Covariant Quantisation of N=1, D=5 Supersymmetric Yang-Mills Theories in 4D Superfield Formalism

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Diploma Thesis

September 29, 2003

Abstract

We derive the Feynman rules for N=1, D=5 supersymmetric Yang-Mills theory expressed in 4D superfields. As an application we calculate the one-loop contribution to the vector superfield propagator and derive the β-function.

Contents

1 Introduction

The unification of the fundamental forces of nature is one of the great aims of high energy physics. After the establishment of the electroweak unification of Glashow, Salam and Weinberg and the development of Quantum Chromodynamics as a gauge theory describing the strong interaction much effort has been put to find a Grand Unified Theory (GUT), a quantum field gauge theory with a simple gauge group from which the electroweak and the strong interaction originate via spontanious symmetry breaking down to the Standard Model gauge group $G_{SM} \equiv SU(3)_C \times SU(2)_L \times U(1)_Y$.

The best discussed theory is the Georgi-Glashow model using the smallest possible G_{SM} embedding gauge group $SU(5)$. The model suffers from the fact that the the weak mixing angle θ_W disagrees with experiment. The theory is severely restricted as the baryon number violating decay of the proton has to be suppressed to agree with experiment. Furthermore, from neutrino oscillation experiments Super-Kamiokande, SNO and KamLAND there is strong evidence for right-handed neutrinos which cannot be directly included in the matter multiplets $\bar{5}$ and 10 of $SU(5)$. These problems can be overcome by choosing larger gauge groups. The best studied group is $SO(10)$, others are E_6, E_7 and E_8 which all contain $SU(5)$ as a subgroup. However there are two problems common to all GUTs mentioned above. The Higgs fields have to be chosen in a representation of the gauge group. Considering the Standard Model gauge groups which remain after symmetry breaking, the Higgs multiplet gets split into weakly charged Higgs fields as well as colour charged Higgs fields. The former induce spontaneous symmetry breaking for the electroweak sector and from unitarity bounds it follows that the electroweak Higgs has to have a mass of the order of $\lesssim 10^3$GeV while the latter have to have a mass of the order of 10^{15}GeV in order to suppress baryon number violating interactions yielding proton decay. The Higgs masses therefore have to be fine tuned over 12 orders of magnitude. This problem is referred to as the triplet doublet splitting problem.[1]

The second generic problem is also related to the small electroweak Higgs mass. The Higgs mass gets renormalised by radiative corrections in every loop order if there is no underlying symmetry forbidding the correction. The radiative correction is of the order of the cutoff scale which for a GUT is given by the the unification scale at $\sim 10^{16}$GeV, another fine tuning is required which adjusts the divergences to almost cancel from $\sim 10^{16}$GeV to $\sim 10^3$GeV *in every loop order*.

[1]The name originates from the the Higgs color triplet and weak doublet in the minimal $SU(5)$ theory.

This so-called hierachy problem can be solved in the framework of supersymmetric GUTs. Here, the supersymmetry between bosons and fermions yields cancelations between quadratic divergences appearing in the calculation of loop contributions. Another advantage compared to non-supersymmetric theories is that the unification of the couplings and therefore the Weinberg angle calculated from the theories in the framework of the minimal supersymmetric extension of the standard model (MSSM) agrees with experiment.

In supersymmetric theories, a new question arises. A priori particles and their superpartners have the same mass which is a direct consequence of the supersymmetry algebra. As this mass degeneracy is not observed, supersymmetry must be broken. It is desirable to break supersymmetry such that the non-renormalisation theorem still holds, canceling the quadratic divergences. Such a soft supersymmetry breaking can be achieved by the Fayet-Iliopoulos and/or the O'Raifeartaigh mechanism. However, these mechanisms are not able to reproduce a mass spectrum in which all superpartners are heavier than the standard model particles as a generalised mass sum rule applies to such spontaneously broken theories.

One way to realise supersymmetric theories in which the standard model particles are lighter than their superpartners is to break supersymmetry in a hidden sector and to mediate the breaking to the MSSM sector. In supersymmetry this can be achieved by a mediator which carries a charge and mediates the breaking via gauge interactions to the MSSM sector (gauge mediated supersymmetry breaking) [1]. In the local version of supersymmetry, supergravity, the gravity multiplet contains possible candidates for the mediating particle (gravity mediated supersymmetry breaking) [2, 3, 4].

Three years ago, it has been proposed to consider a five dimensional supersymmetric GUT compactified on the orbifold $S_1/Z_2 \times Z_2'$ [5, 6].[2] The compactification yields Kaluza-Klein towers of states of which only the zero modes are light while the masses of the excited states are proportional to the inverse of the compactification radius. By the discrete symmetries on the orbifold, the zero modes of the color charged Higgs can be projected out, making them heavy and solving the triplet doublet splitting problem. The breaking of the gauge group down to the standard model group can be achieved by projections of the discrete symmetries, as well, instead of using a Higgs mechanism

[2]Manifolds from which a discrete symmetry is divided out. The concept of compactifications on orbifolds is known for a long time from string theory as well as the possibility to solve the triplet doublet splitting by it [7]. However, it has not been considered in a purely field theoretic setup until recently.

at the GUT scale.[3]

Orbifold GUTs provide a natural possibility for mediated supersymmetry breaking, using one orbifold fixed point (brane) to locate the MSSM, a different one to break supersymmetry and using a bulk field to mediate the breaking (cf. [9] for an $SU(5)$ supersymmetric GUT with the gaugino as mediator). An extension of orbifold GUTs to supergravity theories is possible including mediated supersymmetry breaking (cf. [10]).

Due to these advantages orbifold GUTs attract a lot of attention since 2000. Beyond the studies on $SU(5)$ [6, 9, 11, 12, 13], $SO(10)$ theories in six dimensions compactified to four have been considered [14, 15, 16].

For $N{=}1$, $D{=}4$ supersymmetry the superfield formalism exists which has proven very useful. Superfields describe quantum fields and their superpartners as well as auxiliary fields (which can be introduced to linearise the supersymmetry transformations) as a single object inhabiting the supersymmetric structure. In superfield formalism, supersymmetry is manifest without the need of using infinitesimal supersymmetric field transformations. Therefore most non-renormalisation theorems for $N{=}1$ have been proven in superfields. Calculations in superfields are simpler than in components as every superfield graph contains several component field graphs which otherwise would have to be computed separately. For loop calculations only scalar divergent integrals occur as the superfields used for the supersymmetric extension of gauge bosons, fermions and Higgs bosons are scalar. Having a manifest superfield formulation for five (or higher) dimensional orbifold theories would mean a great simplification for calculation as well as a conceptually powerful tool.

Unfortunately there is no known way to generalise superfields to higher spacetime dimensions in full generality. However it is known how to formulate a ten dimensional supersymmetric Yang-Mills theory in 4D superfields [17], including the derivation of its Feynman rules. In the development of orbifold GUTs, a similar formulation for a 5D vector superfield has been developed [18], followed by the superfield formulation for vector and matter supermultiplets in $D{=}5$ to $D{=}10$ [19]. So far gauge covariance has not been manifest for higher dimensional super Yang-Mills theories. This inconvenience can be solved by introducing a covariant derivative in x_5 direction for 5D theories [20].

