y)]_{x^o = y^o} \sim \delta(\vec{x} - \vec{y}) , \tag{5.48}$$

i.e. it does not contain a derivative of a δ-function term, such that from (5.47) we learn that $[j^o, j^i]$ contains no higher than first-order space derivatives of a δ-function. The space integral over the j^o-argument gives zero for this commutator, i.e. it is a pure "Schwinger term".

Altogether we have learnt that j^o commutes with itself and with all canonical variables of the photon field, i.e. has no hidden dependence on photon field variables, while j^i may contain A^k in a hidden manner.

It must be stressed that all these conclusions are somewhat formal in the sense that the existence of ETC and their termwise differentiability is taken for granted, both of which assumptions are open to debate. In general the justifiability of those formal manipulations depends on what matrix elements one is considering, because the strength of the singularities depends on the states between which the operators are sandwiched. The reason for this can easily be seen from the more satisfactory discussion of ETCR due to Wilson [24] which is based in the analysis of the small distance behaviour of operator products:

$$A(x)B(o) = \sum_i C_i(x)\, O_i(o) \quad + \quad \text{nonsingular operator}$$

where the $C_i(x)$ are covariant, at $x \rightarrow 0$ singular c-number functions. This expansion implies [24]

$$[A(x),\, B(o)] = \sum_i E_i(x)\, O_i(o) \quad + \text{nonsingular operator}$$

where the singular functions $E_i(x)$ are simply related to the $C_i(x)$. Up to every finite order of perturbation theory the functions $C_i(x)$ give rise to singularities (apart from logs) of order $\left(\frac{1}{x}\right)^{d_A + d_B - d_i}$ where the d's are the mass dimensions of the fields, provided the local operators $O_i(o)$ do have non-zero matrix elements between the two states considered. E.g., the most singular coefficient $E_o(x)$ is in general the one of the unit operator such that it does contribute only to the VEV of the commutator (if one only considers connected diagrams).

In addition, we have neglected the presumed need for renormalization, i.e. assumed that the scale factors it involves are finite. Most of the relations we have so derived have been obtained in lowest order of perturbation theory for renormalized QED by R. Brandt [25] , however the commutator of current and charge density is in lowest order perturbation theory more singular than we obtained, which difference can be traced to infinite renormalization in that order (of the current correlation function).

Next we consider the field equations. We obtain from (5.33) and (5.35)

$$\Box B = - \partial_\mu J^\mu \tag{5.49}$$

i.e. B is a free field with only the external source, if, for variational purposes, it is kept arbitrary instead of restricting it to $\partial_\mu J^\mu = 0$. Using the Green's function introduced in (5.13) we obtain

$$(\delta^{\text{ret}} - \delta^{\text{adv}})\, B(x) = \int D(x-y)\, \partial_\mu \delta J^\mu(y)\, dy$$

whereas Peierls' formula (3.29) gives

$$\text{l.h.s.} = i \int [B(x),(A_\mu(y)\delta J^\mu(y) + B(y)\delta K(y))] \quad dy \quad .$$

Therefrom we find the CR at arbitrary times

$$[B(x), A_\mu(y)] = -i\, \partial_\mu^x D(x-y) \tag{5.50}$$

and

$$[B(x), B(y)] = o \; . \tag{5.51}$$

Equation (5.50) actually follows also from (5.49,37,40,38b and 41b), and thus with (5.34) (5.51) follows if $s \neq o$, while for $s = o$ it appears that (5.51) is an original contribution of the Peierls' principle on this formal level. (5.50) gives with (5.32)

$$[B(x), F^{\mu\nu}(y)] = 0 \tag{5.52}$$

and (5.51) gives with (5.33)

$$[B(x), j^\mu(y)] = 0 \, , \tag{5.53}$$

relations that we will later interpret as showing that $F^{\mu\nu}$ and j^μ are invariant under "gauge transformations of the second kind" (of a very restricted class, however, see (6,28)). B is not an ordinary free field of zero mass since as such it should have a commutator with itself proportional to D(x-y) instead of 0, cf. (5.51). But (5.50) shows that B(x) is not identically zero either.

5.4 Spectral representations

We eliminate $F^{\mu\nu}$ from (5.32,33) and obtain with (5.34)

$$\Box A^\mu = -j^\mu + (s-1)\,\partial^\mu B - J^\mu + \partial^\mu K \tag{5.54}$$

which suggests the definitions

$$ {}^{o}A^{\mu}_{\substack{in \\ out}} = A^{\mu} - \int D_{\substack{ret \\ adv}} \left(-j^{\mu} + (s-1)\partial^{\mu}B - J^{\mu} + \partial^{\mu}K\right) . \tag{5.55} $$

However, for a vector field the commutator even if it is a c-number can have various forms, so in order to investigate ${}^{o}A^{\mu}_{\substack{in \\ out}}$ we will now use a different sort of argument.

We set in (5.31-34) $J^{\mu} \equiv o$, $K \equiv o$, whereupon, assuming that to (5.31) is added a scalar matter Lagrangian and that j^{μ} is a covariant vector field, we have a manifestly Poincaré invariant theory. Assuming that a vacuum $>$ exists with $P^{\mu} > \,=\, M^{\mu\nu} > \,=\, o$ and that all other eigenvalues of P^{μ} and $P^{\mu}P_{\mu}$ satisfy $P^{o\prime} > o$, $(P_{\mu}P^{\mu})' > o$, the vacuum expectation value (VEV) of the vector potential operator must have the form

$$ \langle A^{\mu}(x)\, A^{\nu}(y) \rangle = \langle A^{\mu}(o)\, e^{-i\mathbb{P}_{\lambda}(x-y)^{\lambda}} A^{\nu}(o) \rangle = \tag{5.56a} $$

$$ = \int dK\; e^{-iK(x-y)} \langle A^{\mu}(o)\, \delta(\mathbb{P}-K)\, A^{\nu}(o) \rangle . $$

Now for covariance and spectral reasons we must have

$$ \langle A^{\mu}(o)\, \delta(\mathbb{P}-K)\, A^{\nu}(o) \rangle = \tag{5.56b} $$

$$ = (2\pi)^{-3} \left[-g^{\mu\nu}\, {}^{o}\rho_{1}(K^{2}) + K^{\mu}K^{\nu}\, {}^{o}\rho_{2}(K^{2}) \right] \Theta(K^{o})\, \Theta(K^{2}) $$

with ${}^{o}\rho_{1,2}(K^{2})$ real due to Hermiticity (a concept to be clarified in the present context later). There is no contribution from the vacuum intermediate state for covariance reasons. (Such contribution would have given a $\delta(k)$ and not a $\delta(k^{2})$ singularity.) Inserting (5.56b) into (5.56a) gives

$$ \langle A^{\mu}(x)\, A^{\nu}(y) \rangle = \tag{5.56} $$

$$ = \int_{-o}^{\infty} d\varkappa^{2} \left[-{}^{o}\rho_{1}(\varkappa^{2})\, g^{\mu\nu} - {}^{o}\rho_{2}(\varkappa^{2})\, \partial^{\mu}_{x}\partial^{\nu}_{x} \right] i\, \Delta^{(+)}(x-y, \varkappa^{2}) $$

and therefrom

$$< [A^\mu(x), A^\nu(y)] > =$$

$$= \int_{-o}^{\infty} d\varkappa^2 \, [-{}^\circ\varrho_1(\varkappa^2)\, g^{\mu\nu} - {}^\circ\varrho_2(\varkappa^2)\, \partial_x^\mu \partial_x^\nu] \; i\,\Delta(x-y, \varkappa^2) \tag{5.57}$$

where the lower limit (-o) of integration is to indicate that we expect a singularity at $\varkappa^2 = o$ from one-photon intermediate states which is to be taken in full. (The interchange of integrations in the derivation of (5.56) can be shown to be harmless.) (5.57) shows that the VEV of the commutator of the vector potential vanishes in spacelike distances for covariance reasons alone; on the other hand the anticommutator could not vanish in all spacelike distances without ${}^\circ\varrho_1$ and ${}^\circ\varrho_2$ being identically zero. This is the famous "axiomatic" argument that vector (as also scalar and all tensor) fields cannot be quantized with anti-commutators but only with commutators. That A^μ is relative to the ψ of the Dirac or other matter field also a commuting (rather than anti-commuting) operator follows from the (possibly generalized, i.e. not restricted to spin $\frac{1}{2}$ fields) Dirac equation

$$(i\partial_\mu \gamma^\mu - m)\,\psi = -e\, A_\mu \gamma^\mu \psi$$

since the r.h.s. must have the same even- or oddness property as the l.h.s.

From (5.56) follows with (5.32)

$$< F^{\mu\nu}(x)\, F^{\varkappa\lambda}(y) > =$$

$$= \int_{-o}^{\infty} d\varkappa^2 \; {}^\circ\varrho_1(\varkappa^2)\, (\partial^\nu\partial^\lambda g^{\mu\varkappa} - \partial^\mu\partial^\lambda g^{\nu\varkappa} - \partial^\nu\partial^\varkappa g^{\mu\lambda} + \partial^\mu\partial^\varkappa g^{\nu\lambda}) \cdot i\,\Delta^{(+)}(x-y, \varkappa^2) \tag{5.58}$$

the commutator again being obtained by replacing $i\Delta^{(+)}$ by $i\Delta$.
Since the field strengths $F^{\mu\nu}$ are observable quantities, (5.58) should be interpretable in an ordinary QM Hilbert space, i.e. the metric of intermediate states in (5.58) should be positive definite, for which it is necessary that

$$^\circ\rho_1(\varkappa^2) \geq 0 , \tag{5.59}$$

while no such condition can be imposed on $^\circ\rho_2(\varkappa^2)$ on these grounds, since when adopting (5.31) we did introduce unphysical variables also. Comparing the commutator forms (5.57,58) with (5.36,38) gives

$$\int_{-0}^{\infty} d\varkappa^2 \; ^\circ\rho_1(\varkappa^2) = 1 \tag{5.60}$$

and

$$\int_{-0}^{\infty} d\varkappa^2 \; ^\circ\rho_2(\varkappa^2) = 0 \tag{5.61}$$

while (5.56) compared with (5.50,34) gives

$$^\circ\rho_1(\varkappa^2) - \varkappa^2 \, ^\circ\rho_2(\varkappa^2) = s\,\delta(\varkappa^2) . \tag{5.62}$$

We note that $^\circ\rho_1(\varkappa^2)$ in (5.58) must be the same function whatever formulation of QED one chooses again since $F^{\mu\nu}$ is an observable field, i.e. $^\circ\rho_1(\varkappa^2)$ is "gauge independent", while $^\circ\rho_2(\varkappa^2)$ is not.
The requirement that our theory describe massless photons imposes on $^\circ\rho_1(\varkappa^2)$ the form

$$^\circ\rho_1(\varkappa^2) = Z_3\,\delta(\varkappa^2) + {^\circ\hat{\rho}_1}(\varkappa^2) \tag{5.63a}$$

with

$$0 \leq Z_3 < 1 \tag{5.63b}$$

and

$$ {}^{\circ}\hat{\varrho}_1(\varkappa^2) \geqslant 0 \qquad (5.63c) $$

$$ \int_{+0}^{\infty} d\varkappa^2 \; {}^{\circ}\hat{\varrho}_1(\varkappa^2) = 1 - Z_3 \qquad (5.63d) $$

where we have separated out the singularity at $\varkappa^2 = 0$ this function must have due to the contribution from matrix elements $\langle F^{\mu\nu}(x)|$ one-photon state$\rangle$ in (5.58). These matrix elements are the ones on which the free-photon treatment is based (in connection with (5.13)), and $Z_3 = 0$ would seem to indicate that in the theory there are no photons then. Actually, the intricacy of QFT seems to admit $Z_3 = 0$ even with massless photons as we shall see. $Z_3 = 1$ can be excluded from axiomatic arguments that show that in this case the photons are non-interacting particles, which is uninteresting. The suppression of the one-free-photon contribution in (5.58) due to (5.63a,b) has to be taken into account when scattering states involving real photons are to be normalized correctly. The simple technique of "photon amplitude renormalization" which takes care of this is treated in 5.3.4.

We now can see that in (5.48) the coefficient of the δ-function cannot vanish except in the trivial case of noninteracting photons. Namely,
5 from (4.33,38d,58) and the property

$$ \partial_0 \Delta(x-y)\Big|_{x^0=y^0} = -\,\delta^3(\vec{x}-\vec{y}) $$

we have

$$ \langle [j^k(x), F^{0i}(y)] \rangle\, \delta(x^0-y^0) = -i\, g^{ki}\, \delta^3(\vec{x}-\vec{y}) \int_{-0}^{\infty} d\varkappa^2\; {}^{\circ}\varrho_1(\varkappa^2)\, \varkappa^2 \quad . $$

One has to convince oneself whether the integral on the right hand side exists, if not there might occur derivatives of the δ-function as discussed by Brandt [25] . The nondiagonal matrixelement of this commutator between one-photon states is zero (in contrast to Brandt's calculation).

We now discuss the choice of the parameter s in the Lagrangian (5.31). Equations (5.61,62,63a) give

$$
{}^{o}\varrho_2(\varkappa^2) = \frac{{}^{o}\hat{\varrho}_1(\varkappa^2)}{\varkappa^2} + (s - Z_3)\,\delta'(\varkappa^2) - \delta(\varkappa^2)\int_{+o}^{\infty} d\varkappa'^2 \, \frac{{}^{o}\hat{\varrho}_1(\varkappa'^2)}{\varkappa'^2} \tag{5.64}
$$

with the last integral positive due to (5.63c) and ${}^{o}\hat{\varrho}_1(\varkappa^2) \not\equiv o$.
The first and the last term on the right hand side of (5.64) are "gauge-independent" while the middle term expresses the only gauge (i.e. s-) dependence of ${}^{o}\varrho_2(\varkappa^2)$ in our narrow class of gauges.
The most natural choice would be $s = o$, since then (5.34) would show that in the "physical" case $\mathcal{K} \equiv o$ the Lorentz condition is satisfied for the vector potential. However, this is not a Lorentz gauge in the classical sense since in (5.33) there is a constraint force $\partial^{\mu} B$ that is absent in classical electrodynamics (CED) and only due to which the Lorentz condition can be satisfied. Thus, there is no particular virtue to the choice $s = o$ as compared to others. However, this "Landau gauge" is often used since it looks the most natural in terms of Green's functions. The "physically" simplest gauge, however, is the "Gupta-Bleuler gauge" $s = Z_3$ which makes the bizarre term in (5.64) vanish. It is extensively discussed by Källén in his famous Handbook article [8] and we will henceforth use it unless stated otherwise.

We now form the in and out fields by (5.55) or, more generally,

$$
{}^{o}A^{\mu}_{\substack{in\\out}} = A^{\mu} - \int D_{\substack{ret\\adv}} \, (\Box A^{\mu}) \,. \tag{5.65a}
$$

This shows that the Fourier transforms of ${}^{o}A^{\mu}_{\substack{in\\out}}$ are proportional to $\delta(k^2)$ due to $\Box\, {}^{o}A^{\mu}_{\substack{in\\out}} = o$. However, they differ from each other, and we consider how the singular $A_{\substack{in\\out}}$-part is obtained. In Fourier transforms, (5.65a) is

$$ {}^{\circ}A^{\mu}_{\substack{in\\out}}(k) = A^{\mu}(k) - \frac{1}{-k^2 \mp i\varepsilon k^0}\left[(-k^2)\,A^{\mu}(k)\right] \tag{5.65b} $$

where the last product is nonassociative, since

$$ \frac{1}{x \pm i\varepsilon}\left[x\,\delta(x)\right] = 0 \quad , \quad \left[\frac{1}{x \pm i\varepsilon}\,x\right]\delta(x) = \delta(x) $$

the difference being in coordinate space the boundary term from partial integration in (5.65a).
Now

$$ \frac{1}{-k^2 \mp i\varepsilon k^0} = -\frac{P}{k^2} \pm i\pi\,\mathrm{sign}(k^0)\,\delta(k^2) \tag{5.66} $$

and in words (5.65b) means: look for the principal value and $\delta(k^2)$ parts in $A^{\mu}(k)$, complete the principal value parts in the way prescribed by the second part of (5.65b) and by (5.66), and subtract them from $A^{\mu}(k)$. The $\delta(k^2)$-singularity that survives is the l.h.s. of (5.65b). From (5.63,64) we learn that neither ${}^{\circ}\varrho_1(\varkappa^2)$ nor ${}^{\circ}\varrho_2(\varkappa^2)$ have a principal value singularity at $\varkappa^2 = 0$ (note that the integration in (5.56) goes only over $\varkappa^2 \geqslant 0$), therefore in (5.63a) the whole Z_3 term and the last term in (5.64) are associated with the in and out operator, i.e. in coordinate space using the abbreviation

$$ \int_{+0}^{\infty} d\varkappa^2\, \frac{{}^{\circ}\hat{\varrho}_1(\varkappa^2)}{\varkappa^2} =: a \tag{5.67} $$

we have

$$\langle {}^{\circ}A^{\mu}_{\substack{in\\out}}(x)\, A^{\nu}(y) \rangle = \langle A^{\mu}(x)\, {}^{\circ}A^{\nu}_{\substack{in\\out}} \rangle =$$

$$= \langle {}^{\circ}A^{\mu}_{\substack{in\\out}}(x)\, {}^{\circ}A^{\nu}_{\substack{in\\out}}(y) \rangle = [-g^{\mu\nu} Z_3 + a\, \partial^{\mu}_x \partial^{\nu}_x]\, i\, D^{(+)}(x-y) \tag{5.68}$$

for any combination of subscripts. Assuming that the in-in and out-out commutators are c-numbers as discussed after (4.38), it follows that

$$[\, {}^{\circ}A^{\mu}_{\substack{in\\out}}(x)\,,\ {}^{\circ}A^{\nu}_{\substack{in\\out}}(y)] = [-g^{\mu\nu} Z_3 + \alpha\, \partial^{\mu}_x \partial^{\nu}_x]\, i\, D(x-y) \tag{5.69}$$

for either the upper or the lower subscripts. In analogy to the discussion after (4.33) one would write (5.65a) more elaborately as

$$i \int \bar{f}(x)\, \overleftrightarrow{\partial}_{\nu}\, {}^{\circ}A^{\mu}_{\substack{in\\out}}(x)\, d\sigma^{\nu} = \lim_{\sigma \to \mp\infty} i \int \bar{f}(x)\, \overleftrightarrow{\partial}_{\nu}\, A^{\mu}(x)\, d\sigma^{\nu} \tag{5.70}$$

(where $\overleftrightarrow{\partial} = -\overleftarrow{\partial} + \overrightarrow{\partial}$ and $\Box f(x) = o$) .

As we shall see in the next lecture, if $s \neq Z_3$, the theory contains states that cannot be obtained by operating with ${}^{\circ}A^{\mu(-)}_{\substack{in\\out}}$ on the vacuum. These "dipole ghost" states require further creation operators. For a discussion of this topic we refer to N. Nakanishi [28] . This complication, which in particular takes place in the Landau gauge $s = o$, is the reason why we chose the Gupta-Bleuler gauge $s = Z_3$ as the "physically" simplest.

5.5 Amplitude renormalization

We note that it would be convenient for scattering calculations to convert the factor Z_3 on the r.h.s. of (5.69) into 1. This is done by introducing

$$A^{\mu}_{ren} := Z_3^{-1/2} A^{\mu} \tag{5.71a}$$

to be accompanied, to keep the equations (5.32-34) simple, by

$$F^{\mu\nu}_{ren} := Z_3^{-1/2} F^{\mu\nu} \tag{5.71b}$$

$$B_{ren} := Z_3^{+1/2} B \tag{5.71c}$$

such that (5.65) is replaced by

$$A^{\mu}_{\substack{in\\out}} = Z_3^{-1/2}\, {}^{\circ}A^{\mu}_{\substack{in\\out}} = A^{\mu}_{ren} - \int D_{\substack{ret\\adv}} (\Box A^{\mu}_{ren}) \tag{5.72}$$

or the more correct version from (5.70)

$$\begin{aligned} i \int \bar{f}(x) \overleftrightarrow{\partial_\nu} A^{\mu}_{\substack{in\\out}}(x)\, d\sigma^{\nu} &= \\ &= \lim_{\sigma \to \mp\infty} i \int \bar{f}(x) \overleftrightarrow{\partial_\nu} A^{\mu}_{ren}(x)\, d\sigma^{\nu} \end{aligned} \tag{5.73}$$

with (5.69) replaced by

$$[A^{\mu}_{\substack{in\\out}}(x), A^{\nu}_{\substack{in\\out}}(y)] = (-g^{\mu\nu} + 2M \partial^{\mu}_{x} \partial^{\nu}_{x})\, i\, D(x-y) \tag{5.74}$$

where

$$2\, M := a\, Z_3^{-1} = \int_{+0}^{\infty} d\varkappa^2 \, \frac{Z_3^{-1}\, {}^{\circ}\hat{\varrho}_1(\varkappa^2)}{\varkappa^2} \tag{5.75}$$

We also introduce

$$\rho_{1,2}(\varkappa^2) := Z_3^{-1}\ {}^{\circ}\rho_{1,2}(\varkappa^2)\ , \tag{5.76a}$$

$$\hat{\rho}_1(\varkappa^2) := Z_3^{-1}\ {}^{\circ}\hat{\rho}_1(\varkappa^2) \tag{5.76b}$$

such that

$$\rho_1(\varkappa^2) = \delta(\varkappa^2) + \hat{\rho}_1(\varkappa^2)\ , \tag{5.77a}$$

$$1 < Z_3^{-1} \leq \infty\ , \tag{5.77b}$$

$$\hat{\rho}_1(\varkappa^2) \geq 0\ , \tag{5.77c}$$

$$Z_3^{-1} = 1 + \int_{+0}^{\infty} d\varkappa^2\ \hat{\rho}_1(\varkappa^2)\ . \tag{5.77d}$$

Up to now we considered the covariant theory with $J^\mu \equiv 0$, $K \equiv 0$ only to define Z_3, which we will use in the presence of sources. The equations (5.32-34,50) then become

$$F^{\mu\nu}{}_{ren} = \partial^\mu A^\nu{}_{ren} - \partial^\nu A^\mu{}_{ren}\ , \tag{5.78}$$

$$\partial_\nu F^{\mu\nu}{}_{ren} = Z_3^{-\frac{1}{2}}\, j^\mu + Z_3^{-\frac{1}{2}}\, J^\mu + Z_3^{-1}\, \partial^\mu B_{ren}\ , \tag{5.79}$$

$$\partial_\mu A^\mu{}_{ren} = B_{ren} + Z_3^{-\frac{1}{2}}\, K\ , \tag{5.80}$$

$$[B_{ren}(x), A^\mu{}_{ren}(y)] = -i\, \partial_x^\mu\, D(x-y)\ . \tag{5.81}$$

We now use (5.80) with, since K no longer serves any purpose,

$$K \equiv 0$$

to eliminate B_{ren} from (5.79) and find with (5.78)

$$\Box A^{\mu}_{ren} = - j^{\mu}_{ren} - J^{\mu}_{ext} \tag{5.82}$$

where

$$j^{\mu}_{ren} := Z_3^{\frac{1}{2}} j^{\mu} + (1 - Z_3)\, \underbrace{\partial_{\nu} F^{\mu\nu}}_{}{}_{ren} \tag{5.83a}$$

and

$$J^{\mu}_{ext} := Z_3^{\frac{1}{2}} J^{\mu} . \tag{5.83b}$$

Equation (5.35) leads to

$$\partial_{\mu} j^{\mu}_{ren} = 0 \tag{5.84}$$

while (5.49,51–53) give

$$\Box B_{ren} = - \partial_{\mu} J^{\mu}_{ext} , \tag{5.85}$$

$$[B_{ren}(x), B_{ren}(y)] = 0 , \tag{5.86}$$

$$[B_{ren}(x), F^{\mu\nu}_{ren}(y)] = 0 , \tag{5.87}$$

$$[B_{ren}(x), j^{\mu}_{ren}(y)] = 0 . \tag{5.88}$$

Equation (5.82) with (5.83) has the following features: The J^{μ}_{ext} appearing on the r.h.s. of (5.82) is the coefficient of $A_{ren\,\mu}$ in the Lagrangian density (5.31) such that the Green's functions become particularly simple. For this reason it is J^{μ}_{ext} rather than J^{μ} which we have to identify with the "external current" since the numerical magnitude of such current is not determined by the Lagrangian but by the static or almost static effects of charges, and for this the factor multiplying the renormalized field (which has normed amplitude for one-photon state) in the Lagrangian matters. The renormalized current (5.83a) is on the basis of (5.35) alone conserved without need of using (5.80,85), and is not singular on the photon mass shell $k^2 = o$ in contrast to e.g. the r.h.s. of (5.79) which differs from the r.h.s. of (5.82) by a term $\partial^\mu B_{ren}$ which according to (5.81) is there singular. It follows that due to vacuum polarization, i.e. $Z_3 < 1$, also j^μ is singular on $k^2 = o$, as is also seen from the formula

$$j^\mu_{ren} = Z_3^{-\frac{1}{2}} j^\mu + (Z_3^{-1} - 1)(\partial^\mu B_{ren} + J^\mu_{ext})$$

which is obtained by inserting (5.79) into (5.83a).

From (5.79) and (5.83a) we find

$$[\partial^0 B_{ren}(x), \psi(y)]_{x^0=y^0} = [Z_3^{\frac{1}{2}} j^0(x), \psi(y)]_{x^0=y^0} =$$

$$= [j^0_{ren}(x), \psi(y)]_{x^0=y^0} = -e_{ren}\, \psi(y)\, \delta^3(\vec{x}-\vec{y}) \tag{5.89}$$

with

$$e_{ren} := e\, Z_3^{1/2} \tag{5.90}$$

where we assumed j^0 to be of the form given after (5.2). Together with (5.49) and

$$[B_{ren}(x), \psi(y)]_{x^0=y^0} = 0$$

from the canonical independence of B_{ren} and ψ (5.89) leads to

$$[B_{ren}(x), \psi(y)] = -e_{ren}\, D(x-y)\, \psi(y) \tag{5.91}$$

a relation to be used in the next lecture.

Finally, (5.83a,42,36) give

$$[\, j^0_{ren}(x), A^j(y)]_{x^0=y^0} = [\, Z_3^{\frac{1}{2}} j^0(x) + (1-Z_3)\partial_\kappa F^{0\kappa}(x),\ A^j(y)]_{x^0=y^0} = \tag{5.92a}$$

$$= i\,(1-Z_3)\, Z_3^{-\frac{1}{2}}\, \partial_x^j\, \delta^3(\vec{x}-\vec{y})$$

whereas (5.38f) implies

$$[\, j^0_{ren}(x), A^0(y)]_{x^0=y^0} = 0 \quad . \tag{5.92b}$$

6. STATE SPACE OF QUANTUM ELECTRODYNAMICS

In this lecture we will show that the new degrees of freedom introduced by the choice of the Lagrangian density (5.31) instead of (5.21), or rather the Coulomb gauge Lagrangian derived from it, are physically innocous. To this end we analyse (5.74).

