Analytic Calculation of a 1-loop Effective Action for the $O(\mathcal{N}+1)$-symmetric 2-dimensional σ-model

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Abstract

Starting from the standard discretization of the action of the nonlinear σ-model in two dimensions we compute the Wilson effective action on a coarse lattice in a one loop approximation. Two methods are considered for the integration over the constrained fluctuation field.

The first one was proposed by Polyakov. It allows a straightforward extraction of the result for the running coupling constant. The drawback is a rather involved propagator for the high frequency field. This prevents a transparent extraction of its field dependence.

The other way to perform the integration is the extension of the fluctuation field to the whole $\mathbb{R}^{\mathcal{N}+1}$. This eliminates the constraint on the fluctuation field while keeping the expansion of the propagator manageable, but one has to scrutinize an extended action. An explicit form of the effective action is given. It contains a genuine one loop contribution. Polyakov's result can still be recovered to leading order.

Although the analytical form of the effective action is transparently expressed in terms of the background field, numerical investigations require an explicit dependence on the block spin. Therefore a first approximation of the background field is given, and an iterative procedure is suggested to improve its properties.

A Gaussian block spin is also considered. It allows to improve the locality properties of the effective action.

Zusammenfassung

In dieser Arbeit wird die effektive Wirkung für das nichtlineare σ-Modell in zwei Dimensionen in einer Einschleifennäherung berechnet. Ausgangspunkt ist die Standarddiskretisierung der Kontinuumswirkung. Zwei Methoden zur Ausintegration des Hochfrequenzfeldes werden diskutiert.

Die erste geht auf Polyakov zurück. Sie erlaubt eine einfache Herleitung des Ergebnisses für die gleitende Kopplungskonstante. Ihr Nachteil ist ein komplizierter Propagator für das Hochfrequenzfeld, dessen Feldabhängigkeit sich nicht einfach entwickeln läßt.

Die andere Möglichkeit die Integration auszuführen ist die Erweiterung das Fluktuationsfeldes auf den ganzen Raum $\mathbb{R}^{\mathcal{N}+1}$. Dies beseitigt die Beschränkung für das Hochfrequenzfeld, während die Entwicklung des Propagators möglich bleibt. Allerdings ist eine erweiterte Wirkung zu untersuchen. Ein analytischer Ausdruck für die effektive Wirkung wird angegeben. Er enthält einen echten Einschleifenbeitrag. Polyakovs Ergebnis kann aber zu führender Ordnung extrahiert werden.

Obwohl der analytische Ausdruck für die effektive Wirkung durch ein Hintergrundfeld durchschaubar dargestellt werden kann, benötigen numerische Untersuchungen eine Darstellung in Abhängigkeit von dem Blockspinfeld. Deshalb wird eine erste Näherung für das Hintergrundfeld konstruiert, und es wird eine rekursive Verbesserung der Näherung vorgeschlagen.

Ein Gauß'scher Blockspin wird ebenfalls untersucht. Dieser erlaubt es, die Lokalität der effektiven Wirkung zu verbessern.

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Chapter 1

Introduction

1.1 Euclidean Field Theory

Consider a field theory on Minkowski space-time with $(d-1)$ space dimensions. The space-time is parametrized by coordinates x_{Min} with

$$x_{\mathrm{Min}} = (t, x^1_{\mathrm{Min}}, \ldots, x^{d-1}_{\mathrm{Min}}).$$

The metric

$$(x_{\mathrm{Min}})^2 = -t^2 + \sum_{i=1}^{d-1} (x^i_{\mathrm{Min}})^2$$

is not positive definite.

The field theory is described in terms of the path integral formulation by Wightman functions (N-point correlation functions)

$$\frac{1}{\mathcal{Z}} \int \mathcal{D}\phi \phi(x_{\mathrm{Min};1}) \ldots \phi(x_{\mathrm{Min};N}) e^{iS_{\mathrm{Min}}[\phi]}.$$

$\phi(x_{\mathrm{Min}})$ is the field which is defined on Minkowski space-time, S_{Min} is the action defining the considered theory, and

$$\mathcal{Z} \int \mathcal{D}\phi e^{iS_{\mathrm{Min}}[\phi]}$$

is the partition function (generating functional of the correlation functions).

But it is also possible to define expectation values of products of fields in terms of their analytic continuation to imaginary time,

$$t \to -ix_0.$$

Imaginary time fields are called Euclidean fields.

This continuation changes the Minkowski metric to a Euclidean one

$$(x)^2 = \sum_{i=0}^{d-1} (x^i)^2.$$

This metric is positive definite. A field theory is defined by an action $S[\phi]$ with

$$S[\phi] = -iS_{\text{Min}}[\phi].$$

We get the new partition function

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S[\phi]}$$

and in analogy to the Wightman functions the Euclidean correlation functions

$$\langle \phi(x_1) \dots \phi(x_N) \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi \phi(x_1) \dots \phi(x_N) e^{-S[\phi]}.$$

They are also called Schwinger functions or Euclidean Greens functions. These equations can also be interpreted as the partition function and the correlation functions of a system in *classical* statistical mechanics because all distances in a Euclidean space are space like.

So the quantum theory of fields in $(d-1)$ space dimensions is transformed into classical statistical mechanics of fields in d Euclidean dimensions [1]. The Euclidean action plays the role of $\frac{E}{k_B T}$, where E is the classical energy of a d dimensional configuration.

The action which defines a field theory via the partition function $\mathcal{Z}$ has to respect certain axioms, see [15].

One example for such theories is the free field theory. It describes particles that do not interact. Although this seems to be a very unphysical situation (a noninteracting particle could not even be detected), free fields play a major role in the description of interacting fields. They are the starting point for the calculation or construction of all field theories. As free field theories are exactly solvable, other theories are treated in perturbation theory by expanding the path integrals around the associated free theory. In statistical mechanics, the Euclidean free field theory is known as the Gaussian model.

The free field with mass m is described by the action

$$\begin{aligned} S_{\text{free}}[\phi] &= \int d^d x \left(\frac{1}{2} \nabla_\mu \phi(x) \nabla_\mu \phi(x) + \frac{1}{2} m^2 \phi^2(x) \right) \\ &= \int d^d x \frac{1}{2} \phi(-\triangle + m^2) \phi(x) \\ &= \langle \phi, v^{-1} \phi \rangle. \end{aligned}$$

[1] The continuum limit of a Euclidean lattice field theory reveals even profounder analogies with critical phenomena in statistical mechanics. This will be explained in section 1.2.1.

v is called the free propagator. The partition function of the free field theory is an integral over a Gaussian measure, see appendix A.

$$\begin{aligned} \mathcal{Z}_{\text{free}} &= \int \mathcal{D}\phi e^{-\langle \phi, v^{-1}\phi\rangle} \\ &= (\det 2\pi v)^{\frac{1}{2}} \\ &= e^{\frac{1}{2}\operatorname{Tr}\ln(2\pi v)} \end{aligned}$$

Since this is just a constant, the partition function may be normalized to 1 by a factor $\mathcal{N}^{-1} = (\det 2\pi v)^{-\frac{1}{2}}$,

$$\begin{aligned} \mathcal{Z}_{\text{free}} &= \mathcal{N}^{-1} \int \mathcal{D}\phi e^{-\langle \phi, v^{-1}\phi\rangle} \\ &= \int d\mu_v(\phi) = 1. \end{aligned}$$

A theory with an interaction can be described by an action S_{int} which contains a potential term $V(\phi)$,

$$S_{\text{int}} = S_{\text{free}} + V(\phi).$$

The partition function can be treated in perturbation theory

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S_{\text{int}}[\phi]} = (\det 2\pi v)^{-\frac{1}{2}} \int d\mu_v(\phi) e^{-V(\phi)},$$

if the coupling constants in the potential are small.

This lays the ground for the determination of the associated correlation functions and observables.

1.2 Lattice Field Theory

In field theory, the variables are associated with space-time points. The path integrals describing the theory are not necessarily finite. Therefore some regularization is needed. A regularization renders the integrals finite by introducing some cut-off in the space of field configurations. This excludes high frequency configurations from the path integral which cause the divergencies. But the arbitrary cut-off is an unphysical scale. It has to be removed by a process called renormalization. In a renormizable theory the infinities which stem from the short distance behaviour of the theory can be absorbed in a finite number of physical parameters.

One way to regularize a field theory is to introduce a hypercubic lattice Λ as a defining framework,

$$\Lambda = a\mathbb{Z}^d.$$

d denotes the dimension of the (Euclidean) space the fields live on. a is called the lattice spacing or lattice constant. It is the shortest distance defined in a lattice theory.

The fields are no longer defined on the continuum but on the lattice Λ,

$$\phi : \Lambda \to \phi(z); \qquad z \in \Lambda.$$

The functional integral with the measure $\mathcal{D}\phi$ becomes a finite dimensional integral if the volume is finite,

$$\mathcal{D}\phi \to \prod_{z \in \Lambda} d\phi(z).$$

The lattice regularization has several advantages over other conventional regularization schemes used in perturbation theory. As path integrals regularized by a lattice become sets of integrals, [2] they are ready for numerical procedures in case of nonperturbative problems.

It is a very natural regularization for theories with gauge invariance or with constrained variables. An example for the latter is the nonlinear σ-model.

Like every other scheme, lattice regularization breaks certain symmetries. Especially chiral symmetry is broken. The discretization of space-time also breaks $O(d)$ symmetry in the Euclidean case, or Lorentz symmetry of Minkowski space-time. But the remaining discrete symmetry is expected to have $O(d)$ symmetry restored when the cut-off is removed in the limit $a \to 0$. For free field theory this is true.

To define an action for fields living on the lattice points, it is necessary to use derivatives and integrals.

The continuum derivative is discretized by

$$\nabla_\mu \phi(z) = \frac{1}{a} \left[\phi(z + \hat{\mu}) - \phi(z) \right]. \tag{1.1}$$

$\hat{\mu}$ denotes the vector pointing in direction μ of length a. This definition of a derivative is not unique. Any other which reproduces the continuum derivative in the limit $a \to 0$ could be used.

We also define a backward derivative

$$\nabla_{-\mu} \phi(z) = \frac{1}{a} \left[\phi(z - \hat{\mu}) - \phi(z) \right]. \tag{1.2}$$

$\nabla_{-\mu}\phi$ tends to the negative continuum derivative in the limit $a \to 0$. The two derivatives are related by

$$-\nabla_{-\mu} \phi(z + \hat{\mu}) = \nabla_\mu \phi(z).$$

[2] Often path integrals are even introduced as the limit of a set of integral. In this case a temporary mesh is used anyway.

A Laplace operator is introduced by

$$\begin{aligned} -\triangle\phi(z) &= \sum_{\mu=0}^{d-1} \nabla_{-\mu}\nabla_{\mu}\phi(z) \\ &= \frac{1}{a^2}\sum_{\mu=0}^{d-1}\left[2\phi(z) - \phi(z-\hat{\mu}) - \phi(z+\hat{\mu})\right]. \end{aligned} \tag{1.3}$$

An equivalent definition is

$$\triangle\phi(z) = \frac{1}{a}\sum_{\pm\mu}\nabla_{\mu}\phi(z). \tag{1.4}$$

The Leibnitz rule takes the form

$$\nabla_{\mu}\left(\phi\cdot\psi\right)(z) = \nabla_{\mu}\phi(z)\cdot\psi(z) + \phi(z+\hat{\mu})\cdot\psi(z). \tag{1.5}$$

The shifted argument in the second term is a consequence of our definition of the lattice derivative.

On the lattice, integrals are replaced by sums,

$$\int_{\mathbb{R}^d} \to a^d \sum_{z\in\Lambda}\phi(z) = \int_z \phi(z). \tag{1.6}$$

This allows the definition of a scalar product of functions on the lattice,

$$\langle\phi,\psi\rangle = \int_z \bar{\phi}(z)\cdot\psi(z).$$

With respect to this scalar product, the backward derivative $\nabla_{-\mu}$ is the adjoint of ∇_{μ},

$$\begin{aligned} \langle\phi,\nabla_{\mu}\psi\rangle &= \int_z \bar{\phi}(z)\cdot\nabla_{\mu}\psi(z) = \int_z \frac{1}{a}\bar{\phi}(z)\cdot[\psi(z+\hat{\mu})-\psi(z)] \\ &= \int_z \frac{1}{a}[\bar{\phi}(z-\hat{\mu})\cdot\psi(z) - \bar{\phi}(z)\cdot\psi(z)] \\ &= \int_z \nabla_{-\mu}\bar{\phi}(z)\cdot\psi(z) = \langle\nabla_{-\mu}\phi,\psi\rangle. \end{aligned}$$

As a consequence

$$\langle\nabla_{\mu}\phi,\nabla_{\mu}\psi\rangle = \langle\nabla_{-\mu}\nabla_{\mu}\phi,\psi\rangle = -\langle\triangle\phi,\psi\rangle.$$

This shows that $-\langle\triangle\phi,\phi\rangle \geq 0$. So $(-\triangle)$ is a nonnegative operator, just like its continuum counterpart.

The lattice Leibnitz rule implies another form of this integration by parts formula

$$\begin{aligned}\int_z \nabla_\mu \phi(z) \cdot \psi(z) &= \int_z \nabla_\mu \left(\phi \cdot \psi\right)(z) - \int_z \phi(z+\hat{\mu}) \cdot \nabla_\mu \psi(z) \\ &= -\int_z \phi(z+\hat{\mu}) \cdot \nabla_\mu \psi(z).\end{aligned}$$

This equality holds because integrals over total divergencies vanish.

To define path integrals on the lattice, a definition of functional integrals and derivatives is needed. Path integrals on the continuum are often even introduced as the limit of sets of integrals over variables on a discretized Euclidean space. The measure

$$\mathcal{D}\phi = \prod_{z \in \Lambda} d\phi(z)$$

is well defined.

A functional derivative is given by

$$\frac{\delta}{\delta \phi(z)} = a^{-d} \frac{\partial}{\partial \phi(z)}. \tag{1.7}$$

This gives

$$\frac{\delta}{\delta \phi(z)} \phi(w) = \delta(z-w),$$

with

$$\delta(z-w) = a^{-d} \delta_{zw}.$$

The last expression converges to the continuum δ-function in the limit $a \to 0$.

An interesting light is shed on lattice field theory by the work of A. Dimakis and F. Müller-Hoissen [9]. They show that lattice field theory can be obtained from a continuum theory via deformation of the differential calculus.

Assuming

$$[x, dx] = a \cdot dx \tag{1.8}$$

in the one dimensional case, one obtains a deformation of the ordinary differential calculus of differential forms with a deformation parameter a.

As a consequence,

$$f(x)dx = dx f(x+a)$$

for functions f. Furthermore,

$$\begin{aligned} df &\equiv (\partial_{+x} f) dx \\ &= \frac{1}{a} \left[f(x+a) - f(x)\right] dx. \end{aligned}$$

So the left partial derivative turns out to be the discrete derivative. Defining a right partial derivative ∂_{-x} by

$$df = dx(\partial_{-x} f),$$

it can be proven that

$$(\partial_{-x} f) = \frac{1}{a}\left[f(x-a) - f(x)\right].$$

Using the integral defined by

$$\int df = f + \text{const.},$$

one shows the property

$$\int_{x_0}^{x_0+a} dx f(x) = a f(x_0).$$

In d dimensions, the integral over a rectangular cell with side length a acts as a δ-function.

So assuming (1.8) implies our definitions of derivatives and integrals on the lattice.

1.2.1 The Continuum Limit

As the lattice of Euclidean space points is just a temporary framework to regularize the theory, it has to be removed to make contact with continuum physics. For example the lattice spacing a has to be send to zero.

The continuum limit of a theory exists if it is possible to tune the bare couplings of the action during this process in a way that keeps physical observables fixed. Assume, we can solve our lattice regularized theory at some set of bare couplings. Physical predictions will be dimensionless if we absorb appropriate powers of the lattice spacing a.

Consider for example the kinetic term of the action, $\int_z \nabla_\mu \phi \nabla_\mu \phi$. In d Euclidean dimensions an integral has dimension $[a]^d$, while derivatives have always dimension $[a]^{-1}$. Since the action must be dimensionless, the fields must have dimension

$$[\phi] = [a]^{\frac{-(d-2)}{2}}.$$

Thus, the mass m has dimension $[a]^{-1}$.

Any quantity Q with dimension s measured in mass units defines the associated dimensionless quantity $\hat{Q}$ by

$$Q \to \hat{Q} = a^s Q.$$

a is the only scale in the lattice theory. To relate it to a physical scale one needs to compare observables with measurements. The length of a cannot be predicted from lattice theory.

The most prominent observable is the correlation length ξ. It is determines from the long range decay of the two point function.

$$\langle \phi(x)\phi(0)\rangle \sim e^{-\frac{|x|}{\xi}}.$$

It is a function of the bare couplings. The correlation length $\xi = M^{-1}$ is the length scale over which a massive particle with mass M can propagate with significant amplitude,

$$\xi = a\hat{\xi} = \frac{1}{M} = \frac{1}{a^{-1}\hat{M}}.$$

So the observable mass can be written as

$$M = \frac{1}{a\hat{\xi}}.$$

To keep the physical mass M finite in the limit $a \to 0$, the correlation length measured in units of a, $\hat{\xi}$, must diverge. This process leaves the physical correlation length $\xi = \frac{1}{M}$ finite.

A trivial rescaling which multiplies all length by the same factor maps theories into distinct but equivalent theories. Then the continuum limit $a \to 0$ is equivalent to keeping a fixed and sending the correlation length ξ to infinity. In statistical mechanics, a system with diverging correlation length is called critical.

So considering the continuum limit of a field theory is the same as tuning the couplings to their critical values. This is the process which removes the lattice because in either case the observable correlation length becomes much larger than the lattice spacing. This corresponds to a good resolution of the discretized theory. The presence of the regularization will not distort physical predictions anymore.

Theories in which this can be achieved by tuning a finite number of parameters are called renormizable. This criterion is identical to that introduced in perturbation theory.

Lattice regularizations are not only used for analytical purposes but also for numerical simulations that may reveal nonperturbative effects. As a computer can only simulate lattice field theories with a finite lattice spacing a, the results contain also lattice artefacts which stem from the discretization. To reduce their effects and to make physical predictions one has to simulate the theory near the critical limit. This requires rather large lattices since the correlation length determines the extension of the lattice. Furthermore the updating processes for Monte Carlo simulations becomes less and less efficient the larger the correlation length becomes. This problem is known as *critical slowing down.*

There are different ways to circumvent these difficulties. One is, of course, to use fast computers. Another one is a thoughtful design of the updating algorithms. But the renormalization group described below offers solutions which exploit the fact that the discretized forms of continuum actions are not unique. While the physical predictions of a theory are universal for a large class of actions, the cut-off effects are not. Using appropriate actions one may reduce the latter.

1.3 The Renormalization Group

In the renormalization group one starts with a fundamental theory defined one the lattice or continuum and determines a sequence of effective actions for fields on coarser lattice. An effective action describes an effective field theory which encodes the physics of the fundamental theory at low energies. The effective theory can have a fairly low cut-off.

Given a fundamental theory, one has to find a procedure to calculate the associated effective actions. The idea of Wilson's renormalization group is to integrate out the high frequency components of the fundamental fields. This integration incorporates the effects of the high frequency modes of the fundamental theory in the couplings of the effective action. Therefore the results for low energy observables calculated with the effective theory agree with the predictions of the fundamental theory. Low energy observables are those which only involve the degrees of freedom of the effective theory. As the cut-off of an effective theory is low, it can not be used to calculate any physical quantities concerning the high energy effects of the fundamental theory. Above a certain energy the predictions of the effective theory are no longer valid. It is even possible that the theory develops divergencies for high energy observables.

An example may elude this point. The electroweak theory shall play the role of a fundamental theory. An associated effective theory is Fermi theory. While the fundamental action contains degrees of freedoms for massive vector particles which mediate the interaction of fermions, the effective theory assumes point like interactions between those fermions. It knows nothing about the intermediate vector bosons showing up at higher energies. Nevertheless Fermi theory is a valid approximation to the low energy behaviour of the weak interaction.

The example also demonstrates that an effective action must not be renormizable. Renormizability would mean that the theory describes physics correctly for all energy scales. But this is not the aim of the effective theory. The low cut-off guarantees that loop integrals are small because the phase space is small. Physical predictions can thus be read off the effective action on tree level.

While Fermi theory was proposed to explain the physics of the weak interaction without any knowledge about the existence of vector bosons responsible for the mediation of the interaction, we are interested in the construction of effective actions from fundamental ones. Fundamental does not mean that the starting point of the construction must be an renormizable theory living on the continuum. It suffices to start with an action which is by itself an effective action with a higher cut-off. An action with a higher cut-off will be called the fundamental one, the action derived from it with the lower cut-off the effective action.

The effective actions can be derived in momentum or real space. In this work, only the latter case will be considered. The method used is called the real space renormalization group. It is conceptually clear but suffers from errors introduced by truncations of the effective action which are necessary for reasons of practicality.

1.3.1 Effective Actions

As the effective action is computed from the fundamental one by integrating out high frequency modes, it will depend on low frequency fields Φ. These are defined as functions of the fundamental fields ϕ,

$$\Phi = C(\phi).$$

Φ is called the block spin. It is not necessarily of the same type as the fundamental field, and it may be composite.

C is some kind of average operator. Given the fundamental action $S[\phi]$, defined for fields on a fundamental lattice Λ, we introduce an effective action $S_{\text{eff}}[\Phi]$ as

$$e^{-S_{\text{eff}}[\Phi]} = \int \mathcal{D}\phi P(C(\phi), \Phi) e^{-S[\phi]}, \tag{1.9}$$

with

$$\int \mathcal{D}\Phi P(C(\phi), \Phi) = 1. \tag{1.10}$$

P is called the weight functional. It imposes a block spin condition on configurations ϕ.

Integrating (1.9) over Φ and exchanging the sequence of integration, we get with the help of (1.10)

$$\begin{aligned} \mathcal{Z}_{\text{eff}} = \int \mathcal{D}\Phi e^{-S_{\text{eff}}[\Phi]} &= \int \mathcal{D}\phi \int \mathcal{D}\Phi P(C(\phi), \Phi) e^{-S[\phi]} \\ &= \int \mathcal{D}\phi e^{-S[\phi]} = \mathcal{Z}. \end{aligned}$$

The partition functions of the fundamental and effective actions are equal.

$P(C(\phi), \Phi)$ as a function of $C(\phi)$ should be centered around Φ. It should respect the symmetries of the fundamental action if the effective action is supposed to preserve them.

Different choices are possible. A good one is a δ-function

$$P(C(\phi), \Phi) = \delta(C(\phi) - \Phi),$$

because then expectation values of observables which depend on ϕ only through Φ must be identical whether computed with S or S_{eff}. This prepares best for accurate tests of the effective action.

Another choice is a Gaussian block spin condition. There one uses a Gaussian δ_κ instead of the δ-function,

$$P(C(\phi), \Phi) = \delta_\kappa(C(\phi) - \Phi) \tag{1.11}$$

$$= \left(\frac{\kappa a^d}{2\pi}\right)^{\frac{L}{2}} e^{-\frac{\kappa}{2}\langle C(\phi)-\Phi, C(\phi)-\Phi\rangle}. \tag{1.12}$$

a denotes the lattice spacing of the fundamental lattice Λ, d is the dimension of the Euclidean space and L is the number of lattice points.

While the δ-function ensures $C(\phi) = \Phi$, in the Gaussian case the configurations of the fundamental field with average $C(\phi)$ deviating from Φ are also taken into account. The width of the Gaussian, which determines how important deviations of $C(\phi)$ from Φ are, is governed by the parameter κ. In the limit $\kappa \to \infty$, the δ-block spin is restored.

So the effective action is defined by choosing

- a block spin field Φ,
- the averaging operator C,
- the weight functional $P(C(\phi), \Phi)$.

A bad choice of one of these ingredients of (1.9) can be detected by a lack of locality of the effective action.

It is usually not possible to solve (1.9) exactly. Some approximations are needed.

1.3.2 Calculation of an Effective Action

One starts with an action $S[\phi]$ defined on the fundamental lattice Λ with spacing a. The lattice sites $z \in \Lambda$ are grouped to form disjoint (hypercubic) blocks of sidelength $s \cdot a$, $s \in \mathbb{N}$. These blocks are identified with sites x of a coarser lattice $\tilde{\Lambda}$ with lattice spacing $\tilde{a}$, called the block lattice. We write

$$z \in x,$$

if $z \in \Lambda$ belongs to the block x identified with site $x \in \tilde{\Lambda}$. The block containing z is also called $[z]$.

$$z \in x \Leftrightarrow [z] = x.$$

The block spin field lives on the block lattice $\tilde{\Lambda}$, hence its name.

To solve (1.9), some parametrization of the fundamental field ϕ in terms of a low frequency part Ψ, called the background field, and a high frequency field ζ is needed. Both are defined on Λ.

Consider first a linear mapping $C(\phi)$ [3],

$$C(\phi)(x) = \int_z C(x,z)\phi(z) \equiv C\phi(x).$$

The simplest choice is some kind of block average, denoted $\mathcal{C}$,

$$\mathcal{C}\phi(x) = \operatorname*{av}_{z \in x} \phi(z), \tag{1.13}$$

[3] For the σ-model we will use a nonlinear C. But a linearization of the block spin condition will take us back to linear mappings.

with kernel

$$\mathcal{C}(x,z) = \begin{cases} \tilde{a}^{-d} & \text{if } z \in x \\ 0 & \text{otherwise.} \end{cases}$$

The background field Ψ should be determined by the block spin field Φ,

$$\Psi = \Psi[\Phi].$$

A field ζ adds the high frequency part of ϕ to Ψ,

$$\phi = \phi[\Psi[\Phi], \zeta].$$

The effective action is defined by an auxiliary theory,

$$e^{-S_{\text{eff}}[\Phi]} = \int \mathcal{D}\zeta P(C(\phi[\Psi[\Phi],\zeta]),\Phi) e^{-S[\phi[\Psi[\Phi],\zeta]]}. \tag{1.14}$$

Φ enters as an external field which couples to the fluctuation field ζ.

There are different ways to determine the background field Ψ.

- According to Kupiainen and Gawedzki the background field Ψ is obtained from the block spin field Φ by interpolation with a kernel $\mathcal{A}_{\text{KG}}$,

  $$\Psi(z) = \int_{x \in \tilde{\Lambda}} \mathcal{A}_{\text{KG}}(z,x)\Phi(x) \equiv \mathcal{A}_{\text{KG}}\Phi(z).$$

  The field split reads

  $$\phi(z) = \mathcal{A}_{\text{KG}}\Phi(z) + \zeta(z),$$

  if ϕ takes its values in $\mathbb{R}^D$.