With this formalism of [20] it is possible to build up 5D supersymmetric GUTs using superfields only. However, as far as we know, superfields have not been used above the level of constructing the action

[3] For a complete classification of possible projections and low energy groups resulting from this cf. [8].

in five dimensional supersymmetric theories.[4] Loop corrections have been calculated in component fields after the compactified dimension has been integrated out.

The aim of this dipoma thesis is to formulate a five dimensional super Yang-Mills theory in 4D superfields, including a generalised Faddeev-Popov procedure to derive its Feynman rules and to calculate one-loop corrections in this formalism, thereby providing a basis for 5D supersymmetric orbifold GUT calculations in superfields. The orbifolding of the theory is not carried out in this thesis but will be adressed elsewhere in detail. However, following [10] it is straightforward to extend our results to 5D theories on orbifolds.

This thesis is structured as follows. In Chapter 2 we give a brief review of D=4, N=1 supersymmetry, following Wess and Bagger [21].[5] In Chapter 3 we quantise the N=1, D=4 supersymmetric Yang-Mills theory with chiral matter superfields. For this purpose we use the Faddeev-Popov method outlined in [22, 23] and derive the Feynman rules. Apart from the gauge fixing, the main new tools compared to the calculation in a non-supersymmetric theory are superspace integrals and, related to it, the use of D-algebra.[6] We use methods of [22, 24] to deal with this.[7]

As an application of the Feynman rules derived we calculate the β-function of the theory at one-loop level. The result for the theory without matter is well known (cf. [25]). This calculation therefore has to be regarded as a check of our Feynman rules as well as a guide to the five dimensional generalisation.

In Chapter 4 we use the formalism and the 5D action in superfields of [20] and quantise a N=1, D=5 super Yang-Mills theory, following the same procedure as in the 4D theory, i.e. using the Faddeev-Popov method, defining the generating functional, calculating the superfield propagators and reading off the vertices to derive the Feynman rules. As in 4D we derive the β-function of the five dimensional theory and compare it to the 4D result.

[4] The only exception we know is the supergravity one-loop calculation of [10] which has been published while this thesis has been written.

[5] Their conventions are summarised in Appendix A.1. We will use them throughout this thesis.

[6] algebraic relations of supersymmetry covariant derivatives in superspace

[7] A collection of D-algebra rules is given in Appendix B.2.

2 Supersymmetry and Superfields

In the development of Quantum Field Theory it was a very important issue to study symmetries for the S matrix by which all observables are determined. In 1967, after several no-go theorems were found, the Coleman-Mandula Theorem was published [26], saying that any symmetry group of the S-Matrix has to be locally isomorphic to a direct product of the Poincaré Group and an internal symmetry group under some very general assumptions for a Quantum Field Theory and the implicit assumption that all generators of the symmetry groups form a commutator algebra. The direct consequence of the Coleman-Mandula theorem is that there are no symmetries in a Quantum Field Theory which change the spin of any state.

However, relaxing the assumption of the commutator algebra by allowing anticommutators, as well, it has been shown that it is possible to construct so called graded Lie algebras consistent with the other assumptions of a Quantum Field Theory [27, 28]. Here, the symmetry group is not a direct product of internal and spacetime symmetries. The first example of such a theory was then constructed by Wess and Zumino [29]. In 1975 Haag, Łopuszański and Sohnius derived the most general graded Lie algebra [30] for a so-called supersymmetric Quantum Field Theory.

From that time on, supersymmetry has been studied intensively as a direct possible extension to the Standard Model (the Minimal Supersymmetric Model, MSSM), in its localised version (supergravity) which naturally includes gravity, in the context of Grand Unification as well as in cosmology[8] and as a necessary ingredient of String theory to include fermions (yielding Superstring theory).

In the following chapter we give a brief review on the supersymmetry algebra and N=1 supersymmmetry in superfields. We follow the conventions of Wess and Bagger [21] which are summarised in Appendix A.1.

2.1 The Supersymmetry Algebra

Following the Haag-Łopuszański-Sohnius theorem, the possible symmetries and therefore the algebra of the symmetry group generators of a non-trivial supersymmetric theory is severely restricted.

The most general supersymmetry algebra is

$$[P_m, P_n] = 0 \tag{2.1a}$$

[8]The lightest supersymmetric particle, if stable, provides a candidate for dark matter. Another field of studies are supersymmetric models for inflation.

$$[M_{mn}, M_{rs}] = \mathrm{i}(\eta_{nr} M_{ms} - \eta_{ns} M_{mr} - \eta_{mr} Mns + \eta_{ms} M_{nr}) \tag{2.1b}$$

$$[P_m, M_{rs}] = \mathrm{i}(\eta_{mr} P_s - \eta_{ms} P_r) \tag{2.1c}$$

$$[T^a, T^b] = \mathrm{i} f^{abc} T^c \tag{2.1d}$$

$$[P_m, T^a] = 0 = [M_{mn}, T^a] \tag{2.1e}$$

$$[P_m, \mathrm{Q}^L_\alpha] = 0 = [P_m, \bar{\mathrm{Q}}_{\dot{\alpha}L}] \tag{2.1f}$$

$$[\mathrm{Q}^L_\alpha, M_{mn}] = \frac{1}{2}(\sigma_{mn})_\alpha{}^\beta \mathrm{Q}^L_\beta \tag{2.1g}$$

$$[\bar{\mathrm{Q}}_{\dot{\alpha}L}, M_{mn}] = -\frac{1}{2}\bar{\mathrm{Q}}_{L\dot{\beta}}(\bar{\sigma}_{mn})^{\dot{\beta}}{}_{\dot{\alpha}} \tag{2.1h}$$

$$\{\mathrm{Q}^L_\alpha, \bar{\mathrm{Q}}_{\dot{\alpha}M}\} = 2\sigma^m_{\alpha\dot{\alpha}} P_m \delta^L_M \tag{2.1i}$$

$$\{\mathrm{Q}^L_\alpha, \mathrm{Q}^M_\beta\} = \epsilon_{\alpha\beta} X^{[LM]} \tag{2.1j}$$

$$\{\bar{\mathrm{Q}}_{\dot{\alpha}L}, \bar{\mathrm{Q}}_{\dot{\beta}M}\} = \epsilon_{\dot{\alpha}\dot{\beta}} X^\dagger_{[LM]} \tag{2.1k}$$

$$[X^{[LM]}, \text{anything}] = 0 \tag{2.1l}$$

$$[\mathrm{Q}^L_\alpha, T^a] = S^{aL}_M \mathrm{Q}^M_\alpha \tag{2.1m}$$

$$[\bar{\mathrm{Q}}_{\alpha L}, T^a] = -S^{*aM}_L \bar{\mathrm{Q}}_{\alpha M} \tag{2.1n}$$

$$X^{[L,M]} = a^{a[LM]} T^a \tag{2.1o}$$

where $m, n, \ldots$ are spacetime indices, $\alpha, \dot{\alpha}, \ldots$ are Weyl spinor indices (cf. Appendix A.1), $a, b, \ldots$ are indices of the internal symmetry group and $L, M, \ldots = 1 \ldots N$ label the supersymmetry generators. P_m are the momentum generators, M_{mn} are the generators of the Lorentz group and T^a are the generators of the internal symmetry group. The central charges $X^{[LM]}$ are antisymmetric in the supersymmetry indices L and M and therefore vanish for N=1.