We first remark that it is simple to write the theory such that in (5.74) the 2M-term is absent. One may define

$$\check{A}^\mu_{ren}(x) := A^\mu_{ren}(x) + M\, \partial^\mu B_{ren}(x) \tag{6.1}$$

such that

$$\check{A}^\mu_{\substack{in\\out}}(x) = A^\mu_{\substack{in\\out}}(x) + M\, \partial^\mu B_{ren}(x)$$

(assuming for simplicity $\partial_\mu J^\mu_{ext.} = o$ in (5.85)). Then (5.74) becomes with (5.81) and (5.86)

$$[\check{A}^\mu_{\substack{in\\out}}(x), \check{A}^\nu_{\substack{in\\out}}(y)] = -g^{\mu\nu}\, i\, D(x-y) \quad . \tag{6.2}$$

The relation (6.1) is an operator gauge transformation (cf. (5.17)). It is accompanied by a corresponding operator phase transformation (cf. (6.28)) of the matter fields operators, the Dirac (or generalized Dirac) equation for these is then restored. The new operators serve for all physical purposes as well as the former ones, moreover, one can write for the new operators (in their unrenormalized version) a Lagrangian density differing of course from (5.31) but being as canonically sound as (5.31) is, with A^o a dependent variable. The operator gauge transformation mentioned here has been pointed out by Rollnik, Stech, and Nunnemann [29]. The Lagrangian formulation of this gauge is contained in the family of Lagrangians proposed by Nakanishi [28] , which also involve state spaces different from the Gupta-Bleuler one while the above special transformation does, of course, not alter the state space. We will, however, analyse (5.74) instead of (6.2) since it is at least as natural as (6.2), and if we do not fix the value of M by (5.77), more general.

6.1 Gupta-Bleuler gauge

We now describe the structure of the state space of the photons. We proceed in three steps. First we construct a space $\mathcal{H}_{\gamma GB}$ which necessarily must have an indefinite metric. Then we restrict in an unambiguous Lorentz invariant way to a subspace $\mathcal{H}_{\gamma P}$ of states with positive semi-definite norm. The state space of physical photons is built up by equivalence classes of $\mathcal{H}_{\gamma P}$ as state vectors.

6.1.1 Construction of $\mathcal{H}_{\gamma GB}$

Let k be a positive lightlike four-vector, i.e. $k^2 = o$, $k^o > o$. We choose two spacelike unit vectors $e^{(1)}(k)$, $e^{(2)}(k)$ orthogonal to k and to each other (in most of the following we will suppress the argument k of the vectors introduced):

$$(e^{(1)})^2 = (e^{(2)})^2 = -1 \quad ; \qquad e^{(1)} e^{(2)} = 0 \ , \tag{6.3a}$$

$$k\, e^{(1)} = k\, e^{(2)} = 0 \ . \tag{6.3b}$$

There is a unique $\hat{k}$ such that

$$\hat{k}^2 = 0 \ , \quad k\hat{k} = 1 \ , \quad \hat{k} e^{(1)} = \hat{k} e^{(2)} = 0 \ . \tag{6.4}$$

The vectors

$$e^{(0)} := \frac{\alpha k + \alpha^{-1} \hat{k}}{\sqrt{2}} \quad , \quad e^{(3)} := \frac{\alpha k - \alpha^{-1} \hat{k}}{\sqrt{2}} \tag{6.5}$$

with some $\alpha > 0$ form together with $e^{(1)}$ and $e^{(2)}$ an orthonormal system of four-vectors. Then

$$\begin{aligned} g_{\mu\nu} &= e^{(0)}_\mu e^{(0)}_\nu - e^{(1)}_\mu e^{(1)}_\nu - e^{(2)}_\mu e^{(2)}_\nu - e^{(3)}_\mu e^{(3)}_\nu = \\ &= k_\mu \hat{k}_\nu + \hat{k}_\mu k_\nu - e^{(1)}_\mu e^{(1)}_\nu - e^{(2)}_\mu e^{(2)}_\nu \ . \end{aligned} \tag{6.6}$$

The most general set of four-vectors $e^{(1)}$, $e^{(2)}$, $\hat{k}$ so related to k is obtained from any special one by applying a transformation from the "little group" of k (i.e. the proper Lorentz transformations that leave k invariant) to any such set (when the orientation of $\vec{e}^{(1)}$, $\vec{e}^{(2)}$, $\vec{e}^{(3)}$ is fixed).

We now set

$$\begin{aligned} A^\mu_{\substack{in\\out}}(x) = \frac{1}{(2\pi)^{3/2}} \int dk\, \Theta(k^0)\, \delta(k^2) \Big\{ & e^{-ikx} \Big[\sum_{i=1}^{2} a_i(k)\, e^{(i)\mu}(k) + b(k)\, \hat{k}^\mu(k) + c(k)\, k^\mu \Big] + \\ & + e^{ikx} \Big[\sum_{i=1}^{2} a_i(k)^+ e^{(i)\mu}(k) + b(k)^+ \hat{k}^\mu(k) + c(k)^+ k^\mu \Big] \Big\} \end{aligned} \tag{6.7}$$

(where we supress the $\underset{\text{out}}{\text{in}}$ -index of the a,b,c), i.e.

$$a_i(k) := -i \frac{e_\mu^{(i)}(k)}{(2\pi)^{3/2}} \int e^{ikx} \overleftrightarrow{\partial_\nu} A^\mu_{\substack{in\\out}}(x)\, d\sigma^\nu , \qquad i = 1,2 \tag{6.8a}$$

$$b(k) := i \frac{k_\mu}{(2\pi)^{3/2}} \int e^{ikx} \overleftrightarrow{\partial_\nu} A^\mu_{\substack{in\\out}}(x)\, d\sigma^\nu , \tag{6.8b}$$

$$c(k) := i \frac{\hat{k}_\mu}{(2\pi)^{3/2}} \int e^{ikx} \overleftrightarrow{\partial_\nu} A^\mu_{\substack{in\\out}}(x)\, d\sigma^\nu , \tag{6.8c}$$

whereby we have used (6.6) and

$$A^{\mu\,(\pm)}_{\substack{in\\out}}(x) = - \int D^{(\pm)}(x-y) \overleftrightarrow{\partial^y_\nu} A^\mu_{\substack{in\\out}}(y)\, d\sigma^\nu .$$

From (6.8), or more quickly by inserting (6.7) into (5.74), we find

$$[a_i(k), a_j(k')^+] = 2k^0 \delta^3(\vec{k}-\vec{k}')\, \delta_{ij} , \tag{6.9a}$$

$$[b(k), c(k')^+] = [c(k), b(k')^+] = -2k^0 \delta^3(\vec{k}-\vec{k}') , \tag{6.9b}$$

$$[c(k), c(k')^+] = -2M\, 2k^0 \delta^3(\vec{k}-\vec{k}') , \tag{6.9c}$$

and all other commutators vanishing. Here $k^0 = |\vec{k}|$, and the combinations on the r.h.s. are the familiar ones that appear in our covariant normalization $\Theta(k^o)\ \delta(k^2)\ dk$. From

$$[P^\mu_{\substack{in\\out}}, A^\nu_{\substack{in\\out}}(x)] = -i\partial^\mu A^\nu_{\substack{in\\out}}(x) \tag{6.10}$$

(which is a consequence of covariance with $P^\mu_{in} = P^\mu_{out} = P^\mu$ if $J^\mu = o$, and essentially the definition of the photon-variables part of $P^\mu_{\substack{in\\out}}$ otherwise) we find

$$[P^\mu_{\substack{in\\out}}, a_i(k)] = -k^\mu a_i(k), \tag{6.11a}$$

$$[P^\mu_{\substack{in\\out}}, b(k)] = -k^\mu b(k), \tag{6.11b}$$

$$[P^\mu_{\substack{in\\out}}, c(k)] = -k^\mu c(k) \tag{6.11c}$$

and the same relations with + signs on the r.h.s. for a_i^+, b^+, c^+. It follows in the familiar way that a_i^+, b^+, c^+ acting on an eigenstate of P^μ or $P^\mu_{\substack{in\\out}}$ change the eigenvalue by $+ k^\mu$, and similarly a_i, b, c change it by $- k^\mu$. The part of P^μ or $P^\mu_{\substack{in\\out}}$ which does not commute with the photon variables is therefore, using (6.9),

$$P^\mu_{\substack{in\\out}\ photon} = \int dk\ \Theta(k^o)\ \delta(k^2) [\sum_{i=1}^{2} a_i(k)^+ a_i(k) - b(k)^+ c(k) - c(k)^+ b(k) + 2M\ b(k)^+ b(k)]\ k^\mu \tag{6.12}$$

where we have already ordered the operators such that if we define the vacuum state with respect to the $\substack{in\\out}$-photon-field by

$$a_i(k) >_{\substack{in\\out}} = b(k) >_{\substack{in\\out}} = c(k) >_{\substack{in\\out}} = o \tag{6.13a}$$

for all k

normalized by

$$ {}_{\substack{\text{in}\\ \text{out}}}\langle \ | \ \rangle_{\substack{\text{in}\\ \text{out}}} = 1 \tag{6.13b} $$

this will be the eigenstate to lowest energy and satisfy

$$ P^{\mu}_{\substack{\text{in}\\ \text{out}}} \ \text{photon} \ \rangle_{\substack{\text{in}\\ \text{out}}} = 0 \tag{6.14} $$

The equation (6.13) can be written on account of (6.7) as

$$ A^{\mu\,(+)}_{\substack{\text{in}\\ \text{out}}}(x) \ \rangle_{\substack{\text{in}\\ \text{out}}} = 0 \ . \tag{6.15} $$

We now form the linear space $\mathfrak{h}_{GB}$ from the basis vectors that are obtained by applying a_i^+, b^+, c^+ repeatedly on the vacuum. The scalar products between any basis states so obtained can be evaluated using (6.9,13) alone. States obtained by using a_1^+, a_2^+ only behave in the manner familiar from free zero-mass scalar particles. States that involve besides $a^+_{1,2}$ one or more b^+ have scalar product zero with all states that do not involve c^+. States involving besides $a^+_{1,2}$ and b^+ also one or more c^+ yield indefinite scalar products.

6.1.2 Restriction to $\mathfrak{h}_P$

Thus, if we define a linear subspace $\mathfrak{h}_P$ of $\mathfrak{h}_{GB}$ by the condition

$$ b(k)\,\mathfrak{h}_P = 0 \quad \text{for all } k \tag{6.16} $$

which due to (6.9b) is the subspace of $\mathfrak{h}_{GB}$ spanned by the basis vectors not involving c^+, the scalar product in $\mathfrak{h}_P$ is positive semidefinite. In fact, each vector of $\mathfrak{h}_P$ can be decomposed uniquely into a linear combination of vectors involving $a^+_{1,2}$ (applied on the vacuum) only, and a linear combination of vectors each involving one or more b^+, and to scalar products only the first part contributes.

Now, in contrast to $a_{1,2}(k)$ and $c(k)$, according to (6.8b) $b(k)$ does not involve any arbitrariness. In fact we can write

$$
\begin{aligned}
b(k) &= \frac{1}{(2\pi)^{3/2}} \int (\partial_\mu e^{ikx}) \overleftrightarrow{\partial_\nu} A^\mu_{\substack{in\\out}}(x)\, d\sigma^\nu = \\
&= \frac{1}{(2\pi)^{3/2}} \int \partial_\mu \left(e^{ikx} \overleftrightarrow{\partial_\nu} A^\mu_{\substack{in\\out}}(x) \right) d\sigma^\nu - \qquad (6.17) \\
&\quad - \frac{1}{(2\pi)^{3/2}} \int e^{ikx} \overleftrightarrow{\partial_\nu} \partial_\mu A^\mu_{\substack{in\\out}}(x)\, d\sigma^\nu
\end{aligned}
$$

whereof the first term vanishes due to

$$
\begin{aligned}
\int \partial_\mu \left(e^{ikx} \overleftrightarrow{\partial^\nu} A^\mu_{\substack{in\\out}}(x) \right) d\sigma_\nu &= \\
&= \int \partial_\nu \left(e^{ikx} \overleftrightarrow{\partial^\nu} A^\mu_{\substack{in\\out}}(x) \right) d\sigma_\mu
\end{aligned}
$$

and

$$
\Box\, e^{ikx} = \Box\, A^\mu_{\substack{in\\out}} = 0
$$

where Stokes' theorem has been used which is applicable here since $b(k)$ should actually be considered integrated with a smooth function on $k^2=0$ which replaces e^{ikx} by a fast-decreasing function in x. Furthermore,

$$
\begin{aligned}
\partial_\mu A^\mu_{\substack{in\\out}}(x) &= \partial_\mu A^\mu_{ren}(x) + \int D_{\substack{ret\\adv}}(x-y)\, \overleftrightarrow{\partial^y_\mu} \left[j^\mu_{ren}(y) + J^\mu_{ext}(y) \right] dy = \\
&= B_{ren}(x) + \lim_{\sigma \to \mp\infty} \int D(x-y) \left[j^\mu_{ren}(y) + J^\mu_{ext}(y) \right] d\sigma_\mu(y) = \qquad (6.18) \\
&= B_{ren}(x)
\end{aligned}
$$

because of (5.84) and the assumption we now make,

$$\partial_\mu J^\mu_{ext} = 0 \quad , \tag{6.19}$$

and the last integral in (6.18) vanishes since $j^\mu + J^\mu_{ext}$ is (here we utilize the Gupta-Bleuler choice $s = Z_3$) not singular on $k^2 = 0$, cf. (5.82). As B_{ren} is due to (6.19) and (5.85) a free field, (6.17,18) allow to rewrite (6.16) as

$$B_{ren}(x)^{(+)} \; \mathfrak{h}_{\gamma P} = 0 \tag{6.20}$$

which is the famous Gupta-Bleuler subsidiary condition of physical states with positive semi-definite metric. Clearly, $\mathfrak{h}_{\gamma P}$ like $\mathfrak{h}_{\gamma GB}$ is defined in a Lorentz invariant manner. The scalar product in $\mathfrak{h}_{\gamma GB}$ is invariant since its evaluation is based on (5.74) only and so is its restriction to $\mathfrak{h}_{\gamma P}$, such that the positive semi-definiteness of the metric in $\mathfrak{h}_{\gamma P}$ is also an invariant statement. The intermediate use of the systems $e^{(i)}(k)$ served for convenience only; the scalar product between any $\substack{\text{in}\\\text{out}}$-photon-states $|\varphi\rangle, |\varphi'\rangle$ can also be manifestly invariantly evaluated by the formula

$$\begin{aligned} \langle\varphi|\varphi'\rangle = \sum_0^\infty{}_n \frac{i^n}{n!} \int\cdots\int \langle\varphi| A^{\mu_1(-)}_{\substack{in\\out}}(x_1)\cdots A^{\mu_n(-)}_{\substack{in\\out}}(x_n)\rangle_{\substack{in\\out}} \cdot \\ \cdot \overleftarrow{\partial}^{x_1}_{\varsigma_1}\cdots\overleftarrow{\partial}^{x_n}_{\varsigma_n} \prod_1^n{}_i \left(-g_{\mu_i\nu_i} - 2M\overleftarrow{\partial}^{x_i}_{\mu_i}\overrightarrow{\partial}^{x_i}_{\nu_i}\right) \cdot \\ \cdot {}_{\substack{in\\out}}\langle A^{\nu_1(+)}_{\substack{in\\out}}(x_1)\cdots A^{\nu_n(+)}_{\substack{in\\out}}(x_n)|\varphi'\rangle \, d\sigma^{\varsigma_1}\cdots d\sigma^{\varsigma_n} \end{aligned} \tag{6.21}$$

(Exercise: check this)

6.1.3 Space of equivalence classes

In order to obtain a Hilbert space we proceed by completing $\mathcal{H}_P$ abstractly to a pre-Hilbert space. But there are still states with vanishing norm. As discussed in Sect. 6.1.1, each vector of $\mathcal{H}_P$ can be decomposed uniquely into two parts, one of them with positive norm and the other with norm zero, lying in $\mathcal{H}_+$ and $\mathcal{H}_o$, say. Unfortunately this decomposition of $\mathcal{H}_P$ is not Lorentz invariant. Whereas $\mathcal{H}_P$ and $\mathcal{H}_o$ remain invariant under application of a Lorentz transformation $\mathcal{H}_+$ does not, i.e. a vector belonging to $\mathcal{H}_+$ in a certain reference frame may get a component in $\mathcal{H}_o$ by a Lorentz transformation.
(Exercise: look for an example)
From the validity of the Schwarz inequality in $\mathcal{H}_P$ we know that also $\mathcal{H}_o$ is a linear space. The factor space

$$\mathcal{H}_{Fock} = \mathcal{H}_P / \mathcal{H}_o \tag{6.22}$$

is the linear space of equivalence classes of vectors from $\mathcal{H}_P$ whose elements differ only by vectors from $\mathcal{H}_o$. In our realization with a_i^+, b^+, c^+ applied on the vacuum $\mathcal{H}_{GB}$ contains all vectors, $\mathcal{H}_P$ only those not involving c^+, and the elements of $\mathcal{H}_{Fock}$ are in one-to-one correspondence with the vectors of that space that is formed from basis vectors involving a_1^+, a_2^+ applied on the vacuum only. $\mathcal{H}_{Fock}$ is invariantly defined. It is the Hilbert space of transversely polarized incoming (respectively outgoing) photons, i.e. of photons allowed by the Maxwell equations without charges, cf. the remarks after (5.13).
A broad exposition of the concepts involved here can be found in an article by F. Strocchi [30] .

6.2 Gauge invariance

Operators that commute with $B_{ren}(x) = \partial_\mu A^\mu_{ren}(x)$ we call <u>strictly gauge invariant</u>. Such operators are the S operator, since

$$B_{ren} = B_{in} = B_{out} \text{ , i.e. } \quad [S, B_{ren}] = 0, \tag{6.23}$$

the operators $F^{\mu\nu}$, j^{μ}_{ren}, and B_{ren} itself (cf. (5.86-88)). Operators that, together with their adjoints, map $\mathfrak{h}_P$ into $\mathfrak{h}_P$ we call <u>gauge invariant.</u> Operators that, together with their adjoints, map $\mathfrak{h}_o$ into $\mathfrak{h}_o$ we call <u>weakly gauge invariant.</u>

From (6.20) follows that strictly gauge invariant operators are also gauge invariant, and from the validity of the Schwarz inequality in $\mathfrak{h}_P$ follows that gauge invariant operators are also weakly gauge invariant, but the inverse inclusions do not hold. The matrix elements of weakly gauge invariant operators between states from $\mathfrak{h}_P$ are equivalence class functions only, i.e. for such applications these operators are functions on $\mathfrak{h}_{Fock} \otimes \mathfrak{h}_{Fock}$. $P^{\mu}_{\substack{in\\out}, photon}$ due to its form (6.12) is gauge invariant but not strictly gauge invariant, and the same holds for $M^{\mu\nu}_{\substack{in\\out}, photon}$.

If one evaluates the matrix elements in $\mathfrak{h}_P$ of a product of gauge invariant operators, only the intermediate states of $\mathfrak{h}_P$ contribute, and all matrix elements involved depend only on the $\mathfrak{h}_{Fock}$ equivalence classes of all states. (This does not hold for only weakly gauge invariant operators.) Thus, the positive definiteness requirement we derived from (5.58) is now justified. The matrix elements of B_{ren} are, moreover, zero between states of $\mathfrak{h}_P$ such that due to (5.78,80,82) (with K = o) the Maxwell equations (5.5a,b) hold as equations among strictly gauge invariant operators.

As (5.91) shows, ψ and of course also ψ^+ are not strictly gauge invariant, and not even gauge invariant since for $\phi \in \mathfrak{h}_P$ e.g.

$$B^{(+)}_{ren}(x)\, \psi(y) \,|\phi\rangle = - e_{ren}\, D^{(+)}(x-y)\, \psi(y)\, |\phi\rangle \neq 0$$

in general. However, the operator ψ_{in} appears to be strictly gauge invariant due to the following argument: One would expect that (with a scale factor inserted as in (4.23))

$$Z_2^{\frac{1}{2}}\, \psi_{in}(x) = \psi(x) - \int dy\; S_{ret}(x-y)\,(-i\partial\!\!\!/+M)\,\psi(y)$$

such that

$$Z_2^{1/2}\,[\,\psi_{in}(x), B_{ren}(z)] = -\,e_{ren}\;\psi(x)\;D(x-y)\; +$$

$$+\; e_{ren}\int dy\; S_{ret}(x-y)(-i\partial\!\!\!/^{y}+M)[\psi(y)\;D(y-z)] =$$

$$= -i\, e_{ren} \lim_{\sigma\to\mp\infty} \int d\sigma_\mu(y)\; S(x-y)\;\gamma^\mu\;\psi(y)\;D(y-z)$$

which should only be nonzero if $\psi(y)D(y-z)$ is singular on the mass shell $p^2 = M^2$ for the electron. But ψ is only singular on that mass shell, and for $k^2 = o$, $p^2 = M^2$,

$$(p+k)^2 = M^2 + 2pk \neq M^2.$$

Unfortunately, this argument does apply only if one gives the photon a small but finite mass; for strictly zero-mass photons the asymptotic electron operators $\psi_{\substack{in\\out}}$, $\psi^{+}_{\substack{in\\out}}$ do even not exist in the usual sense in which e.g. $A^{\mu}_{\substack{in\\out}}$ exist. This difficulty is that of the infrared catastrophy, for which a completely satisfactory Hilbert space treatment has not yet been given (as remarked after (5.38)). However, the zero-mass photons Hilbert space properties are more interesting than those for finite-mass photons, and the infrared problem being beyond the scope of these lectures, we conclude that "apart from the infrared problem" the Hilbert space $\mathfrak{h}_{QED}$ of QED is

$$\mathfrak{H}_{\substack{in\\out}, \text{ electrons}} \otimes \mathfrak{H}_{\substack{in\\out}, \text{ positrons}} \otimes \mathfrak{H}_{\substack{in\\out} \text{ Fock}}$$

the last factor referring to photons as analysed above. Since $\psi_{\substack{in\\out}}, \psi^{+}_{\substack{in\\out}}$ are, apart from the difficulty just mentioned, strictly gauge invariant,

$$\mathfrak{H}_{QED} = \mathfrak{H}_{P} / \mathfrak{H}_{0} \tag{6.24}$$

where we now allow in $\mathfrak{H}_{P}$ also the $\psi_{\substack{in\\out}}, \psi^{+}_{\substack{in\\out}}$ as creation operators on the vacuum. In $\mathfrak{H}_{QED}$ $P^{\mu}_{\substack{in\\out}}$ (or, if $J^{\mu} \equiv 0$, simply P^{μ}) and equally $M^{\mu\nu}$ are gauge invariant in the obvious sense. Concerning the state space problem in QED, see e.g. [56] and the references given there.

6.3 Gauge transformations

Gauge transformations of the first kind are defined by

$$\psi \to U \psi U^{-1} = e^{-i\alpha e_{ren}} \psi , \qquad A^{\mu} \to A^{\mu}$$

where the unitary operator U is of the form $U = e^{i\alpha Q}$ and α is a constant. The CR of the generator Q with the fields are

$$[Q, \psi] = -e_{ren} \psi , \qquad [Q, A^{\mu}] = 0 . \tag{6.25}$$

From (5.89,92) we see that Q is given by

$$Q = \int d\sigma_{\mu} \, j^{\mu}_{ren} . \tag{6.26}$$

However, the current j^{μ}_{ren} is not completely determined by (6.25,26) as it may contain in addition e.g. terms of the form $\partial_{\nu} S^{\mu\nu}$, with $S^{\mu\nu} = - S^{\nu\mu}$, as for non-minimal electromagnetic coupling, which do not contribute in (6.26) because of Stokes' theorem.

While (5.5) together with the possibly generalized Dirac equation

$$(-i\alpha^\mu\partial_\mu + m\beta)\,\psi = e\,\alpha_\mu A^\mu\,\psi = e_{ren}\,\alpha_\mu A^\mu_{ren}\,\psi \tag{6.27}$$

remains invariant under gauge transformations of the second kind

$$(-i)(-i)\,e\,\partial_\mu\Lambda \qquad\qquad e\,\partial_\mu\Lambda$$

$$A^\mu \rightarrow A^\mu + \partial^\mu\Lambda \quad , \qquad \psi \rightarrow e^{-ie\Lambda}\,\psi$$

where Λ now may depend arbitrarily on x, (5.32-34) and (6.27) remain invariant under

$$A^\mu \rightarrow A^\mu + \partial^\mu\Lambda\,, \qquad B \rightarrow B + \tfrac{1}{s}\Box\Lambda \quad , \qquad \psi \rightarrow e^{-ie\Lambda}\,\psi$$

only if

$$\Box\,\Lambda = \text{constant} \quad . \tag{6.28}$$

The only meaningful solution of this is $\Box\Lambda = 0$. If Λ commutes with all operators appearing in the theory and with itself, then the change can be implemented by (we replace Λ by $\Lambda Z_3^{1/2}$)

$$A^\mu_{ren} \rightarrow e^{\,i\int\Lambda\overleftrightarrow{\partial_\nu}B_{ren}\,d\sigma^\nu}\cdot A^\mu_{ren}\cdot e^{-i\int\Lambda\overleftrightarrow{\partial_\nu}B_{ren}\,d\sigma^\nu} =$$

$$= A^\mu_{ren} + \partial^\mu\Lambda$$

where (5.81) is used, and similarly for the other operators, whereby for ψ and ψ^+ we use (5.91) and its Hermitean conjugate rather than the Dirac equation. This time independent canonical transformation, under which strictly gauge invariant operators are invariant but merely gauge invariant ones not, maps every state vector within its own equivalence class, e.g. the vacuum is changed within its equivalance class $\{|\ \rangle_{\substack{in\\out}}\}$ which (with respect to the electromagnetic field) is the vacuum of $\mathfrak{h}_{Fock}$ and characterized by

$$F^{\mu\nu(+)}_{\substack{in\\out}}\ \{|\ \rangle_{\substack{in\\out}}\} = 0 \quad , \qquad \{|\ \rangle_{\substack{in\\out}}\} \subset \mathfrak{h}_P \ .$$

However, the special gauge (6.15) is useful to obtain still covariant Green's functions also when non-gauge-invariant operators are involved. For Λ = const. the above integrals are not meaningful. In this case the gauge transformations of the second kind reduce to those of the first kind which are generated by the renormalized charge (6.26). If the gauge function Λ is an operator, we obtain an "operator gauge transformation", a fairly trivial example of which we encountered in (6.1). A more significant such transformation is the one from the Gupta-Bleuler gauge to the Coulomb (or radiation) gauge to be presented now.

