Supersymmetry Breaking at Finite Temperature

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Abstract

The mechanism of supersymmetry breaking at finite temperature is still only partly understood. Though it has been proven that temperature always breaks supersymmetry, the spontaneous nature of this breaking remains unclear, in particular the role of the Goldstone fermion.

The aim of this work is to unify two existing approaches to the subject. From a hydrodynamic point of view, it has been argued under very general assumptions that in any supersymmetric quantum field theory at finite temperature there should exist a massless fermionic collective excitation, named phonino because of the analogy to the phonon. In the framework of a self-consistent resummed perturbation theory, it is shown for the example of the Wess-Zumino model that this mode fits very well into the quantum field theoretical framework pursued by earlier works. Interpreted as a bound state of boson and fermion, it contributes to the supersymmetric Ward-Takahashi identities in a way showing that supersymmetry is indeed broken spontaneously with the phonino playing the role of the Goldstone fermion.

The second part of the work addresses the case of supersymmetric quantum electrodynamics. It is shown that also here the phonino exists and must be interpreted as the Goldstone mode. This knowledge allows a generalization to a wider class of models.

Zusammenfassung

Der Mechanismus der Supersymmetriebrechung bei endlicher Temperatur ist nach wie vor nur unvollständig verstanden. Es scheint klar zu sein, dass die Supersymmetrie hierbei stets gebrochen ist, nicht jedoch, ob es sich tatsächlich um eine spontane Brechung handelt, die sich unter anderem in der Existenz eines Goldstone-Fermions äußern müsste.

In dieser Arbeit sollen zwei in der Literatur vorhandene Ansätze vereinigt werden. Von einem hydrodynamischen Standpunkt aus ist vorhergesagt worden, dass in supersymmetrischen Quantenfeldtheorien bei endlicher Temperatur stets ein masseloses Fermion als kollektive Anregung auftreten sollte, das wegen seiner Analogie zum Phonon auch Phonino genannt wurde. Es soll hier am Beispiel des Wess-Zumino-Modells im Rahmen einer selbstkonsistenten resummierten Störungstheorie gezeigt werden, dass sich diese kollektive Anregung, interpretiert als gebundener Zustand aus Boson und Fermion, perfekt in den quantenfeldtheoretischen Rahmen früherer Arbeiten einfügt und in einer Form zu den supersymmetrischen Ward-Takahashi-Identitäten beiträgt, die beweist, dass die Supersymmetrie in der Tat spontan gebrochen ist und das Phonino das zugehörige Goldstone-Teilchen darstellt.

Der zweite Teil der Arbeit befasst sich mit der Erweiterung auf die supersymmetrische Quantenelektrodynamik. Es wird gezeigt, dass auch hier das Phonino als kollektive Anregung auftritt und die Rolle des Goldstone-Fermions spielt. Diese Ergebnisse erlauben die Verallgemeinerung auf eine größere Klasse von Modellen.

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Introduction

Physics of elementary particles is in a very interesting stage. On the one hand, we have the very successful standard model that is in amazing agreement with experiment. On the other hand, it is quite clear for theoretical reasons and there are also the first experimental indications that this is not the final theory of nature that physicists long for. The extensions of the standard model that presently attract most attention all involve supersymmetry in some way. This has a number of reasons.

First of all, supersymmetry is an intriguing mathematical concept. In a seminal paper by Haag, Łopuszański and Sohnius [1], it was shown that it is the only fundamentally new kind of symmetry that we may find in the S-matrix of particle physics, besides the well-known bosonic internal symmetries and Poincaré and conformal symmetry.

Moreover, also from the point of view of phenomenology, supersymmetric quantum field theories have some very attractive properties. Even though supersymmetry is still lacking direct experimental evidence, there is a number of indirect phenomenological indications that motivate physicists to search for supersymmetry with extraordinary efforts [2]. For example, it allows for the unification of gauge couplings in supersymmetric grand unified theories and provides a solution for the hierarchy and naturalness problems.

Also in string theories, believed to be promising candidates for theories that unify gravitation with the other known forces, the concept of supersymmetry arises very naturally.

On the other hand, it is obvious that unbroken supersymmetry is not realized in nature, and simple attempts to break global supersymmetry spontaneously are phenomenologically ruled out. Though there exist very attractive breaking mechanisms like in the framework of supergravity, many alternative scenarios have been proposed and no unique picture has emerged yet. Hence, any mechanism that naturally breaks supersymmetry is worth a further investigation.

In a separate development, our insight into the evolution of the universe from the big bang to its present state has made an enormous progress over the last decades, and there has been a stimulating interchange of ideas between cosmology and quantum field theory. Thermal field theory was de-

veloped as a tool to describe quantum field theory at finite temperature and has been used to investigate the properties of a hot plasma like it is believed to have existed shortly after the big bang. For a thorough understanding of the early universe, it is essential to know how symmetries evolved during the cooling of the universe, in particular to understand possible phase transitions that may have left traces in the present state of the universe. For example, it is well-known that the spontaneously broken gauge symmetry of the standard model is restored at high temperature [3], a phenomenon that appears quite generally for global and gauge symmetries [4, 5].

If we believe that supersymmetry was realized in the hot early stages of the universe in one or the other way, it is indispensable to understand the mechanism of supersymmetry breaking at high temperatures in order to find out what influence this may have had on the evolution of the universe.

It is by now understood that supersymmetry is always broken at finite temperature [6]. This is in fact not surprising, since in thermal field theory the ground state is a statistical ensemble in which bosons and fermions behave differently as the distribution function depends on the statistics. As a consequence, also the effective masses, determined by the interaction with the heat bath, are in general different for bosons and fermions. Another illustrative reason for the breakdown of supersymmetry is the fact that the Hamiltonian, which is basically the square of the supercharge, has a non-vanishing expectation value at finite temperature. Moreover, for these reasons supersymmetry is broken at any non-vanishing temperature and, in contrast to internal symmetries, never restored at high temperatures.

Since the thermal ground state is responsible for the breakdown of supersymmetry, it has much in common with a spontaneous breaking. Hence, a natural question to ask is whether it is associated with the existence of a massless Goldstone fermion. Indeed, the Goldstone theorem can be applied at finite temperature, and it can be shown that there must be a zero-energy Goldstone state. However, since the rest frame of the heat bath also breaks Lorentz invariance, it is not obvious that this mode must be associated with a propagating Goldstone particle [7, 8].

Yet, for the simple case of the Wess-Zumino model, it could be shown that there is such a massless particle [9] which appears as a bound state rather than a fundamental fermion. Furthermore, arguments have been given to predict the general existence of such an excitation that should be regarded as a collective excitation of the supersymmetric plasma similar to sound waves in a medium with spontaneously broken Lorentz invariance. As these fermionic excitations are a supersymmetry analogue of phonons, they were baptized 'phoninos'.

From the many aspects that are still left to explore, we would like to focus on two points. Firstly, we will again study the Wess-Zumino model which is relatively simple in that it describes only one chiral supermultiplet.

Though it has already been investigated in a number of papers, there is still room for some clarification. We will give an explicit proof of the existence of the phonino and show that it fits perfectly into the quantum field theoretical framework of earlier studies. In particular, with the contributions from the phonino it will be possible to explicitly verify the Ward-Takahashi identities of spontaneously broken supersymmetry. This shows that the phonino really plays the role of the Goldstone fermion.

Secondly, we would like to proceed to more realistic models involving also vector multiplets. As a first step, we will investigate supersymmetric quantum electrodynamics and show that also here the phonino exists as a bound state, in this case of electron and scalar electron, and must be interpreted as the Goldstone particle associated with the spontaneous breakdown of supersymmetry.

With these explicit results in mind, we will then take a new look at the general situation and try to clarify why supersymmetry is broken at finite temperature in a model independent way.

In the following chapter, we will recall some basic facts that are needed throughout this work before we review the development of the subject of supersymmetry breaking at finite temperature. The next two chapters constitute the main part of the work and contain our results on the Wess-Zumino model and supersymmetric quantum electrodynamics. We conclude with some generalizations and an outlook.

Chapter 1

Quantum field theoretical preliminaries

In this work, we will make use of thermal field theory as the combination of statistical mechanics and quantum field theory that is appropriate for the description of a plasma in thermal equilibrium. In order to be self-contained, we would like to collect some basic knowledge from both of these topics that will be needed in the later chapters, since not everyone may be familiar with them.

1.1 Statistical mechanics

In statistical mechanics, as is well known, a system in thermal equilibrium can be characterized by the grand canonical ensemble. For a system with a Hamiltonian H and conserved charges N_i, such as baryon number or electric charge, the equilibrium state is described by the density operator

$$\rho = \frac{e^{-\beta(H-\mu_i N_i)}}{Z}.$$

Here, β is the inverse temperature

$$\beta = \frac{1}{k_B T}$$

and μ_i are the chemical potentials which are Lagrange multipliers for the conserved charges N_i. In the rest of this work, we will only consider the case of vanishing chemical potentials. For convenience, we work in units where Boltzmann's constant is $k_B = 1$.

The partition function

$$Z(\beta) = \operatorname{tr} e^{-\beta(H-\mu_i N_i)}$$

normalizes the density operator such that

$$\operatorname{tr}\rho = 1,$$

where the trace is taken over all physical states of the system.

The thermal expectation value of an observable $\mathcal{O}$ is then given by

$$\langle\mathcal{O}\rangle_T = \operatorname{tr}\mathcal{O}\rho = Z^{-1}\operatorname{tr}\mathcal{O}e^{-\beta(H-\mu_i N_i)}. \tag{1.1}$$

It should be noted that this expression makes sense only with some ultraviolet and finite volume cutoffs. In the thermodynamic limit of infinite volume, neither Z nor the trace are defined, and the appropriate limit must be taken.

All macroscopic properties of the system can be calculated from the partition function. For example, the equation of state is given by

$$PV = T\,\ln Z$$

with the pressure P and volume V.

1.2 Free quantum fields

Let us consider the case of a free scalar field of mass m in thermal equilibrium. In terms of creation and annihilation operators, it can be written as

$$\phi(x) = \int\frac{d^3p}{(2\pi)^3\,2E_p}\left[a(\vec{p})\,e^{-ipx} + a^\dagger(\vec{p})\,e^{ipx}\right] \tag{1.2}$$

with $p_0 = E_p = \sqrt{\vec{p}^2 + m^2}$. We have chosen the normalization in order to satisfy the usual commutation relations,

$$[a(\vec{p}), a^\dagger(\vec{p}\,')] = (2\pi)^3\,2E_p\,\delta^{(3)}(\vec{p}-\vec{p}\,'). \tag{1.3}$$

Now, what is the influence of the thermal background on the time-ordered propagator, in other words, what is

$$\langle T\phi(x)\phi(y)\rangle_T\ ?$$

According to Wick's theorem, we can write

$$T\phi(x)\phi(y) = \langle 0|T\phi(x)\phi(y)|0\rangle + :\phi(x)\phi(y):$$

where the first term on the right hand side is the vacuum Feynman propagator. While the expectation value of the normal-ordered second term vanishes in the vacuum, at finite temperature this is no longer the case. Instead, we know that the expectation value of the particle number operator is determined by the Bose-Einstein distribution function

$$n_B(p_0) = \frac{1}{e^{\beta|p_0|}-1}, \tag{1.4}$$

such that

$$\langle a^\dagger(\vec{p})a(\vec{p}')\rangle_T = (2\pi)^3\, 2E_p\, \delta^{(3)}(\vec{p}-\vec{p}')\, n_B(E_p)$$

in the normalization (1.3). Herewith, we can calculate the thermal expectation value of the normal product by making use of the representation (1.2),

$$\begin{aligned}\langle :\phi(x)\phi(y):\rangle_T &= \int \frac{d^3p}{(2\pi)^3\, 2E_p}\left[n_B(p_0)e^{-ip(x-y)} + n_B(p_0)e^{ip(x-y)}\right] \\ &= \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, 2\pi\, \delta(p^2-m^2)\, n_B(p_0).\end{aligned}$$

Altogether, we find that the thermal propagator is given by

$$D(x-y) = \langle T\phi(x)\phi(y)\rangle_T = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)} D(p)$$

with

$$D(p) = \frac{i}{p^2-m^2+i\epsilon} + 2\pi\, \delta(p^2-m^2)\, n_B(p_0). \tag{1.5}$$

The free fermion propagator can be derived in an analogous manner and is given by

$$S(x-y) = \langle T\psi(x)\overline{\psi}(y)\rangle_T = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)} S(p)$$

with

$$S(p) = \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} - (\not{p}+m)\, 2\pi\, \delta(p^2-m^2)\, n_F(p_0), \tag{1.6}$$

where the Fermi-Dirac distribution function is defined as

$$n_F(p_0) = \frac{1}{e^{\beta|p_0|}+1}. \tag{1.7}$$

Compared to the bosonic propagator, the different statistics changes, besides the distribution function, the sign of the thermal contribution.

1.3 Real and imaginary time formalisms

Up to now, we have only described the free theory. The quantization of interacting quantum field theories in a thermal background is slightly more involved. As we have seen above, annihilation operators acting on the ground

state do not vanish and therefore Wick's theorem cannot be applied without care.

Fortunately, for practical calculations, simple Feynman rules have been derived also for finite temperature. In fact, there are two approaches on the market. The Matsubara imaginary time formalism, for an introduction see [10, 11], is adequate mainly for the calculation of static properties. It is based on the observation that thermal two-point functions of bosons (fermions) obey periodic (anti-periodic) boundary conditions in imaginary time. This can be seen as follows. If one views the Boltzmann factor in equation (1.1) as a translation in imaginary time, one can easily derive the relation (for bosonic operators A and B)

$$\langle B(t')A(t+i\beta)\rangle_T = \langle A(t)B(t')\rangle_T.$$

This so-called Kubo-Martin-Schwinger condition is also the starting point for a more algebraic approach to quantum statistical mechanics [12]. The main drawback of this approach is however that it works in imaginary time and the physically relevant real-time Green functions can only be obtained by analytic continuation which in practice is difficult or impossible to perform. Furthermore, it explicitly breaks Lorentz and supersymmetry because frequencies are treated as discrete variables that are moreover different for bosons and fermions. For these reasons, we will not make use of this formalism here, although it has been used to check some of our results.

Better suited for our purposes is the second, real time formalism, also called thermo field dynamics, which was developed by Takahashi, Umezawa and collaborators [13], for a good introduction see [10, 14]. The aim of this approach is to calculate thermal Green functions directly for real-time arguments as expectation values in a temperature-dependent vacuum state,

$$\langle \mathcal{O}\rangle_T = \langle\, 0(\beta)\,|\,\mathcal{O}\,|\,0(\beta)\,\rangle.$$

However, since the thermal state is not a pure state but described by a density matrix, this is not possible in the original Hilbert space. The simplest possibility to find a representation like this is by doubling the degrees of freedom. One introduces for each field ϕ_i a ghost field $\tilde{\phi}_i$ whose interaction is given by the same Lagrangian as for the physical fields, however with opposite sign,

$$\tilde{\mathcal{L}}(\{\tilde{\phi}_i\}) = -\mathcal{L}(\{\tilde{\phi}_i\}).$$

Perturbation theory can then be built up by using the same Feynman diagrams as in the vacuum theory. Now two kinds of interaction vertices occur, the usual one for the physical particles, and a second one involving the corresponding ghost fields with a coupling of opposite sign. The transition between both types of vertices is made possible by the non-diagonal

elements of the free propagator which reads, for a free scalar field of mass m,

$$\begin{aligned}\widetilde{D}(p) = &\begin{pmatrix} \dfrac{i}{p^2 - m^2 + i\epsilon} & 0 \\ 0 & -\dfrac{i}{p^2 - m^2 - i\epsilon} \end{pmatrix} \\ &+ 2\pi\, \delta(p^2 - m^2)\, n_B(p_0) \begin{pmatrix} 1 & e^{\beta|p_0|/2} \\ e^{\beta|p_0|/2} & 1 \end{pmatrix}.\end{aligned} \tag{1.8}$$

The (11) component of this matrix is again the propagator for the physical particles that was derived in equation (1.5) above. The (22) component propagates the ghosts, and the non-diagonal components can transform one into the other. One can clearly see that the effect of the heat bath is given by the temperature dependent second part of the propagator which contains the thermal distribution function. In the zero temperature limit, this contribution vanishes and the ghost fields decouple.

For a fermion of mass m, the free propagator is given by

$$\begin{aligned}\widetilde{S}(p) = &(\not{p} + m) \begin{pmatrix} \dfrac{i}{p^2 - m^2 + i\epsilon} & 0 \\ 0 & -\dfrac{i}{p^2 - m^2 - i\epsilon} \end{pmatrix} \\ &- 2\pi\, (\not{p} + m)\, \delta(p^2 - m^2)\, n_F(p_0) \begin{pmatrix} 1 & e^{\beta|p_0|/2} \\ -e^{\beta|p_0|/2} & 1 \end{pmatrix}.\end{aligned} \tag{1.9}$$

Using these ingredients, one can now perturbatively calculate real-time thermal expectation values as Green functions of the physical 1-fields.

In practical calculations, there is often no need to include the ghost fields, but there are cases in which their neglection would lead to inconsistencies. In our calculations, we will later see that there is fortunately no need to include these ghost contributions.

1.4 Higher order corrections

It is important to understand how higher order corrections modify the thermal propagators. At first glance, it is not obvious that the structure of the free propagators given above remains unchanged when one includes thermal self energy corrections.

First note that the matrix propagator (1.8) can be written as

$$\widetilde{D}(p) = M_b(p_0) \begin{pmatrix} D_F(p) & 0 \\ 0 & D_F^*(p) \end{pmatrix} M_b(p_0),$$

where

$$M_b(p_0) = \begin{pmatrix} \cosh\theta(p_0) & \sinh\theta(p_0) \\ \sinh\theta(p_0) & \cosh\theta(p_0) \end{pmatrix} \quad \text{with} \quad \sinh^2\theta(p_0) = n_B(p_0),$$

and

$$D_F(p) = \frac{i}{p^2 - m^2 + i\epsilon}$$

is the usual Feynman propagator.

It can be shown that this structure is a consequence of *CPT* and translation invariance and must therefore be valid also for the full propagator,

$$\widetilde{\mathcal{D}}(p) = M_b(p_0) \begin{pmatrix} \mathcal{D}_F(p) & 0 \\ 0 & \mathcal{D}_F^*(p) \end{pmatrix} M_b(p_0).$$

Full and free propagator are now related by a Dyson-Schwinger equation,

$$\widetilde{\mathcal{D}}^{-1}(p) = \widetilde{D}^{-1}(p) + i\widetilde{\Pi}(p),$$

where $\widetilde{\Pi}$ is the self energy matrix that clearly must have a similar structure,

$$\widetilde{\Pi}(p) = M_b(p_0)^{-1} \begin{pmatrix} \Pi(p) & 0 \\ 0 & -\Pi^*(p) \end{pmatrix} M_b(p_0)^{-1}, \tag{1.10}$$

where the only independent component Π must satisfy the Dyson-Schwinger equation

$$\mathcal{D}_F^{-1}(p) = D_F^{-1}(p) + i\Pi(p).$$

In practice, one now calculates perturbatively the (11) component of the self energy matrix $\widetilde{\Pi}$. From this, one obtains the real and imaginary parts of the function Π by means of the following relations which can easily be derived as a consequence of the matrix structure (1.10),

$$\begin{aligned} \operatorname{Re}\Pi(p) &= \operatorname{Re}\widetilde{\Pi}_{11}(p), \\ \operatorname{Im}\Pi(p) &= \tanh(\beta|p_0|/2)\operatorname{Im}\widetilde{\Pi}_{11}(p). \end{aligned} \tag{1.11}$$

In the simplest cases, the thermal self energy Π is real and small compared to the squared mass such that it only leads to a small, temperature-dependent mass shift. The physical (11) propagator is then modified by the usual replacement

$$p^2 - m^2 \to p^2 - m^2 - \Pi(p)$$

while the overall structure

$$\mathcal{D}(p) = \frac{i}{p^2 - m^2 - \Pi(p) + i\epsilon} + 2\pi\,\delta\big(p^2 - m^2 - \Pi(p)\big)\,n_B(p_0) \tag{1.12}$$

remains unchanged.

One should note that there is now an ambiguity in defining the mass. Because of the breaking of Lorentz invariance, the self energy $\Pi(p)$ not only depends on p^2 but also on $\vec{p}$. Therefore, the value of the propagator at zero momentum or the pole for vanishing three-momentum lead to different notions of mass while they coincide at zero temperature. For reasons that will become clear later, we will adopt the definition of mass as the pole of the full propagator in the limit of vanishing three-momentum,

$$m^2(T) = m^2 + \Pi(m, 0).$$

In general, the situation is of course more complicated. For massless particles or at ultra-high temperatures, the full propagator may differ considerably from the zero-temperature one. In particular, it is well known from hot gauge theories that the naive perturbative treatment may break down and a more involved resummation becomes necessary. Furthermore, the self energy in general acquires an imaginary part which leads to damping. For a full discussion, see e.g. [10].

The fermionic case appears quite similar. Here, the matrix propagator (1.9) can be written as

$$\widetilde{S}(p) = M_f(p_0) \begin{pmatrix} S_F(p) & 0 \\ 0 & \overline{S}_F(p) \end{pmatrix} M_f(p_0),$$

where

$$M_f(p_0) = \begin{pmatrix} \cos\theta(p_0) & -\sin\theta(p_0) \\ \sin\theta(p_0) & \cos\theta(p_0) \end{pmatrix} \quad \text{with} \quad \sin^2\theta(p_0) = n_F(p_0).$$

As usual,

$$S_F(p) = \frac{i}{\not{p} - m + i\epsilon}$$

is the Feynman propagator and

$$\overline{S}_F = \gamma^0 S_F^\dagger \gamma^0.$$

From the Dyson-Schwinger equation,

$$\widetilde{\mathcal{S}}^{-1}(p) = \widetilde{S}^{-1}(p) + i\widetilde{\Sigma}(p),$$

it follows that the self energy has the matrix structure

$$\widetilde{\Sigma}(p) = M_f(p_0)^{-1} \begin{pmatrix} \Sigma(p) & 0 \\ 0 & -\overline{\Sigma}(p) \end{pmatrix} M_f(p_0)^{-1}. \tag{1.13}$$

The components of the propagator must then satisfy a similar equation,

$$\mathcal{S}_F^{-1}(p) = S_F^{-1}(p) + i\Sigma(p),$$

where Σ can be related to the (11) component of the self energy matrix. To this end, one must bear in mind that the component has itself a matrix structure, but by defining the dispersive and absorptive parts of the self energy as

$$\begin{aligned}\operatorname{Re}\Sigma(p) &= \frac{1}{2}\left(\Sigma(p)+\overline{\Sigma}(p)\right),\\ \operatorname{Im}\Sigma(p) &= \frac{1}{2i}\left(\Sigma(p)-\overline{\Sigma}(p)\right),\end{aligned}$$

one can derive the following relations from the matrix structure (1.13),

$$\begin{aligned}\operatorname{Re}\Sigma(p) &= \operatorname{Re}\widetilde{\Sigma}_{11}(p),\\ \operatorname{Im}\Sigma(p) &= \coth(\beta|p_0|/2)\operatorname{Im}\widetilde{\Sigma}_{11}(p).\end{aligned} \tag{1.14}$$

In the simplest cases, where the thermal self energy Σ is real and small compared to the mass, it only leads to a temperature-dependent shift

$$\not{p} - m \to \not{p} - m - \Sigma(p)$$

such that the full Feynman propagator can be approximated as

$$\mathcal{S}_F(p) = \frac{i(\not{p}+m+\Sigma'(p))}{p^2-m^2-\Pi_F(p)+i\epsilon},$$

where

$$\Pi_F(p) = \tfrac{1}{2}\operatorname{tr}\left[(\not{p}+m)\Sigma(p)\right]$$

and

$$\Sigma' = \tfrac{1}{2}\operatorname{tr}\Sigma - \Sigma.$$

The overall structure of the physical (11) propagator thereby remains the same as for the free propagator,

$$\begin{aligned}\mathcal{S}(p) = &\left(\not{p}+m+\Sigma'(p)\right)\cdot\\ &\cdot\left(\frac{i}{p^2-m^2-\Pi_F(p)+i\epsilon} - 2\pi\,\delta\big(p^2-m^2-\Pi_F(p)\big)\,n_F(p_0)\right).\end{aligned} \tag{1.15}$$

For vanishing three-momentum, the self energy correction appears as a thermal mass shift,

$$m^2(T) = m^2 + \Pi_F(m,0).$$

1.5 Renormalization

When one calculates higher order corrections in a relativistic quantum field theory, it is always important to ensure finite results by identifying divergences and removing them via a renormalization procedure.

In finite temperature field theory, it turns out that the ultraviolet divergences can be absorbed by the same redefinition of the parameters as in the vacuum theory. This is easy to understand if one remembers that the thermal part of the propagator has support on the mass shell and decays rapidly with growing momentum because of the distribution function. Hence, it does not bring along new ultraviolet divergences.

This knowledge is particularly useful for the case of supersymmetry. If we consider a theory with unbroken supersymmetry at vanishing temperature, the non-renormalization theorems also hold at finite temperature. This means that the superpotential is not renormalized.

For phenomenological purposes, it may be useful to pick a renormalization scheme where thermal corrections are included in the definition of the renormalized masses, see e.g. [15]. Since Lorentz invariance is broken, there is however some arbitrariness in this approach. For conceptual clarity, it is advantageous to renormalize at vanishing temperature.

In the following, we will only be concerned with the calculation of the finite temperature-dependent corrections. Also if it is not stated explicitly, when we calculate some Green functions, we always mean the temperature-dependent contributions and neglect the higher order vacuum contributions that are small after renormalization.

1.6 Collective excitations

In section 1.4, we had seen how the propagators of the fundamental particles are modified by the interaction with the heat bath. In the simplest case, this is just a shift in the mass, while in general also the dispersion law is changed.

In addition, due to the complex dynamics of the matter background, there may however appear completely new features in the particle propagators. New poles may show up that are not present in the vacuum, describing the propagation of quasiparticles that may be regarded as collective excitations of the medium. These have characteristic dispersion laws, usually accompanied by imaginary contributions meaning that these excitations are damped.

The investigation of the particle structure of a hot plasma is therefore slightly more involved than usually in relativistic quantum field theory where the physical particles are mostly put in from the outset. One proceeds by identifying quasiparticles and investigating their properties, adding them to the effective description of the system until it emerges a self-consistent picture of the dynamics. This process sometimes obscures the systematic perturbative expansion one is used to in quantum field theory, but it comes very close to the actual physics of the system and allows also the inclusion of nonperturbative effects.

1.7 Symmetries

Symmetries play an important role in all parts of physics, and it is particularly interesting to study their behavior at finite temperature, since the interaction with the heat bath may lead to a restoration or breaking of symmetries.

What do we mean by a symmetry? We say that a physical system is symmetric if the expectation value of all physical quantities remains unchanged when the system undergoes the symmetry transformation δ, i.e.,

$$\langle \delta \mathcal{O} \rangle = 0$$

for all observables $\mathcal{O}$. In such a case, the symmetry is called manifest, otherwise it is broken. At this point, this is meant in a completely general, algebraic sense. Depending on the physical situation, the observable can stand for a function on the classical phase space or in quantum theory for an operator whose expectation value is taken in the vacuum or a thermal state.

In case the basic equations of the theory possess a symmetry, in particular the Hamiltonian is invariant,

$$\delta H = 0,$$

it is still meaningful to speak of a symmetry even if the particular representation, the ground state, does not respect it. In this case, the symmetry, though not realized manifestly, will in general leave some distinct traces. We then say that the symmetry is broken spontaneously.

If even this is not the case, the symmetry is broken explicitly.