In this thesis we only consider N=1 supersymmetry in four dimensions and N=1 supersymmetry in five dimensions which can be represented by a four dimensional N=2 theory without central charges (cf. Appendix A.2 and Chapter 4.1). Then, $S^{aL}_M \equiv 0 \equiv a^{a[LM]}$ and the only non vanishing algebraic relations are (2.1b), (2.1c), (2.1d), (2.1g), (2.1h) and (2.1i) where the last imposes supersymmetry i.e. a symmetry between bosons and fermions which has been absent in theories considering commutator relations as symmetry algebras only. Often (2.1i) is called the supersymmetry algebra on its own.

Irreducible representations of the supersymmetry algebra are constructed by defining a Clifford ground state and applying the (properly normalised) supersymmetry generators to it.

For the Lorentz algebra (2.1a) to (2.1c) there are two Casimir operators, P^2 and $\mathcal{W}^2$, where

$$\mathcal{W}^m \equiv -\frac{1}{2}\epsilon^{mnrs} P_n M_{rs} \tag{2.2}$$

which yield equal masses and spin for an irreducible Lorentz representation. In a non-trivial representation of the supersymmetry algebra, particles of different spin occur due to $[Q^L_\alpha, \mathcal{W}^2] \neq 0$. The degeneracy in mass is still valid as $[Q^L_\alpha, P^2] = 0$.

The degeneracy in mass is not observed in nature as no superpartners to the Standard Model particles have been observed and therefore must be heavier. It can be lifted by spontanious supersymmetry breaking by a Fayet-Iliopoulos and/or a O'Raifertaigh term. This is however not sufficient to lift the masses of all superpartners beyond those of Standard model particles as the generalised mass sumrule

$$\mathrm{STr}\, M^2 \equiv \sum_{\text{particles}} (-1)^{2J}(2J+1)\mathrm{Tr} M_J^2 = -2g(\mathrm{Tr} T^a) D^a \tag{2.3}$$

applies (cf. e.g. [31]), where the sum is over the real on-shell degrees of freedom of all particles, J is the spin of the particle, M_J is its mass and D^a is the vaccuum expectation value of the chiral auxiliary supermultiplet. Therefore, as stated in the introduction, more sophisticated mechanisms have to cure this problem.

2.2 Superspace

In the last section, the supersymmetry algebra has been presented as a graded Lie algebra including commutators as well as anticommutators. By introducing anticommuting spinorial 2-component parameters θ_α and $\bar{\theta}_{\dot{\alpha}}$ it can be rewritten as an algebra of commutators only. For N=1 four dimensional supersymmetry it is possible to generate the group of supersymmetry transformations from it in an analogous way as elements of Lie Groups are generated by its Lie Algebra. But in superspace, the group elements depend on spacetime as well as the anticommuting parameters.

The supersymmetry group element can be defined by

$$G(x, \theta, \bar{\theta}) = \mathrm{e}^{\mathrm{i}(-x^m P_m + \theta \mathrm{Q} + \bar{\theta}\bar{\mathrm{Q}})} \ , \tag{2.4}$$

where Q_α and $\bar{\mathrm{Q}}^{\dot{\alpha}}$ are the supersymmetry generators[9] One representation for the supersymmetry generators is found by realising that the

[9] If we do not write out the spinor indices, undotted indices are contracted from top to bottom and dotted indices are contracted from bottom to top (cf. Appendix A.1).

left multiplication of $G(0,\zeta,\bar{\zeta})$ to a group element is generated by the differential operators

$$\begin{aligned} \mathrm{Q}_\alpha &= \frac{\partial}{\partial\theta^\alpha} - \mathrm{i}\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m \\ \bar{\mathrm{Q}}_{\dot{\alpha}} &= -\frac{\partial}{\partial\theta^{\dot{\alpha}}} + \mathrm{i}\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\partial_m \ . \end{aligned} \tag{2.5}$$

Q_α and $\bar{\mathrm{Q}}_{\dot{\alpha}}$ indeed satisfy the supersymmetry relation

$$\{\mathrm{Q}_\alpha, \bar{\mathrm{Q}}_{\dot{\alpha}}\} = 2\mathrm{i}\sigma^m_{\alpha\dot{\alpha}}\partial_m \tag{2.6}$$

as expected for supersymmetry generators acting on the representation space.

The right group multiplication can be generated by

$$\begin{aligned} \mathrm{D}_\alpha &= \frac{\partial}{\partial\theta^\alpha} + \mathrm{i}\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m \\ \bar{\mathrm{D}}_{\dot{\alpha}} &= -\frac{\partial}{\partial\theta^{\dot{\alpha}}} - \mathrm{i}\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\partial_m \end{aligned} \tag{2.7}$$

which anticommute with Q_α and $\bar{\mathrm{Q}}^{\dot{\alpha}}$ [10] and furthermore satisfy

$$\begin{aligned} \{\mathrm{D}_\alpha, \bar{\mathrm{D}}_{\dot{\alpha}}\} &= -2\mathrm{i}\sigma^m_{\alpha\dot{\alpha}}\partial_m \\ \{\mathrm{D}_\alpha, \mathrm{D}_\beta\} &= \{\bar{\mathrm{D}}_{\dot{\alpha}}, \bar{\mathrm{D}}_{\dot{\beta}}\} = 0 \ . \end{aligned} \tag{2.8}$$

Superfields $F(z = (x,\theta,\bar{\theta}))$ are the elements of the representation space of the supersymmetry group. As θ and $\bar{\theta}$ are Grassmann variables, the power series expansion of F in these variables is finite. As the infinitesimal supersymmetry transformation is given by $\zeta\mathrm{Q} + \bar{\zeta}\bar{\mathrm{Q}}$ and as the Q's are first order differential operators, linear combinations and products of superfields are superfields. As the D's pairwise anticommute with the Q's, constrained fields with $\bar{\mathrm{D}}_{\dot{\alpha}}F = 0$ or $\mathrm{D}_\alpha F = 0$ are superfields and their constraint is conserved under supersymmetry transformations. Superfields which fulfill this constraint are called chiral and anti-chiral superfields and are discussed in Chapter 2.3. They are of importance as they can be seen as supersymmetric generalisation of fermions as well as of the Higgs. As the D's are differential operators, products and linear combinations of (anti-)chiral fields are (anti-)chiral. Another constraint which is conserved by supersymmetry transformations is $V = V^\dagger$. It defines the vector superfield. Vector superfields can be seen as the supersymmetric generalisation of gauge bosons and are discussed in Chapter 2.4 and 2.5 for the Abelian and non-Abelian case.