6.4 Coulomb gauge

We define

$$A^\mu_c(x) = A^\mu_{ren}(x) - \partial^\mu[(-\Delta)^{-1}\partial_i A^i_{ren}](x) \tag{6.29}$$

with

$$\Delta = -\partial_i\partial^i$$

such that

$$A^i_c = A^{iT}_{ren}$$

and thus

$$\partial_i A^i_c = 0 \tag{6.30}$$

and

$$\psi_c(x) = e^{i e_{ren} [(-\Delta)^{-1} \partial_i A^i_{ren}](x)} \psi(x) , \qquad (6.31a)$$

$$\psi_c^+(x) = e^{-i e_{ren} [(-\Delta)^{-1} \partial_i A^i_{ren}](x)} \psi^+(x) \qquad (6.31b)$$

where the exponent commutes with ψ, ψ^+ as it contains independent canonical variables.

These Coulomb gauge operators (in line with our not treating the renormalization of the matter field in this lecture, we do not attempt to put (6.31) in a mathematically less objectionable form) are strictly gauge invariant:

$$[B_{ren}(x), A^\mu_c(y)] = -i \partial^\mu_x D(x-y) + $$
$$+ i \partial^\mu_x \partial^y_i \partial_x^i (-\Delta)^{-1}_y D(x-y) = 0 \qquad (6.32)$$

with use of (5.81), and with (5.91)

$$[B_{ren}(x), \psi_c(y)] = -e_{ren} (-\Delta)^{-1}_y \partial^x_i \partial^i_y D(x-y) \psi_c(y) +$$
$$+ e_{ren} D(x-y) \psi_c(y) = 0 \qquad (6.33a)$$

and likewise

$$[B_{ren}(x), \psi_c^+(y)] = 0 . \qquad (6.33b)$$

This implies that in calculations involving these operators and other strictly gauge invariant ones, only states of $\mathcal{H}_P$ and even only sample states from the equivalence classes that are the elements of $\mathcal{H}_{QED}$ need be inserted. This applies in particular to the field equations which take, by using (6.29,32), in the Coulomb gauge the form

$$\Box A_c^i = - j_{ren}^{i\,T} - J_{ren}^{i\,T} \quad , \tag{6.34}$$

$$A_c^0 = -(-\Delta)^{-1} \left(j_c^0 + J_{ext}^0 \right) , \tag{6.35}$$

$$(-i\gamma^\mu \partial_\mu + M)\psi_c = e_{ren}\, \gamma_\mu A_c^\mu \psi_c \quad , \tag{6.36}$$

where

$$A_c^i = A_{ren}^{i\,T} \quad , \tag{6.37}$$

$$j_c^0 = j_{ren}^0 + \partial^0 B_{ren} \quad , \tag{6.38}$$

and

$$[A_c^i(x), \psi_c(y)]_{x^0=y^0} = 0 \quad . \tag{6.39}$$

We have from (5.83a)

$$j_{ren}^{i\,T} = e_{ren} \left(\psi^+ \alpha^i \psi \right)^T + (1 - Z_3)\, \partial_j F^{ij} \quad . \tag{6.40}$$

The ETCR's for the Coulomb gauge operators in the present renormalized form, together with the unchanged ones, are

$$[A^0_c, A^0_c] = [A^0_c, A^i_c] = [A^i_c, A^j_c] =$$

$$= [A^i_c, \psi_c] = [A^i_c, \psi_c^+] = [j^0_c, A^0_c] =$$

$$= [j^0_c, F^{0i}_{ren}] = [F^{0i}_{ren}, F^{0j}_{ren}] = [F^{0i}_{ren}, A^0_c] = \tag{6.41}$$

$$= [j^0_c, A^i_c] = [j^i_{ren}, A^j_c] = [j^{iT}_{ren}, A^0_c] = 0$$

and

$$[A^0_c(x), \psi_c(y)]_{x^0=y^0} = -Z_3^{-1}(-\Delta)^{-1} e_{ren}\, \delta^3(\vec{x}-\vec{y})\, \psi_c , \tag{6.42a}$$

$$[A^0_c(x), \psi^+_c(y)]_{x^0=y^0} = Z_3^{-1}(-\Delta)^{-1} e_{ren}\, \delta^3(\vec{x}-\vec{y})\, \psi^+_c , \tag{6.42b}$$

$$[F^{0i}_{ren}(x), \psi_c(y)]_{x^0=y^0} = Z_3^{-1}\partial^i(-\Delta)^{-1}\, \delta^3(\vec{x}-\vec{y})\, \psi_c \tag{6.42c}$$

$$[F^{0i}_{ren}(x), \psi^+_c(y)]_{x^0=y^0} = -Z_3^{-1}\partial^i(-\Delta)^{-1}\, \delta^3(\vec{x}-\vec{y})\, \psi^+_c \tag{6.42d}$$

$$[F^{0i}_{ren}(x), A^j_c(y)]_{x^0=y^0} = i\, Z_3^{-1}\, (g^{ij} - \partial^i\partial^j(-\Delta)^{-1})\, \delta^3(\vec{x}-\vec{y}), \tag{6.42e}$$

$$\{\alpha^0\psi_c, \psi^+_c\alpha^0\}_{x^0=y^0} = i\, \delta^3(\vec{x}-\vec{y})\, \alpha^0 . \tag{6.42f}$$

The occurrence of the transverse projector in (6.34) makes all operators except the already originally strictly gauge invariant ones j^μ and $F^{\mu\nu}$ not commute (or anticommute) in spacelike distances, in fact, already the ETC's (6.42a-e) are all nonlocal.

As remarked before, for matrix elements in $\mathfrak{H}_P$, and $\mathfrak{H}_P$ is invariant under all Coulomb gauge operators as introduced here, the B_{ren}-term in (6.38) may be omitted. Moreover, for picturing the Coulomb gauge in a particularly simple manner one may restrict the state space further.

One may choose the polarization vectors in (6.3) and the parameter α in (6.5) such that $e^{(1)o}$, $e^{(2)o}$, and $e^{(3)o}$ vanish while $e^{(o)} = (1,\vec{o})$. Then the omission of the $\mathcal{H}_P$ -part of the state vectors as described after (6.16) can be expressed as the restriction to $\mathcal{H}_{rad,\,{in \atop out}}$ with

$$A^{o\,(+)}_{{in \atop out}} \; \mathcal{H}_{rad,\,{in \atop out}} = 0 \quad , \quad \mathcal{H}_{rad,\,{in \atop out}} \subset \mathcal{H}_P \quad . \tag{6.43}$$

These two spaces are not invariant under the Coulomb gauge operators, however, and for either the $\mathcal{H}_{rad,\,in}$ or $\mathcal{H}_{rad,\,out}$ one may without any change in the equations (6.34-41) (except that the B_{ren}-term in (6.38) is to be omitted) supply the above Coulomb gauge operators on both sides with the restrictions to the spaces defined by (6.43). These restrictions are, due to the indefinite metric, not projection operators in a Hilbert space. The fact that $\mathcal{H}_{rad,\,in} \neq \mathcal{H}_{rad,\,out}$ however, makes this transition somewhat artificial. We leave it as an exercise to investigate the relation of our renormalized Coulomb gauge to the elementary one discussed after (5.24), and whose commutation relations are given e.g. in Bjorken and Drell [2] .

We finally comment on the physical adequacy of the covariant theory based on (6.40) for QED. We have established that this theory is related to Coulomb gauge QED. For the latter we only omitted to supply the P^μ and $M^{\mu\nu}$ and to show that they obey the structure relations of the Poincaré group and transform the field operators covariantly up to the gauge transformation that takes one from one Coulomb gauge to the other. The gauge function for this transformation can be obtained from (5.19) or more directly from the formula

$$A^\mu_c = A^\mu_{ren} - \partial^\mu \frac{\partial_\nu - \partial_\lambda n^\lambda n_\nu}{\partial_g \partial^g - \partial_\sigma \partial_\tau n^\sigma n^\tau} A^\nu_{ren}$$

which gives explicitly the normal (i.e. time axis) dependence of A_c^μ
For these matters see e.g. Bjorken and Drell [2] or B. Zumino [22].
We exhibited in (6.29,31) the operator gauge transformation that relates the covariant theory to the Coulomb gauge theory. That such transformation does not affect scattering amplitudes should follow from a suitable equivalence theorem though e.g. neither the ones considered by Kamefuchi, O'Raifeartaigh, and Salam [31] nor the ones of Borchers [32] cover the present case. In view of this, one would directly show that the singularities of the appropriate Green's functions are the same. An obstacle to a completely satisfactory demonstration of this is the lack so far of a renormalized Coulomb gauge theory, apart from the infrared problem; formal proofs of equivalence abound in the literature (see, e.g. [33]). The situation is the same if, in order to avoid the infrared problem, one gives the photon a small but nonzero mass, the theory dealt with in section 7.

6.5 Remarks

6.5.1 Causality

How do we know that QED is a causal theory in the relativistic sense? The field equations and commutation relations in the Coulomb gauge certainly are nonlocal. But for causality it suffices that the observable local quantities like $F^{\mu\nu}$, j^μ commute in spacelike distances. They are gauge invariant in the ordinary sense and in the sense of commuting with B, and so from their commuting in spacelike distances due to manifest covariance in the Gupta-Bleuler gauge it follows that they also commute if the state space is restricted to the one of the Coulomb gauge (recall the need to scrutinize the intermediate states when operator products like $[F^{\mu\nu}(x), j^\varrho(y)]$ are considered).

Another expression of causality are by common understanding dispersion relations. In QED scattering amplitudes involving photons only exist, and, neglecting technical difficulties in rigorous proofs, dispersion relations hold for them. In the proof the indefinite metric would play no role since the S-operator relates to physical states only. Scattering amplitudes involving charged particles exist only if the photon mass were finite though arbitrarily small. In that case the analog of the Coulomb gauge is the Yukawa gauge (cf. e.g. [1] , [58]) but also covariant gauges exist, with and even without indefinite metric and lead, of course, to the same covariant scattering amplitudes as the noncovariant Yukawa gauge. Since for these scattering amplitudes dispersion relations can (e.g. for vector meson - vector meson scattering) be proven, it follows that the fact that a theory can be formulated in a nonlocal and noncovariant manner is not in contradiction to causal behaviour. This also applies to QED.

6.5.2 Use of indefinite metric

In all our discussion we did not comment on the fact that we did not work in a Hilbert space but in a larger space with indefinite or, upon imposing (6.20), semidefinite metric for which the concepts "Hermitean adjoint" etc. are not explained. Our procedure was to develop, as proposed by S.N. Gupta [34] , the adequate concepts intrinsically: After introducing a scalar product $\langle \phi | \phi' \rangle$ linear in $|\phi'\rangle$ and antilinear in $|\phi\rangle$ one defines the pseudoadjoint of an operator A by

$$\langle \phi | A^{+} | \phi' \rangle = \overline{\langle \phi' | A | \phi \rangle}$$

and proves that a pseudo-selfadjoint operator has real eigenvalues apart from the eigenvalues to eigenstates of norm zero, while its expectation value is always real. In our case, the scalar product (6.21) was dictated by the commutation relations of the in-operators; for equality with the scalar products evaluated using out-operators we seem to have to rely, if $J^{\mu} \equiv 0$, on a suitable generalized version of the TCP-theorem. The use, as is done in most textbooks, of a metric operator, which is by necessity noninvariant, does not seem to have much merit in this respect except in connection with the anyway noncovariant interaction representation, which has, however, other serious shortcomings.

We finally emphasize that the need of either introducing an indefinite metric, or of giving up manifest covariance, if one wants to have a vector potential, is unavoidable, and for the reasons given before (5.15) a vector potential appears indispensable. For a simple proof of this (based on (5.56,58) and (5.63a) with $Z_3 > o$) and also of the impossibility to have the free Maxwell equations with covariant vector potential valid irrespective of the metric, we refer to R. Strocchi [30].

7. GREEN'S FUNCTIONS IN QUANTUM ELECTRODYNAMICS AND NEUTRAL VECTOR MESON THEORY

We now continue our discussion of Green's functions. From the equations of motion we derive several relations among them which will be useful for renormalization procedure. Since in this lecture we are not interested in the state space we discuss (massive) neutral vector meson theory rather than QED.

7.1 Stueckelberg gauge

We use the Lagrangian (5.31) but amend it by a "photon" mass term, and also write the Dirac Lagrangian out to later obtain the complete set of Green's functions equations:

$$L = -\frac{1}{2} F_{\mu\nu}(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{m_u^2}{2} A_\mu A^\mu - B\,\partial^\mu A_\mu + \frac{s}{2} B^2 +$$

$$+ \bar\psi (i\partial\!\!\!/ + e A\!\!\!/ - M_u)\psi + \tag{7.1}$$

$$+ J_\mu A^\mu + \bar\psi \eta + \bar\eta \psi + KB .$$

The field equations derived from the Stueckelberg Lagrangian (7.1.) are

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad , \tag{7.2}$$

$$\partial_\nu F^{\mu\nu} - m_u^2 A^\mu = e\,\bar\psi \gamma^\mu \psi + \partial^\mu B + J^\mu \tag{7.3}$$

$$=: j^\mu + \partial^\mu B + J^\mu \quad ,$$

$$\partial_\mu A^\mu = s\,B + K \qquad , \tag{7.4}$$

$$(-i\not\partial + M_u)\,\psi = \eta + e\not A\,\psi \qquad , \tag{7.5}$$

$$\bar\psi\,(i\overleftarrow{\not\partial} + M_u) = \bar\eta + e\,\bar\psi\not A \qquad . \tag{7.6}$$

With $\bar\eta,\ \eta \neq o$, the current j^μ is not conserved, but

$$\partial_\mu j^\mu = ie\,\bar\psi\eta - ie\,\bar\eta\psi \tag{7.7}$$

from (7.5,6). Actually, (7.7) can also be derived for a more refined definition of the current, since $\bar\psi\eta + \bar\eta\psi$ is the part in the Lagrangian (7.1) that is not invariant under gauge transformations of the first kind (cf. sect. 6.3) [35] . All CCR are the same as in the case of vanishing mass, (5.36-38) and the familiar ones involving the Dirac field. If $\bar\eta = \eta = K = o$ and $\partial_\mu J^\mu = o$ the theory admits the gauge transformations of the second kind

$$A^\mu \to A^\mu + \partial^\mu\Lambda\ ,\ B \to B - m_u^2\Lambda\ ,\ \psi \to e^{-ie\Lambda}\psi,\ \bar\psi \to e^{ie\Lambda}\bar\psi \tag{7.8a}$$

provided

$$(\Box + m_u^2 s)\ \Lambda = o \qquad . \tag{7.8b}$$

If Λ is a c-number function, not only the field equations but also the CCR remain hereby unchanged.

From (7.3,4,7) we have

$$(\Box + m_u^2 s)\ B = -\ \partial_\mu J^\mu - m_u^2 K - i\,e\,\bar\psi\eta + i\,e\,\bar\eta\psi \tag{7.9}$$

from which by Peierls' method (cf. (5.50,51)) the CR of B with all operators at all times follow:

$$[B(x),\ A^{\mu}(y)] = -i\ \partial_x^{\mu} \Delta(x-y, sm_u^2), \tag{7.10}$$

$$[B(x),\ B(y)] = -i\ m_u^2\ \Delta(x-y, sm_u^2), \tag{7.11}$$

$$[B(x),\ \psi(y)] = -e\ \ \Delta(x-y, sm_u^2)\ \psi(y), \tag{7.12a}$$

$$[B(x),\ \bar{\psi}(y)] = +e\ \ \Delta(x-y, sm_u^2)\ \bar{\psi}(y). \tag{7.12b}$$

In addition the equations of motion (7.2,3) then give

$$[B(x),\ F^{\mu\nu}(y)] = o\ , \tag{7.13}$$

$$[B(x),\ j^{\mu}(y)] = o\ . \tag{7.14}$$

(7.11) shows that the metric of the free field B is indefinite. In contrast to the case $m_u^2 = o$ discussed in lecture 5, for $m_u^2 > o$ the indefinite metric can be removed from the theory in a covariant way by an operator gauge transformation of the type (7.9).

7.2 Proca gauge

Setting formally $\Lambda(x) = \frac{1}{m_u^2} B(x)$, the gauge transformation (7.9)

leads to new variables

$$U^{\mu}(x) = A^{\mu}(x) + \frac{1}{m_u^2}\partial^{\mu} B(x) \tag{7.15}$$

such that $\partial_{\mu} U^{\mu}(x) = o$ if $\partial_{\mu} J^{\mu} = K = o$, and

$$\varphi(x) = e^{-i \frac{e}{m_u^2} B(x)}\ \psi(x)\ , \tag{7.16a}$$

$$\bar{\varphi}(x) = e^{+i \frac{e}{m_u^2} B(x)}\ \bar{\psi}(x)\ . \tag{7.16b}$$

(This transformation is discussed in greater detail by W. Zimmermann [36]).

One easily shows that B commutes with these variables at all times

$$[U^\mu(x), B(y)] = [\varphi(x), B(y)] = [\bar{\varphi}(x), B(y)] = o \qquad (7.17)$$

and that the field equations take the form

$$F^{\mu\nu} = \partial^\mu U^\nu - \partial^\nu U^\mu \quad , \qquad (7.18)$$

$$\partial_\nu F^{\mu\nu} - m_u^2 U^\mu = e\bar{\varphi}\gamma^\mu\varphi + J^\mu \equiv j^\mu + J^\mu , \qquad (7.19)$$

$$(-i\not{\partial} + M_u)\varphi = \frac{e}{2}\gamma_\mu\{U^\mu, \varphi\} \quad , \qquad (7.20)$$

$$\bar{\varphi}(i\overleftarrow{\not{\partial}} + M_u) = \frac{e}{2}\{\bar{\varphi}, U^\mu\}\gamma_\mu \qquad (7.21)$$

which would follow a Proca Lagrangian

$$L = -\frac{1}{2}F_{\mu\nu}(\partial^\mu U^\nu - \partial^\nu U^\mu) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m_u^2}{2}U_\mu U^\mu + J_\mu U^\mu +$$

$$+ i\bar{\varphi}\not{\partial}\varphi - \bar{\varphi}M_u\varphi + \frac{e}{2}\bar{\varphi}\gamma_\mu\{U^\mu, \varphi\} \qquad (7.22)$$

which would be just a Maxwell + Dirac Lagrangian except for the "photon" mass term.

In contrast to the Maxwell - Dirac case, however, where we have the gauge freedom and canonical quantization is not possible, (7.22) admits canonical quantization since the constraint equation

$$\partial_i F^{oi} - m_u^2 U^o = e\bar{\varphi}\gamma^o\varphi + J^o = j^o + J^o \qquad (7.23)$$

can be solved for U^o due to $m_u^2 > o$. The CCR derived from (7.22) are the same that one derives for U^μ, $F^{\mu\nu}$, $\bar{\varphi}$, φ from the way these variables were introduced before, starting from the CCR derived from (7.1) as in QED. There is an interesting difference, however: The operator gauge transformation way gives

$$\left[U^{o}(x),\ U^{i}(y)\right]_{x^{o}=y^{o}} = i\ \frac{1}{m_u^2}\ \partial^{i}\delta^{3}(\vec{x}-\vec{y}) \qquad (7.24)$$

directly using the definition of U^{μ} and the known CR for the constituents. On the other hand, the Lagrangian (7.22) leads upon canonical quantization to (7.24) only upon assuming

$$[j^{o},\ U^{i}] \ = \ o \qquad (7.25)$$

since for this Lagrangian U^{o} is a dependent field to be calculated from (7.23), where $e\,\bar{\varphi}\gamma^{o}\varphi$ is rather the symbolic form for j^{o}. The reason for this seems to be the following one: On the form of j^{μ} which in the Stueckelberg gauge approach is a functional of $\bar{\psi}$, ψ, and A^{μ} (because of (7.14) it does not depend on B) two constraints are imposed:

i. the divergence equation (7.7),

ii. in the transition (7.15,16) to the Proca gauge it has to become a functional of $\bar{\varphi}$, φ, and U^{μ} alone, since (7.17,18,19) again imply this functional to be independent of B.

These requirements may well restrict the form of j^{μ} to such an extent that (7.25) is a consequence.

The operator gauge transformation (7.8), with for zero sources B a free field with indefinite metric quantization as follows from (7.11), shows that the state space of the Stueckelberg theory (7.1) is

$$\mathfrak{h}_{St} = \mathfrak{h}_{Pr} \otimes \mathfrak{h}_{B} \qquad (7.26)$$

with $\mathfrak{h}_{Pr}$ the positive-definite-metric Hilbert space of the Proca theory (7.22), and $\mathfrak{h}_{B}$ the indefinite-metric "Hilbert"-space of the free B field. In fact, substituting (7.15,16) in (7.1) (with $\bar{\eta} = \eta = \kappa = o$, $\partial_{\mu}J^{\mu} = o$) decompose the Lagrangian and also the generators of the Poincaré transformations P^{μ} and $M^{\mu\nu}$ into a sum of two terms, one of them containing only U^{μ}, $\bar{\varphi}$, φ variables, the other containing only the B variables but with an opposite overall sign as would be the case for a free scalar field, this reflecting the indefinite metric.

From (7.11,16) follows

$$\psi(x) = e^{i\frac{e}{m_u^2}B(x)}\varphi(x) = e^{\frac{e^2}{4m_u^2}\Delta^{(1)}(0,sm_u^2)} :e^{i\frac{e}{m_u^2}B(x)}: \varphi(x) \quad (7.27)$$

Using (7.17,26) we therefore find

$$\langle\psi(x)\,\tilde{\psi}(y)\rangle = e^{\frac{e^2}{2m_u^2}\Delta^{(1)}(0,sm_u^2)}\, e^{-i\frac{e^2}{m_u^2}\Delta^{(+)}(x-y)} \langle\varphi(x)\,\tilde{\varphi}(y)\rangle \quad (7.28)$$

where the infinite constant factor is to be absorbed into a renormalization of the Stueckelberg fields against the Proca fields. Similar explicit relations hold between other VEV and Green's functions of these theories. From (7.28) the two-point Wightman functions $\langle\psi(x)\,\tilde{\psi}(y)\rangle$ and $\langle\varphi(x)\,\tilde{\varphi}(y)\rangle$ cannot both be tempered distributions in the sense of L. Schwartz [37] . Since in perturbation theory the former ones lead to tempered distributions of order-independent growth in momentum space, the conjecture is reasonable, that the Stueckelberg theory, if it exists, yields tempered distributions. Then the Proca theory leads to the more general "strictly localizable" distributions of A. Jaffe [38], and in perturbation theory to a growth that is the stronger the higher the order. The interested reader is referred to [38] , [39] , and the literature given there.

7.3 Two-point functions

In this section we discuss briefly the spectral representations of both the vector meson and the spinor fields two-point functions and give explicit expressions of the inverse propagators which will be used in the amputation procedure of Green's functions.

7.3.1 Vector meson propagator

The discussion of the spectral representation of the two-point function $\langle A^\mu(x)A^\nu(y)\rangle$ for the theory without sources follows the same pattern as in QED. The formulas (5.56) up to (5.61) remain unchanged. Because of (7.10), however, (5.62) is to be replaced by

$$^\circ\varrho_1(\varkappa^2) - \varkappa^2\, {}^\circ\varrho_2(\varkappa^2) = s\,\delta(\varkappa^2 - s\,m_u^2)\ . \tag{7.29}$$

Assuming the existence of a neutral vector meson of mass m (5.63a) becomes

$$^\circ\varrho_1(\varkappa^2) = Z_3\,\delta(\varkappa^2 - m^2) + {}^\circ\hat{\varrho}_1(\varkappa^2) \tag{7.30a}$$

with

$$0 \leq Z_3 < 1 \quad , \tag{7.30b}$$

$$^\circ\hat{\varrho}_1(\varkappa^2) \geq 0 \quad , \tag{7.30c}$$

$$\int_{m_s^2}^{\infty} d\varkappa^2\, {}^\circ\hat{\varrho}_1(\varkappa^2) = 1 - Z_3 \quad , \tag{7.30d}$$

where m_s is the threshold for $^\circ\hat{\varrho}_1$ -contributions, while (5.64) is to be replaced by

$$^\circ\varrho_2(\varkappa^2) = \frac{^\circ\hat{\varrho}_1(\varkappa^2)}{\varkappa^2} + Z_3\frac{\delta(\varkappa^2 - m^2)}{m^2} - \frac{1}{m_u^2}\,\delta(\varkappa^2 - s\,m_u^2) - a\,\delta(\varkappa^2) \tag{7.31}$$

with

$$a := \frac{Z_3}{m^2} + \int_{m_s^2}^{\infty} d\varkappa^2\, \frac{^\circ\hat{\varrho}_1(\varkappa^2)}{\varkappa^2} - \frac{1}{m_u^2}\ . \tag{7.31a}$$

The third term on the r.h.s. of (7.31) is due to the B-part, the others come from the U-(Proca)-theory, cf. (7.11,15,26). If now $a \neq o$, U^μ would contain the derivative of a scalar zero-mass field (or a vector field), of pathological behaviour, since we assume there to be a stable vector particle of finite mass m, and there would not be an apparent reason to prevent a stable particle to decay into such "a-particle". In perturbation theory at least, we find we can get away without such particle, which means that the theory possesses a reasonable formal solution with

$$a = o \tag{7.32}$$

which we will henceforth assume.

According to (7.11,31) the choice of the parameter s solely reigns the mass of the B-particle and, as follows from the discussion after (7.26), has no other effect. For convenience we chose

$$s = \frac{m^2}{m_u^2} \tag{7.33}$$

which makes the two δ -functions in (7.31) coincide, such that the mass of the B-particle coincides with the physical vector meson mass. From (7.31,32) we learn

$$Z_3 \, m_u^2 < m^2 < m_u^2 \quad , \tag{7.34}$$

as an equality sign can hold only if $\circ \hat{\S}_1 \equiv o$, which, however, leads to a theory without interaction. Therefore s has the property

$$o \leq s < 1$$

with $s = o$ only if m_u^2 diverges.