  Consider a free field theory with action

  $$\begin{aligned} S_0[\phi] &= \frac{1}{2}\langle \phi, v^{-1}\phi \rangle \\ &= \int_z \int_w \phi(z) v^{-1}(z,w) \phi(w). \end{aligned}$$

Inserting S_0 in (1.9) yields for the exponent in the case of a Gaussian block spin

$$\begin{aligned}
& -\frac{\kappa}{2}\langle \mathcal{C}\phi - \Phi, \mathcal{C}\phi - \Phi\rangle - \frac{1}{2}\langle \phi, v^{-1}\phi\rangle \\
= \; & -\frac{\kappa}{2}\langle \mathcal{C}\mathcal{A}_{\mathrm{KG}}\Phi + \mathcal{C}\zeta - \Phi, \mathcal{C}\mathcal{A}_{\mathrm{KG}}\Phi + \mathcal{C}\zeta - \Phi\rangle - \frac{1}{2}\langle \mathcal{A}_{\mathrm{KG}}\Phi + \zeta, v^{-1}(\mathcal{A}_{\mathrm{KG}}\Phi + \zeta)\rangle \\
= \; & -\frac{\kappa}{2}\langle \mathcal{C}\mathcal{A}_{\mathrm{KG}}\Phi, \mathcal{C}\mathcal{A}_{\mathrm{KG}}\Phi\rangle - \frac{\kappa}{2}\langle \mathcal{C}\zeta, \mathcal{C}\zeta\rangle - \frac{\kappa}{2}\langle \Phi, \Phi\rangle \\
& -\kappa\langle \mathcal{C}\mathcal{A}_{\mathrm{KG}}\Phi, \mathcal{C}\zeta\rangle + \kappa\langle \mathcal{C}\mathcal{A}_{\mathrm{KG}}\Phi, \Phi\rangle + \kappa\langle \mathcal{C}\zeta, \Phi\rangle \\
& -\frac{1}{2}\langle \mathcal{A}_{\mathrm{KG}}\Phi, v^{-1}\mathcal{A}_{\mathrm{KG}}\Phi\rangle - \frac{1}{2}\langle \zeta, v^{-1}\zeta\rangle - \langle \zeta, v^{-1}\mathcal{A}_{\mathrm{KG}}\Phi\rangle \\
= \; & -\frac{1}{2}\left(\kappa\langle \mathcal{C}\mathcal{A}_{\mathrm{KG}}\Phi, \mathcal{C}\mathcal{A}_{\mathrm{KG}}\Phi\rangle + \kappa\langle \Phi, \Phi\rangle - 2\kappa\langle \mathcal{C}\mathcal{A}_{\mathrm{KG}}\Phi, \Phi\rangle + \langle \mathcal{A}_{\mathrm{KG}}\Phi, v^{-1}\mathcal{A}_{\mathrm{KG}}\Phi\rangle\right) \\
& -\frac{1}{2}\left(\kappa\langle \mathcal{C}\zeta, \mathcal{C}\zeta\rangle + \langle \zeta, v^{-1}\zeta\rangle\right) \\
& -\frac{1}{2}\left(2\kappa\langle \mathcal{C}\mathcal{A}_{\mathrm{KG}}\Phi, \mathcal{C}\zeta\rangle - 2\kappa\langle \mathcal{C}\zeta, \Phi\rangle + 2\langle \zeta, v^{-1}\mathcal{A}_{\mathrm{KG}}\Phi\rangle\right)
\end{aligned}$$

The propagator Γ_{KG} for the fluctuation field ζ can be read off as

$$\Gamma_{\mathrm{KG}} = \left(v^{-1} + \kappa\mathcal{C}^\dagger\mathcal{C}\right)^{-1}. \tag{1.15}$$

Γ_{KG} is self adjoint.

One demands that the mixed terms involving ζ and Φ vanish,

$$\langle \zeta, [\kappa\mathcal{C}^\dagger\mathcal{C}\mathcal{A}_{\mathrm{KG}} + v^{-1}\mathcal{A}_{\mathrm{KG}} - \kappa\mathcal{C}^\dagger]\Phi\rangle = 0,$$

for any configuration Φ and ζ. Therefore,

$$\underbrace{[\kappa\mathcal{C}^\dagger\mathcal{C} + v^{-1}]}_{\Gamma_{\mathrm{KG}}^{-1}}\mathcal{A}_{\mathrm{KG}} - \kappa\mathcal{C}^\dagger = 0.$$

This yields for $\mathcal{A}_{\mathrm{KG}}$,

$$\mathcal{A}_{\mathrm{KG}} = \kappa\Gamma_{\mathrm{KG}}\mathcal{C}^\dagger.$$

Given the free propagator v^{-1} and the averaging kernel $\mathcal{C}$ this fixes our choice of $\mathcal{A}_{\mathrm{KG}}$ for any κ.

The form of the effective action is the same as the one we started with because the integral over the fluctuation field ζ is a Gaussian integral with covariance Γ_{KG}, and this contributes only a field independent constant to the effective action. The new propagator, the block spin propagator u, comes out as

$$\begin{aligned}
u^{-1} &= \kappa(\mathcal{C}\mathcal{A}_{\mathrm{KG}})^\dagger\mathcal{C}\mathcal{A}_{\mathrm{KG}} + \kappa - 2\kappa\mathcal{C}\mathcal{A}_{\mathrm{KG}} + \mathcal{A}_{\mathrm{KG}}^\dagger v^{-1}\mathcal{A}_{\mathrm{KG}} \\
&= \underbrace{\mathcal{A}_{\mathrm{KG}}^\dagger\Gamma_{\mathrm{KG}}^{-1}}_{\kappa\mathcal{C}}\mathcal{A}_{\mathrm{KG}} + \kappa - 2\kappa\mathcal{C}\mathcal{A}_{\mathrm{KG}} \\
&= \kappa\left(1 - \mathcal{C}\mathcal{A}_{\mathrm{KG}}\right).
\end{aligned}$$

The expressions for $\mathcal{A}_{\rm KG}$ and u^{-1} are not useful in the limit $\kappa \to \infty$. But we are also interested in the case of a δ-block spin. Therefore we have to examine the terms further.

$$\begin{aligned}
\mathcal{A}_{\rm KG} &= \kappa\Gamma_{\rm KG}\mathcal{C}^\dagger \\
\Leftrightarrow \Gamma_{\rm KG}^{-1}\mathcal{A}_{\rm KG} &= \kappa\mathcal{C}^\dagger \\
\Leftrightarrow v\Gamma_{\rm KG}^{-1}\mathcal{A}_{\rm KG} &= \kappa v\mathcal{C}^\dagger \\
\Leftrightarrow (1-\kappa v\mathcal{C}^\dagger\mathcal{C})\mathcal{A}_{\rm KG} &= \kappa v\mathcal{C}^\dagger \\
\Leftrightarrow \mathcal{A}_{\rm KG} &= \kappa v\mathcal{C}^\dagger(1-\mathcal{C}\mathcal{A}_{\rm KG}) \\
&= v\mathcal{C}^\dagger u^{-1}
\end{aligned}$$

This can be inserted in u^{-1},

$$\begin{aligned}
u^{-1} &= \kappa(1-\mathcal{C}v\mathcal{C}^\dagger u^{-1}) \\
\Leftrightarrow u &= \mathcal{C}v\mathcal{C}^\dagger + \frac{1}{\kappa}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\Gamma_{\rm KG}^{-1} &= v^{-1} + \kappa\mathcal{C}^\dagger\mathcal{C} \\
\Leftrightarrow v &= \Gamma_{\rm KG} + \kappa\Gamma_{\rm KG}\mathcal{C}^\dagger\mathcal{C}v,
\end{aligned}$$

leads to

$$\begin{aligned}
\Gamma_{\rm KG} &= v - \kappa\Gamma_{\rm KG}\mathcal{C}^\dagger\mathcal{C}v \\
&= v - \mathcal{A}_{\rm KG}u\mathcal{A}_{\rm KG}^\dagger.
\end{aligned}$$

As a consequence

$$\begin{aligned}
v^{-1}\Gamma_{\rm KG} &= 1 - v^{-1}\mathcal{A}_{\rm KG}u\mathcal{A}_{\rm KG}^\dagger \\
&= 1 - \mathcal{C}^\dagger\mathcal{A}_{\rm KG}^\dagger,
\end{aligned}$$

and

$$\Gamma_{\rm KG}v^{-1} = 1 - \mathcal{A}_{\rm KG}\mathcal{C}.$$

The various results can be summarized,

$$\begin{aligned}
\mathcal{A}_{\rm KG} &= v\mathcal{C}^\dagger u^{-1}, \\
u &= \mathcal{C}v\mathcal{C}^\dagger + \frac{1}{\kappa}, \\
\Gamma_{\rm KG} &= \left(v^{-1} + \kappa\mathcal{C}^\dagger\mathcal{C}\right)^{-1} \\
&= (1-\mathcal{A}_{\rm KG}\mathcal{C})v \\
&= v(1-\mathcal{C}^\dagger\mathcal{A}_{\rm KG}^\dagger).
\end{aligned} \tag{1.16}$$

The general properties of these kernels and their Fourier representation are well known [13, 18, 32] for the choice (1.13) of the averaging operator $\mathcal{C}$.

In the limit $\kappa \to \infty$ we get

$$\begin{aligned} \mathcal{C}\Gamma_{\text{KG}} &= \frac{1}{\kappa}\mathcal{A}^{\dagger}_{\text{KG}} \to 0, \\ \Gamma_{\text{KG}}\mathcal{C}^{\dagger} &= \frac{1}{\kappa}\mathcal{A}_{\text{KG}} \to 0, \\ \mathcal{C}\mathcal{A}_{\text{KG}} &= 1 - \frac{1}{\kappa}u^{-1} \to 1. \end{aligned} \tag{1.17}$$

Once the background field and hence the fluctuation field are determined, one may proceed in the following way for interacting theories with potential $V[\phi]$.

$$\begin{aligned} e^{-S_{\text{eff}}[\Phi]} &= \int \mathcal{D}\phi\delta_{\kappa}(\mathcal{C}\phi - \Phi)e^{-S[\phi]} \\ &= e^{-\frac{1}{2}\langle\Phi,u^{-1}\Phi\rangle}\int \mathcal{D}\zeta e^{-\frac{1}{2}\langle\zeta,\Gamma^{-1}_{\text{KG}}\zeta\rangle}e^{-V[\mathcal{A}_{\text{KG}}\Phi+\zeta]} \\ &= e^{-\frac{1}{2}\langle\Phi,u^{-1}\Phi\rangle}\mathcal{N}^{-1}\int d\mu_{\Gamma_{\text{KG}}}(\zeta)e^{-V[\mathcal{A}_{\text{KG}}\Phi+\zeta]} \\ &= e^{-\frac{1}{2}\langle\Phi,u^{-1}\Phi\rangle - V_{\text{eff}}[\mathcal{A}_{\text{KG}}\Phi]}. \\ \mathcal{Z}_{\text{eff}} &= \mathcal{N}'^{-1}\int d\mu_u(\Phi)e^{-V_{\text{eff}}[\mathcal{A}_{\text{KG}}\Phi]}. \end{aligned}$$

The kinetic term of the effective action is fixed by the definition of $\mathcal{A}_{\text{KG}}$. $V_{\text{eff}}[\mathcal{A}_{\text{KG}}\Phi]$ is an effective potential defined by

$$V_{\text{eff}}[\mathcal{A}_{\text{KG}}\Phi] = -\ln\int d\mu_{\Gamma_{\text{KG}}}(\zeta)e^{-V[\mathcal{A}_{\text{KG}}\Phi+\zeta]} = -\ln\langle e^{-V[\mathcal{A}_{\text{KG}}\Phi+\zeta]}\rangle_{\Gamma_{\text{KG}}},$$

where $\langle . \rangle_{\Gamma_{\text{KG}}}$ stands for a Gaussian expectation value with respect to ζ and covariance Γ_{KG}. Its perturbation expansion in compact notation,

$$e^{-V_{\text{eff}}[\mathcal{A}_{\text{KG}}\Phi]} = e^{\frac{1}{2}\langle\frac{\delta}{\delta\zeta},\Gamma_{\text{KG}}\frac{\delta}{\delta\zeta}\rangle}e^{-V[\mathcal{A}_{\text{KG}}\Phi+\zeta]}\Big|_{\zeta=0},$$

is explained in appendix B.

One can interpret the choice of $\mathcal{A}_{\text{KG}}$ in the Gawedzki-Kupiainen formalism in the following way. The condition that the mixed terms in the expansion of the free action and the contributions from the Gaussian block spin condition vanish, is equivalent to the statement that the background field $\Psi = \mathcal{A}_{\text{KG}}\Phi$ extremizes the free action plus the contributions of the Gaussian weight functional (1.12).

Using a δ-block spin, one enforces the condition $\mathcal{C}\Psi = \Phi$ or equivalently $\mathcal{C}\mathcal{A}_{\text{KG}} = 1$. Then the background field extremizes the free action under the constraint $\mathcal{C}\Psi = \Phi$.

- Another way to determine the background field Ψ is to extremize the *full* action of a theory. A Gaussian block spin adds again some terms to the action S in (1.9) which have to be taken into account in this procedure, while a δ-block spin demands a field Ψ extremizing the full action with an average equal to the block spin.

  For the nonlinear σ-model this choice is appropriate. The interaction of this model is not mediated through a potential term, but by a constraint the fields must fulfill. This makes the split of the action in a free and interacting part not feasible without some assumptions about the fields.

  Gauge theories are another example for theories where this extremalization of the full action is the starting point of the calculation of an effective action. There the expansion of the full action around the minimum configurations are used for reasons of gauge invariance.

  So we proceed in two steps. First one computes the classical perfect action

  $$S_{\mathrm{cl}}[\Phi] = S[\Psi[\Phi]] = \mathrm{extr}.S[\phi].$$

  This fixes the background field $\Psi[\Phi]$ for given block spin.

  Given again a split of the fundamental field, $\phi[\Psi, \zeta]$, one computes loop corrections by expanding around the background field,

  $$S[\phi] = S[\Psi] + \frac{1}{2}\zeta S''[\Psi]\zeta + \ldots.$$

  A one loop calculation yields

  $$\begin{aligned} S_{\mathrm{eff}}[\Phi] &= S_{\mathrm{eff}}[\Psi[\Phi]] \\ &= S_{\mathrm{cl}}[\Psi] - \frac{1}{2}\mathrm{tr}\,\ln\Gamma[\Psi], \end{aligned}$$

  with

  $$\Gamma[\Psi] = \left(S''[\Psi] + \kappa\mathcal{C}^\dagger\mathcal{C}\right)^{-1}. \tag{1.18}$$

  In [18] it is shown that it is possible to truncate the effective action in a way which allows a recursion of this saddle point approximation. The iteration involves recursion relations for the first and second derivative of the action evaluated at *constant* fields.

The extremalization of the full action of the nonlinear σ-model will determine the background field Ψ in this work. Since the fundamental variables of this model are subject to the constraint $\phi^2 = 1$, the background field has to be constructed as an element of the sphere, too.

One should notice that (1.15) and (1.18) both describe high frequency propagators with an infrared cut-off which suppresses long range correlations. The term $\kappa\mathcal{C}^\dagger\mathcal{C}$ plays the role of a mass.

Although we will not use the Gawedzki-Kupiainen formalism as a starting point, the high frequency propagator of the nonlinear σ-model turns out to be in the fashion of (1.15). But there are field dependent modifications of the infrared cut-off. To extract this field dependence, one is forces to expand the full high frequency propagator around Γ_{KG}. This expansion involves derivatives of Γ_{KG} that can be expressed in terms of $\mathcal{A}_{\text{KG}}$. This motivates the introduction of the interpolation kernel.

1.3.3 The Renormalization Group Flow

The procedure described so far allows to calculate an effective action S_{eff} from a given fundamental action S with higher cut-off. It is called a renormalization group step. In general new couplings between the fields are introduced by integrating over the high frequency modes of the fundamental fields. Since this integration is done perturbatively in most cases, it is convenient to perform the integral in many steps. One introduces a sequence of lattices

$$\Lambda \equiv \Lambda^0 \to \Lambda^1 \to \cdots \to \Lambda^N \equiv \tilde{\Lambda},$$

with increasing lattice spacings

$$a \equiv a^0 \to a^1 \to \cdots \to a^N \equiv \tilde{a}.$$

The lattice spacings are related by

$$a^{i+1} = s \cdot a^i, \qquad s \in \mathbb{N}.$$

Starting with the fundamental action S, renormalization group steps induce a sequence of actions S^i with lower and lower cut-off,

$$S \equiv S^0 \to S^1 \to \cdots \to S^N \equiv S_{\text{eff}}.$$

As mentioned before in section 1.2.1, the lattice spacing a^i can be set to 1 by rescaling, if one considers only one such theory at a time. The actions can be compared, since they all live on the unit lattice. To parametrize the sequence of actions one introduces running coupling constants.

Each action S^i is described by a set of coupling constants $\{K^i_j\}$,

$$S^i[\phi_i] = \sum_j K^i_j \theta_j(\phi^i).$$

$\theta_j(\phi^i)$ is a function of the field ϕ^i which describes the nature of the coupling. A general example for $\theta(\phi)$ is $[\phi]^m[\nabla_\mu \phi]^n$.

Renormalization group steps map the set of couplings $\{K^i_j\}$ of action S^i on a new set $\{K^{i+1}_j\}$ describing the (effective) action S^{i+1}. Thus the iteration of renormalization group steps leads to a flow of the coupling constants, hence the name running coupling constants.

$$\{K^0_j\} \to \{K^1_j\} \to \cdots \to \{K^N_j\}. \tag{1.19}$$

This flow is called the renormalization group flow. In the infinite dimensional space of couplings, the renormalization group flow traces a trajectory. Since all the points on this renormalization group trajectory are connected by renormalization group steps, they describe the same physics.

But because of the rescaling, the correlation length is reduced by a factor s^{-1} in each step,

$$\xi \to \frac{\xi}{s}.$$

Thus the system is driven away from criticality. One should remember that the high frequency propagator contains an infrared regulator. This, and the fact that the block spin field which couples to the high frequency field ζ in (1.14) is an external field make the integral (1.14) noncritical. That is expected because the block spin field constrains the average of fundamental fields via the weight functional and disrupts long range order.

So if the theory on the finest lattice Λ is near its critical limit, it describes continuum physics. But it is connected to a noncritical system by a sequence of renormalization group steps performed by noncritical integrals. In principle this allows to simulate continuum physics even on coarse lattice. Coarse means that on these lattices the correlation length measured in units of the lattice spacing is small. No critical slowing down will be encountered.

Although the renormalization group flow introduces new couplings making the simulation of the action on the coarse lattice more involved, there can be a net gain.

While these ideas follow directly from Wilson's renormalization group transformation, it is very difficult to construct an effective action which is connected to the associated continuum theory. The integral (1.14) cannot be solved in general without some approximations [4] .

First of all one has to use perturbation theory. But perturbation theory can be made sufficiently accurate by using small steps in length scale. This is true because the phase space of fluctuation fields increases for large steps.

The second problem is the parametrization of the sequence of effective actions. All kind of couplings may appear in the renormalization group flow. So truncations of the effective action will be needed. It is not save to neglect the contributions of an effective action which is small in a perturbative sense, for example higher powers of fields.

A truncation exploiting the smoothness of the background field is proposed in [18]. There one neglects higher orders of $\nabla_\mu \Psi$, but not derivatives of higher order $\nabla^n \Psi$. In contrast to the saddle point approximation, this truncation becomes more and more accurate for large scaling factors s. This is due to the fact that Ψ minimizes the full action for given block spin. The kinetic term in the action requires smooth minimizing background fields on length scale a^i for large $s = a^{i+1}/a^i$.

[4] As the integral is noncritical, it may be solved by Monte-Carlo simulations. This strategy is called Monte-Carlo renormalization group (MCRG)

1.4 The Nonlinear σ-model in 2 Dimensions

The $O(\mathcal{N}+1)$-symmetric nonlinear σ-model is defined by the action

$$S[\phi] = \frac{\beta}{2} \int \partial_\mu \phi(z) \cdot \partial_\mu \phi(z) dz.$$

The field ϕ is a normalized $\mathcal{N}+1$ component vector. It is subject to the constraint

$$\phi^2(z) = 1, \tag{1.20}$$

or $\phi \in S^{\mathcal{N}}$.

A standard lattice discretization of the action is given by substituting lattice derivatives for ∂_μ,

$$\begin{aligned} S[\phi] &= \frac{\beta}{2} \int_z \nabla_\mu \phi(z) \cdot \nabla_\mu \phi(z) \\ &= -\int_z \phi \cdot \triangle \phi(z). \end{aligned} \tag{1.21}$$

This discretized version of the σ-model is also known as the (generalized) Heisenberg model.

While the action has the form of the one known from massless free field theories, the constraint introduces a nontrivial interaction. The factor in front of the integral in (1.21) is a coupling. It cannot be removed by rescaling of the field, since this merely leads to a shift of β from the action to the constraint.

One can easily convince oneself that solving the constraint with respect to one field component we arrive at a nontrivial interaction between the remaining components.

The partition function is

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S[\phi]}.$$

The interaction is mediated by the measure $\mathcal{D}\phi$ which is the uniform measure on the sphere,

$$\mathcal{D}\phi = \prod_{z \in \Lambda} d\phi(z) \delta(\phi^2(z) - 1).$$

There are two ways to make the interaction explicit.

Solving the constraint: Introduce a basis and solve the constraint for ϕ^0.

$$\phi(z) = \begin{pmatrix} \sqrt{1-\pi^2(z)} \\ \pi(z) \end{pmatrix}$$

The derivative of the zeroth component becomes

$$\nabla_\mu \left(\sqrt{1-\pi^2(z)} \right) = \frac{-\pi^i(z) \nabla_\mu \pi^i(z)}{\sqrt{1-\pi^2(z)}}.$$

An explicit interaction is seen in the action,

$$\begin{aligned} S[\pi] &= \frac{\beta}{2}\nabla_\mu\pi^0\nabla_\mu\pi^0 + \nabla_\mu\pi^i\nabla_\mu\pi^i \\ &= \frac{\beta}{2}\nabla_\mu\pi^i\left(\delta^{ij} + \frac{\pi^i\pi^j}{1-(\pi)^2}\right)\nabla_\mu\pi^j. \end{aligned}$$

The measure is

$$\prod_{z\in\Lambda}\frac{d\pi(z)}{\sqrt{1-(\pi(z))^2}} = e^{-\frac{1}{2}\sum_{z\in\Lambda}\ln(1-(\pi(z)^2)}.$$

It is well defined on the lattice.

Introduction of a Lagrange multiplier $\alpha(z)$: One can represent the δ-function in the measure by an interaction of ϕ with a Lagrange multiplier α. In the partition function an additional integral over α is introduced. But as the integral over ϕ is a Gaussian integral, it can be solved. The partition function becomes an integral over α with a nontrivial action. This is normally the starting point for investigations concerned with large $\mathcal{N}$.

In this work we will derive an effective action for the $O(\mathcal{N}+1)$-symmetric nonlinear σ-model in two dimensions. The case $d=2$ is of particular interest for several reasons.

Although the σ-model is of no direct relevance for particle physics, it has been a favorite theoretical laboratory for a long time. This is due to the striking similarities with $SU(\mathcal{N}+1)$ Yang-Mills theories in four space-time dimensions. First of all, it is renormizable.

For $\mathcal{N}>1$ the model is known to be asymptotically free [29]. Moreover, it has a mass gap which appears in the exponential decay of the long distance behaviour of the two point function.

Because of this and due to the existence of rigorous solutions, σ-models served as testing facilities for several nonperturbative techniques, like $\frac{1}{\mathcal{N}}$-expansion or operator product expansions.

Two special members of this class of models have to be mentioned.

While for $\mathcal{N}>1$ no phase transition for finite β is encountered, the $O(2)$ model exhibits such a transition at some finite $\beta_{\text{cr.}}$. This phase transition, known as the Kosterlitz-Thouless transition, is of a quite astonishing nature. It seems to be in conflict with the Mermin-Wagner theorem which allow no spontaneous symmetry breaking in two dimensions. But it can be explained as an unbinding process for vortex-antivortex pairs. Thus no naive spontaneous symmetry breaking is involved.

The other one is the $O(3)$ model. It is conformaly invariant, exhibits nontrivial topological structure and admits instanton solutions on a classical level. On quantum level its vacuum structure is nontrivial.

Last but not least, for $\mathcal{N}\to\infty$ the $O(\mathcal{N}+1)$ model is exactly solvable.

An effective action for the 2-dimensional nonlinear σ-model in 1-loop approximation was first computed by Polyakov [29]. He computed those terms which diverge as $\ln(\tilde{a}/a)$

when $a \mapsto 0$. These terms do not depend on the details of the infrared cut-off of the auxiliary theory and are therefore independent of a choice of block spin. In this work we compute also the finite parts. They depend on the choice of block spin through a quantity $M(\Phi)$ which plays the role of a mass in the auxiliary theory but is field dependent. It turns out that this field dependence is easier to handle for Gaussian block spin.

Polyakov's result has the property that it differs from the tree approximation to the effective action by a coupling constant renormalization only. A few years ago the asymptotic freedom of the model caused new interest in them. In [19] the authors proposed a *perfect* action for the $O(\mathcal{N}+1)$ model. They claimed on the basis of general arguments that even the full one loop approximation merely changes β. But this claim is not correct, and it was later withdrawn [21].

In contrast to a perfect action which reproduces the results of a theory with higher cut-off on a coarse lattice (see section 1.3.3), the authors stated that their action shows no lattice artefacts at all.

It was said above that such actions exist as a consequence of Wilson's renormalization group philosophy. Actions on the renormalized trajectory, connected to continuum physics by an infinite number of renormalization group steps, should be free of artefacts. The problem is to find and parametrize actions which are close enough to the renormalized trajectory for practical purposes. Hasenfratz and Niedermayer suggest the following construction for asymptotically free theories.

Consider again the renormalization group flow of couplings (1.19). A fixed point of the transformation is a set of couplings with the property

$$K_j^{*i+1} = K_j^{*i}.$$

The couplings do not change under a renormalization group step.

They define a fixed point action S^* which is thus invariant under a RG step,

$$S^*[\phi] \to S^*[\Phi].$$

Since the correlation length ξ is decreased by a factor s^{-1} in each step, a fixed point action has either vanishing or infinite correlation length.

Assume that we are on the critical surface with $\xi = \infty$. For suitable parameters we arrive after an infinite number of renormalization group steps at a fixed point action (FPA).

Now let us perform a tiny step away from the fixed point action in a relevant direction, still iterating the renormalization group procedure. Then we follow the renormalized trajectory which leads us to shorter correlation length.

The fixed point of a renormalization group transformation and the renormalized trajectory along which the action runs under repeated steps form a perfect quantum action without lattice artefacts. The fixed point action itself reproduces all the important properties of the classical continuum action. It is a classical perfect action.

The authors of [19] claim that for asymptotically free theories the fixed point action is a sufficient approximation for the renormalized trajectory. These theories have a critical

surface at $\beta = \infty$. For large β a renormalization group step

$$e^{-\beta' \mathcal{H}'[\Phi]} = \int \mathcal{D}\phi e^{-\beta[\mathcal{H}[\phi]+T[\Phi,\phi]]}$$

is exactly performed in a saddle point approximation by minimizing

$$\mathcal{H}[\phi] + T[\Phi, \phi] \tag{1.22}$$

for a given Φ. We use the notation $S[\phi] = \beta\mathcal{H}[\phi]$ which extracts an over all factor β from the action. T is the contribution from the weight functional. In the limit $\beta \to \infty$ loop corrections can be neglected.

In conclusion, a classical fixed point action is determined by

$$\mathcal{H}^*[\Phi] = \min_{\phi}.[\mathcal{H}^*[\phi] + T[\Phi, \phi]]. \tag{1.23}$$

This action describes continuum physics on tree level. The solution of (1.23) is a problem of classical field theory. In [19] an action is stated which solves (1.23) and involves only a manageable number of couplings.