In quantum field theory, a symmetry transformation is primarily meant to be a linear mapping on the space of polynomials in the fundamental fields. A certain model is said to be invariant under a symmetry group if the classical action (and thereby the Hamiltonian) is invariant,

$$\delta \int d^4x \, \mathcal{L} = 0.$$

In this work, we will not be concerned with anomalies but assume that the symmetry is then also realized on the quantum level.

Due to the Noether theorem, any continuous symmetry is associated with a current $J^\mu(x)$ that is Lorentz-covariant and conserved,

$$\partial_\mu J^\mu(x) = 0,$$

such that the change of the operator $\mathcal{O}$ under a symmetry transformation is given by

$$\delta \mathcal{O} = -i\varepsilon \int d^3x \, [J^0(x), \mathcal{O}],$$

with an infinitesimal transformation parameter ε. We may formally write this as

$$\delta\mathcal{O} = -i\varepsilon[Q, \mathcal{O}]$$

where we have defined the charge Q as

$$Q = \int d^3x\, J^0(x). \tag{1.16}$$

As a consequence of the conservation of the current, the charge is independent of time.

Formally, we may always let the charge act on some operator $\mathcal{O}$ in the sense of

$$[Q, \mathcal{O}] = \int d^3x\, [J^0(x), \mathcal{O}],$$

while the actual existence of the charge, i.e., the convergence of the integral (1.16), is already a special case. If it exists, the charge necessarily destroys the ground state because of translation invariance,

$$Q\,|0\rangle = 0,$$

from which it follows that

$$\langle 0|\delta\mathcal{O}|0\rangle = -i\varepsilon\langle 0|[Q, \mathcal{O}]|0\rangle = 0$$

for all $\mathcal{O}$. Therefore, the symmetry is manifest. Note that this reasoning holds for the vacuum in an obvious manner, but also for a thermal ground state, since in this case

$$\langle\delta\mathcal{O}\rangle_T = -i\varepsilon Z^{-1}\,\mathrm{tr}\,[Q, \mathcal{O}]e^{-\beta H} = 0$$

because of the cyclicity of the trace, if the Hamiltonian commutes with the charge. The spontaneous breaking of a symmetry, characterized by the appearance of some nontrivial vacuum (or thermal) expectation value

$$\langle\delta\mathcal{O}\rangle \neq 0,$$

is therefore accompanied by the nonexistence of the charge.

1.8 Supersymmetry

In this work, we are most interested in supersymmetry which transforms bosonic operators into fermionic ones and vice versa. This makes it necessary to modify some of the formulae of the previous section where the symmetry was implicitly assumed to be of bosonic nature.

In the framework of relativistic quantum field theory in four spacetime dimensions, and restricting ourselves to a single pair of supercharges ($N = 1$

supersymmetry), the supersymmetry algebra is unique. In terms of the generators Q that can be chosen to have the transformation properties of a Majorana spinor, it reads

$$\{Q, \overline{Q}\} = 2\gamma^\mu P_\mu, \tag{1.17}$$

where P_μ is the momentum operator. This displays that supersymmetry is intimately related with Poincaré symmetry. For reference, we have collected our conventions concerning spinor calculus in appendix A.

For a systematic introduction to the construction of supersymmetric quantum field theories, we must refer to the literature, see e.g. [16–21]. Here, we can only collect some formulae that we will utilize later on, but fortunately, what will be needed does not go beyond textbook knowledge.

For illustration, let us recall the simplest representation of the supersymmetry algebra, the chiral multiplet. Starting from a Weyl fermion ψ and a complex scalar ϕ, the supersymmetry transformations must be of the form

$$\begin{aligned}
\delta\phi &= \sqrt{2}\,\xi\psi \\
\delta\psi &= \sqrt{2}\,\xi F - i\sqrt{2}\,\partial^\mu\phi\,\sigma_\mu\overline{\xi} \\
\delta F &= -i\sqrt{2}\,\overline{\xi}\overline{\sigma}^\mu\partial_\mu\psi,
\end{aligned}$$

where ξ is an infinitesimal fermionic transformation parameter. The auxiliary scalar field F must be introduced in order to close the algebra.

For the spontaneous breakdown of supersymmetry, one of the right hand sides of the transformation laws must acquire a nontrivial expectation value. In this case, this is obviously only possible for the term involving the auxiliary field. If

$$\langle F\rangle \neq 0,$$

it follows that the fermion field transforms inhomogeneously,

$$\langle\delta\psi\rangle \neq 0,$$

and supersymmetry is broken spontaneously.

The requirement of supersymmetry of a model is very restrictive. In particular, unbroken supersymmetry implies that all members of one supermultiplet have the same mass. Characteristic of spontaneous supersymmetry breaking in the vacuum is a remaining relation between the masses of one supermultiplet. It reads

$$\sum_J (-1)^{2J}(2J+1)\,m_J^2 = 0, \tag{1.18}$$

where the sum is taken over the spins J of the members of the multiplet.

Like any continuous symmetry, also supersymmetry is associated with a conserved current that can be calculated as follows. While the action

of a supersymmetric model is by definition invariant under supersymmetry transformations, the Lagrangian in general is not. Instead, it transforms into a total derivative,

$$\delta\mathcal{L} = \partial_\mu \kappa^\mu.$$

Together with the Noether current

$$N^\mu = -\sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i,$$

one can define the supercurrent as

$$J^\mu = \kappa^\mu + N^\mu.$$

Note that the supercurrent is a fermionic operator as well as the supercharge which is formally defined as

$$Q = \int d^3x\, J^0(x).$$

In terms of the supercharge, one can express the supersymmetry transformations as

$$\delta\phi_i = -i\bar{\xi}[Q, \phi_i]_\pm,$$

where the commutator (anticommutator) applies for bosonic (fermionic) operators ϕ_i.

Of great importance will be the transformation behavior of the supercurrent. It is given by

$$\{Q, \overline{J}^\mu\} = 2\, T^{\mu\nu}\gamma_\nu + \text{S.T.} \tag{1.19}$$

which is basically the local form of equation (1.17) since the energy-momentum tensor $T^{\mu\nu}$ is the current which generates the translations P^μ. Unless we state the additional Schwinger terms and specify which form of the energy-momentum tensor we mean, this equation is only of limited use. We will refrain from entering into this discussion but note that the ambiguity lies only in the addition of a total derivative whose expectation value vanishes in any translation invariant state [19, 20]. Hence, we may use this relation in vacuum or thermal expectation values.

1.9 Ward-Takahashi identities

The symmetry of a quantum field theory displays itself in relations between different Green functions. Particularly important relations can be derived for the time-ordered correlation functions involving the symmetry current.

Let us consider the case where the current J^μ is fermionic as well as the operator Ψ. Because of current conservation, one can calculate

$$\begin{aligned}\partial^x_\mu \langle T J^\mu(x)\Psi(y)\rangle &= \partial^x_\mu \big(\theta(x_0-y_0)\langle J^\mu(x)\Psi(y)\rangle - \theta(y_0-x_0)\langle \Psi(y)J^\mu(x)\rangle\big)\\ &= \delta(x_0-y_0)\langle\{J^0(x),\Psi(y)\}\rangle.\end{aligned}$$

Introducing the charge, we write

$$\delta(x^0-y^0)\{J^0(x),\Psi(y)\} = \delta^{(4)}(x-y)\{Q,\Psi(y)\}$$

and obtain

$$\partial^x_\mu \langle T J^\mu(x)\Psi(y)\rangle = \delta^{(4)}(x-y)\langle\{Q,\Psi(y)\}\rangle. \tag{1.20}$$

For higher n-point functions, this relation, called Ward-Takahashi identity, generalizes as follows,

$$\begin{aligned}&\partial^x_\mu \langle T J^\mu(x)\mathcal{O}_1(y_1)\dots\mathcal{O}_n(y_n)\rangle\\ &\qquad = \sum_{i=1}^n \delta^{(4)}(x-y_i)\langle T\mathcal{O}_1(y_1)\dots[Q,\mathcal{O}_i(y_i)]_\pm\dots\mathcal{O}_n(y_n)\rangle,\end{aligned} \tag{1.21}$$

where the plus (minus) sign applies for fermionic (bosonic) operators $\mathcal{O}_i$.

Also the generalization to tensor quantities, in particular the symmetry currents, is straightforward. In this case, one must however pay attention to a covariant time-ordering, see e.g. [22], and the above derivation should only be taken as an illustration.

It is important to note that the Ward-Takahashi identities are basically operator identities and are therefore valid even if the symmetry is spontaneously broken. In contrast, if a symmetry is broken explicitly, there is no need for these identities to hold.

For the same reason, the identities are valid in the vacuum as well as in a thermal background. Hence, they are suited for the investigation of spontaneous symmetry breaking also at finite temperatures.

1.10 Goldstone theorem

Consider a Ward-Takahashi identity of the general form

$$\partial^x_\mu \langle T J^\mu(x)\mathcal{O}_1(y)\rangle = \delta^{(4)}(x-y)\langle\mathcal{O}_2(y)\rangle. \tag{1.22}$$

If the symmetry is broken spontaneously, there exist some operators $\mathcal{O}_1$ and $\mathcal{O}_2$ such that the right hand side is nontrivial. This has an important consequence for the left hand side.

Transformed into momentum space, setting

$$\Gamma^\mu_{J\mathcal{O}_1}(k) = \int d^4x\, e^{ik(x-y)}\langle T J^\mu(x)\mathcal{O}_1(y)\rangle,$$

one can rewrite the identity (1.22) as

$$-ik_\mu \Gamma^\mu_{J\mathcal{O}_1}(k) = \langle \mathcal{O}_2(y) \rangle \neq 0.$$

In order to satisfy this equation for all momenta k, the Fourier transformed Green function $\Gamma^\mu_{J\mathcal{O}_1}$ on the left hand side must have a pole in the limit $k \to 0$. In the vacuum, it follows from Lorentz covariance that there must be a pole for all lightlike momenta $k^2 = 0$, that is, there must exist a massless Goldstone particle.

One cannot however draw this conclusion at finite temperature. In this case, Lorentz invariance is broken and there can in principle be an isolated pole for vanishing momentum without the need for a Goldstone particle. Therefore, it requires some more effort to verify the existence of a propagating Goldstone particle in a specific model.

In the above discussion, we did not make any assumptions about the nature of the current. Therefore, since a non-vanishing right hand side may only appear for a bosonic operator $\mathcal{O}_2$, the spontaneous breakdown of a bosonic symmetry implies the existence of a bosonic Goldstone mode while the breaking of supersymmetry results in the existence of a Goldstone fermion, also called Goldstino.

Chapter 2

Supersymmetry at finite temperature

Supersymmetry at finite temperature is not a new subject. Over more than twenty years, there have been several attempts to understand in which way, if at all, supersymmetry is broken by finite temperature effects. Yet there are still many aspects left to explore.

Let us first try to trace the remarkably 'nonlinear' evolution of the topic.

2.1 Broken or not?

In the early days, starting with [23], it was observed from the investigation of the effective potential that finite temperature self energy corrections may result in nontrivial thermal expectation values of scalar fields. This indicates the spontaneous breakdown of supersymmetry. Furthermore, it was found that supersymmetry is not restored at high temperatures like a gauge symmetry.

As a consequence of a non-vanishing thermal expectation value, the masses of the particles coupling to this field are shifted. This leads to a mass splitting between the members of one supermultiplet of order

$$m_F^2 - m_B^2 \sim \begin{cases} \dfrac{g^2}{m}\, e^{-m/T} \left(\dfrac{Tm}{2\pi}\right)^{3/2} & T \ll m, \\ g^2 T^2 & T \gg m \end{cases} \tag{2.1}$$

(where m is the common tree-level mass) as the most obvious sign for the breakdown of supersymmetry.

Later it was realized that there is much more to particle dynamics in a hot plasma than the effective potential. It was shown in [24] that the mass corrections coming from the coupling to the expectation value of the scalar field are of the same order of magnitude as the remaining one-loop

self energy corrections. For consistency, all these diagrams must be taken into account if one calculates the thermal masses. As a consequence, also the mass relation

$$\sum_J (-1)^{2J}(2J+1)\, m_J^2 = 0$$

known from ordinary spontaneous supersymmetry breaking in the vacuum is no longer valid at finite temperature.

In [25], general quantum field theoretical arguments were given to support the idea that supersymmetry is always broken at finite temperature, basically because there usually exists some nontrivial thermal expectation value such that

$$\langle \delta\psi \rangle_T \sim \langle F \rangle_T \neq 0.$$

While the authors of this study made use of the imaginary time formalism that seems to break supersymmetry explicitly by imposing different boundary conditions on bosons and fermions, the same results could later also be obtained in the real-time formalism [7].

Only later it was recognized [26] that thermal expectation values of the fundamental F fields are not the only possible order parameters of supersymmetry breaking. It was observed that in certain models the composite boson-fermion operator $A\Psi$ is transformed into something with a nontrivial thermal expectation value,

$$\langle \{Q, A(x)\Psi(y)\} \rangle_T = \alpha \langle A(x)A(y) \rangle_T + \beta \langle \Psi(x)\overline{\Psi}(y) \rangle_T \neq 0.$$

This shows that the breakdown of supersymmetry is a more general phenomenon that occurs even when there are no thermal expectation values of the fundamental scalar fields, for example because of a chiral symmetry.

In a separate development, it was observed [27] that though nontrivial thermal expectation values

$$\langle \delta\mathcal{O} \rangle_T = \mathrm{tr}\, \rho\, \delta\mathcal{O} \neq 0$$

always occur as a sign of supersymmetry breaking, the Klein operator inserted expectation values

$$\langle \delta\mathcal{O} \rangle_K = \mathrm{tr}\, (-1)^F \rho\, \delta\mathcal{O} = 0$$

vanish for all temperatures if they do so in the vacuum, where the Klein operator $(-1)^F$ has eigenvalues $+1$ on bosonic and -1 on fermionic states. Several papers were devoted to this interesting topic, and some confusion arose if this observation is of physical relevance, starting a rather dialectic debate whether or not supersymmetry should be considered broken, see [28] and the references therein. It became however clear that these 'superthermal expectation values' describe an interesting relation between the bosonic

and fermionic sub-ensembles, but not the thermodynamic properties of the system under consideration.

Probably the most concise analysis is due to Buchholz and Ojima [6] who, by making use of the algebraic approach to quantum field theory, could show in a model independent way that supersymmetry is broken in any thermal state. Precisely, their theorem states that if supercharges as generators of supersymmetry transformations can be defined in a spatially homogeneous physical state, then this state describes the vacuum. They also showed that this breakdown of supersymmetry is in a way even stronger then the usual spontaneous breakdown of a bosonic internal symmetry insofar as it is not even possible to construct symmetrized states on which an action of supercharges can be defined. For a compact symmetry group, in contrast, such a symmetric state can easily be defined by averaging over the symmetry group. Consequently, they introduced the new term 'spontaneous collapse' of supersymmetry. Although this result is probably the last word on the question if supersymmetry is broken at finite temperature, it does not yet tell us much about the phenomenological consequences thereof.

2.2 Is there a Goldstone mode?

It is not only interesting to understand if supersymmetry is broken, but also to understand the mechanism of the breaking. Though the thermal breaking obviously has much in common with a spontaneous breaking, we have already described some important differences to well-known examples of spontaneous symmetry breaking like the absence of a mass relation and the non-restoration at high temperatures. The crucial feature that can discriminate between spontaneous and explicit symmetry breaking is the validity of the Ward-Takahashi identities and as a consequence the existence of a Goldstone fermion.

Already in [25] the possible appearance of a Goldstone mode was investigated. The authors made however use of the imaginary time formalism where boundary contributions to the Ward-Takahashi identities appear because of the different boundary conditions imposed on bosons and fermions. This fact obscures the detection of such a mode.

In the real-time formalism, it was later established that a Goldstone mode exists and should be seen in real-time Green functions. Two independent analyses came to this conclusion by investigating the Ward-Takahashi identities, for good reviews see [29, 30].

The first work [7] found, in certain interacting theories, the appearance of a zero-energy singularity in the propagator of the elementary fermion that was therefore interpreted as the Goldstone fermion. In a subsequent paper [26], these results were generalized to other models and it was noticed

that the Goldstone mode is not always the fundamental field but can also appear in the composite boson-fermion channel.

In the second work [8], followed by [31], it was demonstrated from the Ward-Takahashi identity for the composite operator $A\Psi$, which they termed the 'thermal superpair', that the two-point function of this operator should have a zero-momentum singularity. The physical interpretation was given that one particle can be replaced by its superpartner of the same energy and momentum, which is a nontrivial process even in the limit of vanishing momentum but does not cost energy because of the mass degeneracy.

Both authors found the Goldstone pole only for exactly vanishing momentum. In addition, they observed that the Ward-Takahashi identities imply a zero-momentum singularity even in the free theory where a Goldstone particle obviously does not exist. Hence, the authors denied the interpretation as a propagating Goldstone particle.

Already some years earlier, Kapusta [32] showed in an explicit calculation for the conformally invariant Wess-Zumino model that a soft mode can be found in the fermion propagator, not only in the zero momentum limit, but with a dispersion relation

$$k_0 = \frac{1}{3}|\vec{k}|. \tag{2.2}$$

He interpreted it as a bound state of fermion and boson or a long-wavelength collective oscillation of the plasma. This result was obtained by a self-consistent solution of the Dyson-Schwinger equations. The fact that this result cannot be reached perturbatively thereby explains the failure of the various other attempts to find the Goldstino. However, the author attributed the existence of the soft fermion to chiral symmetry and not to the breakdown of supersymmetry.

In the same spirit, Chia [33] found, in some special models, a zero-momentum pole in the propagator of the fundamental fermion without regarding it as a general phenomenon or investigating the dispersion law.

In [34], the Goldstino was again taken seriously as a particle and its dispersion law for low temperatures was investigated for the first time, the result being

$$k_0 = \frac{T}{m}|\vec{k}| \tag{2.3}$$

for the Wess-Zumino model and a supersymmetric σ-model. The calculation made use of a resummed perturbation theory where the different thermal masses for bosons and fermions were included in the propagators in the calculation of self energy diagrams. Though this work contained a number of inaccuracies that had to be corrected by the authors of [9], the result was essentially confirmed. In the latter work, it became clear that the Goldstino pole for low temperatures can only be obtained by an elaborate resummation procedure which was performed by solving the Bethe-Salpeter

equations. Therefore the existence of the Goldstone particle is really a non-perturbative result. In view of this, it should be interpreted as a bound state of boson and fermion or a collective excitation rather than a fundamental Goldstone fermion. Furthermore, it was argued that the appearance of such an excitation is similar to the existence of sound waves and just as general. Accordingly, they introduced the term 'phonino' to stress the analogy to the phonon which can be viewed as the Goldstone particle of thermally broken Lorentz symmetry.

These two studies also calculated the damping rates and found that the phonino is, in the range of small momenta where it exists, a long-lived excitation.

Altogether, the physical picture that has emerged by now can be described as follows. Supersymmetry is inevitably broken at finite temperature, and the effective propagators of bosons and fermions interacting with the heat bath differ considerably. Quite generally, the phonino appears as a collective excitation of the supersymmetric plasma rather than as a fundamental Goldstone fermion. Its properties can be studied nonperturbatively in the framework of a self-consistent resummed perturbation theory.

From this historical development, it becomes clear that it would be desirable to unify the various developments, in particular the approach from [9] inspired by kinetic theory and the quantum field theoretical framework of [26, 31]. On the other hand, it would be valuable to obtain a model independent picture of supersymmetry breaking at finite temperature and to proceed to more realistic models, as up to now most studies were restricted to toy models with only chiral superfields.

2.3 Hydrodynamics of the supersymmetric plasma

Remarkably, the existence of a slow-moving collective excitation in the supersymmetric plasma is as general as the existence of sound waves, and it can be proved in a similar way. We would like to recall the derivation from [9].

The linear response of a Lorentz symmetric system to the influence of an external field $h^{\mu\nu}$ coupled to the energy-momentum tensor $T_{\mu\nu}$ is given by

$$\delta\langle T_{\mu\nu}(x)\rangle_T = -i\int d^4x'\,\langle T_{\mu\nu}(x)T_{\rho\sigma}(x')\rangle_T\,h^{\rho\sigma}(x')$$

such that the propagation of the external disturbance is determined by the (retarded, to be precise) correlation function $\langle T_{\mu\nu}T_{\rho\sigma}\rangle_T$.

For a relativistic perfect fluid, the energy-momentum tensor is given by

$$\langle T^{\mu\nu}\rangle_T = (\rho + p)u^\mu u^\nu - pg^{\mu\nu},$$

where p is the pressure and ρ the energy density while

$$u^\mu = (1 - \vec{v}^2)^{-1/2} (1, \vec{v})$$

is the four-velocity of the fluid. In equilibrium, we take $\vec{v} = 0$, in other words, we go to the rest frame of the heat bath.

After a perturbation, fast relaxation processes take place and local thermodynamic equilibrium is established. The local deviation from the global average can be described by two parameters, the temperature $\Delta T(x,t)$ and velocity $\vec{v}(x,t)$. In a linear approximation, the components of the energy-momentum tensor are then given by

$$\begin{aligned}
\langle T_{00}(x,t)\rangle &= \left(1 + \Delta T(x,t)\,\frac{\partial}{\partial T}\right)\rho,\\
\langle T_{0i}(x,t)\rangle &= v_i(x,t)(\rho + p),\\
\langle T_{ij}(x,t)\rangle &= \delta_{ij}\left(1 + \Delta T(x,t)\,\frac{\partial}{\partial T}\right)p.
\end{aligned}$$

Since the energy-momentum tensor is conserved, $\partial^\mu T_{\mu\nu} = 0$, the parameters v and ΔT are related by

$$\begin{aligned}
\partial_0 \Delta T\,\frac{\partial \rho}{\partial T} - \partial_i v_i(x,t)(\rho + p) &= 0,\\
\partial_0 v_i(x,t)\,(\rho + p) - \partial_i \Delta T\,\frac{\partial p}{\partial T} &= 0.
\end{aligned}$$

By substituting v, one finds the wave equation

$$\left(\frac{\partial \rho}{\partial T}\,\partial_0^2 - \frac{\partial p}{\partial T}\,\vec{\partial}^2\right)\Delta T(x,t) = 0,$$

describing the propagation of sound waves with the velocity

$$v_S^2 = \frac{\partial p/\partial T}{\partial \rho/\partial T}. \tag{2.4}$$

Hence, as a consequence of the spontaneous breakdown of Lorentz symmetry in the thermal bath, perturbations in the energy-momentum tensor propagate as sound waves. Their quanta, the phonons, can be viewed as the Goldstone bosons associated with the spontaneous symmetry breaking.

Now, let us come to the supersymmetric case. Here, we have the conserved supercurrent J^μ. Consider a small, space and time dependent supersymmetry variation characterized by a parameter $\xi(x,t)$ which leads to a local deviation of the supercurrent from the global average, $\delta_\xi J^\mu(x,t)$. Since we know that by such a supersymmetry transformation the supercurrent is transformed into the energy-momentum tensor, its variation can be expressed as

$$\delta_\xi J_\mu = -2i\gamma^\nu \xi\, T_{\mu\nu}.$$

The conservation equation, $\partial^\mu J_\mu = 0$, then tells us that also the local variation of the supercurrent must be conserved, giving

$$\partial^\mu \langle \delta_\xi J_\mu(x,t) \rangle = -2i\gamma^\nu \partial^\mu \xi(x,t) \langle T_{\mu\nu}(x,t) \rangle = 0.$$

In terms of the components of the energy-momentum tensor, this can be written as

$$(\rho\,\gamma_0 \partial_0 + p\,\gamma_i \partial_i)\,\xi(x,t) = 0,$$

which is nothing but the Dirac equation for a massless fermion propagating with the velocity

$$v_{SS} = \frac{p}{\rho}. \tag{2.5}$$

Therefore, in a supersymmetric system whose supersymmetry is broken spontaneously by the thermal bath, there should exist 'supersymmetric sound'. Its quanta, interpreted as the Goldstone fermions associated with the spontaneous symmetry breaking, would naturally be called phoninos. Of course there is no classical picture of these fermionic waves, but in the framework of thermal field theory it should be possible to verify the existence of such a fermionic collective excitation. Just as we expect sound waves to appear as poles in the correlator $\langle T_{\mu\nu}(x) T_{\rho\sigma}(y) \rangle_T$, supersymmetric sound should lead to poles in the correlation function $\langle J_\mu(x) \overline{J}_\nu(y) \rangle_T$.

Both of these types of waves have a very characteristic dispersion law. In the relativistic limit, when the temperature is much higher than the particle masses, pressure and energy density are related by $p = \frac{1}{3}\rho$, and therefore

$$v_S^2 = v_{SS} = \frac{1}{3}, \qquad T \gg m. \tag{2.6}$$

In the non-relativistic limit of a massive theory, we have $p = \frac{T}{m}\rho$, resulting in

$$v_S^2 = v_{SS} = \frac{T}{m}, \qquad T \ll m. \tag{2.7}$$

In the intermediate range, $T \simeq m$, v_S^2 and v_{SS} differ by a few percent [9].

So far, the derivation is completely general and model independent. In an interacting theory, the fluid is of course not perfect and one will in general get corrections to the dispersion law and also damping. This naturally depends strongly on the model. In any case, sound and supersymmetric sound waves can exist only for wavelengths greater than the mean free path so that there is time to establish local thermodynamic equilibrium. At higher frequencies, these waves are strongly damped.

Chapter 3

The Wess-Zumino model

Already a number of studies has dealt with the Wess-Zumino model at finite temperature because of its pretended simplicity. The authors come however to rather different conclusions, see [7–9,24,25,33,34] as a sample. We believe that a new look at this model may clarify some aspects of the breakdown of supersymmetry at finite temperature.

In the first sections of this chapter we will review the model and its supersymmetry. We will then investigate the structure of the effective thermal propagators. These will be used to give a new, explicit proof for the existence of the phonino which was predicted in [9, 34] as a bound state of boson and fermion. With these results, we will validate the Ward-Takahashi identities for various modes in order to show that supersymmetry is broken spontaneously with the phonino playing the role of the Goldstone fermion of thermally broken supersymmetry.

3.1 The model

The Wess-Zumino model [35] is the simplest supersymmetric quantum field theory. It describes a single self-interacting chiral superfield

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta\mathcal{F}(y),$$

whose component fields are a scalar field ϕ, a Weyl fermion ψ and an auxiliary scalar field $\mathcal{F}$. The interaction is determined by the superpotential which we take as

$$W(\Phi) = \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3.$$

The Lagrangian reads

$$\mathcal{L} = \Phi^\dagger\Phi\big|_{\theta\theta\overline{\theta}\overline{\theta}} + \left(W(\Phi)\big|_{\theta\theta} + \text{h.c.}\right).$$

Without dwelling on the superspace formalism, we rewrite it in terms of the component fields,

$$\begin{aligned}\mathcal{L} = {} & \tfrac{1}{2}\overline{\Psi}(i\partial\!\!\!/ - m)\Psi + \tfrac{1}{2}\partial_\mu A\,\partial^\mu A + \tfrac{1}{2}\partial_\mu B\,\partial^\mu B + \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2 \\ & + m(AF - BG) + g(A^2 - B^2)F - 2gABG - g\overline{\Psi}(A - i\gamma^5 B)\Psi.\end{aligned} \tag{3.1}$$

Here, we have defined the Majorana spinor

$$\Psi = \begin{pmatrix}\psi \\ \overline{\psi}\end{pmatrix},$$

and we have rewritten the fields ϕ and $\mathcal{F}$ in terms of their real components

$$\begin{aligned} A &= \tfrac{1}{\sqrt{2}}\,(\phi + \phi^\dagger), & F &= \tfrac{1}{\sqrt{2}}\,(\mathcal{F} + \mathcal{F}^\dagger), \\ B &= -\tfrac{i}{\sqrt{2}}(\phi - \phi^\dagger), & G &= -\tfrac{i}{\sqrt{2}}\,(\mathcal{F} - \mathcal{F}^\dagger). \end{aligned}$$

The auxiliary fields can be eliminated through their algebraic equations of motion,

$$\begin{aligned} F &= -mA - g(A^2 - B^2), \\ G &= mB + 2gAB, \end{aligned} \tag{3.2}$$

such that the Lagrangian becomes

$$\begin{aligned}\mathcal{L} = {} & \tfrac{1}{2}\overline{\Psi}(i\partial\!\!\!/ - m)\Psi + \tfrac{1}{2}(\partial_\mu A\,\partial^\mu A - m^2A^2) + \tfrac{1}{2}(\partial_\mu B\,\partial^\mu B - m^2B^2) \\ & - gmA(A^2 + B^2) - \tfrac{1}{2}g^2(A^2 + B^2)^2 - g\overline{\Psi}(A - i\gamma^5 B)\Psi.\end{aligned} \tag{3.3}$$

This Lagrangian describes a scalar field A and a pseudoscalar field B interacting with the Majorana fermion Ψ. All fields have equal mass m, and also the couplings are fixed in terms of the parameters g and m because of supersymmetry.