It turns out that the most convenient <u>amplitude renormalization</u> is given by

$$A^{\mu}_{ren} := s^{-1/2} A^{\mu} , \qquad F^{\mu\nu}_{ren} := s^{-1/2} F^{\mu\nu} , \tag{7.35}$$

$$B_{ren} := s^{+1/2} B \quad , \qquad J^{\mu}_{ren} := s^{+1/2} J^{\mu} .$$

Correspondingly the <u>renormalized charge</u> is defined to be

$$e_{ren} := s^{+1/2} e \tag{7.36}$$

such that

$$e_{ren} A^{\mu}_{ren} = e A^{\mu} .$$

According to (7.35) the renormalized spectral functions are

$$\varrho_{1,2}(\varkappa^2) := s^{-1} \, {}^{\circ}\varrho_{1,2}(\varkappa^2) \, , \quad \hat{\varrho}_1(\varkappa^2) := s^{-1} \, {}^{\circ}\hat{\varrho}_1(\varkappa^2).$$

Thus (7.31a) with (7.32) becomes

$$\frac{1}{m^2} = \frac{Z_3}{s} \cdot \frac{1}{m^2} + \int_{m_s^2}^{\infty} d\varkappa^2 \, \frac{\hat{\varrho}_1(\varkappa^2)}{\varkappa^2} \tag{7.37}$$

while (5.60) with (7.30a) now reads

$$\frac{1}{s} = \frac{Z_3}{s} + \int_{m_s^2}^{\infty} d\varkappa^2 \; \hat{\varrho}_1(\varkappa^2) \; . \tag{7.38}$$

As the coefficient of the δ -function in (7.30a) by this renormalization becomes ($\frac{Z_3}{s}$) the A^{μ}_{in}-field part of A^{μ}_{ren} defined in analogy to (5.65a) is now normalized to $(\frac{Z_3}{s})^{1/2}$, i.e. when calculating scattering amplitudes from Green's functions one has to multiply the Green's functions by $(\frac{s}{Z_3})^{r/2}$ when r "photons" are involved to be able to use a reduction formula as usual. Since, however, the calculation of the scattering amplitude is the last and simple step after completed calculation of the Green's function, the convenience of the more complicated Green's function calculation is decisive. (This renormalization convention is the one adopted by Kroll, Lee, and Zumino [40]). For $m^2 \longrightarrow o$, according to (7.37) $\frac{Z_3}{s} \longrightarrow 1$ from below, such that we recover the Gupta-Bleuler gauge (or, in [40] , the QED-Landau gauge).

With our choice of parameters (7.32,33) the two-point function of renormalized vector meson fields has the representation

$$\langle A^{\mu}_{ren}(x)A^{\nu}_{ren}(o)\rangle = \int_o^{\infty} d\varkappa^2 \; \varrho_1(\varkappa^2)\left(-g^{\mu\nu} - \frac{\partial^{\mu}\partial^{\nu}}{\varkappa^2}\right) i\,\Delta^{(+)}(x,\varkappa^2) + \frac{1}{m^2}\partial^{\mu}\partial^{\nu}\, i\,\Delta^{(+)}(x,m^2) \tag{7.39a}$$

with

$$\varrho_1(\varkappa^2) = \frac{Z_3}{s}\,\delta(\varkappa^2 - m^2) + \hat{\varrho}_1(\varkappa^2) \; . \tag{7.39b}$$

Since in our (Stueckelberg) gauge the vector potentials are canonically independent variables the Green's function

$$\langle (A^{\mu}_{ren}(x)A^{\nu}_{ren}(o))_+ \rangle =: \Delta'_F{}^{\mu\nu}(x) =$$

$$= \int d\varkappa^2\, \varrho_1(\varkappa^2)\left(-g^{\mu\nu} - \frac{\partial^{\mu}\partial^{\nu}}{\varkappa^2}\right)\Delta_F(x,\varkappa^2) + \frac{1}{m^2}\partial^{\mu}\partial^{\nu}\Delta_F(x,m^2) \tag{7.40}$$

coincides with $\langle T(A^{\mu}_{ren}(x)A^{\nu}_{ren}(o))\rangle$.

This can easily be shown:

from $\quad \Theta(x^o)\,\partial^{\mu}\partial^{\nu} = \partial^{\mu}\partial^{\nu}\,\Theta(x^o) - g^{\nu o}\partial^{\mu}\delta(x^o) - g^{\mu o}\delta(x^o)\,\partial^{\nu}$

one finds

$$\Theta(x^o)[\partial^{\mu}\partial^{\nu}\, i\,\Delta^{(+)}(x)] + \Theta(-x^o)[-\partial^{\mu}\partial^{\nu}\, i\,\Delta^{(-)}(x)] = \partial^{\mu}\partial^{\nu}\Delta_F + i\,g^{\mu o}g^{\nu o}\,\delta(x)$$

(note that our Δ_F is called $i\Delta_F$ by Bjorken and Drell [2]), and the noncovariant terms cancel due to (7.37).

Passing to Fourier transforms (which for convenience are denoted by the same symbols)

$$\Delta'^{\mu\nu}_F(k) = \int dx\, e^{ikx}\, \Delta'^{\mu\nu}_F(x) =$$

$$= \int_0^\infty d\varkappa^2\, \varrho_1(\varkappa^2)\left[-g^{\mu\nu} + \frac{k^\mu k^\nu}{\varkappa^2}\right]\frac{i}{k^2-\varkappa^2+i\varepsilon} \quad - \frac{1}{m^2}\, k^\mu k^\nu \frac{i}{k^2-m^2+i\varepsilon}$$

we find after some calculation by again using (7.37)

$$\Delta'^{\mu\nu}_F(k) = \Delta^{\mu\nu}_F(k) +$$

$$+ i\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2+i\varepsilon}\right)\frac{k^2}{k^2-m^2+i\varepsilon}\int_{m_s^2}^\infty d\varkappa^2\, \frac{\hat{\varrho}_1(\varkappa^2)\,(m^2-\varkappa^2)}{\varkappa^2(k^2-m^2+i\varepsilon)} \qquad (7.41)$$

i.e. in the Stueckelberg gauge the "free" propagator $\Delta_F^{\mu\nu}(k)$ is corrected by a term that is transverse but has a pole at $k^2 = m^2$ with residuum proportional to m^2, and vanishes at $k = o$. That the pole term is not the free one is characteristic for the non-zero mass case; in fact, for $m^2 \to o$, i.e. $\frac{Z_3}{s} \to 1$, there remains only the free pole term at $k^2 = o$.

From (7.41) we have, in the sense of matrix inverse,

$$[(\Delta'_F(k)^{-1}]^{\mu\nu} = i g^{\mu\nu}(k^2-m^2) +$$

$$+ i(k^2-m^2)(g^{\mu\nu}k^2 - k^\mu k^\nu)\, \frac{\displaystyle\int d\varkappa^2\, \frac{\hat{\varrho}_1(\varkappa^2)(m^2-\varkappa^2)}{\varkappa^2(k^2-\varkappa^2+i\varepsilon)}}{\displaystyle 1 - k^2\int d\varkappa^2\, \frac{\hat{\varrho}_1(\varkappa^2)(m^2-\varkappa^2)}{\varkappa^2(k^2-\varkappa^2+i\varepsilon)}} \qquad (7.42)$$

$$=: [(\Delta_F(k))^{-1}]^{\mu\nu} - \Pi^{\mu\nu}(k) \quad .$$

From (7.42) these properties of $\Pi^{\mu\nu}(k)$ follow:

(a) it is transverse,
(b) it vanishes at k = o of second order,
(c) it vanishes at $k^2 = m^2$, (7.43)
(d) for $m^2 = o$, it vanishes of fourth order as $k \to o$.

(The denominator in (7.42) could vanish for some k^2 between m^2 and m_s^2; such kinematical (in this case "Castillejo-Dalitz-Dyson") singularity will however, never be noticed in perturbation theory which implies expansion of the ratio in (7.42) in powers of $\hat{g}_1$).

We mentioned before that due to the renormalization (7.35) instead of the one with s replaced by Z_3, in the calculation of scattering amplitudes from Green's functions one needs the value of $\frac{s}{Z_3}$. Instead of using (7.37) it is more convenient to directly obtain it from $\Pi^{\mu\nu}$.

<u>Exercise:</u> Prove

$$\frac{s}{Z_3} = 1 + \frac{i}{6m^2} k^\lambda \left. \frac{\partial \Pi_\mu{}^\mu(k)}{\partial k^\lambda} \right|_{k^2 = m^2}$$

7.3.2 Electron propagator

Analogous considerations to those that led to (5.56) give for the Dirac field

$$\langle \psi(x) \bar{\psi}(0) \rangle = \int_0^\infty d\varkappa^2 \left(i\,{}^\circ\sigma_1(\varkappa^2)\, \partial^x + {}^\circ\sigma_2(\varkappa^2) \right) i\, \Delta^{(+)}(x, \varkappa^2) .$$

If the metric is positive, e.g. in the Proca gauge, $|\varkappa|\, {}^\circ\sigma_1(\varkappa^2) \geq$ $\geq |{}^\circ\sigma_2(\varkappa^2)|$. Introducing

$${}^\circ\sigma(\pm|\varkappa|) := {}^\circ\sigma_\pm(\varkappa^2) := \frac{1}{2}\left({}^\circ\sigma_1(\varkappa^2) \pm \frac{1}{|\varkappa|}\, {}^\circ\sigma_2(\varkappa^2) \right)$$

we have

$$\langle \psi(x)\,\bar{\psi}(0)\rangle = \int_{-\infty}^{+\infty} d\varkappa\,(i\gamma^{x}+\varkappa)\,{}^{\circ}\sigma(\varkappa)\; i\Delta^{(+)}(x,\varkappa^{2}) \tag{7.44}$$

with, in the positive-metric case ${}^{\circ}\sigma(\varkappa) \geq 0$.

From the canonical anticommutation relation one finds

$$\int_{-\infty}^{\infty} d\varkappa\;{}^{\circ}\sigma(\varkappa) = 1\,. \tag{7.45}$$

Analogously to (7.30a) we should now have

$${}^{\circ}\sigma(\varkappa) = Z_2\,\delta(\varkappa-M) + {}^{\circ}\hat{\sigma}(\varkappa) \tag{7.46}$$

where M is the renormalized mass of the electron.
In the Proca gauge one has $0 \leq Z_2 < 1$, ${}^{\circ}\hat{\sigma}(\varkappa) \geq 0$ while in the Stueckelberg gauge Z_2 need not be bounded by 1. The fact that the property $Z_2 \geq 0$ remains true arises as follows: In the massive vector meson case there is no difficulty of principle in constructing one-charged-particle in and out states as there is in QED, i.e. there exist $\psi_{\substack{in\\out}} = \varphi_{\substack{in\\out}}$, since the arguments preceding (6.24) now do apply, and $[\psi_{\substack{in\\out}}, B_{ren}] = 0$ implies, because of (7.11,26), that $\psi_{\substack{in\\out}}$ operate in $\mathfrak{h}_{P_r}$ only and thus are equal to $\varphi_{\substack{in\\out}}$ at least after an adjustment of phases. But then the contributions from the one-particle intermediate states in (7.44) are the same as when calculated in $\mathfrak{h}_{P_r}$ alone, and have positive-definite metric leading to $Z_2 \geq 0$. It is easily shown that the contributions in (7.44) from states with the same or opposite parity relative to the free one-particle states are associated with positive respectively negative $\varkappa$, and a δ-contribution in (7.44) from $\varkappa = -M$ also would imply a one-particle-state degeneracy that does at

least not occur in perturbation theory.

Amplitude renormalization of the Dirac field is achieved by defining

$$\psi_{ren} := Z_2^{-1/2}\,\psi \quad , \qquad \bar{\psi}_{ren} = \bar{\psi}\, Z_2^{-1/2} \; ,$$

$$\bar{\eta}_{ren} = \bar{\eta}\, Z_2^{+1/2} \quad , \qquad \eta_{ren} = Z_2^{+1/2}\,\eta \; , \tag{7.47}$$

and

$$\sigma(\varkappa) := Z_2^{-1}\; {}^{o}\sigma(\varkappa) = \delta(\varkappa - M) + \hat{\sigma}(\varkappa) \; . \tag{7.48}$$

From (7.44) the full electron propagator becomes

$$\langle (\psi_{ren}(x)\;\bar{\psi}_{ren}(o))_+ \rangle \;=:\; S_F'(x) =$$

$$= \int_{-\infty}^{+\infty} d\varkappa\; \sigma(\varkappa)\,(i\not{\partial} + \varkappa)\;\Delta_F(x, \varkappa^2) \; , \tag{7.49}$$

which again coincides with $\langle T(\psi_{ren}(x)\;\bar{\psi}_{ren}(o))\rangle$ since

$$\theta(x^o)\,\partial^\mu = \partial^\mu\,\theta(x^o) - g^{\mu o}\,\delta(x^o)$$

and

$$\Delta(x)\Big|_{x^o=o} = o \; .$$

According to (7.48) the Fourier transform of $S_F'(x)$ is

$$S_F'(p) = \int dx\; e^{ipx}\, S_F'(x) =$$

$$= i \left[\frac{1}{\not p - M + i\varepsilon} + \int_{-\infty}^{+\infty} d\varkappa \, \frac{\hat{\sigma}(\varkappa)}{\not p - \varkappa + i\varepsilon} \right] , \tag{7.50}$$

such that its inverse reads

$$S'_F(p)^{-1} = -i(\not p - M) + i(\not p - M) \frac{\int_{-\infty}^{+\infty} d\varkappa \, \frac{\hat{\sigma}(\varkappa)}{\not p - \varkappa + i\varepsilon}}{1 + \int_{-\infty}^{+\infty} d\varkappa \, \frac{\hat{\sigma}(\varkappa)}{\not p - \varkappa + i\varepsilon} (\not p - M)} (\not p - M) \tag{7.51}$$

$$=: S_F(p)^{-1} \qquad - \Sigma(p)$$

where as well as $S'_F(p)$ also $\Sigma(p)$ is a function $\Sigma(\not p)$ of the matrix $\not p$ only, which, as is easily seen from (7.51), vanishes of second order at $\not p = M$, i.e.

$$\Sigma(\not p) \Big|_{\not p = M} = 0 \qquad \text{and} \qquad \frac{\partial}{\partial \not p} \Sigma(\not p) \Big|_{\not p = M} = 0 \quad . \tag{7.52}$$

Without need of bringing $\Sigma(p)$ explicitly into the form of a function of $\not p$ (7.52) can be replaced by

$$(\not p + M) \, \Sigma(p) = \Sigma(p) \, (\not p + M) = 0 \quad \text{if } p^2 = M^2, \tag{7.53a}$$

and

$$(\not p + M) \left(\frac{\partial}{\partial p_\mu} \Sigma(p) \right) (\not p + M) = 0 \quad \text{if } p^2 = M^2. \tag{7.53b}$$

The ratio in (7.51) can be freed of matrices standing in the denominator in the familiar way. This is left as an exercise to the reader.

7.4 Green's functions

We are now in position to write down the renormalized field equations. From (7.3,4,35) we find (setting K = o as it does no longer serve any purpose)

$$(\Box + m^2)\, A^\mu_{ren} = - j^\mu_{ren} - J^\mu_{ren} \tag{7.54a}$$

with

$$j^\mu_{ren} := s^{+\frac{1}{2}} j^\mu + (1-s)\, \partial_\nu F^{\mu\nu}_{ren}$$

whereas (7.5,6,47) give

$$(-i\not\partial + M)\, \psi_{ren} = Z_2\, e_{ren} \not A_{ren} \psi_{ren} + \eta_{ren} + \tag{7.54b}$$

$$+ (1-Z_2)(-i\not\partial + M)\, \psi_{ren} + Z_2\, \delta M\, \psi_{ren}$$

and

$$\bar\psi_{ren} (i\overleftarrow{\not\partial} + M) = \bar\psi_{ren} \not A_{ren}\, e_{ren} Z_2 + \bar\eta_{ren} + \tag{7.54c}$$

$$+ \bar\psi_{ren} (i\overleftarrow{\not\partial} + M)(1-Z_2) + \bar\psi_{ren}\, \delta M\, Z_2$$

with

$$\delta M := M - M_u \;.$$

Since from now on we shall use only the renormalized charge and fields we omit the subscript "ren" for brevity.

Now from the generating functional of disconnected Green's functions

$$\hat G_{disc} \{\bar\eta, \eta, J\} \quad := \quad {}_{out}\langle \;|\; \rangle_{in}$$

we define in analogy to (4.7,8)

$$\hat{G}^{\mu_1\cdots\mu_r}_{disc}(x_1\cdots x_n, y_1\cdots y_m, z_1\cdots z_r) := \hat{G}^{\mu_1\cdots\mu_r}_{disc}(x_1\cdots x_n, y_1\cdots y_m, z_1\cdots z_r; \bar{\eta}, \eta, J\} \equiv \tag{7.55}$$

$$\equiv {}_{out}\langle (\psi(x_1)\cdots\psi(x_n)\,\bar{\psi}(y_1)\cdots\bar{\psi}(y_m)\,A^{\mu_1}(z_1)\cdots A^{\mu_r}(z_r))_+ \rangle_{in} \equiv$$

$$= \frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x_1)}\cdots\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x_n)}\left(-\frac{1}{i}\right)\frac{\delta}{\delta\eta(y_1)}\cdots\left(-\frac{1}{i}\right)\frac{\delta}{\delta\eta(y_m)}\,\frac{1}{i}\frac{\delta}{\delta J_{\mu_1}(z_1)}\cdots\frac{1}{i}\frac{\delta}{\delta J_{\mu_r}(z_r)}\;\hat{G}_{disc}\{\bar{\eta},\eta,J\}\;.$$

Since in the Stueckelberg (and Gupta-Bleuler) gauge the fields $A^\mu, \psi, \bar{\psi}$ are all independent canonical variables we have from (4.8)

$${}_{out}\langle (\psi(x_1)\cdots A^{\mu_r}(z_r))_+ \rangle_{in} = {}_{out}\langle T(\psi(x_1)\cdots A^{\mu_r}(z_r))\rangle_{in}\;. \tag{7.55a}$$

Setting $\bar{\eta} = \eta = J = 0$ we obtain the (disconnected) Green's functions

$$G^{\mu_1\cdots\mu_r}_{disc}(x_1\cdots x_n, y_1\cdots y_m, z_1\cdots z_r) := \hat{G}^{\mu_1\cdots\mu_r}_{disc}(x_1\cdots x_n, y_1\cdots y_m, z_1\cdots z_r; 0,0,0\} \equiv \tag{7.56}$$

$$\equiv \langle (\psi(x_1)\cdots\psi(x_n)\,\bar{\psi}(y_1)\cdots\bar{\psi}(y_m)\,A^{\mu_1}(z_1)\cdots A^{\mu_r}(z_r))_+ \rangle$$

which are covariant by definition (recall the discussion after (4.8)). From (7.55a) we see that the $(\ldots)_+$-product (sometimes also called T^*-product) in the Stueckelberg gauge coincides with the (naive) T-product. The connected parts $\hat{G}^{\mu_1\cdots\mu_r}(x_1\cdots x_n, y_1\cdots y_m, z_1\cdots z_r)$ and $G^{\mu_1\cdots\mu_r}(x_1\cdots x_n, y_1\cdots y_m, z_1\cdots z_r)$ are obtained similarly to (7.55,56) by differentiation of

$$\hat{G}\{\bar{\eta},\eta,J\} := \ln \hat{G}_{disc}\{\bar{\eta},\eta,J\}\;. \tag{7.57}$$

Due to gauge invariance of the first kind (charge conservation) only Green's functions with the same number of ψ - and $\bar{\psi}$ -variables, i.e. n=m, are different from zero, and invariance under charge conjugation implies all Green's functions with an odd number of vector particles arguments but without spinor or antispinor particle arguments to vanish (Furry's theorem).

Taking n=m=o, r=2 or n=m=1, r=o (7.56,57) give the one-particle propagators (7.40) and (7.49), respectively:

$$\Delta'^{\mu\nu}_F(x-y) = G^{\mu\nu}_{disc}(,,xy) = G^{\mu\nu}(,,xy) \;, \tag{7.58a}$$

$$S'_F(x-y) = G_{disc}(x,y,) = G(x,y,) \;. \tag{7.58b}$$

The <u>amputation operation,</u> indicated by underlining the argument, is defined by the convolution with the corresponding (in the convolution sence) inverse-propagator matrix:

$$\hat{G}^{\cdot\cdot\mu\cdot\cdot}(\ldots,\ldots,\ldots\underline{x}\ldots) = \int dy\, G^{\mu\nu}(,,xy)^{-1}\hat{G}^{\cdot\cdot}{}_{\nu}{}^{\cdot\cdot}(\ldots,\ldots,\ldots y\ldots) \;, \tag{7.59a}$$

$$\hat{G}^{\cdots}(\ldots\underline{x}\ldots,\ldots,\ldots) = \int dy\, G(x,y,)^{-1}\hat{G}^{\cdots}(\ldots y\ldots,\ldots,\ldots) \;, \tag{7.59b}$$

$$\hat{G}^{\cdots}(\ldots,\ldots\underline{x}\ldots,\ldots) = \int dy\, \hat{G}^{\cdots}(\ldots,\ldots y\ldots,\ldots)\, G(y,x,)^{-1} \;, \tag{7.59c}$$

and amputation of the one-particle propagators (7.58) gives by definition

$$G^{\mu\nu}(,,\underline{x}y) = G^{\mu\nu}(,,x\underline{y}) = g^{\mu\nu}\,\delta(x-y) \;, \tag{7.60a}$$

$$G(\underline{x},y,) = G(x,\underline{y},) = 1\cdot\delta(x-y) \;. \tag{7.60b}$$

We introduce <u>Fourier transforms</u> by

$$\int dx_1\cdots dx_n\, dy_1\cdots dy_n\, dz_1\cdots dz_r\; e^{ip_1x_1+\ldots+ip_nx_n-iq_1y_1-\ldots-iq_ny_n+ik_1z_1+\ldots+ik_rz_r}$$

$$\times\, G^{\mu_1\cdots\mu_r}(x_1\cdots x_n, y_1\cdots y_n, z_1\cdots z_r) =: \tag{7.61}$$

$$(2\pi)^4\,\delta\Big(\sum_1^n{}_i\, p_i - \sum_1^n{}_i\, q_i + \sum_1^r{}_i\, k_i\Big)\; G^{\mu_1\cdots\mu_r}(p_1\cdots p_n, q_1\cdots q_n, k_1\cdots k_r)$$

with $G^{\mu_1\cdots\mu_r}(p_1 \ldots k_r)$ defined only where the argument of the δ-function vanishes. (For convenience we use the same symbols both in configuration space and in momentum space.) The amputation operation then reads

$$G(\underline{p_1}\cdots\underline{p_n}, \underline{q_1}\cdots\underline{q_n}, \underline{k_1}\cdots\underline{k_r})\ \delta(\Sigma p - \Sigma q + \Sigma k) =$$

$$= \prod_{i=1}^{n} S_F'(p_i)^{-1} \cdot \prod_{i=1}^{r} \Delta_F'(k_i)^{-1}\ G(p_1\cdots p_n, q_1\cdots q_n, k_1\cdots k_r) \prod_{i=1}^{n} S_F'(q_i)^{-1}\ \ \delta(\Sigma p - \Sigma q + \Sigma k)$$

where we have suppressed all indices. From (7.42,37) we find

$$\left.\frac{[\Delta_F'(k)^{-1}]^{\mu\nu}}{(-k^2+m^2)}\right|_{k^2=m^2} = -i\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}\right)\frac{s}{Z_3} - i\ \frac{k^\mu k^\nu}{m^2} \qquad (7.62)$$

and from (7.51,52)

$$\left.\frac{(S_F'(p))^{-1}}{(-\not p + M)}\right|_{\not p = M} = i \qquad (7.63)$$

such that amputation on the mass shell in coordinate space (up to a factor i) for "photon" arguments is equivalent to application of the Klein-Gordon operator apart from a factor $\frac{s}{Z_3}$ for the transverse part, and for spinor arguments is equivalent to application of the Dirac operator.

In terms of the generating functionals the renormalized field equations (7.54) take the form (cf. (4.14,15))

$$(\Box + m^2)\frac{1}{i}\frac{\delta}{\delta J_\mu}\hat{G}_{disc} = \Big[\mathrm{Tr}\left(e Z_2 \gamma^\mu \frac{1}{i}\frac{\delta}{\delta\bar\eta}\left(-\frac{1}{i}\right)\frac{\delta}{\delta\eta}\right) - J^\mu - \qquad (7.64a)$$

$$- (1-s)\left(\partial^\mu\partial_\varsigma - \Box g^\mu{}_\varsigma\right)\frac{1}{i}\frac{\delta}{\delta J_\varsigma}\Big]\ \hat{G}_{disc} \quad ,$$

$$(-i\not\partial + M)\frac{1}{i}\frac{\delta}{\delta\bar{\eta}}\hat{G}_{disc} = [\frac{1}{i}\frac{\delta}{\delta J_\rho} e Z_2 \gamma_\rho \frac{1}{i}\frac{\delta}{\delta\bar{\eta}} + \eta + \tag{7.64b}$$

$$+ (1-Z_2)(-i\not\partial + M)\frac{1}{i}\frac{\delta}{\delta\bar{\eta}} + Z_2 \delta M \frac{1}{i}\frac{\delta}{\delta\bar{\eta}}]\hat{G}_{disc} ,$$

$$(-\frac{1}{i})\frac{\delta}{\delta\eta}\hat{G}_{disc}(i\overleftarrow{\not\partial} + M) = (-\frac{1}{i})\frac{\delta}{\delta\eta} e Z_2 \gamma_\rho \frac{1}{i}\frac{\delta}{\delta J_\rho}\hat{G}_{disc} + \bar{\eta}\hat{G}_{disc} + \tag{7.64c}$$

$$+ (-\frac{1}{i})\frac{\delta}{\delta\eta}\hat{G}_{disc}(i\overleftarrow{\not\partial} + M)(1-Z_2) + (-\frac{1}{i})\frac{\delta}{\delta\eta}\hat{G}_{disc}\,\delta M Z_2 ,$$

and similarly for the connected parts. We can integrate these equations with the free field Green's functions $\Delta_F^{\mu\nu}(x) = - g^{\mu\nu} \Delta_F(x)$ and $S_F(x)$ which do not give rise to boundary terms (cf. the discussion after (4.31)) and satisfy the differential equations

$$(\Box_x + m^2)\Delta_F(x-y) = (-i\not\partial_x + M)S_F(x-y) = S_F(x-y)(i\overleftarrow{\not\partial}_y + M) = -i\,\delta(x-y).$$

Then they read

$$\hat{G}^\mu(,,x) = \int dy\, \Delta^\mu_{F\nu}(x-y)[-\mathrm{Tr}(ieZ_2\gamma^\nu\hat{G}(y,y,)) + \hat{G}(,y,)\; ieZ_2\gamma^\nu\hat{G}(y,,) +$$

$$+ iJ^\nu(y) + i(1-s)(\partial^\nu\partial_\rho - \Box\, g^\nu{}_\rho)\hat{G}^\rho(,,y)] , \tag{7.65a}$$

$$\hat{G}(x,,) = \int dy\; S_F(x-y)[ieZ_2\gamma_\rho\hat{G}^\rho(y,,y) + ieZ_2\gamma_\rho\hat{G}(y,,)\hat{G}^\rho(,,y) +$$

$$+ i\eta(y) + i(1-Z_2)(-i\not\partial + M)\hat{G}(y,,) + iZ_2\delta M\hat{G}(y,,)] , \tag{7.65b}$$

$$\hat{G}(,x,) = \int dy \left[\hat{G}^{\varsigma}(,y,y) ieZ_2 \gamma_{\varsigma} + \hat{G}(,y,)\, \hat{G}^{\varsigma}(,,y)\, ieZ_2 \gamma_{\varsigma} + \right. \tag{7.65c}$$

$$\left. + i\bar{\eta}(y) + \hat{G}(,y,)\, i(i\overleftarrow{\not\partial} + M)(1-Z_2) + \hat{G}(,y,)\, i\,\delta M Z_2 \right] S_F(y-x) \quad .$$

Further differentiation w.r.t. the sources yields similar equations for Green's functions with an arbitrary number of arguments. The reader is suggested to look up the corresponding equations in case of theories with ϕ^3- and ϕ^4-coupling in [41] and [42] , respectively. There graphical representations are given which may be helpful for understanding.