[10] This is a simple consequence of the associativity of group multiplication.

To construct Lagrangians from superfields it is useful to introduce integrals over Grassmann variables by

$$\int \mathrm{d}\eta = 0 \qquad , \qquad \int \mathrm{d}\eta \eta = 1 \ . \tag{2.9}$$

This definition can be extended to spinorial Grassmann variables by defining the measure by

$$\begin{aligned} \mathrm{d}^2\theta &= -\frac{1}{4}\mathrm{d}\theta^\alpha \mathrm{d}\theta^\beta \varepsilon_{\alpha\beta} \\ \mathrm{d}^2\bar{\theta} &= -\frac{1}{4}\mathrm{d}\bar{\theta}_{\dot{\alpha}} \mathrm{d}\bar{\theta}_{\dot{\beta}} \varepsilon^{\dot{\alpha}\dot{\beta}} \\ \mathrm{d}^4\theta &= \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \end{aligned} \tag{2.10}$$

leading to

$$\begin{aligned} \int \mathrm{d}^2\theta = 0 \qquad &, \qquad \int \mathrm{d}^2\theta\theta^2 = 1 \\ \int \mathrm{d}^2\bar{\theta} = 0 \qquad &, \qquad \int \mathrm{d}^2\bar{\theta}\bar{\theta}^2 = 1 \end{aligned} \tag{2.11}$$

and

$$\delta(\theta) = \theta^2 \qquad , \qquad \delta(\bar{\theta}) = \bar{\theta}^2 \tag{2.12}$$

act like delta distributions on functions in ordinary integrals.

To construct Lagrangians from superfields, an object is needed which leads to a supersymmetry invariant action. Demanding the Lagrangian to be invariant under supersymmetry transformation is a too strong condition as this is fulfilled by constant fields only.[11] The only possibility left to get a supersymmetry invariant action[12] is a Lagrangian, which transforms like a spacetime density. For a general superfield, this is given for the $\theta^2\bar{\theta}^2$-component, while for chiral superfields it is true for the θ^2-component and respectively, for anti-chiral fields it is given for the $\bar{\theta}^2$-component.

The most general action for N=1, D=4 supersymmetry can therefore be written in superfields by

$$\begin{aligned} S &= \int \mathrm{d}^4x \left\{ \left(\prod_{\text{any fields}} \right)\Bigg|_{\theta^2\bar{\theta}^2} + \left[\left(\prod_{\text{chiral fields}} \right)\Bigg|_{\theta^2} + \text{h.c.} \right] \right\} \\ &= \int \mathrm{d}^8z \left\{ \left(\prod_{\text{any fields}} \right) + \left[\left(\prod_{\text{chiral fields}} \right) \delta(\bar{\theta}) + \text{h.c.} \right] \right\} \end{aligned} \tag{2.13}$$

where $\mathrm{d}^8z = \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}$.

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[11] Considering the anticommutator of the supersymmetry generators, supersymmetry invariance implies $\sigma^m_{\alpha\dot{\alpha}}\partial_m\mathcal{L} = 0$.

[12] up to surface terms

2.3 Chiral Superfields

Chiral superfields are defined to fulfill the constraint

$$\bar{\mathrm{D}}_{\dot{\alpha}}\Phi = 0\ . \tag{2.14}$$

Applying the constraint to the component expansion of a general superfield, it follows that

$$\begin{aligned}\Phi &= A(y) + \sqrt{2}\theta\psi(y) + \theta^2 F\,(y) \\ &= A(x) + \mathrm{i}\theta\sigma^m\bar{\theta}\partial_m A(x) + \frac{1}{4}\theta^2\bar{\theta}^2\Box A(x) + \sqrt{2}\theta\psi(x) \\ &\quad - \frac{\mathrm{i}}{\sqrt{2}}\theta^2\partial_m\psi(x)\sigma^m\bar{\theta} + \theta^2 F(x)\end{aligned} \tag{2.15}$$

where A and F are complex fields, ψ is a Weyl spinor and $y^m \equiv x^m + \mathrm{i}\theta\sigma^m\bar{\theta}$.[13]

Analogously for anti-chiral superfields the constraint is

$$\mathrm{D}_\alpha\bar{\Phi} = 0\ , \tag{2.16}$$

which yields the component expansion

$$\begin{aligned}\bar{\Phi} &= A^*(y^\dagger) + \sqrt{2}\bar{\theta}\bar{\psi}(y^\dagger) + \bar{\theta}^2 F^*(y^\dagger) \\ &= A^*(x) - \mathrm{i}\theta\sigma^m\bar{\theta}\partial_m A^*(x) + \frac{1}{4}\theta^2\bar{\theta}^2\Box A^*(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) \\ &\quad + \frac{\mathrm{i}}{\sqrt{2}}\bar{\theta}^2\theta\sigma^m\partial_m\bar{\psi}(x) + \bar{\theta}^2 F^*(x) \\ &= A^*(y) - 2\mathrm{i}\theta\sigma^m\bar{\theta}\partial_m A^*(y) + \theta^2\bar{\theta}^2\Box A^*(y) + \sqrt{2}\bar{\theta}\bar{\psi}(y) \\ &\quad + \sqrt{2}\mathrm{i}\bar{\theta}^2\theta\sigma^m\partial_m\bar{\psi}(y) + \bar{\theta}^2 F^*(y)\end{aligned} \tag{2.17}$$

where ${y^\dagger}^m = x^m - \mathrm{i}\theta\sigma^m\bar{\theta}$.

The most general renormalisable action of N chiral fields therefore is[14]

$$S = \int \mathrm{d}^4x \left\{ \bar{\Phi}_i\Phi_i\Big|_{\theta^2\bar{\theta}^2} + \left[\left(\frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3!}\lambda_{ijk}\Phi_i\Phi_j\Phi_k \right)\Big|_{\theta^2} + \mathrm{h.c.} \right] \right\} \tag{2.18}$$

with $i, j, k = 1..N$ and m_{ij} and λ_{ijk} symmetric.

Rewriting the action as a superspace integral leads to

$$S = \int \mathrm{d}^8z \left\{ \bar{\Phi}_i\Phi_i + \left[\left(\frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3!}\lambda_{ijk}\Phi_i\Phi_j\Phi_k \right)\delta(\bar{\theta}) + \mathrm{h.c.} \right] \right\}\ . \tag{2.19}$$

[13] Choosing y instead of x as super coordinate is convenient for calculations which involve chiral superfields only, because in these coordinates, $\bar{\mathrm{D}}_{\dot{\alpha}}$ becomes $\bar{\mathrm{D}}^y_\alpha = \frac{\partial}{\partial\theta^\alpha}$ which simplifies computations.

[14] The contributing component for terms coupling more than 4 chiral fields are non-renormalisable because they have mass dimension > 4.