Still one needs to proceed to finite correlation length on the renormalized trajectory. The uncontrolled assumption of [19] is that it suffices to tune β to finite values while keeping the other couplings of the classically perfect action $\mathcal{H}^*$ fixed. This assumes that the classical perfect action is close to the renormalized trajectory for finite β too.

The present calculation was initiated to test this statement.

We will calculate a renormalization group step for the nonlinear σ-model. The result will be an effective action in Wilson's sense. Using a normalized $\mathcal{N}+1$-component block spin field Φ, it is defined as

$$e^{-S_{\text{eff}}[\Phi]} = \int \mathcal{D}\phi \prod_x \delta(C\phi(x), \Phi(x)) e^{-S[\phi]} \; ; \tag{1.24}$$

$$\mathcal{D}\phi = \prod_z d\phi(z). \tag{1.25}$$

$d\phi$ is the uniform measure on the sphere $S^{\mathcal{N}}$. The δ-term is the $\mathcal{N}$-dimensional δ-function on the sphere. It is given by

$$\int d\phi_1 f(\phi_1)\delta(\phi_1, \phi_2) = f(\phi_2)$$

for test functions on the sphere.

A block spin Φ of unit length is the appropriate choice as long as β is not very small. One may imagine that $S_{eff}[\Phi]$ is obtained from another effective action which depends on a field of fluctuating length by integrating out the radial modes. This is legitimate, i.e. produces no nonlocalities, so long as the radial modes have mass above the cutoff $\tilde{a}^{-1}$. The

exact Wilson effective action depends on a choice of the operator C. Since $\Phi(x)$ is a unit vector, C is necessarily nonlinear. We choose

$$\Phi(x) = C\phi(x) \equiv \frac{\mathcal{C}\phi(x)}{|\mathcal{C}\phi(x)|} \ , \tag{1.26}$$

where $\mathcal{C}$ is the linear operator which averages over blocks, see (1.13). This special choice of $\mathcal{C}$ will be used to linearize the block spin condition in the next section 2.1.

To evaluate (1.24) we will use a saddle point approximation. In general this method is valid for small s. But we are interested in the effective action for large β. While the classical action is of order β, loop corrections are of order β^0. This makes the approximation accurate in the limit $\beta \to \infty$.

The exact result to one loop order has to be simplified for numerical investigations. This will be done in the fashion of [18]. Contributions of higher order in $\nabla\Psi$ will be neglected, but no polynomial approximation is necessary.

Thus our approximations are valid for large s. Nevertheless, further simplifications require smoothness of the block spin field.

A Gaussian block spin is considered too. It is defined by substituting δ_κ in (1.24).

1.4.1 Note on the Large Field Problem

Soon after rigorous work on the renormalization group started 20 years ago, it was recognized that one could not expect that the effective action would be local for completely arbitrary block spin configurations. This was termed the *large field problem*. A device to overcome this difficulty was proposed by Benfatto et al [5], and subsequently implemented in the work of Kupiainen and Gawedzki [14] and of Balaban [3]. It involves proofs that *large fields* in the above sense are very improbable. Fermi fields have no large field problem [11].

A *large field problem* can appear for fields which are not large in a naive sense. For instance, in 2-dimensional ϕ^4-theory with a distinct mexican hat potential (pronounced maximum at $\phi = 0$), the block spin $\Phi(x) \equiv 0$ is in the large field domain. Since $\Phi = 0$ is translation invariant, the auxiliary theory, whose field is the fluctuation field, ought to respect symmetry under block lattice translations, with symmetry group $(\tilde{a}\mathbb{Z})^2$. Numerical work by Grießl [17] showed that the symmetry was spontaneously broken (to $(2\tilde{a}\mathbb{Z})^2$). Such long range order in the auxiliary theory is incompatible with locality of the effective action.

The σ-model also has a large field problem for $\mathcal{N} > 1$. Divide the block lattice in black and white squares in a checkerboard fashion and consider the configuration $\Phi(x)$ which points "up" (in +0 -direction) on white squares, and down on the black ones. A particular extremizing background field Ψ has components

$$\begin{aligned} \Psi^0(z) &= \cos(\pi(z^1 + z^2)/\tilde{a}), \\ \Psi^1(z) &= \sin(\pi(z^1 + z^2)/\tilde{a}), \\ \Psi^i(z) &= 0 \qquad (i > 1) \ . \end{aligned} \tag{1.27}$$

But this is not unique. Continuous rotations in the space orthogonal to the 0-direction yield degenerate extrema. Therefore the auxiliary theory has a zero mode, its correlation functions will not decay quickly, and one cannot expect a local effective action for block spins very close to Φ. But note that these are "energetically" the most unfavorable block spin configurations of all; they are near maxima of the effective action.

Chapter 2

S_{eff} in a Saddle Point Approximation

2.1 Linearization of the Constraint

The Wilson effective action for the nonlinear σ-model is defined by (1.24).

A perturbative calculation is not straightforward because the argument of the δ-function is a nonlinear function of the field. It is nonlinear because of the definition of C.

Therefore a parametrization of an arbitrary fundamental field ϕ in terms of the background field Ψ and a fluctuation field ζ is needed such that the constraint becomes linear in ζ.

$$\phi = \phi[\Psi, \zeta].$$

As the fluctuation field describes the deviation of a fundamental field $\phi \in S^{\mathcal{N}}$ from a smooth field $\Psi \in S^{\mathcal{N}}$, it takes its values in a tangent space of the sphere $S^{\mathcal{N}}$ and adds the high frequency part of ϕ to Ψ. It has $\mathcal{N}$ components, and we will choose it so that

$$\zeta(z) \perp \Phi(x) \text{ for } z \in x \ . \tag{2.1}$$

Using this block wise unique tangent space, a linearization of the constraint is feasible. The linearization is possible because we consider only *disjoint* blocks.

Now we decompose $\phi(z)$ into components $\phi^{||}(z)$ and $\phi^{\perp}(z)$ parallel and perpendicular to $\Phi(x)$ for $z \in x$.

$$\phi^{||}(z) || \Phi(x) \text{ for } z \in x$$

and

$$\phi^{\perp}(z) \perp \Phi(x) \text{ for } z \in x \ .$$

The operator C was defined as

$$C\phi(x) \equiv \mathcal{C}\phi(x)/|\mathcal{C}\phi(x)| = \Phi(x),$$

with $\mathcal{C}$ being the average over *one* block. Thus

$$\mathcal{C}\phi(x) - \Phi|\mathcal{C}\phi|(x) = 0 \tag{2.2}$$

and

$$\mathcal{C}\phi^{\perp}(x) = 0. \tag{2.3}$$

The average of the components of ϕ perpendicular to Φ vanishes.

On the other hand, the eq. (2.2) can be read as a condition which defines the block spin Φ. The new block spin condition is

$$\mathcal{C}\phi - \Phi(\Phi \cdot \mathcal{C}\phi) = 0. \tag{2.4}$$

There is a Jacobian $J[\Psi, \zeta]$ to the transformation to this new condition.

(2.3) motivates the introduction of a fluctuation field ζ with

$$\zeta(z) \in T_{\Phi(x)}S^{\mathcal{N}} \text{ for } z \in x \ .$$

We parametrize

$$\begin{aligned} \phi^{\perp}(z) &= \Psi^{\perp}(z) + \zeta(z), \\ \phi^{||}(z) &= \left(1 - \phi^{\perp}(z)^2\right)^{\frac{1}{2}} \Phi(x). \end{aligned} \tag{2.5}$$

$C\Psi(x) = \Phi(x)$ implies $\mathcal{C}\Psi^{\perp}(x) = 0$ and therefore

$$\mathcal{C}\phi^{\perp}(x) = \mathcal{C}\Psi^{\perp}(x) + \mathcal{C}\zeta(x) = 0$$

yields

$$\mathcal{C}\zeta(x) = 0. \tag{2.6}$$

This constraint is linear in ζ. Later, a further linear transformation to variables $\xi \perp \Psi$ is considered.

With the parametrization (2.5) and the Jacobian $J[\Psi, \zeta]$ the effective action is determines by

$$e^{-S_{\rm eff}[\Phi]} = \int \prod_z d\zeta(z)\delta(\mathcal{C}\zeta)J(\Psi, \zeta)e^{-S(\phi[\Psi,\zeta])}. \tag{2.7}$$

Notation: It is convenient to use the following notation. Here and everywhere we use

$$\tilde{\Phi}(z) = \Phi(x) \text{ for } z \in x.$$

as a block wise constant interpolation of Φ to the fundamental lattice Λ. As we will need the product of $\tilde{\Phi}$ with Ψ, we write

$$\cos\theta(z) = \tilde{\Phi}(z) \cdot \Psi(z)$$

for this local scalar product.

ϕ^T will denote the transpose of a column vector ϕ. It is a row vector. $\phi_1^T(z)\phi_2(z)$ is a scalar, while $\phi_1(z)\phi_2^T(z)$ denotes a matrix.

The projector on $T_{\Psi(z)}S^{\mathcal{N}}$ is called $\pi(z)$,

$$\pi(z) = 1 - \Psi\Psi^T(z).$$

2.1.1 The Parametrization in Terms of ζ in a Moving Frame

The parametrization (2.5) can be expressed in a moving frame f

$$\begin{aligned} f_0(z) &= \tilde{\Phi}(z), \\ f_1(z) &= \frac{\Psi^\perp(z)}{|\Psi^\perp(z)|}, \\ \vec{f}(z) &= (f_2(z), \dots f_{\mathcal{N}}(z)) \text{ orthogonal.} \end{aligned} \tag{2.8}$$

As $\Psi^\perp(z)$ is the component of $\Psi(z)$ perpendicular to $\tilde{\Phi}(z)$, we see

$$f_1(z) = \frac{\Psi(z) - \tilde{\Phi}(z)(\tilde{\Phi}(z) \cdot \Psi(z))}{\sin\theta(z)}.$$

The moving frame $\{f_\alpha(z)\}$ depends on z as Ψ depends on z.

ζ can be expanded in this local basis

$$\zeta = (0, \zeta^1, \dots \zeta^{\mathcal{N}}) = (0, \zeta^1, \vec{\zeta}).$$

The parametrization reads

$$\begin{aligned} \phi^\perp &= (\sin\theta + \zeta^1) f_1 + \vec{\zeta} \cdot \vec{f}, \\ \phi^{||} &= \left[\cos\theta - \tan\theta\zeta^1 - \frac{1}{2}\frac{1}{\cos\theta}(\zeta)^2 - \frac{1}{2}\frac{\tan^2\theta}{\cos\theta}(\zeta^1)^2\right] f_0 + O\left((\zeta)^3\right) \end{aligned} \tag{2.9}$$

Or covariantly

$$\begin{aligned} \phi &= \Psi + \zeta - \frac{\Psi^\perp \cdot \zeta}{\cos\theta}\tilde{\Phi} \\ &\quad -\frac{1}{2}\left[\frac{(\zeta)^2}{\cos\theta} + \frac{(\Psi^\perp \cdot \zeta)^2}{\cos^3\theta}\right]\tilde{\Phi} + O\left((\zeta)^3\right). \end{aligned}$$

This parametrization is quite complicated. If one chooses another with fluctuation fields in the tangent space of Ψ, the parametrization is simpler. But the constraint on the fluctuation field becomes more involved. Nevertheless it is still linear.

2.1.2 The Jacobian J

The block spin condition

$$\Phi(x) = \frac{\mathcal{C}\phi(x)}{|\mathcal{C}\phi(x)|}$$

is equivalent to

$$F(\phi|\Phi) = \mathcal{C}\phi(x) - \Phi(x)(\Phi(x)\cdot\mathcal{C}\phi(x)) = 0.$$

Because of $\Phi \cdot F(\phi|\Phi) = 0$, there are only $\mathcal{N}$ independent conditions.

Here we determine the full Jacobian $J[\Psi,\zeta]$ associated to the parametrization which leads to this linear condition. It includes contributions from the measure.

As the block spin definition is implemented by δ-functions which are centered on blocks

$$\prod_x \delta(C\phi(x), \Phi(x)), \tag{2.10}$$

we can choose a local basis with

$$\Phi(x) = (1, 0, \ldots, 0)\,.$$

We denote by F^i the components of the new block spin condition.

$$F^i = \mathcal{C}\phi^i - \Phi^i\,(\mathcal{C}\phi\cdot\Phi) = 0, \qquad i = 1,\ldots\mathcal{N}.$$

The Jacobian is given by

$$J_0(\mathcal{C}\phi) = |\det\frac{\delta F^i}{\delta\Phi^j}|(\phi).$$

With

$$\frac{\delta\Phi^0}{\delta\Phi^j} = \frac{\delta\sqrt{1-\vec{\Phi}^2}}{\delta\Phi^j} = -\frac{\Phi^j}{\Phi^0}$$

we compute

$$\frac{\delta F^i}{\delta\Phi^j} = -\delta^{ij}\Phi^0\mathcal{C}\phi^0 + \Phi^i\frac{\Phi^j}{\Phi^0}\mathcal{C}\phi^0.$$

In our local basis the last term of this result vanishes

$$\frac{\delta F^i}{\delta\Phi^j} = -\delta^{ij}\Phi^0\mathcal{C}\phi^0.$$

Or covariantly

$$\frac{\delta F^i}{\delta\Phi^j} = -\delta^{ij}\Phi\cdot\mathcal{C}\phi.$$

Therefore

$$\begin{aligned} J_0(\phi) &= |\det \frac{\delta F^i}{\delta \Phi^j}|(\phi) = e^{\mathrm{tr}\,\ln(-\delta^{ij}\Phi\cdot\mathcal{C}\phi)} = e^{\mathcal{N}\ln\Phi\cdot\mathcal{C}\phi} \\ &= |\Phi\cdot\mathcal{C}\phi|^{\mathcal{N}}. \end{aligned} \tag{2.11}$$

In a 1-loop approximation, this result will only be needed to order $(\zeta)^0$.

A contribution from the measure on the sphere $d\phi$ is combined with J_0 to the full Jacobian J. The result is

$$J(\Psi,\zeta) = J^0(\mathcal{C}\phi)\prod_{z\in x}\left(1-(\Psi^\perp+\zeta)^2\right)^{-\frac{1}{2}}, \tag{2.12}$$

$$J^0(\mathcal{C}\phi) = \prod_x J_0(\mathcal{C}\phi(x)) = \prod_x(\Phi(x)\cdot\mathcal{C}\phi(x))^{\mathcal{N}} \tag{2.13}$$

As usual, Ψ is regarded as determined by Φ. In a 1-loop calculation, $J_0(\mathcal{C}\phi(x))$ is approximated by $J_0(\mathcal{C}\Psi(x))$. The last factor in J will be canceled when we transform to the ξ-variables in the next section.

For finite κ the Jacobian will be κ-dependent.

2.1.3 Parametrization of the Fundamental Field

The fluctuation field ζ allows a parametrization of the fundamental field in such a way that the block spin condition is linear in ζ. But the action of the σ-model involves derivatives of ϕ which contribute derivatives of ζ if one uses this field split. The constraint

$$\zeta(z)\cdot\Phi(x)=0, \text{ for } z\in x$$

causes jumps in the values of ζ at the block boundaries. These jumps carry over to the derivatives. To avoid this obstacle, we will make another linear transformation from $\zeta(z)\perp\Phi(x)$ to variables ξ with $\xi(z)\perp\Psi(z)$. The fundamental action $S(\phi)$ in (2.7) will be expanded in these variables. The block spin constraint $\delta(\mathcal{C}\zeta)$ can be expressed as a linear constraint for the variables ξ.

One introduces a new moving frame e with $e_0(z)=\Psi(z)$. An example for such a frame is

$$\begin{aligned} e_0(z) &= \Psi(z), \\ e_1(z) &= \frac{-\tilde\Phi^\perp(z)}{|\tilde\Phi^\perp(z)|} = \frac{-\tilde\Phi+\Psi(\Psi\cdot\tilde\Phi)}{\sin\theta}(z), \\ \vec e(z) &= (e_2(z),\dots e_{\mathcal{N}}(z)) = \vec f(z). \end{aligned} \tag{2.14}$$

$\tilde\Phi^\perp$ denotes the component of $\tilde\Phi$ perpendicular to Ψ.

We will use this special moving frame to derive a parametrization of ϕ in terms of ξ and the transformation from ζ to ξ. But the results can be stated in a covariant form. So this explicit basis and the basis f can be regarded as convenient temporary frameworks.

The fluctuation field ξ has components

$$\xi = (0, \xi_1, \ldots \xi_{\mathcal{N}}) = (0, \xi_1, \vec{\xi})$$

in the local basis e because

$$\xi \in T_{\Psi(z)} S^{\mathcal{N}}.$$

A change from the basis f (2.8) to basis e (2.14) is a simple rotation in the $\tilde{\Phi}$-Ψ-plane.

$$\begin{aligned} f_0 &= \cos\theta e_0 - \sin\theta e_1 \\ f_1 &= \sin\theta e_0 + \cos\theta e_1 \\ \vec{f} &= \vec{e} \end{aligned}$$

Inserting this in (2.9) yields

$$\begin{aligned} \phi &= e_0 + \frac{1}{\cos\theta}\zeta^1 e_1 + \vec{\zeta}\cdot\vec{e} \\ &\quad -\frac{1}{2}(\zeta)^2 e_0 - \frac{1}{2}\tan\theta(\zeta^1)^2 e_1 \\ &\quad +\frac{1}{2}\tan\theta(\zeta)^2 e_1 + \frac{1}{2}\tan^3\theta(\zeta^1)^2 e_1 + o(\zeta^3) \end{aligned}$$

One defines a linear transformation

$$\xi = Q^{-1}\zeta \tag{2.15}$$

with

$$\begin{aligned} \xi^1 &= \frac{1}{\cos\theta}\zeta^1, \\ \vec{\xi} &= \vec{\zeta}. \end{aligned}$$

Then

$$(\zeta)^2 + \tan^2\theta(\zeta^1)^2 = (\xi)^2,$$

and we can write

$$\begin{aligned} \phi &= e_0 + \xi^1 e_0 + \vec{\xi}\cdot\vec{e} \\ &\quad -\frac{1}{2}(\xi)^2 e_0 + \frac{1}{2}\tan\theta(\xi)^2 e_1 + o(\xi^3). \end{aligned}$$

This is the parametrization of the fundamental field ϕ in terms of Ψ and $\xi \in T_\Psi S^{\mathcal{N}}$.

The covariant form of this parametrization is

$$\phi = \Psi + \xi - \frac{1}{2}\frac{1}{\cos\theta}\xi^2\tilde{\Phi} + o(\xi^3). \tag{2.16}$$

It is independent of the moving frame with $e_0 = \Psi$.

2.1.4 The Linear Transformation $Q : T_\Psi S^{\mathcal{N}} \mapsto T_{\tilde\Phi} S^{\mathcal{N}}$

The map Q defined above depends on $\Psi(z)$ and $\tilde\Phi(z)$. It is a linear transformation between different tangent spaces of the sphere.

$$Q(z) : T_{\Psi(z)} S^{\mathcal{N}} \mapsto T_{\Phi(x)} S^{\mathcal{N}} \quad \text{for } z \in x.$$

We define

$$\vec{e} \cdot \vec{f}^T = \sum_{i=2}^{\mathcal{N}} e^i f^{iT}.$$

From (2.15) it follows that

$$\begin{aligned}
Q^{-1} &= \frac{1}{\cos\theta} e_1 f_1^T + \vec{e} \cdot \vec{f}^T \\
&= \frac{1}{\cos\theta} e_1 f_1^T - e_1 e_1^T + e_1 e_1^T + \vec{e} \cdot \vec{f}^T \\
&= e_1 \left[\frac{1}{\cos\theta} f_1 - e_1 \right]^T + \underbrace{1_{T_\Psi S^{\mathcal{N}}}}_{\pi} \\
&= \frac{-\tilde\Phi + \Psi\cos\theta}{\sin\theta} \left[\frac{1}{\cos\theta} \frac{\Psi - \Psi\cos^2\theta}{\sin\theta} \right]^T + \pi \\
&= (-\tilde\Phi + \Psi\cos\theta) \left[\frac{1}{\cos\theta} \Psi \right]^T + \pi \\
&= -\frac{1}{\cos\theta} \tilde\Phi \Psi^T + \Psi\Psi^T + \pi \\
&= 1 - \frac{1}{\cos\theta} \tilde\Phi \Psi^T. \qquad (2.17)
\end{aligned}$$

Similarly for Q

$$\begin{aligned}
Q &= \cos\theta f_1 e_1^T + \vec{f} \cdot \vec{e}^T \\
&= \cos\theta f_1 e_1^T - e_1 e_1^T + e_1 e_1^T + \vec{f} \cdot \vec{e}^T \\
&= (\cos\theta f_1 - e_1) e_1^T + \underbrace{1_{T_\Psi S^{\mathcal{N}}}}_{\pi} \\
&= \left[\cos\theta \frac{\Psi - \tilde\Phi\cos\theta}{\sin\theta} - \frac{-\tilde\Phi + \Psi\cos\theta}{\sin\theta} \right] \left[\frac{-\tilde\Phi + \Psi\cos\theta}{\sin\theta} \right]^T + \pi \\
&= \left[\frac{1 - \cos^2\theta}{\sin^2\theta} \tilde\Phi \right] \left[-\tilde\Phi + \Psi\cos\theta \right]^T + \pi \\
&= \pi + \tilde\Phi(\Psi^T \cos\theta - \tilde\Phi^T) \qquad (2.18)
\end{aligned}$$

The results for Q and Q^{-1} are independent of the bases e and f.

THE JACOBIAN $|\det Q|$

We also need the Jacobian associated with the transformation from ζ-variables to ξ-variables.

Since

$$\begin{aligned} Qe_1 &= f_1 \cos\theta, \\ Qe_i &= f_i \text{ for } i = 2, \ldots \mathcal{N}, \end{aligned}$$

we define a Matrix $Q^i{}_j$ with

$$Qe_j = \sum_{i=1}^{\mathcal{N}} f_i Q^i{}_j.$$

The $\mathcal{N} \times \mathcal{N}$ matrix $(Q^i{}_j)$ is diagonal if we use the bases f and e defined above. It has a single eigenvalue $\cos\theta$ which is distinct from 1. Therefore

$$|\det Q^i{}_j| = \cos\theta. \tag{2.19}$$

The integration variables ζ^i are the coefficients of ζ in the orthonormal basis f, while the variables ξ^i are the coefficients of ξ in the orthonormal basis e. Since $Q\xi = \zeta$,

$$\begin{aligned} \prod_{i=1}^{\mathcal{N}} d\zeta^i &= |\det Q^i{}_j| \prod_{i=1}^{\mathcal{N}} d\xi^i \\ &= \cos\theta \prod_{i=1}^{\mathcal{N}} d\xi^i. \end{aligned} \tag{2.20}$$

2.2 THE ONE LOOP APPROXIMATION

Using the expansion (2.16) one can determine the effective Boltzmann factor in a saddle point approximation. The fundamental action $S(\phi)$ has to be expanded to second order in ξ around an extremizing background field Ψ,

$$S(\phi) = S(\Psi) + \left.\frac{\delta S(\phi)}{\delta \phi}\right|_{\phi=\Psi} \xi + \frac{1}{2}\xi \left.\frac{\delta^2 S(\phi)}{\delta \phi^2}\right|_{\phi=\Psi} \xi + o(\xi^3). \tag{2.21}$$

Because of the extremizing properties of the background field Ψ, the linear term in ξ vanishes. The leading order $S(\Psi[\Phi])$ will be called $S_{\rm cl}(\Psi)$. It is a classical perfect action in Wilson's sense if one regards it as a function of Φ.

The one loop approximation is an approximation to order β^0. Thus the jacobian factor has to be expanded only to zeroth order in the fluctuation field. As

$$\begin{aligned} d\zeta(z) &= |\det Q(z)| d\xi(z) \\ &= \cos\theta(z) d\xi(z), \end{aligned}$$

the factor

$$\begin{aligned}\left(1-(\Psi^{\perp}+\zeta)^2\right)^{-\frac{1}{2}} &= \left(1-(\Psi^{\perp})^2\right)^{-\frac{1}{2}}+o(\zeta)\\ &= \frac{1}{(1-\sin^2\theta)^{\frac{1}{2}}}+o(\zeta)\\ &= \frac{1}{\cos\theta}+o(\zeta)\end{aligned}$$

multiplying J^0 in the full Jacobian J cancels out to this order.

The constraint $\delta(\mathcal{C}\zeta)$ can be expressed as $\delta(\mathcal{C}Q\xi)$.

So, to one loop order the effective Boltzmann factor reads

$$e^{-S_{\rm eff}[\Phi]} = e^{-S_{\rm cl}[\Psi]}J^0(\mathcal{C}\Psi)\int\prod_z d\xi(z)\delta(\mathcal{C}Q\xi)e^{-\int_z\frac{\beta}{2}\xi S''\xi}. \tag{2.22}$$

The integration of the variable $\xi(z)$ is over the $\mathcal{N}$-dimensional tangent space $T_{\Psi(z)}S^{\mathcal{N}}$. It is subject to the constraint

$$\Psi(z)\cdot\xi(z)=0. \tag{2.23}$$

The term $\delta(\mathcal{C}Q\xi)$ can be regarded as the limit of a Gaussian in the case of a δ-block spin.

As a result, one obtains the effective action as a sum of the classical action $S_{\rm cl}(\Psi)$, the jacobian term $-\ln J^0(\mathcal{C}\Psi)$ and a $\mathrm{Tr}\ln\Gamma$ term. The propagator Γ is the covariance of the above mentioned Gaussian measure. It includes the contributions from the second derivative of the action and the representation of $\delta(\mathcal{C}Q\Psi)$ as a Gaussian.

The formula is not very illuminating because the full propagator Γ has a complicated dependence on the background field Ψ. The dependence has three sources:

- the constraint (2.23) on ξ,
- the Ψ-dependence of Q,
- the Ψ-dependence of the second derivative of the action S''.

To get a useful formula one needs to make certain simplifications. They are possible if one exploits the smoothness of the background field Ψ on length scale a. Ψ is always smooth on this length scale, whether the block spin Φ is smooth or not, provided the blocks are chosen large enough. A basic reason for this is that there are no domain walls in a 2-dimensional ferromagnet with continuous symmetry, because the free energy of such domain walls would decrease by making them wider. This is an old argument by M. Fisher [12] which was made mathematically precise by Dobrushin and Shlosman in their version of the proof of the absence of spontaneous breaking of continuous symmetries in 2 dimensions [10]. In such an approximation which exploits the smoothness of Ψ it is not

necessary to consider terms of higher order than second order in $\nabla_\mu \Psi$. As S'' contains field dependent terms of first and second order in $\nabla_\mu \Psi$, one needs to treat them in first and second order perturbation theory respectively. This extracts the field dependence of S'' from the propagator.

The full propagator depends on Q also. The field dependence of Q reflects the choice of block spin. Its contributions are not of order $\ln \frac{\tilde{a}}{a}$ and are therefore *not* included in Polyakov's result [29].

Assumptions about the smoothness of the block spin field Φ are needed, if one wants to derive explicit formulas. Smoothness of Φ on the block lattice with lattice spacing $\tilde{a}$ means that Ψ is also smooth on the fine lattice on length scale $\tilde{a}$. But the extremizing properties of the background field guarantee smoothness of Ψ only on length scale a. Therefore the assumption of a smooth block spin field restricts the validity of the more explicit formulas for the effective action.