The iteration of these equations generates the perturbation theoretical expansion of the Green's functions in pre-renormalized form, i.e. the counter terms on the r.h.s. of (7.54) are supplied by the operators

$$\varphi^{\mu\nu}(xy) := i(1-s)(\partial^{\mu}_{x}\partial^{\nu}_{x} - \Box_x g^{\mu\nu})\ \delta(x-y) \quad , \tag{7.66a}$$

$$\varepsilon(xy) := \left[i(1-Z_2)(-i\overrightarrow{\not\partial}_x + M) \quad + \ i\, Z_2\, \delta M \right] \delta(x-y) \tag{7.66b}$$

and by the factor Z_2 in the bare vertex operator

$$\gamma^{\mu}(x,y,z) := ieZ_2\, \gamma^{\mu}\, \delta(x-z)\ \delta(z-y) \quad . \tag{7.66c}$$

In the final lecture we show that they can be absorbed in a convergent calculation, mainly with the help of the Ward identities, which will be derived in the next section. The order in which one uses the equations (7.65) does not matter since in the form of functional differential equations (7.64) they are integrable as one easily proves by constructing a formal solution by the technique of lecture 4.

7.5 Ward identities

It is the aim of renormalization theory to show how the divergent expressions encountered when one solves (7.65) by iteration can be converted into convergent ones by suitable choice of the constants Z_2, s, and δM. It will turn out, however, that our equations are somewhat too formal to enable to cope straightforwardly with all exigencies that will arise. As a help hereto, we have to use that the main requirement for the current j^μ is that (in the absence of sources) it is conserved and, of course, transforms as a vector, and satisfies the ETC

$$[j^0(x), \psi(y)]_{x^0=y^0} = -e\ \psi(x)\ \delta(x-y)\ . \qquad (7.67)$$

The consequences of this for Green's functions are the Ward-Takahashi identities.

In order to obtain them we take the matrix element of the divergence of (7.54a) between the states ${}_{out}\langle|$ and $|\rangle_{in}$. Upon using (7.7) we then have

$$(\Box + m^2)\,\partial_\mu \hat{G}^\mu_{disc}(,,x) = -\partial_\mu J^\mu\, \hat{G}_{disc} - \qquad (7.68)$$

$$- ie\ \hat{G}_{disc}(,x,)\ \eta(x) + i\,e\,\bar{\eta}(x)\hat{G}_{disc}(x,,)$$

and similarly for $G\{\bar{\eta},\eta,J\}$ the usefullness of which is that it does not involve Green's functions with coinciding arguments and thus will help to circumvent some of the ambiguities of the latter. The integrated form of (7.68) is obtained by using the Green's function Δ_F:

$$i\,\partial_\mu \hat{G}^\mu_{disc}(.,z) = \int dx\, \Delta_F(z-x)\,[\,\partial_\mu J^\mu(x)\hat{G}_{disc}\{\bar{\eta},\eta,J\} - \qquad (7.69)$$

$$- ie\ \bar{\eta}(x)\hat{G}_{disc}(x,,) + ie\hat{G}_{disc}(,x,)\ \eta(x)\ ]\ .$$

Because of the importance of (7.68,69) we briefly indicate an alternative and more elementary derivation. Consider

$$\partial^z_\mu \langle(\psi(x_1)\cdots\psi(x_n)\,\bar{\psi}(y_1)\cdots\bar{\psi}(y_n)\ A^\mu(z)\,A^{\mu_1}(z_1)\cdots A^{\mu_r}(z_r))_+\rangle = \qquad (7.70)$$

$$= \langle(\psi(x_1)\cdots\psi(x_n)\,\bar{\psi}(y_1)\cdots\bar{\psi}(y_n)\ B(z)\ A^{\mu_1}(z_1)\cdots A^{\mu_r}(z_r))_+\rangle$$

which holds since from (7.4,35) $\partial_\mu A^\mu(z) = B(z)$ and the commutativity at equal times of $A^o(z)$ with all operators that occur above. Now the commutators of $B(z)$ with all other operators on the r.h.s. of (7.70) are known, namely (cf. (7.10,12,35,36,47)

$$[B^{(\pm)}(z), A^\mu(x)] = -i\partial_z^\mu \, \Delta^{(\pm)}(z-x,m^2) \quad ,$$

$$[B^{(\pm)}(z), \psi(x)] = -e\,\Delta^{(\pm)}(z-x,m^2)\ \psi(x) \quad ,$$

$$[B^{(\pm)}(z), \bar\psi(y)] = +e\,\Delta^{(\pm)}(z-y,m^2)\ \bar\psi(y) \quad .$$

Thus we can decompose $B = B^{(+)} + B^{(-)}$, and move in (7.70) $B^{(+)}$ to the right, $B^{(-)}$ to the left, until they reach the vacua and annihilate them. Which operators are hereby to be passed depends on the time relation just as in the elementary proof of Wick's theorem for free fields. E.g. we obtain from

$$\langle (\dots\ B(z)\ A^{\mu_1}(z_1)\cdots)_+ \rangle$$

the term

$$\langle (\cdots)_+ \rangle\, [-\Theta(z^o-z_1^o)\ i\partial_z^{\mu_1}\ \Delta^{(+)}(z-z_1) + \Theta(z_1^o-z^o)\ i\partial_z^{\mu_1}\ \Delta^{(-)}(z-z_1)] =$$

$$= \quad -\langle (\dots)_+ \rangle\ \partial_z^{\mu_1}\ \Delta_F(z-z_1)$$

which is just what is obtained from the first term on the r.h.s. of (7.69), and similarly we obtain the other terms. In this proof of (7.68,69), (7.67) appears not to have been needed; however, it is required (in unrenormalized form) to obtain (7.12) from the CCR and the equation of motion (7.3) for $\mu = o$ and the free field property (7.9) of B for sources being zero.

If we pass to connected Green's functions, amputate the "photon" argument, and use the relation

$$\partial_\mu \Delta'^{\mu\nu}_F(x-y) = \partial_\mu \Delta^{\mu\nu}_F(x-y) = -\partial^\nu \Delta_F(x-y) \quad , \tag{7.71}$$

which follows from (7.69), (7.68) becomes

$$i\partial_\mu \hat{G}^\mu(,,\underline{z}) = -\partial_\mu J^\mu(z) + ie\bar{\eta}(z)\hat{G}(z,,) - ie\hat{G}(,z,)\eta(z) \ . \tag{7.72}$$

(7.72) gives in a very compact form the familiar Ward identities

$$i\partial^z_\mu G^{\mu\mu_1\cdots\mu_r}(\underline{x}_1\cdots\underline{x}_n,\underline{y}_1\cdots\underline{y}_n,\underline{z}\,\underline{z}_1\cdots\underline{z}_r) = -\delta_{n0}\delta_{r1}\ \partial^{\mu_1}_z(\Box_z+m^2)\ \delta(z-z_1) +$$

$$+\sum_{\kappa=1}^{n} e\ G(x_\kappa,z_,)^{-1}\ G^{\mu_1\cdots\mu_r}(\underline{x}_1\cdots z\cdots\underline{x}_n,\underline{y}_1\cdots\underline{y}_n,\underline{z}_1\cdots\underline{z}_r) - \tag{7.73}$$

$$-\sum_{\kappa=1}^{n} e\ G^{\mu_1\cdots\mu_r}(x_1\cdots\underline{x}_n,\underline{y}_1\cdots z\cdots\underline{y}_n,\underline{z}_1\cdots\underline{z}_r)\ G(z,y_{\kappa},)^{-1}$$

where the first term on the r.h.s. can easily be calculated from (7.42).

If all spinor particle arguments in (7.72) or (7.73) are put on the mass shell, the last two terms do not contribute, since the singularities in these arguments then are removed, and thus the r.h.s. vanish except for the trivial case when there are no spinor and only vector particles. (This argument seems to fail if the momentum of the "photon" is exactly zero, since then the singularity is preserved. However, at this point in momentum space we have only a finite and no $\delta(k)$-type contribution such that it can be omitted since the momenta must always be thought to be smeared over slightly.) Therefore the Green's functions with all spinor particle arguments put on the mass shell are transverse in all vector particle arguments on and off the mass shell, except for the "photon" propagator.

In the simplest case of the vertex function

$$\Gamma^\mu(x,y,z) := G^\mu(\underline{x},\underline{y},\underline{z}) \tag{7.74a}$$

(7.73) reduces upon use of (7.60) to

$$i\,\partial^z_\mu\,\Gamma^\mu(x,y,z) = e\left[\,S'_F(x-z)^{-1}\,\delta(z-y)\;-\;\delta(x-z)S'_F(z-y)^{-1}\right] \quad , \qquad (7.75a)$$

and writing

$$\Gamma^\mu(p,q) := G^\mu(\underline{p},\underline{q},\underline{k})\;,\quad k = q - p \qquad (7.74b)$$

(7.75a) becomes

$$(q_\mu - p_\mu)\;\Gamma^\mu(p,q) = e\left[S'_F(p)^{-1} - S'_F(q)^{-1}\right] \quad , \qquad (7.75b)$$

the generalized Ward identity due to Takahashi [43] . Differentiating it w.r.t. q and letting $q \rightarrow p$ gives

$$\Gamma^\mu(p,p) = -\,e\,\frac{\partial}{\partial p_\mu}\,S'_F(p)^{-1} \qquad (7.76)$$

provided

$$\lim_{q\rightarrow p}\;(q_\nu - p_\nu)\,\frac{\partial}{\partial q_\mu}\,\Gamma^\nu(p,q) = 0$$

a regularity assumption that is verifiable in perturbation theory. From (7.51) or (7.53b) we obtain

$$(\not p+M)\,\Gamma^\mu(p,p)(\not p+M)\Big|_{p^2=M^2} = ie(\not p+M)\,\gamma^\mu\,(\not p+M)\Big|_{p^2=M^2} =$$

$$= i\,e\,2\,p^\mu(\not p+M)\Big|_{p^2=M^2} \quad , \qquad (7.77a)$$

which is mostly abbreviated in the symbolic form

$$\Gamma^\mu(p,p)\Big|_{\not p=M} = i\,e\,\gamma^\mu \quad . \qquad (7.77b)$$

7.6 Gauge independence of S-matrix elements

In section 4.4 we described the reduction technique to convert T-products between out-states on the left and in-states on the right into integrals over Green's functions. From the discussion after (7.73) we then learn that an S-matrix element w.r.t. the vector field arguments is an expression of the form (the e_μ are polarization vectors)

$$e_{\mu_1}(k_1)\dots e_{\mu_r}(k_r) \quad T^{\mu_1\cdots\mu_r}(p_1\dots p_n, q_1\dots q_n, k_1\dots k_r) \tag{7.78}$$

where

$$(k_i)_{\mu_i} \quad T^{\mu_1\cdots\mu_i\cdots\mu_r} = 0 \tag{7.79}$$

such that the change of a polarization vector by a vector proportional to the corresponding (mass shell) momentum k has no effect. This is what is commonly understood as gauge invariance of the S-matrix. The subsidiary condition (6.16) on the in- and out-state furthermore yields

$$e_{\mu_i}(k_i) \quad (k_i)^{\mu_i} = 0 \qquad \text{for all } i \quad , \tag{7.80}$$

which for the massless case expresses the absence of (not covariantly separable) c^+-type photons, and for the massive case says that we do not consider the scattering of the (covariantly separable) scalar B-mesons which in fact are free particles. In the massive case (7.80) is, in contrast to the massless case, actually unnecessary since in the former one every polarization vector that does not satisfy (7.80) can be replaced by a "gauge change" by

$$\hat{e}_\mu(k) = e_\mu(k) - \frac{ek}{m^2} k_\mu$$

which is orthogonal to k, and it is $\hat{e}$ rather than e that should be subjected to the normalization condition $e^2 = -1$.

The general solution of (7.78) is obtained as follows:
Choose a timelike vector a that is (for preservation of manifest covariance) one of the mass shell momenta p or q or some convenient linear combination of them and change the gauge of all polarization operators by

$$e_\mu \longrightarrow e'_\mu = e_\mu - \frac{ea}{ka} k_\mu \tag{7.81}$$

such that all e' are orthogonal to a. (7.78) then reads

$$e'_{\mu_1} \ldots e'_{\mu_r} \; T'^{\mu_1 \ldots \mu_r} \tag{7.82}$$

where $T'^{\mu_1 \cdots \mu_r}$ is any properly covariant tensor of rank r that can be formed from the available vectors (k, p, q, γ) but which need not involve a since all polarization vectors are orthogonal to it. But otherwise $T'^{\mu_1 \cdots \mu_r}$ underlies no restrictions, since if we substitute (7.81) into (7.82) the resulting tensor $T^{\mu_1 \cdots \mu_r}$ satisfies (7.79) and as the derivation shows is the most general solution of (7.79). In the massive case $T'^{\mu_1 \cdots \mu_r}$ may have terms proportional to $(k_i)^{\mu_i}$, however, from (7.81,80) one sees that they give contributions proportional to $k_i^2 = m^2$, and thus do not contribute in the massless case. Therefore in the massless case the most general $T'^{\mu_1 \cdots \mu_r}$ (and $T^{\mu_1 \cdots \mu_r}$, too) will involve fewer scalar functions of invariants (which are the coefficients of the covariant tensors formed from the available vectors) than are necessary in the massive case. (Cf. e.g. [44]).

In unitarity sums (i.e. summation over final or averaging over initial polarization states) one encounters the expression

$$\sum_{i=1}^{3} e^{(i)}_\mu(k)\, e^{(i)}_\nu(k) = - g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2} \tag{7.83a}$$

in the massive case and according to (6.16), if m = o,

$$\sum_{i=1}^{2} e^{(i)}_\mu(k)\, e^{(i)}_\nu(k) = - g_{\mu\nu} + \hat{k}_\mu k_\nu + k_\mu \hat{k}_\nu \quad . \tag{7.83b}$$

Due to (7.79) in both cases only the $g^{\mu\nu}$-term need to be kept. The interesting point is that in (7.83a) the full contribution arises from three, in (7.83b) only from two terms. In order to compare both cases we must choose a frame of reference in which e.g. $e_o^{(i)} = o$, $i = 1,2,3$ resp. 1,2, and $\vec{e}^{(i)} \perp \vec{k}$, $\vec{e}^{(1)} \perp \vec{e}^{(2)}$ for both (7.83a) and (7.83b). Then $e^{(3)}$ is given by $e_o^{(3)} = \frac{|\vec{k}|}{m}$, $\vec{e}^{(3)} = \frac{k^o}{m} \frac{\vec{k}}{|\vec{k}|}$ such that

$$e^{(3)\mu} = \frac{|\vec{k}|}{mk_o} k^\mu + \left(o, \frac{m}{k_o} \frac{\vec{k}}{|\vec{k}|}\right) .$$

The first part gives no contribution due to (7.79), and the last part, describing the three-dimensionally-longitudinal polarization, does not give a contribution when $m \to o$ due to its explicit factor m. This is the formal reason why, in a fixed reference frame, the $e^{(3)}$-polarized meson for $m \to o$ are ultimately neither absorbed nor emitted, and thus allow the two transverse-polarized photons only to interact physically. For the above transition to be possible, however, (7.80) is necessary which is, of course, satisfied by the polarization vectors we choose, i.e. photons with polarization vectors violating (7.80) would be "absorbed" and "emitted" were it not for their explicit exclusion due to imposing (7.80).

It is instructive to see the gauge independence of the S-matrix also in terms of Green's functions rather than only via the operator gauge transformations to the Coulomb gauge etc., which is difficult to implement at least. The Ward identity (7.68) is a special case of

$$\left[e\left(\eta \frac{\delta}{\delta\eta} - \bar{\eta}\frac{\delta}{\delta\bar{\eta}}\right) - \partial J + (\Box f)^{-1} \partial \frac{\delta}{\delta J} \right] \hat{G}^f_{disc}\{\bar{\eta}, \eta, J\} \qquad (7.84)$$

(cf. e.g. [45]), where $\hat{G}^f_{disc}\{\bar{\eta}, \eta, J\}$ is the generating functional of Green's functions in the gauge where the longitudinal part of the "photon" propagator is

$$G^{f\,\mu\nu}(,,xy)_{long} = \partial_x^\mu\,\partial_y^\nu\ f(xy) \quad ,$$

i.e.

$$f(xy) = o \qquad \text{in the Landau gauge}$$

and

$$f(xy) = -i\ \Box_x^{-1}(\Box_x + m^2)^{-1}\ \delta(x-y) \text{ in the Stueckelberg gauge.}$$

The generating functional $\hat{G}^{f'}_{disc}\{\bar\eta, \eta, J\}$ which satisfies (7.84) with f replaced by another gauge function f' can be expressed in terms of $\hat{G}^{f}_{disc}\{\bar\eta, \eta, J\}$ by the formulas

$$\hat{G}^{f'}_{disc}\{\bar\eta,\eta,J\} = e^{-\frac{1}{2}(e\eta\frac{\delta}{\delta\eta} - e\bar\eta\frac{\delta}{\delta\bar\eta} - \partial J)\cdot\Delta f\cdot(e\eta\frac{\delta}{\delta\eta} - e\bar\eta\frac{\delta}{\delta\bar\eta} - \partial J)} \times \hat{G}^{f}_{disc}\{\bar\eta,\eta,J\}$$

or

$$\hat{G}^{f'}_{disc}\{\bar\eta,\eta,J\} = e^{-\frac{1}{2}\partial J\cdot\Delta f\cdot\partial J}\ e^{\frac{1}{2}\frac{\delta}{\delta h}\cdot\Delta f\cdot\frac{\delta}{\delta h}}\ .$$

$$\times\ \hat{G}^{f}_{disc}\{e^{-ieh - e\Delta f\cdot\partial J}\,\bar\eta,\ e^{ieh - e\Delta f\,\partial J}\,\eta,\ J\}\Big|_{h=0}$$

(with $\Delta f = f' - f$) which can be derived from a formal solution of (7.84) similar to (4.43).

The electron propagator in the new gauge then reads

$$G^{f'}(x,y,) = e^{e^2\Delta f(xy) - \frac{e^2}{2}\Delta f(xx) - \frac{e^2}{2}\Delta f(yy)} \times G^{f}(x,y,)\ .$$

Of practical interest are only gauges where $f(xy) = f((x-y)^2)$, such that, if $\Delta f((x-y)^2) \to 0$ for $(x-y)^2 \to \infty$, an asymptotically unchanged normalization of the Fermi fields is obtained by absorbing the constant factors $\exp[-\frac{e^2}{2}\Delta f(0)]$ into the fields. The Green's function that is relevant for e.g. Compton scattering then is

$$
\begin{aligned}
G^{f'\,\mu\nu}_{disc}(x,y,zu) = e^{e^2\Delta f(xy)} \Big[& G^{f\,\mu\nu}_{disc}(x,y,zu) + \\
& + ie\,\partial^\mu_z\big(\Delta f(zx) - \Delta f(zy)\big)\cdot G^{f\,\nu}_{disc}(x,y,u) + \\
& + ie\,\partial^\nu_u\big(\Delta f(ux) - \Delta f(uy)\big)\cdot G^{f\,\mu}_{disc}(x,y,z) + \\
& + \partial^\mu_z\partial^\nu_u\,\Delta f(zu)\cdot G^{f}_{disc}(x,y,) \Big].
\end{aligned}
$$

Due to the transversality of the photons, only the first term in the square bracket contributes to scattering amplitudes, and asymptotically the exponential factor is replaceable by one. (In momentum space, in the expansion of the exponential all but the first term would give rise to convolutions which remove the singularity from the pole term.) *)

8. RENORMALIZATION

As can be seen from simple solvable models [46] the "bare" parameters (masses, coupling constants) of a Lagrangian field theory differ from the corresponding "observed" ones if the fields are in interaction. Adjusting the bare parameters such that the observed ones have assigned values is called renormalization. In the physically most interesting cases this renormalization involves infinite quantities. There are, however, theories in which all infinities arising in e.g. a perturbation theoretical treatment are consistently removed by renormalization.

*) More general nontransverse parts of the photon propagator than considered here, as relevant for the Coulomb gauge, are considered in [33].

We begin by determining the (ultraviolet) divergence character of a Feynman integral in neutral vector meson theory or QED. The well-known power counting (cf. e.g. [2] , sect. 19.10) gives the degree of "superficial divergence"

$$D = d - n(d - 1) - r(\frac{d}{2} - 1) + N_{\gamma}(\frac{d}{2} - 2) \tag{8.1}$$

where n is the number of external electron-positron pairs, r the number of external "photon" lines, N_{γ} the number of vertices, and d the number of space-time dimensions (physically, $d = 4$). A theory is called renormalizable if D depends only on the external lines; it is superrenormalizable or nonrenormalizable if for fixed number of external lines D decreases resp. increases with increasing order (number of vertices).

Hence QED in four dimensions is renormalizable. The only graphs that give rise to superficially divergent ($D \geqslant o$) "renormalization functions" are listed in the following table:

$n = o$, $r = 2$, $D = 2$: photon self energy $\Pi^{\mu\nu}$
$n = 1$, $r = o$, $D = 1$: electron self energy Σ
$n = 1$, $r = 1$, $D = o$: vertex Γ^{μ}
$n = o$, $r = 4$, $D = o$: photon-photon scattering $X^{\mu\nu\varrho\sigma}$

(The tadpole graph with $n = o$, $r = 1$, and the 3-photon-vertex with $n = o$, $r = 3$ vanish due to Furry's theorem).

8.1 Regularization

It is necessary to specify the way how to deal with these divergent quantities in formal manipulations. We find it most satisfactory to introduce regulator fields into the Lagrangian from the beginning [47]. Our Stueckelberg Lagrangian (7.1) can be expressed in terms of re-normalized fields as

$$L = -\frac{1}{2}\ \partial_\mu A_\nu\ \partial^\mu A^\nu \ + \frac{m^2}{2}\ A_\mu A^\mu + \frac{1}{4}(1-s)(\partial_\mu A_\nu - \partial_\nu A_\mu)\ (\partial^\mu A^\nu - \partial^\nu A^\mu) +$$

$$+ \overline{\psi}(i\partial\!\!\!/ - M)\psi + \overline{\psi} Z_2 e A\!\!\!/ \psi \ + \overline{\psi}(Z_2-1)(i\partial\!\!\!/ \ -M)\psi \ + \ \overline{\psi} Z_2\ \delta M\, \psi \ +$$

$$+ \overline{\psi}\eta \qquad + \overline{\eta}\psi \qquad + J_\mu A^\mu \ . \tag{8.2}$$

We replace it by

$$L = L_o + L_I + L_S \tag{8.3}$$

with

$$L_o = \sum_{a=o}^{N} c_a^{-1} \left[-\frac{1}{2}\,\partial_\mu A_{a\nu}\ \partial^\mu A_a^{\ \nu} + \frac{m_a^2}{2}\ A_{a\mu}\ A_a^{\ \mu}\right] \ +$$

$$+ \sum_{f=o}^{N'} \sum_{k=1}^{n_f} \quad \overline{\psi}_{fk}(i\partial\!\!\!/ \ - M_f)\ \ \psi_{fk}\ , \tag{8.3a}$$

$$L_I = \sum_{f,k} \ \overline{\psi}_{fk} Z_2^f e \underline{A\!\!\!/}\ \psi_{fk} \ +$$

$$+ \frac{1}{4}\ (1-s)(\partial_\mu \underline{A}_\nu \ - \partial_\nu \underline{A}_\mu)(\partial^\mu \underline{A}^\nu \ - \partial^\nu \underline{A}^\mu) \ +$$

$$+ \sum_{f,k}\ \overline{\psi}_{fk}\ \left[\ (Z_2^f-1)(i\,\partial\!\!\!/ \ - M_f) + Z_2^f\ \delta M_f\right]\ \psi_{fk}\ , \tag{8.3b}$$

$$L_S = J_\mu\ \underline{A}^\mu \ + \quad \sum_{f,k} \left[\ \overline{\psi}_{fk}\ \eta_{fk} + \overline{\eta}_{fk}\ \psi_{fk}\right] \quad , \tag{8.3c}$$

where $\quad \underline{A}^\mu = \sum_a \ A_a^{\ \mu}$.

The field equations then read

$$c_a^{-1}(\Box + m_a^2)A_a^{\mu} = - j^{\mu} - J^{\mu} \tag{8.4a}$$

with

e_{ren}

$$j^{\mu} = \sum_{f,k} \bar{\psi}_{fk} Z_2^f e \gamma^{\mu} \psi_{fk} + (1-s)\, \partial_{\nu}(\partial^{\mu} \underline{A}^{\nu} - \partial^{\nu} \underline{A}^{\mu})$$

and

$$(-i\not{\partial} + M_f)\, \psi_{fk} = \eta_{fk} + (1-Z_2^f)(-i\not{\partial} + M_f)\psi_{fk} + Z_2^f \delta M_f\, \psi_{fk} +$$

$$+ Z_2^f e \underline{\not{A}}\, \psi_{fk}\, , \tag{8.4b}$$

$$\bar{\psi}_{fk}(i\overleftarrow{\not{\partial}} + M_f) = \bar{\eta}_{fk} + \bar{\psi}_{fk}(i\overleftarrow{\not{\partial}} + M_f) + \bar{\psi}_{fk}\delta M_f Z_2^f +$$

$$+ \bar{\psi}_{fk} Z_2^f e \underline{\not{A}}\, . \tag{8.4c}$$

In (8.3,4) the zeroth member of the set of "photon" fields corresponds to the physical field of (8.2), i.e. $A_o^{\mu} = A^{\mu}$, $c_o = 1$, $m_o = m$. We have introduced N auxiliary "photon" fields of masses m_a. The constants c_a are subjected to the constraints

$$\sum_{a=o}^{N} c_a\, m_a^{2L} = o \quad \text{for} \quad L = o,1,\ldots,N-1\, , \tag{8.5}$$

the general solution of which is

$$c_a = \frac{\underline{c}}{\prod_{b\neq a}(m_b^2 - m_a^2)} \quad \text{with} \quad \underline{c} = \prod_{a\neq o} (m_a^2 - m_o^2) \quad . \tag{8.6}$$

A simple realization of (8.5,6) is given by

$$m_a^2 = m^2 + a\,\Lambda^2, \; c_a = (-1)^N \binom{N}{a}, \qquad \underline{c} = N!\;\Lambda^{2N} \quad . \tag{8.7}$$

From $\sum_a \frac{c_a}{\Box + m_a^2} = \underline{c} \prod_a \frac{1}{\Box + m_a^2}$ and the independence of the r.h.s. of (8.4a) of the index a follows

$$\left[\sum_a \frac{c_a}{\Box + m_a^2}\right]^{-1} \underline{A}^\mu = \underline{c}^{-1} \prod_a (\Box + m_a^2)\, \underline{A}^\mu = - j^\mu - J^\mu ,$$

and therefore, using (8.4a) again, every constituent of $\underline{A}$ can be regained from it by application of a differential operator:

$$A_a = \prod_{b\neq a} \frac{\Box + m_b^2}{(-m_a^2 + m_b^2)} \; \underline{A} \quad . \tag{8.8}$$

The effect of this arrangement of regulator fields is the following: Since the electrons are coupled to all components of $\underline{A}^\mu$ equally every internal "photon" propagator of the theory (8.2) is to be replaced by the regularized propagator

$$\Delta_{F\,\mathrm{reg}}^{\mu\nu} = \sum_a c_a \;\Delta_{F\,a}^{\mu\nu} = i g^{\mu\nu}\, \underline{c} \prod_a \frac{1}{\Box + m_a^2}$$

such that by suitable choice of the number of auxiliary fields there will be no divergence difficulties in the internal "photon" momentum integrations as long as the regulator masses remain finite. External "photon" lines are obtained by application of the operator

$\prod_{a\neq o} \frac{\Box + m_a^2}{(-m^2+m_a^2)}$ to $\Delta_{F\,reg}^{\mu\nu}$ which becomes unity if the auxiliary masses tend to infinity. In order to obtain the lowest order contribution of the "photon" propagator itself the differential operator must be applied only once.