The Feynman rules can easily be read off from the Lagrangian. Firstly, we have the usual real-time propagators for massive particles, as displayed in figure 3.1. Dashed lines will always denote A-bosons, wiggly lines indicate

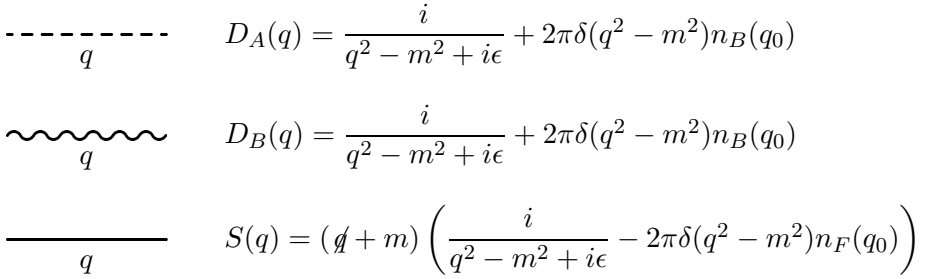


Figure 3.1: Real-time propagators for the Wess-Zumino model

B-bosons, while solid lines stand for the fermion. We will often split the propagators into thermal and vacuum parts, namely

$$S(q) = (\not{q} + m)\left(D_m(q) + \tilde{S}^T_m(q)\right),$$
$$D_{A/B}(q) = D_m(q) + D^T_m(q).$$

Here,

$$D_m(q) = \frac{i}{q^2 - m^2 + i\epsilon}$$

is the Feynman propagator for a scalar boson of mass m, and the thermal parts are given by

$$\tilde{S}^T_m(q) = -2\pi\delta(q^2 - m^2)n_F(q_0),$$
$$D^T_m(q) = 2\pi\delta(q^2 - m^2)n_B(q_0).$$

The interaction vertices are summarized in figure 3.2.

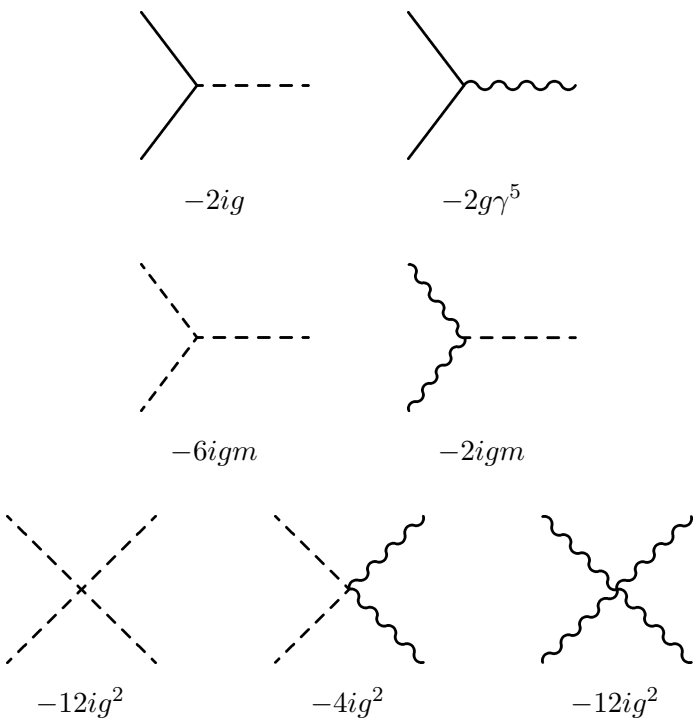


Figure 3.2: Interaction vertices of the Wess-Zumino model

3.2 Supersymmetry of the model

The action formed by the Lagrangian (3.1) is invariant under the supersymmetry transformations

$$\begin{aligned}
\delta A &= \overline{\xi}\Psi \\
\delta B &= \overline{\xi} i\gamma^5 \Psi \\
\delta \overline{\Psi} &= \overline{\xi} i \partial\!\!\!/ (A - i\gamma^5 B) + \overline{\xi}(F - Gi\gamma^5) \\
\delta \Psi &= -i \partial\!\!\!/ (A + i\gamma^5 B)\xi + (F - i\gamma^5 G)\xi \\
\delta F &= -i\overline{\xi}\gamma^\mu \partial_\mu \Psi \\
\delta G &= -\overline{\xi}\gamma^5\gamma^\mu \partial_\mu \Psi .
\end{aligned} \tag{3.4}$$

It is a simple task to calculate the variation of the Lagrangian under these transformations,

$$\delta\mathcal{L} = \overline{\xi}\partial_\mu \kappa^\mu,$$

which, together with the Noether current

$$\overline{\xi} N^\mu = -\sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i,$$

allows the definition of the conserved supercurrent

$$J^\mu = \kappa^\mu + N^\mu .$$

Explicit calculation gives

$$J^\mu = -\partial\!\!\!/ \left(A - i\gamma^5 B\right)\gamma^\mu \Psi + i\left(F - i\gamma^5 G\right)\gamma^\mu \Psi . \tag{3.5}$$

The supercharge

$$Q = \int d^3x\, J^0(x)$$

then allows us to express the above supersymmetry transformations as the commutator (anticommutator) with the charge,

$$\delta\phi = -i\overline{\xi}[Q, \phi]_\pm .$$

In the following, we will need

$$\begin{aligned}
[Q, A] &= i\Psi \\
[Q, B] &= -\gamma^5 \Psi \\
\{Q, \overline{\Psi}\} &= -\partial\!\!\!/ (A - i\gamma^5 B) + i(F - Gi\gamma^5).
\end{aligned}$$

With this knowledge, we can establish the Ward-Takahashi identities for the various modes we are interested in by inserting these transformation laws into the general formula (1.21). For the fundamental fermion, we simply find

$$\partial^x_\mu \langle T J^\mu(x) \overline{\Psi}(y)\rangle = i\delta^{(4)}(x - y)\langle F(y)\rangle . \tag{3.6}$$

For the composed $A\Psi$ and $B\Psi$ modes, we get

$$\begin{aligned}
&\partial^x_\mu \langle T J^\mu(x) A(y) \overline{\Psi}(z)\rangle \\
&\quad = \delta^{(4)}(x-y)\, i\langle T\Psi(y)\overline{\Psi}(z)\rangle - \delta^{(4)}(x-z)\,\langle T A(y)\partial\!\!\!/ A(z)\rangle \\
&\qquad + \delta^{(4)}(x-z)\, i\langle T A(y) F(z)\rangle
\end{aligned} \tag{3.7}$$

and likewise

$$\begin{aligned}
&\partial^x_\mu \langle T J^\mu(x) B(y) \overline{\Psi}(z)\rangle \\
&\quad = -\delta^{(4)}(x-y)\,\gamma^5\langle T\Psi(y)\overline{\Psi}(z)\rangle + \delta^{(4)}(x-z)\, i\gamma^5\langle T B(y)\partial\!\!\!/ B(z)\rangle \\
&\qquad + \delta^{(4)}(x-z)\,\gamma^5\langle T B(y) G(z)\rangle.
\end{aligned} \tag{3.8}$$

It is worth to study how these identities are satisfied at finite temperature. Let us switch off the interaction for a while. The first identity (3.6) is then of course trivially satisfied like at zero temperature with both sides of the equation vanishing. The situation for the composite operator is different. We consider the special case $y = z$ and calculate the Fourier transformed right hand side of equation (3.7). Replacing the F field by means of equation (3.2), we get

$$\begin{aligned}
&\int d^4x\, e^{ip(x-y)} RHS \\
&\qquad = \int \frac{d^4q}{(2\pi)^4}\,\big(iS(q) + (-iq\!\!\!/ - im)D(q)\big) \\
&\qquad = \int \frac{d^4q}{(2\pi)^4}\,\left[i(q\!\!\!/ + m)\left(\frac{i}{q^2-m^2+i\epsilon} - 2\pi n_F(q_0)\delta(q^2-m^2)\right)\right. \\
&\qquad\qquad \left. - i(q\!\!\!/ + m)\left(\frac{i}{q^2-m^2+i\epsilon} + 2\pi n_B(q_0)\delta(q^2-m^2)\right)\right] \\
&\qquad = \int \frac{d^4q}{(2\pi)^4}\, i(q\!\!\!/ + m)(-2\pi)\big(n_F(q_0)+n_B(q_0)\big)\delta(q^2-m^2).
\end{aligned}$$

As is well known, the zero temperature contributions from the bosonic and fermionic propagators cancel because of the equality of their masses, and the Ward-Takahashi identity is trivially satisfied. At finite temperature, however, we see that the different thermal propagators for boson and fermion do not cancel and leave a nontrivial right hand side.

The left hand side of (3.7) gives

$$\begin{aligned}
LHS &= \partial^x_\mu \langle T(-\partial\!\!\!/ - im) A(x)\gamma^\mu \Psi(x) A(y)\overline{\Psi}(y)\rangle \\
&= \partial^x_\mu \int \frac{d^4q'}{(2\pi)^4}\, e^{-iq'(x-y)}(iq\!\!\!/' - im)D(q')\gamma^\mu \int \frac{d^4q}{(2\pi)^4}\, e^{-iq(x-y)} S(q) \\
&= \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)} \int \frac{d^4q}{(2\pi)^4}\,(ip\!\!\!/ - iq\!\!\!/ - im)D(p-q)(-ip\!\!\!/)S(q).
\end{aligned}$$

The loop integral gets two contributions from the thermal part of either of the propagators. The fact that these have support on the mass shell helps us to calculate the algebra in the numerators that finally exactly cancel their denominators,

$$\begin{aligned}
&\int d^4x\, e^{ip(x-y)} LHS \\
&\quad = \int \frac{d^4q}{(2\pi)^4} \big(p^2 - 2qp + \not{p}(\not{q} - m)\big) i(\not{q} + m) \cdot \\
&\qquad\qquad \cdot \left(\frac{-2\pi n_F(q_0)\delta(q^2 - m^2)}{(p-q)^2 - m^2} + \frac{2\pi n_B(q_0)\delta\big((p-q)^2 - m^2\big)}{q^2 - m^2} \right) \\
&\quad = \int \frac{d^4q}{(2\pi)^4} \Big(i(\not{q} + m)(-2\pi) n_F(q_0)\delta(q^2 - m^2) \\
&\qquad\qquad + i(\not{q} + m - \not{p})(-2\pi) n_B(q_0)\delta\big((p-q)^2 - m^2\big) \Big) \\
&\quad = \int \frac{d^4q}{(2\pi)^4}\, i(\not{q} + m)(-2\pi)\big(n_F(q_0) + n_B(q_0)\big)\delta(q^2 - m^2),
\end{aligned}$$

leaving us with the same result as for the right hand side. Thus the Ward-Takahashi identity is indeed fulfilled.

For future reference, it is useful to explicitly calculate the resulting integral. Here and in the following, we distinguish between the two cases of non-relativistic and relativistic temperatures that allow a simple calculation by means of the formulae given in appendix B. We thereby neglect corrections suppressed by factors of order T/m and m/T, respectively, as long as they are not relevant. Without difficulty, one finds

$$\begin{aligned}
&\int \frac{d^4q}{(2\pi)^4}\, i(\not{q} + m)(-2\pi)\big(n_F(q_0) + n_B(q_0)\big)\delta(q^2 - m^2) \\
&\qquad\qquad = \begin{cases} -2ie^{-m/T}\left(\dfrac{Tm}{2\pi}\right)^{3/2} & T \ll m \\ -im\dfrac{T^2}{8} & T \gg m \end{cases}
\end{aligned}$$

such that the Ward-Takahashi identities become

$$\begin{aligned}
\partial^x_\mu \langle T J^\mu(x) A(y) \overline{\Psi}(y)\rangle &= \partial^x_\mu \langle T J^\mu(x) B(y) \overline{\Psi}(y)\rangle (-i\gamma^5) \\
&= \delta^{(4)}(x-y) \cdot \begin{cases} -2ie^{-m/T}\left(\dfrac{Tm}{2\pi}\right)^{3/2} & T \ll m, \\ -im\dfrac{T^2}{8} & T \gg m. \end{cases}
\end{aligned} \tag{3.9}$$

In the vacuum, the presence of a nontrivial right hand side of some Ward-Takahashi identity usually implies that there must be a pole in the correlation function on the left hand side for vanishing momentum and thereby

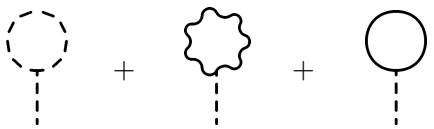
for all lightlike momenta, implying the existence of a massless Goldstone particle. At finite temperature, the situation is different. Here, the Ward-Takahashi identity is non-trivially satisfied, but without the need for a Goldstone fermion that is obviously not present in the free theory. The correlation function has rather an isolated pole for vanishing momentum which is allowed since Lorentz invariance is broken by the thermal bath.

In the interacting theory, we will later see that higher order corrections have the effect that the Ward-Takahashi identities are satisfied in a different way, and there is indeed a contribution from a Goldstone particle.

3.3 One-loop corrections

Let us now study how the interaction with the thermal background affects the physics of our model. In the following, we are only interested in the finite-temperature corrections. We assume that the theory is renormalized at vanishing temperature and do not consider contributions from the vacuum.

The first thing to investigate is the appearance of expectation values of the fundamental fields. The only candidate to develop such a nontrivial one-point function is the scalar field A. Indeed, there are three similar tadpole contributions, and one can calculate

$\langle A\rangle =$  $+$ $+$

$$
\begin{aligned}
&= \frac{i}{-m^2}\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\left(-6igmD_A(q) - 2igmD_B(q) + 2ig\,\mathrm{tr}\,S(q)\right)\\
&= -\frac{4g}{m}\int\frac{d^4q}{(2\pi)^4}\left(D_m^T(q) - \tilde{S}_m^T(q)\right)\\
&= -\frac{4g}{m}\int\frac{d^4q}{(2\pi)^4}\,2\pi\delta(q^2-m^2)\big(n_B(q_0)+n_F(q_0)\big)\\
&= -\frac{4g}{m}\int\frac{d^3q}{(2\pi)^3}\,\frac{n_B(E_q)+n_F(E_q)}{E_q}
\end{aligned}
$$

where $E_q = \sqrt{m^2+\vec{q}^2}$. In the limiting cases of low and high temperatures, we obtain

$$
\langle A\rangle = \begin{cases} -\dfrac{m\alpha}{g} & T \ll m,\\[2ex] -\dfrac{gT^2}{2m} & T \gg m, \end{cases} \tag{3.10}
$$

where we have introduced the small parameter

$$
\alpha = 8\frac{g^2}{m^3}\int\frac{d^3q}{(2\pi)^3}\,e^{-(m+\frac{\vec{q}^2}{2m})/T} = g^2e^{-m/T}\left(\frac{2T}{\pi m}\right)^{3/2}. \tag{3.11}
$$

Thus, at non-vanishing temperature, the A field carries a nontrivial thermal expectation value which is the first sign that in this model supersymmetry is broken at finite temperature. From the calculation, the reason for this behavior becomes clear. At vanishing temperature, the three diagrams cancel each other due to supersymmetry. At finite temperature, however, the difference $D_m^T(q) - \tilde{S}_m^T(q)$ of the thermal propagators for bosons and fermions no longer vanishes which causes a nontrivial expectation value.

Our result is in agreement with the literature [9] where it is calculated more thoroughly by minimizing the effective potential.

At this point, we will not use the expectation value to rearrange the Lagrangian like it is often done at zero temperature, but we will continue to work with the supersymmetric tree-level Lagrangian for a while and treat the coupling to the thermal expectation value as a small correction.

Let us now come to the one-loop corrections to the propagators of the fundamental fields. As described in section 1.4, the most important effect of the interaction with the heat bath is the displacement of the propagator poles, leading to a modification of the corresponding dispersion relations. For the moment, we are only interested in the real parts of the poles, neglecting possible damping effects.

In order to study the exact position of the poles, we will calculate the one-loop self energy corrections up to order g^2 near the mass shell, that is, for momenta $p^2 = m^2 + \mathcal{O}(g^2)$. As long as we only have corrections that are small compared to the mass, it is reasonable to make the following ansatz for the full propagators,

$$\mathcal{S}(p) = (\not{p} + m)\left(\frac{i}{p^2 - m^2 - \Pi_F(p) + i\epsilon} - 2\pi\delta\big(p^2 - m^2 - \Pi_F(p)\big)n_F(p_0)\right)$$

for the fermion propagator, where

$$\Pi_F(p) = \tfrac{1}{2}\,\mathrm{tr}\,\big[(\not{p} + m)\Sigma(p)\big],$$

and

$$\mathcal{D}_{A/B}(p) = \frac{i}{p^2 - m^2 - \Pi_{A/B}(p) + i\epsilon} + 2\pi\delta\big(p^2 - m^2 - \Pi_{A/B}(p)\big)n_B(p_0)$$

for the bosonic one. We will only be concerned with the real parts of the self energies Σ and $\Pi_{A/B}$. As discussed in section 1.4, they can be calculated as the real part of the (11) component of the respective self energy matrix.

The one-loop Feynman diagrams contributing to the self energies of the fermion and A and B bosons, respectively, are displayed in figures 3.3 to 3.5. Firstly, we have the coupling to the expectation value of the A field that was calculated above. Also the other diagrams can easily be calculated in a very similar way and give contributions of the same order of magnitude.

It is an interesting result of the calculation that, to leading order, none of these self energy diagrams shows a dependence on the three-momentum $\vec{p}$, as long as $p^2 = m^2$. This is obvious only for the tadpole diagrams. Thus, the self energy corrections basically show up as a small displacement of the masses,

$$m_i^2 = m^2 + \Pi_i(p)\big|_{p^2=m^2}$$

without changing the dispersion relation

$$p_0^2 = \vec{p}^2 + m_i^2.$$

We have listed the results of our calculation for the thermal mass shifts $\delta m_i^2 = \Pi_i(p)\big|_{p^2=m^2}$ in table 3.1. Explicitly, the thermal masses have the values

$$\begin{aligned} m_A^2 &= \begin{cases} m^2 - \frac{20}{3} m^2 \alpha & T \ll m, \\ m^2 - 2g^2 T^2 & T \gg m, \end{cases} \\ m_B^2 &= m^2, \\ m_F^2 &= \begin{cases} m^2 - 2m^2 \alpha & T \ll m, \\ m^2 - g^2 T^2 & T \gg m. \end{cases} \end{aligned} \tag{3.12}$$

We find that the mass degeneracy of the three fields is lifted by the one-loop self energy corrections. There appears a small mass splitting which again displays the breakdown of supersymmetry at finite temperature. As expected, it vanishes in the limit of zero temperature. The assumption that

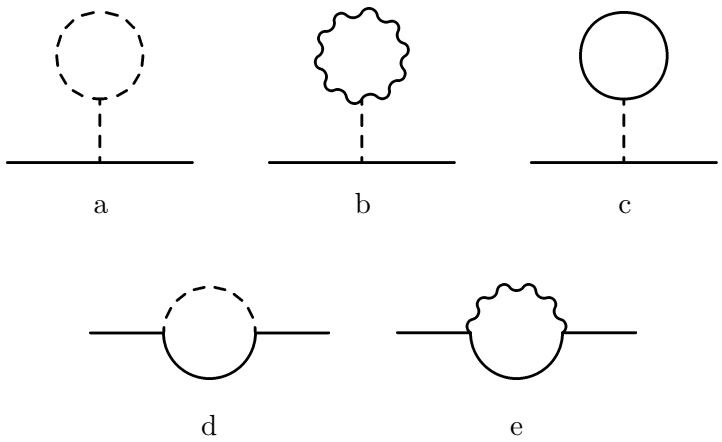


Figure 3.3: One-loop contributions to the fermion self energy

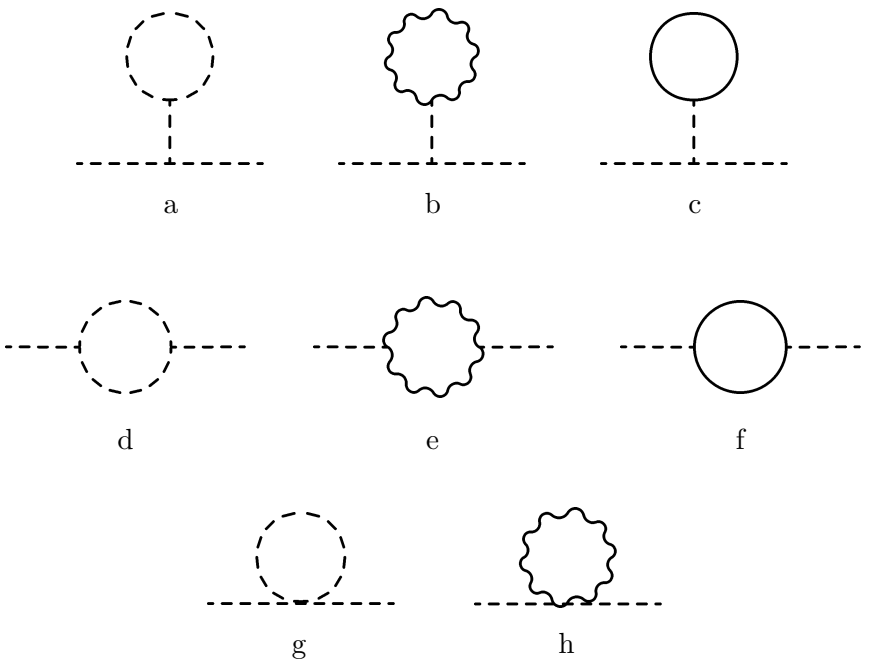


Figure 3.4: One-loop contributions to the A boson self energy

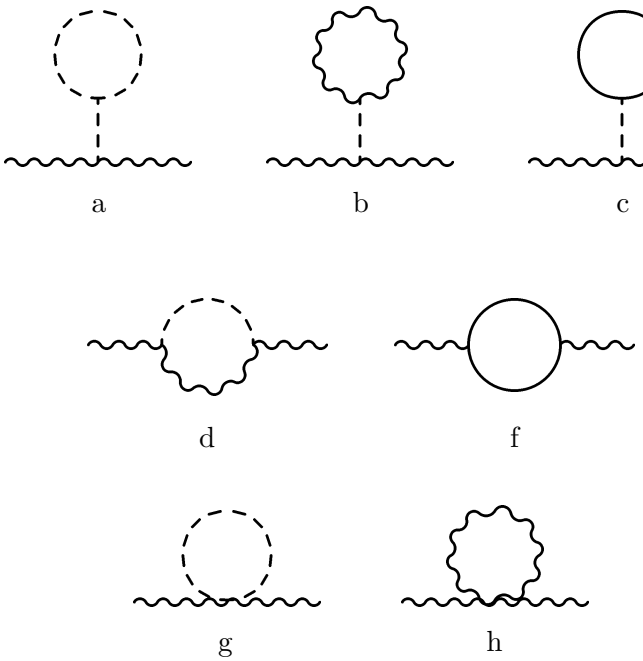


Figure 3.5: One-loop contributions to the B boson self energy

	Fig. 3.3 — F		Fig. 3.4 — A		Fig. 3.5 — B	
	$T \ll m$	$T \gg m$	$T \ll m$	$T \gg m$	$T \ll m$	$T \gg m$
a+b+c	$-4m^2\alpha$	$-2g^2T^2$	$-6m^2\alpha$	$-3g^2T^2$	$-2m^2\alpha$	$-g^2T^2$
d+e+f	$2m^2\alpha$	g^2T^2	$-\frac{5}{3}m^2\alpha$	$\frac{1}{3}g^2T^2$	$m^2\alpha$	$\frac{1}{3}g^2T^2$
g+h			$m^2\alpha$	$\frac{2}{3}g^2T^2$	$m^2\alpha$	$\frac{2}{3}g^2T^2$
δm_i^2	$-2m^2\alpha$	$-g^2T^2$	$-\frac{20}{3}m^2\alpha$	$-2g^2T^2$	0	0

Table 3.1: Contributions to the thermal mass shifts

the mass shifts are small compared to the mass limits our discussion to a temperature range $gT \ll m$.

In a model where supersymmetry is broken spontaneously already in the vacuum, the masses of the members of one supermultiplet are related by the mass formula (1.18). Here we see that this relation is no longer valid, at least for low temperatures, since $m_A^2 + m_B^2 \neq 2m_F^2$. This raises the question whether the heat bath breaks supersymmetry spontaneously or rather explicitly.

Our results have been verified in the imaginary time formalism and are also in agreement with the results by Lebedev and Smilga [9]. Earlier calculations, however, come to different results [25]. The reason lies in the fact that at finite temperature particle masses can be defined in different ways, since Lorentz invariance is broken. For example, the mass obtained from the inverse propagator for vanishing momentum or the mass derived from the effective potential are in general different from the pole mass that we are using here. In any case, it is this pole mass that will enter in all our further calculations.

3.4 The phonino

3.4.1 The full fermion propagator

As a first step in the investigation of the thermal propagators, we have calculated the one-loop self energies on the mass shell, leading to a small modification of the masses of the fundamental fields. However, according to the arguments given in chapter 2, we suspect the existence of collective excitations that may appear as completely new features in the thermal background. In order to find them, we have to analyze all poles of the full propagators.

The poles of the full fermion propagator are characterized by the condition

$$\det \mathcal{S}^{-1}(p) = -i \det\big(\not{p} - m - \Sigma(p)\big) = 0, \tag{3.13}$$

therefore we must know the self energy $\Sigma(p)$ as a function of momentum. To this end, we evaluate the one-loop diagrams drawn in figure 3.3, giving

$$\Sigma(p) = 2g\langle A\rangle + 8g^2 \int \frac{d^4q}{(2\pi)^4} \frac{2\pi\delta(q^2-m^2)}{(p-q)^2 - m^2 + i\epsilon} \big(\not{q}\, n_F(q_0) + (\not{q} - \not{p}) n_B(q_0)\big). \tag{3.14}$$

To simplify matters, we take the limit of vanishing three-momentum and calculate

$$\Sigma(p_0, 0) = 2g\langle A\rangle + \frac{8g^2\gamma^0}{p_0} \int \frac{d^3q}{(2\pi)^3} \left(\frac{n_B(E_q)}{E_q} \frac{2E_q^2 - p_0^2}{4E_q^2 - p_0^2} + \frac{n_F(E_q)}{E_q} \frac{2E_q^2}{4E_q^2 - p_0^2} \right).$$

One observes that the second part develops a pole for vanishing momentum. The reason for this behavior is quite clear. In the limit $p \to 0$, the poles of both propagators in the loop coincide, leading to an anomalously big self energy.