2.4 Vector Superfields (Abelian case)

Another constraint on superfields which leads to a supermultiplet of phenomenological interest is reality condition

$$V = V^{\dagger} \ . \tag{2.20}$$

Expanding the field in its θ-components yields

$$\begin{aligned} V =& C(x) + \mathrm{i}\theta\chi(x) - \mathrm{i}\bar{\theta}\bar{\chi}(x) \\ &+ \frac{\mathrm{i}}{2}\theta^2\left[M(x) + \mathrm{i}N(x)\right] - \frac{\mathrm{i}}{2}\bar{\theta}^2\left[M(x) - \mathrm{i}N(x)\right] - \theta\sigma^m\bar{\theta}v_m(x) \\ &+ \mathrm{i}\theta^2\bar{\theta}\left[\bar{\lambda}(x) + \frac{\mathrm{i}}{2}\bar{\sigma}^m\partial_m\chi(x)\right] - \mathrm{i}\bar{\theta}^2\theta\left[\lambda(x) - \frac{\mathrm{i}}{2}\sigma^m\partial_m\bar{\chi}(x)\right] \\ &+ \frac{1}{2}\theta^2\bar{\theta}^2\left[D(x) + \frac{1}{2}\Box C(x)\right] \end{aligned} \tag{2.21}$$

where C, D, M and N are real scalar fields, χ is a Weyl spinor and v_m is a real vector field which will be identified with the gauge boson.

A supersymmetric generalisation of the Abelian gauge transformation is given by

$$V \to V + \mathrm{i}(\Lambda - \bar{\Lambda}) \tag{2.22}$$

where Λ is a chiral field.

The gauge transformations in component fields are

$$\begin{aligned} C &\to C + \mathrm{i}(A - A^*) \\ \chi &\to \chi + \sqrt{2}\psi \\ M + \mathrm{i}N &\to M + \mathrm{i}N + 2F \\ v_m &\to v_m + \partial_m(A + A^*) \\ \lambda &\to \lambda \\ D &\to D \end{aligned} \tag{2.23}$$

so that the vector component transforms as needed for a gauge field.

By fixing all degrees of freedom of the chiral field except the $(A + A^*)$-component, it is possible to gauge away C, $M + \mathrm{i}N$ and χ, leaving the vector superfield with v_m as the lowest component in the θ-expansion, the gaugino λ and the auxiliary field D. The only remaining gauge freedom is the ordinary gauge transformation for the vector field component v_m. One advantage of this so called Wess-Zumino gauge is that all powers in V higher than V^2 vanish, which simplifies calculations because matter fields couple non-polynomial to the gauge field (cf. (2.27)).

The remaining powers of V are

$$\begin{aligned} V &= -\theta\sigma^m\bar{\theta}v_m(x) + \mathrm{i}\theta^2\bar{\theta}\bar{\lambda}(x) - \mathrm{i}\bar{\theta}^2\theta\lambda(x) + \frac{1}{2}\theta^2\bar{\theta}^2 D(x) \\ V^2 &= -\frac{1}{2}\theta^2\bar{\theta}^2 v_m v^m \ . \end{aligned} \tag{2.24}$$

However this gauge breaks supersymmetry in the sense that a supersymmetry transformation does not preserve the Wess-Zumino gauge.

The supersymmetry generalisation of the field strength tensor is given by the (anti-)chiral, gauge invariant superfields

$$\begin{aligned} W_\alpha &= -\frac{1}{4}\bar{\mathrm{D}}^2\mathrm{D}_\alpha V \\ \bar{W}_{\dot{\alpha}} &= -\frac{1}{4}\mathrm{D}^2\bar{\mathrm{D}}_{\dot{\alpha}} V \ , \end{aligned} \tag{2.25}$$

which include the standard field strength tensor F_{mn} as their θ- (or respectively $\bar{\theta}$-) component in their component expansion.

To couple chiral superfields to the vector superfield, they have to transform under the gauge group according to

$$\begin{aligned} \Phi'_l &= \mathrm{e}^{-\mathrm{i}t_l\Lambda}\Phi_l \\ \bar{\Phi}'_l &= \mathrm{e}^{\mathrm{i}t_l\bar{\Lambda}}\bar{\Phi}_l \end{aligned} \tag{2.26}$$

where t_l is the charge of the l^{th} chiral superfield. With these transformations, the kinetic term of the chiral superfields $\Phi\bar{\Phi}$ is not invariant under gauge transformations. This can be cured by modifying it to[15]

$$\mathcal{L}_{\text{kin}} = \bar{\Phi}_l \mathrm{e}^{2t_l V}\Phi_l \ . \tag{2.27}$$

The most general renormalisable action with an Abelian gauge group is therefore given by

$$\begin{aligned} S = \int \mathrm{d}^4x \Bigg\{ &\frac{1}{4}\left(W^\alpha W_\alpha\Big|_{\theta^2} + \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\Big|_{\bar{\theta}^2}\right) + \bar{\Phi}_l\mathrm{e}^{2t_l V}\Phi_l\Big|_{\theta^2\bar{\theta}^2} \\ &+ \left[\left(\frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3!}\lambda_{ijk}\Phi^i\Phi_j\Phi_k\right)\Big|_{\theta^2} + \text{h.c.}\right]\Bigg\} \ . \end{aligned} \tag{2.28}$$

Here m_{ij} and λ_{ijk} are restricted by demanding gauge invariance.

[15] Choosing a factor $\mathrm{i}t_l$ in the exponent instead of $\mathrm{i}2t_l$ would just rescale the vector superfield. We choose it as in equation (2.26) as this leads to the textbook Langrangian for the vector field component part.

2.5 Vector Superfields (Non-Abelian) and Coupling to Chiral Superfields

In this chapter, the concepts of the last chapter are generalised to a non-Abelian compact gauge group $\mathcal{G}$, which is generated from the Lie algebra

$$[T^a, T^b] = \mathrm{i} f^{abc} T^c \ . \tag{2.29}$$

Here, the T^a are the hermitian, traceless generators of the gauge group, f^{abc} are completely antisymmetric real structure constants and the generators in the adjoint representation are chosen such that

$$\mathrm{Tr}\left(T^a T^b\right) = k\delta^{ab} \tag{2.30}$$

with k real.

The transformation of chiral superfields generalises to

$$\begin{aligned} \Phi_i' &= \mathrm{e}^{-\mathrm{i}\Lambda_{ij}} \Phi_j \\ \bar{\Phi}_i' &= \mathrm{e}^{\mathrm{i}\bar{\Lambda}_{ij}} \bar{\Phi}_j' \end{aligned} \tag{2.31}$$

with

$$\Lambda_{ij} = T^a_{ij} \Lambda^a \ , \tag{2.32}$$

where g is coupling constant and the T^a_{ij} are chosen in the representation appropriate for the chiral field multiplet.