Similarly we have introduced in (8.3) N' multiplets of auxiliary "electron" fields. Each multiplet contains n_f fields of mass M_f which are quantized with anticommutators or commutators according to $\delta_f = \pm 1$. The constants $d_f = n_f \cdot \delta_f$ are subjected to constraints similar to (8.5). The simplest choice of parameters corresponding to (8.7) is

$$M_f^2 = M^2 + f\Lambda'^2, \; n_f = \binom{N'}{f}, \quad \delta_f = (-1)^f \quad .$$

The auxiliary electron fields give rise to additional contributions to closed loops. However, due to their normal or anomalous commutation character and multiplicity the multiplets contribute with different signs δ_f and weights n_f. Therefore every electron loop contribution of the theory (8.2) is to be replaced by

$$\sum_{f=o}^{N'} d_f \;\left[\text{loop contribution with (electron mass)}^2 = M_f^2\right] .$$

The constraints on the d_f then again guarantee the loop integrations to be convergent.

The numbers N,N' of regulator fields depend on the highest degree D of superficial divergence occuring in the calculations. Since from (8.1) we know that in QED and in neutral vector meson theory Feynman integrals are at most quadratically divergent, it is sufficient to choose $N = 1$, $N' = 2$. In the following we will keep our regularization procedure in mind but will not indicate it explicitly in order not to mar the equations with additional indices.

8.2 Skeleton expansion of Green's functions

We are now in position to present a brief discussion of renormalization of neutral vector meson theory, since all manipulations can be done unambigously as long as the auxiliary masses are finite. (The renormalization of QED is not possible by simply setting $m = 0$, since the electron propagator cannot be normalized on the mass shell due to the infrared divergence. This complication can be circumvented by use of the intermediate renormalization (cf. e.g. [2], sect. 19.9): then (7.52) is to be replaced by

$$\Sigma(\not p)\Big|_{\not p=M} = 0 \qquad \text{and} \qquad \frac{\partial}{\partial \not p}\,\Sigma(\not p)\Big|_{p=0} = 0$$

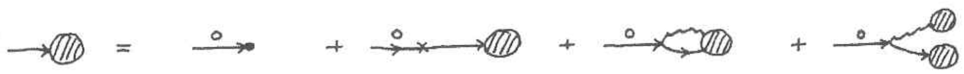
and correspondingly in (7.77b). The normalization of the photon propagator (7.43) need not be changed. For the calculation of observable quantities in QED, namely differential cross sections, we refer e.g. to [48] , [49] , [50] .)

We will now show by the method of [41] that in the calculation of Green's functions all quantities can be eliminated which would diverge if the regulator masses go to infinity. The first step will be to eliminate the counterterms (7.66) from all Green's functions other than the renormalization functions Π , Σ , Γ , X.

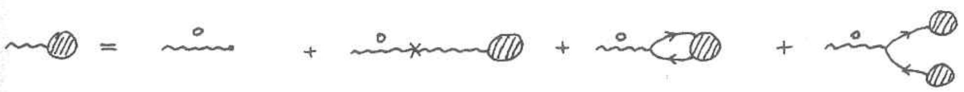
We rewrite (7.65b,c,a) in the form

$$\hat{G}_{x,,} = S_{Fxy}\ \left[i\eta(y) + \varepsilon_{yx'}\hat{G}_{x',,} + \gamma_{yx'z}(\hat{G}_{x',,z} + \hat{G}_{x',,}\ \hat{G}_{,,z}) \right] \tag{8.9a}$$

$$\hat{G}_{,y,} = S_{Fxy}\left[-i\bar{\eta}(x) + \varepsilon_{y'x}\ \hat{G}_{,y',} + \gamma_{y'xz}(\hat{G}_{,y',z} + \hat{G}_{,y',}\ \hat{G}_{,,z}) \right] \tag{8.9b}$$



$$\hat{G}_{,,z} = \Delta_{Fzz'}\left[iJ(z') + \varphi_{z'u}\ \hat{G}_{,,u} + \gamma_{yxz'}(\hat{G}_{x,y,} + \hat{G}_{x,,}\ \hat{G}_{,y,}\ ) \right] \tag{8.9c}$$



where we have used a graphical notation similar to that of [41] , [42]. In writing (8.9) we have omitted all indices as we will do throughout this lecture whenever permissable. Furthermore we introduced a matrix notation for the space-time arguments and extended the summation convention to them, i.e. repeated arguments are thought to be integrated over. However, because of the anticommutivity of the (physical) electron fields we have to keep in mind the following <u>sign rule</u>:
Whenever an antispinor argument is to be summed over with a preceding spinor argument one encounters a minus sign (cf. the last terms in (8.9c) and the corresponding ones in (7.65a)). We will incorporate these signs into the summation convention rather than marring the formulas with them.

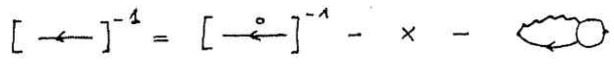
According to (7.58) the one-particle propagators are calculated from (8.9) by another functional derivation and then setting the sources equal to zero. After some manipulations one obtains

$$S'^{-1}_{Fxy} = S^{-1}_{Fxy} - \Sigma_{xy} \tag{8.10a,b}$$

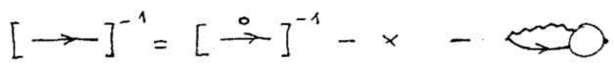
$$\Delta'^{-1}_{Fzu} = \Delta^{-1}_{Fzu} - \Pi_{zu} \tag{8.10c}$$

with

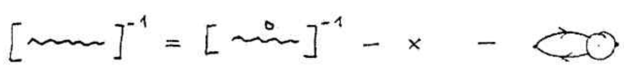
$$\Sigma_{xy} = \varepsilon_{xy} + \gamma_{xx'z} \, G_{x',\underline{y},z} \tag{8.11a}$$



$$\Sigma_{xy} = \varepsilon_{xy} + \gamma_{y'xz} \, G_{\underline{y},y',z} \tag{8.11b}$$



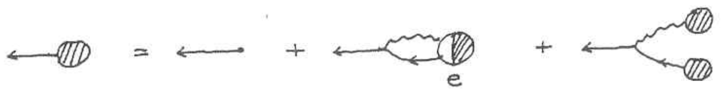
$$\Pi_{zu} = \varphi_{zu} + \gamma_{yxz} \, G_{x,y,\underline{u}} \tag{8.11c}$$



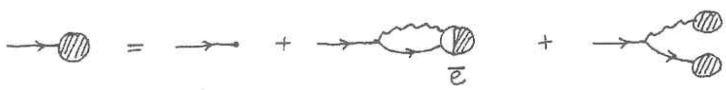
Comparing (7.42,51) with (8.10,11) we see that we can use the renormalization conditions (7.43,53) as a directive for the choice of s, Z_2, and δM in (7.66).

We now insert the expressions obtained in (8.10,11) for ε , φ back into (8.9):

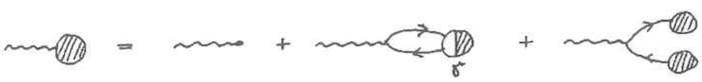
$$\hat{G}_{\underline{x},,} = i\eta(x) + \gamma_{xx'z}(\hat{G}^{e}_{x',,z:} + \hat{G}_{x',,} \quad \hat{G}_{,,z}) \tag{8.12a}$$



$$\hat{G}_{,\underline{y},} = i\bar{\eta}(y) + \gamma_{y'yz}(\hat{G}^{\bar{e}}_{,y',z:} + \hat{G}_{,y',} \quad \hat{G}_{,,z}) \tag{8.12b}$$

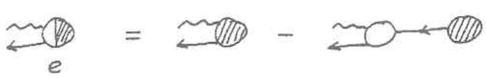


$$\hat{G}_{,,\underline{z}} = iJ(z) + \gamma_{yxz}(\hat{G}^{\gamma}_{x,y,:} + \hat{G}_{x,,} \quad \hat{G}_{,y,}) \tag{8.12c}$$

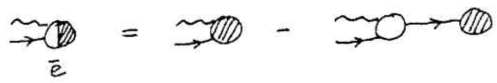


with the functionals

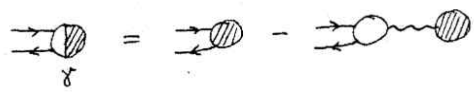
$$\hat{G}^{e}_{x,,z:} = \hat{G}_{x,,z} - G_{x,\underline{y},z} \quad \hat{G}_{y,,} \tag{8.13a}$$



$$\hat{G}^{\bar{e}}_{,y,z:} = \hat{G}_{,y,z} - G_{\underline{x},y,z}\,\hat{G}_{,x,} \tag{8.13b}$$



$$\hat{G}^{\gamma}_{x,y,:} = \hat{G}_{x,y,} - G_{x,y,\underline{z}}\,\hat{G}_{,,z} \tag{8.13c}$$



being irreducible between the arguments in front of the colon (vertical bar) and those (to be created) behind it w.r.t. the particle lines indicated by superscripts. (The pictures in (8.12) correspond to unamputated functionals.)

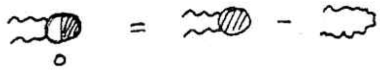
(8.12) reduce to identities in the case of the one-particle propagators (cf. (7.59)), since

$$G^{e}_{x,,z:,y,} = G^{\bar{e}}_{,y,z:x,,} = G_{x,y,:,,z} = 0$$

by definition.

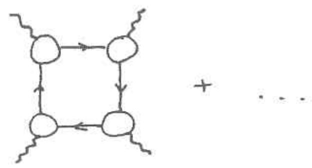
For later use we define two further functionals, one of them being "vacuum-irreducible" and the other one-photon irreducible in the sense of (8.13c):

$$\hat{G}^{o}_{,,zz':} = \hat{G}_{,,zz'} - G_{,,zz'} \tag{8.13d}$$



$$\hat{G}^{\gamma}_{,,zz'z'':} = \hat{G}_{,,zz'z''} - G_{,,zz'z''\underline{u}}\ \hat{G}_{,,u} \qquad (8.13e)$$

Having in (8.12) eliminated the counterterms ε and φ from the Green's functions with more than two external legs we proceed to eliminate the bare vertex operator $\gamma_{xyz} = ieZ_2\gamma\,\delta_{xz}\,\delta_{zy}$ in favour of the full vertex $\Gamma_{xyz} \equiv G_{\underline{x},\underline{y},\underline{z}}$ and the photon-photon-scattering amplitude $X_{zuvw} \equiv G_{,,\underline{zuvw}}$. Although the latter can be expressed in terms of S_F' and Γ, e.g. as

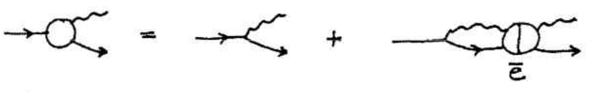


the introduction of X will enable us to write the integral equations (8.12) in such a form that the regularization can be removed.

From (8.12) we obtain by functional differentiation

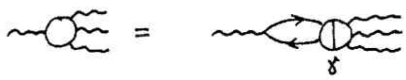
$$\Gamma_{xyz} = \gamma_{xyz} + \gamma_{xx'z'}\, G^{e}_{x',,z':,\underline{y},\underline{z}} \qquad (8.14a)$$

$$\Gamma_{xyz} = \gamma_{xyz} + \gamma_{y'yz'}\, G^{\bar{e}}_{,y',z':\underline{x},,\underline{z}} \tag{8.14b}$$



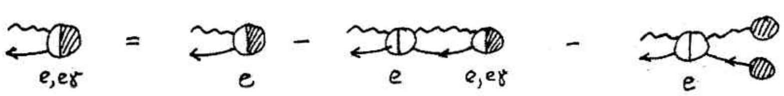
$$\Gamma_{xyz} = \gamma_{xyz} + \gamma_{y'x'z}\, G^{\gamma}_{x',y',:\underline{x},\underline{y},} \tag{8.14c}$$

$$X_{zz'z''z'''} = \gamma_{yxz}\, G^{\gamma}_{x,y,:,,\underline{z'z''z'''}} \tag{8.14d}$$

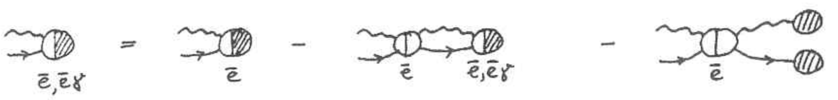


We now define in turn a set of two- or three-particle irreducible functionals as (e.g. iterative) solutions of the following integral equations (cf. [41] , [42]):

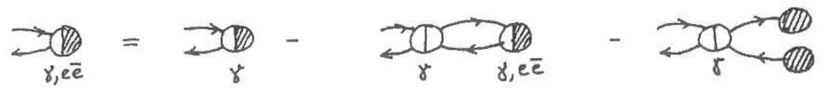
$$\hat{G}^{e,e\gamma}_{x,,z:} = \hat{G}^{e}_{x,,z:} - G^{e}_{x,,z:,\underline{x'},\underline{z'}} \left(\hat{G}^{e,e\gamma}_{x',,z':} + \hat{G}_{x',,}\ \hat{G}_{,,z'}\right) \tag{8.15a}$$



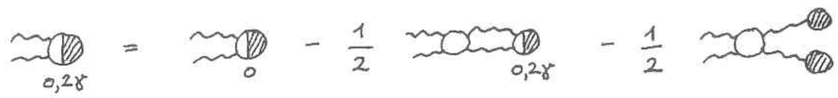
$$\hat G^{\bar e,\bar e\gamma}_{,y,z:} = \hat G^{\bar e}_{,y,z:} - G^{\bar e}_{,y,z:\underline{y'},\underline{z'}} \left(\hat G^{\bar e,\bar e\gamma}_{,y',z':} + \hat G_{,y'} \; \hat G_{,,z'}\right) \tag{8.15b}$$



$$\hat G^{\gamma,e\bar e}_{x,y,:} = \hat G^{\gamma}_{x,y,:} - G^{\gamma}_{x,y,:\underline{y'},\underline{x'}} \left(\hat G^{\gamma,e\bar e}_{x',y',:} + \hat G'_{x,,} \; \hat G_{,y',}\right) \tag{8.15c}$$

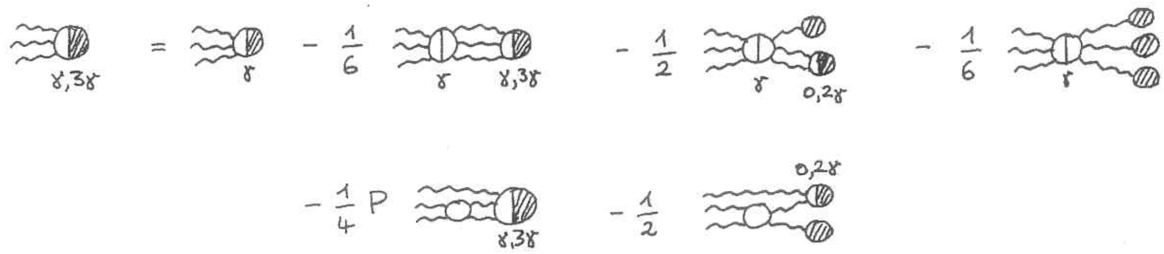


$$\hat G^{o,2\gamma}_{,,zz':} = \hat G^{o}_{,,zz':} - \frac{1}{2} G_{,,zz'\underline{uu'}} \left(\hat G^{o,2\gamma}_{,,uu':} + \hat G_{,,u} \; \hat G_{,,u'}\right) \tag{8.15d}$$



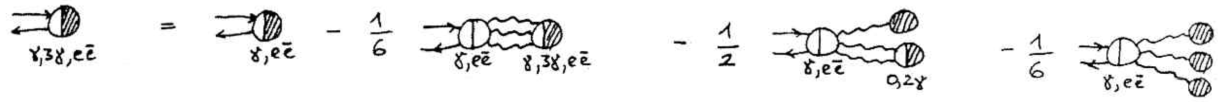
$$\hat G^{\gamma,3\gamma}_{,,zz'z'':} = \hat G^{\gamma}_{,,zz'z'':} - \frac{1}{6} G^{\gamma}_{,,zz'z'':,,\underline{uu'u''}} \left(\hat G^{\gamma,3\gamma}_{,,uu'u'':} + 3\, \hat G_{,,u} \; \hat G^{o,2\gamma}_{,,u'u'':} + \hat G_{,,u} \; \hat G_{,,u'} \; \hat G_{,,u''}\right) -$$

$$- \frac{1}{4} P_{zz'z''} \, G_{,,z'z''\underline{uu'}} \left(\hat G^{\gamma,3\gamma}_{,,zuu':} + 2\, \hat G^{o,2\gamma}_{,,zu':} \; \hat G_{,,u''}\right) \tag{8.15e}$$

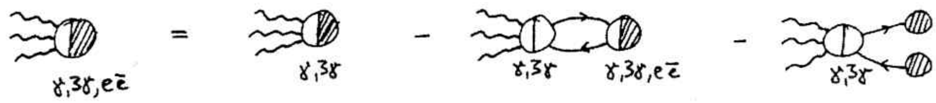


(8.15f)

$$\hat G^{\gamma,3\gamma,e\bar e}_{x,y:} = \hat G^{\gamma,e\bar e}_{x,y,:} - \frac{1}{6} G^{\gamma,e\bar e}_{x,y,:,,\underline{zz'z''}} \left(\hat G^{\gamma,3\gamma,e\bar e}_{,,zz'z'':} + 3\,\hat G_{,,z}\,\hat G^{o,2\gamma}_{,,z'z'':} + \hat G_{,,z}\,\hat G_{,,z'}\,\hat G_{,,z''}\right)$$



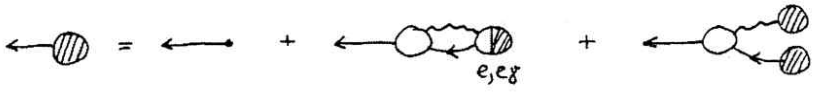
$$\hat G^{\gamma,3\gamma,e\bar e}_{,,zz'z'':} = \hat G^{\gamma,3\gamma}_{,,zz'z'':} - G^{\gamma,3\gamma}_{,,zz'z'':\underline{y},\underline{x},} \left(\hat G^{\gamma,3\gamma,e\bar e}_{x,y,:} + \hat G_{x,,}\,\hat G_{,y,}\right) \qquad (8.15g)$$



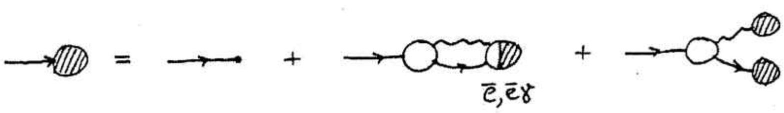
(8.15f,g) form a system of coupled integral equations. In (8.15d,e) we have used (8.13d) which gives $G^{o}_{,,zz':,,uu'} = G_{,,zz'uu'}$. The symbol $P_{zz'z''}$ in (8.15e) denotes summation over the permutations of z,z',z''.

Using these functionals and the equations (8.14) for the vertex functions, (8.12) can be written in the form

$$\hat G_{\underline{x},,} = i\eta(x) + \Gamma_{xx'z}\left(\hat G^{e,e\gamma}_{x',,z:} + \hat G_{x',,}\,\hat G_{,,z}\right) \qquad (8.16a)$$

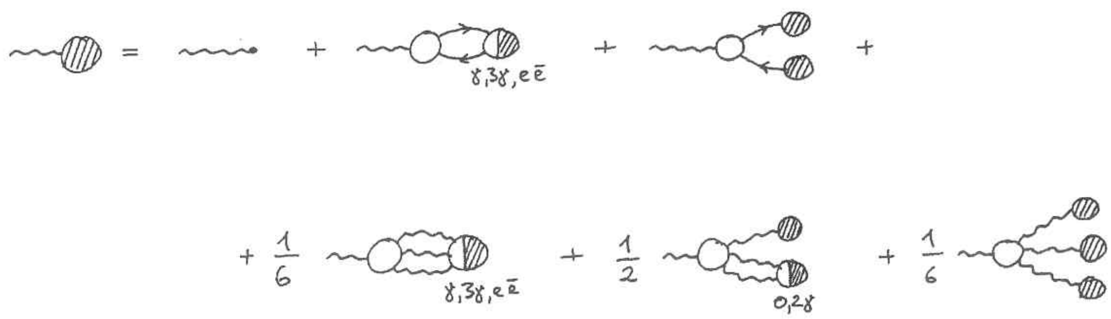


$$\hat G_{,\underline{y},} = i\bar\eta(y) + \Gamma_{y'yz}\left(\hat G^{\bar e,\bar e\gamma}_{,y',z:} + \hat G_{,y',}\,\hat G_{,,z}\right) \qquad (8.16b)$$



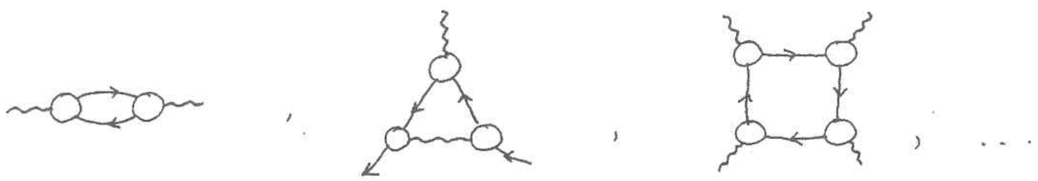
$$\hat{G}_{,,\underline{z}} = iJ(z) + \Gamma_{yxz}\,(\hat{G}^{\gamma,3\gamma,e\bar{e}}_{x,y,:} + \hat{G}_{x,,}\,\hat{G}_{,y,}) + \qquad (8.16c)$$

$$+ \frac{1}{6}\,X_{zuu'u''}\,(\hat{G}^{\gamma,3\gamma,e\bar{e}}_{,,uu'u'':} + 3\,\hat{G}_{,,u}\,\hat{G}^{0,2\gamma}_{,,u'u'':} + \hat{G}_{,,u}\,\hat{G}_{,,u'}\,\hat{G}_{,,u''})$$



In view of (8.15) these equations reduce to identities in the case of the renormalization functions.

The equations (8.16) together with (8.13,15) allow to expand all Green's functions in terms of Δ_F' , S_F' , Γ , and X only, whereby in this expansion no self energy or vertex correction parts, e.g.



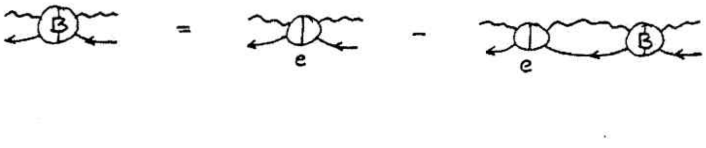
arise. This statement can be proven by induction w.r.t. the number of vertex functions: Assume all Green's functions expansions in terms of the renormalization functions to be given up to order $\Gamma^n X^m$, say. Then (8.13,15) give the expansion of the irreducible functions (in the sense of Dyson [61]) up to the same order. Insertion into (8.16) then gives the Green's functions up to $\Gamma^{n+1} X^m$ and $\Gamma^n X^{m+1}$. Due to the irreducibility character of the functionals in (8.16) no self energy or vertex correction parts arise in this iterative construction, i.e. we obtain the "skeleton expansions" of all Green's functions.

Since all integrals occurring in these expansions are superficially convergent, i.e. the integrands behave like p^{-n}, with $n \geqslant 5$ (apart from logarithms), for all (combinations of) integration momenta, the regularization of the electron lines can be removed from the lines connecting the renormalization functions. This can be seen as follows: From every integral corresponding to a loop of auxiliary electron lines at least one power of Λ' in the denominator can be factored out after a scaling of the integration momentum: $p = p'\Lambda'$. Since the remaining integral is finite in the limit $\Lambda' \to \infty$ the whole contribution vanishes in the regularization limit. After having eliminated the auxiliary electron fields from the skeleton expansions they are only needed for the calculation of the renormalization functions Π, Σ, Γ, and X themselves.

8.3 Bethe-Salpeter equations

The analysis of the equations (8.15) is aided by introducing Bethe-Salpeter kernels by the following integral equations, which define these kernels uniquely at least in terms of the mentioned skeleton expansions:

$$B^{e,e\gamma}_{x,,z:,y',z'} = G^{e}_{\underline{x},,\underline{z}:,\underline{y}',\underline{z}'} - G^{e}_{\underline{x},,\underline{z}:,x'',z''} B^{e,e\gamma}_{x'',,z'':,y',z'} \qquad (8.17a)$$



$$B^{\bar{e},\bar{e}\gamma}_{,y,z:x',,z'} = G^{\bar{e}}_{,\underline{y},\underline{z}:\underline{x}',,\underline{z}'} - G^{\bar{e}}_{,\underline{y},\underline{z}:y'',,z''} B^{\bar{e},\bar{e}\gamma}_{,y'',z'':x',,z'} \tag{8.17b}$$

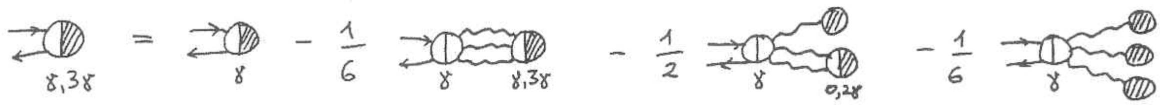
We may write (8.17) collectively as

$$B = I \quad - I\ S\ B \tag{8.18}$$

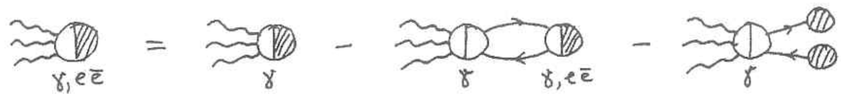
where I refers to the totally amputated one-particle irreducible functions and S symbolizes the two-particle propagators $\Delta_F' \cdot S_F'$.