In the relevant limits of low and high temperatures (in the second case for small momenta $p_0 \ll T$), we can perform the integration. In each case, the result can be written in terms of $\langle A\rangle$ (compare (3.10)) as

$$\Sigma(p_0, 0) = 2g\langle A\rangle - \frac{mg\langle A\rangle}{p_0}\gamma^0.$$

Once we insert this result into condition (3.13) that characterizes the poles, we get the dispersive equation

$$\left(p_0 + \frac{mg\langle A\rangle}{p_0}\right)^2 = \big(m + 2g\langle A\rangle\big)^2$$

with the solutions

$$|p_0| = m + g\langle A\rangle + \mathcal{O}(g^2\langle A\rangle^2),$$
$$|p_0| = g\langle A\rangle + \mathcal{O}(g^2\langle A\rangle^2).$$

The first solution reproduces the small shift in the fermion mass that was already calculated in the previous section. The existence of a second solution indicates another excitation with a very tiny mass. In fact, such a pole is to be expected according to the Goldstone theorem and the general arguments given in chapter 2, albeit with exactly vanishing mass. It thus seems that our simple one-loop calculation is not sufficient.

The following sections show that some more effort is necessary to uncover a massless pole in the fermion propagator. Firstly, a consistent calculation of the full self energy should also involve the full propagators in the internal lines, that is, we should work with resummed propagators that take into account the fact that the mass degeneracy between bosons and fermion is lifted by the interaction with the thermal background. For the high-temperature case, this is indeed the way to find the desired massless pole. For low temperatures, in contrast, there is also a class of many-loop diagrams that give important contributions and must duly be summed up before one obtains the correct result.

3.4.2 The high temperature case

Let us now calculate the fermion self energy more carefully. First, we restrict ourselves to the relativistic case where $T \gg m$.

We have already learned that in this case the propagators are modified by a small, constant mass correction. Therefore, we will work with the one-loop resummed propagators

$$\mathcal{S}(q) = \left(\not{q} + m + \Sigma'(q)\right)\left(\frac{i}{q^2 - m_F^2 + i\epsilon} - 2\pi\delta(q^2 - m_F^2)n_F(q)\right),$$
$$\mathcal{D}_{A/B}(q) = \frac{i}{q^2 - m_{A/B}^2 + i\epsilon} + 2\pi\delta(q^2 - m_{A/B}^2)n_B(q),$$

where we have calculated

$$\begin{aligned} m_F^2 &= m^2 - g^2T^2, \\ m_A^2 &= m^2 - 2g^2T^2, \\ m_B^2 &= m^2. \end{aligned} \tag{3.15}$$

As one can check in equation (3.14), the one-loop self energy Σ splits into a constant part, coming from the tadpole diagrams, and a momentum dependent part,

$$\Sigma(q) = \Sigma_{\mathbb{1}} + \Sigma_\gamma(q),$$

where $\Sigma_{\mathbb{1}}$ is proportional to the unit matrix while Σ_γ is traceless and an odd function of the momentum. Hence,

$$\Sigma'(q) = \tfrac{1}{2}\operatorname{tr}\Sigma(q) - \Sigma(q) = \Sigma_{\mathbb{1}} - \Sigma_\gamma(q).$$

Our task is again to evaluate the self energy diagrams drawn in figure 3.3, now making use of the resummed propagators. We are interested in the behavior of $\Sigma(k)$ for small momenta, therefore we make a linear approximation in k.

If we take the thermal part of the fermion propagator and the vacuum part of the A boson propagator in diagram 3.3 d, we obtain a contribution to $-i\Sigma(k)$ that can be calculated as follows,

$$\begin{aligned}
&-4g^2\int\frac{d^4q}{(2\pi)^4}\,(-2\pi)\delta(q^2-m_F^2)n_F(q_0)\frac{i(\not{q}+m+\Sigma'(q))}{(k-q)^2-m_A^2}\\
&\quad=4ig^2\int\frac{d^3q}{(2\pi)^3}\,\frac{n_F(E_q)}{2E_q}\sum_{\pm}\frac{\not{q}+m+\Sigma'(q)}{m_F^2-m_A^2-2kq}\bigg|_{q_0=\pm E_q}\\
&\quad=4ig^2\int\frac{d^3q}{(2\pi)^3}\,\frac{n_F(E_q)}{E_q}\left(\frac{m+\Sigma_{\mathbb{1}}}{m_F^2-m_A^2}+\frac{2kq(\not{q}-\Sigma_\gamma(q))}{(m_F^2-m_A^2)^2}\right)\bigg|_{q_0=E_q}.
\end{aligned}\tag{3.16}$$

In addition, we have similar contributions from the thermal part of the bosonic propagator as well as from the B boson. Summing up all these contributions, not forgetting the tadpole diagram, we find

$$\begin{aligned}
\Sigma(k)=&-\frac{4g^2}{m_F^2-m_A^2}(m+\Sigma_{\mathbb{1}})\int\frac{d^3q}{(2\pi)^3}\,\frac{n_B(E_q)+n_F(E_q)}{E_q}\\
&-\frac{4g^2}{m_F^2-m_B^2}(-m-\Sigma_{\mathbb{1}})\int\frac{d^3q}{(2\pi)^3}\,\frac{n_B(E_q)+n_F(E_q)}{E_q}\\
&-8g^2\left(k_0\gamma^0+\tfrac{1}{3}\vec{k}\vec{\gamma}\right)\left(\frac{1}{(m_F^2-m_A^2)^2}+\frac{1}{(m_F^2-m_B^2)^2}\right)\cdot\\
&\quad\cdot\int\frac{d^3q}{(2\pi)^3}\,E_q\big(n_B(E_q)+n_F(E_q)\big)\\
&+\Sigma_{\mathbb{1}}.
\end{aligned}$$

We can now insert the mass splittings resulting from equation (3.15),

$$m_F^2-m_A^2=m_B^2-m_F^2=g^2T^2,$$

and obtain

$$\begin{aligned}
\Sigma(k)=&-\frac{8}{T^2}(m+\Sigma_{\mathbb{1}})\int\frac{d^3q}{(2\pi)^3}\,\frac{n_B(E_q)+n_F(E_q)}{E_q}\\
&-\frac{16}{g^2T^4}\left(k_0\gamma^0+\tfrac{1}{3}\vec{k}\vec{\gamma}\right)\int\frac{d^3q}{(2\pi)^3}\,E_q\big(n_B(E_q)+n_F(E_q)\big)\\
&+\Sigma_{\mathbb{1}}.
\end{aligned}$$

The integration can be performed by making use of equations (B.1) to (B.4) in the appendix, and the whole expression simplifies to

$$\Sigma(k)=-m-\frac{\pi^2}{g^2}\left(k_0\gamma^0+\tfrac{1}{3}\vec{k}\vec{\gamma}\right).$$

Using this result, we can write down the full inverse propagator

$$\mathcal{S}^{-1}(k) = -i\big(\not{k} - m - \Sigma(k)\big) = -i\left(\not{k} + \frac{\pi^2}{g^2}\left(k_0\gamma^0 + \tfrac{1}{3}\vec{k}\vec{\gamma}\right)\right). \tag{3.17}$$

We find that the self energy exactly cancels the Lagrangian mass such that the momentum dependent contribution from the loop diagrams dominates the inverse propagator,

$$\mathcal{S}^{-1}(k) \simeq -i\frac{\pi^2}{g^2}\left(k_0\gamma^0 + \tfrac{1}{3}\vec{k}\vec{\gamma}\right).$$

Thus, the full fermion propagator is singular for vanishing momentum as well as on the dispersion curve

$$k_0 = \pm\frac{1}{3}|\vec{k}|, \tag{3.18}$$

displaying the existence of a massless collective excitation propagating with velocity $v = \frac{1}{3}$, just as one would expect from the general arguments given in chapter 2. Consequently, we identify this pole in the fermion propagator as the phonino.

Note that our derivation is valid only for small momenta $k \ll g^2T$, since in equation (3.16) we assumed that $kq \ll m_F^2 - m_A^2 = g^2T^2$ with the thermal momentum $q = \mathcal{O}(T)$. For larger momenta, we expect nonlinear corrections to the dispersion law as well as damping, such that the existence of the phonino is limited to small momenta. This has been discussed in [9].

We would like to mention that this mechanism to cancel the mass of the fermion by loop corrections goes back to a widely unnoticed paper by Kapusta [32] who performed a similar calculation for the somewhat simpler case of the massless Wess-Zumino model.

For the exact cancellation of the Lagrangian mass, it was crucial to have the relation $m_F^2 - m_B^2 = m_A^2 - m_F^2$. Otherwise, the result would have been of order, but not exactly m. For small temperatures, this relation no longer holds (see equation (3.12)), and some more efforts will have to be undertaken in order to find the zero-momentum pole.

3.4.3 The low temperature case

At high temperature, a resummation of the boson and fermion propagators with subsequent calculation of the self energy diagrams like the one drawn in figure 3.6 a was necessary in order to find a zero momentum pole in the fermion propagator. At low temperatures, things are even more complicated since also higher order vertex corrections have to be resummed.

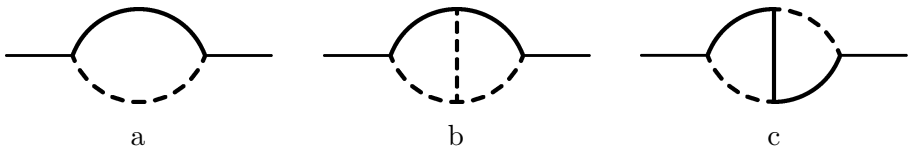


Figure 3.6: Typical one- and two-loop contributions to the fermion self energy

Take a look at the two-loop diagram 3.6 b. Consider the left part of the diagram as a correction to the $\overline{\Psi}\Psi A$ vertex. For vanishing external momentum and low temperature, it gives a contribution of order

$$g^3 m \int \frac{d^3q}{(2\pi)^3} \frac{n_F(E_q) + n_B(E_q)}{E_q} \frac{m}{m_F^2 - m_A^2} \frac{1}{m^2} \sim g,$$

meaning that its contribution is as relevant as that from the one-loop diagram 3.6 a. For high temperature, in contrast, we get

$$g^3 m \int \frac{d^3q}{(2\pi)^3} \frac{n_F(E_q) + n_B(E_q)}{E_q} \frac{T}{m_F^2 - m_A^2} \frac{1}{T^2} \sim \frac{m}{T} g,$$

so it does not give a relevant contribution, the reason being the Yukawa coupling proportional to $m \ll T$.

Diagram 3.6 c is equally important for low temperatures, while for high temperatures the Dirac algebra of the numerators,

$$(\not{q} + m)(\not{p} + \not{q} + m)(\not{p} + m) = 3m(\not{q} + m)(\not{p} + m)$$

if $q^2 = p^2 = m^2$, shows that it is again suppressed by a factor of m/T and not relevant for our purposes. Thus, for high temperature there is no need to proceed beyond one-loop order.

Now, for small temperatures, what are the diagrams that can give such important contributions? In principle, each order of the perturbative expansion is suppressed by a factor g^2 and a factor of n_F or n_B from the thermal propagators. The reason why the contribution of a higher order diagram can nevertheless become large is the appearance of the small mass difference in the denominator that cancels these suppression factors. As long as each intermediate vertex is followed by a pair of propagators of particles with a tiny mass difference, the diagram can become relevant. Altogether, we must add up all the ladder diagrams like the one drawn in figure 3.7, with all possible intermediate states taken into account.

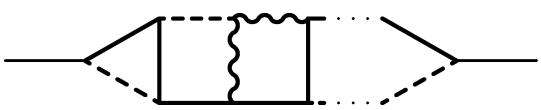
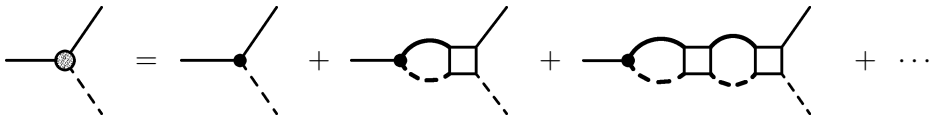


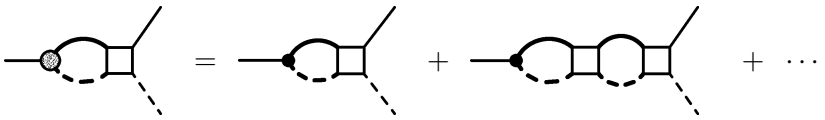
Figure 3.7: Typical many-loop diagram giving a relevant contribution to the fermion self energy for small temperature

The resummation will be performed in the following way. We first calculate the full one-particle irreducible $\overline{\Psi}\Psi A$ and $\overline{\Psi}\Psi B$ vertex functions by making use of the Bethe-Salpeter equations. With this knowledge, we can then evaluate the fermion self energy which finally leads us to the full fermion propagator.

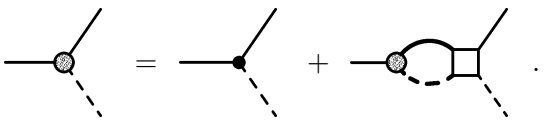
Let us shortly explain the idea of the Bethe-Salpeter equations in a diagrammatic language. The perturbative expansion of the full vertex (denoted by a shaded spot) reads



Here, the empty square stands for any proper scattering subdiagram, and a summation over all possible intermediate states is implied. The thick internal lines denote full propagators. Extension by another loop leads to

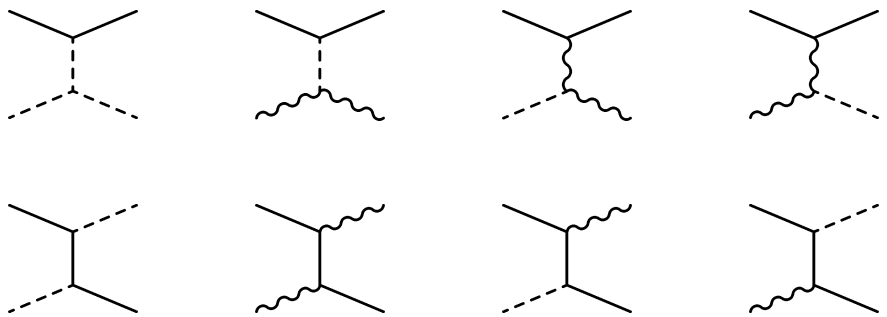


which can be substituted into the first equation, giving the Bethe-Salpeter equation,

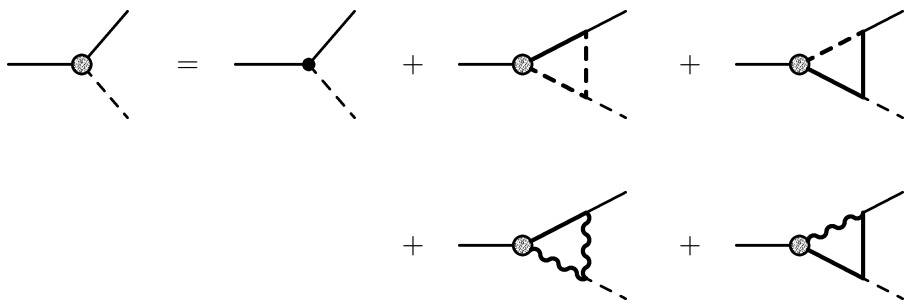


The Bethe-Salpeter equation can be applied in two rather different ways. Firstly, it allows the resummation of ladder diagrams in order to obtain the full vertex function. Secondly, this will be needed later on, it can be used to identify poles in the full vertex function. In this case, the tree level vertex is irrelevant compared to the divergent full vertex and one can leave away the first part on the right hand side, thereby obtaining the homogeneous Bethe-Salpeter equation.

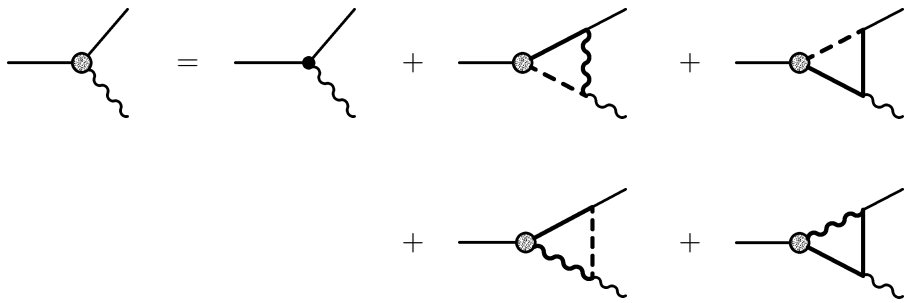
In our particular case of the Wess-Zumino model, the intermediate state in the Bethe-Salpeter equation can consist of a fermion with either of the two bosons. The square then stands for any of the following subdiagrams that contribute to the irreducible $A\Psi$ and $B\Psi$ scattering amplitude:



Thus, the Bethe-Salpeter equation for the vertex function involving the A boson reads

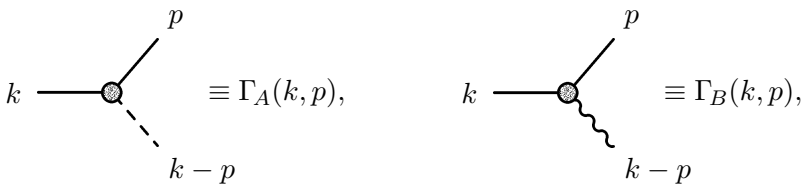


and similarly for the B boson,



Let us now translate these diagrammatic equations into mathematical language. The amputated full proper vertex functions will be denoted

by $\Gamma_A(k,p)$ and $\Gamma_B(k,p)$, respectively,



which we want to calculate for small incoming momenta k. The thermal contributions to the Bethe-Salpeter equations come from loop momenta on the mass shell, therefore we consider only on-shell momenta $p^2 = m^2$. We will again approximate the full propagators in the internal lines by one-loop resummed propagators, that is, we take the free thermal propagators with the thermal masses as calculated in section 3.3.

All of these diagrams have a very similar structure. We again find expressions like those in the calculation for high temperature, equation (3.16), with small denominators of order of the mass differences. For simplicity, we write

$$\Delta_i = m_F^2 - m_i^2,$$

or, in numbers,

$$\begin{aligned} \Delta_A &= \frac{14}{3} m^2 \alpha, \\ \Delta_B &= -2m^2\alpha. \end{aligned} \tag{3.19}$$

The propagators of the exchanged particles depend on the relative sign of the zero components of outgoing and loop momenta and lead to denominators of $3m^2$ and $-m^2$, respectively. We can safely neglect small corrections of order $\vec{p}/m$ or $\vec{q}/m$ (where q is the momentum of the fermion in the loop), since $\vec{p}$ and $\vec{q}$ are thermal momenta of order $\sqrt{Tm}$ which, for low temperature, is negligible compared to the mass.

Altogether, the Bethe-Salpeter equation for the A boson vertex reads

$$\begin{aligned} \Gamma_A(k,p) = -2ig - 4g^2 \int \frac{d^3q}{(2\pi)^3} \frac{e^{-E_q/T}}{E_q} \cdot \\ \cdot \Bigg\{ \left(\frac{\not p + 2m}{3m^2} + \frac{3m}{-m^2} \right) \frac{\not q + m}{\Delta_A - 2qk} \Gamma_A(k,q) \Big|_{q_0 = E_q \cdot \operatorname{sign} p_0} \\ + \left(\frac{\not p + 2m}{-m^2} + \frac{3m}{3m^2} \right) \frac{\not q + m}{\Delta_A - 2qk} \Gamma_A(k,q) \Big|_{q_0 = -E_q \cdot \operatorname{sign} p_0} \\ - \left(\frac{-\not p + 2m}{3m^2} + \frac{m}{-m^2} \right) \frac{-\not q + m}{\Delta_B - 2qk} i\gamma^5 \Gamma_B(k,q) \Big|_{q_0 = E_q \cdot \operatorname{sign} p_0} \\ - \left(\frac{-\not p + 2m}{-m^2} + \frac{m}{3m^2} \right) \frac{-\not q + m}{\Delta_B - 2qk} i\gamma^5 \Gamma_B(k,q) \Big|_{q_0 = -E_q \cdot \operatorname{sign} p_0} \Bigg\}. \end{aligned}$$

For the B boson vertex, we find

$$
\begin{aligned}
i\gamma^5\Gamma_B(k,p) = -2ig - 4g^2 \int \frac{d^3q}{(2\pi)^3}\, \frac{e^{-E_q/T}}{E_q} \cdot \\
\cdot \Bigg\{ \left(\frac{\not p}{3m^2} + \frac{-m}{-m^2}\right) \frac{\not q + m}{\Delta_A - 2qk} \Gamma_A(k,q)\Big|_{q_0 = E_q \cdot \operatorname{sign} p_0} \\
+ \left(\frac{\not p}{-m^2} + \frac{-m}{3m^2}\right) \frac{\not q + m}{\Delta_A - 2qk} \Gamma_A(k,q)\Big|_{q_0 = -E_q \cdot \operatorname{sign} p_0} \\
- \left(\frac{-\not p}{3m^2} + \frac{m}{-m^2}\right) \frac{-\not q + m}{\Delta_B - 2qk} i\gamma^5 \Gamma_B(k,q)\Big|_{q_0 = E_q \cdot \operatorname{sign} p_0} \\
- \left(\frac{-\not p}{-m^2} + \frac{m}{3m^2}\right) \frac{-\not q + m}{\Delta_B - 2qk} i\gamma^5 \Gamma_B(k,q)\Big|_{q_0 = -E_q \cdot \operatorname{sign} p_0} \Bigg\}.
\end{aligned}
$$

In order to solve these seemingly complex coupled integral equations, it helps to introduce new functions

$$
\begin{aligned}
\tilde\Gamma_A(k,p) &= \frac{\not p + m}{\Delta_A - 2pk}\, \Gamma_A(k,p), \\
\tilde\Gamma_B(k,p) &= -i\gamma^5 \frac{\not p + m}{\Delta_B - 2pk}\, \Gamma_B(k,p)
\end{aligned}
\tag{3.20}
$$

which simplify our equations considerably. In terms of these functions, the Bethe-Salpeter equations can be written as

$$
\begin{aligned}
\tilde\Gamma_A(k,p) = \frac{\not p + m}{\Delta_A - 2pk} \Bigg\{ -2ig + \frac{4g^2}{m^2} \int \frac{d^3q}{(2\pi)^3}\, e^{-E_q/T}. \qquad (3.21) \\
\cdot \left[2\left(\tilde\Gamma_A(k,q) + \tilde\Gamma_A(k,-q)\right) + \frac{2}{3}\left(\tilde\Gamma_B(k,q) + \tilde\Gamma_B(k,-q)\right)\right]_{q_0 = E_q} \Bigg\}, \\
\tilde\Gamma_B(k,p) = \frac{\not p - m}{\Delta_B - 2pk} \Bigg\{ -2ig + \frac{4g^2}{m^2} \int \frac{d^3q}{(2\pi)^3}\, e^{-E_q/T}. \qquad (3.22) \\
\cdot \left[-\frac{2}{3}\left(\tilde\Gamma_A(k,q) + \tilde\Gamma_A(k,-q)\right) + \frac{2}{3}\left(\tilde\Gamma_B(k,q) + \tilde\Gamma_B(k,-q)\right)\right]_{q_0 = E_q} \Bigg\}.
\end{aligned}
$$

In view of the symmetry of the integrands, we introduce the functions

$$
V_{A,B}(k,\vec p) = \left[\tilde\Gamma_{A,B}(k,p) + \tilde\Gamma_{A,B}(k,-p)\right]_{p_0 = E_p}.
$$

From equation (3.21), one easily derives the following relation,

$$
\frac{\not p}{m}\left[\tilde\Gamma_A(k,p) - \tilde\Gamma_A(k,-p)\right]_{p_0=E_p} = \left[\tilde\Gamma_A(k,p) + \tilde\Gamma_A(k,-p)\right]_{p_0=E_p} = V_A(k,\vec p),
$$

allowing us to reconstruct the function $\tilde{\Gamma}_A$ from V_A by means of

$$\tilde{\Gamma}_A(k,p) = \frac{\not{p}+m}{2m} V_A(k,\vec{p}). \tag{3.23}$$

Analogously, one derives

$$\tilde{\Gamma}_B(k,p) = \frac{-\not{p}+m}{2m} V_B(k,\vec{p}). \tag{3.24}$$

In order to solve for V_A and V_B, we make a linear approximation for small momenta,

$$V_{A,B}(k,\vec{p}) = V_{A,B}(0,\vec{p}) + k_\mu C^\mu_{A,B}(\vec{p}) \tag{3.25}$$

which we can now insert into equations (3.21) and (3.22).

Let us first consider the case $k = 0$. In this limit, the Bethe-Salpeter equations for $V_{A,B}(0,\vec{p})$ are independent of the three-momentum $\vec{p}$. Hence, one can easily perform the integration by making use of the relations

$$\frac{4g^2}{m^2}\int \frac{d^3q}{(2\pi)^3}\, e^{-E_q/T} = \frac{3}{28}\frac{\Delta_A}{m} = -\frac{1}{4}\frac{\Delta_B}{m} \tag{3.26}$$

that follow immediately from equations (3.19) and (3.11). We find the system of linear equations

$$\begin{aligned} V_A(0,\vec{p}) &= \frac{2m}{\Delta_A}\left[-2ig + \frac{3}{28}\frac{\Delta_A}{m}\left(2V_A(0,\vec{p}) + \frac{2}{3}V_B(0,\vec{p})\right)\right] \\ V_B(0,\vec{p}) &= -\frac{2m}{\Delta_B}\left[-2ig - \frac{1}{4}\frac{\Delta_B}{m}\left(-\frac{2}{3}V_A(0,\vec{p}) + \frac{2}{3}V_B(0,\vec{p})\right)\right] \end{aligned}$$

which is easy to solve. The solution is

$$V_A(0,\vec{p}) = V_B(0,\vec{p}) = \frac{2m}{\Delta_B}\cdot 2ig. \tag{3.27}$$

As an aside, we remark that the homogeneous system of equations, obtained by leaving away the tree level vertex, does not have a nontrivial solution. This means that no poles appear in the vertex function we are considering which will be important later on.