To ensure gauge invariance of the kinetic term of the chiral action $\mathcal{L}_{\mathrm{kin}}$ in (2.27), the vector superfield multiplet has to transform as

$$\mathrm{e}^{2gV'} = \mathrm{e}^{-\mathrm{i}\bar{\Lambda}} \mathrm{e}^{2gV} \mathrm{e}^{\mathrm{i}\Lambda} \tag{2.33}$$

where

$$V = T^a V^a \tag{2.34}$$

which by using Hausdorff's formula leads to

$$\delta(2gV) = \mathrm{i}\mathcal{L}_{gV}\left[\left(\Lambda + \bar{\Lambda}\right) + \coth \mathcal{L}_{gV}\left(\Lambda - \bar{\Lambda}\right)\right] \tag{2.35}$$

as infinitesimal transformation of V. Here $\coth \mathcal{L}_{gV}$ denotes the power series in $\mathcal{L}_{gV}$ and $\mathcal{L}_{gV}\left(\Lambda + \bar{\Lambda}\right) = \left[gV, \Lambda + \bar{\Lambda}\right]$.

Using

$$x \coth(x) = 1 + \frac{1}{3}x^2 - \frac{1}{45}x^4 + O\left(x^6\right) \tag{2.36}$$

it follows that

$$\begin{aligned} \delta(2gV) =& \mathrm{i}\left\{\left(\Lambda - \bar{\Lambda}\right) + \left[gV, \Lambda + \bar{\Lambda}\right] + \frac{1}{3}\left[gV, \left[gV, \Lambda - \bar{\Lambda}\right]\right]\right. \\ &\left. -\frac{1}{45}\left[gV, \left[gV, \left[gV, \Lambda - \bar{\Lambda}\right]\right]\right] + O(g^4)\right\} \ . \end{aligned} \tag{2.37}$$

This expansion will be needed in Chapter 3 to calculate the 3- and 4-vertices of super Yang-Mills theory.

The supersymmetric field strength has to transform according to

$$\begin{aligned} W_\alpha &\to W'_\alpha = e^{-\mathrm{i}\Lambda} W_\alpha e^{\mathrm{i}\Lambda} \\ \bar{W}_{\dot\alpha} &\to \bar{W'}_{\dot\alpha} = e^{\mathrm{i}\bar\Lambda} \bar{W}_{\dot\alpha} e^{-\mathrm{i}\bar\Lambda} \end{aligned} \tag{2.38}$$

and is given by

$$\begin{aligned} W_\alpha &= -\frac{1}{4}\bar{\mathrm{D}}^2 \mathrm{e}^{-2gV} \mathrm{D}_\alpha \mathrm{e}^{2gV} \\ \bar{W}_{\dot\alpha} &= -\frac{1}{4}\mathrm{D}^2 \mathrm{e}^{-2gV} \bar{\mathrm{D}}_{\dot\alpha} \mathrm{e}^{2gV} \ . \end{aligned} \tag{2.39}$$

Therefore, the most general, renormalisable action for a super Yang-Mills Theory takes the form

$$\tilde{S}_{\mathrm{SYM}} = \tilde{S}_{\mathrm{gauge}} + S_{\mathrm{mat}} \tag{2.40}$$

with[16]

$$\begin{aligned} \tilde{S}_{\mathrm{gauge}} &= \int \mathrm{d}^4x \frac{\mathrm{Tr}}{16kg^2} \left(W^\alpha W_\alpha \Big|_{\theta^2} + \bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha} \Big|_{\bar\theta^2} \right) \\ &= \int d^8z \frac{\mathrm{Tr}}{16kg^2} \left(W^\alpha W_\alpha \delta(\bar\theta) + \bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha} \delta(\theta) \right) \end{aligned} \tag{2.41}$$

and

$$\begin{aligned} S_{\mathrm{mat}} &= \int \mathrm{d}^4x \left\{ \Phi_l \mathrm{e}^{2gV} \Phi_l \Big|_{\theta^2\bar\theta^2} + \left[\left(\frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3!} \lambda_{ijk} \Phi_i \Phi_j \Phi_k \right) \Big|_{\theta^2} + \mathrm{h.c.} \right] \right\} \\ &= \int d^8z \left\{ \bar\Phi_l \mathrm{e}^{2gV} \Phi_l + \left[\frac{1}{2} m_{ij} \Phi_i \Phi_j \delta(\bar\theta) + \frac{1}{3!} \lambda_{ijk} \Phi_i \Phi_j \Phi_k \delta(\bar\theta) + \mathrm{h.c.} \right] \right\} \ . \end{aligned} \tag{2.42}$$

Here, as in the abelian case m_{ij} and λ_{ijk} are restricted by demanding gauge invariance in the way that they have to be invariant tensors of the gauge group. For a theory with massive matter, the only way of fulfilling this constraint is to choose a real representation of the gauge group with $m_{ij} = m\delta_{ij}$. For a massless theory, a complex representation can be chosen as well.

For later use, we split S_{mat} into a free and an interacting part. The free part is

$$S_{0,\mathrm{mat}} = \int d^8z \left\{ \bar\Phi_l \Phi_l + \left[\frac{1}{2} m\delta_{ij} \Phi_i \Phi_j \delta(\bar\theta) + \mathrm{h.c.} \right] \right\} \ . \tag{2.43}$$

[16]We write $\tilde{S}_{\mathrm{SYM}}$ and $\tilde{S}_{\mathrm{gauge}}$ because these actions will be modified in the next chapter according to a generalised Faddeev-Popov procedure.

The interacting part is

$$S_{\mathrm{INT,mat}} = \int d^8z \left\{ \bar{\Phi}_l \left(\mathrm{e}^{2gV} - 1 \right) \Phi_l + \left[\frac{1}{3!} \lambda_{ijk} \Phi_i \Phi_j \Phi_k \delta(\bar{\theta}) + \mathrm{h.c.} \right] \right\} . \tag{2.44}$$

3 Quantisation of Super Yang-Mills Theories in Superfields

The aim of this chapter is to quantise super Yang-Mills theories in superfields. To quantise the theory, the gauge has to be fixed. Then we rewrite the action in terms of chiral and anti-chiral ghost supermultiplets, applying a generalisation of the Faddeev-Popov procedure proposed by [24]. This is done, using the conventions of [21, 23] and by using Minkowskian path integrals. We also give the BRS transformations.

As a next step, the Feynman rules of the theory are derived, including rules of D-algebra [22] in our conventions. These will then be used to do a one-loop calculation for the gauge superfield propagator leading to the β-function of super Yang-Mills theory (cf. [22, 25]).

3.1 Generalised Faddeev-Popov Procedure and BRS Transformations

The action of a super Yang-Mills theory has been given in (2.41). Expanding the supersymmetric field strength in the vector superfield according to (2.39), and using (2.8) gives

$$\begin{aligned} \tilde{S}_{\mathrm{gauge}} = \int d^8z \frac{\mathrm{Tr}}{k} \bigg\{ & \frac{1}{8} V \mathrm{D}^\alpha \bar{\mathrm{D}}^2 \mathrm{D}_\alpha V + \frac{1}{4} g (\bar{\mathrm{D}}^2 \mathrm{D}^\alpha V)[V, \mathrm{D}_\alpha V] \\ & - \frac{1}{8} g^2 [V, (\mathrm{D}^\alpha V)] \bar{\mathrm{D}}^2 [V, (\mathrm{D}_\alpha V)] - \frac{1}{6} g^2 (\bar{\mathrm{D}}^2 \mathrm{D}^\alpha V)[V, [V, \mathrm{D}_\alpha V]] + O(g^3) \bigg\} . \end{aligned} \tag{3.1}$$

To quantise the theory, we follow a generalised Faddeev-Popov procedure as outlined in [24] and in more detail in [22].