There exists an extension of Q to an $(\mathcal{N}+1) \times (\mathcal{N}+1)$ matrix $\hat{Q}$. When the block spin field is smooth, $\hat{Q}$ is close to 1, and one can derive a power series expansion of the propagator in powers of $\hat{Q} - 1$. The terms in this expansion contain the well known Gawedzki-Kupiainen high frequency propagator Γ_{KG}.

So the dependence of S'' and Q on Ψ can be handled, if the background field and the block spin field are smooth respectively.

There remains the constraint $\Psi(z) \cdot \xi(z) = 0$ on the integration variable $\xi(z)$.

There are two ways to handle this.

2.2.1 The Constraint on the Fluctuation Field

The constraint (2.23) can be implemented either with the help of a local basis or an extension to a $\mathcal{N}+1$-dimensional integration.

- **Polyakov's method:**
  One expands the fields in a basis $e_1(z), \ldots e_{\mathcal{N}}(z)$ for the tangent space $T_{\Psi(z)} S^{\mathcal{N}}$. In differential geometry such a basis is called a moving frame. S'' becomes a $\mathcal{N} \times \mathcal{N}$ matrix in this basis.

  Polyakov's method has the advantage that the origin of the characteristic factor $\mathcal{N}-1$ in the formula for the running coupling constant emerges in a very transparent fashion from the form of S''. Therefore the details are shown in the next section. The result agrees with Polyakov's to order $\ln \frac{\tilde{a}}{a}$. Corrections to higher order are found.

  But the disadvantage of Polyakov's method is that the expansion of the propagator in powers of $\hat{Q} - 1$ would be very cumbersome. The effective action was thus also determined in a different way.

- **$\mathcal{N}+1$-dimensional integration**

One inserts a 1 in the form of a Gaussian integral over an additional integration variable $\xi^0(z) \in \mathbb{R}$. This new integration is combined with the $\mathcal{N}$-dimensional integration over $\xi(z)$ to an $\mathcal{N}+1$-dimensional integration over $\varphi(z) = \xi(z) + \xi^0(z)\Psi(z) \in \mathbb{R}^{\mathcal{N}+1}$.

In this formulation the power series expansion in $\hat{Q} - 1$ is straightforward, and an explicit formula for the effective action can be derived. But Polyakov's result must be extracted by evaluating the singular part of a 1-loop diagram.

This method can be used to extend the results to Gaussian block spin.

2.2.2 Expansion of the Action

To integrate out the fluctuation field ξ in (2.22), we have to expand the action $S(\phi)$ in terms of ξ around the classical action $S_{\text{cl}}(\Psi)$. This yields an expression for S''.

Using (2.16), one can expand the action as

$$\begin{aligned}
\frac{\beta}{2}\int_z |\nabla_\mu \phi|^2(z) &= \frac{\beta}{2}\int_z |\nabla_\mu \Psi + \nabla_\mu \xi - \frac{1}{2}\nabla_\mu(\frac{1}{\cos\theta}\xi^2\tilde{\Phi})|^2(z) + o(\xi^3) \\
&= \frac{\beta}{2}\int_z |\nabla_\mu \Psi|^2(z) + |\nabla_\mu \xi|^2(z) - \nabla_\mu \Psi \cdot \nabla_\mu(\frac{1}{\cos\theta}\xi^2\tilde{\Phi})(z) \\
&\quad + \text{linear terms in } \xi + o(\xi^3) \\
&= \frac{\beta}{2}\int_z |\nabla_\mu \Psi|^2(z) + |\nabla_\mu \xi|^2(z) + \frac{1}{2}\frac{1}{\cos\theta}\xi^2\tilde{\Phi} \cdot \triangle\Psi(z) \\
&\quad + \text{linear terms in } \xi + o(\xi^3) \qquad (2.24)
\end{aligned}$$

In the last step an integration by parts was performed. But the integration by parts formula derived from the exact Leibnitz rule on the lattice was used.

The linear term vanishes because Ψ extremizes the action. $\frac{\beta}{2}\int_z |\nabla_\mu \Psi|^2(z)$ is the classical action S_{cl}. This term is independent of ξ and does not play a role in the integration over the fluctuation field.

It remains to consider the terms quadratic in ξ. They make up $\xi S''\xi$. The last term involves Ψ explicitly, while the kinetic term for ξ contains such a dependence only through the constraint (2.23) on ξ.

For the two methods of performing the integration mentioned above in section (2.2.1) on needs to treat these terms differently.

Chapter 3

The Action in a Gauge Covariant Form

Here we discuss a method to extract Polyakov's result using a moving frame. The action will be presented in an arbitrary local basis. This extends the global $O(\mathcal{N}+1)$ symmetry to a local one. Using Polyakov's special moving frame, we can derive the characteristic $(\mathcal{N}-1)$-term for the running coupling constant. The result is extended by finite corrections which are due to the details of the infrared cut-off.

A δ-block spin is assumed in this chapter.

3.1 The Potential A_μ

It is convenient to write the action in a gauge covariant form by introducing an arbitrary z-dependent local basis. The basis consists of an orthonormal set of vectors

$$e_\alpha(z), \quad \alpha = 0, \ldots \mathcal{N}$$

for every site $z \in \Lambda$ of the fine lattice. The basis spans $\mathbb{R}^{\mathcal{N}+1}$, so that

$$e_\alpha(z) \cdot e_\beta(z) = \delta_{\alpha\beta}. \tag{3.1}$$

The fundamental field can be expanded in this basis as

$$\phi(z) = \sum_{\alpha=0}^{\mathcal{N}} \underline{\phi}^\alpha(z) e_\alpha(z) = \underline{\phi}^\alpha(z) e_\alpha(z). \tag{3.2}$$

Similarly for ξ, Ψ and $\tilde{\Phi}$. The expansion coefficients are assembled in $\mathcal{N}+1$-dimensional column vectors $\underline{\phi}$, $\underline{\Psi}$, $\underline{\tilde{\Phi}}$ and $\underline{\xi}$.

$$\underline{\phi}(z) = \begin{pmatrix} \underline{\phi}^0(z) \\ \vdots \\ \underline{\phi}^{\mathcal{N}}(z) \end{pmatrix}.$$

Using the Leibnitz rule on the lattice (1.5), one finds

$$\begin{aligned} \nabla_\mu \phi(z) &= \nabla_\mu \left(\underline{\phi}^\alpha(z) e_\alpha(z) \right) \\ &= \left(\nabla_\mu \underline{\phi}^\alpha(z) \right) e_\alpha(z+\hat{\mu}) + \underline{\phi}^\alpha(z) \nabla_\mu e_\alpha(z). \end{aligned} \tag{3.3}$$

The action takes the covariant form

$$\begin{aligned} (\nabla_\mu \phi)^2(z) &= [\nabla_\mu \underline{\phi}^\alpha(z) e_\alpha(z+\hat{\mu}) + \underline{\phi}^\beta(z) \nabla_\mu e_\beta(z)] \cdot [\nabla_\mu \underline{\phi}^\sigma(z) e_\sigma(z+\hat{\mu}) + \underline{\phi}^\rho(z) \nabla_\mu e_\rho(z)] \\ &= \nabla_\mu \underline{\phi}^\alpha(z) \underbrace{e_\alpha(z+\hat{\mu}) \cdot e_\sigma(z+\hat{\mu})}_{\delta_{\alpha\sigma}} \nabla_\mu \underline{\phi}^\sigma(z) \\ &\quad + \nabla_\mu \underline{\phi}^\alpha(z) e_\alpha(z+\hat{\mu}) \cdot \nabla_\mu e_\rho(z) \underline{\phi}^\rho(z) \\ &\quad + \underline{\phi}^\beta(z) \nabla_\mu e_\beta(z) \cdot e_\sigma(z+\hat{\mu}) \nabla_\mu \underline{\phi}^\sigma(z) \\ &\quad + \underline{\phi}^\beta(z) \nabla_\mu e_\beta(z) \cdot \nabla_\mu e_\rho(z) \underline{\phi}^\rho(z). \end{aligned}$$

The second and third term are identical

Introducing matrices $A_\mu(z)$ by

$$A_{\alpha\beta\mu}(z) = e_\alpha(z+\hat{\mu}) \cdot \nabla_\mu e_\beta(z), \tag{3.4}$$

the action reads

$$\begin{aligned} (\nabla_\mu \phi)^2(z) &= (\nabla_\mu \underline{\phi}^\alpha)^2(z) + 2 \nabla_\mu \underline{\phi}^\alpha A_{\alpha\beta\mu} \underline{\phi}^\beta(z) \\ &\quad + \underline{\phi}^\alpha A_{\rho\alpha\mu} A_{\rho\beta\mu} \underline{\phi}^\beta(z) \\ &= \left[(\nabla_\mu + A_\mu) \underline{\phi} \right] \cdot \left[(\nabla_\mu + A_\mu) \underline{\phi} \right](z) \\ &= \left[(\nabla_\mu + A_\mu) \underline{\phi} \right]^2(z). \end{aligned} \tag{3.5}$$

In a constant basis, $A_\mu(z)$ vanishes.

Using

$$\nabla_\mu (e_\alpha(z) \cdot e_\beta(z)) = 0,$$

it follows that

$$e_\alpha(z+\hat{\mu})\cdot\nabla_\mu e_\beta(z) = -\nabla_\mu e_\alpha(z)\cdot e_\beta(z). \tag{3.6}$$

This yields the following substitute for antisymmetry in the indices α, β

$$\begin{aligned} A_{\alpha\beta\mu}(z) &= e_\alpha(z+\hat{\mu})\cdot\nabla_\mu e_\beta(z) \\ &= e_\beta(z)\cdot(-\nabla_\mu e_\alpha(z)) \\ &= e_\beta(z)\cdot\nabla_{-\mu} e_\alpha(z+\hat{\mu}) \\ &= A_{\beta\alpha-\mu}(z+\hat{\mu}). \end{aligned} \tag{3.7}$$

The extension of ordinary derivatives ∇_μ to covariant derivatives $\nabla_\mu + A_\mu$ reminds of the covariant derivatives in gauge theories. Indeed, the form follows from an extension of the global $O(\mathcal{N}+1)$ symmetry of the σ-model to a local one. But the potential A_μ is not a dynamical field which couples to ϕ, it is merely the potential needed to make derivatives covariant. No kinetic term for the field A is introduced.

3.2 Polyakov's Method

In the case of Polyakov's method, one expands the terms containing the fluctuation field ξ in (2.24) in an arbitrary moving frame.

$$|\nabla_\mu \xi|^2(z) = |(\nabla_\mu + A_\mu)\underline{\xi}|^2(z), \tag{3.8}$$

as in (3.5). The last term in (2.24) involves only local scalar products of the field ξ on one lattice site. This can be expressed in the moving frame, too.

$$\xi^2 = \underline{\xi}^2.$$

We conclude

$$\underline{\xi} S''_A \underline{\xi}(z) = |(\nabla_\mu + A_\mu)\underline{\xi}|^2(z) + \frac{1}{2}\frac{1}{\cos\theta}[\tilde{\Phi}\cdot\triangle\Psi]\underline{\xi}^2(z). \tag{3.9}$$

Now the Polyakov's special moving frame is introduced. It sets the component $\underline{\xi}^0$ to zero and allows a perturbative treatment of the term $|(\nabla_\mu + A_\mu)\underline{\xi}|^2$ which contains also a dependence of S''_A on Ψ.

Polyakov [29] used a special moving frame e with

$$e_0(z) = \Psi(z).$$

The other basis vectors $e_1(z), \ldots e_{\mathcal{N}}(z)$ span the tangent space $T_{\Psi(z)}S^{\mathcal{N}}$. Therefore the ξ-field has no 0-component in this frame.

$$\underline{\xi}(z) = \begin{pmatrix} 0 \\ \underline{\xi}^1(z) \\ \vdots \\ \underline{\xi}^{\mathcal{N}}(z) \end{pmatrix}$$

There is a residual arbitrariness in the choice of the basis. The $O(\mathcal{N})$ group of those rotations which leave $\Psi(z)$ invariant form a symmetry group of gauge transformations. The $\mathcal{N} \times \mathcal{N}$ matrices

$$a_\mu(z) = (A_{ij\mu}(z), \quad i,j = 1, \ldots \mathcal{N}) \tag{3.10}$$

which depend only on the basis of $T_{\Psi(z)}S^{\mathcal{N}}$ transform like gauge fields under these gauge transformations.

On the other hand,

$$A_{0i\mu}(z) = -(\nabla_\mu \Psi(z))^i$$

transform like $\mathcal{N}$-vector fields.

The field strength tensor for the vector potential a_μ is given by [1]

$$\begin{aligned} F_{ij\mu\nu}(z) &= \nabla_\mu a_{ij\nu}(z) - \nabla_\nu a_{ij\mu}(z) \\ &\quad + a_{ik\mu}(z+\hat{\nu}) a_{kj\nu}(z) - a_{ik\nu}(z+\hat{\mu}) a_{kj\mu}(z) \\ &= \nabla_\mu a_{ij\nu} + a_{ik\mu}(z+\hat{\nu}) a_{kj\nu}(z) - (\mu \leftrightarrow \nu) \end{aligned} \tag{3.11}$$

One computes

$$\begin{aligned} \nabla_\mu a_{ij\nu}(z) &= \nabla_\mu (e_i(z+\hat{\nu}) \cdot \nabla_\nu e_j(z)) \\ &= e_i(z+\hat{\mu}+\hat{\nu}) \cdot \nabla_\mu \nabla_\nu e_j(z) + \nabla_\mu e_i(z+\hat{\nu}) \cdot \nabla_\nu e_j(z), \end{aligned} \tag{3.12}$$

and similarly

$$\begin{aligned} \nabla_\nu a_{ij\mu}(z) &= \nabla_\nu (e_i(z+\hat{\mu}) \cdot \nabla_\mu e_j(z)) \\ &= e_i(z+\hat{\mu}+\hat{\nu}) \cdot \nabla_\nu \nabla_\mu e_j(z) + \nabla_\nu e_i(z+\hat{\mu}) \cdot \nabla_\mu e_j(z). \end{aligned} \tag{3.13}$$

Using (3.6) and the completeness of the basis in the form

$$\sum_{i=1}^{\mathcal{N}} e_i(z) e_i^T(z) = 1 - \Psi(z)\Psi^T(z),$$

[1] This is the field strength tensor which one gets by use of noncommutative differential calculus. It was shown by Dimakis, Müller-Hoissen and Striker [9] that the conventional lattice gauge theory formalism is equivalent to a noncommutative differential geometry. In this formulation, the lattice Leibnitz rule takes the standard form $d(fg) = (df)g + fdg$, and all the familiar formula of continuum gauge field theory remain valid on the lattice. See also section 1.2

it follows

$$\begin{aligned} a_{ik\mu}(z+\hat{\nu})a_{kj\nu}(z) &= \underbrace{e_i(z+\hat{\mu}+\hat{\nu})\cdot\nabla_\mu e_k(z+\hat{\nu})}_{-\nabla_\mu e_i(z+\hat{\nu})\cdot e_k(z+\hat{\nu})} e_k(z+\hat{\nu})\cdot\nabla_\nu e_j(z) \\ &= -\nabla_\mu e_i(z+\hat{\nu})(1-\Psi(z+\hat{\nu})\Psi^T(z+\hat{\nu}))\nabla_\nu e_j(z) \\ &= -\nabla_\mu e_i(z+\hat{\nu})\cdot\nabla_\nu e_j(z) \\ &\quad + \underbrace{\nabla_\mu e_i(z+\hat{\nu})\cdot\Psi(z+\hat{\nu})}_{-e_i(z+\hat{\mu}+\hat{\nu})\cdot\nabla_\mu\Psi(z+\hat{\nu})}\underbrace{\Psi(z+\hat{\nu})\cdot\nabla_\nu e_j(z)}_{-\nabla_\nu\Psi(z)\cdot e_j(z)} \\ &= -\nabla_\mu e_i(z+\hat{\nu})\cdot\nabla_\nu e_j(z) \\ &\quad + e_i(z+\hat{\mu}+\hat{\nu})\cdot\nabla_\mu\Psi(z+\hat{\nu})\nabla_\nu\Psi(z)\cdot e_j(z). \end{aligned}$$

Interchanging μ and ν yields

$$a_{ik\nu}(z+\hat{\mu})a_{kj\mu}(z) = -\nabla_\nu e_j(z+\hat{\mu})\cdot\nabla_\mu e_i(z) + e_j(z+\hat{\mu}+\hat{\nu})\cdot\nabla_\nu\Psi(z+\hat{\mu})\nabla_\mu\Psi(z)\cdot e_i(z).$$

Adding the terms, one concludes

$$F_{ij\mu\nu}(z) = e_i(z+\hat{\mu}+\hat{\nu})\cdot\nabla_\mu\Psi(z+\hat{\nu})\nabla_\nu\Psi(z)\cdot e_j(z) - (\mu\leftrightarrow\nu). \tag{3.14}$$

We see that the field strength tensor is of order $(\nabla\Psi)^2$. It follows that the vector potential a_μ in Lorentz gauge,

$$\nabla_\mu a_\mu = 0,$$

is also of order $(\nabla\Psi)^2$. The commutator $[a_\mu, a_\nu]$ in (3.11) is negligible, and to leading order the vector potential could be recovered as

$$a_\nu(z) = -\int_w \nabla_\mu v_{\mathrm{Cb}}(z-w)F_{\nu\mu}(w)\ .$$

Although the Coulomb potential v_{Cb} in 2 dimensions does not exist, its derivative is well defined.

Now we can separate the terms involving a_μ and $A_{0i\mu} = \Psi(z+\hat{\mu})\cdot\nabla_\mu e_i(z)$. Using the antisymmetry (3.7) of A_μ and the relation

$$\nabla_{-\mu}\Psi(z+\hat{\mu}) = -\nabla_\mu\Psi(z)$$

we obtain

$$\begin{aligned} |(\nabla_\mu + A_\mu)\underline{\xi}|^2 &= (\nabla_\mu\underline{\xi}^i)^2 + 2\nabla_\mu\underline{\xi}^i a_{ij\mu}\underline{\xi}^j \\ &\quad + \underline{\xi}^i a_{ki\mu}a_{kj\mu}\underline{\xi}^j \\ &\quad + \underline{\xi}^i A_{0i\mu}A_{0j\mu}\underline{\xi}^j \\ &= |(\nabla_\mu + a_\mu)\underline{\xi}|^2 + \underline{\xi}^i(-e_i\cdot\nabla_\mu\Psi)(-e_j\cdot\nabla_\mu\Psi)\underline{\xi}^j \\ &= |(\nabla_\mu + a_\mu)\underline{\xi}|^2 + |\underline{\xi}\cdot\underline{\nabla_\mu\Psi}|^2. \end{aligned}$$

The last term involves the components of $[\nabla_\mu \Psi]^i$ of $\nabla_\mu \Psi$ with respect to the moving frame, $\underline{\nabla_\mu \Psi}$, and not $\nabla_\mu \underline{\Psi}^i$. $\nabla_\mu \underline{\Psi}^i$ would be zero in a Polyakov basis!

One can split $\tilde{\Phi}$ into components parallel and perpendicular to Ψ.

$$\begin{aligned} \tilde{\Phi} &= \tilde{\Phi}^\perp + \tilde{\Phi}^{\|} \\ &= \tilde{\Phi}^\perp + (\Psi \cdot \tilde{\Phi})\tilde{\Phi} \\ &= \tilde{\Phi}^\perp + \cos\theta \tilde{\Phi}. \end{aligned} \tag{3.15}$$

Inserting this in (3.9), we see that

$$\frac{1}{\cos\theta}\xi^2[\tilde{\Phi} \cdot \triangle \Psi] = \xi^2 \Psi \cdot \triangle \Psi + \frac{1}{\cos\theta}\xi^2[\tilde{\Phi}^\perp \cdot \triangle \Psi]$$

In conclusion

$$\underline{\xi} S''_A \underline{\xi} = \int_z |(\nabla_\mu + a_\mu)\underline{\xi}|^2(z) + \underline{\xi} S''_I \underline{\xi}$$

with

$$\underline{\xi} S''_I \underline{\xi} = \int_z [\underline{\xi} \cdot \underline{\nabla_\mu \Psi}]^2(z) + \underline{\xi}^2(z)[\Psi \cdot \triangle \Psi](z) + \frac{1}{\cos\theta}\underline{\xi}^2(z)[\tilde{\Phi}^\perp \cdot \triangle \Psi](z). \tag{3.16}$$

We will see later that the last term is very small for smooth enough block spin fields. This is a consequence of the extremizing properties of Ψ. The term is $\triangle\Psi$ multiplied with an expression of order $(\Psi^\perp)^3$, and it turns out not to contribute at all to order $\ln(\frac{\tilde{a}}{a})$.

The effective Boltzmann factor becomes

$$e^{-S_{\rm eff}[\Phi]} = e^{-S_{\rm cl}[\Psi]} J^0(\mathcal{C}\Psi) \int \prod_z d\underline{\xi}(z) \delta(\mathcal{C}Q\underline{\xi}) e^{-\frac{\beta}{2}\int_z |(\nabla_\mu + a_\mu)\underline{\xi}|^2(z) + \underline{\xi} S''_I \underline{\xi}}. \tag{3.17}$$

The volume element is given by

$$d\underline{\xi}(z) = d\underline{\xi}^1(z) \dots d\underline{\xi}^{\mathcal{N}}(z).$$

Because of the choice of the vectors $e_1(z), \dots e_{\mathcal{N}}(z) \in T_{\Psi(z)} S^{\mathcal{N}}$, the constraint (2.23) is implemented. Now one has to solve a Gaussian integral without a constraint. A rescaling $\underline{\xi} \to \underline{\xi}/\beta^{-\frac{1}{2}}$ is possible.

Let us write $[z] \in \tilde{\Lambda}$ for the block x which contains z. Remember that $Q(z)$ is a map from $T_{\Psi(z)} S^{\mathcal{N}}$ to $T_{\tilde{\Phi}(z)} S^{\mathcal{N}}$. $\mathcal{C}$ defines a map of functions on the fine lattice Λ with values in $T_{\tilde{\Phi}(z)} S^{\mathcal{N}}$ to functions on the block lattice with values in $T_{\Phi([z])} S^{\mathcal{N}}$. Therefore the operator $Q^T \mathcal{C}^\dagger \mathcal{C} Q$ maps functions with values in $T_{\Psi(z)} S^{\mathcal{N}}$ into functions of the same kind. The Polyakov basis elements $e_i(z), i = 1, \dots \mathcal{N}$ are a basis of $T_{\Psi(z)} S^{\mathcal{N}}$.

We denote by $P(z,w) = (P_{ij}(z,w); i,j = 1, \dots \mathcal{N})$ the matrix of the kernel of $Q^T \mathcal{C}^\dagger \mathcal{C} Q$ with respect to this basis.

$$\int_w P_{ij}(z,w) \underline{\xi}^j(w) = e_i(z) \cdot \left(Q^T \mathcal{C}^\dagger \mathcal{C} Q \xi\right)(z).$$

$P(z,w)$ is only nonzero when z and w belong to the same block $[z]=[w]$. The δ-function becomes a Gaussian (or a limit of a Gaussian) as follows

$$\begin{aligned}\delta(\mathcal{C}Q\xi) &= (\lim_{\kappa\to\infty})N_\kappa e^{-\frac{\beta\kappa}{2}\int_z\int_w \underline{\xi}^i(z)P_{ij}(z,w)\underline{\xi}^j(w)}\\ N_\kappa &= \left(\frac{\kappa a^d}{2\pi}\right)^{\frac{\mathcal{N}}{2}}.\end{aligned}$$

One defines the high frequency propagator for the fluctuation field as

$$\Gamma^e_\kappa = \left([\nabla_\mu + a_\mu]^2 + \kappa P\right)^{-1} \tag{3.18}$$

in the Polyakov basis with $e_0(z)=\Psi(z)$.

Then one can evaluate the Gaussian integral (3.17) for S_{eff}. The result is

$$S_{\text{eff}}(\Phi) = -S_{\text{cl}}(\Psi) - \ln J^0(\mathcal{C}\Psi) + \frac{1}{2}\text{tr}\, S_I''\Gamma^e_\kappa - \frac{1}{2}\text{Tr}\,\ln\Gamma^e_\kappa + \text{const.}. \tag{3.19}$$

3.3 Recovery of Polyakov's Result

Polyakov [29] determined the contributions to the effective action (3.19) which are of order $\ln(\frac{\tilde{a}}{a})$. They do not depend on the detailed form of the block spin which fixes the infrared cut-off in the auxiliary theory with fields ξ. The term κP in the high frequency propagator has the effect of an infrared cut-off. This has been discussed in detail in the work of Kupiainen and Gawedzki [13]. Although κP depends on the block spin Φ which is a detail of the infrared cut-off, one can replace κP by a mass term M^2, where M is of order $o(\tilde{a}^{-1})$. This yields the results that do not depend on the details of the cut-off.

The high frequency propagator also has a dependence on the $O(\mathcal{N})$-gauge field a_μ. We show that this can be neglected by exploiting the smoothness of the background field Ψ. We need to consider only terms up to order $o\left([\nabla_\mu\Psi]^2\right)$. The result is gauge invariant. In Lorentz gauge, a_μ is $o\left([\nabla_\mu\Psi]^2\right)$ as we saw. A perturbation expansion in a_μ shows that $\text{Tr}\,\ln\Gamma^e_\kappa$ is $o(a_\mu^2)$. Therefore the a_μ dependence of this term can be neglected. S_I'' is evidently $o\left([\nabla_\mu\Psi]^2\right)$, so that the a_μ dependence in the propagator multiplying it can also be neglected. The high frequency propagator matrix can therefore be replaced by

$$\Gamma^e_\kappa \approx \delta_{ij}v_M(z-w).$$

v_M is the Yukawa potential in 2 dimensions with mass M of order $o(\tilde{a}^{-1})$,

$$v_M = (-\triangle + M^2)^{-1}.$$

In this approximation $\text{Tr}\,\ln\Gamma^e_\kappa$ is a constant. The Jacobian is not ultraviolet divergent and is therefore not a feature of the details of the infrared cut-off.

Inserting S_I'' (3.16), we get the result to the desired approximation.

$$S_{\text{eff}}(\Phi) = \frac{1}{2}\left(\beta - (\mathcal{N}-1)v_M(0)\right)\int_z [\nabla_\mu\Psi]^2(z) + S_{\text{isZero}} \tag{3.20}$$

with

$$S_{\text{isZero}} = \frac{1}{2}\mathcal{N} v_M(0) \int_z [\cos\theta]^{-1} [\tilde{\Phi}^{\perp} \cdot \triangle \Psi](z). \tag{3.21}$$

Here as everywhere

$$\begin{aligned} \cos\theta &= \tilde{\Phi}(z) \cdot \Psi(z), \\ \tilde{\Phi}^{\perp}(z) &= \tilde{\Phi}(z) - \Psi(z)(\Psi(z) \cdot \tilde{\Phi}(z)). \end{aligned}$$

Except for the term S_{isZero} this is Polyakov's result. In the next section we show that S_{isZero} is actually zero in the present approximation. This is a consequence of the extremality properties of the background field Ψ.

3.4 Vanishing of the Correction Term to Polyakov's Result to Order $\ln\left(\frac{\tilde{a}}{a}\right)$

It remains to examine S_{isZero}. The term appears as a correction to Polyakov's result. Here we want to show that it vanishes to order $\ln\left(\frac{\tilde{a}}{a}\right)$. Therefore Polyakov's result is reproduced in his approximation.