It is illustrative to express this result in terms of the amputated vertex functions we started with by means of equations (3.23) and (3.24) together with the definitions (3.20). The result is

$$\begin{aligned} \Gamma_B(0,p) &= -2ig(-i\gamma^5), \\ \Gamma_A(0,p) &= -2ig\left(-\frac{\Delta_A}{\Delta_B}\right). \end{aligned} \tag{3.28}$$

Interestingly, the Yukawa coupling to the B boson comes out of the whole

resummation without a change while the coupling to the A boson is enhanced by a factor of

$$-\frac{\Delta_A}{\Delta_B} = \frac{7}{3}.$$

Let us now proceed to the more difficult task, the calculation of the terms linear in the external momentum k. Once we substitute our ansatz (3.25) into the Bethe-Salpeter equations (3.21) and (3.22) that we also approximate linearly in k, the coefficients can be read off as follows,

$$\begin{aligned}
C_A^\mu(\vec{p}) &= 2\not{p}\,\frac{2p^\mu}{\Delta_A^2}\bigg|_{p_0=E_p} \cdot \\
&\quad\cdot\left\{-2ig + \frac{4g^2}{m^2}\int\frac{d^3q}{(2\pi)^3}\,e^{-E_q/T}\left[2V_A(0,\vec{q}) + \frac{2}{3}V_B(0,\vec{q})\right]\right\} \\
&\quad + \frac{2m}{\Delta_A}\frac{4g^2}{m^2}\int\frac{d^3q}{(2\pi)^3}\,e^{-E_q/T}\left[2C_A^\mu(\vec{q}) + \frac{2}{3}C_B^\mu(\vec{q})\right], \\
C_B^\mu(\vec{p}) &= 2\not{p}\,\frac{2p^\mu}{\Delta_B^2}\bigg|_{p_0=E_p} \cdot \\
&\quad\cdot\left\{-2ig + \frac{4g^2}{m^2}\int\frac{d^3q}{(2\pi)^3}\,e^{-E_q/T}\left[-\frac{2}{3}V_A(0,\vec{q}) + \frac{2}{3}V_B(0,\vec{q})\right]\right\} \\
&\quad - \frac{2m}{\Delta_B}\frac{4g^2}{m^2}\int\frac{d^3q}{(2\pi)^3}\,e^{-E_q/T}\left[-\frac{2}{3}C_A^\mu(\vec{q}) + \frac{2}{3}C_B^\mu(\vec{q})\right].
\end{aligned}$$

With the solution (3.27) for $V_{A,B}(0,\vec{q})$ and again using the relations (3.26), one can evaluate the curly brackets and finds

$$\begin{aligned}
k\cdot C_A(\vec{p}) &= -\frac{14}{3}ig\frac{4\not{p}(k\cdot p)}{\Delta_A^2}\bigg|_{p_0=E_p} \\
&\quad + \frac{8g^2}{m\Delta_A}\int\frac{d^3q}{(2\pi)^3}\,e^{-E_q/T}\left[2\,k\cdot C_A(\vec{q}) + \frac{2}{3}\,k\cdot C_B(\vec{q})\right], \\
k\cdot C_B(\vec{p}) &= -2ig\frac{4\not{p}(k\cdot p)}{\Delta_B^2}\bigg|_{p_0=E_p} \\
&\quad - \frac{8g^2}{m\Delta_B}\int\frac{d^3q}{(2\pi)^3}\,e^{-E_q/T}\left[-\frac{2}{3}\,k\cdot C_A(\vec{q}) + \frac{2}{3}\,k\cdot C_B(\vec{q})\right].
\end{aligned} \tag{3.29}$$

At this point, it is no longer possible to neglect the momentum dependence since in particular the vector components $\vec{C}_{A/B}(\vec{p})$ get an important contribution that rises quadratically with $|\vec{p}|$. In order to solve the system of equations, we multiply by $e^{-E_q/T}$ and integrate. After application of the

integrals (B.6) to (B.8) and straightforward simplification, we obtain the linear system

$$\begin{aligned}\int\frac{d^3q}{(2\pi)^3}\,e^{-E_q/T}\,k\cdot C_A(\vec{q}) &= -\frac{im^2}{2g\Delta_A}\left(m\,k_0\gamma^0+T\,\vec{k}\vec{\gamma}\right)\\ &\quad+\int\frac{d^3q}{(2\pi)^3}\,e^{-E_q/T}\left[\frac{3}{7}\,k\cdot C_A(\vec{q})+\frac{1}{7}\,k\cdot C_B(\vec{q})\right]\\ \int\frac{d^3q}{(2\pi)^3}\,e^{-E_q/T}\,k\cdot C_B(\vec{q}) &= \frac{im^2}{2g\Delta_B}\left(m\,k_0\gamma^0+T\,\vec{k}\vec{\gamma}\right)\\ &\quad+\int\frac{d^3q}{(2\pi)^3}\,e^{-E_q/T}\left[-\frac{1}{3}\,k\cdot C_A(\vec{q})+\frac{1}{3}\,k\cdot C_B(\vec{q})\right].\end{aligned}$$

The solution is given by

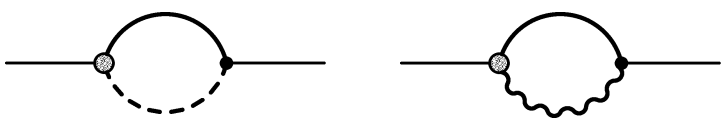
$$\begin{aligned}\int\frac{d^3q}{(2\pi)^3}\,e^{-E_q/T}\,k\cdot C_A(\vec{q}) &= \int\frac{d^3q}{(2\pi)^3}\,e^{-E_q/T}\,k\cdot C_B(\vec{q})\\ &= \frac{im^2}{2g\Delta_B}\left(m\,k_0\gamma^0+T\,\vec{k}\vec{\gamma}\right).\end{aligned}\tag{3.30}$$

Now one could proceed and calculate the functions $C^{\mu}_{A/B}(\vec{q})$ by inserting this result into equation (3.29). We will skip this last step since the result will be of no importance in the following.

Remember that our actual goal is the calculation of the full fermion propagator. With our knowledge of the full one-particle irreducible vertex functions, we can now calculate the full proper self energy. In the limit of small momentum, the main contribution comes from the diagrams drawn in figure 3.8 a and 3.8 b, where we can now utilize our results for the full resummed vertex functions and propagators.

In fact, the calculation becomes very simple since one can easily convince oneself that the self energy is given by

$$\Sigma(k) = -2ig\int\frac{d^3q}{(2\pi)^3}\,\frac{e^{-E_q/T}}{E_q}\big(V_A(k,\vec{q})+V_B(k,\vec{q})\big).$$



a b

Figure 3.8: The dominant full proper self energy diagrams

All the necessary ingredients are supplied by equations (3.27) and (3.30), and one can calculate

$$\begin{aligned}\Sigma(k) &= -2ig\int\frac{d^3q}{(2\pi)^3}\,e^{-E_q/T}\frac{8ig}{\Delta_B} - \frac{2ig}{m}\frac{im^2}{g\Delta_B}\left(m\,k_0\gamma^0 + T\,\vec{k}\vec{\gamma}\right)\\ &= -m + \frac{2m^2}{\Delta_B}\left(k_0\gamma^0 + \frac{T}{m}\vec{k}\vec{\gamma}\right).\end{aligned}$$

Thus, the self energy $\Sigma(0) = -m$ exactly cancels the Lagrangian mass term in the full inverse propagator which thereby becomes

$$i\mathcal{S}^{-1}(k) = \not{k} - m - \Sigma(k) = \not{k} - \frac{2m^2}{\Delta_B}\left(k_0\gamma^0 + \frac{T}{m}\vec{k}\vec{\gamma}\right). \tag{3.31}$$

Just as it was the case for high temperatures, the inverse propagator for small momenta is dominated by the contribution from the self energy,

$$\mathcal{S}^{-1}(k) \simeq i\frac{2m^2}{\Delta_B}\left(k_0\gamma^0 + \frac{T}{m}\vec{k}\vec{\gamma}\right), \tag{3.32}$$

and we find that the full propagator is singular for vanishing momentum as well as on the dispersion curve

$$k_0 = \pm\frac{T}{m}|\vec{k}|. \tag{3.33}$$

This proves the existence of a massless fermionic excitation propagating with velocity $v = \frac{T}{m}$. Since this is exactly what was predicted in chapter 2, we identify it as the phonino.

Like it was the case for high temperatures, the existence of the phonino pole is limited to small momenta. In this case, we have made the approximation $kq \ll \Delta_A$ such that the derivation is restricted to momenta $|\vec{k}| \ll \frac{\Delta_A}{\sqrt{Tm}}$.

The existence of the phonino which apparently appears as a Goldstone fermion strongly indicates the spontaneous nature of the thermal supersymmetry breaking. The phonino pole in the fermion propagator is present at any finite temperature, only the residue and dispersion law change with temperature. In the limit $T \to 0$, the residue vanishes and the Goldstone mode disappears, since supersymmetry is restored.

The low-temperature phonino pole and dispersion law were already found by Lebedev and Smilga [9] by investigating the homogeneous Bethe-Salpeter equations for the full (i.e., not one-particle irreducible) vertex functions. They did however not calculate the full fermion propagator. For the verification of the Ward-Takahashi identities that we aim at in order to obtain a closed picture of the spontaneous nature of the breakdown of supersymmetry, this outcome of our lengthy calculation is essential.

3.5 Verification of the Ward-Takahashi identities

Now that we have calculated all the relevant elements of the model, the resummed propagators and vertices, we can address the verification of the Ward-Takahashi identities for the full interacting theory.

In section 3.2, we have derived the identity

$$\partial_\mu^x \langle T J^\mu(x) \overline{\Psi}(y) \rangle = i \langle F(y) \rangle \, \delta^{(4)}(x-y), \tag{3.34}$$

whose right hand side was evaluated to leading order in the coupling in section 3.3,

$$\langle F \rangle = -m \langle A \rangle = \begin{cases} \dfrac{m^2 \alpha}{g} & T \ll m, \\ \dfrac{g T^2}{2} & T \gg m. \end{cases} \tag{3.35}$$

For the verification of this identity to leading order, we must evaluate the main contributions to the left hand side which come from the diagrams drawn in figure 3.9. Here, the encircled cross denotes the supercurrent

$$J^\mu(x) = (-\partial\!\!\!/ - im) A(x) \gamma^\mu \Psi(x) + (\partial\!\!\!/ - im) B(x) i\gamma^5 \gamma^\mu \Psi(x) + \mathcal{O}(g),$$

and the cross stands for the thermal expectation value of the A field. The shaded spot denotes the full vertex, where for low temperatures one must take the resummed vertex function that was calculated in the previous section. Also for the propagators we must make use of the full resummed thermal propagators.

The evaluation of the three diagrams thus gives

$$\begin{aligned} &\langle T J^\mu(x) \overline{\Psi}(y) \rangle = \\ &\quad \int \frac{d^4k}{(2\pi)^4} \, e^{-ik(x-y)} \Big\{ -im \langle A \rangle \, \gamma^\mu \mathcal{S}(k) \\ &\qquad + \int \frac{d^4q}{(2\pi)^4} \, (i q\!\!\!/ - im) \, \gamma^\mu \mathcal{D}_A(q) \mathcal{S}(k-q) \Gamma_A(k, k-q) \mathcal{S}(k) \\ &\qquad + \int \frac{d^4q}{(2\pi)^4} \, (-i q\!\!\!/ - im) \, \gamma^\mu \mathcal{D}_B(q) (-i\gamma^5) \mathcal{S}(k-q) \Gamma_B(k, k-q) \mathcal{S}(k) \Big\}. \end{aligned}$$

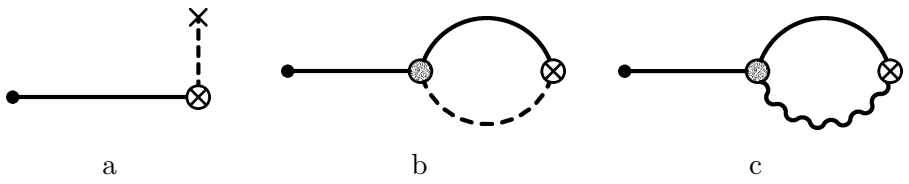


Figure 3.9: Contributions to the Ward-Takahashi identity (3.34)

We are mainly interested in the investigation of the role played by the Goldstone mode. Therefore, we calculate the Fourier transform for small momenta $k \to 0$. First, we amputate the fermion propagator and set

$$\Gamma^{\mu}_{J\overline{\Psi}}(k) = \int d^4x\, e^{ik(x-y)} \langle T J^{\mu}(x) \overline{\Psi}(y) \rangle\, \mathcal{S}^{-1}(k)$$

which we can calculate as

$$\begin{aligned}
&\Gamma^{\mu}_{J\overline{\Psi}}(k) \\
&= i \int \frac{d^3q}{(2\pi)^3} \sum_{q_0=\pm E_q} \Bigg\{ (\not{q} - m)\, \gamma^{\mu}\, (-\not{q} + m)\, \frac{n_B(E_q) + n_F(E_q)}{2E_q} \frac{-i}{\Delta_A} \Gamma_A(0,q) \\
&\qquad + (-\not{q} - m)\, \gamma^{\mu}\, \left(-i\gamma^5\right) (-\not{q} + m)\, \frac{n_B(E_q) + n_F(E_q)}{2E_q} \frac{-i}{\Delta_B} \Gamma_B(0,q) \Bigg\} \\
&\quad - im\gamma^{\mu} \langle A \rangle + \mathcal{O}(k) \\
&= \int \frac{d^3q}{(2\pi)^3} \sum_{q_0=\pm E_q} (-2q^{\mu} \not{q})\, \frac{n_B(E_q) + n_F(E_q)}{2E_q} \frac{\Gamma_A(0,q)}{\Delta_A} \\
&\qquad + \int \frac{d^3q}{(2\pi)^3} \sum_{q_0=\pm E_q} (-2q^{\mu} \not{q})\, \frac{n_B(E_q) + n_F(E_q)}{2E_q} \left(-i\gamma^5\right) \frac{\Gamma_B(0,q)}{\Delta_B} \\
&\quad - im\gamma^{\mu} \langle A \rangle + \mathcal{O}(k).
\end{aligned} \tag{3.36}$$

These are by now well-known integrals. For $T \gg m$, we obtain, separating timelike and spacelike components,

$$\Gamma^{\mu}_{J\overline{\Psi}}(0) = -im\langle A \rangle \left(\gamma^{\mu} + \frac{\pi^2}{g^2} \left\{ \begin{array}{c} \gamma^0 \\ -\frac{1}{3}\gamma^i \end{array} \right\} \right), \tag{3.37}$$

while for $T \ll m$

$$\Gamma^{\mu}_{J\overline{\Psi}}(0) = -im\langle A \rangle \left(\gamma^{\mu} - \frac{2m^2}{\Delta_B} \left\{ \begin{array}{c} \gamma^0 \\ -\frac{T}{m}\gamma^i \end{array} \right\} \right). \tag{3.38}$$

In momentum space, the Ward-Takahashi identity (3.34) we want to verify reads

$$-ik_{\mu}\Gamma^{\mu}_{J\overline{\Psi}}(k) = -im\langle A \rangle\, \mathcal{S}^{-1}(k), \tag{3.39}$$

and indeed, for $T \gg m$ we find, in our linear approximation,

$$\frac{k_{\mu}\Gamma^{\mu}_{J\overline{\Psi}}(k)}{m\langle A \rangle} = -i \left\{ \not{k} + \frac{\pi^2}{g^2} \left(k_0\gamma^0 + \frac{1}{3}\vec{k}\vec{\gamma} \right) \right\},$$

which is nothing but the full inverse propagator we have calculated earlier, see (3.17).

Similarly, for low temperatures we find that

$$\frac{k_\mu \Gamma^\mu_{J\overline{\Psi}}(k)}{m\langle A\rangle} = -i\left\{ \not{k} - \frac{2m^2}{\Delta_B}\left(k_0\gamma^0 + \frac{T}{m}\vec{k}\vec{\gamma}\right)\right\}$$

gives exactly the inverse propagator we obtained in equation (3.31).

Thus, the Ward-Takahashi identity (3.34) is fulfilled through the contribution from the phonino.

Once we have convinced ourselves that the Ward-Takahashi identity is satisfied at finite temperature in our resummed approach, we can in turn use it to obtain more information about the full propagator. If one keeps also the linear terms in the calculation of $\Gamma^\mu_{J\overline{\Psi}}(k)$, equation (3.36), one can deduce the nonlinear corrections to the phonino dispersion law. In agreement with [9], they are found to be small in the region where the phonino exists.

Let us proceed to the second identity,

$$\begin{aligned}\partial^x_\mu \langle T J^\mu(x) A(y)\overline{\Psi}(y)\rangle \\ = \delta^{(4)}(x-y)\left(i\langle T\Psi(y)\overline{\Psi}(y)\rangle - im\langle TA(y)A(y)\rangle\right),\end{aligned} \quad (3.40)$$

whose right hand side was also calculated in section 3.2,

$$i\langle T\Psi(y)\overline{\Psi}(y)\rangle - im\langle TA(y)A(y)\rangle = \left\{ \begin{array}{ll} -im\frac{m^2\alpha}{4g^2} & T \ll m \\ -im\frac{T^2}{8} & T \gg m \end{array} \right\} = i\frac{m^2}{4g}\langle A\rangle\,.$$

Again, we have in principle a lot of diagrams like those drawn in figure 3.10. In the limit of vanishing momentum we are interested in, most of these diagrams are finite and hence negligible compared to the contribution from the Goldstone mode. For the diagrams in figure 3.10 a and b this is obvious, while for diagram 3.10 c we must distinguish between two cases. Firstly, the four-point function which we denoted by a hatched box gets a class of one-particle irreducible contributions. These can be summed up by means of the same Bethe-Salpeter equation that we used for calculating the one-particle irreducible three-point function in the previous section. At this point, it is only important that this resummation does not introduce new

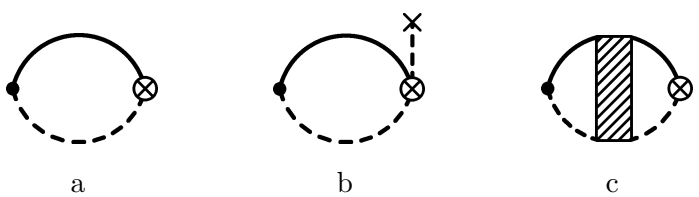
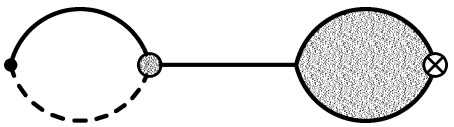


Figure 3.10: Contributions to the Ward-Takahashi identity (3.40)

poles, in other words that the homogeneous Bethe-Salpeter equation does not have any nontrivial solution, as we remarked following equation (3.27). Hence, also this class of diagrams is finite in the limit of low momentum.

Thus, the dominant contribution for small momenta comes from the remaining one-particle reducible diagrams with a single intermediate fermion, schematically



where the big smudge stands for the sum of all diagrams discussed above in connection with the first Ward-Takahashi identity. The contributions from diagrams of this structure to the Fourier transformed correlation function reads, in the low-momentum limit,

$$\begin{aligned}\Gamma^{\mu}_{JA\overline{\Psi}}(k) &\equiv \int d^4x\, e^{ik(x-y)} \langle T J^{\mu}(x) A(y) \overline{\Psi}(y) \rangle \\ &= \Gamma^{\mu}_{J\overline{\Psi}}(k) \mathcal{S}(k) \int \frac{d^4q}{(2\pi)^4}\, \Gamma_A \mathcal{D}_A(q) \mathcal{S}(k-q) \\ &= \Gamma^{\mu}_{J\overline{\Psi}}(k) \mathcal{S}(k) \left(-\frac{m}{4g}\right).\end{aligned}$$

The numerical factor is the same for low and high temperatures.

We can now make use of the first Ward-Takahashi identity (3.39) and obtain

$$-ik_{\mu}\Gamma^{\mu}_{JA\overline{\Psi}}(k) = i\frac{m^2}{4g}\langle A \rangle \tag{3.41}$$

which is nothing but the Ward-Takahashi identity (3.40) we wanted to prove, written in momentum space.

Let us compare the ways the Ward-Takahashi identity is fulfilled for the free and interacting theories. In the free theory, the loop diagram in figure 3.10 a is the only one that contributes to the left hand side of the identity. Because of the equality of the masses of both propagators, it becomes singular in the vanishing momentum limit, thereby saturating the Ward-Takahashi identity non-trivially without the need for a massless mode as was proven in section 3.2. In contrast, in the interacting theory there appears a mass splitting between fermion and boson, and diagrams 3.10 a and b are finite even in the limit of vanishing momentum. Hence, there must be a massless Goldstone mode that contributes to the left hand side in order to saturate the Ward-Takahashi identity. Our calculation shows that the phonino pole in the fermion propagator exactly does this job.

The identity for the B boson can be checked in the same manner, and one obtains, as expected,

$$\partial_\mu^x \langle T J^\mu(x) B(y) \overline{\Psi}(y) \left(-i\gamma^5\right)\rangle = \partial_\mu^x \langle T J^\mu(x) A(y) \overline{\Psi}(y)\rangle. \tag{3.42}$$

Altogether, we have shown the validity of all three Ward-Takahashi identities, in particular we have found that they can only be fulfilled through the contribution from the phonino.

If we compare the relative couplings of the three modes that we have investigated to the supercurrent,

$$\begin{gathered}\int d^4x\, \partial_\mu^x \langle T J^\mu(x) A \overline{\Psi}(y)\rangle = \\ \int d^4x\, \partial_\mu^x \langle T J^\mu(x) B \overline{\Psi}(y)(-i\gamma^5)\rangle = \\ -\frac{m}{4g}\int d^4x\, \partial_\mu^x \langle T J^\mu(x) \overline{\Psi}(y)(-i\gamma^5)\rangle = \frac{im^2}{4g}\langle A\rangle\,,\end{gathered}$$

we can identify the linear combination

$$\chi = A\Psi - i\gamma^5 B\Psi \tag{3.43}$$

as the Goldstone mode. In the fermion propagator, the Goldstone pole appears because of the Yukawa coupling to this mode,

$$\mathcal{L}_Y = -g\overline{\Psi}(A - i\gamma^5 B)\Psi.$$

It is perhaps not surprising that it is this linear combination if one goes back to the superspace formalism. Rewritten in terms of the original chiral superfield, one has

$$\left(A - i\gamma^5 B\right)\Psi = \begin{pmatrix} \Phi^2\big|_\theta \\ \Phi^{\dagger 2}\big|_{\overline{\theta}} \end{pmatrix}, \tag{3.44}$$

that is, the Goldstone mode is the fermionic component field of the composite superfield Φ^2.

Let us verify one last Ward-Takahashi identity that requires almost no calculation but turns out to be very illuminating. Consider the supercurrent itself. According to the transformation law (1.19), the associated Ward-Takahashi identity reads

$$\partial_\mu^x \langle T J^\mu(x) \overline{J}^\nu(y)\rangle = 2\delta^{(4)}(x-y)\langle T^{\mu\nu}(y)\rangle\gamma_\mu. \tag{3.45}$$

The left hand side of this identity gets contributions from the bilinear operators $A\Psi$ and $B\Psi$ in the same way as we have calculated above, with a

single fermion intermediate state. Hence, we can write, in momentum space,

$$\begin{aligned}\Gamma^{\mu\nu}_{J\overline{J}}(k) &\equiv \int d^4x\, e^{ik(x-y)}\langle T J^\mu(x)\overline{J}^\nu(y)\rangle \\ &= \Gamma^\mu_{J\overline{\Psi}}(k)\mathcal{S}(k)\Gamma^\nu_{\Psi\overline{J}}(k).\end{aligned}$$

By making use of the Ward-Takahashi identity (3.39), we find

$$-ik_\mu\Gamma^{\mu\nu}_{J\overline{J}}(k) = -im\langle A\rangle\,\Gamma^\nu_{\Psi\overline{J}}(k).$$

This can now be calculated by using our earlier results for $\Gamma^\mu_{J\overline{\Psi}}(k)$.

First, in the limit of high temperature, we make use of (3.37) and obtain the left hand side of the Ward-Takahashi identity as

$$-ik_\mu\Gamma^{\mu\nu}_{J\overline{J}}(k) = m^2\langle A\rangle^2\,\frac{\pi^2}{g^2}\begin{Bmatrix}\gamma^0\\ -\frac{1}{3}\gamma^i\end{Bmatrix} = \frac{\pi^2}{4}T^4\begin{Bmatrix}\gamma^0\\ -\frac{1}{3}\gamma^i\end{Bmatrix}.$$

For the right hand side, we must evaluate the thermal expectation value of the energy-momentum tensor. In a thermal equilibrium state, it reads

$$\langle T^{\mu\nu}\rangle = \mathrm{diag}(\rho,p,p,p),$$

where in the high-temperature limit the pressure is given by $p=\frac{1}{3}\rho$, and the energy density can be calculated as

$$\rho = \int\frac{d^3q}{(2\pi)^3}\big(2n_F(E_q)+2n_B(E_q)\big)E_q = \frac{\pi^2}{8}T^4. \tag{3.46}$$

Hence, for the right hand side of the identity (3.45) we find

$$2\langle T^{\mu\nu}\rangle\gamma_\mu = 2\begin{Bmatrix}\rho\,\gamma^0\\ -p\,\gamma^i\end{Bmatrix} = \frac{\pi^2}{4}T^4\begin{Bmatrix}\gamma^0\\ -\frac{1}{3}\gamma^i\end{Bmatrix},$$

thereby proving that the Ward-Takahashi identity

$$-ik_\mu\Gamma^{\mu\nu}_{J\overline{J}}(k) = 2\langle T^{\mu\nu}\rangle\gamma_\mu$$

is indeed fulfilled.

For low temperatures, one calculates in the same way, using (3.38),

$$\begin{aligned}-ik_\mu\Gamma^{\mu\nu}_{J\overline{J}}(k) &= -m^2\langle A\rangle^2\,\frac{2m^2}{\Delta_B}\begin{Bmatrix}\gamma^0\\ -\frac{T}{m}\gamma^i\end{Bmatrix} \\ &= 8me^{-m/T}\left(\frac{Tm}{2\pi}\right)^{3/2}\begin{Bmatrix}\gamma^0\\ -\frac{T}{m}\gamma^i\end{Bmatrix}.\end{aligned}$$

Since in this temperature range the pressure is given by $p = \frac{T}{m}\rho$ and the energy density becomes

$$\rho = \int \frac{d^3q}{(2\pi)^3} \big(2n_F(E_q) + 2n_B(E_q)\big) E_q = 4me^{-m/T} \left(\frac{Tm}{2\pi}\right)^{3/2},$$

one finds

$$2\langle T^{\mu\nu}\rangle\gamma_\mu = 8me^{-m/T} \left(\frac{Tm}{2\pi}\right)^{3/2} \left\{ \begin{array}{c} \gamma^0 \\ -\frac{T}{m}\,\gamma^i \end{array} \right\} = -ik_\mu \Gamma^{\mu\nu}_{J\overline{J}}(k),$$

again proving the Ward-Takahashi identity.

The verification of the Ward-Takahashi identities also in the resummed interacting theory is a striking evidence for the consistency of our approach. Though at first our resummation program, accompanied by a somewhat opaque reshuffling of the perturbative expansion, may have seemed questionable, we take our results as a justification and believe that our physical picture of thermal supersymmetry breaking is at least self-consistent. Even though the existing attempts to prove the Goldstone theorem at finite temperature are inconclusive from a mathematical point of view, our explicit calculation shows that at least in the Wess-Zumino model there exists a massless fermion whose contribution to the Ward-Takahashi identities allows us to identify it as the Goldstino.

To summarize, we have shown that in the Wess-Zumino model supersymmetry is broken spontaneously by the thermal bath, the most obvious signs being the appearance of a thermal expectation value of a scalar field and a mass splitting between the members of the supermultiplet. In addition, we have shown that the phonino exists as a collective excitation with the expected dispersion law and really plays the role of the Goldstone fermion in that it saturates the supersymmetric Ward-Takahashi identities.

Chapter 4

Supersymmetric quantum electrodynamics

As a first step to generalize the results of the previous chapter to more realistic models, we will study a simple model in which the matter fields, described by chiral superfields, have an abelian gauge interaction with a vector superfield.

We will proceed in a way very similar to our investigation of the Wess-Zumino model. After a presentation of the model, the thermal one-loop corrections will be calculated. With this knowledge, one can set up and solve the Bethe-Salpeter equations that will prove the existence of the phonino as a bound state of electron and scalar electron. Again, the phonino will be interpreted as the Goldstone fermion of spontaneously broken supersymmetry.

To our knowledge, such a model has not been studied at finite temperature before.