As in an ordinary Yang-Mills-Theory, the gauge has to be fixed to calculate the propagator (cf. e.g. [32], [33]). In Wess-Zumino gauge, terms of $O(V^5)$ vanish up to spacetime derivatives as the lowest field component is the $\theta\bar{\theta}$ component and due to the superspace integral and two D's and $\bar{\mathrm{D}}$'s acting, the only contributing part of the Lagrangian is the $\theta^4\bar{\theta}^4$ component. However, as pointed out in Chapter 2.4, Wess-Zumino gauge breaks supersymmetry, so we follow a different path.

Starting with the path integral for super Yang-Mills theory

$$Z_{\text{gauge}} = \int \mathcal{D}V \mathrm{e}^{\mathrm{i}\tilde{S}_{\text{gauge}}[V]} \ , \tag{3.2}$$

the gauge fixing is done by defining the gauge invariant integral over the gauge group by

$$\Delta(V) = \int \mathcal{D}\Lambda\mathcal{D}\bar{\Lambda}\delta[F(V,\Lambda,\bar{\Lambda}) - f]\delta[\bar{F}(V,\Lambda,\bar{\Lambda}) - \bar{f}] \tag{3.3}$$

and inserting a factor of $\Delta^{-1}(V)\Delta(V)$ into (3.2). In (3.3), F is a chiral function of the vector superfield and the gauge parameters such that for any chiral $f(x,\theta)$ there is one and only one $(V,\Lambda,\bar{\Lambda})$ with $F = f$ and analogously for anti-chiral $\bar{F}$ and $\bar{f}$. An appropriate choice for F and $\bar{F}$ is

$$\begin{aligned} F(V,\Lambda,\bar{\Lambda}) &= -\frac{1}{4}\bar{\mathrm{D}}^2 2gV \\ \bar{F}(V,\Lambda,\bar{\Lambda}) &= -\frac{1}{4}\mathrm{D}^2 2gV \ . \end{aligned} \tag{3.4}$$

Averaging over f and $\bar{f}$ with a weighting factor $\mathcal{D}f\mathcal{D}\bar{f}e^{\frac{2\mathrm{i}\mathrm{Tr}}{\gamma k}\int \mathrm{d}^8 z \bar{f} f}$ results in

$$Z_{\text{gauge}} = \int \mathcal{D}V\Delta^{-1}(V)\mathrm{e}^{\mathrm{i}(\tilde{S}_{\text{gauge}}[V] + S_{\text{GF}}[V])} \tag{3.5}$$

where

$$S_{\text{GF}}[V] = \frac{1}{8\gamma}\int d^8 z \frac{\mathrm{Tr}}{k}((\bar{\mathrm{D}}^2 V)(\mathrm{D}^2 V)) \tag{3.6}$$

To rewrite the inverse gauge group integral, we express the δ-distributions as integrals over Λ' and $\bar{\Lambda}'$:

$$\begin{aligned} \Delta(V) &= \int \mathcal{D}\bar{\Lambda}'\mathcal{D}\Lambda'\mathcal{D}\bar{\Lambda}\mathcal{D}\Lambda \exp\left[\frac{\mathrm{Tr}}{k}\int d^8 z \ \left\{\Lambda'\left(\frac{\delta F}{\delta\Lambda}\Lambda + \frac{\delta F}{\delta\bar{\Lambda}}\bar{\Lambda}\right)\delta(\bar{\theta})\right.\right. \\ &\qquad\qquad \left.\left. + \bar{\Lambda}'\left(\frac{\delta \bar{F}}{\delta\Lambda}\Lambda + \frac{\delta \bar{F}}{\delta\bar{\Lambda}}\bar{\Lambda}\right)\delta(\theta)\right\}\right] \\ &= \int \mathcal{D}\bar{\Lambda}'\mathcal{D}\Lambda'\mathcal{D}\bar{\Lambda}\mathcal{D}\Lambda \exp\left[\frac{\mathrm{Tr}}{k}\int d^8 z \ \left\{\Lambda'(-\frac{1}{4}\bar{\mathrm{D}}^2)(\delta(2gV))\delta(\bar{\theta})\right.\right. \\ &\qquad\qquad \left.\left. + \bar{\Lambda}'(-\frac{1}{4}\mathrm{D}^2)(\delta(2gV))\delta(\theta)\right\}\right] \\ &= \int \mathcal{D}\bar{\Lambda}'\mathcal{D}\Lambda'\mathcal{D}\bar{\Lambda}\mathcal{D}\Lambda \exp\left[\frac{\mathrm{Tr}}{k}\int d^8 z \ (\Lambda' + \bar{\Lambda}')(\delta(2gV))\right] \ . \end{aligned} \tag{3.7}$$

Then we use (2.35) and replace the (anti-)chiral gauge parameters by (anti-)chiral anticommuting ghost fields, using that for Grassmann variables the *inverse* functional determinant can be expressed by

$$\Delta^{-1}(V) = \int \mathcal{D}\bar{c}'\mathcal{D}c'\mathcal{D}\bar{c}\mathcal{D}c\ \mathrm{e}^{\mathrm{i}S_{\mathrm{FP}}[V,\bar{c}',c',\bar{c},c]} \tag{3.8}$$

with

$$\begin{aligned} S_{\mathrm{FP}}[V,\bar{c}',c',\bar{c},c] &= \frac{\mathrm{Tr}}{k}\int d^8z(c'+\bar{c}')(-\mathrm{i})(\delta(2gV)) \\ &= \frac{\mathrm{Tr}}{k}\int d^8z(c'+\bar{c}')\Big\{ (c-\bar{c}) + [gV, c+\bar{c}] \\ &\qquad + \frac{1}{3}[gV,[gV,c-\bar{c}]] + O(g^3)\Big\} \end{aligned} \tag{3.9}$$

The generating functional for super Yang-Mills theory, including gauge fixing and ghosts is therefore

$$Z_{\mathrm{gauge}} = \int \mathcal{D}V\mathcal{D}\bar{c}'\mathcal{D}c'\mathcal{D}\bar{c}\mathcal{D}c\ \mathrm{e}^{\mathrm{i}S_{\mathrm{gauge}}} \tag{3.10}$$

with

$$S_{\mathrm{gauge}} = \tilde{S}_{\mathrm{gauge}} + S_{\mathrm{GF}} + S_{\mathrm{FP}} \tag{3.11}$$

which are given in (3.1), (3.6) and (3.9). For later use, we split this action in a free and an interacting part.