S_{isZero} is given by the integral

$$S_{\text{isZero}} \sim \int_z [\cos\theta(z)]^{-1} \tilde{\Phi}^{\perp} \cdot \triangle \Psi(z).$$

As a consequence of the extremality properties of Ψ, the integral is actually zero.

Remark: *$v_M(0)$ will be replaced by a matrix $\Gamma_Q(z,z)$ in the exact result. Then the term will not vanish exactly. But it can still be neglected. The arguments will be presented in section 4.2.3*

As we saw in section (2.1), general fields ϕ which satisfy the block spin condition can be parametrized in terms of fields $\zeta(z) \perp \tilde{\Phi}(z)$. The ζ-fields then satisfy the constraint

$$\mathcal{C}\zeta(x) = 0. \tag{3.22}$$

The parametrization reads to first order in ζ

$$\phi(z) = \Psi(z) + \zeta(z) - \frac{1}{\Psi \cdot \tilde{\Phi}(z)} (\Psi \cdot \zeta(z)) \tilde{\Phi}(z) + o(\zeta^2).$$

In this formula the constraint $\zeta \perp \tilde{\Phi}$ has been abandoned because a component $\zeta^0 \tilde{\Phi}$ of ζ

in the direction of $\tilde{\Phi}$ does not contribute anything

$$\begin{aligned}\phi^{\|}(z) &= \Psi^{\|}(z) + \zeta^0\tilde{\Phi}(z) - \frac{1}{\Psi\cdot\tilde{\Phi}(z)}(\Psi^{\perp}\cdot\zeta(z))\tilde{\Phi}(z) \\ &\quad \underbrace{- \frac{1}{\Psi\cdot\tilde{\Phi}(z)}(\Psi\cdot\tilde{\Phi}(z))\zeta^0\tilde{\Phi}(z)}_{\zeta^0\tilde{\Phi}(z)} + o(\zeta^2) \\ &= \Psi^{\|}(z) - \frac{1}{\Psi\cdot\tilde{\Phi}(z)}(\Psi^{\perp}\cdot\zeta(z))\tilde{\Phi}(z) + o(\zeta^2).\end{aligned}$$

The extremality condition reads therefore

$$\begin{aligned}\zeta^T S'(\Psi) &= 0, \\ S'(\Psi) &= \frac{\delta}{\delta\zeta}S(\phi)\Big|_{\phi=\Psi},\end{aligned}$$

for fields ζ with constraint (3.22).

This is equivalent to

$$S'(\Psi) = \mathcal{C}^{\dagger}\lambda \tag{3.23}$$

with a Lagrange multiplier λ. $\lambda(x)$ is a field on the coarse lattice $\tilde{\Lambda}$. With

$$S(\phi) = -\frac{\beta}{2}\int_z \phi\cdot\triangle\phi(z)$$

we find that

$$S(\phi[\Psi,\zeta]) = -\frac{\beta}{2}\int_z \left(\Psi\cdot\triangle\Psi(z) + 2\zeta\cdot\triangle\Psi(z) - 2\frac{\triangle\Psi\cdot\tilde{\Phi}}{\Psi\cdot\tilde{\Phi}}(\Psi\cdot\zeta)(z)\right) + o(\zeta^2).$$

Then

$$S'(\Psi) = -\frac{\beta}{2}(2\triangle\Psi - 2\frac{\triangle\Psi\cdot\tilde{\Phi}}{\Psi\cdot\tilde{\Phi}}\Psi)$$

or

$$(-\triangle\Psi + \frac{\triangle\Psi\cdot\tilde{\Phi}}{\Psi\cdot\tilde{\Phi}}\Psi) = \mathcal{C}^{\dagger}\lambda. \tag{3.24}$$

Since $\tilde{\Phi}$ is constant on blocks, we find that

$$\tilde{\Phi}\cdot\mathcal{C}^{\dagger}\lambda = (-\tilde{\Phi}\cdot\triangle\Psi + \frac{\triangle\Psi\cdot\tilde{\Phi}}{\Psi\cdot\tilde{\Phi}}\tilde{\Phi}\cdot\Psi) = 0.$$

This yields

$$\Phi(x) \cdot \lambda(x) = 0. \tag{3.25}$$

By definition

$$\tilde{\Phi}^{\perp}(z) \cdot \Psi(z) = 0.$$

This allows us to reexpress the term $\tilde{\Phi}^{\perp} \cdot \triangle\Psi(z)$ as

$$\begin{aligned} -\tilde{\Phi}^{\perp} \cdot \triangle\Psi(z) &= -\tilde{\Phi}^{\perp} \cdot (\triangle\Psi - \Psi \frac{\triangle\Psi \cdot \tilde{\Phi}}{\Psi \cdot \tilde{\Phi}})(z) \\ &= (\tilde{\Phi}^{\perp} \cdot \mathcal{C}^{\dagger}\lambda)(z). \end{aligned}$$

Thus one computes by inserting the definition of $\tilde{\Phi}^{\perp}$

$$\begin{aligned} \int_z [\cos\theta(z)]^{-1} \tilde{\Phi}^{\perp} \cdot \triangle\Psi(z) &= -\int_z \frac{1}{\cos\theta(z)} [\tilde{\Phi} - \Psi(\Psi \cdot \tilde{\Phi})](z) \cdot (\mathcal{C}^{\dagger}\lambda)(z) \\ &= -\int_x \mathcal{C}\left(\frac{1}{\cos\theta}\tilde{\Phi} - \Psi\right)(x) \cdot \lambda(x). \end{aligned} \tag{3.26}$$

The integrand is zero. This is shown as follows

$$\mathcal{C}\Psi(x) = \rho(x)\Phi(x)$$

with some real $\rho(x)$. This is a consequence of the block spin condition (1.26). Moreover, because $\tilde{\Phi}$ is constant on blocks, it follows that

$$\mathcal{C}\left(\frac{1}{\cos\theta}\tilde{\Phi}\right)(x) = \Phi(x) \operatorname*{av}_{z \in x}\left(\frac{1}{\cos\theta}\right)(x).$$

This is again a multiple of the vector $\Phi(x)$. But according to (3.25), $\Phi(x) \cdot \lambda(x) = 0$. The integrand in (3.26) vanishes, and S_{isZero} is actually zero.

Polyakovs's result [29] has been recovered.

Chapter 4

$(\mathcal{N}+1)$-dimensional Integration

Here we present an alternative method to Polyakov's moving frame for evaluating the Gaussian integral (2.22) with the constraint (2.23) for the effective action. This method allows an expansion of the result in terms of $(\hat{Q}-1)$ for smooth block spin and yields an explicit expression for the effective action. Polyakov's result for the running coupling constant can be recovered, although the recovery is more involved.

We still consider a δ-block spin. So a limit $\kappa \to \infty$ is understood. The Gaussian block spin will be discussed in chapter 6.

We will use the notation ϕ^T for the transpose of a column vector extensively. Scalar products will be denoted by $\phi_1^T \phi_2$.

4.1 Expansion of the Integration

The idea is to expand the integration from $T_{\Psi(z)}S^{\mathcal{N}}$ to the whole $\mathbb{R}^{\mathcal{N}+1}$. This can be achieved by inserting an extra integration variable

$$\xi^0(z) \in \mathbb{R}$$

by insertion of

$$1 = N_\kappa \int \Pi d\xi^0 e^{-\frac{\beta}{2}(\int_z |\nabla_\mu \xi^0|^2 + \kappa \xi^0 \mathcal{C}^\dagger \mathcal{C} \xi^0)}. \tag{4.1}$$

N_κ is a suitable field independent constant which normalizes the integral. $\mathcal{C}$ is our block average defined in (1.13). κ is the same parameter that appears in the representation of the δ-function imposing the block spin condition in (2.22). For δ-block spin it is send to infinity.

A new fluctuation field φ is introduced by combining the integration variable ξ^0 with the old fluctuation field ξ,

$$\varphi(z) = \xi(z) + \xi^0(z)\Psi(z). \tag{4.2}$$

ξ^0 can be recovered as

$$\xi^0(z) = \Psi^T(z)\varphi(z). \tag{4.3}$$

With $\pi(z) = 1 - \Psi\Psi^T(z)$ it holds true that

$$\xi(z) = \pi(z)\varphi(z). \tag{4.4}$$

$\xi(z)$ is now considered as an element of $\mathbb{R}^{\mathcal{N}+1}$, still satisfying the constraint $\Psi^T(z)\xi(z) = 0$. But here we expand in a *constant* basis $e_0, \dots e_{\mathcal{N}}$, independent of z.

The expansion of the action (2.24) is extended by adding the $|\nabla_\mu \xi^0|^2$ term.

$$\begin{aligned} S_{\text{ext}}(\Psi|\varphi) &= \frac{\beta}{2}\int_z |\nabla_\mu \xi^0|^2(z) + S(\phi) \\ &= S_{\text{cl}}(\Psi) + \frac{\beta}{2}\int_z \left(|\nabla_\mu \xi|^2 + |\nabla_\mu \xi^0|^2 + \frac{1}{\cos\theta}\tilde{\Phi}^T \triangle \Psi \xi^2\right)(z). \end{aligned} \tag{4.5}$$

The linear terms in ξ are still omitted because Ψ extremizes the action.

We want to reexpress the kinetic terms for the fluctuation fields ξ^0 and ξ as a kinetic term for the combined field φ. Thus

$$|\nabla_\mu \xi|^2 + |\nabla_\mu \xi^0|^2 = |\nabla_\mu \varphi|^2 + \Delta L.$$

ΔL is a remainder which will be combined with

$$\frac{1}{\cos\theta}\tilde{\Phi}^T \triangle \Psi \xi^2 = \frac{1}{\cos\theta}\tilde{\Phi}^T \triangle \Psi \varphi^T \pi \varphi$$

to an expression $\varphi^T S''_{\text{ext,I}} \varphi$. $S''_{\text{ext,I}}$ is given in (4.15).

The expression for the effective action contains also the term $\delta(\mathcal{C}Q\xi)$. This term is regarded as the limit of a Gaussian. By inserting (4.1) its contribution to the effective action is combined with the factor

$$e^{-\frac{\beta}{2}\int_z \kappa \xi^0 \mathcal{C}^\dagger \mathcal{C} \xi^0}.$$

This can be done in the following fashion. One extends Q (2.18) to a map

$$\hat{Q} : \mathbb{R}^{\mathcal{N}+1} \mapsto \mathbb{R}^{\mathcal{N}+1}.$$

Q has the property that it annihilates $\Psi(z)$ and maps $T_{\Psi(z)}S^{\mathcal{N}}$ to $T_{\tilde{\Phi}(z)}S^{\mathcal{N}} \subset \mathbb{R}^{\mathcal{N}+1}$. We add to this the operator

$$Q_N(z) = \tilde{\Phi}(z)\Psi^T(z) \tag{4.6}$$

which annihilates $T_{\Psi(z)}S^{\mathcal{N}}$ and maps the ray through $\Psi(z)$ into the ray through $\tilde{\Phi}(z)$. This gives

$$\hat{Q}(z) = 1 - \Psi\Psi^T(z) + \tilde{\Phi}(z)\left(\Psi^T(z)[1+\cos\theta(z)] - \tilde{\Phi}^T(z)\right). \tag{4.7}$$

The ranges of the various maps are

$$\begin{aligned} Q(z) &: T_{\Psi(z)}S^{\mathcal{N}} \mapsto T_{\tilde{\Phi}(z)}S^{\mathcal{N}}, \\ Q_N(z) &: \xi^0(z)\Psi(z) \mapsto \zeta^0(z)\tilde{\Phi}(z); \qquad \xi^0, \zeta^0 \in \mathbb{R}, \\ \hat{Q}(z) = Q(z) + Q_N(z) &: \mathbb{R}^{\mathcal{N}+1} \mapsto \mathbb{R}^{\mathcal{N}+1}. \end{aligned}$$

Because of

$$\hat{Q}\varphi = Q\xi + Q_N\Psi\xi^0,$$

we see that

$$\begin{aligned} \varphi^T\hat{Q}^T\mathcal{C}^\dagger\mathcal{C}\hat{Q}\varphi &= \xi^T\hat{Q}^T\mathcal{C}^\dagger\mathcal{C}\hat{Q}\xi + \xi^0\Psi^T\hat{Q}^T\mathcal{C}^\dagger\mathcal{C}\hat{Q}\Psi\xi^0 \\ &= \xi^T Q^T\mathcal{C}^\dagger\mathcal{C}Q\xi + \xi^0\underbrace{\Psi^T\Psi}_{1}\tilde{\Phi}^T\mathcal{C}^\dagger\mathcal{C}\tilde{\Phi}\underbrace{\Psi^T\Psi}_{1}\xi^0 \\ &= \xi^T Q^T\mathcal{C}^\dagger\mathcal{C}Q\xi + \xi^0\mathcal{C}^\dagger\mathcal{C}\xi^0. \end{aligned}$$

So the implementation of the block spin condition and the κ-dependent part of (4.1) are expressed in terms of φ.

4.1.1 Definition of a High Frequency Propagator for the Extended Integration

After the insertion of (4.1) in (2.22) and the various combinations of expressions containing ξ and ξ^0 to terms only involving φ, we can rewrite the formula for the effective action

$$e^{-S_{\text{eff}}(\Phi)} = e^{-S_{\text{cl}}(\Psi)}J^0(\mathcal{C}\Psi)\int\prod_z d\varphi(z)e^{-\frac{\beta}{2}\int_z\left([|\nabla_\mu\varphi(z)|^2+\kappa\varphi^T\hat{Q}^T\mathcal{C}^\dagger\mathcal{C}\hat{Q}\varphi(z)]+\varphi^T S''_{\text{ext,I}}\varphi(z)\right)}. \tag{4.8}$$

A limit $\kappa\to\infty$ is understood. In contrast to (2.22) there is no constraint on φ. Thus, the Gaussian integral can be performed. But the result depends on Ψ and Φ through $\hat{Q}$ and $S''_{\text{ext,I}}$. A way to extract this dependence is needed.

We define a new high frequency propagator

$$\Gamma_Q = \lim_{\kappa\to\infty}\left(-\triangle + \kappa\hat{Q}^T\mathcal{C}^\dagger\mathcal{C}\hat{Q}\right)^{-1}. \tag{4.9}$$

Γ_Q depends on Ψ through $\hat{Q}$. When we want to make the dependence explicit, we write $\Gamma_{\hat{Q}[\Psi]}$.

$\Gamma_Q(z,w)$ is a map $\mathbb{R}^{\mathcal{N}+1} \mapsto \mathbb{R}^{\mathcal{N}+1}$. Thus it is a $(\mathcal{N}+1)\times(\mathcal{N}+1)$ matrix. Its only Ψ-dependence is in $\hat{Q}$. In section 4.2.2 we will show how to expand in a power series in $\hat{Q}-1$. In zeroth order, Γ_Q agrees with the Kupiainen-Gawedzki high frequency propagator,

$$(\Gamma_Q)_{\alpha\beta} = \Gamma_{\text{KG}}\delta_{\alpha\beta} + o(\hat{Q}-1).$$

Using this propagator, the effective action can be computed by perturbation theory. Because of the smoothness of Ψ, we only need to consider terms to order $|\nabla_\mu\Psi|^2$. But $S''_{\text{ext,I}}$ contains terms of order $\nabla_\mu\Psi$ and $|\nabla_\mu\Psi|^2$. The latter have to be treated to first order in perturbation theory only, while the others have to be included also to second order.

The result is

$$\begin{aligned} S_{\text{eff}}(\Phi) &= S_{\text{cl}}(\Psi) - \ln J^0(\mathcal{C}\Psi) - \frac{1}{2}\text{Tr}\,\ln\Gamma_{Q[\Psi]} \\ &\quad \underbrace{+\frac{1}{2}\langle\int_z \varphi^T S''_{\text{ext,I}}\varphi(z)\rangle}_{S^{(1)}_{\text{eff}}} \underbrace{-\frac{1}{8}\langle\left[\int_z \varphi^T S''_{\text{ext,I}}\varphi(z)\right]^2\rangle_{\text{c}}}_{S^{(2)}_{\text{eff}}}. \end{aligned} \tag{4.10}$$

$\langle . \rangle$ denotes the expectation value in a free field theory with propagator $\Gamma_{Q[\Psi]}$.

$$\langle\left[\int_z \varphi^T S''_{\text{ext,I}}\varphi(z)\right]^2\rangle_{\text{c}} = \langle\left[\int_z \varphi^T S''_{\text{ext,I}}\varphi(z)\right]^2\rangle - \langle\int_z \varphi^T S''_{\text{ext,I}}\varphi(z)\rangle^2$$

is the connected part of the diagrams contributing to the expectation value. Only the connected parts will contribute to the effective action.

The term $\frac{1}{2}\text{Tr}\,\ln\Gamma_{Q[\Psi]}$ is *not* a constant. It depends on the block spin field Φ through $\hat{Q}$ and Ψ.

4.1.2 The Extended Second Derivative of the Action

We need to determine ΔL which is defined by

$$\Delta L = |\nabla_\mu\xi|^2 + |\nabla_\mu\xi^0|^2 - |\nabla_\mu\varphi|^2.$$

Using (4.2) and (4.3) the calculation is straightforward. We use the lattice Leibnitz rule. So shifted arguments appear. Unwritten arguments are z. The kinetic term for ξ reads

$$\begin{aligned}
|\nabla_\mu \xi(z)|^2 &= |\nabla_\mu(\pi\varphi(z))|^2 \\
&= |\pi(z)\nabla_\mu\varphi(z) + (\nabla_\mu\pi)(z)\varphi(z+\hat\mu)|^2 \\
&= |\nabla_\mu\varphi(z) - \Psi\Psi^T\nabla_\mu\varphi(z) - (\nabla_\mu\Psi(z)\Psi^T(z+\hat\mu) + \Psi(z)\nabla_\mu\Psi^T(z))\varphi(z+\hat\mu)|^2 \\
&= |\nabla_\mu\varphi|^2 - \nabla_\mu\varphi^T\Psi\Psi^T\nabla_\mu\varphi \\
&\quad +\varphi^T(z+\hat\mu)\Psi(z+\hat\mu)\nabla_\mu\Psi^T\nabla_\mu\Psi\Psi^T(z+\hat\mu)\varphi(z+\hat\mu) \\
&\quad +\varphi^T(z+\hat\mu)\nabla_\mu\Psi\nabla_\mu\Psi^T\varphi(z+\hat\mu) \\
&\quad -2\nabla_\mu\varphi^T\nabla_\mu\Psi\Psi^T(z+\hat\mu)\varphi(z+\hat\mu) \\
&\quad +2\nabla_\mu\varphi^T\Psi\Psi^T\nabla_\mu\Psi\Psi^T(z+\hat\mu)\varphi(z+\hat\mu) \\
&\quad +2\varphi^T(z+\hat\mu)\Psi(z+\hat\mu)\underbrace{\nabla_\mu\Psi^T\Psi}_{-\frac{a}{2}|\nabla_\mu\Psi|^2}\nabla_\mu\Psi^T\varphi(z+\hat\mu). \qquad (4.11)
\end{aligned}$$

Unwritten arguments are z. The last term is of order $(\nabla_\mu\Psi)^3$ and is neglected. (4.1) adds a contribution

$$\begin{aligned}
|\nabla_\mu\xi_0^2(z)|^2 &= |\nabla_\mu(\Psi^T(z)\varphi(z))|^2 \\
&= |\nabla_\mu\Psi^T(z)\varphi(z+\hat\mu) + \Psi^T(z)\nabla_\mu\varphi(z)|^2 \\
&= \varphi^T(z+\hat\mu)\nabla_\mu\Psi\nabla_\mu\Psi^T\varphi(z+\hat\mu) \\
&\quad +2\varphi^T(z+\hat\mu)\nabla_\mu\Psi\Psi^T\nabla_\mu\varphi \\
&\quad +\nabla_\mu\varphi^T\Psi\Psi^T\nabla_\mu\varphi. \qquad (4.12)
\end{aligned}$$

Adding (4.11) and (4.12) and subtracting $|\nabla_\mu\varphi|^2$ we can determine ΔL as

$$\begin{aligned}
\Delta L &= \varphi^T(z+\hat\mu)\left[2\nabla_\mu\Psi\nabla_\mu\Psi^T + \Psi(z+\hat\mu)\nabla_\mu\Psi^T\nabla_\mu\Psi\Psi^T(z+\hat\mu)\right]\varphi(z+\hat\mu) \\
&\quad +2\nabla_\mu\varphi^T\left[\Psi\nabla_\mu\Psi^T - \nabla_\mu\Psi\Psi^T(z+\hat\mu) + \Psi\Psi^T\nabla_\mu\Psi\Psi^T(z+\hat\mu)\right]\varphi(z+\hat\mu). \qquad (4.13)
\end{aligned}$$

One can drop the shifted arguments in the $\Psi(z+\hat\mu)\nabla_\mu\Psi^T\nabla_\mu\Psi\Psi^T(z+\hat\mu)$ term because the deviation from $\Psi\nabla_\mu\Psi^T\nabla_\mu\Psi\Psi$ is of higher order in $\nabla_\mu\Psi$. Defining

$$j_\mu = \Psi\nabla_\mu\Psi^T - \nabla_\mu\Psi\Psi^T(z+\hat\mu) + \Psi\underbrace{\Psi^T\nabla_\mu\Psi}_{-\frac{a}{2}|\nabla_\mu\Psi|^2}\Psi^T(z+\hat\mu), \qquad (4.14)$$

we conclude

$$\begin{aligned}
\varphi^T S''_{\mathrm{ext,I}}\varphi &= \frac{1}{\cos\theta}\tilde\Phi^T\triangle\Psi\varphi^T\pi\varphi \\
&\quad +\varphi^T(z+\hat\mu)\left[2\nabla_\mu\Psi\nabla_\mu\Psi^T + \Psi\nabla_\mu\Psi^T\nabla_\mu\Psi\Psi^T\right]\varphi(z+\hat\mu) \\
&\quad +2\nabla_\mu\varphi^T j_\mu\varphi(z+\hat\mu). \qquad (4.15)
\end{aligned}$$

The last term in the definition of j_μ is of order $-\frac{a}{2}|\nabla_\mu\Psi|^2$. Therefore the relation

$$j_\mu(z) = j^T_{-\mu}(z+\hat{\mu}) \tag{4.16}$$

holds because

$$\begin{aligned} j^T_{-\mu}(z+\hat{\mu}) &= \nabla_{-\mu}\Psi(z+\hat{\mu})\Psi^T(z+\hat{\mu}) - \Psi(z)\nabla_{-\mu}\Psi^T(z+\hat{\mu}) + o(a) \\ &= -\nabla_\mu\Psi(z)\Psi^T(z+\hat{\mu}) + \Psi(z)\nabla_\mu\Psi^T(z) + o(a) \\ &= j_\mu(z) + o(a). \end{aligned}$$

4.2 The Effective Action

Here we determine the contributions of the different terms of the effective action (4.10). The expectation values can be evaluated using appendix B. Expanding in powers of $\hat{Q}-1$, the field dependence of $\mathrm{Tr}\,\ln\Gamma_{Q[\Psi]}$ is extracted.

4.2.1 The Contribution of the Expanded Kinetic Term

Inserting (4.15) in (4.8), we have to solve the Gaussian integral

$$e^{-S_{\mathrm{eff}}(\Phi)} = e^{-S_{\mathrm{cl}}(\Psi)} J^0(\mathcal{C}\Psi) \int \prod_z d\varphi(z) e^{-\frac{\beta}{2}\int_z \varphi^T \Gamma_Q^{-1}\varphi + \varphi^T S''_{\mathrm{ext,I}}\varphi}.$$

This integral is split into a Gaussian integral with covariance Γ_Q and a perturbation $e^{-\frac{\beta}{2}\int_z \varphi^T S''_{\mathrm{ext,I}}\varphi}$. As φ is not subject to a constraint, one can rescale this variable

$$\varphi \to \beta^{-\frac{1}{2}}\varphi.$$

Then the factor β in front of the integral in the exponent can be dropped.

The split yields

$$\begin{aligned} \int \prod_z d\varphi(z) e^{-\frac{1}{2}\int_z \varphi^T \Gamma_Q^{-1}\varphi + \varphi^T S''_{\mathrm{ext,I}}\varphi} &= \det(2\pi\Gamma_Q)^{\frac{1}{2}} \int d\mu_{\Gamma_Q}(\varphi) e^{-\frac{1}{2}\int_z \varphi^T S''_{\mathrm{ext,I}}\varphi} \\ &= e^{\frac{1}{2}\mathrm{Tr}\,\ln\Gamma_Q} \left\langle e^{-\frac{1}{2}\int_z \varphi^T S''_{\mathrm{ext,I}}\varphi} \right\rangle \\ &= e^{\frac{1}{2}\mathrm{Tr}\,\ln\Gamma_Q} e^{\frac{1}{2}\langle \frac{\delta}{\delta\varphi}, \Gamma_Q \frac{\delta}{\delta\varphi}\rangle} e^{-\frac{1}{2}\int_z \varphi^T S''_{\mathrm{ext,I}}\varphi}\Big|_{\varphi=0}, \end{aligned}$$

with

$$\langle \frac{\delta}{\delta\varphi}, \Gamma_Q \frac{\delta}{\delta\varphi}\rangle = \int_u \int_v \Gamma_Q(u,v) \frac{\delta}{\delta\varphi(u)} \frac{\delta}{\delta\varphi(v)},$$

see appendix B. Because of the smoothness of Ψ, it is sufficient to expand the exponential with the $S''_{\text{ext,I}}$ dependence to second order. To first order we obtain a contribution

$$S^{(1)}_{\text{eff}} = \frac{1}{2}\langle \int_z \varphi^T S''_{\text{ext,I}} \varphi \rangle,$$

and the second order adds a term

$$S^{(2)}_{\text{eff}} = -\frac{1}{8}\langle \int_z \varphi^T S''_{\text{ext,I}} \varphi(z) \int_w \varphi^T S''_{\text{ext,I}} \varphi(w) \rangle_{\text{c}}.$$

In the last expression only the j_μ-dependent part of $S''_{\text{ext,I}}$ is needed because only j_μ is of order $\nabla_\mu \Psi$, and we are only interested in the result to order $(\nabla_\mu \Psi)^2$.

This yields the results for the effective action (4.10) if one considers only the connected diagrams.