4.1 The model

A supersymmetric abelian gauge theory generalizing quantum electrodynamics was first constructed by Wess and Zumino [36]. It is built up of two chiral superfields

$$\Phi_\pm(y,\theta) = \phi_\pm(y) + \sqrt{2}\theta\psi_\pm(y) + \theta\theta F_\pm(y),$$

each containing as component fields a scalar field $\phi_\pm$, a Weyl fermion $\psi_\pm$ and an auxiliary scalar field $F_\pm$. In addition, there is a vector superfield that in Wess-Zumino gauge can be written as

$$V(x,\theta,\overline{\theta}) = \theta\sigma^\mu\overline{\theta}A_\mu(x) + i\theta\theta\overline{\theta}\overline{\lambda}(x) - i\overline{\theta}\overline{\theta}\theta\lambda(x) + \tfrac{1}{2}\theta\theta\overline{\theta}\overline{\theta}D(x)$$

which describes a vector field A^μ, the photon, and a Weyl fermion λ, the photino, where D is an auxiliary scalar field.

The Lagrangian reads

$$\begin{aligned}\mathcal{L} = &\Phi_+^\dagger e^{-2eV}\Phi_+\big|_{\theta\theta\overline{\theta}\overline{\theta}} + \Phi_-^\dagger e^{2eV}\Phi_-\big|_{\theta\theta\overline{\theta}\overline{\theta}} \\ &+ \frac{1}{32}W^\alpha W_\alpha\big|_{\theta\theta} + m\left(\Phi_+\Phi_-\big|_{\theta\theta} + \text{h.c.}\right)\end{aligned} \tag{4.1}$$

where the supersymmetric field strength is given by

$$W_\alpha = (\overline{D}\,\overline{D})D_\alpha V.$$

Again, we will not work in the superspace formalism but rewrite the Lagrangian in terms of the component fields,

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\Psi}(i\not{D} - m)\Psi + \tfrac{i}{2}\overline{\Lambda}\not{\partial}\Lambda + \tfrac{1}{2}D^2 \\ &+ (D_\mu B)^\dagger(D^\mu B) + (D_\mu C)^\dagger(D^\mu C) + F_-^\dagger F_- + F_+^\dagger F_+ \\ &+ \tfrac{m}{\sqrt{2}}\left[F_-(B^\dagger - iC^\dagger) + F_-^\dagger(B + iC) - F_+(B - iC) - F_+^\dagger(B^\dagger + iC^\dagger)\right] \\ &- ieD(B^\dagger C - C^\dagger B) \\ &+ e\overline{\Lambda}(C^\dagger + i\gamma^5 B^\dagger)\Psi + e\overline{\Psi}(C + i\gamma^5 B)\Lambda\end{aligned} \tag{4.2}$$

with the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$.

Here, we have combined the two Weyl spinors $\psi_\pm$ to form one massive Dirac spinor, the electron field

$$\Psi = \begin{pmatrix}\psi_- \\ \overline{\psi}_+\end{pmatrix},$$

and we have defined the photino Majorana spinor

$$\Lambda = \begin{pmatrix}\lambda \\ \overline{\lambda}\end{pmatrix}.$$

We have also rewritten the fields $\phi_\pm$ in terms of new complex scalar fields B and C of definite parity,

$$\begin{aligned}B &= \tfrac{1}{\sqrt{2}}(\phi_+^\dagger - \phi_-), \\ C &= -\tfrac{i}{\sqrt{2}}(\phi_+^\dagger + \phi_-).\end{aligned}$$

The Lagrangian can be simplified by replacing the auxiliary fields by means of their algebraic equations of motion,

$$\begin{aligned}F_+ &= \frac{m}{\sqrt{2}}(B^\dagger + iC^\dagger), \\ F_- &= -\frac{m}{\sqrt{2}}(B + iC),\end{aligned}$$

and

$$D = ie(B^\dagger C - C^\dagger B).$$

It thereby becomes

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\Psi}(i\not{D} - m)\Psi + \tfrac{i}{2}\overline{\Lambda}\not{\partial}\Lambda \\ &+ (D_\mu B)^\dagger(D^\mu B) - m^2 B^\dagger B + (D_\mu C)^\dagger(D^\mu C) - m^2 C^\dagger C \\ &+ \tfrac{1}{2}e^2(B^\dagger C - C^\dagger B)^2 \\ &+ e\overline{\Lambda}(C^\dagger + i\gamma^5 B^\dagger)\Psi + e\overline{\Psi}(C + i\gamma^5 B)\Lambda.\end{aligned} \tag{4.3}$$

The Lagrangian contains the usual gauge invariant parts from quantum electrodynamics (QED) and scalar QED, that is, it describes a massive charged fermion Ψ and two massive charged bosons B and C coupled to the photon A_μ. In addition, we have a massless Majorana fermion Λ, the photino. Moreover, supersymmetry requires equal masses for charged bosons and fermions and fixes the photino Yukawa couplings as well as the scalar couplings in terms of the electric charge e.

The Feynman rules are easily read off from the Lagrangian. The real-time propagators are displayed in figure 4.1. Dashed lines will always denote scalar electrons, while directed solid lines stand for electrons. Undirected solid lines denote photinos, and a wiggly line stands for the photon.

The photon propagator depends of course on the chosen gauge which leads to different forms of the tensor $G_{\mu\nu}$. At finite temperature, the Coulomb gauge is the most physical one, since only the transverse degrees of freedom are taken into account and no unphysical degrees of freedom are thermalized. The disadvantage it usually has, not to be Lorentz invariant, does not really matter here, since Lorentz invariance is broken anyway. In

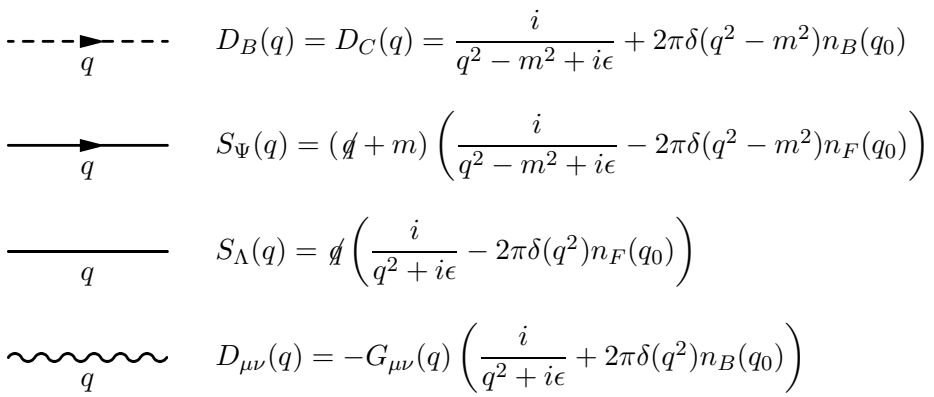


Figure 4.1: Real-time propagators for supersymmetric QED

the Coulomb gauge, the tensor $G_{\mu\nu}$ reads

$$G_{00}(q) = -\frac{q^2}{\vec{q}^2}, \qquad G_{0i}(q) = 0, \qquad G_{ij}(q) = -\delta_{ij} + \frac{q_i q_j}{\vec{q}^2}.$$

This can also be written in the more covariant form

$$G_{\mu\nu}(q) = g_{\mu\nu} + \frac{q_\mu q_\nu}{\vec{q}^2} - \frac{q_0}{\vec{q}^2}(q_\mu u_\nu + u_\mu q_\nu)$$

with $u_\mu = (1, 0)$. Besides, we will work in the covariant gauge with

$$G_{\mu\nu}(q) = g_{\mu\nu} - (1 - \xi)\frac{q_\mu q_\nu}{q^2},$$

where the special case $\xi = 1$ is called the Feynman gauge and often the most convenient one. Finally, in the temporal gauge, we have

$$G_{0\mu}(q) = 0, \qquad G_{ij}(q) = -\delta_{ij} + \frac{q_i q_j}{q_0^2}.$$

The interaction vertices are summarized in figure 4.2.

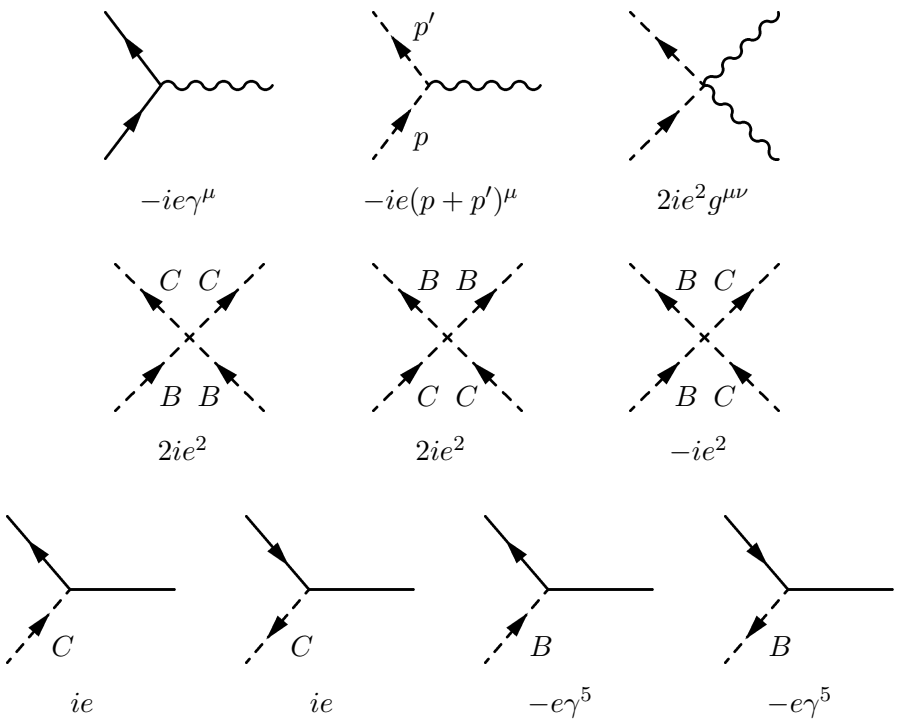


Figure 4.2: Interaction vertices for supersymmetric QED

4.2 Supersymmetry of the model

Since electron and scalar electron are built from chiral superfields, their supersymmetry transformations are very similar to those for the fields in the Wess-Zumino model that we have studied earlier. After replacing the auxiliary fields, they read

$$\begin{aligned}
[Q, B] &= i\gamma^5\Psi \\
[Q, B^\dagger] &= -i\gamma^5\Psi^c \\
[Q, C] &= \Psi \\
[Q, C^\dagger] &= -\Psi^c \\
\{Q, \overline{\Psi}\} &= i\partial\!\!\!/(C^\dagger - i\gamma^5 B^\dagger) - m(C^\dagger + i\gamma^5 B^\dagger) \\
\{Q, \overline{\Psi^c}\} &= -i\partial\!\!\!/(C - i\gamma^5 B) + m(C + i\gamma^5 B),
\end{aligned} \tag{4.4}$$

the corresponding contribution to the supercurrent being

$$\begin{aligned}
J^\mu_{ch}(x) = &(i\partial\!\!\!/ - m)C^\dagger(x)\gamma^\mu\Psi(x) - (i\partial\!\!\!/ - m)C(x)\gamma^\mu\Psi^c(x) \\
&- (i\partial\!\!\!/ + m)B^\dagger(x)i\gamma^5\gamma^\mu\Psi(x) + (i\partial\!\!\!/ + m)B(x)i\gamma^5\gamma^\mu\Psi^c(x).
\end{aligned} \tag{4.5}$$

With this knowledge, we can easily derive the Ward-Takahashi identities. Since the chiral superfields carry electric charge, only the neutral composite operators can give a nontrivial identity. In analogy to (3.7) and (3.8), one finds

$$\begin{aligned}
\partial^x_\mu\langle TJ^\mu(x)C(y)\overline{\Psi}(z)\rangle = -\partial^x_\mu\langle TJ^\mu(x)C^\dagger(y)\overline{\Psi^c}(z)\rangle = \\
\delta^{(4)}(x-y)S(y-z) + \delta^{(4)}(x-z)(-i\partial\!\!\!/ - m)D(y-z)
\end{aligned}$$

and

$$\begin{aligned}
\partial^x_\mu\langle TJ^\mu(x)B(y)\overline{\Psi}(z)\rangle = -\partial^x_\mu\langle TJ^\mu(x)B^\dagger(y)\overline{\Psi^c}(z)\rangle = \\
i\gamma^5\left(\delta^{(4)}(x-y)S(y-z) + \delta^{(4)}(x-z)(-i\partial\!\!\!/ - m)D(y-z)\right).
\end{aligned}$$

Again, the vacuum contributions to these equations cancel, leaving a non-vanishing contribution from the thermal propagators. For the case $y = z$, we copy the result for the right hand side from our calculation for the Wess-Zumino model such that we can summarize the four identities as follows,

$$\left.\begin{array}{r}
\partial^x_\mu\langle TJ^\mu(x)C(y)\overline{\Psi}(y)\rangle \\
-\partial^x_\mu\langle TJ^\mu(x)C^\dagger(y)\overline{\Psi^c}(y)\rangle \\
-\partial^x_\mu\langle TJ^\mu(x)B(y)\overline{\Psi}(y)\rangle i\gamma^5 \\
\partial^x_\mu\langle TJ^\mu(x)B^\dagger(y)\overline{\Psi^c}(y)\rangle i\gamma^5
\end{array}\right\} = \delta^{(4)}(x-y)\cdot\left\{\begin{array}{ll}
-2e^{-\frac{m}{T}}\left(\frac{Tm}{2\pi}\right)^{3/2} & T \ll m, \\
-m\frac{T^2}{8} & T \gg m.
\end{array}\right. \tag{4.6}$$

The nontrivial right hand side shows that supersymmetry is broken at finite temperature also in this model.

Now that we know how the various modes couple to the supercurrent, we can read off the linear combination of these operators with the maximal coupling,

$$\chi = -iC^\dagger\Psi + iC\Psi^c - \gamma^5 B^\dagger\Psi + \gamma^5 B\Psi^c, \tag{4.7}$$

where the normalization was chosen in order to give a Majorana spinor. If we express this linear combination in terms of the original chiral superfields, we find that it can be written as

$$\chi = \begin{pmatrix} \Phi_+\Phi_-\big|_\theta \\ \Phi_+^\dagger\Phi_-^\dagger\big|_{\overline{\theta}} \end{pmatrix}, \tag{4.8}$$

in other words, it is the fermionic component field of the chiral superfield $\Phi_+\Phi_-$ which determines the mass term in the superpotential, see (4.1).

Since this mode has to satisfy a nontrivial Ward-Takahashi identity,

$$\partial_\mu^x \langle T J^\mu(x)\chi(y)\rangle \neq 0,$$

it must be considered as the chiral superfield's contribution to the Goldstone mode associated with the breakdown of supersymmetry.

Let us now move on to the vector superfield. The supersymmetry transformations of the physical fields read

$$\begin{aligned} \{Q, \overline{\Lambda}\} &= -\frac{i}{4} F_{\mu\nu}\left[\gamma^\mu, \gamma^\nu\right], \\ [Q, F_{\mu\nu}] &= -\left(\partial_\mu\gamma_\nu - \partial_\nu\gamma_\mu\right)\Lambda, \end{aligned} \tag{4.9}$$

the corresponding contribution to the supercurrent being

$$J^\mu_{vec}(x) = -\frac{i}{4} F_{\nu\lambda}(x)[\gamma^\nu, \gamma^\lambda]\gamma^\mu\Lambda(x). \tag{4.10}$$

With these transformation laws, we can again write down the Ward-Takahashi identities. First, for the photino field, we find

$$\partial_\mu^x \langle T J^\mu(x)\overline{\Lambda}(y)\rangle = -\delta^{(4)}(x-y)\frac{i}{4}\langle F_{\mu\nu}\rangle\left[\gamma^\mu, \gamma^\nu\right] = 0 \tag{4.11}$$

which vanishes since also at finite temperature the field strength tensor does not develop a nontrivial expectation value because of charge conjugation invariance. This means that the photino, though neutral and massless, does not contribute to the Goldstone mode of broken supersymmetry. If we look at the photino Yukawa interaction,

$$\mathcal{L}_Y = e(C\overline{\Psi} + C^\dagger\overline{\Psi^c} + B\overline{\Psi}i\gamma^5 + B^\dagger\overline{\Psi^c}i\gamma^5)\Lambda,$$

we consistently find that the linear combination of the charged fields coupling to the photino field is orthogonal to the Goldstone mode χ defined above. The reason for this becomes clear if one remembers that the vector superfield couples only to the field $\Phi_+^\dagger\Phi_+ + \Phi_-^\dagger\Phi_-$ while the Goldstone mode is a component of $\Phi_+\Phi_-$.

The second possible contribution to the Goldstone mode comes from the composite operator $F_{\nu\lambda}\Lambda$ present in the supercurrent. The Ward-Takahashi identity for this mode reads

$$\begin{aligned}\partial_\mu^x\langle TJ^\mu(x)F_{\nu\lambda}(y)\overline{\Lambda}(z)\rangle = \delta^{(4)}(x-y)\langle-(\partial_\nu\gamma_\lambda-\partial_\lambda\gamma_\nu)\Lambda(y)\overline{\Lambda}(z)\rangle \\ -\tfrac{i}{4}\delta^{(4)}(x-z)\langle F_{\nu\lambda}(y)F_{\rho\sigma}(z)\rangle[\gamma^\rho,\gamma^\sigma].\end{aligned} \tag{4.12}$$

Also this identity is worth an explicit verification. Making use of the relation

$$\langle F_{\nu\lambda}(x)F_{\rho\sigma}(y)\rangle[\gamma^\rho,\gamma^\sigma] = 4\,\partial_\rho\big(\partial_\lambda\langle A_\nu(x)A_\sigma(y)\rangle - \partial_\nu\langle A_\lambda(x)A_\sigma(y)\rangle\big)\gamma^\rho\gamma^\sigma,$$

we can evaluate the right hand side of equation (4.12). In the Feynman gauge, one easily obtains, for $y=z$,

$$\begin{aligned}&\int d^4x\, e^{ip(x-y)}RHS \\ &\quad= \int\frac{d^4q}{(2\pi)^4}\left[i(q_\nu\gamma_\lambda-q_\lambda\gamma_\nu)S_\Lambda(q) + i\not{q}(q_\lambda D_{\nu\sigma}(q)-q_\nu D_{\lambda\sigma}(q))\gamma^\sigma\right] \\ &\quad= \int\frac{d^4q}{(2\pi)^4}\left[i(q_\nu\gamma_\lambda-q_\lambda\gamma_\nu)\not{q}\left(\frac{i}{q^2+i\epsilon}-2\pi n_F(q_0)\delta(q^2)\right)\right. \\ &\qquad\qquad\left.-i\not{q}\,(q_\lambda g_{\nu\sigma}-q_\nu g_{\lambda\sigma})\,\gamma^\sigma\left(\frac{i}{q^2+i\epsilon}+2\pi n_B(q_0)\delta(q^2)\right)\right] \\ &\quad= \int\frac{d^4q}{(2\pi)^4}\,i(q_\nu\gamma_\lambda-q_\lambda\gamma_\nu)\not{q}(-2\pi)\big(n_F(q_0)+n_B(q_0)\big)\delta(q^2).\end{aligned}$$

Again, we observe the cancellation of the zero temperature parts.

The left hand side of (4.12) can be transformed to the same expression in close analogy to the calculation for the chiral superfield. Hence, also for the free vector superfield at finite temperature the Ward-Takahashi identity is satisfied nontrivially without the contribution from a Goldstone fermion.

Let us now calculate the remaining integral. We first go over to the whole supercurrent of the vector superfield, as given in equation (4.10), by contraction with $-\frac{i}{4}\gamma^\mu[\gamma^\nu,\gamma^\lambda]$. After having performed the Dirac algebra,

$$(q_\nu\gamma_\lambda-q_\lambda\gamma_\nu)\not{q}\,\gamma^\mu[\gamma^\nu,\gamma^\lambda] = -16\,q^\mu\not{q}$$

(as long as $q^2=0$), we can calculate the timelike and spacelike components of the whole expression as

$$\begin{aligned}\int d^4x\,\partial_\mu^x\langle TJ^\mu(x)\overline{J}^\rho_{vec}(y)\rangle &= \int\frac{d^4q}{(2\pi)^4}\,4\,q^\rho\not{q}\;2\pi\big(n_F(q_0)+n_B(q_0)\big)\delta(q^2) \\ &= \frac{\pi^2T^4}{8}\begin{cases}2\,\gamma^0\\ -\frac{2}{3}\,\gamma^i\end{cases}.\end{aligned}$$

Of course, the timelike component is, up to a factor of $2\gamma^0$, nothing but the energy density of two bosonic and two fermionic relativistic degrees of freedom, $\rho = \frac{\pi^2}{8}T^4$, as we have calculated in equation (3.46), while the spacelike components are proportional to the pressure $p = \frac{1}{3}\rho$. Thus, we have verified the Ward-Takahashi identity for the vector superfield's contribution to the supercurrent,

$$\partial_\mu^x \langle T J^\mu(x) \overline{J}^\nu_{vec}(y)\rangle = 2\,\delta^{(4)}(x-y)\langle T^{\mu\nu}_{vec}(y)\rangle\gamma_\mu. \tag{4.13}$$

In summary, our investigation of the Ward-Takahashi identities has come to the following result. The operator χ defined in (4.7) should describe the contribution of the chiral superfields to the Goldstone mode. In the free theory, this mode satisfies a nontrivial Ward-Takahashi identity though it is only composed of free massive fields. In the interacting theory, in view of our results for the Wess-Zumino model, we would expect the appearance of a massless bound state in this mode, the phonino. The explicit verification of this anticipation will be the task of the following sections.

Also the supercurrent of the vector superfield obeys a nontrivial Ward-Takahashi identity, so that its two-point function should as well show the phonino pole in the full interacting theory. Since photon and photino do not interact on tree level, this pole should appear via the coupling to the chiral superfields.

4.3 One-loop corrections

4.3.1 Electron and scalar electron

In order to understand the influence of the thermal background on the physics of our model, we need to know the thermal self energy corrections to the propagators.

Let us start with the charged massive particles. Again, we include the relevant self energy corrections by making the ansatz

$$\mathcal{S}(p) = (\not{p} + m)\left(\frac{i}{p^2 - m^2 - \Pi_F(p) + i\epsilon} - 2\pi\delta\big(p^2 - m^2 - \Pi_F(p)\big)n_F(p_0)\right)$$

for the full fermion propagator, where

$$\Pi_F(p) = \tfrac{1}{2}\,\mathrm{tr}\left[(\not{p} + m)\Sigma(p)\right]$$

with $\Sigma(p)$ the real part of the (11) component of the thermal self energy matrix. It is assumed to be small compared to the mass and basically leads to a small shift in the effective electron mass.

The charged boson propagator is taken to be

$$\mathcal{D}(p) = \frac{i}{p^2 - m^2 - \Pi_B(p) + i\epsilon} + 2\pi\delta\big(p^2 - m^2 - \Pi_B(p)\big)n_B(p_0)$$

with the self energy Π_B, shifting the mass of the scalar electron.

The physical masses of the particles in the thermal background are then given by the poles of their propagators in the limit of vanishing spacelike momentum,

$$m_F^2 = m^2 + \Pi_F(m,0),$$
$$m_B^2 = m^2 + \Pi_B(m,0).$$

The one-loop Feynman diagrams contributing to the self energies are displayed in figures 4.3 and 4.4. Possible additional contributions from photon tadpole diagrams vanish identically also at finite temperature because of charge conjugation invariance.

We will evaluate these diagrams in the real time formalism. Applying the Feynman rules given above, we obtain

$$\Pi_F^a(p) = -ie^2 \int \frac{d^4q}{(2\pi)^4}\, \tfrac{1}{2}\,\mathrm{tr}\big[(\not{p}+m)\,\gamma^\mu S_\Psi(q)\gamma^\nu\big] D_{\mu\nu}(p-q)$$

$$\Pi_F^b(p) = -2ie^2 \int \frac{d^4q}{(2\pi)^4}\, \tfrac{1}{2}\,\mathrm{tr}\big[(\not{p}+m)\; S_\Lambda(q)\big] D_B(p-q)$$

$$\Pi_B^a(p) = -ie^2 \int \frac{d^4q}{(2\pi)^4}\; (p+q)^\mu\, D_B(q)\, (p+q)^\nu\, D_{\mu\nu}(p-q)$$

$$\Pi_B^b(p) = -e^2 \int \frac{d^4q}{(2\pi)^4}\, g^{\mu\nu} D_{\mu\nu}(q)$$

$$\Pi_B^c(p) = ie^2 \int \frac{d^4q}{(2\pi)^4}\; \mathrm{tr}\big[S_\Psi(q) S_\Lambda(q-p)\big]$$

$$\Pi_B^d(p) = e^2 \int \frac{d^4q}{(2\pi)^4}\, D_B(q).$$

In order to simplify these expressions, we split the propagators into their

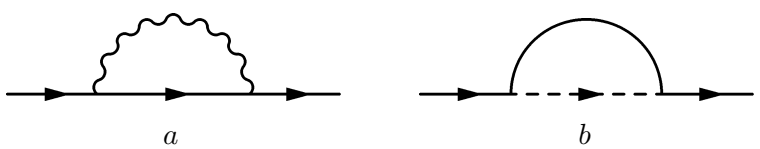


Figure 4.3: One-loop electron self energy diagrams

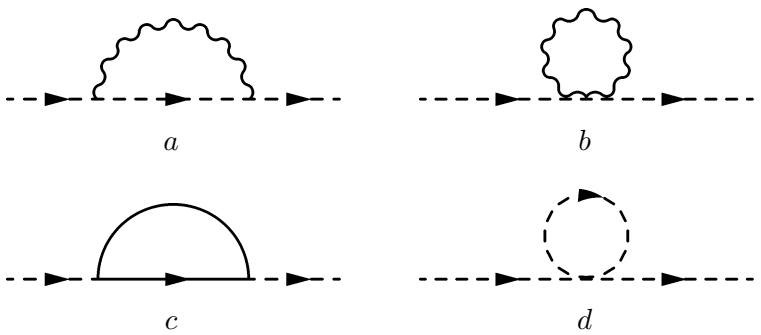


Figure 4.4: One-loop self energy diagrams for the scalar electron

vacuum and thermal parts, namely

$$\begin{aligned}
S_\Psi(q) &= (\not{q} + m)\left(D_m(q) + \tilde{S}_m^T(q)\right) \\
S_\Lambda(q) &= \not{q}\left(D_0(q) + \tilde{S}_0^T(q)\right) \\
D_B(q) &= D_m(q) + D_m^T(q) \\
D_{\mu\nu}(q) &= -G_{\mu\nu}(q)\Big(D_0(q) + D_0^T(q)\Big).
\end{aligned}$$

Here,

$$D_\mu(q) = \frac{i}{q^2 - \mu^2 + i\epsilon}$$

is the free bosonic Feynman propagator of mass μ and

$$\begin{aligned}
\tilde{S}_\mu^T(q) &= -2\pi\,\delta(q^2 - \mu^2)\,n_F(q_0), \\
D_\mu^T(q) &= 2\pi\,\delta(q^2 - \mu^2)\,n_B(q_0).
\end{aligned}$$

Contributions to the real part of the thermal self energies arise if one takes the thermal part of one and the vacuum part of the other propagator in each loop, and accordingly only the thermal one for the tadpole diagrams.