The treatment of (8.15c-f) is more complicated due to the 3 γ-intermediate states. We must define two new functionals:

$$\hat{G}^{\gamma,3\gamma}_{x,y,:} = \hat{G}^{\gamma}_{x,y,:} - \frac{1}{6} G^{\gamma}_{x,y,:,,\underline{zz'z''}}(\hat{G}^{\gamma,3\gamma}_{,,zz'z'':} + 3\ \hat{G}_{,,z}\ \hat{G}^{o,2\gamma}_{,,z'z'':} + \hat{G}_{,,z}\ \hat{G}_{,,z'}\ \hat{G}_{,,z''}) \tag{8.19a}$$

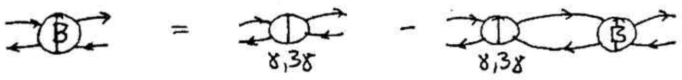


$$\hat{G}^{\gamma,e\bar{e}}_{,,zz'z'':} = \hat{G}^{\gamma}_{,,zz'z'':} - G^{\gamma}_{,,zz'z'':\underline{y},\underline{x},}(\hat{G}^{\gamma,e\bar{e}}_{x,y,:} + \hat{G}_{x,,}\ \hat{G}_{,y,}) \tag{8.19b}$$

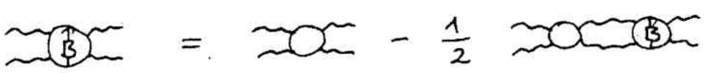


Now we are in position to define some further Bethe-Salpeter kernels:

$$B^{\gamma,3\gamma,e\bar{e}}_{x,y,:y',x'} = G^{\gamma,3\gamma}_{\underline{x},\underline{y},:\underline{y}',\underline{x}'} - G^{\gamma,3\gamma}_{\underline{x},\underline{y},:y'',x''}\, B^{\gamma,3\gamma,e\bar{e}}_{x'',y'',:y',x'} \tag{8.20a}$$



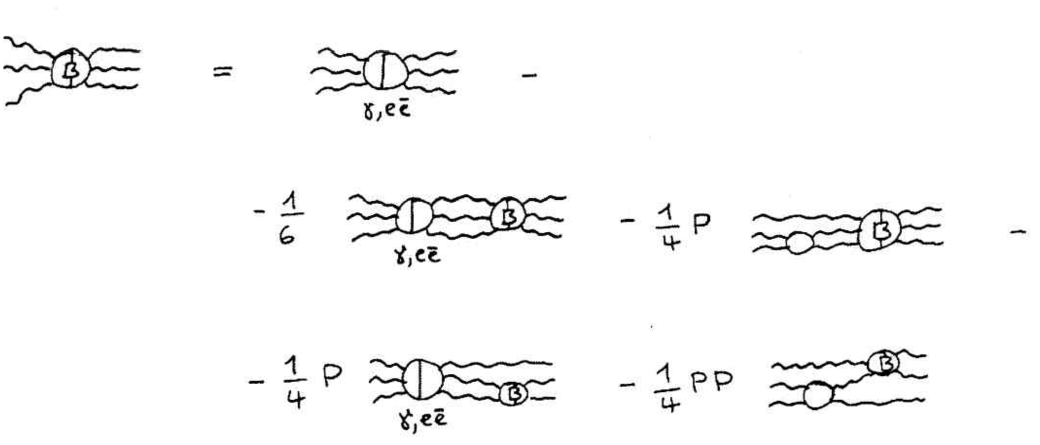
$$B^{o,2\gamma}_{,,zz':,,uu'} = G_{,,\underline{zz'uu'}} - \frac{1}{2}\, G_{,,\underline{zz'}vv'}\, B^{o,2\gamma}_{,,vv':,,uu'} \tag{8.20b}$$



$$B^{\gamma,3\gamma,e\bar{e}}_{,,zz'z'':,,uu'u''} = G^{,e\bar{e}}_{,,\underline{zz'z''}:,,\underline{uu'u''}} - \tag{8.20c}$$

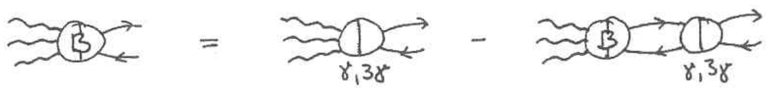
$$- \frac{1}{6}\, G^{\gamma,e\bar{e}}_{,,\underline{zz'z''}:,,vv'v''}\, B^{\gamma,3\gamma,e\bar{e}}_{,,vv'v'':,,uu'u''} - \frac{1}{4}\, P_{zz'z''}\, G_{,,\underline{zz'}vv'}\, B^{\gamma,3\gamma e\bar{e}}_{,,zv'v'':,,uu'u''} -$$

$$- \frac{1}{4}\, P_{uu'u''}\, G^{\gamma,e\bar{e}}_{,,\underline{zz'z'}:,,\underline{u}v'v''}\, B^{o,2\gamma}_{,,v'v'':,,u'u''} - \frac{1}{4}\, P_{zz'z''}\, P_{uu'u''}\, G_{,,\underline{zz''}v\underline{u''}}\, B^{o,2\gamma}_{,,zv:,,uu'}$$



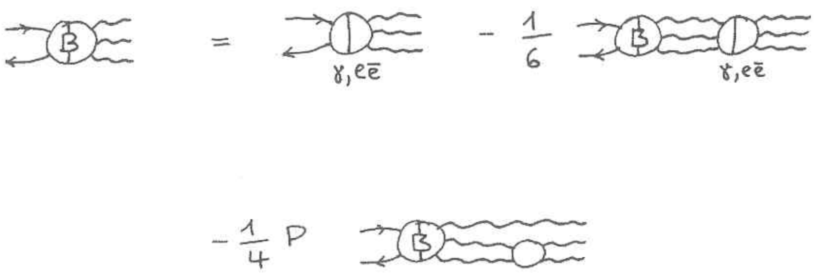
(For later use we state that the(superficially logarithmically divergent) Bethe-Salpeter kernel $B^{o,2\gamma}$ can be eliminated from the last two terms of (8.20c), cf. equ. (40) of ref. [42] .)

$$B^{\gamma,3\gamma,e\bar{e}}_{,,zz'z'':y',x',} = G^{\gamma,3\gamma}_{,,\underline{zz'z''}:\underline{y'},\underline{x'},} - B^{\gamma,3\gamma,e\bar{e}}_{,,zz'z'':y,x,} \quad G^{\gamma,3\gamma}_{x,y,:\underline{y'},\underline{x'},} \qquad (8.20d)$$



$$B^{\gamma,3\gamma,e\bar{e}}_{x,y,:,,zz'z''} = G^{\gamma,e\bar{e}}_{\underline{x},\underline{y},:,,\underline{zz'z''}} - \frac{1}{6}\, B^{\gamma,3\gamma,e\bar{e}}_{x,y,:,,uu'u''}\, G^{\gamma,e\bar{e}}_{,,uu'u'':,,\underline{zz'z''}} -$$

$$- \frac{1}{4}\, P_{zz'z''}\, B^{\gamma,3\gamma,e\bar{e}}_{x,y,:,,zu'u''}\, G_{,,u'u''\underline{z'z''}} \qquad (8.20e)$$



A somewhat lengthy calculation shows that (8.20) again can be written in the form (8.18) with B, I, and S now being 2x2 matrices. Their elements are given by:

$$B_{11} = B^{\gamma,3\gamma,e\bar{e}}_{x,y,:y',x',} \qquad B_{12} = B^{\gamma,3\gamma,e\bar{e}}_{x,y,:,,uu'u''}$$

$$B_{21} = B^{\gamma,3\gamma,e\bar{e}}_{,,zz'z'':y',x',} \qquad B_{22} = \frac{3}{2} P_{zz'z''}\, \triangle'^{-1}_{Fzu}\, B^{o,2\gamma}_{,,z'z'':,,u'u''} + B^{\gamma,3\gamma,e\bar{e}}_{,,zz'z'':,,uu'u''}$$

$$I_{11} = G^{\gamma}_{\underline{x},\underline{y},:\underline{y}',\underline{x}',} \qquad I_{12} = G^{\gamma}_{\underline{x},\underline{y},:,,\underline{uu'u''}}$$

$$I_{21} = G^{\gamma}_{,,\underline{zz'z''}:\underline{y}',\underline{x}',} \qquad I_{22} = \frac{3}{2}\, P_{zz'z''}\, \triangle'^{-1}_{Fzu}\, G_{,,\underline{z'z''u'u''}} + G^{\gamma}_{,,\underline{zz'z''}:,,\underline{uu'u''}}$$

and the propagator matrix S is diagonal:

$$S_{11} = S'_{Fxx'}\, S'_{Fy'y} \qquad S_{22} = P_{zz'z''}\, \triangle'_{Fzu} \cdot \triangle'_{Fzu} \cdot \triangle'_{Fzu}$$

The Bose symmetry factors $\frac{1}{n!}$ which take into account the equivalence of the Photon lines are incorporated into the summation convention.

8.4 Renormalization functions

We now can solve (8.14) for the bare vertices. Namely, multiplying (8.18) from the left by S gives

$$(1 + S\ I)(1 - S\ B) = 1 \tag{8.21a}$$

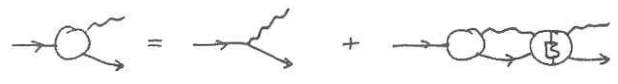
(where, of course, 1 stands for a matrix with products of δ-functions in its diagonal), and similarily

$$(1 - S\ B)(1 + S\ I) = 1 \tag{8.21b}$$

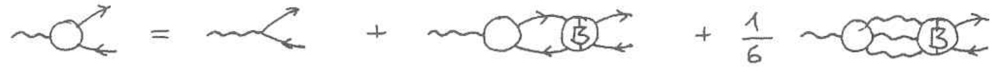
since in the second term on the r.h.s. of (8.18) I and B may be interchanged which can be seen at least from the iterative solution of (8.18). Inserting (8.21a) into (8.14) gives

$$\Gamma_{xyz} = \gamma_{xyz} + G_{\underline{x},x',z'} B^{e,e\gamma}_{x',,z':,y,z} \tag{8.22a}$$

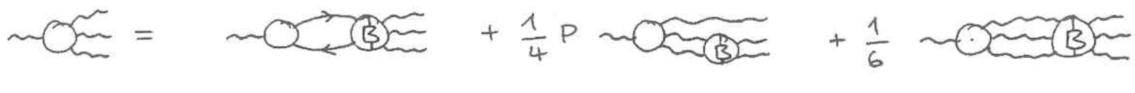
$$\Gamma_{xyz} = \gamma_{xyz} + G_{y',\underline{y},z'} B^{\bar{e},\bar{e}\gamma}_{,y',z':x,,z} \tag{8.22b}$$



$$\Gamma_{xyz} = \gamma_{xyz} + G_{y',x',\underline{z}} B^{\gamma,3\gamma,e\bar{e}}_{x',y',:x,y,} + \frac{1}{6} G_{,,\underline{z}uu'u''} B^{\gamma,3\gamma,e\bar{e}}_{,,uu'u'':x,y,} \tag{8.22c}$$



$$X_{zuu'u''} = G_{y,x,\underline{z}} B^{\gamma,3\gamma,e\bar{e}}_{x,y,:,,uu'u''} + G_{,,\underline{z}vv'v''}\left(\frac{1}{4} P_{v'v''} \Delta'^{-1}_{Fvu} B^{o,2\gamma}_{,,v'v'':,,u'u''} + \frac{1}{6} B^{\gamma,3\gamma,e\bar{e}}_{,,vv'v'':,,uu'u''}\right) \tag{8.22d}$$



or in abbreviated notation

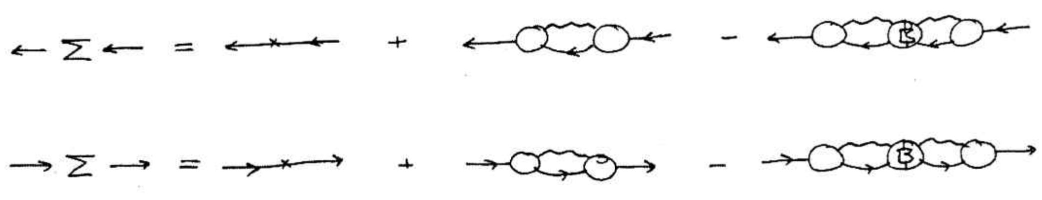
$$\Gamma = \gamma + \Gamma \; S \; B \tag{8.22}$$

where (8.22c,d) have been combined by introducing two-dimensional vectors γ, Γ with components

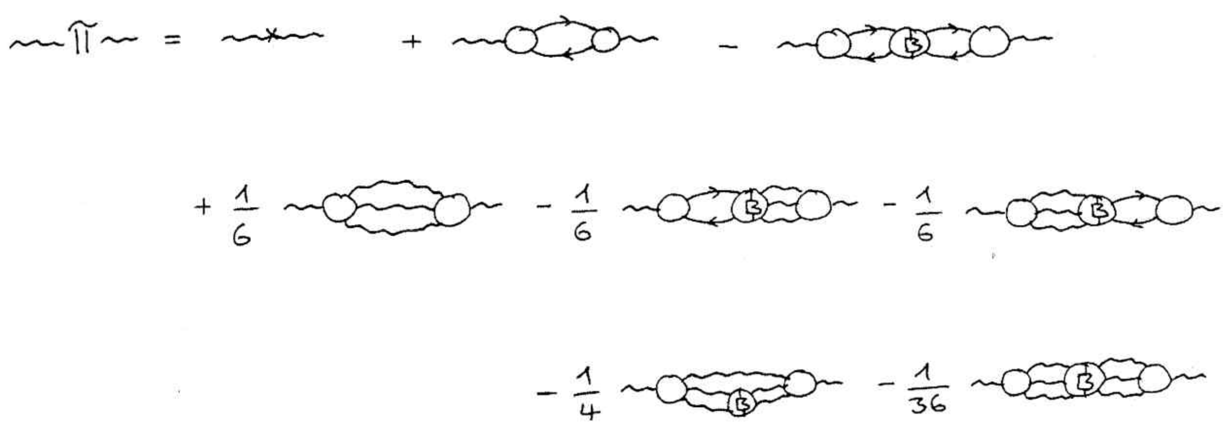
$$\gamma_1 = \gamma \quad , \quad \gamma_2 = 0 \quad , \quad \Gamma_1 = \Gamma \quad , \quad \Gamma_2 = X \quad . \tag{8.23}$$

Finally, inserting (8.22) into (8.11) gives

$$\Sigma = \varepsilon + \gamma S \Gamma = \varepsilon + \Gamma S \Gamma - \Gamma S B S \Gamma \qquad (8.24a,b)$$



$$\Pi = \varphi + \gamma S \Gamma = \varphi + \Gamma S \Gamma - \Gamma S B S \Gamma \qquad (8.24c)$$



We still have to demonstrate that the renormalization functions in (8.22,24) can be calculated in terms of convergent integrals only. Then the regulator fields can be eliminated from them, too, without occurrence of any divergent quantity.

8.4.1 Vertex function and photon-photon scattering amplitude

We now proceed to eliminate the bare vertex γ from the vertex equation (8.22). The method to be shown was first used by B. Ferretti [62]. In momentum space (8.22) reads

$$\overset{k}{\Gamma}p = \gamma + \overset{k}{\Gamma}q \quad q_S^k q \; q_B^k p \tag{8.25}$$

where we have indicated the momentum dependence in an obvious way such that k denotes the momentum transfer from right to left and the symbols at the right (left) collectively denote the incoming (outgoing) momenta which are restricted by momentum conservation. In (8.25) an integration over q is understood (including factors of $(2\pi)^{-4}$) and we still have to keep in mind our sign rule and the Bose symmetry factors.

Subtraction at $k = o$ then gives

$$\Gamma - \overset{o}{\Gamma} = (\Gamma - \overset{o}{\Gamma})SB + \overset{o}{\Gamma}(S-\overset{o}{S})B + \overset{oo}{\Gamma S}(B-\overset{o}{B}) \tag{8.26}$$

where we furthermore suppressed general momenta and those to be integrated over. Using (8.21b,18) we obtain

$$\Gamma - \overset{o}{\Gamma} = \overset{o}{\Gamma}(S-\overset{o}{S})I + \overset{oo}{\Gamma S}(B-\overset{o}{B})(1+SI) \; . \tag{8.27}$$

Subtraction of (8.25) at $k = p = o$ gives

$$\overset{o}{\Gamma} - \overset{o}{\Gamma}{}^{o} = \overset{oo}{\Gamma S}(\overset{o}{B}-\overset{o}{B}{}^{o}) \tag{8.28}$$

where $\overset{o}{\Gamma}{}^{o}$ is the vertex function with all momenta zero.
For covariance (parity, and time reversal invariance) reasons we have

$$\overset{\circ}{\Gamma}{}_1^{\mu o} = i\, e\, z_1 \gamma^\mu \tag{8.29a}$$

while transversality of the (regularized) photon-photon scattering amplitude (cf. the discussion after (7.73)) implies

$$\overset{\circ}{\Gamma}_2 = o \quad \text{and especially} \quad \overset{\circ}{\Gamma}{}_2^{o} = o \ . \tag{8.29b}$$

The constant z_1 can be calculated (recursively to higher and higher powers of e) from (8.28) and the renormalization condition (7.77a):

$$4\, i\, e\, (1-z_1)\, p^\mu = \mathrm{Tr}\left[\overset{\circ}{\Gamma}{}_1^{\mu}\overset{\circ}{S}_{11}(\overset{\circ}{B}{}^s_{11}-\overset{\circ}{B}{}^o_{11})(\not{p}+M)\right] \tag{8.30}$$

for $p^2 = M^2$, where "s" means putting the two arguments on the r.h.s. on the mass shell at p. Then (8.28) can be solved for $\overset{\circ}{\Gamma}_1$ ($\overset{\circ}{\Gamma}_2$ is already known to vanish)

$$\overset{\circ}{\Gamma} = \overset{\circ}{\Gamma}{}^o\left[1 - \overset{\circ}{S}(\overset{\circ}{B}-\overset{\circ}{B}{}^o)\right]^{-1} = \overset{\circ}{\Gamma}{}^o + \overset{\circ}{\Gamma}{}^o\,\overset{\circ}{S}(\overset{\circ}{B}-\overset{\circ}{B}{}^o) + \dots \tag{8.31}$$

which can be inserted into (8.27). From (8.27) we then obtain the equations for the vertex function and for the photon-photon scattering amplitude, respectively:

$$\begin{aligned}\Gamma_1 = \overset{\circ}{\Gamma}_1 &+ \overset{\circ}{\Gamma}_1(S_{11}-\overset{\circ}{S}_{11})I_{11} + \overset{\circ}{\Gamma}_1\overset{\circ}{S}_{11}(B_{11}-\overset{\circ}{B}_{11}) + \\ &+ \overset{\circ}{\Gamma}_1\overset{\circ}{S}_{11}(B_{11}-\overset{\circ}{B}_{11})S_{11}I_{11} + \overset{\circ}{\Gamma}_1\overset{\circ}{S}_{11}(B_{12}-\overset{\circ}{B}_{12})S_{22}I_{21}\end{aligned} \tag{8.32a}$$

$$\Gamma_2 = \mathring{\Gamma}_1(S_{11}-\mathring{S}_{11})I_{12} + \mathring{\Gamma}_1\mathring{S}_{11}(B_{12}-\mathring{B}_{12}) +$$

$$+ \mathring{\Gamma}_1\mathring{S}_{11}(B_{11}-\mathring{B}_{11})S_{11}I_{12} + \mathring{\Gamma}_1 S_{11}(B_{12}-\mathring{B}_{12})S_{22}I_{22} \quad . \tag{8.32b}$$

So far we have obtained (8.32) only in regularized form. But it can easily be verified that all integrals occurring in (8.30,31,32) are convergent (in perturbation theory) in virtue of the differences $\mathring{B}-\mathring{B}^o$, $B-\mathring{B}$, and $S-\mathring{S}$ in the integrands. Due to the scaling argument discussed after (8.16) therefore the vertex function Γ and the photon-photon scattering amplitude X can be calculated from the subtracted equations (8.32) without need of electron regulator fields.

In our determination of the vertex function we have used (8.22c,d) in order to obtain formulas needed in the calculation of Σ below. If one is interested in e.g. the vertex as form factor, i.e. with the photon momentum nonzero but with the Fermion momenta on the mass shell, one may use (8.22a,b) in a similar way as we used (8.22c,d). However, then the analog of $\mathring{\Gamma}^o$ is $\mathring{\Gamma}^s$ which has a more complicated structure than (8.29a).

8.4.2 Electron self energy

The discussion of the electron self energy operator Σ is simplified by use of the Ward identity. (Although we have derived the Ward identities (7.73) only on a formal level, their validity can easily be established from the regularized theory by removal of the cutoffs, $\Lambda, \Lambda' \to \infty$)

According to (7.76) and (7.51) the vertex function $\mathring{\Gamma}$ can be written in the form

$$\Gamma^\mu(p,p) = i\, e\left[A(p^2)\gamma^\mu + B(p^2)p^\mu + 2p^\mu A'(p^2)(\not p-M)\right] \tag{8.33}$$

with, from (7.77a)

$$A(M^2) + M\,B(M^2) = 1 \;. \tag{8.34}$$

As is clear from (8.33) the functions $A(p^2)$ and $B(p^2)$ can be calculated from $\mathrm{Tr}\,\mathring{\Gamma}^{\mu}$ and $\mathrm{Tr}(\gamma^{\nu}\mathring{\Gamma}^{\mu})$ where (8.31) has to be inserted. Due to (7.76,53a) $S_F'(p)^{-1}$ then can be obtained from (8.33) by integration as

$$S_F'(p)^{-1} = -i\left[\, A(p^2)(\not{p}-M) + \frac{1}{2}\int_{M^2}^{p^2} du\, B(u) \,\right] \;. \tag{8.35a}$$

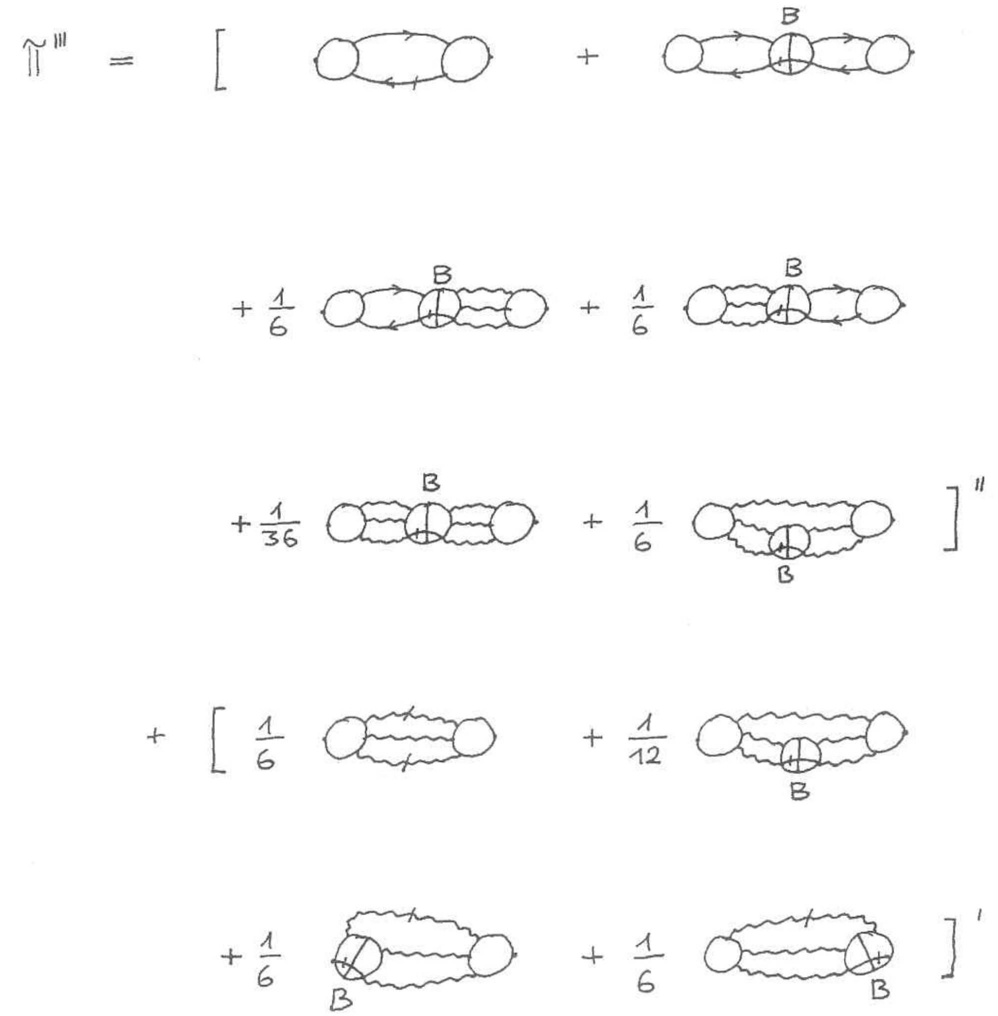
Alternatively, $S_F'(p)^{-1}$ can be derived from (7.75b) without integration. After a short calculation one obtains

$$S_F'(p)^{-1} = \frac{1}{3e}\,\frac{\partial}{\partial p_o^{\nu}}\left[\,(p_o-p)_{\mu}\,\Gamma^{\mu}(p,p_o)(\not{p}_o+M)\,\right]\,(\gamma^{\nu}-\frac{p_o^{\nu}\not{p}_o}{M^2}) \tag{8.35b}$$

where p_o is a momentum on the mass shell, i.e. $p_o^2 = M^2$.

8.4.3 Photon self energy

The photon self energy operator Π can be calculated from (8.24c) by two slightly different methods.
The differentiation method consists in first calculating the third derivative Π''' from (8.24c) and then regaining Π from it by integration whereby the integration constants are chosen such that the renormalization conditions (7.43) are fulfilled. For brevity we give the final result for Π''' only in graphical form (cf. equ. (39') of ref. [42]):



(8.36)

In (8.36) all "overlapping divergences" that mar (8.24c) have been disentangled and all integrals become convergent after the remaining differentiations have been performed.