Inserting $S''_{\text{ext,I}}$, the first order correction is

$$\begin{aligned} S^{(1)}_{\text{eff}} = \frac{1}{2}\langle \int_z \varphi^T S''_{\text{ext,I}} \varphi \rangle &= \frac{1}{2}\int_{u,v} \Gamma_Q(u,v) \frac{\delta}{\delta\varphi(u)} \frac{\delta}{\delta\varphi(v)} \int_z \varphi^T S''_{\text{ext,I}} \varphi \Big|_{\varphi=0} \\ &= \frac{1}{4}\cdot 2 \int_z \text{tr}\, \left[2\nabla_\mu \Psi \nabla_\mu \Psi^T + \Psi \nabla_\mu \Psi^T \nabla_\mu \Psi \Psi^T \right] \Gamma_Q(z+\hat{\mu}, z+\hat{\mu}) \\ &\quad + \frac{1}{4}\cdot 2 \cdot 2 \int_z \text{tr}\; j_\mu \nabla_\mu \Gamma_Q(z,w)|_{w=z+\hat{\mu}} \\ &\quad + \frac{1}{4}\cdot 2 \int_z \text{tr}\, \pi \Gamma_Q(z,z) \frac{1}{\cos\theta} \tilde{\Phi}^T \triangle \Psi. \end{aligned}$$

The second order correction $S^{(2)}_{\text{eff}}$ is obtained by expanding $e^{\frac{1}{2}\langle \frac{\delta}{\delta\varphi}, \Gamma_Q \frac{\delta}{\delta\varphi} \rangle}$ to second order.

$$\begin{aligned} S^{(2)}_{\text{eff}} &= -\frac{1}{8}\cdot 4 \langle \int_z \nabla_\mu \varphi^T(z) j_\mu(z) \varphi(z+\hat{\mu}) \int_w \nabla_\nu \varphi^T(w) j_\nu(w) \varphi(w+\hat{\nu}) \rangle_{\text{c}} \\ &= -\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} \langle \frac{\delta}{\delta\varphi}, \Gamma_Q \frac{\delta}{\delta\varphi} \rangle \frac{1}{2} \langle \frac{\delta}{\delta\varphi}, \Gamma_Q \frac{\delta}{\delta\varphi} \rangle \\ &\quad \int_z \nabla_\mu \varphi^T(z) j_\mu(z) \varphi(z+\hat{\mu}) \int_w \nabla_\nu \varphi^T(w) j_\nu(w) \varphi(w+\hat{\nu}) - \text{disconnected} \\ &= -\frac{1}{16} \int_u \int_v \int_l \int_m \frac{\delta}{\delta\varphi(u)} \Gamma_Q(u,v) \frac{\delta}{\delta\varphi(v)} \frac{\delta}{\delta\varphi(l)} \Gamma_Q(l,m) \frac{\delta}{\delta\varphi(m)} \\ &\quad \int_z \nabla_\mu \varphi^T(z) j_\mu(z) \varphi(z+\hat{\mu}) \int_w \nabla_\nu \varphi^T(w) j_\nu(w) \varphi(w+\hat{\nu}) - \text{disconnected} \end{aligned}$$

Only the first two terms in the definition of j_μ (4.14) are needed here because the last term is $o(a|\nabla_\mu \Psi|^2)$ and will not contribute to order $(\nabla_\mu \Psi)^2$ in the second order correction.

Let ∇^z denote derivatives with respect to z. $\overleftarrow{\nabla}^w$ denotes derivatives acting to the left with respect to w. The result [1] is

$$S_{\text{eff}}^{(2)} = -\frac{1}{2}\int_z\int_w \text{tr}\,\Big[\nabla_\mu^z\Gamma_Q(z,w)\,\overleftarrow{\nabla}_\nu^w\, j_\nu(w)\Gamma_Q(w+\hat\nu,z+\hat\mu)j_\mu^T(z) + \nabla_\mu^z\Gamma_Q(z,w+\hat\nu)j_\nu^T(w)\nabla_\nu^w\Gamma_Q(w,z+\hat\mu)j_\mu^T(z)\Big] \tag{4.17}$$

$S_{\text{eff}}^{(2)}$ is singular in the limit $a\to 0$.

$$S_{\text{eff}}^{(2)} = S_{\text{eff,finite}}^{(2)} + S_{\text{eff,singular}}^{(2)}.$$

So we have to extract the singular part which is proportional to $\ln\frac{\tilde a}{a}$ because it will contribute to Polyakov's result. Using

$$j_\mu(z) = j_{-\mu}^T(z+\hat\mu) \approx -j_\mu^T(z),$$

we see by partial integration that the singular parts of

$$\nabla_\mu^z\Gamma_Q(z,w)\,\overleftarrow{\nabla}_\nu^w\, j_\nu(w)\Gamma_Q(w+\hat\nu,z+\hat\mu)j_\mu^T(z)$$

and

$$\nabla_\mu^z\Gamma_Q(z,w+\hat\nu)j_\nu^T(w)\nabla_\mu^w\Gamma_Q(w,z+\hat\mu)j_\mu^T(z)$$

are equal. It suffices to extract the singular part of the 1-loop Feynman diagram

$$\mathcal{G}_{\mu\nu}^{\alpha\beta,\gamma\delta}(z,w) = \nabla_\mu^z\Gamma_Q^{\alpha\beta}(z,w)\,\overleftarrow{\nabla}_\nu^w\,\Gamma_Q^{\gamma\delta}(w+\hat\nu,z+\hat\mu). \tag{4.18}$$

Its singular part does not depend on the details of the cut-off. Thus it is independent of Q and the choice of block spin, and we can replace the full high frequency propagators again by Yukawa potentials $v_M(z-w)\delta^{\alpha\beta}$ with mass M of order $\tilde a^{-1}$ as in section 3.3. This restores the translation invariance of the propagators on the fine lattice Λ.

The form of the singular part is

$$G\delta_{\mu\nu}\delta_{\alpha\beta}\delta_{\gamma\delta}\delta(z-w)$$

with an yet unknown coefficient G. This follows from power counting and rotational invariance.

[1] It can be expressed with arguments $z+\hat\mu$ and $w+\hat\nu$ only

$$S_{\text{eff}}^{(2)} = -\frac{1}{2}\int_{z'}\int_{w'} \text{tr}\,\Big[\nabla_{\mu'}^{z'}\Gamma_Q(z',w')\,\overleftarrow{\nabla}_{\nu'}^{w'}\, j_{\nu'}^T(w')\Gamma_Q(w',z')j_{\mu'}(z') + \nabla_{\mu'}^{z'}\Gamma_Q(z',w')j_{\nu'}(w')\nabla_{\nu'}^{w'}\Gamma_Q(w',z')j_{\mu'}(z')\Big].$$

The notation $\mu'=-\mu$, $\nu'=-\nu$ was used.

Consider $\delta_{\mu\nu}\mathcal{G}_{\mu\nu}$. Since

$$-\triangle\Gamma = 1 + \text{nonsingular},$$

G is determined as

$$G = \frac{1}{2}.$$

The singular part turns out to be

$$S^{(2)}_{\text{eff,singular}} \sim v_M(0) j_\mu(z) j^T_\mu(z).$$

Inserting this yields

$$S^{(2)}_{\text{eff}} = \text{nonsingular} - \frac{1}{2}\int_z\int_w 2\cdot\frac{1}{2}\delta_{\mu\nu}\delta(z-w) j_\nu(w)\Gamma_Q(w+\hat{\nu}, z+\hat{\mu}) j^T_\mu(z). \tag{4.19}$$

Integrating over w and setting $\nu=\mu$, we obtain to order $(\nabla_\mu\Psi)^2$

$$\begin{aligned}
S^{(2)}_{\text{eff,singular}} &= -\frac{1}{2}\int_z\int_w \delta_{\mu\nu}\delta(z-w)\text{tr}\, j_\nu(w)\Gamma_Q(w+\hat{\nu}, z+\hat{\mu}) j^T_\mu(z) \\
&= -\frac{1}{2}\int_z \text{tr}\, j_\mu(z)\Gamma_Q(z+\hat{\mu}, z+\hat{\mu}) j^T_\mu(z) \\
&= -\frac{1}{2}\int_z \text{tr}\, \big[(\nabla_\mu\Psi(z)\Psi^T(z) - \Psi(z+\hat{\mu})\nabla_\mu\Psi^T(z)) \\
&\qquad (\Psi(z)\nabla_\mu\Psi^T(z) - \nabla_\mu\Psi(z)\Psi^T(z+\hat{\mu}))\big]\,\Gamma_Q(z+\hat{\mu}, z+\hat{\mu}) \\
&= -\frac{1}{2}\int_z \text{tr}\,[\nabla_\mu\Psi(z)\Psi^T(z)\Psi(z)\nabla_\mu\Psi^T(z) \\
&\qquad \underbrace{-\Psi(z+\hat{\mu})\nabla_\mu\Psi^T(z)\Psi(z)\nabla_\mu\Psi^T(z)}_{\text{lattice artefact}} \\
&\qquad \underbrace{-\nabla_\mu\Psi(z)\Psi^T(z)\nabla_\mu\Psi(z)\Psi^T(z+\hat{\mu})}_{\text{lattice artefact}} \\
&\qquad +\Psi(z+\hat{\mu})\nabla_\mu\Psi^T(z)\nabla_\mu\Psi(z)\Psi^T(z+\hat{\mu})]\Gamma_Q(z+\hat{\mu}, z+\hat{\mu}) \\
&= -\frac{1}{2}\int_z \text{tr}\,\big[\nabla_\mu\Psi(z)\nabla_\mu\Psi^T(z) \\
&\qquad +\Psi(z+\hat{\mu})\nabla_\mu\Psi^T(z)\nabla_\mu\Psi(z)\Psi^T(z+\hat{\mu})\big]\,\Gamma_Q(z+\hat{\mu}, z+\hat{\mu})
\end{aligned} \tag{4.20}$$

The singular term is subtracted from $S^{(2)}_{\text{eff}}$ to make the diagram finite. It is added to the

term $S^{(1)}_{\text{eff}}$. The final result is

$$\begin{aligned} S^{(1)}_{\text{eff,ren}} &= \frac{1}{2}\int_z \operatorname{tr}\left[\nabla_\mu \Psi \nabla_\mu \Psi^T\right]\Gamma_Q(z+\hat{\mu}, z+\hat{\mu}) \\ &\quad + \frac{1}{2}\int_z \operatorname{tr}\frac{1}{\cos\theta}[\tilde{\Phi}^T \triangle \Psi]\pi\Gamma_Q(z,z) \\ &\quad + \int_z j_\mu(z)\nabla_\mu \Gamma_Q(z,w)|_{w=z+\hat{\mu}}, \end{aligned} \tag{4.21}$$

$$\begin{aligned} S^{(2)}_{\text{eff,ren}} &= -\frac{1}{2}\int_z\int_w \operatorname{tr}\Big[\nabla^z_\mu \Gamma_Q(z,w)\,\overleftarrow{\nabla}^w_\nu\, j_\nu(w)\Gamma_Q(w+\hat{\nu}, z+\hat{\mu}) j^T_\mu(z) \\ &\qquad + \nabla^z_\mu \Gamma_Q(z, w+\hat{\nu}) j^T_\nu(w)\nabla^w_\nu \Gamma_Q(w, z+\hat{\mu}) j^T_\mu(z) \\ &\qquad - \delta_{\mu\nu}\delta(z-w) j_\nu(w)\Gamma_Q(w+\hat{\nu}, z+\hat{\mu}) j^T_\mu(z)\Big] \end{aligned} \tag{4.22}$$

Evaluation of the Lattice Correction Term

To get a simplified result in zeroth order in $(\hat{Q}-1)$, we also need to evaluate the lattice artefact in $S^{(1)}_{\text{eff}}$. Γ_Q can be replaced by Γ_{KG} to this order. The artefact is

$$\int_z \operatorname{tr} j_\mu(z)\nabla_\mu \Gamma_{\text{KG}}(z,w)|_{w=z+\hat{\mu}}\,. \tag{4.23}$$

Consider the derivative of $\Psi^T\Psi = 1$.

$$\begin{aligned} & \nabla_\mu\left(\Psi^T\Psi\right) = 0 \\ \Leftrightarrow\quad & \nabla_\mu \Psi^T\Psi(z+\hat{\mu}) + \Psi^T\nabla_\mu\Psi = 0. \end{aligned}$$

On the other hand,

$$\Psi(z+\hat{\mu}) = \Psi(z) + a\nabla_\mu\Psi(z).$$

Thus,

$$\nabla_\mu \Psi^T\Psi = -\frac{a}{2}|\nabla_\mu\Psi|^2. \tag{4.24}$$

This can be used to calculate the trace,

$$\begin{aligned} \operatorname{tr} j_\mu &= \nabla_\mu\Psi^T\Psi - \underbrace{\Psi^T(z+\hat{\mu})\nabla_\mu\Psi}_{-\nabla_\mu\Psi^T\Psi} + (1+a\nabla_\mu\Psi^T\Psi)\nabla_\mu\Psi^T\Psi \\ &= 3\nabla_\mu\Psi^T\Psi = -\frac{3}{2}a|\nabla_\mu\Psi|^2. \end{aligned}$$

So (4.23) is really a lattice artefact. In the continuum limit $\nabla_\mu\Psi^T\Psi = 0$ [2].

[2] As Ψ is a vector of unit length, this is evident because $\nabla_\mu\Psi(z) \in T_{\Psi(z)}S^{\mathcal{N}}$ on the continuum. But here we need the exact form of the lattice artefact.

Nevertheless the artefact can not be neglected because

$$\begin{aligned}\nabla_\mu \Gamma_{\rm KG}(z,w)|_{w=z+\hat\mu} &= -\nabla_{-\mu}\Gamma_{\rm KG}(z+\hat\mu,w)|_{w=z+\hat\mu} \\ &= -\frac{a}{4}\triangle\Gamma_{\rm KG}(z+\hat\mu,w)\Big|_{w=z+\hat\mu} \\ &= \frac{1}{4a}+O(1).\end{aligned}$$

This holds true because the singular part of $\Gamma_{\rm KG}$ is translation invariant and because $\nabla_\mu\Gamma_{\rm KG}(z,w)|_{z=w}$ is independent of $\pm\mu$ by lattice symmetry. As $\sum_{\pm\mu}\nabla_\mu f = a\triangle f$ and $\triangle\Gamma\sim\frac{1}{a^2}$, the statement is proven. Therefore the lattice artefact comes out as

$$\mathrm{tr}\, j_\mu(z)\ \nabla_\mu\Gamma_{\rm KG}(z,w)|_{w=z+\hat\mu} = -\frac{3}{8}|\nabla_\mu\Psi|^2(z)+O(a). \tag{4.25}$$

4.2.2 Field Dependence of the High Frequency Propagator for $\kappa\to\infty$

The effective action contains a term $\mathrm{Tr}\ln\Gamma_Q$. It has a field dependence through $\hat Q$. To extract this dependence we need to expand the propagator around the familiar high frequency propagator $\Gamma_{\rm KG}$ in powers of $\hat Q-1$. To obtain a result in the limit $\kappa\to\infty$ we use a propagator Γ_α which depends on a real parameter α,

$$\Gamma_\alpha = (-\triangle+\kappa\mathcal{C}_\alpha^\dagger\mathcal{C}_\alpha)^{-1} \tag{4.26}$$

where $\mathcal{C}_\alpha$ is a (matrix valued) α-dependent block averaging operator. A limit $\kappa\mapsto\infty$ should be taken in the end, if desired. In our application

$$\mathcal{C}_\alpha = \mathbf{1}\mathcal{C}+\alpha\mathcal{C}(\hat Q-1). \tag{4.27}$$

We will use a formula which gives the derivative Γ'_α of Γ_α with respect to α.

Let $v=\triangle^{-1}$. It is known from the work of Kupiainen and Gawedzki, that Γ_α admits the following representation

$$\Gamma_\alpha = (1-\mathcal{A}_\alpha\mathcal{C}_\alpha)v \tag{4.28}$$

where $\mathcal{A}_\alpha$ is an interpolation operator which maps functions on the coarse lattice into smooth functions on the fine lattice, and which obeys

$$\mathcal{C}_\alpha\mathcal{A}_\alpha = 1-\frac{1}{\kappa}u_\alpha^{-1}. \tag{4.29}$$

For finite κ,

$$\begin{aligned}\mathcal{A}_\alpha &= v\mathcal{C}_\alpha^\dagger u_\alpha^{-1}, \\ u_\alpha &= \mathcal{C}_\alpha v\mathcal{C}_\alpha^\dagger+\frac{1}{\kappa}.\end{aligned} \tag{4.30}$$

Since $(u_\alpha^{-1})' = -u_\alpha^{-1} u'_\alpha u_\alpha^{-1}$, one obtains by straightforward differentiation with respect to α

$$\begin{aligned} \Gamma'_\alpha &= -(\Gamma_\alpha \mathcal{C}_\alpha^{\dagger\prime} \mathcal{A}_\alpha^\dagger + \mathcal{A}_\alpha \mathcal{C}'_\alpha \Gamma_\alpha) && (4.31) \\ \mathcal{A}'_\alpha &= \Gamma_\alpha \mathcal{C}_\alpha^{\dagger\prime} u_\alpha^{-1} - \mathcal{A}_\alpha \mathcal{C}'_\alpha \mathcal{A}_\alpha \ . && (4.32) \end{aligned}$$

In our application, $\mathcal{C}'_\alpha = \mathcal{C}(\hat{Q}-1)$ independent of α, and $\mathcal{A}_{\alpha=0}$ is the Kupiainen-Gawedzki interpolation operator $\mathcal{A}_{\rm KG}$ multiplied by the $(\mathcal{N}+1)\times(\mathcal{N}+1)$ unit matrix $\mathbf{1}$. Therefore the expansion of Tr $\ln \Gamma_Q$ to second order in $\hat{Q}-1$ reads

$$\text{Tr } \ln \Gamma_Q = \text{Tr } \ln \Gamma_{KG} + \text{Tr } \left(\Gamma'_\alpha \Gamma_\alpha^{-1}\right)|_{\alpha=0} + \frac{1}{2}\text{Tr } \left(\Gamma'_\alpha \Gamma_\alpha^{-1}\right)'|_{\alpha=0} + \ldots \tag{4.33}$$

with

$$\text{Tr } \left(\Gamma'_\alpha \Gamma_\alpha^{-1}\right) = -\text{Tr } \Big[\mathcal{A}_\alpha \mathcal{C}(\hat{Q}-1) + \text{h.c.}\Big] \tag{4.34}$$

and

$$\begin{aligned} \text{Tr } \left(\Gamma'_\alpha \Gamma_\alpha^{-1}\right)' &= -\text{Tr } [\mathcal{A}'_\alpha \mathcal{C}'_\alpha + \text{h.c.}] && (4.35) \\ &= -\text{Tr } \Big[\Gamma_\alpha (\mathcal{C}_\alpha^\dagger)' u_\alpha^{-1} \mathcal{C}'_\alpha - \mathcal{A}_\alpha \mathcal{C}'_\alpha \mathcal{A}_\alpha \mathcal{C}'_\alpha + \text{h.c.}\Big]_{\alpha=0} && (4.36) \end{aligned}$$

The first term in eq. (4.33) is a field independent constant.

The kernels $\mathcal{A}_{\alpha=0}$ and $\mathcal{C}$ are proportional to the unit matrix, therefore the first order term involves

$$\text{tr}\,(\hat{Q}-1) = \cos^2\theta + \cos\theta - 2. \tag{4.37}$$

As a result

$$-\frac{1}{2}\text{Tr } \left(\Gamma'_\alpha \Gamma_\alpha^{-1}\right) = \int_x \int_z \mathcal{A}_{\rm KG}(z,x)\mathcal{C}(x,z)[\cos^2\theta(z) + \cos\theta(z) - 2] + \text{const} \tag{4.38}$$

It can now be inserted into the result for Tr $\ln \Gamma_Q$ (4.33).

It remains to examine the second order term. To order $\epsilon = \Psi - \tilde{\Phi}$ we can write

$$(\hat{Q}-1)(z) = \cos\theta(z)\tilde{\Phi}(z)\Psi^T(z) - \Psi(z)\tilde{\Phi}^T(z) + \ldots$$

The second order term is an integral whose integrand contains factors

$$\begin{aligned} \text{tr}\,[(\hat{Q}(w)-1)^T(\hat{Q}(z)-1)] &= (1+\cos\theta(z)\cos\theta(w))(\tilde{\Phi}(z)\cdot\tilde{\Phi}(w))(\Psi(z)\cdot\Psi(w)) \\ &\quad -(\cos\theta(z)+\cos\theta(w))(\Psi(w)\cdot\tilde{\Phi}(z))(\Psi(z)\cdot\tilde{\Phi}(w)) \end{aligned}$$

and

$$\begin{aligned} \text{tr}\,[(\hat{Q}(z)-1)(\hat{Q}(w)-1)] &= (1+\cos\theta(z)\cos\theta(w))(\Psi(w)\cdot\tilde{\Phi}(z))(\Psi(z)\cdot\tilde{\Phi}(w)) \\ &\quad -(\cos\theta(z)+\cos\theta(w))(\tilde{\Phi}(z)\cdot\tilde{\Phi}(w))(\Psi(z)\cdot\Psi(w)). \end{aligned}$$

They may now be inserted into eq.(4.36) to yield the result for the second order contribution

$$\begin{aligned}
-\frac{1}{4}\mathrm{Tr}\,\left(\Gamma'_\alpha\Gamma^{-1}_\alpha\right)'_{\alpha=0} &= \frac{1}{2}\int_{z,w}\int_{x,y}[(1+\cos\theta(z)\cos\theta(w))(\tilde\Phi(z)\cdot\tilde\Phi(w))(\Psi(z)\cdot\Psi(w)) \\
&\quad -(\cos\theta(z)+\cos\theta(w))(\Psi(w)\cdot\tilde\Phi(z))(\Psi(z)\cdot\tilde\Phi(w))]\cdot \\
&\quad \cdot[\Gamma_{KG}(z,w)\mathcal{C}^\dagger(w,x)u^{-1}(x,y)\mathcal{C}(y,z)] \\
&\quad -[(1+\cos\theta(z)\cos\theta(w))(\Psi(w)\cdot\tilde\Phi(z))(\Psi(z)\cdot\tilde\Phi(w)) \\
&\quad -(\cos\theta(z)+\cos\theta(w))(\tilde\Phi(z)\cdot\tilde\Phi(w))(\Psi(z)\cdot\Psi(w))]\cdot \\
&\quad \cdot[\mathcal{A}_{\mathrm{KG}}(z,x)\mathcal{C}(x,w)\mathcal{A}_{\mathrm{KG}}(w,y)\mathcal{C}(y,z)]. \qquad (4.39)
\end{aligned}$$

These results are also valid for finite κ.

Summing the two terms we obtain the desired expansion of $-\frac{1}{2}\mathrm{Tr}\,\ln\Gamma_Q$, see eq.(4.40).

4.2.3 The Result and some Simplifications

Here we collect the different contributions of the effective action introduced in (4.10),

$$\begin{aligned}
S_{\mathrm{eff}}(\Phi) &= S_{\mathrm{cl}}(\Psi)-\ln J^0(\mathcal{C}\Psi)-\frac{1}{2}\mathrm{Tr}\,\ln\Gamma_{Q[\Psi]} \\
&\quad +S^{(1)}_{\mathrm{eff}}+S^{(2)}_{\mathrm{eff}}.
\end{aligned}$$

$S^{(1)}_{\mathrm{eff}}$ and $S^{(2)}_{\mathrm{eff}}$ are the contributions of the second derivative of the extended action, $S''_{\mathrm{ext,I}}$, in first and second order perturbation theory. They are expectation values defined with respect to the covariance

$$\Gamma_Q=\lim_{\kappa\to\infty}\left(-\triangle+\kappa\hat Q^T\mathcal{C}^\dagger\mathcal{C}\hat Q\right)^{-1}.$$

The first term is the classical perfect action $S_{\mathrm{cl}}[\Psi[\Phi]]$. It will be made explicit in chapter 5 because there a construction of an extremizing field Ψ for given block spin Φ is proposed.

The Jacobian $J^0(\mathcal{C}\Psi)$ is given in (2.13) as

$$J^0(\mathcal{C}\Psi)=\prod_x(\Phi(x)\cdot\mathcal{C}\Psi(x))^{\mathcal{N}}.$$

Its logarithm may be combined with the result for $\frac{1}{2}\mathrm{Tr}\,\ln\Gamma_Q$,

$$\begin{aligned}
-\frac{1}{2}\mathrm{Tr}\,\ln\Gamma_Q &= \int_x\int_z\mathcal{A}_{\mathrm{KG}}(z,x)\mathcal{C}(x,z)[\cos^2\theta(z)+\cos\theta(z)-2] \\
&+ \frac{1}{2}\int_{z,w}\int_{x,y}[(1+\cos\theta(z)\cos\theta(w))(\tilde\Phi(z)\cdot\tilde\Phi(w))(\Psi(z)\cdot\Psi(w)) \\
&\quad -(\cos\theta(z)+\cos\theta(w))(\Psi(w)\cdot\tilde\Phi(z))(\Psi(z)\cdot\tilde\Phi(w))]\cdot \\
&\quad \cdot[-\Gamma_{KG}(z,w)\triangle\mathcal{A}_{\mathrm{KG}}(w,y)\mathcal{C}(y,z)] \\
&\quad -[(1+\cos\theta(z)\cos\theta(w))(\Psi(w)\cdot\tilde\Phi(z))(\Psi(z)\cdot\tilde\Phi(w)) \\
&\quad -(\cos\theta(z)+\cos\theta(w))(\tilde\Phi(z)\cdot\tilde\Phi(w))(\Psi(z)\cdot\Psi(w))]\cdot \\
&\quad \cdot[\mathcal{A}_{\mathrm{KG}}(z,x)\mathcal{C}(x,w)\mathcal{A}_{\mathrm{KG}}(w,y)\mathcal{C}(y,z)]. \qquad (4.40)
\end{aligned}$$

The relation $\mathcal{C}^\dagger u^{-1} = v^{-1}\mathcal{A}_{\rm KG} = -\triangle\mathcal{A}_{\rm KG}$ was used to eliminate the block spin propagator u in $\mathrm{Tr}\,\ln\Gamma_Q$.

The remaining one loop contributions are $S^{(1)}_{\rm eff,ren}$ and $S^{(2)}_{\rm eff,ren}$. They were renormalized by the singular part of $-\frac{1}{8}\langle\big[\int_z \varphi^T S''_{\rm ext,I}\varphi(z)\big]^2\rangle_{\rm c}$. From the first, a lattice artefact has been extracted,

$$\mathrm{tr}\, j_\mu(z)\ \nabla_\mu\Gamma_{\rm KG}(z,w)|_{w=z+\hat\mu} = -\frac{3}{8}|\nabla_\mu\Psi|^2(z).$$

The complete $S^{(1)}_{\rm eff,ren}$ will renormalize the coupling constant β. Splitting again $\tilde\Phi$ like in (3.15), we can write

$$\begin{aligned} S^{(1)}_{\rm eff,ren} &= \frac{1}{2}\int_z \nabla_\mu\Psi^T\delta\beta^1_{\rm eff}(z+\hat\mu)\nabla_\mu\Psi \\ &\quad +\frac{1}{2}\int_z \delta\beta^2_{\rm eff}(z)[\Psi^T\triangle\Psi] + \delta\beta^2_{\rm eff}(z)\frac{1}{\cos\theta}[(\Phi^T)^\perp\triangle\Psi] \\ &\quad +\int_z \mathrm{tr}\, j_\mu(z)\nabla_\mu\Gamma_Q(z,w)|_{w=z+\hat\mu}, \end{aligned}$$

with

$$\delta\beta^1_{\rm eff}(z) = \Gamma_Q(z,z), \tag{4.41}$$

$$\delta\beta^2_{\rm eff}(z) = \mathrm{tr}\,[1-\Psi\Psi^T]\Gamma_Q(z,z). \tag{4.42}$$

These coupling constant renormalizations both have a dependence on Ψ through $\hat Q$. This makes them dependent on Φ. $\delta\beta^1_{\rm eff}$ is a $(\mathcal{N}+1)\times(\mathcal{N}+1)$ matrix, while $\delta\beta^2_{\rm eff}$ is a scalar. The term containing the part of Φ perpendicular to Ψ, $(\Phi^T)^\perp$, will not contribute to the effective action to order $\ln\frac{\tilde a}{a}$, see section 3.4. There is a very small remainder in the exact result because here $v_M(0)$ is replaced by $\Gamma_Q(z,z)$ which is not diagonal in order $\hat Q-1$. However, because $\Phi^\perp$ is also small, this term is negligible in first order in $\hat Q-1$. It will not enter the simplified result below.