We will perform the calculation in the Feynman gauge where

$$G_{\mu\nu}(q) = g_{\mu\nu}.$$

If we now put everything together and evaluate the traces, we are left

with

$$\begin{aligned}
\Pi_F^a(p) &= -ie^2 \int \frac{d^4q}{(2\pi)^4} \left(4\,pq - 8\,m^2\right) \Big(\tilde{S}_m^T(q) D_0(p-q) + D_m(q) D_0^T(p-q)\Big) \\
\Pi_F^b(p) &= -ie^2 \int \frac{d^4q}{(2\pi)^4}\, 4\,pq \Big(\tilde{S}_0^T(q) D_m(p-q) + D_0(q) D_m^T(p-q)\Big) \\
\Pi_B^a(p) &= ie^2 \int \frac{d^4q}{(2\pi)^4}\, (p+q)^2 \Big(D_m^T(q) D_0(p-q) + D_m(q) D_0^T(p-q)\Big) \\
\Pi_B^b(p) &= e^2 \int \frac{d^4q}{(2\pi)^4}\, 4\, D_0^T(q) \\
\Pi_B^c(p) &= ie^2 \int \frac{d^4q}{(2\pi)^4}\, 4\,q(q-p) \Big(\tilde{S}_m^T(q) D_0(q-p) + D_m(q) \tilde{S}_0^T(q-p)\Big) \\
\Pi_B^d(p) &= e^2 \int \frac{d^4q}{(2\pi)^4}\, D_m^T(q).
\end{aligned} \tag{4.14}$$

Since we are mainly interested in the behavior of the propagators close to their poles, we will restrict ourselves to momenta near the mass shell, that is, $p^2 - m^2 \lesssim \mathcal{O}(\alpha m^2)$, where as usual $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant, and calculate the self energies to first order in α.

Apart from the tadpole diagrams, all integrals contain a product of one massive and one massless propagator, where the thermal one has support on the respective mass shell. This fact allows a simplification of the vacuum propagators as follows. For $p^2 = m^2$, we notice that

$$D_m(p-q)\Big|_{q^2=0} = \frac{i}{-2pq + i\epsilon}, \qquad D_0(p-q)\Big|_{q^2=m^2} = \frac{i}{2m^2 - 2pq + i\epsilon},$$

what can be used to simplify our expressions considerably.

Let us first extract the dominant contributions to the self energies. For low temperatures, they come from the massless particles in the thermal background, photon and photino, while for high temperatures the particle masses can safely be neglected and all contributions are of order

$$\begin{aligned}
\Pi(p) &\sim e^2 \int \frac{d^4q}{(2\pi)^4}\, 2\pi\delta(q^2) n_{B/F}(q_0) \\
&\sim \alpha\pi T^2.
\end{aligned}$$

The quantitative numbers can now be calculated without difficulty, our results are displayed in table 4.1. The photon diagrams have also been calculated in the Coulomb and temporal gauges, giving the same results. However, only the sum of diagrams 4.4 a and 4.4 b is independent of the gauge.

	Π_F^a	Π_F^b	Π_B^a	Π_B^b	Π_B^c	Π_B^d
$T \ll m$	$\frac{2}{3}\alpha\pi T^2$	$\frac{1}{3}\alpha\pi T^2$	$-\frac{2}{3}\alpha\pi T^2$	$\frac{4}{3}\alpha\pi T^2$	$\frac{1}{3}\alpha\pi T^2$	
$T \gg m$	$\alpha\pi T^2$	$\alpha\pi T^2$	$-\frac{1}{3}\alpha\pi T^2$	$\frac{4}{3}\alpha\pi T^2$	$\frac{2}{3}\alpha\pi T^2$	$\frac{1}{3}\alpha\pi T^2$

Table 4.1: Contributions to the thermal mass shifts

The results for the pure QED graph 4.3 a and the scalar QED graphs 4.4 a and b are in agreement with earlier results in the literature [15]. Also the contributions from the photino and tadpole diagrams 4.3 b and 4.4 c and d have been calculated before [37], although in this paper the high-temperature values of diagrams 4.4 a, b and d are given wrongly.

Altogether, we obtain the thermal masses

$$m_F^2 = m_B^2 = \begin{cases} m^2 + \alpha\pi T^2 & T \ll m, \\ m^2 + 2\alpha\pi T^2 & T \gg m. \end{cases} \tag{4.15}$$

Up to this precision, we find that the masses of electron and scalar electron remain degenerate, contrary to our expectation to find a small mass difference as a manifestation of the breakdown of supersymmetry. However, so far we have only calculated the leading contributions from the massless particles in the heat bath, neglecting corrections due to the particle masses.

The resulting mass shifts of order αT^2 show that we must restrict ourselves to a temperature range $eT \ll m$ so as to satisfy the assumption that the self energy corrections are small compared to the mass.

In order to find a mass splitting between the bosons and the fermion, we have to refine our calculation. We define

$$\Delta(p) = \Pi_F^a(p) + \Pi_F^b(p) - \Pi_B^a(p) - \Pi_B^b(p) - \Pi_B^c(p) - \Pi_B^d(p)$$

which we compute by inserting equations (4.14). Remarkably, it turns out that the supersymmetry of the model leads to a cancellation of nearly all of these terms. For $p^2 = m^2$, without calculating a single integral, the sum shrinks to

$$\begin{aligned} \Delta(p) &= 4ie^2m^2 \int \frac{d^4q}{(2\pi)^4} \left[\tilde{S}_m^T(q) - D_m^T(q)\right] D_0(p-q) \\ &= 4e^2m^2 \int \frac{d^4q}{(2\pi)^4}\, 2\pi\delta(q^2-m^2) \frac{n_F(q_0) + n_B(q_0)}{(p-q)^2 + i\epsilon}. \end{aligned} \tag{4.16}$$

For small temperatures, this integral gives

$$\Delta(p) = -4me^2 \int \frac{d^3q}{(2\pi)^3} \frac{e^{-E_q/T}}{(\vec{p}-\vec{q})^2}.$$

In the limit $|\vec{p}| \to 0$, we obtain

$$\Delta(m,0) = -4me^2 e^{-m/T} \frac{\sqrt{mT}}{(2\pi)^{3/2}},$$

while Δ is decaying like $1/|\vec{p}|^2$ for $|\vec{p}| \to \infty$,

$$\Delta(E_p, \vec{p}) \to -\frac{4me^2}{|\vec{p}|^2} e^{-m/T} \left(\frac{mT}{2\pi}\right)^{3/2}.$$

Thus, in contrast to the Wess-Zumino model, the breakdown of supersymmetry does not only lead to a displacement of boson and fermion masses, but moreover, the difference $\Delta(p)$ between the fermionic and bosonic self energies shows a dispersive character. This must carefully be taken into account in all further calculations.

4.3.2 Photino and photon

The photino self energy is not difficult to calculate since there is only the diagram drawn in figure 4.5, where of course both bosons and charge conjugated states contribute. It reads

$$\begin{aligned}\Pi_\Lambda(p) &= \tfrac{1}{2} \operatorname{tr}\left[\not{p}\, \Sigma_\Lambda(p)\right] \\ &= -4ie^2 \int \frac{d^4q}{(2\pi)^4}\, 2pq \left(\tilde{S}_m^T(q) D_m(p-q) + D_m(q) D_m^T(p-q)\right).\end{aligned}$$

Again, we are interested in the behavior near the poles, $p^2 = 0$, where the equality of boson and fermion masses allows a simplification by making use of the relations

$$D_m(q)\Big|_{(p-q)^2=m^2} = \frac{i}{2pq + i\epsilon}, \qquad D_m(p-q)\Big|_{q^2=m^2} = \frac{i}{-2pq + i\epsilon},$$

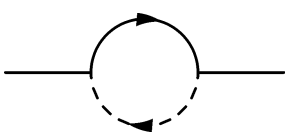


Figure 4.5: One-loop photino self energy

leaving

$$\begin{aligned}\Pi_\Lambda(p) &= -4e^2 \int \frac{d^4q}{(2\pi)^4} \left(\tilde{S}_m^T(q) - D_m^T(q)\right) \\ &= \begin{cases} 8e^2 \frac{e^{-m/T}}{m} \left(\frac{mT}{2\pi}\right)^{3/2} & T \ll m, \\ 2\alpha\pi T^2 & T \gg m. \end{cases}\end{aligned} \tag{4.17}$$

Thus, also the photino acquires a tiny mass. Its exact value will however be of no importance in the following calculations.

The photon self energy can be calculated analogously by evaluating the diagrams drawn in figure 4.6. The resulting effective photon mass is gauge invariant and of the same order of magnitude as that for the photino. We will skip the calculation since the explicit result is not needed in the following, but we note that, as a consequence of the breakdown of supersymmetry, there appears a small mass splitting also between the members of the vector supermultiplet.

4.4 The phonino

As we have argued in section 4.2, we expect a massless bound state, the phonino, to show up in the composite operator

$$\chi = -iC^\dagger\Psi + iC\Psi^c - \gamma^5 B^\dagger\Psi + \gamma^5 B\Psi^c, \tag{4.18}$$

that is, we expect a zero-momentum pole in the amplitude

$$\langle T\chi(x)\overline{\chi}(y)\rangle.$$

We would like to show the existence of such a pole by setting up and solving the Bethe-Salpeter equations for this mode. For this purpose, we split the points and the contributions from both bosons by defining

$$\begin{aligned}\chi_C(x_F, x_B) &= -iC^\dagger(x_B)\Psi(x_F) + iC(x_B)\Psi^c(x_F), \\ \chi_B(x_F, x_B) &= -\gamma^5 B^\dagger(x_B)\Psi(x_F) + \gamma^5 B(x_B)\Psi^c(x_F),\end{aligned}$$

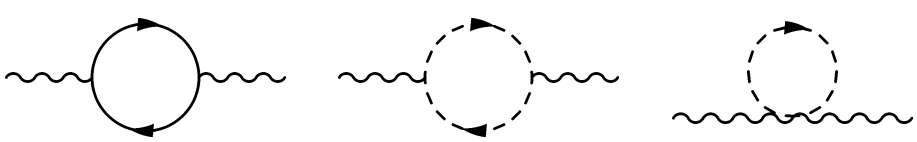


Figure 4.6: Photon self energy diagrams

such that

$$\chi(x) = \chi_B(x,x) + \chi_C(x,x).$$

In the following, we will investigate the amplitudes

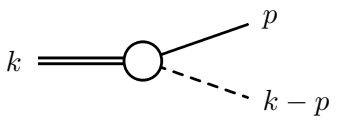
$$\Gamma_C(x_F, x_B, y) = \langle T\chi_C(x_F, x_B)\overline{\chi}(y)\rangle,$$
$$\Gamma_B(x_F, x_B, y) = i\gamma^5\langle T\chi_B(x_F, x_B)\overline{\chi}(y)\rangle i\gamma^5,$$

where on tree level we simply have $\Gamma_C = \Gamma_B$.

Transforming into momentum space, we introduce the amputated vertex functions $\Gamma_B(k,p)$ and $\Gamma_C(k,p)$ such that

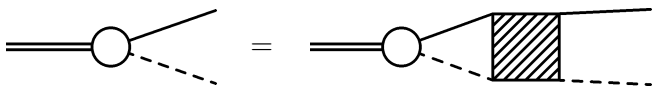
$$\Gamma_i(x_F, x_B, y) = \int\frac{d^4p}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}\, e^{-ip(x_F-x_B)+ik(y-x_B)}\mathcal{S}(p)\mathcal{D}(k-p)\Gamma_i(k,p). \quad (4.19)$$

It is illustrative to switch to a diagrammatic language and write the vertex function as

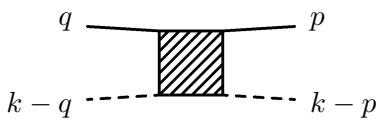


$= \Gamma_i(k,p).$

For the existence of a bound state, it must satisfy the homogeneous Bethe-Salpeter equation which pictorially reads



where the kernel



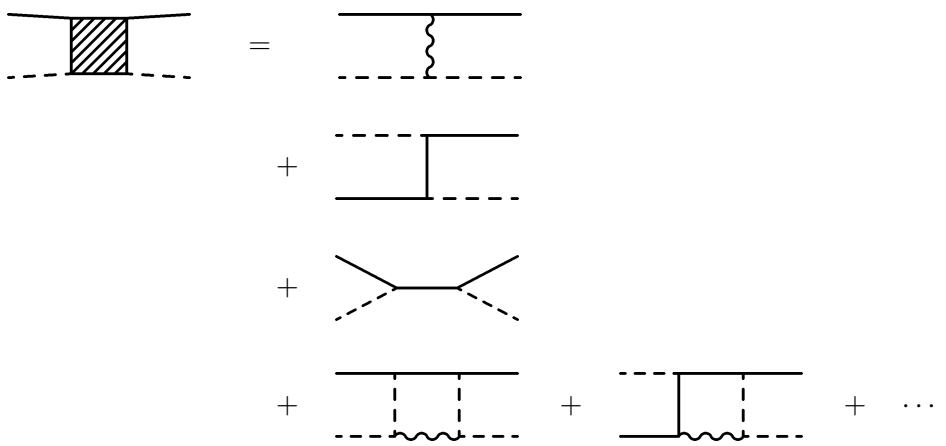
$= K_{ij}(k,p,q)$

is the sum of all two-particle irreducible sub-diagrams that contribute to the scattering process.

In formulae, the Bethe-Salpeter equations read

$$\Gamma_i(k,p) = \int \frac{d^4q}{(2\pi)^4}\, K_{ij}(k,p,q)\mathcal{S}(q)\mathcal{D}(k-q)\Gamma_j(k,q). \tag{4.20}$$

The structure of the diagrams contributing significantly to the kernels is very similar to those we encountered for the case of the Wess-Zumino model. In order to give a thermal contribution, the intermediate lines must be nearly on-shell, that is, $p^2 \simeq q^2 \simeq m^2 + \mathcal{O}(\Delta(p))$. Then, for small momenta $k \ll \Delta(0)/m$, a sizeable contribution arises if parallel lines have a nearly degenerate mass such that the small mass splitting between the members of a supermultiplet appears in the denominator of the scattering amplitude. Hence, the following ladder diagrams come into question as relevant contributions to the kernels.



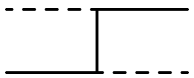
We have not marked the electron lines by arrows, since there are contributions from both charge conjugated states. This should not lead to confusion if one remembers that the incoming and outgoing fermions are always electrons, while the intermediate lines denote photinos.

Let us discuss these diagrams one by one.

Case 1:

The contribution from this diagram to the kernel of the Bethe-Salpeter equation can be written down straightforwardly since the photon coupling is diagonal in the contributions of both bosons. We find

$$K^{(1)}_{ij}(k,p,q) = -e^2\gamma^\mu \mathcal{D}_{\mu\nu}(p-q)(p+q-2k)^\nu \delta_{ij}. \tag{4.21}$$

Case 2: 

Here we have to take more care, since at first view the photino coupling is not diagonal. One can check, however, that the two bosons are not transformed into each other, since in this case the Dirac algebra of the fermion propagators would give

$$(\not{p} + m)i\gamma^5(\not{p} + \not{q})(\not{q} + m) = 0 \quad \text{for} \quad p^2 = q^2 = m^2.$$

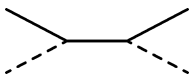
Apart from this, the contribution is equal for both bosons since

$$i\gamma^5 S_\Lambda(p+q)i\gamma^5 = S_\Lambda(p+q).$$

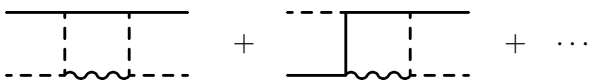
Furthermore, one has to take care of the relative sign of the charge conjugated contributions to the state χ, as the photino exchange transforms the electron into its charge conjugated state.

To make a long story short, the contribution to the kernel turns out to be

$$K^{(2)}_{ij}(k,p,q) = e^2 \mathcal{S}_\Lambda(p+q-k)\delta_{ij}. \tag{4.22}$$

Case 3: 

This diagram does not give a contribution, since the linear combination that constitutes the state χ does not couple to the photino, as was shown in section 4.2.

Case 4: 

Also the photon-photino intermediate state does not contribute to the Goldstone mode. This can be seen as follows:

First, we have to introduce a function $\Gamma^\mu(k,p)$ describing the coupling of the photon-photino state to the Goldstone mode. It is connected with the functions $\Gamma_i(k,p)$ by its own Bethe-Salpeter equation which gets contributions only from the electron and scalar electron exchange like in the left half of the diagrams drawn above. By explicit calculation, one can easily show that the sum of both contributions results in

$$\Gamma^\mu(k,p) \sim p^\mu.$$

The contribution from the photon-photino state to the Bethe-Salpeter equations for the Goldstone mode is then given by

$$\delta\Gamma_i(k,p) = \int \frac{d^4q}{(2\pi)^4}\, K_i^\mu(k,p,q)\mathcal{S}_\Lambda(q)\mathcal{D}_{\mu\nu}(k-q)\Gamma^\nu(k,q),$$

where one finds that also this kernel is of the form

$$K_i^\mu(k,p,q) \sim q^\mu.$$

The whole integrand is then proportional to

$$q^\mu \mathcal{D}_{\mu\nu}(k-q)q^\nu \mathcal{S}_\Lambda(q) \sim q^2$$

which vanishes (up to terms of higher order in α) since the thermal propagators have their support for momenta $q^2 \simeq 0$.

Thus, there is no relevant contribution from these diagrams.

Altogether, the full Bethe-Salpeter equation reads

$$\Gamma_i(k,p) = -e^2 \int \frac{d^4q}{(2\pi)^4} \Big[\gamma^\mu \mathcal{D}_{\mu\nu}(p-q)(p+q-2k)^\nu - \mathcal{S}_\Lambda(p+q-k)\Big]\cdot \\ \cdot \mathcal{D}(k-q)\mathcal{S}(q)\Gamma_i(k,q). \quad (4.23)$$

Since it is the same for both bosons, we deduce that $\Gamma_B = \Gamma_C$ even for the full vertex so that we can drop the index henceforth.

We are interested in solutions of the Bethe-Salpeter equation for small momenta k. To this end, we only keep corrections linear in k to terms that are of order α as well, neglecting higher order corrections. First we note that, in this approximation, the product $\mathcal{D}(k-q)\mathcal{S}(q)$ can be simplified as follows,

$$\begin{aligned}
\mathcal{D}(k-q)\mathcal{S}(q) &= (\not{q}+m)\left(\frac{i}{q^2-2qk-m^2-\Pi_B(q)}\tilde{\mathcal{S}}^T(q)\right. \\
&\qquad\qquad \left. + \mathcal{D}^T(k-q)\frac{i}{q^2-m^2-\Pi_F(q)}\right) \\
&= (\not{q}+m)\left(\frac{i}{\Pi_F(q)-2qk-\Pi_B(q)}\tilde{\mathcal{S}}^T(q)\right. \\
&\qquad\qquad \left. + \mathcal{D}^T(q)\frac{i}{2qk+\Pi_B(q)-\Pi_F(q)}\right) \\
&= (\not{q}+m)\,\frac{i}{\Delta(q)-2qk}\left(\tilde{S}_m^T(q) - D_m^T(q)\right),
\end{aligned}$$

since the thermal parts of the full propagators, $\tilde{\mathcal{S}}^T(q)$ and $\mathcal{D}^T(k-q)$, have their support for momenta $q^2 = m^2+\Pi_F(q)$ and $(k-q)^2 = m^2+\Pi_B(q)$, respectively.

In the Feynman gauge, the kernel can be written as

$$\gamma^\mu \mathcal{D}_{\mu\nu}(p-q)(p+q)^\nu - \mathcal{S}_\Lambda(p+q) \\ = -(\not{p}+\not{q})D_0(p-q) - (\not{p}+\not{q})D_0(p+q).$$

After these replacements, the Bethe-Salpeter equation reads

$$\Gamma(k,p) = ie^2 \int \frac{d^4q}{(2\pi)^4}\,(\not{p}+\not{q})\left[D_0(p-q)+D_0(p+q)\right]\cdot \\ \cdot \frac{\tilde{S}_m^T(q) - D_m^T(q)}{\Delta(q)-2qk}\,(\not{q}+m)\,\Gamma(k,q). \quad (4.24)$$

Though we have only worked in the Feynman gauge up to now, it should be mentioned that this form of the Bethe-Salpeter equation can also be obtained in the Coulomb gauge.

In order to obtain a solution, it proved useful already in the Wess-Zumino model to introduce the function

$$\tilde{\Gamma}(k,p) = \frac{\not{p}+m}{\Delta(p)-2pk}\Gamma(k,p)$$

which must then satisfy

$$\tilde{\Gamma}(k,p) = ie^2 \frac{\not{p}+m}{\Delta(p)-2pk}\int \frac{d^4q}{(2\pi)^4}\,2m\left[D_0(p-q)+D_0(p+q)\right]\cdot \\ \cdot \left(\tilde{S}_m^T(q) - D_m^T(q)\right)\tilde{\Gamma}(k,q).$$

One immediately finds that its matrix structure is given by

$$\tilde{\Gamma}(k,p) = (\not{p}+m)V(k,p).$$

In the limit of vanishing momentum $k \to 0$, the remaining equation for $V(0,p)$ reads

$$V(0,p) = \Delta(p)^{-1}\,2ie^2m^2 \int \frac{d^4q}{(2\pi)^4}\left[D_0(p-q)+D_0(p+q)\right]\cdot \\ \cdot \left(\tilde{S}_m^T(q) - D_m^T(q)\right)V(0,q). \quad (4.25)$$

Comparing with (4.16), we observe that $V(0,p) = const.$ is actually a solution of this equation, since

$$2ie^2m^2 \int \frac{d^4q}{(2\pi)^4}\left[D_0(p-q)+D_0(p+q)\right]\left(\tilde{S}_m^T(q) - D_m^T(q)\right) = \Delta(p).$$

Thus, we have shown that a nontrivial solution of the Bethe-Salpeter equation for vanishing momentum indeed exists, which proves that there

is a zero-momentum pole in the amplitude $\langle T\chi(x)\overline{\chi}(y)\rangle$ corresponding to a massless bound state of electron and scalar electron.

What remains to be done is to show that this pole really corresponds to a particle state with the expected dispersion relation, the phonino.

To this end, it will be useful that we have calculated the momentum dependence of the vertex function. In terms of the vertex function we started with, our solutions reads

$$\Gamma(k,p) = \Delta(p)\cdot V(k), \tag{4.26}$$

where the dependence on the small momentum k encoded in the function V is still unknown.

4.5 Ward-Takahashi identities

We can now use the result obtained from the solution of the Bethe-Salpeter equations to verify the Ward-Takahashi identity for the Goldstone mode χ. As we derived earlier, see (4.6), it reads

$$\partial^x_\mu \langle T J^\mu(x)\overline{\chi}(y)\rangle = \delta^{(4)}(x-y)\cdot \begin{cases} -8ie^{-\frac{m}{T}}\left(\frac{Tm}{2\pi}\right)^{3/2} & T\ll m, \\ -\frac{i}{2}mT^2 & T\gg m. \end{cases} \tag{4.27}$$

In section 4.2, we verified this identity for the free theory. In the full theory, however, we have found that electron and scalar electron obtain a mass difference by the interaction with the heat bath. Hence, the same reasoning can no longer be applied when we work with the resummed propagators. Instead, in order to satisfy the Ward-Takahashi identity, there must be a contribution from the massless bound state.

Let us calculate the left hand side of (4.27). We can rewrite the contribution from the chiral superfields to the supercurrent as

$$J^\mu_{ch}(x) = i\lim_{x_B\to x}\left((i\partial\!\!\!/_{x_B} - m)\gamma^\mu\chi_C(x_B,x) + i\gamma^5(i\partial\!\!\!/_{x_B} - m)\gamma^\mu i\gamma^5\chi_B(x_B,x)\right).$$

Remember also that we had introduced the function $\Gamma(k,q)$ as the amputated correlation function,

$$\begin{aligned}&\langle T\chi_C(x_F,x_B)\overline{\chi}(y)\rangle \\ &\qquad = \int\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\,e^{-ik(x_B-y)}e^{-iq(x_F-x_B)}\mathcal{S}(q)\mathcal{D}(k-q)\Gamma(k,q),\end{aligned}$$

and similarly for the contribution from the B boson. We can then write the Fourier transformed left hand side of (4.27), in a linear approximation in k,

as follows,

$$\begin{aligned}\int d^4x\, e^{ik(x-y)} LHS \\ &= \int \frac{d^4q}{(2\pi)^4} \Big[(\not{k} - \not{q} - m) \not{k}\, \mathcal{S}(q)\mathcal{D}(k-q)\Gamma(k,q) \Big] - i\gamma^5 \Big[\cdots \Big] i\gamma^5 \\ &= 2i \int \frac{d^4q}{(2\pi)^4}\, 2\pi\delta(q^2 - m^2) \big(n_B(q_0) + n_F(q_0) \big)\, 2kq\, \frac{\not{q} + m}{\Delta(q)} \Gamma(k,q).\end{aligned}$$

With our knowledge of the q-dependence of $\Gamma(k,q)$,

$$\Gamma(k,q) = \Delta(q) V(k),$$

we can calculate, first for low temperatures $T \ll m$,

$$\begin{aligned}\int d^4x\, e^{ik(x-y)} LHS &= 4i \int \frac{d^4q}{(2\pi)^4}\, 2\pi\delta(q^2 - m^2) \big(n_B(q_0) + n_F(q_0) \big)\, kq\, \not{q} V(k) \\ &= 8im\, e^{-m/T} \left(\frac{mT}{2\pi} \right)^{3/2} \left(k_0\gamma^0 + \frac{T}{m} \vec{k}\vec{\gamma} \right) V(k).\end{aligned}$$

Thus, in order to fulfill the Ward-Takahashi identity (4.27), the function $V(k)$ must read

$$V(k)^{-1} = -m \left(k_0\gamma^0 + \frac{T}{m} \vec{k}\vec{\gamma} \right).$$

With this result, we can finally calculate the propagator of the Goldstone mode,

$$\begin{aligned}\int d^4x\, e^{ik(x-y)} \langle \chi(x) \overline{\chi}(y) \rangle &= 4 \int \frac{d^4q}{(2\pi)^4}\, \mathcal{S}(q)\mathcal{D}(k-q)\Gamma(k,q) \\ &= e^{-m/T} \left(\frac{mT}{2\pi} \right)^{3/2} \frac{2i}{m \left(k_0\gamma^0 + \frac{T}{m} \vec{k}\vec{\gamma} \right)}. \qquad (4.28)\end{aligned}$$

This expression is very similar to that derived for the full fermion propagator in the Wess-Zumino model, equation (3.32), and shows that the Goldstone mode χ describes a phonino moving with the velocity

$$v = \frac{T}{m}. \qquad (4.29)$$

Because of the linear approximation in k, this result is valid in the momentum range $|\vec{k}| \ll \Delta(m,0)/\sqrt{Tm}$.

For high temperatures $T \gg m$, a similar calculation gives

$$V(k)^{-1} = -\frac{\pi^2 T^2}{2m} \left(k_0\gamma^0 + \frac{1}{3} \vec{k}\vec{\gamma} \right),$$

and

$$\int d^4x\, e^{ik(x-y)} \langle \chi(x) \overline{\chi}(y) \rangle = \frac{im^2}{4\pi^2 \left(k_0 \gamma^0 + \frac{1}{3} \vec{k} \vec{\gamma} \right)}. \tag{4.30}$$

In this case, the phonino velocity is

$$v = \frac{1}{3}. \tag{4.31}$$

Thus, we have shown that the massless bound state in the electron-scalar electron channel has the properties of the phonino.

One point that needs some clarification is the following. According to the general derivation in chapter 2, the phonino velocity should be given by

$$v = \frac{p}{\rho}.$$

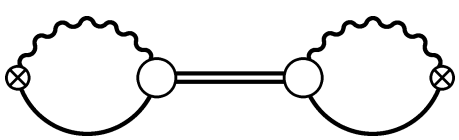
In supersymmetric quantum electrodynamics, also in the 'non-relativistic' limit $T \ll m$, the massless photons and photinos form a relativistic plasma and therefore dominate pressure and energy density, leading to a phonino velocity $v = \frac{1}{3}$. What we found is rather the non-relativistic velocity $v = \frac{T}{m}$ that is expected if all particles are massive.