The subtraction method for obtaining Π consists in first calculating the auxiliary tensor

$$\Pi^{\mu\nu}_{tr}(k) = \Pi^{\mu\nu}(k) - \Pi^{\mu\nu}(o) - k_\lambda \left(\frac{\partial}{\partial k_\lambda} \Pi^{\mu\nu}(k) \right)_{k=o} -$$

$$- \frac{1}{2} k_\lambda k_\varkappa \left(\frac{\partial^2}{\partial k_\lambda \, \partial k_\varkappa} \Pi^{\mu\nu}(k) \right)_{k=o} \qquad (8.37)$$

and then regaining Π from it again by use of the renormalization conditions but without need of integration. Π_{tr} is transverse as is Π (in its regularized form). Since the counterterm φ in (8.24c) is quadratic in the momentum k it does not contribute in (8.37) and therefore we need consider only

$$\Pi_u = \Pi - \varphi = \Gamma S \Gamma - \Gamma S B S \Gamma \qquad (8.38)$$

in forming Π_{tr}. In performing the subtractions in (8.37) the essential point is that differences and derivatives of Γ and I (which is introduced via (8.27)) are always eliminated in favour of differences and derivatives of B and S, whereby again the overlapping divergences are disentangled. The final formula looks similar to (8.36) with, however, some derivatives replaced by differences. Of course both methods are related to each other since Π_{tr} is the remainder of a Taylor series which can be calculated from Π''' by a well-known formula and vice versa.

According to our regularization prescription every contribution to (8.37) arising from an electron loop has to be replaced by a weighted sum over loop contributions with different electron masses. However, we may reorder these terms in such a way that one obtains differences of the form (8.37) for every type of electron separately. Due to the subtractions then the contributions of the physical and the auxiliary electrons are separately finite. Moreover, the contribution of each electron regulator field can be seen to vanish because of the scaling argument already given above at the end of section 8.2. Therefore one ends up with only the contributions of the physical electron. Although the subtraction terms in the regularized version of (8.37) were transverse, the contributions from the physical electron alone are not as can easily be verified from the lowest order contribution to (8.37), i.e. the elimination of the auxiliary electron fields must be paid for by allowing, on the counter term level, quadratically (and also logarithmically) divergent counter terms that are not transverse.

Having obtained $\widetilde{\Pi}_{tr}(k)$, which is transverse and vanishes of fourth order as $k \to o$, it is easy to form the photon self energy operator $\widetilde{\Pi}(k)$ from it, that satisfies the renormalization conditions (7.43):

$$\widetilde{\Pi}^{\mu\nu}(k) = \widetilde{\Pi}^{\mu\nu}_{tr}(k) - (g^{\mu\nu}k^2 - k^{\mu}k^{\nu})\left(\frac{\widetilde{\Pi}_{\lambda}{}^{\lambda}{}_{tr}(k)}{3\ k^2}\right)_{k^2=m^2} \tag{8.39}$$

(The subtraction term is identically zero if $m^2 = o$.)

Thus the renormalization of charge, mass, and photon amplitude (these three are directly related in the convention (7.33,35,36)) is distributed over two steps, (8.37) and (8.39), in the latter only finite quantities are involved. The factor $\frac{s}{Z_3}$ which is needed for the calculation of the correctly normalized scattering amplitudes, cf. the discussion after (7.43), is to be determined from the finite expression (8.39) of $\widetilde{\Pi}^{\mu\nu}$.

Now our renormalization scheme is complete: In (8.16) we have obtained the skeleton expansions of all Green's functions other than the renormalization functions. The irreducible functions occurring in (8.16) can be calculated from (8.13,15). The renormalization functions Γ, X, Σ, and $\widetilde{\Pi}$ can be calculated iteratively from (8.32a), (8.32b), (8.35), and (8.39), respectively, whereby the Bethe-Salpeter kernels occurring in these expansions are determined by (8.17) and (8.20).

We have shown that the regularization of the electron lines can be removed from these equations. However, we did not prove that the photon regulators can be eliminated, too. Since every photon line is regularized separately there are no complications due to gauge invariance, and the problem is similar to e.g. the one in ϕ^4-theory. Since a somewhat laborious power counting shows that all integrals of the above system of equations are convergent in virtue of the judiciously arranged subtractions, the photon regulators are not necessary in that system. In fact, this system is merely a resummed form of the prescriptions given by Bogoliubov and Shirkov [3] for the calculation of renormalized Green's functions adapted to QED and neutral vector meson theory.

8.5 Vanishing electron mass *)

8.5.1 γ_5-invariance

The renormalization conditions (7.43,53) were given for electron mass different from zero and for photon mass either zero or different from zero. If the electron mass vanishes but not the photon mass, it suffices to let M = o in (7.53). In this case the theory possesses γ_5-invariance, i.e. the substitution

$$\psi \longrightarrow e^{i\alpha\gamma_5}\,\psi$$

$$\overline{\psi} \longrightarrow \overline{\psi}\, e^{i\alpha\gamma_5}$$

$$A \longrightarrow A$$

leaves the Lagrangian unchanged. More precisely, in the regularized Lagrangian (8.3) the γ_5-substitution must only be carried out on the physical electron-positron field ψ_o, $\overline{\psi}_o$, with M_o = o. This implies that the corresponding conserved Noether current is

$$j_5^{\mu} = Z_2^{o}\, \overline{\psi}_o\, \gamma_5 \gamma^{\mu}\, \psi_o \, . \tag{8.40}$$

The γ_5-invariance holds also for the renormalized theory: It suffices **) that any Green's function has in the corresponding graphs an odd number of γ-matrices along any electron arc. From the renormalized

*) Section added in Fall 1970. **) cp. footnote on p. 174

equations (8.32a,35a) it is immediately seen that γ-oddness of the electron propagator (due to M = o) and of the vertex to lowest perturbation theoretical order implies recursively this property to all orders, since in (8.32a) only a linear combination of γ-odd terms is formed and B $\equiv$ o in (8.35a). Indeed, the current (8.40) has a limit as the regularization is removed; this renormalized current is conserved (as implied by the appropriate Ward identities) but not gauge invariant. A gauge invariant current in the limit could only have been obtained by taking instead of (8.40)

$$j_{5\,g.i.}^{\mu} = Z_5 \sum_{f,k} Z_2^f \overline{\Psi}_{fk} \gamma_5 \gamma^{\mu} \Psi_{fk} \tag{8.41}$$

with Z_5 a suitable (in the limit not finite) constant, in analogy to the vector current in (8.4a). However, the space integral

$$Q_5 = \int d\sigma_{\mu}\, j_5^{\mu}$$

is gauge invariant [63] .

8.5.2 Massless quantum electrodynamics

More interesting is the case of electron and photon mass both equal to zero. Then (8.35) is not applicable since this would require normalization of the electron propagator on the mass shell, which leads to UR-divergence as in finite-mass-QED. For this reason also the vertex cannot then be normalized at zero electron momenta. Moreover, the renormalization conditions (7.43) cannot be imposed*) since the second derivative of the inverse photon propagator is singular at zero momentum due to the vanishing electron mass, which makes zero momentum the threshold for pair production. The difficulty can also be seen as follows: if the

*) Strictly speaking, this holds also for vector meson mass m > o, when an adaption of the method described after (8.53) should be used.

conditions (7.43,53) with $M = m = o$ could be imposed, the theory, if thereby determined, would not involve any mass parameter, such that it would have to be scale invariant, and (7.43,53) would imply that the electron and photon propagators were the free ones. But then it follows [64] that there cannot be any interaction at all.

It is therefore necessary to treat the renormalization of massless QED separately. Let us denote in this section the vertex function and the negative inverse propagators collectively by Γ, i.e.

$$\Gamma(\not p) = - S'_F(\not p)^{-1} \tag{8.42a}$$

$$\Gamma^\mu(p,q) = \Gamma^\mu(p,q) \tag{8.42b}$$

$$\Gamma^{\mu\nu}(k) = - [\Delta'_F(k)^{-1}]^{\mu\nu} \tag{8.42c}$$

Let the unrenormalized form of $\Gamma(\not p)$, i.e. the one before the final subtraction (cf. (8.24a,b)), be $\Gamma_u(p)$. We set

$$\Gamma(\not p) = \Gamma(p,p_o) - A(p_o) - p_\mu B^\mu(p_o) \tag{8.43a}$$

with

$$\Gamma(p,p_o) = \Gamma_u(p) - \Gamma_u(p_o) - (p-p_o)_\mu \frac{\partial \Gamma_u(p_o)}{\partial p_{o\mu}} \tag{8.43b}$$

which is finite due to $D = 1$ of the electron self energy part, for any p_o with $p_o^2 \neq o$, whereby manipulations analogous to those we discussed in connection with (8.37) are implied. From the zero-mass condition

$$\Gamma(o) = o \tag{8.44}$$

follows

$$A(p_o) = \Gamma(o,p_o) \; , \tag{8.45}$$

and from

$$\frac{\partial \Gamma(\not p)}{\partial p_\mu} = \frac{\partial \Gamma(p,p_o)}{\partial p_\mu} - B^\mu(p_o)$$

by setting $p = p_o$

$$B^\mu(p_o) = -\frac{\partial \Gamma(\not p_o)}{\partial p_{o\mu}} \quad . \tag{8.46}$$

Now, for covariance reasons

$$\Gamma(\not p) = \not p\, U(p^2) + V(p^2) \tag{8.47}$$

such that

$$B^\mu(p_o) = -\gamma^\mu U(p_o^2) - 2\, p_o^\mu \left[\not p_o U'(p_o^2) + V'(p_o^2)\right] . \tag{8.48}$$

To find $U'(p_o^2)$ and $V'(p_o^2)$, we form the second derivative of (8.43a) at $p = p_o$:

$$\frac{\partial^2 \Gamma(\not p_o)}{\partial p_{o\mu}\partial p_{o\nu}} = \frac{\partial^2 \Gamma_u(p_o)}{\partial p_{o\mu}\partial p_{o\nu}} = 2\,(\gamma^\mu p_o^\nu + \gamma^\nu p_o^\mu + \not p_o g^{\mu\nu})\, U'(p_o^2) +$$

$$+ 2g^{\mu\nu}V'(p_o^2) + 4p_o^\mu p_o^\nu\left[\not p_o U''(p_o^2) + V''(p_o^2)\right] . \tag{8.49}$$

Here all but the terms proportional to $g^{\mu\nu}$ are removed by transverse projection, and forming traces gives $U'(p_o^2)$ and $V'(p_o^2)$ for insertion in (8.48), with the final result for (8.43a)

$$\Gamma(\not{p}) = \Gamma_u(p,p_o) - \Gamma_u(o,p_o) +$$

$$+ \frac{pp_o}{12p_o^2} \left\{ \not{p}_o \, \mathrm{Tr} \left[\not{p}_o \left(g^{\mu\nu} - \frac{p_o^\mu p_o^\nu}{p_o^2}\right) \frac{\partial^2 \Gamma_u(p_o)}{\partial p_o{}^\mu \partial p_o{}^\nu} \right] + \right.$$

$$\left. + p_o^2 \, \mathrm{Tr} \left[\left(g^{\mu\nu} - \frac{p_o{}^\mu p_o{}^\nu}{p_o^2}\right) \frac{\partial^2 \Gamma_u(p_o)}{\partial p_o{}^\mu \partial p_o{}^\nu} \right] \right\} + \not{p}\, U(p_o^2) \qquad (8.50)$$

The constant $U(p_o^2)$ in the last term is arbitrary except from being

imaginary for $p_o^2 < o$ as can be seen from (7.51). It determines the normalization of the electron propagator and, thereby, of the renormalized electron fields. In (8.50) one may also use

$$\frac{\partial^2 \Gamma_u(p_o)}{\partial p_o{}^\mu \partial p_o{}^\nu} = \left. \frac{\partial^2 \Gamma(p,p_o)}{\partial p^\mu \partial p^\nu} \right|_{p=p_o} .$$

In order to determine the renormalized vertex function, in view of $D = o$, from the unrenormalized $\Gamma_u(p,q)$, i.e. the one before the final subtraction, we use Ward's identity

$$\Gamma^\mu(p,p) = - e \, \frac{\partial \Gamma(\not{p})}{\partial p_\mu} \qquad (8.51)$$

such that

$$\Gamma^{\mu}(p,q) = \Gamma_u^{\mu}(p,q) - \Gamma_u^{\mu}(p,p) - e \frac{\partial \Gamma(\not{p})}{\partial p_{\mu}} \tag{8.52}$$

where the difference of the first two terms is finite and to be calculated in analogy to (8.27), and the last term is to be obtained from (8.50).

As mentioned before, the negative inverse photon propagator in (8.42c) cannot be normalized at k = o due to an UR-singularity of the second derivative there. Since the definition of, e.g., Gupta-Bleuler gauge in sections 5.4 and 7.3.1 rests on the possibility of separating from $\Gamma^{\mu\nu}(k)$ a part proportional to $g^{\mu\nu}$, which dominates at k = o and determines the normalization there, that gauge does not exist in massless QED, the only intrinsically defined gauge (in the class of gauges obtainable from (7.1)) being the Landau gauge

$$\Gamma^{\mu\nu}(k) = (g^{\mu\nu}k^2 - k^{\mu}k^{\nu})\, \Gamma(k^2) \,. \tag{8.53}$$

(Of course, one may add to (8.53) a term of e.g. the form $k^{\mu}k^{\nu}\cdot$const., however, this cannot yield Gupta-Bleuler gauge, since $\Gamma(o)$ does not exist.)

Thus we set in analogy to (8.43)

$$\Gamma^{\mu\nu}(k) = \Pi^{\mu\nu}(k,k_o) - A^{\mu\nu}(k\ ) - k_{\alpha}B^{\mu\nu\alpha}(k_o) - \frac{1}{2}\, k_{\alpha}k_{\beta}C^{\mu\nu\alpha\beta}(k_o) \tag{8.54a}$$

with

$$\Gamma^{\mu\nu}(k,k_o) = \Gamma_u^{\mu\nu}(k) - (k-k_o)_\alpha \frac{\partial \Gamma_u^{\mu\nu}(k_o)}{\partial k_{o\alpha}} -$$

$$- \frac{1}{2}(k-k_o)_\alpha (k-k_o)_\beta \frac{\partial^2 \Gamma_u^{\mu\nu}(k_o)}{\partial k_{o\alpha} \partial k_{o\beta}} \qquad (8.54b)$$

which is finite, due to D = 2 of the inverse photon propagator before final subtraction with $k_o^2 \neq 0$, and to be calculated again in the manner discussed in connection with (8.37). The zero-mass condition

$$\Gamma^{\mu\nu}(o) = o \qquad (8.55a)$$

and the trivial one

$$\frac{\partial}{\partial k_\alpha} \Gamma^{\mu\nu}(k) \Big|_{k=o} = o \qquad (8.55b)$$

give

$$A^{\mu\nu}(k_o) = \Gamma^{\mu\nu}(o,k_o) \qquad (8.56a)$$

and

$$B^{\mu\nu\alpha}(k_o) = \frac{\partial \Gamma^{\mu\nu}(k,k_o)}{\partial k_{o\alpha}} \Big|_{k=o} . \qquad (8.56b)$$

Now

$$C^{\mu\nu\alpha\beta}(k_o) = - \frac{\partial^2 \Gamma^{\mu\nu}(k_o)}{\partial k_{o\alpha} \partial k_{o\beta}} \tag{8.57}$$

yields, with (8.53),

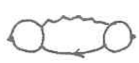
$$- \frac{1}{2} k_\alpha k_\beta C^{\mu\nu\alpha\beta}(k_o) = (g^{\mu\nu}k^2 - k^\mu k^\nu) \; \Gamma(k_o^2) +$$

$$+ \left[2 k^\mu k_o^\nu (kk_o) + 2 k_o^\mu k^\nu (kk_o) - 4 g^{\mu\nu}(kk_o)^2 + k_o^\mu k_o^\nu k^2 - g^{\mu\nu} k^2 k_o^2 \right] \Gamma'(k_o^2) -$$

$$- 2 (g^{\mu\nu} k_o^2 - k_o^\mu k_o^\nu)(kk_o)^2 \; \Gamma''(k_o^2) \; . \tag{8.58}$$

Insertion of (8.56a,b,58) into (8.54a) and multiplying by k_μ, using (8.53), allows to solve for $\Gamma'(k_o^2)$ and $\Gamma''(k_o^2)$ provided $(kk_o)^2 - k^2 k_o^2 \neq o$, i.e. k and k_o are not parallel. The, for $k_o^2 < o$ imaginary (cf. (7.42)), constant $\Gamma(k_o^2)$ remains undetermined and fixes the normalization of the photon propagator.

γ_5-invariance of the theory so obtained follows as before: It is secured if the electron propagator and the vertex are γ-odd. To lowest order, this is trivial. Let it hold up to order e^{2n} for $\Gamma(\not p)$ and up to order e^{2n+1} for $\Gamma^\mu(p,q)$. Then it holds for $\Gamma_u(p)$ to order e^{2n+2} due to

Γ_u =  - 

and the skeleton expansion for the Bethe-Salpeter kernel, and thus from (8.50) with (8.43b) also for $\Gamma(\not p)$, and for $\Gamma^\mu_u(p,q)$ to order

e^{2n+3} due to

$$\Gamma^{\mu}_{u} =$$ 

and thus from (8.52) and the result on $\Gamma(\not{p})$ just obtained also for $\Gamma^{\mu}(p,q)$. The induction is complete. Thus, in (8.47) $V(p^2) \equiv o$.

Finally, we consider on which parameter the theory so constructed actually depends. We have introduced $U(p_o^2)$, $\mathcal{G}(k_o^2)$, for some $p_o^2 < o$, resp. $k_o^2 < o$, and e. For simplicity, we choose $k_o^2 = p_o^2$. By re-normalizing the photon and the electron propagators, we can make $\Gamma(p_o^2) = i$ and $U(p_o^2) = i$. (We take the sign in the latter case, concerning which no rigorous result is known, as that suggested by perturbation theory for small p_o^2.) The coupling constant e associated with this choice by (8.51) and designated by $e(p_o^2)$ is thus the only physically significant parameter, since the operator normalizations chosen here are arbitrary conventions in the absence of an intrinsic mass and in view of the impossibility of normalizing the propagators at zero momentum. Now, due to non-scale-invariance, $e(p_o^2)$ depends on p_o^2 nontrivially for the unchanged theory, or, equivalently, a length scale change for the theory changes $e(p_o^2)$. Thus the relations we have given define a one-parameter family of physically distinct theories, all of them, however, related to each other by dilatation, with no intrinsic dimensionless parameter definable. In particular, e in (8.51) is not a renormalized charge in the conventional sense, due to the impossibility of normalizing the photon propagator in the conventional manner. It can be shown [65] that if the massless theory exists, as here formulated, as the limit of conventional finite-electron-mass QED (and in this sense it does exist in perturbation theory) then one must let the conventional charge go to zero in that limiting process. The rate at which one lets it go to zero relative to the electron mass determines which member of the one-parameter family of zero-mass theories just described is obtained. The absence of any intrinsic dimensionless parameter of the zero-mass theory (in perturbation theory) is the root of the famous result of Gell-Mann and Low [66] that the bare charge, defined by a certain large-momentum limit, is independent of the renormalized one.

Textbooks

[1] A. Akhiezer, V.B. Bereztetski: Quantum Electrodynamics, New York, Wiley, 1963

[2] J.D. Bjorken, S.D. Drell: Relativistic Quantum Fields, New York, Mc Graw - Hill, 1965

[3] N.N. Bogoliubov, D.V. Shirkov: Introduction to the Theory of Quantized Fields, New York, Interscience, 1959

[4] P.A.M. Dirac: The Principles of Quantum Mechanics, Oxford, Claredon, 1958

[5] F.J. Dyson: Advanced Quantum Mechanics, Cornell University, 1951

[6] J.M. Jauch, F. Rohrlich: The Theory of Photons and Electrons, Cambridge (Mass.), Addison-Wesely, 1955

[7] R. Jost: The General Theory of Quantized Fields, Providence, Rhode Island, American Mathematical Society, 1965

[8] G. Källen: in Handbuch der Physik, Vol. V,1, Edt. S. Flügge, Berlin, Springer, 1958

[9] P. Roman: Advanced Quantum Theory, Reading (Mass.), Addison-Wesley, 1965

[10] S. Schweber: An Introduction to Relativistic Quantum Field Theory, New York, Harper and Row, 1961

[11] R.F. Streater, A.S. Wightman: PCT, Spin and Statistics, and all that, New York, Benjamin, 1964

[12] W. Thirring: Principles of Quantum Electrodynamics, New York, Academic Press, 1958

References

[1] J. Schwinger in: Lectures on Particles and Field Theory, Vol. II, Eds. S. Deser, K.W. Ford; Englewood Cliffs, Prentice-Hall, 1965

[2] F.R. Gantmacher: Matrizenrechnung, Vol. I, Sect. IX, § 13, Berlin, VEB Dt. Verl. d. Wiss., 1958

[3] - Sect. VII, § 5

[4] F.A. Berezin: The Method of Second Quantization, New York, Academic Press, 1966

[5] J. Schwinger: Phys. Rev. 82, 914 (1951); reprinted in: Quantum Electrodynamics, Ed. J. Schwinger, New York, Dover, 1958

[6] C.S. Lam: Nuovo Cimento 38, 1755 (1965)

[7] J. Schwinger: Proc. Nat. Ac. Sc. 46, 883, 1401 (1960)

[8] R.E. Peierls: Proc. Roy. Soc. (London) A 214, 143 (1952)

[9] E.P. Wigner: Annals of Math. 40, 149 (1939)

[10] A.J. Macfarlane: J. Math. Phys. 3, 1116 (1962)

[11] R. Arnowitt, S. Deser: Math. Phys. 3, 637 (1962)

[12] W. Pauli, M. Fierz: Nuovo Cimento 15, 167 (1938)

[13] M.A. Neumark: Lineare Darstellungen der Lorentzgruppe, Berlin, VEB Dt. Verl. d. Wiss., 1963

[14] I.M. Gelfand, R.A. Minlos, Z.Ya.Shapiro: Representations of the Lorentz Group and their Applications, Oxford, Pergamon, 1963

[15] Y. Takahashi: An Introduction to Field Quantization, Oxford, Pergamon, 1969

[16] D.J. Gross, R. Jackiw: Phys. Rev. 184, 1577 (1969)

[17] T.D. Lee, C.N. Yang: Phys. Rev. 128, 885 (1962)

[18] L.S. Brown: Phys. Rev. 150, 1338 (1966)

[19] D.G. Boulware, S. Deser: Phys. Rev. 151, 1278 (1966)

[20] D.G. Boulware, L.S. Brown: Phys. Rev. 156, 1724 (1967)

[21] H. Goldstein: Classical Mechanics, Reading (Mass.), Addison-Wesley, 1959

[22] B. Zumino: J. Math. Phys. 1, 1 (1960)

[23] S. Ciccariello, R. Gatto, G. Sartori, M. Tonin: Univ. Padova preprint IFPTH - 7/70

[24] K.G. Wilson: Phys. Rev. 179, 1499 (1969)

[25] R.A. Brandt: Phys. Rev. 166, 1795 (1968)
[26] R. Haag: Phys. Rev. 112, 669 (1958)
[27] K. Hepp in: Axiomatic Field Theory, Vol. I, Eds. M. Chretien, S. Deser, New York, Gordon and Breach, 1966
[28] N. Nakanishi: Progr. Theoret. Phys. 38, 881 (1967)
[29] H. Rollnik, B. Stech, E. Nunnemann: Z. Physik 159, 482 (1960)
[30] F. Strocchi: Phys. Rev. 162, 1429 (1967)
[31] S. Kamefuchi, L.O'Raifeartaigh, A. Salam: Nucl. Phys. 28, 529 (1961)
[32] H.J. Borchers: Nuovo Cimento 15, 784 (1961)
[33] I. Bialynicki-Birula: Phys. Rev. D2, 2877 (1970)
[34] S.N. Gupta: Can. J. Phys. 31, 961 (1957)
[35] S. Adler, R. Dashen: Current Algebras, New York Benjamin, 1968
[36] W. Zimmermann: Commun. Math. Phys. 8, 66 (1968)
[37] L. Schwartz: Théorie des distributions, Paris, Hermann, 1950/51
[38] A. Jaffe: Phys. Rev. 158, 1454 (1967)
[39] F. Constantinescu: Nuovo Cimento Lett. 1, 849 (1969)
[40] N.M. Kroll, T.D. Lee, B. Zumino: Phys. Rev. 157, 1376 (1967)
[41] K. Symanzik in: Lectures on High Energy Physics,Ed. B. Jaksić, Zagreb, 1961. Reprinted: New York, Gordon and Breach, 1966
[42] R.W. Johnson: J. Math. Phys. 11, 2161 (1970)
[43] Y. Takahashi: Nuovo Cimento 6 370 (1957)
[44] N. Dombey: Nuovo Cimento 32, 1696 (1964)
[45] I. Bialynicki-Birula: Nuovo Cimento 17, 951 (1960)
[46] A.S.Wightman in:Cargèse Lectures 1964,Ed.M.Lévy,NewYork,Gordon&Breach 1967
[47] S.N. Gupta: Proc. Roy. Soc. (London) A 66, 129 (1953)
[48] D. Yennie, S. Frautschi, H. Suura: Ann. Phys. (N.Y.) 13, 379 (1961)
[49] K.E. Eriksson: Nuovo Cimento 19, 1010 (1961)
[50] B.O. Enflo: Nuovo Cimento A 67, 595 (1970)
[51] K. Johnson, E.C.G. Sudarshan: Ann. Phys. (N.Y.) 13, 126 (1961)
[52] T.D. Lee: Phys. Rev. 128, 899 (1962)
[53] S. Weinberg: Phys. Rev. 138 B, 988 (1965)
[54] G. Velo, D. Zwanziger: Phys. Rev. 186, 1337, 188, 2218 (1969)
[55] R.A. Brandt: Ann. Phys. (N.Y.) 52, 122 (1969)
[56] K.E. Eriksson: Physica Scripta 1, 3 (1970)
[57] P.P. Kulish, L.D. Faddeev: Theoret. and Math. Phys. 4, 153 (1970)
[58] E.C.G. Stueckelberg: Helv. Phys. Acta 30, 209 (1957)
[59] A. Salam, J. Strathdee: Phys. Rev. D2, 2869 (1970)

[60] I.S. Gerstein, R. Jackiw, B.W. Lee, S. Weinberg: preprint MIT-CTP 161

[61] F.J. Dyson: Phys. Rev. 75, 486, 1736 (1949)

[62] B. Ferretti: Nuovo Cimento (L) 12, 457 (1954)

[63] S. Adler: Phys. Rev. 177, 2424 (1969)

[64] K. Pohlmeyer: Comm. Math. Phys. 12, 204 (1969)

[65] K. Symanzik in: Springer Tracts in Modern Physics, Vol. 57

[66] M. Gell-Mann, F.E. Low: Phys. Rev. 95, 1300 (1954)