Except from the coupling constant renormalization, there is a genuine one loop correction.

$$\begin{aligned} S^{(2)}_{\rm eff,ren} &= -\frac{1}{2}\int_z\int_w \mathrm{tr}\,\Big[\nabla^z_\mu\Gamma_Q(z,w)\overleftarrow{\nabla}^w_\nu\, j^T_\nu(w)\Gamma_Q(w+\hat\nu,z+\hat\mu)j_\mu(z) \\ &\qquad +\nabla^z_\mu\Gamma_Q(z,w+\hat\nu)j_\nu(w)\nabla^w_\nu\Gamma_Q(w,z+\hat\mu)j_\mu(z) \\ &\qquad -\delta_{\mu\nu}\delta(z-w)j^T_\nu(w)\Gamma_Q(w+\hat\nu,z+\hat\mu)j_\mu(z)\Big]. \end{aligned} \tag{4.43}$$

It depends on the matrix $\Psi\nabla_\mu\Psi^T$ at two different sites z,w.

For practical purposes one needs some simplification. Later we will see that a Gaussian block spin definition allows a treatment of (4.8) without the introduction of Γ_Q as a Ψ-dependent propagator. But here we need the assumption that Φ is smooth to make further approximations. This has already been exploited to expand $\mathrm{Tr}\,\ln\Gamma_Q$ in powers of $(\hat Q-1)$.

To leading order, the terms of order $(\hat{Q}-1)$ will be neglected except in the Tr $\ln \Gamma_Q$-term. We use

$$\Gamma_Q = \Gamma_{\text{KG}} \mathbf{1} + o(\hat{Q}-1).$$

The simplified result is

$$\begin{aligned} S_{\text{eff}} &= -\sum_x \ln J_0(\mathcal{C}\Psi(x)) - \frac{1}{2}\text{Tr } \ln \Gamma_Q + \frac{1}{2}\int_z \underbrace{(\beta - \mathcal{N}\delta\beta_{\text{eff}}(z) + \delta\beta_{\text{eff}}(z+\hat{\mu}))\,|\nabla_\mu \Psi(z)|^2}_{\beta_{\text{eff}}(z)|\nabla_\mu\Psi(z)|^2} \\ &\quad + \text{renormalized 1-loop diagram} + \text{lattice artefact.} \end{aligned} \tag{4.44}$$

The sum of the first two terms will be called the *augmented Jacobian*, the Jacobian proper is given by (2.13). The renormalized 1-loop diagram is the same as $S^{(2)}_{\text{eff,ren}}$ above, except that Γ_{KG} is substituted for Γ_Q; moreover the last term in the definition (4.14) of j_μ may be dropped in $S^{(2)}_{\text{eff,ren}}$ because it is of order a.

The coupling constant is renormalized by

$$\delta\beta_{\text{eff}}(z) = \Gamma_{KG}(z,z). \tag{4.45}$$

Γ_{KG} has only translational symmetry under translations by block lattice vectors. Therefore the renormalized coupling fluctuates with z. But nevertheless $\Gamma_{KG}(z,z)$ is very nearly constant except near block boundaries. We expect that the deviation of $\delta\beta_{\text{eff}}(z)$ from its block average can be neglected. Finally

$$\begin{aligned} \text{lattice artefact} &= \int_z \text{tr}\, j_\mu(z) \nabla_\mu \Gamma_{KG}(z,w)|_{w=z+\hat{\mu}} \\ &= -\frac{3}{8}\int_z |\nabla_\mu \Psi|^2(z) + O(a). \end{aligned}$$

as $a \to 0$. The lattice artefacts would vanish if $\Psi^T \nabla_\mu \Psi$ were zero as is true in the continuum. On the lattice it is $o(a)$, but nevertheless there remains a contribution when $a \to 0$ because of the singularity of $\nabla_\mu \Gamma_{KG}$ at coinciding points. It amounts to a finite subtraction from the bare coupling constant.

Only the β_{eff}-Term contributes to order $\ln \tilde{a}/a$, and we recover again Polyakov's result in this approximation.

The formula for Tr $\ln \Gamma_Q$ to second order in $(\hat{Q}-1)$ was given above in (4.40).

Chapter 5

The Background Field

The effective action S_{eff} has to be expressed in terms of the block spin field Φ. To derive such an expression, one has to determine the background field Ψ as a function of Φ.

Given the block spin field on the coarse lattice, we seek the field $\phi = \Psi$ on the fine lattice which extremizes the action $S[\phi]$ of the σ-model subject to the constraint

$$C\Psi = \Phi. \tag{5.1}$$

The extremality condition leads to a nonlinear equation for Ψ (see eq.(3.24) above). It is nonlinear because $\Psi(z)$ must have length 1.

The strategy is to start with an approximation $\Psi^{(0)}$ which satisfies the block spin condition exactly ($\mathcal{C}\Psi^{(0)\perp} = 0$), which has the expected smoothness properties of the exact Ψ except for discontinuities of the normal derivatives at block boundaries, and which reduces to the exact extremum when Φ is constant. For reasonably smooth Φ the discontinuities of the normal derivatives will be small.

Using $\Psi^{(0)}$ as a first approximation, we can derive improved approximations $\Psi^{(k)}$, $k = 1, 2, ...$ by iterating a procedure described below. We will see that the smoothness of Ψ can again be exploited to argue that a single iteration with result $\Psi^{(1)}$ is enough if Φ is reasonably smooth. The formula for $\Psi^{(1)}$ will involve the high frequency propagator $\Gamma_{Q[\Psi^{(0)}]}$ defined in (4.9). This propagator contains a dependence on $\Psi^{(0)}$ through Q.

5.1 Smooth Interpolation of Block Spin Fields

Here we construct a first approximation $\Psi^{(0)}(z)$ on the continuum which has a given block spin

$$\Phi(x) = \underset{z \in x}{\text{av}}\, \Psi(z)/(modulus) \equiv \mathcal{C}\Psi(x)/(modulus).$$

It is continuous and smooth except for (small) discontinuities of the normal derivative on block boundaries, and it is close to $\tilde{\Phi}(z)$ if Φ is smooth. Remember that the average $\underset{z\in x}{\text{av}}$ is over the *lattice points* inside the square. The lattice field $\Psi^{(0)}$ is obtained by restriction to points z on the lattice.

We assume that the block spin Φ is reasonably smooth so that $\Phi(x)\cdot\Phi(y) > 0$ when x, y are nearest or next nearest neighbours. This restriction removes some sign arbitrariness which could otherwise lead to discontinuities.

The continuum is divided into squares x of sidelength $\tilde{a}$; the lattice points inside the squares form a block.

The field $\Psi^{(0)}$ is constructed as follows.

1. **Preliminary definition of $\Psi^{(0)}$ on the corners of the squares:** We determine the field at the corners z_c of the squares,

$$\Psi^{(0)}(z_c) = (\sum_{x_i(z_c)} \Phi(x_i))/(modulus)\ . \tag{5.2}$$

The sum goes over the four squares $x_i(z_c)$ with corner z_c.

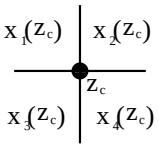

Figure 5.1: The figure elucidates the labeling of the blocks adjacent to z_c.

2. **Preliminary definition of $\Psi^{(0)}$ on the sides of the squares:** We consider the interpolations of the values of the function $\Psi^{(0)}(z_c)$ at the corners to functions on the sides between two adjacent corners. In this way, $\Psi^{(0)}$ is defined on the whole boundary of every square x, and is close to $\tilde{\Phi}(z)$ when Φ is smooth.

   Consider the side with endpoints z_0 and z_1 which separates squares x and y. Let the 4 squares with joint corner z_0 be x, y, x_0, y_0. The squares with joint corner z_1 are called x, y, x_1, y_1. The side is parametrized by $t = 0 \ldots 1$, with $z_0 = z(0)$, $z_1 = z(1)$. The interpolation is

$$\begin{aligned}\Psi^{(0)}(t) \;=\; & (\Phi(x)+\Phi(y) \\ & +(1-t)[\Phi(x_0)+\Phi(y_0)] + t[\Phi(x_1)+\Phi(y_1)])/(modulus)\end{aligned} \tag{5.3}$$

3. **Preliminary interpolation $\tilde{\Psi}^{(0)}$ to the inside of the squares:** Again we consider one square x at a time and construct a preliminary interpolation $\tilde{\Psi}^{(0)}$ which interpolates $\Psi^{(0)}$ from the boundaries to the inside, such that it is smooth inside and takes the prescribed values (5.3) on the boundary.

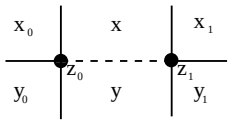

Figure 5.2: Labeling of the blocks mentioned in (5.3). $\Psi^{(0)}$ is now defined on the dashed line.

The resulting function on the whole continuum is smooth except for discontinuities of the normal derivatives across the boundaries of the squares.

Introduce the notation

$$\Psi_x^\perp(z) = \Psi^{(0)}(z) - \tilde{\Phi}(z)(\tilde{\Phi}(z) \cdot \Psi^{(0)}(z))$$

Given $\Psi_x^\perp(z)$, the field $\Psi^{(0)}(z)$ for $z \in x$ can be recovered by [1]

$$\Psi^{(0)}(z) = \Psi_x^\perp(z) + \left(1 - |\Psi_x^\perp(z)|^2\right)^{\frac{1}{2}} \tilde{\Phi}(z). \tag{5.4}$$

Let the points z in the closed square x be parametrized by (t_1, t_2), $0 \le t_1, t_2 \le 1$. The four sides of the square have $t_1 = 0$ or $t_1 = 1$ or $t_2 = 0$ or $t_2 = 1$ respectively. Regard $\Psi_x^\perp$ etc. as a function of (t_1, t_2). We consider first the linear interpolation $\tilde{\Psi}^\perp$ of the boundary values of $\Psi^\perp$ to the inside of the square,

$$\begin{aligned}
\tilde{\Psi}^\perp(t_1,t_2) &= (1-t_1)[\Psi^\perp(0,t_2) - \frac{1}{2}(1-t_2)\Psi^\perp(0,0) - \frac{1}{2}t_2\Psi^\perp(0,1)] \\
&\quad + t_1[\Psi^\perp(1,t_2) - \frac{1}{2}(1-t_2)\Psi^\perp(1,0) - \frac{1}{2}t_2\Psi^\perp(1,1)] \\
&\quad + (1-t_2)[\Psi^\perp(t_1,0) - \frac{1}{2}(1-t_1)\Psi^\perp(0,0) - \frac{1}{2}t_1\Psi^\perp(1,0)] \\
&\quad + t_2[\Psi^\perp(t_1,1) - \frac{1}{2}(1-t_1)\Psi^\perp(0,1) - \frac{1}{2}t_1\Psi^\perp(1,1)].
\end{aligned}$$

4. Restoration of the block spin condition: Adjust the value of the block spin while retaining the values of $\Psi^{(0)}$ at the boundaries and maintaining the smoothness. Again this is done separately for the squares x, using local coordinates (t_1, t_2) as above.

Let $\tilde{a} = s \cdot a$. There are s^2 lattice points per square. $\mathcal{C}$ takes the average over the lattice sites inside the square x similarly as before. Then

$$\begin{aligned}
\Psi_x^\perp(t_1,t_2) &= \tilde{\Psi}_x^\perp(t_1,t_2) - \alpha k(t_1,t_2), \\
k(t_1,t_2) &= \sin(\pi t_1)\sin(\pi t_2) \\
\alpha &= s^2 \sin^2\left(\frac{\pi}{2s}\right) \mathcal{C}\tilde{\Psi}_x^\perp(x).
\end{aligned}$$

[1] In general, $\Psi_x^\perp(z) \neq \Psi_y^\perp(z)$ for z on the side separating squares x and y, because of the jump of Φ. This is why a linear interpolation of $\Psi^\perp$ could not be used on the sides. Note however that there are no lattice points on the sides; every lattice point belongs to a unique square.

The real function k vanishes at the boundaries of the square. Its block average is

$$\mathcal{C}k = \left[s \sin(\frac{\pi}{2s})\right]^{-2}.$$

The average $\mathcal{C}$ is an average over the sites of a block. It is not an integral over a function defined on the continuum.

So the factor α ensures the block spin condition,

$$\begin{aligned} \mathcal{C}\Psi_x^{\perp} &= \mathcal{C}\tilde{\Psi}_x^{\perp} - \mathcal{C}(\alpha k) = \mathcal{C}\tilde{\Psi}_x^{\perp} - \alpha \mathcal{C}k \\ &= \mathcal{C}\tilde{\Psi}_x^{\perp} - \left[s \sin(\frac{\pi}{2s})\right]^2 \mathcal{C}\tilde{\Psi}_x^{\perp} \left[s \sin(\frac{\pi}{2s})\right]^{-2} \\ &= 0. \end{aligned}$$

This completes the construction of the component of $\Psi^{(0)}$ perpendicular to $\tilde{\Phi}$. Imposing the condition $(\Psi^{(0)})^2 = 1$ fixes the values of $\Psi^{(0)}$ by (5.4).

The result satisfies all the requirements. Since the z-coordinates are $z_\mu = ast_\mu + z_\mu^0$ where z_μ^0 are the coordinates of the lower left corner of square x, the discontinuities in $\nabla\Psi^{(0)}$ across boundaries are of order s^{-1} if the blocks are large.

Note that the construction is local. For $z \in x$, $\Psi^{(0)}(z)$ depends only on the value of the block spin Φ at x and the eight nearest and next to nearest neighbours of x.

So $\Psi^{(0)}$ is an explicit functions of these nine values by virtue of the formulas above. But it is a nonpolynomial function of Ψ because of the normalizing factor 1/modulus in (5.3) and the square root in (5.4). Nevertheless, for sufficiently smooth Φ the factors could be expanded to obtain a polynomial approximation.

5.2 Iterative Improvement of the Approximation to the Background Field

We proceed with the iteration step. Given any approximate extremum $\Psi^{(0)}$ of the action (not necessarily the $\Psi^{(0)}$ constructed above) , we parametrize an arbitrary field ϕ with the desired block spin with ξ-variables $\xi \perp \Psi^{(0)}$. This is similar to the parametrization of the fundamental field (2.16). But now $\Psi^{(0)}$ is substituted for the exact background field Ψ, while Ψ itself is parametrized.

The parametrization to first order in ξ reads

$$\Psi = \Psi^{(0)} + \xi. \tag{5.5}$$

ξ is subject to the constraint

$$\mathcal{C}Q[\Psi^{(0)}]\xi = 0, \tag{5.6}$$

because this ensures $C\Psi = \Phi$. Since Ψ should extremize the action, the equation

$$\xi^T S'(\Psi) = 0$$

holds for arbitrary $\xi \perp \Psi^{(0)}$ which satisfies the constraint (5.6). This is equivalent to

$$S'(\Psi) = Q^T \mathcal{C}^\dagger \lambda.$$

with a Lagrange multiplier λ. λ is a field on the coarse lattice. Power series expansion to first order in ξ around $\Psi^{0)}$ gives

$$S'(\Psi) = S'(\Psi^{(0)}) + S''(\Psi^{(0)})\xi + \ldots \tag{5.7}$$

Using (5.5) one computes

$$S'(\Psi^{(0)}(z)) = -\triangle \Psi^{(0)}(z).$$

As this is only valid as a linear form on the tangent space $T_{\Psi^{(0)}} S^{\mathcal{N}}$, i.e. when contracted with arbitrary $\xi(z) \perp \Psi^{(0)}(z)$, it is convenient to write

$$S'(\Psi^{(0)})(z) = -\pi^{(0)}(z) \triangle \Psi^{(0)}(z),$$

with the projector $\pi^{(0)}$ on the tangent space $T_{\Psi^{(0)}} S^{\mathcal{N}}$.

Inserting this into the expansion (5.7) we get a linear equation for ξ,

$$S''(\Psi^{(0)})\xi = \pi^{(0)} \triangle \Psi^{(0)} + Q^T \mathcal{C}^\dagger \lambda.$$

The Lagrange multiplier λ ensures the constraint (5.6). It is a standard result known from the work of Kupiainen and Gawedzki [13] that the solution of such a linear equation can be written in the form

$$\begin{aligned} \xi &= \Gamma \pi^{(0)} \triangle \Psi^{(0)}, \\ \Gamma &= \lim_{\kappa \mapsto \infty} (S'' + \kappa Q^T \mathcal{C}^\dagger \mathcal{C} Q)^{-1} \end{aligned}$$

Γ agrees with the full high frequency propagator in the background field $\Psi^{(0)}$.

Next we recall the fact that the full high frequency propagator in a background field Ψ agrees with $\pi \Gamma_{Q[\Psi]} \pi$ to zeroth order in $\nabla \Psi$. To the desired accuracy we can therefore replace Γ by $\pi^{(0)} \Gamma_Q$.

We record the final result for the background field

$$\begin{aligned} \Psi &= (\Psi^{(0)} + \xi)/(\text{modulus}), \\ \xi &= \pi^{(0)} \Gamma_{Q[\Psi^{(0)}]} \pi^{(0)} \triangle \Psi^{(0)}, \\ \pi^{(0)}(z) &= 1 - \Psi^{(0)}(z) \Psi^{(0)T}(z) \end{aligned}$$

with $\Psi^{(0)}$ from section 5.1 above. The division by the modulus is to ensure exact validity of $|\Psi|^2 = 1$; note that (modulus) $= 1 + O(\xi^2)$.

Note that small discontinuities of normal derivatives of $\Psi^{(0)}$ at block boundaries give small contributions. On kinematical grounds, the derivatives are proportional s^{-1} times a small factor if Φ is smooth. On the other hand, the length of the boundary is proportional to s for blocks with s^2 lattice points.

Therefore ξ will be small if Φ is reasonably smooth.

Chapter 6

The Effective Action for Gaussian Block Spin

One would like to improve the locality properties of the effective action derived from the $(\mathcal{N}+1)$-dimensional integration. To make this feasible, we will discuss the Gaussian block spin condition also. That means the use of a Gaussian as the weight functional in (1.9) for the nonlinear σ-model.

The choice of the weight functional was one of the properties of the definition of the effective action in 1.3.1 that determined the form, and thus the locality, the action takes. As we have fixed the other two ingredients of (1.9), this is the only one left we can tune.

The kind of block spin to discuss was chosen as a field of the same type as the fundamental field, a $(\mathcal{N}+1)$-dimensional vector of unit length, because we did not expect the nature of the field to change under a small decrease of the cut-off. This resulted in a problem to linearize the block spin condition $C\phi = \Phi$ in the case of δ-block spins since we needed a necessarily nonlinear averaging operator C. The arbitrariness to take almost any linear operator $\mathcal{C}$ in the definition $C\phi(x) = \frac{\mathcal{C}\phi(x)}{|\mathcal{C}\phi(x)|}$ was exploited by the linearization of the constraint in section 2.1. We had to use the average of the fundamental field over *one* block, so that we could express $C\phi = \Phi$ as (2.4), viz.

$$\mathcal{C}\phi(x) - \Phi(x)\,(\Phi(x)\cdot\mathcal{C}\phi(x)) = 0.$$

A Jacobian J_0 was introduced to implement this new block spin condition.

So alternatives to our choice of the weight functional are needed if we want to improve the locality of the action.

There are two reasons to scrutinize Gaussian block spins.

- The effective action was derived with a δ-block spin. But such a block spin condition is implemented by a δ-function which can be represented as the limit of a Gaussian. So a great deal of the results in the previous sections remain valid.

- The idea to use a Gaussian as a weight functional goes back to T. Bell and K. Wilson [4]. Hasenfratz and Niedermayer implemented block spin conditions by Gaussians in their work, see [19] and sequent work of the authors. They pointed out that the use of a certain width of the Gaussian optimizes the locality properties of their action. This inspired for example [23, 22, 6] to proceed in the same fashion.

One should admit that there could be a better way to improve locality properties of effective actions in general. The tuning of the width of the Gaussian is a rather arbitrary procedure. If it is possible, one should try to regard different averaging operators $\mathcal{C}$ instead. This alters the fall of properties of interpolation kernels $\mathcal{A}_{\mathrm{KG}}$ and high frequency propagators Γ_{KG}, too. But one still enforces the condition $C\phi = \Phi$ which allows stringent tests of the effective action, a condition abandoned by weight functionals deviating from a δ-function.

Furthermore, this separates high and low frequency modes clearly. Although our implementation of the δ-block spin condition which fixes $\mathcal{C}$ and leaves no room for optimization, does not allow an improvement of the locality of our effective action, we recorded the results for this weight functional first. This is because the sources of error in real space renormalization group do not seem to be well understood. The option to make clean tests is therefore desired.

The alternative Gaussian block spin is discussed here to provide an action with a tunable parameter for numerical investigations.

This parameter is κ which determines the width of the Gaussian weight functional in the substitute for $\delta(\mathcal{C}\zeta)$ in (2.7),

$$\delta_\kappa \left(\mathcal{C}^\perp[\Phi]\phi(x)\right) = \mathcal{N}_\kappa e^{-\frac{\kappa}{2}|\mathcal{C}^\perp[\Phi]\phi(x)|^2}.$$

$\mathcal{N}_\kappa$ is a field independent constant and will be omitted. $\mathcal{C}^\perp[\Phi]$ is a linear averaging operator in the spirit of (2.4) which depends parametrically on Φ,

$$\mathcal{C}^\perp[\Phi]\phi(x) = \mathcal{C}\phi(x) - \Phi(x)\left(\Phi(x)\cdot\mathcal{C}\phi(x)\right) = \mathcal{C}\phi^\perp(x). \tag{6.1}$$

The κ-dependent effective action is defined as follows.

$$e^{-S_{\mathrm{eff}}[\Phi]} = \int \mathcal{D}\phi \prod_x \left(J_0(\mathcal{C}\phi(x)) e^{-\frac{\kappa}{2}|\mathcal{C}^\perp[\Phi]\phi(x)|^2}\right) e^{-S[\phi]}, \tag{6.2}$$

where J_0 is a κ-dependent Jacobian which ensures that

$$\int \mathcal{D}\Phi \prod_x \left(J_0(\mathcal{C}\phi(x)) e^{-\frac{\kappa}{2}|\mathcal{C}^\perp[\Phi]\phi(x)|^2}\right) = 1 \tag{6.3}$$

for all ϕ.

The calculation proceeds exactly as before, and the result is the same as for $\kappa = \infty$, except for the following changes.

- The Jacobian J_0 now contains a mild κ-dependence.

- As said before in 1.3.2 the background field is determined by a new saddle point condition which contains terms that stem from the Gaussian weight functional.

  If it is the aim to improve the locality properties of the classical perfect action, one should choose κ of order β,

  $$\kappa = \beta\tilde{\kappa}\tilde{a}^2.$$

  The result is that our previous analytical approximation for Ψ remains valid for large enough $\tilde{\kappa}$, except that the high frequency propagator (4.9) for finite κ has to be substituted. A small $\tilde{\kappa}$ approximation will be mentioned. Both approximations become exact when the block spin tends to a constant.

- The result for $\mathrm{Tr}\ln\Gamma_Q$ remains valid. But finite $\tilde{\kappa}$ of order 1 admits an alternative treatment without additional assumptions concerning the smoothness of Φ.

  This is important, since one would like to use the results for nonperturbative investigations, where integrals over arbitrary fields are involved.

6.1 The Jacobian J_0 for Gaussian Block Spin

The Jacobian J_0 is defined by (6.3). $\mathcal{C}^\perp[\Phi]$ depends on Φ only through $\Phi(x)$. It is block wise constant.

According to (6.3), we must calculate

$$J_0^{-1}\left(\mathcal{C}\phi(x)\right) = \int d\Phi e^{-\frac{\kappa}{2}|\mathcal{C}^\perp[\Phi]\phi(x)|^2}.$$

Let $\Xi(x) = \mathcal{C}\phi(x)$. Then

$$|\mathcal{C}^\perp[\Phi]\phi(x)|^2 = |\Xi(x)|^2 - |\Phi(x)\cdot\Xi(x)|^2.$$

If we choose a basis so that Ξ points in the 0-direction, and write

$$\Phi = \left(\sqrt{1-\chi^2}, \chi\right),$$

we see that

$$\begin{aligned} |\mathcal{C}^\perp[\Phi]\phi(x)|^2 &= |\Xi(x)|^2 - ||\Xi|\sqrt{1-\chi^2}|^2(x) \\ &= |\Xi|^2\cdot|\chi|^2(x). \end{aligned}$$

Using the standard representation of the uniform measure $d\Phi$ on the sphere in terms of coordinates χ, we get

$$\begin{aligned} J_0^{-1}\left(\Xi\right) &= \int \frac{d^{\mathcal{N}}\chi}{\sqrt{1-\chi^2}} e^{-\frac{\kappa}{2}|\Xi|^2\cdot|\chi|^2} \\ &= \mathrm{const}\cdot\left(|\Xi|^2 - \frac{1}{\kappa} + \ldots\right)^{-\frac{\mathcal{N}}{2}}. \end{aligned} \tag{6.4}$$

The last expression is valid for large κ. In the limit $\kappa\to\infty$, it agrees with (2.11).

6.2 The Background Field Ψ and the Classical Perfect Action for Gaussian Block Spin

On introduces a background field $\Psi(z)$ which extremizes the exponent of (6.2), viz.

$$S[\phi] + \frac{\kappa}{2}\sum_x |\mathcal{C}^\perp[\Phi]\phi(x)|^2 = \text{Extr.}, \tag{6.5}$$

at $\phi = \Psi$. Then one parametrizes the field ϕ in terms of a fluctuation field ζ as before in eq. (2.5),

$$\phi^\perp = \Psi^\perp + \zeta.$$

It follows that

$$\begin{aligned}\frac{\kappa}{2}\sum_x |\mathcal{C}^\perp[\Phi]\phi(x)|^2 &= \frac{\beta\tilde{\kappa}}{2}\tilde{a}^2\sum_x |\mathcal{C}\phi^\perp(x)|^2 \\ &= \frac{\beta\tilde{\kappa}}{2}\int_x |\mathcal{C}\Psi^\perp(x)|^2 + \frac{\beta\tilde{\kappa}}{2}\int_z 2\zeta\cdot\mathcal{C}^\dagger\mathcal{C}\Psi^\perp(z) + \zeta\cdot\mathcal{C}^\dagger\mathcal{C}\zeta(z).\end{aligned}$$

We make the transition to ξ-variables, $\xi = Q^{-1}\zeta$ (see (2.15)), and expand in powers of ξ. By (6.5), the linear term must vanish. We obtain the new saddle point condition

$$S'[\Psi] = -\beta\tilde{\kappa}Q^T\mathcal{C}^\dagger\mathcal{C}\Psi^\perp. \tag{6.6}$$

The new classical perfect action, which is the value of expression (6.5) at the extremum $\xi = 0$, comes out as

$$S_{\text{cl}}[\Psi] = S[\Psi] + \frac{\beta\tilde{\kappa}}{2}\int_x |\mathcal{C}\Psi^\perp(x)|^2. \tag{6.7}$$

Our definition does *not* include the Jacobian J_0. The classical perfect action is of order β, while the logarithm of the Jacobian J_0 is of order β^0.