The reason for this is the following. Up to now, we have only considered the coupling of the mode χ to the chiral superfields' contribution to the supercurrent. As a consequence, the Goldstone mode so far does not 'see' the energy-momentum tensor of the massless particles. If we included the coupling to the vector superfield's contribution to the supercurrent as well,

$$\partial_\mu^x \langle T J_{vec}^\mu(x) \overline{\chi}(y) \rangle,$$

we should obtain the right relativistic dispersion law.

According to our discussion in section 4.2, the phonino pole should also appear in the two-point function of the supercurrent which, at low temperature, is dominated by the contribution from the vector superfield. Since there is no direct interaction between photon und photino, this pole must originate in the interaction with the chiral superfields, too. The main contribution should therefore come from diagrams of the general structure



with an intermediate phonino line.

Altogether, we find that supersymmetric quantum electrodynamics behaves quite similar to the Wess-Zumino model. Supersymmetry is broken spontaneously by the thermal bath. Though no fundamental scalar field acquires a thermal expectation value, we find a mass splitting between the members of the three supermultiplets. The phonino appears as a bound state, and the way it contributes to the supersymmetric Ward-Takahashi identities shows that it must be interpreted as the Goldstone fermion.

Chapter 5

Some generalizations

5.1 Behavior of more general models

Our results for the Wess-Zumino model and supersymmetric quantum electrodynamics allow an immediate generalization to a wider class of models. Let us compare the three possible order parameters that we encountered in our investigation of the Ward-Takahashi identities for the Wess-Zumino model and examine which of them is good for a generalization.

First, resembling the proof of the Goldstone theorem in the vacuum, we have studied the Ward-Takahashi identity for the fundamental fermion which relates its propagator to the expectation value of its scalar superpartner,

$$i\int d^4x\,\partial_\mu\langle TJ^\mu(x)\overline{\Psi}(y)\rangle = m\langle A\rangle \sim gT^2. \tag{5.1}$$

For simplicity, we now restrict ourselves to the high-temperature case. Obviously, the finite-temperature contribution to the right hand side appears only as a higher order effect due to the interaction. Furthermore, it is different from zero only in certain models. In supersymmetric QED, for example, we saw that none of the scalar fields develops a nontrivial thermal expectation value so that no fundamental fermion couples to the supercurrent and could thereby be the Goldstone mode. Hence, thermal expectation values of fundamental fields are not suited to serve as a general order parameter for the breakdown of supersymmetry.

Secondly, we have investigated the composite boson-fermion operators with a Ward-Takahashi identity of the form

$$i\int d^4x\,\partial_\mu\langle TJ^\mu(x)A\overline{\Psi}(y)\rangle = m\langle AA\rangle - \langle\Psi\overline{\Psi}\rangle \sim mT^2 \tag{5.2}$$

which is nontrivial in the case of a non-vanishing mass m. As a consequence, we found for both the Wess-Zumino model and supersymmetric QED that

the Goldstone mode is determined by the field combination appearing in the mass term of the superpotential, see equations (3.44) and (4.8). This will generally be the case in a model with only massive chiral superfields. But what happens if there is no such mass term? As an example, consider the massless Wess-Zumino model. Here, the first identity (5.1) is trivial since chiral symmetry prevents the scalar field from getting a thermal expectation value. But also the right hand side of the second identity (5.2) becomes nontrivial only at a higher order, since a thermal mass is generated by the interaction with the heat bath. However, according to the general arguments given in chapter 2, we would rather expect supersymmetry to be broken at tree level. Thus, also this identity is of limited use for the general case.

Now consider the third Ward-Takahashi identity involving the supercurrent,

$$\int d^4x\, \partial_\mu \langle T J^\mu(x) \overline{J}^\nu(y) \rangle = 2\, \langle T^{\mu\nu} \rangle \gamma_\mu \sim T^4. \tag{5.3}$$

In contrast to the first two identities that are satisfied in a very model-dependent way, this one is universal. It shows that supersymmetry is broken as long as the energy-momentum tensor has a non-vanishing expectation value, which is the case for any supersymmetric model at finite temperature. At high temperature, each boson-fermion pair contributes equally to the energy density such that the right hand side basically counts the number of supermultiplets in the theory. The particle masses and interactions affect it only as a small correction, as long as the interaction is weak. At lower temperatures, not much higher than the masses, particles of different mass contribute of course differently. This is what we observed in our investigation of supersymmetric quantum electrodynamics at low temperatures where the main contribution came from the massless particles in the thermal bath.

Because of the non-vanishing right hand side of the Ward-Takahashi identity, a zero-momentum Goldstone pole must appear in the two-point function of the supercurrent. It should in principle be found as a bound state in all operators that contribute to the supercurrent, where the residues have the same ratio as the corresponding contributions to the energy-momentum tensor. In the picture of supersymmetric sound, this can be made plausible. If the plasma is in thermodynamic equilibrium and oscillates as a whole, each component should carry the ratio of the excitation that corresponds to its share of the total energy density.

It is interesting that the existing works on the subject have investigated the first two Ward-Takahashi identities in depth [26, 31] and come to very model dependent conclusions about the nature of the Goldstone fermion. They did however not clearly see that the breakdown of supersymmetry at finite temperature is a direct consequence of the breakdown of Lorentz symmetry and as universal, as it is stated in equation (5.3). In [9], the general nature of supersymmetry breakdown was realized, but their hydrodynamic

explanation of the phonino remained somewhat obscure from a quantum field theoretical perspective. Note that similar arguments have been given in [38] where the authors did however not enter into explicit calculations.

5.2 Lorentz invariance and supersymmetry

What is known about the behavior of Lorentz symmetry at finite temperature? As was rigorously shown in [41], it is inevitably broken in any thermal state, which is of course no surprise since the heat bath fixes a rest frame. In the same work, it was also argued that a zero-energy Goldstone spectrum should exist that however does not necessarily carry a one-particle structure. For the example of a scalar field, it was shown that the operator $\phi\phi$ has as zero-momentum spectrum consistent with the Goldstone theorem. The same conclusions were shared by the authors of [42] and later [43] who could show that the Ward-Takahashi identity of Lorentz symmetry,

$$\begin{aligned}\partial^\mu \langle T M_{\mu\nu\lambda}(x)\phi(y)\phi(z)\rangle = &-i\delta^{(4)}(x-y)(y_\nu\partial^y_\lambda - y_\lambda\partial^y_\nu)\langle\phi(y)\phi(z)\rangle \\ &- i\delta^{(4)}(x-z)(z_\nu\partial^z_\lambda - z_\lambda\partial^z_\nu)\langle\phi(y)\phi(z)\rangle,\end{aligned} \tag{5.4}$$

where

$$M_{\mu\nu\lambda} = x_\nu T_{\mu\lambda} - x_\lambda T_{\mu\nu} \tag{5.5}$$

is the conserved current that generates Lorentz transformations, are satisfied in the Goldstone realization with the appearance of a zero-momentum singularity in the propagator of the bilinear operator $\phi\phi$.

After what we have seen for the case of supersymmetry breakdown, we immediately realize the analogy. Compared to (5.2), the supercurrent J^μ is replaced by the Lorentz current $M_{\mu\nu\lambda}$, and the operator ϕ^2 replaces the fermionic component field of the same superfield Φ^2. Also this Ward-Takahashi identity can be traced back to a more general one involving the energy-momentum tensor,

$$\begin{aligned}i\int d^4x\, \partial^\mu \langle T M_{\mu\nu\lambda}(x) T_{\rho\sigma}(y)\rangle& \\ = -g_{\nu\rho}\langle T_{\lambda\sigma}\rangle + g_{\lambda\rho}\langle T_{\nu\sigma}\rangle &- g_{\nu\sigma}\langle T_{\lambda\rho}\rangle + g_{\lambda\sigma}\langle T_{\nu\rho}\rangle,\end{aligned} \tag{5.6}$$

where on the right hand side we find a combination of the components of the energy-momentum tensor. The same is however true for the supercurrent exposed to a supersymmetry transformation. Hence, if the energy-momentum tensor acquires a nontrivial expectation value, both of them must satisfy a nontrivial Ward-Takahashi identity, displaying that both symmetries are broken simultaneously. As a consequence, in the case of supersymmetry a Goldstone fermion must appear in the two-point function of the supercurrent, while for Lorentz symmetry a Goldstone boson appears in the two-point function of the energy-momentum tensor.

Interestingly, the authors of the aforementioned more formal works denied the correspondence of the zero-momentum Goldstone mode to a propagating particle, thereby ignoring that this correspondence has actually been known for a long time. The long-range correlations of the energy-momentum tensor resulting from the breakdown of Lorentz invariance are nothing but sound waves that show up as collective excitations of the plasma.

This confusion has a same simple reason, as we discussed for the breakdown of supersymmetry. In the free theory, there is indeed only an isolated zero-momentum pole since sound waves can propagate only for wavelengths long compared to the mean free path which is infinite in the free theory. Hence, sound and supersymmetric sound can appear only as a nonperturbative phenomenon in the interacting theory. In fact, the treatment of sound and supersymmetric sound already goes slightly beyond our framework of equilibrium field theory since it requires a local deviation from thermodynamic equilibrium. Therefore it might very well be that our results escape a rigorous treatment like that pursued in [6].

Apart from the common reason for the appearance of phonons and phoninos, the origin being nothing but the supercurrent multiplet, the notion of a 'supermultiplet of Goldstone modes' would be of little use. Since it is the breakdown of Lorentz and supersymmetry that leads to their existence, phonons and phoninos may behave quite differently, in particular they have different dispersion laws as we have seen in section 2.3.

5.3 Outlook

Though it has by now emerged a consistent picture of the Wess-Zumino model in a thermal background that can be generalized to other models, we can by no means claim that we have fully understood the behavior of supersymmetry at finite temperature, mainly because the models we have studied so far are far from realistic. First of all, it is obvious that thermal effects alone cannot account for the absence of supersymmetry we observe in nature. At moderate temperatures, the mass splitting (2.1) between the members of a supermultiplet is ridiculously small, and for all practical purposes supersymmetry appears unbroken. Therefore, it would be necessary to investigate the combination of thermal and other mechanisms of supersymmetry breaking.

A model where supersymmetry is broken spontaneously already at zero temperature is the classic O'Raifeartaigh model [39]. Here, at finite temperature, we have two competing Goldstone modes. The fundamental Goldstone fermion exists already at zero temperature. At low temperatures, we would expect the phonino to appear dominantly as a bound state of the massless particles, while at high temperature all three supermultiplets should contribute equally to the phonino. At a temperature $gT \simeq m$, the effective

potential will be dominated by thermal effects such that a nontrivial scalar expectation value is no longer generated and, only in this respect, supersymmetry is restored. The role of the Goldstone mode is then taken over by the phonino alone. Attempts to understand the behavior of this model have been undertaken in [33,40], but the explicit proof of the existence of the phonino and the investigation of its properties would hardly be feasible, just remember our lengthy calculation in chapter 3 for the much less complex Wess-Zumino model.

Interesting from the point of view of phenomenology would also be the case of soft supersymmetry breaking. Here, supersymmetry is not an exact symmetry of the theory, and there is no reason to expect a phonino at low temperatures. Nevertheless, for temperatures above the supersymmetry breaking scale, we would expect the soft breaking to become irrelevant compared to the thermal breaking such that some approximate Goldstone mode appears.

Another important generalization towards realistic models would be the investigation of nonabelian gauge theories. In contrast to the abelian model we have studied, the self-interaction of nonabelian gauge fields and their superpartners should lead to the existence of the phonino already in the pure gauge theory. In combination with matter fields, also the interplay between supersymmetry and other symmetries could lead to interesting effects. In this respect, it has been suggested that supersymmetry could trigger the breaking or restoration of global or gauge symmetries at high temperatures [44, 45].

Particularly interesting would also be the investigation of supergravity in a thermal background.

Conclusions

In this work, we have contributed to the understanding of supersymmetry breaking by a thermal background in two respects.

It could be shown that two different approaches to the subject existing in the literature fit very well together. For the case of the Wess-Zumino model, we have shown that the phonino, predicted as a rather peculiar collective phenomenon in [9], really must be interpreted as the Goldstone particle of spontaneously broken supersymmetry since it contributes to the Ward-Takahashi identities in just the right way that proves the spontaneous nature of the breaking. In earlier works, these identities have been studied [8, 26], but the correspondence of the Goldstone pole to a propagating particle has not been realized.

We have also investigated supersymmetric quantum electrodynamics as a simple model where the matter fields have an additional gauge interaction with a vector superfield. It has been shown how the various fields contribute to the Goldstone mode, and an explicit calculation proved the existence of the phonino as a bound state of electron and scalar electron, thereby showing that supersymmetry is broken in a similar way as in the Wess-Zumino model.

Besides these explicit results for two different models, we have tried to clarify the general nature of the thermal breakdown of supersymmetry. Though it has much in common with a spontaneous breaking, we have found considerable differences. As it was suggested in [6] where the term 'spontaneous collapse' of supersymmetry was introduced, one should view it as a qualitatively new mechanism of symmetry breaking.

We believe that the Goldstone fermion we have been dealing with in this work is very interesting in the framework of quantum field theory, but on the other hand, we could only speculate about phenomenological implications of our results. The models considered so far have been chosen in order to allow a clear-cut investigation of the thermal breakdown of supersymmetry, the disadvantage being their remoteness from reality. Phenomenologically acceptable models have of course a much more complicated structure, consider for example the minimal supersymmetric standard model as a spontaneously broken nonabelian gauge theory with soft supersymmetry breaking. Even though our results already allow us to understand the basic behaviour

of more general theories, realistic models must be the subject of a future investigation.

Respecting cosmology, a natural playground for thermally broken supersymmetry, we do not yet see an immediate application of our results. Nevertheless, many seemingly strange particles have found the attention of cosmologist, and phonons for example are believed to play a role in structure formation. One should also keep in mind that most interesting processes that have formed the universe like we observe it today have taken place under extreme conditions, where the system is far from equilibrium and important quantum numbers are broken. It would therefore be necessary to investigate supersymmetry in more general than thermal equilibrium states or at least under the influence of a non-vanishing chemical potential.

In any case, we believe that supersymmetry breaking at finite temperature is a very interesting topic that certainly deserves further attention.

Appendix A

Notations and Conventions

For reference, we would like to collect some of the conventions and useful formulae that we have used throughout this work.

First of all, spacetime is flat and our metric is

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1). \tag{A.1}$$

A.1 Weyl spinors

Building blocks of the spinor calculus are the Weyl spinors transforming under the fundamental representation of the Lorentz group $SL(2, \mathbb{C})$. They are two-component objects denoted by a lower undotted index, like ξ_α. From such an object we can build various other ones, first, we have the complex conjugate spinor

$$\overline{\xi}_{\dot{\alpha}} = \xi^*_\alpha. \tag{A.2}$$

Then, using the matrix

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{A.3}$$

we can define spinors with upper indices,

$$\xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta, \qquad \overline{\xi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\overline{\xi}_{\dot{\beta}}. \tag{A.4}$$

It is easy to show that these objects carry again representations of the Lorentz group,

$$\begin{aligned} &\xi_\alpha \to M_\alpha{}^\beta \xi_\beta, && \overline{\xi}_{\dot{\alpha}} \to (M^*)_{\dot{\alpha}}{}^{\dot{\beta}} \overline{\xi}_{\dot{\beta}}, \\ &\xi^\alpha \to (M^{-1^T})^\alpha{}_\beta \xi^\beta, && \overline{\xi}^{\dot{\alpha}} \to (M^{-1^\dagger})^{\dot{\alpha}}{}_{\dot{\beta}} \overline{\xi}^{\dot{\beta}}, \end{aligned} \qquad M \in SL(2, \mathbb{C}), \tag{A.5}$$

where ^{T} always means the transpose, * the complex conjugate and † the adjoint of a matrix.

Products of two-component spinors as defined by

$$\begin{aligned}\chi\xi &= \chi^{\alpha}\xi_{\alpha} = \epsilon^{\beta\alpha}\chi_{\alpha}\xi_{\beta} = \epsilon_{\alpha\beta}\chi^{\alpha}\xi^{\beta},\\ \overline{\chi}\overline{\xi} &= \overline{\chi}_{\dot{\alpha}}\overline{\xi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\overline{\chi}_{\dot{\alpha}}\overline{\xi}_{\dot{\beta}} = \epsilon_{\dot{\beta}\dot{\alpha}}\overline{\chi}^{\dot{\alpha}}\overline{\xi}^{\dot{\beta}}\end{aligned} \tag{A.6}$$

are then indeed Lorentz scalars. Here we have introduced the inverse ϵ-matrix

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.7}$$

Analyzing the transformation properties of the Pauli matrices,

$$\begin{aligned}\sigma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\\ \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\end{aligned} \tag{A.8}$$

one can deduce that they are well described by the index structure $\sigma^{\mu}{}_{\alpha\dot{\alpha}}$. Raising the spinor indices, one defines

$$\overline{\sigma}^{\mu\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^{\mu}{}_{\beta\dot{\beta}}, \tag{A.9}$$

which can be shown to equal

$$\overline{\sigma}^{\mu} = (\mathbb{1}, -\vec{\sigma}) = \sigma_{\mu}. \tag{A.10}$$

A.2 Dirac spinors

A four-component Dirac spinor is composed of two Weyl spinors,

$$\Psi = \begin{pmatrix} \xi_{\alpha} \\ \overline{\chi}^{\dot{\alpha}} \end{pmatrix}. \tag{A.11}$$

Its conjugate spinor reads

$$\overline{\Psi} = \Psi^{\dagger}\gamma^0 = (\chi^{\alpha}, \overline{\xi}_{\dot{\alpha}}), \tag{A.12}$$

where we have chosen the Dirac matrices in the Weyl representation,

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \overline{\sigma}^{\mu} & 0 \end{pmatrix}, \tag{A.13}$$

with the usual anticommutation relations

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu} \cdot \mathbb{1}. \tag{A.14}$$

We further define

$$\sigma^{\mu\nu} = \tfrac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right] \tag{A.15}$$

as well as

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \tag{A.16}$$

which fulfills

$$\{\gamma^5, \gamma^\mu\} = 0, \quad (\gamma^5)^2 = \mathbb{1}. \tag{A.17}$$

A.3 Majorana spinors

Charge conjugation is supposed to be an antilinear mapping on the space of Dirac spinors,

$$\Psi^c = C\overline{\Psi}^T, \tag{A.18}$$

which transforms a solution of the Dirac equation into one of the opposite charge. This condition requires C to obey

$$C\gamma^{\mu T}C^{-1} = -\gamma^\mu. \tag{A.19}$$

If charge conjugation is to be an anti-unitary operation with the property $(\Psi^c)^c = \Psi$, we must further have

$$C^\dagger = C^{-1}, \quad C^T = -C. \tag{A.20}$$

In the Weyl representation, this can be achieved by the matrix

$$C = -i\gamma^0\gamma^2 = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix}, \tag{A.21}$$

and we get the simple relation

$$\Psi^c = \begin{pmatrix} \xi_\alpha \\ \overline{\chi}^{\dot\alpha} \end{pmatrix}^c = \begin{pmatrix} \chi_\alpha \\ \overline{\xi}_{\dot\alpha} \end{pmatrix}. \tag{A.22}$$

Thus, a Dirac spinor built up of only one Weyl spinor,

$$\Psi = \begin{pmatrix} \xi_\alpha \\ \overline{\xi}^{\dot\alpha} \end{pmatrix}, \tag{A.23}$$

fulfills the Majorana condition

$$\Psi^c = \Psi \tag{A.24}$$

and is called a Majorana spinor.

Generalizing equation (A.19), we can define

$$\Gamma^c = C\Gamma^T C^{-1} \tag{A.25}$$

for any four-by-four matrix Γ. For the usual basis, we have

$$\Gamma^c = \begin{cases} \Gamma & \text{for } \Gamma = \mathbb{1},\ i\gamma^5,\ \gamma^\mu\gamma^5, \\ -\Gamma & \text{for } \Gamma = \gamma^\mu,\ \sigma^{\mu\nu}. \end{cases} \tag{A.26}$$

Applying these relations, together with the identity

$$\psi\sigma^{\mu}\overline{\chi} = -\overline{\chi}\overline{\sigma}^{\mu}\psi, \tag{A.27}$$

one can immediately show the following relations for a Majorana spinor Λ and a Dirac spinor Ψ,

$$\begin{aligned}
\overline{\Psi}\Lambda &= \overline{\Lambda}\Psi^{c}, \\
\overline{\Psi}\gamma^{\mu}\Lambda &= -\overline{\Lambda}\gamma^{\mu}\Psi^{c}, \\
\overline{\Psi}\sigma^{\mu\nu}\Lambda &= -\overline{\Lambda}\sigma^{\mu\nu}\Psi^{c}, \\
\overline{\Psi}\gamma^{\mu}\gamma^{5}\Lambda &= \overline{\Lambda}\gamma^{\mu}\gamma^{5}\Psi^{c}, \\
\overline{\Psi}\gamma^{5}\Lambda &= \overline{\Lambda}\gamma^{5}\Psi^{c}.
\end{aligned} \tag{A.28}$$

Appendix B

Some Integrals

Here, we give a number of integrals containing the high and low temperature limits of thermal distribution functions that occur repeatedly throughout this work.

$$\int \frac{d^3q}{(2\pi)^3} \frac{n_B(|\vec{q}|)}{|\vec{q}|} = \frac{T^2}{12} \tag{B.1}$$

$$\int \frac{d^3q}{(2\pi)^3} \frac{n_F(|\vec{q}|)}{|\vec{q}|} = \frac{T^2}{24} \tag{B.2}$$

$$\int \frac{d^3q}{(2\pi)^3} |\vec{q}|\, n_B(|\vec{q}|) = \frac{\pi^2 T^4}{30} \tag{B.3}$$

$$\int \frac{d^3q}{(2\pi)^3} |\vec{q}|\, n_F(|\vec{q}|) = \frac{7\pi^2 T^4}{240} \tag{B.4}$$

$$\int \frac{d^3q}{(2\pi)^3} \frac{1}{\vec{q}^2} e^{-\beta\frac{\vec{q}^2}{2m}} = \frac{\sqrt{mT}}{(2\pi)^{\frac{3}{2}}} \tag{B.5}$$

$$\int \frac{d^3q}{(2\pi)^3} e^{-\beta\frac{\vec{q}^2}{2m}} = \left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} \tag{B.6}$$

$$\int \frac{d^3q}{(2\pi)^3} \vec{q}^2 e^{-\beta\frac{\vec{q}^2}{2m}} = 3mT \left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} \tag{B.7}$$

$$\int \frac{d^3q}{(2\pi)^3} (\vec{p}\vec{q})(\vec{q}\vec{\gamma}) f(|\vec{q}|) = \vec{p}\vec{\gamma} \int \frac{d^3q}{(2\pi)^3} \frac{\vec{q}^2}{3} f(|\vec{q}|) \tag{B.8}$$

Bibliography

[1] R. Haag, J. T. Łopuszański and M. Sohnius, Nucl. Phys. B **88** (1975) 257.

[2] TESLA Technical Design Report, arXiv:hep-ph/0106314.

[3] D. A. Kirzhnits, JETP Lett. **15** (1972) 529; D. A. Kirzhnits and A. D. Linde, Phys. Lett. B **42** (1972) 471.

[4] L. Dolan and R. Jackiw, Phys. Rev. D **9** (1974) 3320.

[5] S. Weinberg, Phys. Rev. D **9** (1974) 3357.

[6] D. Buchholz and I. Ojima, Nucl. Phys. B **498** (1997) 228.

[7] D. Boyanovsky, Phys. Rev. D **29** (1984) 743.

[8] H. Matsumoto, M. Nakahara, Y. Nakano and H. Umezawa, Phys. Rev. D **29** (1984) 2838.

[9] V. V. Lebedev and A. V. Smilga, Nucl. Phys. B **318** (1989) 669.

[10] N. P. Landsman and C. G. van Weert, Phys. Rept. **145** (1987) 141.

[11] J. I. Kapusta, "Finite Temperature Field Theory," *Cambridge Univ. Pr. (1989).*

[12] R. Haag, N. M. Hugenholtz and M. Winnink, Commun. Math. Phys. **5** (1967) 215.

[13] H. Umezawa, H. Matsumoto and M. Tachiki, "Thermo Field Dynamics And Condensed States," *North-Holland, Amsterdam (1982).*

[14] M. Le Bellac, "Thermal Field Theory," *Cambridge Univ. Pr. (1996).*

[15] J. F. Donoghue, B. R. Holstein and R. W. Robinett, Annals Phys. **164** (1985) 233. [Erratum-ibid. **172** (1986) 483].

[16] H. P. Nilles, Phys. Rept. **110** (1984) 1.

[17] M. F. Sohnius, Phys. Rept. **128** (1985) 39.

[18] D. Bailin and A. Love, "Supersymmetric Gauge Field Theory And String Theory," *IOP, Bristol, UK (1994).*

[19] S. Weinberg, "The Quantum Theory Of Fields. Vol. 3: Supersymmetry," *Cambridge Univ. Pr. (2000).*

[20] J. Wess and J. Bagger, "Supersymmetry And Supergravity," *Princeton Univ. Pr. (1983).*

[21] P. C. West, "Introduction To Supersymmetry And Supergravity," *World Scientific, Singapore (1986).*

[22] C. Itzykson and J. B. Zuber, "Quantum Field Theory," *McGraw-Hill, New York (1980).*

[23] A. Das and M. Kaku, Phys. Rev. D **18** (1978) 4540.

[24] S. Midorikawa, Prog. Theor. Phys. **73** (1985) 1245.

[25] L. Girardello, M. T. Grisaru and P. Salomonson, Nucl. Phys. B **178** (1981) 331.

[26] H. Aoyama and D. Boyanovsky, Phys. Rev. D **30** (1984) 1356.

[27] L. Van Hove, Nucl. Phys. B **207** (1982) 15.

[28] J. Fuchs, Nucl. Phys. B **246** (1984) 279.

[29] D. Boyanovsky, Physica **15D** (1985) 152.

[30] H. Matsumoto, M. Nakahara, Y. Nakano and H. Umezawa, Physica **15D** (1985) 163.

[31] H. Matsumoto, M. Nakahara, H. Umezawa and N. Yamamoto, Phys. Rev. D **33** (1986) 2851.

[32] J. I. Kapusta, Phys. Lett. B **118** (1982) 343 [Erratum-ibid. B **122** (1983) 486].

[33] S. P. Chia, Phys. Rev. D **33** (1986) 2485.

[34] R. Gudmundsdottir and P. Salomonson, Nucl. Phys. B **285** (1987) 1.

[35] J. Wess and B. Zumino, Nucl. Phys. B **70** (1974) 39.

[36] J. Wess and B. Zumino, Nucl. Phys. B **78** (1974) 1.

[37] R. W. Robinett, Phys. Rev. D **31** (1985) 336.

[38] R. G. Leigh and R. Rattazzi, Phys. Lett. B **352** (1995) 20.

[39] L. O'Raifeartaigh, Nucl. Phys. B **96**, 331 (1975).

[40] Y. Leblanc and H. Umezawa, Phys. Rev. D **33** (1986) 2288.

[41] I. Ojima, Lett. Math. Phys. **11** (1986) 73.

[42] H. Aoyama, Phys. Lett. B **171** (1986) 420.

[43] H. Matsumoto, H. Umezawa, N. Yamamoto and N. J. Papastamatiou, Phys. Rev. D **34** (1986) 3217.

[44] A. Masiero, D. V. Nanopoulos and T. Yanagida, Phys. Lett. B **138** (1984) 91.

[45] M. L. Mangano, Phys. Lett. B **147** (1984) 307.

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