$\mathcal{C}\Psi^\perp(x)$ comes out to be of order $\frac{1}{\kappa}$. Therefore the second term in (6.7) vanishes in the limit $\kappa\to\infty$, and we recover the previous result.

One needs to solve (6.5). Suppose that an approximate solution $\Psi^{(0)}$ is at hand. Then an improved solution $\Psi^{(0)} + \xi + O(\xi^2)$ is determined as before in section 5.2.

Expanding to first order in ξ, eq. (6.6) takes the form

$$\left[S''[\Psi^{(0)}] + \beta\tilde{\kappa}Q^T\mathcal{C}^\dagger\mathcal{C}Q\right]\xi = -S'[\Psi^{(0)}] - \beta\tilde{\kappa}Q^T\mathcal{C}^\dagger\mathcal{C}\Psi^{(0)\perp}$$

with approximate solution

$$\Psi = \left(\Psi^{(0)} + \pi^{(0)}\Gamma_{Q_{[\Psi^{(0)}]}}\pi^{(0)}[\triangle\Psi^{(0)} - \beta\tilde{\kappa}Q^T\mathcal{C}^\dagger\mathcal{C}\Psi^{(0)\perp}]\right)/\text{modulus} + O(\xi^2). \tag{6.8}$$

$\Gamma_{Q_{[\Psi^{(0)}]}}$ is the high frequency propagator (4.9) with finite κ.

A first approximation $\Psi^{(0)}$ can be constructed in the same way as in section 5.1, possibly with a different choice of α.

We propose two choices.

large $\tilde{\kappa}$ approximation. We choose α as before, so that $\mathcal{C}\Psi^{(0)\perp} = 0$. Then the $\beta\tilde{\kappa}$-term in eq.(6.8) vanishes and we obtain the same formula for Ψ as before, except for the use of the finite-κ-propagator.

small κ approximation. Choose $\alpha = 0$ and use the full eq.(6.8).

6.3 The High Frequency Propagator Γ_{KG} and the Interpolation Operator $\mathcal{A}_{\text{KG}}$ for Finite κ

The high frequency propagator Γ_{KG} and the interpolation operator $\mathcal{A}_{\text{KG}}$ were recorded for finite κ in section 1.3.2, see (1.16). Some useful relations were summarized in (1.17).

Let us now discuss why the effective action with suitable finite $\tilde{\kappa}$ is expected to have better locality properties than at $\kappa = \infty$. The source of the improved locality is the better falloff property of the high frequency propagator Γ_{KG} and the interpolation operator $\mathcal{A}_{\text{KG}}$.

These locality properties are inherited by the classical perfect action. And the corrections to the classical perfect action also benefit from the improved falloff properties of Γ_{KG} and $\mathcal{A}_{\text{KG}}$.

The falloff properties of Γ_Q, which appears in the analytic formula for the background field, are inherited from those of Γ_{KG}. Since $\mathcal{A}_{\text{KG}} = \kappa\Gamma_{\text{KG}}\mathcal{C}^\dagger$, this follows from the perturbation expansion of Γ_Q in powers of $\hat{Q} - 1$.

Some examples for interpolation operators and high frequency propagators with finite κ are shown below. The figures were produced with a software package written by Max Grießl and Jan Würthner, see [33].

In conclusion, if one wishes to achieve good locality properties in the effective action, $\tilde{\kappa}$ should be so chosen that Γ_{KG} has good locality properties. As we said in the introduction, this has a prize. Systematic tests of the accuracy of various approximations are easier with $\kappa = \infty$.

6.4 An Alternative Treatment of the $\hat{Q}$-Dependency of the Action

Here we consider an alternative treatment of the term $\kappa\varphi^T\hat{Q}^T\mathcal{C}^\dagger\mathcal{C}\hat{Q}\varphi$ in eq. (4.8). For small κ it is not necessary to introduce the propagator Γ_Q (4.9).

We can write the term as

$$\kappa\varphi^T[\hat{Q}^T\mathcal{C}^\dagger\mathcal{C}\hat{Q} - \mathcal{C}^\dagger\mathcal{C}]\varphi + \kappa\varphi^T\mathcal{C}^\dagger\mathcal{C}\varphi. \tag{6.9}$$

If one considers the term $\kappa\varphi^T[\hat{Q}^T\mathcal{C}^\dagger\mathcal{C}\hat{Q} - \mathcal{C}^\dagger\mathcal{C}]\varphi$ as a part of the perturbation which needs to be taken into account only to lowest order, then a simplification is possible. This approximation is only accurate when κ is not to large. Because of the rescaling $\varphi \to \varphi/\beta^{\frac{1}{2}}$ this approximation is valid for large β. It does *not* depend on the smoothness of Φ.

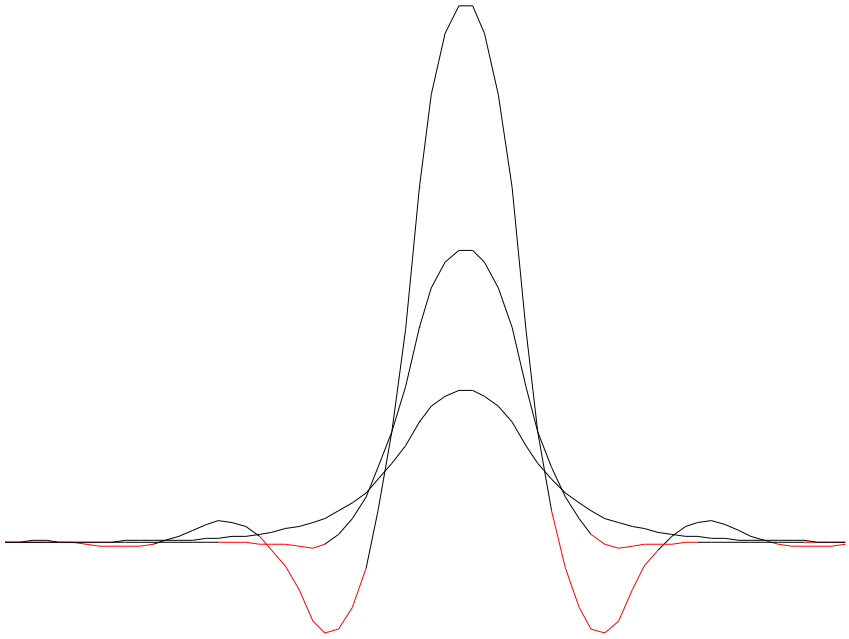

Figure 6.1: This figure shows cross sections through $\mathcal{A}_{\rm KG}$-kernels with κ values of 100 (top), $\frac{1}{8}$ (center) and $\frac{1}{32}$ (bottom). Note that the kernel oscillates for high κ values.

The result has again the form (4.10) but with $\Gamma_{\rm KG}$ substituted for Γ_Q. In particular, the expectation values are now determined in a free field theory with propagator $\Gamma_{\rm KG}$. So $\mathrm{Tr}\,\ln\Gamma_Q$ becomes a field independent constant. But a new perturbation is added to (4.10),

$$\begin{aligned}\frac{1}{2}\langle\kappa\varphi^T[\hat{Q}^T\mathcal{C}^\dagger\mathcal{C}\hat{Q}-\mathcal{C}^\dagger\mathcal{C}]\varphi\rangle_{\Gamma_{\rm KG}} &= \frac{1}{2}\mathrm{Tr}\,\kappa[\hat{Q}^T\mathcal{C}^\dagger\mathcal{C}\hat{Q}-\mathcal{C}^\dagger\mathcal{C}]\Gamma_{KG}\\ &= \frac{1}{2}\kappa\int_x \operatorname*{av}_{z\in x}\operatorname*{av}_{w\in x}\mathrm{tr}\,[\hat{Q}^T(z)\hat{Q}(w)-1]\Gamma_{KG}(w,z)\end{aligned}\quad(6.10)$$

Note that the integral has a κ in front. Therefore the result is not useful in the limit $\kappa\to\infty$. z and w belong to the same block.

Define $\delta\Psi(z)$ as $\delta\Psi(z)=\Psi(z)-\tilde{\Phi}(z)$. The trace over the matrix $\hat{Q}^T(z)\hat{Q}(w)-1$

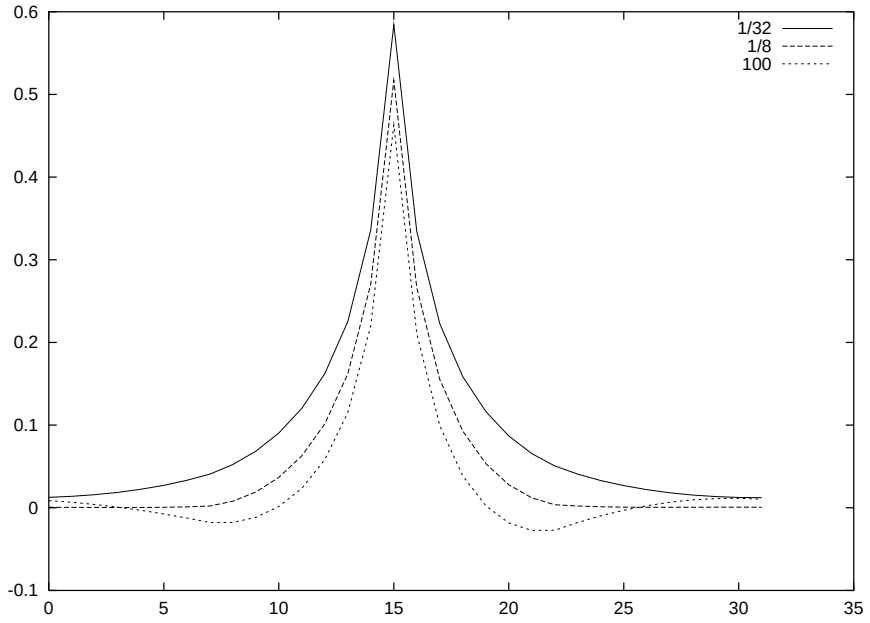

Figure 6.2: Some high frequency propagators $\Gamma_{\rm KG}$ with κ values of 100, $\frac{1}{8}$ and $\frac{1}{32}$. Again the kernel oscillates for high κ values.

becomes

$$\begin{aligned}
\mathrm{tr}\,[\hat{Q}^T(z)\hat{Q}(w)-1] &= (\Psi(z)\cdot\Psi(w))^2-1 \\
&\quad +\Psi(z)\cdot\Psi(w)(1-\cos\theta(z)\cos\theta(w)) \\
&\quad +(\cos^2\theta(z)-1)+(\cos^2\theta(w)-1) \qquad (6.11)\\
&= 3(\cos\theta(z)-1)+3(\cos\theta(w)-1) \\
&\quad +3(\cos\theta(z)-1)(\cos\theta(w)-1) \\
&\quad +(\cos\theta(z)-1)(\cos\theta(z)-\cos^2\theta(w)) \\
&\quad +(\cos\theta(w)-1)(\cos\theta(w)-\cos^2\theta(z)) \\
&\quad +\delta\Psi(z)\cdot\delta\Psi(w)(3-\cos\theta(z)\cos\theta(w)) \\
&\quad +o\left((\delta\Psi^3)\right) \qquad (6.12)
\end{aligned}$$

While (6.11) is the exact result, one assumes that $\delta\Psi$ is small in (6.12). Note that this is again based on the smoothness of Φ.

As the $\hat{Q}$-dependence is extracted by perturbation theory directly from (4.8), we can substitute $\Gamma_{\rm KG}$ for Γ_Q in all our formulas. Using (6.11), it can be done without assuming smooth block spin configurations. This is an important simplification for nonperturbative investigations.

Thus, the substitute for (4.10) becomes

$$\begin{aligned} S_{\text{eff}}(\Phi) &= S_{\text{cl}}(\Psi) - \ln J^0(\mathcal{C}\Psi) - \frac{1}{2}\text{Tr } \ln \Gamma_{\text{KG}} \\ &\quad + S^{(1)}_{\text{eff}} + S^{(2)}_{\text{eff}} \\ &\quad + \frac{1}{2}\langle \kappa\varphi^T[\hat{Q}^T\mathcal{C}^\dagger\mathcal{C}\hat{Q} - \mathcal{C}^\dagger\mathcal{C}]\varphi\rangle_{\Gamma_{\text{KG}}}. \end{aligned} \tag{6.13}$$

The expectation values $S^{(1)}_{\text{eff}}$ and $S^{(2)}_{\text{eff}}$ are now those of a free theory with covariance Γ_{KG}. The constant $\text{Tr } \ln \Gamma_{\text{KG}}$ can be omitted.

(4.44) is valid for small κ, if one adds (6.10) and substitutes the constant $\text{Tr } \ln \Gamma_{\text{KG}}$ for $\text{Tr } \ln \Gamma_Q$,

$$\begin{aligned} S_{\text{eff}} &= -\sum_x \ln J_0(\mathcal{C}\Psi(x)) - \frac{1}{2}\text{Tr } \ln \Gamma_{\text{KG}} + \frac{\beta\tilde{\kappa}}{2}\int_x |\mathcal{C}\Psi^\perp(x)|^2 \\ &\quad + \frac{1}{2}\int_z \underbrace{(\beta - \mathcal{N}\delta\beta_{\text{eff}}(z) + \delta\beta_{\text{eff}}(z+\hat{\mu}))\,|\nabla_\mu \Psi(z)|^2}_{\beta_{\text{eff}}(z)|\nabla_\mu\Psi(z)|^2} \\ &\quad + \text{renormalized 1-loop diagram} + \text{lattice artefact} \\ &\quad + \frac{1}{2}\kappa \int_x \underset{z\in xw\in x}{\text{av av}} \text{tr}\,[\hat{Q}^T(z)\hat{Q}(w) - 1]\Gamma_{KG}(w,z). \end{aligned} \tag{6.14}$$

The background field Ψ is determined by the full equation (6.8). In (4.25) one can find the expression for the lattice artefact. The renormalized one loop diagram is given in (4.22). There again one has to substitute Γ_{KG} for Γ_Q. $\text{tr}\,[\hat{Q}^T(z)\hat{Q}(w) - 1]$ is given in (6.11).

Chapter 7

Conclusion

In this work we calculated a Wilson effective action S_{eff} for the $O(\mathcal{N}+1)$-symmetric nonlinear σ-model in two dimensions in a 1-loop approximation. The fundamental action S is a naive discretization of the continuum theory. It lives on a lattice of spacing a, while the effective action describes the physics of the model on a coarser lattice of larger spacing $\tilde{a}$. We defined the block spin Φ as a field of the same type as the fundamental field. A δ-function constraint was used to implement the block spin condition. This ensures $C\phi = \Phi$. The linearization of the constraint introduces a field dependent high frequency propagator Γ_Q, see (4.9). This makes the extraction of the field dependencies of S_{eff} cumbersome.

The effective action is composed of the classical perfect action with a renormalized coupling constant, an augmented contribution from a Jacobian and further corrections. These results are summarized in section 4.2.3. The classical perfect action is quite simple if one expresses it in terms of an interpolation of Φ minimizing the action S. The jacobian term depends on $\cos\theta(z) = \Psi(z)\cdot\Phi([z])$. The augmentation is the power series expansion of $\mathrm{Tr}\,\ln\Gamma_Q$ (4.40). The further corrections are $\Psi\cdot\Phi$-dependent fluctuations of the renormalized coupling and a genuine 1-loop correction. The coupling constant renormalizations can be found in (4.41) and (4.42). The 1-loop correction (4.43) depends on the matrix $\Psi\nabla_\mu\Psi^T$ at different sites. To complete the construction, an analytic approximation for Ψ is given in chapter 5. Using it one can express the classical perfect action and S_{eff} as a function of the block spin. While Polyakov's result can be recovered, our calculation extends his approximation. It contains contributions which remain finite when the original cut-off a^{-1} is sent to infinity and it yields also the terms which are sensitive to the choice of the block spin.

It was claimed that a 1-loop approximation would merely renormalize the coupling β. This statement is in conflict with the existence of the genuine 1-loop correction, see (4.43).

Simplifications of the effective action depend on the smoothness of Φ. This restricts their validity. To leading order one may substitute the well known Gawedzki-Kupiainen propagator Γ_{KG} for Γ_Q in the renormalized coupling and the 1-loop correction. The result

is given in (4.44).

To improve the locality properties of the effective action, a Gaussian block spin is considered in chapter 6. Most of the results for δ-block spin remain valid. The Jacobian becomes dependent on the width of the Gaussian which is determined by a parameter κ. The construction of the background field has to be altered because a high frequency propagator for finite κ has to be used. The classical action is enlarged by a term involving the component of Ψ perpendicular to the block spin.

For small κ an alternative treatment is possible. If the details of the block spin condition can be regarded as a perturbation, the Ψ-dependent propagator is only needed for the construction of the background field. The propagator which enters the 1-loop calculation is Γ_{KG}. The results are derived and summarized in section 6.4. This simplification does not demand a smooth block spin field.

In all the calculations the exact Leibnitz rule on the lattice was used.

For numerical purposes the Fourier transforms of the various kernels are collected in the appendix C.

In principle, a calculation of the 1-loop effective action for pure gauge theories along the same lines is feasible. The linearization of the block spin constraint and other ingredients were described in Balaban's work [2] for general gauge group. The Fourier expansions of the interpolation kernel and high frequency propagators are also known, for a particular choice of block spin [26].

Appendix A

Gaussian Integrals

Let $a \geq 0$. Then

$$\int e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

and

$$\int e^{-\frac{a}{2}x^2+bx} = e^{\frac{b^2}{2a}}\sqrt{\frac{2\pi}{a}}$$

for all b. These formulas can be generalized as follows.

$$\int \mathcal{D}\phi e^{-\frac{1}{2}\langle\phi,\Gamma^{-1}\phi\rangle} = \det(2\pi\Gamma)^{\frac{1}{2}} \tag{A.1}$$

and

$$\int \mathcal{D}\phi e^{-\frac{1}{2}\langle\phi,\Gamma^{-1}\phi\rangle+\langle J,\phi\rangle} = \det(2\pi\Gamma)^{\frac{1}{2}} e^{\frac{1}{2}\langle J,\Gamma J\rangle} \tag{A.2}$$

with

$$\mathcal{D}\phi = \prod_x d\phi(x).$$

$\langle .,. \rangle$ denotes the scalar product on the space of functions.

This motivates to define a measure $d\mu_\Gamma$, called the Gaussian measure with mean zero and covariance Γ, as

$$d\mu_\Gamma(\phi) = (\det 2\pi\Gamma)^{-\frac{1}{2}} e^{-\frac{1}{2}\langle\phi,\Gamma^{-1}\phi\rangle}\mathcal{D}\phi. \tag{A.3}$$

Because of eq. (A.1) the Gaussian measure has mass one

$$\int d\mu_\Gamma(\phi) = 1.$$

The Fourier transform is called the characteristic function of the measure $d\mu_\Gamma$. By (A.2), we have

$$\int d\mu_\Gamma(\phi) e^{i\langle k,\phi\rangle} = e^{-\frac{1}{2}\langle k,\Gamma k\rangle}. \tag{A.4}$$

The characteristic function can be used to calculate all the moments of the measure as

$$\begin{aligned}\int d\mu_\Gamma(\phi)\phi(x_1)\dots\phi(x_n) &= \left[\frac{1}{i}\frac{\delta}{\delta k(x_1)}\dots\frac{1}{i}\frac{\delta}{\delta k(x_n)}\int d\mu_\Gamma(\phi)e^{i\langle k,\phi\rangle}\right]_{k=0} \\ &= \left[\frac{1}{i}\frac{\delta}{\delta k(x_1)}\dots\frac{1}{i}\frac{\delta}{\delta k(x_n)}e^{-\frac{1}{2}\langle k,\Gamma k\rangle}\right]_{k=0}.\end{aligned}$$

The covariance is given by

$$\langle\phi(x)\phi(y)\rangle_\Gamma = \frac{1}{i}\frac{\delta}{\delta k(x)}\frac{1}{i}\frac{\delta}{\delta k(y)}\int d\mu_\Gamma(\phi)e^{i\langle k,\phi\rangle}\Big|_{k=0} = \Gamma(x,y). \tag{A.5}$$

Appendix B

Perturbation Theory

Let $d\mu_\Gamma$ be a Gaussian measure. For a functional $F[\zeta]$ the equation

$$\int d\mu_\Gamma(\zeta) F[\zeta] = e^{\frac{1}{2}\langle \frac{\delta}{\delta\zeta}, \Gamma \frac{\delta}{\delta\zeta}\rangle} F[\zeta]\Big|_{\zeta=0} \tag{B.1}$$

with

$$\langle \frac{\delta}{\delta\zeta}, \Gamma \frac{\delta}{\delta\zeta}\rangle = \int_x \int_y \Gamma(x,y) \frac{\delta}{\delta\zeta(x)} \frac{\delta}{\delta\zeta(y)}$$

holds.

Proof: $\tilde{F}[k]$ is the Fourier transform of $F[\zeta]$.

$$\begin{aligned}
\int d\mu_\Gamma(\zeta) F[\zeta] &= \int d\mu_\Gamma(\zeta) \int \mathcal{D}k \tilde{F}[k] e^{i\langle k,\zeta\rangle} \\
&= \int \mathcal{D}k \tilde{F}[k] e^{-\frac{1}{2}\langle k, \Gamma k\rangle} \\
&= e^{\frac{1}{2}\langle \frac{\delta}{\delta\zeta}, \Gamma \frac{\delta}{\delta\zeta}\rangle} \int \mathcal{D}k \tilde{F}[k] e^{i\langle k,\zeta\rangle}\Big|_{\zeta=0} \\
&= e^{\frac{1}{2}\langle \frac{\delta}{\delta\zeta}, \Gamma \frac{\delta}{\delta\zeta}\rangle} \; F[\zeta]|_{\zeta=0} \, .
\end{aligned}$$

As an example:

$$\int d\mu_\Gamma(\phi)\phi(x)\phi(y) = e^{\frac{1}{2}\langle \frac{\delta}{\delta\phi}, \Gamma \frac{\delta}{\delta\phi}\rangle} \phi(x)\phi(y)\Big|_{\phi=0} .$$

One expands the exponential function and calculates

$$\begin{aligned}
& (1+\frac{1}{2}\int_z\int_w \Gamma(z,w)\frac{\delta}{\delta\phi(z)}\frac{\delta}{\delta\phi(w)}+\dots)\,\phi(x)\phi(y)|_{\phi=0} \\
= \;& \frac{1}{2}\int_z\int_w \Gamma(z,w)\frac{\delta}{\delta\phi(z)}\left[\phi(x)\delta(w-y)+\phi(y)\delta(w-x)\right]_{\phi=0} \\
= \;& \frac{1}{2}\int_z\int_w \Gamma(z,w)\left[\delta(z-x)\delta(w-y)+\delta(w-x)\delta(z-y)\right]_{\phi=0} \\
= \;& \Gamma(x,y).
\end{aligned}$$

This is the result of equation (A.5).

Appendix C

Fourier Transform of Kupiainen Gawedzki Kernels

Given a lattice Λ of lattice spacing a with points $z, w, ...$ and a block lattice $\tilde{\Lambda}$ of lattice spacing $\tilde{a} = sa$, (s a positive integer) with sites $x, y, ...$, we characterize the points by real coordinates z^μ resp. x^μ etc. The conjugate variables k_μ and p_μ take their values in the duals Λ^* and $\tilde{\Lambda}^*$,

$$-\frac{\pi}{a} < k_\mu \leq \frac{\pi}{a}, \tag{C.1}$$

$$-\frac{\pi}{\tilde{a}} < p_\mu \leq \frac{\pi}{\tilde{a}}. \tag{C.2}$$

If the lattices are infinitely extended, p_μ and k_μ are real variables. If the lattice Λ has extension $La = \tilde{L}\tilde{a}$ instead,

$$p_\mu, k_\mu \in \frac{2\pi}{La}\mathbb{Z} \tag{C.3}$$

We use the notation $\int_k(...) = \int d^2k$ if Λ is infinitely extended , and

$$\int_k(\dots) = \left(\frac{2\pi}{La}\right)^2 \sum_k(\dots)$$

otherwise. The same formulas is used for $\int_p$, only the boundaries of the integration are different according to eq.(C.2). Let

$$D = \{l | l_\mu \in \frac{2\pi}{\tilde{a}}\mathbb{Z},\ -\frac{\pi}{a} < l_\mu \leq \frac{\pi}{a}\}\ .$$

Then every $k \in \Lambda^*$ admits a unique decomposition

$$k = p + l\ ,\ p \in \tilde{\Lambda}^*,\ l \in D. \tag{C.4}$$

The Fourier transform of the massless lattice propagator $v(z-w)$ is

$$\tilde{v}(k) = \left(\sum_{\mu=1,2} \frac{2}{a^2}[1-\cos k_\mu a] \right)^{-1}$$

Because of invariance under translations by lattice vectors of the block lattice, the averaging kernel $\mathcal{C}$, interpolation kernel $\mathcal{A} \equiv \mathcal{A}_{KG}$, block propagator $u = u_{KG}$ and high frequency propagator $\Gamma \equiv \Gamma_{KG}$ admit Fourier expansions

$$\mathcal{C}(x,z) = (2\pi)^{-2} \int_{k\in\Lambda^*} \tilde{\mathcal{C}}(k) e^{ik_\mu(x^\mu - z^\mu)} , \tag{C.5}$$

$$\mathcal{A}(z,x) = (2\pi)^{-2} \int_{k\in\Lambda^*} \tilde{\mathcal{A}}(k) e^{-ik_\mu(x^\mu - z^\mu)} , \tag{C.6}$$

$$u(x-y) = (2\pi)^{-2} \int_{p\in\tilde{\Lambda}^*} \tilde{u}(p) e^{ip(x-y)}, \tag{C.7}$$

$$\Gamma(z,w) = (2\pi)^{-2} \sum_{l,l'\in D} \int_{p\in\tilde{\Lambda}^*} \tilde{\Gamma}_{ll'}(p) e^{i(l'z-lw)} e^{ip(z-w)} \tag{C.8}$$

The averaging kernel $\mathcal{C}(x,z) = \tilde{a}^{-2}$ for $z \in x$ and 0 otherwise. Assuming $z \in x$ iff $-a/2 < z^\mu - x^\mu \leq a/2$ one obtains

$$\tilde{\mathcal{C}}(k) = \prod_{\mu=1,2} \left(\frac{2}{\tilde{a}k_\mu} \sin(k_\mu a/2) \right) \tag{C.9}$$

Notational convention: *When variables k, p appear together in one formula, they are related by the unique decomposition (C.4).*

From eqs.(1.16) one computes

$$\tilde{u}(p) = \sum_{l\in D} \tilde{v}(p+l) |\tilde{\mathcal{C}}(p+l)|^2 , \tag{C.10}$$

$$\tilde{\mathcal{A}}(k) = \tilde{v}(k)\tilde{\mathcal{C}}(-k)\tilde{u}(p)^{-1} , \tag{C.11}$$

$$\tilde{\Gamma}_{ll'}(p) = \delta_{ll'}\tilde{v}(p+l) - \tilde{\mathcal{A}}(p+l')\tilde{\mathcal{C}}(p+l)\tilde{v}(p+l) . \tag{C.12}$$

The variables l, l^* assume s^2 values each.

In d dimensions, the formulas remain valid, except that $\mu = 1, ..., d$, and factors $(2\pi)^d$ have to be substituted for $(2\pi)^2$. The variables l, l' now assume s^d values.

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