The free part is

$$\begin{aligned} S_{0,\mathrm{gauge}} &= \int d^8z \frac{\mathrm{Tr}}{k}\left\{\frac{1}{8}\left[V\mathrm{D}^\alpha\bar{\mathrm{D}}^2\mathrm{D}_\alpha V + \frac{1}{\gamma}((\bar{\mathrm{D}}^2V)(\mathrm{D}^2V))\right] + [\bar{c}'c - c'\bar{c}]\right\} \\ &= \int d^8z \frac{\mathrm{Tr}}{k}\left\{-V\Box(P_T - \frac{1}{\gamma}(P_1+P_2))V + [\bar{c}'c - c'\bar{c}]\right\} \end{aligned} \tag{3.12}$$

where in the first equation we used, that $c'c$ and $\bar{c}'\bar{c}$ vanish under a superspace integral and in the second equation, we did partial integrations on the gauge fixing term and used the projector definitions (B.15).

The interacting part is

$$\begin{aligned} S_{\mathrm{INT,gauge}} = \int d^8z \frac{\mathrm{Tr}}{k}\Big\{&\frac{1}{4}g(\bar{\mathrm{D}}^2\mathrm{D}^\alpha V)[V,\mathrm{D}_\alpha V] + g(c'+\bar{c}')\,[V,c+\bar{c}] \\ &-\frac{1}{8}g^2[V,(\mathrm{D}^\alpha V)]\bar{\mathrm{D}}^2[V,(\mathrm{D}_\alpha V)] - \frac{1}{6}g^2(\bar{\mathrm{D}}^2\mathrm{D}^\alpha V)[V,[V,\mathrm{D}_\alpha V]] \\ &+\frac{1}{3}g^2(c'+\bar{c}')\,[V,[V,c-\bar{c}]] + O(g^3)\Big\}\ . \end{aligned} \tag{3.13}$$

The (anti-)chiral matter superfields and their interaction term with the gauge superfield are not affected by the gauge fixing procedure. Their generating functional is just given by

$$Z_{\mathrm{mat}} = \int \mathcal{D}V\mathcal{D}\bar{\Phi}\mathcal{D}\Phi \mathrm{e}^{\mathrm{i}S_{\mathrm{mat}}} \tag{3.14}$$

with S_{mat} from (2.42).

By gauge fixing, the gauge symmetry is broken. However, similar to ordinary Yang-Mills theory, a global symmetry is left, which is manifested in the BRS transformation invariance [22]. The BRS transformations in our conventions are

$$\begin{aligned}
\mathrm{s}(2gV) &= \delta(2gV)(\Lambda \to c, \bar{\Lambda} \to \bar{c}) \\
&= \mathrm{i}\mathcal{L}_{gV}\left[(c+\bar{c}) + \coth \mathcal{L}_{gV}\,(c-\bar{c})\right] \\
\mathrm{s}\bar{c}' &= \frac{\mathrm{i}}{16(2g)^2\gamma}\bar{\mathrm{D}}^2\mathrm{D}^2(2gV) \\
\mathrm{s}c' &= \frac{\mathrm{i}}{16(2g)^2\gamma}\mathrm{D}^2\bar{\mathrm{D}}^2(2gV) \\
\mathrm{s}\bar{c} &= \bar{c}^2 \\
\mathrm{s}c &= c^2 \\
\mathrm{s}\Phi &= \delta\Phi(\Lambda \to c) = -\mathrm{i}2gc\Phi \\
\mathrm{s}\bar{\Phi} &= \delta\bar{\Phi}(\bar{\Lambda} \to \bar{c}) = \mathrm{i}2g\bar{c}\bar{\Phi}\ .
\end{aligned} \tag{3.15}$$

As for ordinary Yang-Mills theory (cf. [33]), all BRS transformations are nilpotent, except $\mathrm{s}\bar{c}'$ and $\mathrm{s}c'$, which are only nilpotent if the equations of motion are imposed. Otherwise, they only fulfill $\mathrm{s}^3c' = 0 = \mathrm{s}^3\bar{c}'$.

To prove invariance of the action of a super Yang-Mills theory $S_{\mathrm{SYM}} = S_{\mathrm{gauge}} + S_{\mathrm{mat}}$ we observe that the non-gauge-fixed action $\tilde{S}_{\mathrm{gauge}} + S_{\mathrm{mat}}$ is trivially invariant due to gauge invariance of the classical theory. It remains to show that $S_{\mathrm{GF}} + S_{\mathrm{FP}}$ is invariant.

$$\begin{aligned}
\mathrm{s}S_{\mathrm{GF}} &= \mathrm{s}\left\{\frac{1}{(2g)^2 8\gamma}\int d^8z\frac{\mathrm{Tr}}{k}((\bar{\mathrm{D}}^2 2gV)(\mathrm{D}^2 2gV))\right\} \\
&= \frac{1}{(2g)^2 8\gamma}\int d^8z\frac{\mathrm{Tr}}{k}\mathrm{s}(2gV\bar{\mathrm{D}}^2\mathrm{D}^2 2gV)) \\
&= \frac{1}{(2g)^2 8\gamma}\int d^8z\frac{\mathrm{Tr}}{k}\mathrm{s}(2gV\frac{16(2g)^2\gamma}{\mathrm{i}}\mathrm{s}c') \\
&= 2\int d^8z\frac{\mathrm{Tr}}{k}(\mathrm{s}(2gV))\frac{1}{2\mathrm{i}}(\mathrm{s}c' + \mathrm{s}\bar{c}') \\
&= \mathrm{i}\int d^8z\frac{\mathrm{Tr}}{k}(\mathrm{s}(2gV))\mathrm{s}(c' + \bar{c}')\ .
\end{aligned} \tag{3.16}$$

We did several partial integrations and used $s(FG) = (sF)G \pm F(sG)$, where the terms are subtracted if F contains an odd number of grassmann variables.

$$\begin{aligned} \mathrm{s}S_{\mathrm{FP}} &= \mathrm{s}\left\{-\mathrm{i}\int d^8z\frac{\mathrm{Tr}}{k}(c'+\bar{c}')\mathrm{s}(2gV)\right\} \\ &= -\mathrm{i}\int d^8z\frac{\mathrm{Tr}}{k}(\mathrm{s}(2gV))\mathrm{s}(c'+\bar{c}') \end{aligned} \tag{3.17}$$

where we used that $\mathrm{s}(c'+\bar{c}')$ is grassmann-even and therefore commutes with $(\mathrm{s}(2gV))$. Hence, $\mathrm{s}(S_{GF}+S_{FP}) = 0$.

3.2 Derivation of Propagators and Feynman Rules

In this chapter the Feynman rules for N=1, D=4 supersymmetry will be derived, starting from the generating functionals Z_{gauge} and Z_{mat} given in (3.10) and 3.14). We will first review the functional derivative for chiral fields in superfield formalism, as this is the main new feature compared to non-supersymmetric quantum field theory.[17] Then the propagators and the generating functional will be derived for chiral, ghost and vector superfields, following [24]. Using those, the 3-vector-superfield-vertex will be calculated in coordinate- and momentum space. Finally, the 3-chiral-superfield-vertex will be given and a simplification for its Feynman rules will be derived.[18]

To do functional differentiation for chiral superfields, one has to take into account that chiral fields are subject to the constraint $\bar{\mathrm{D}}_{\dot{\alpha}}\Phi_i = 0$. This is done by varying the fields in the y-basis [21] (c.f. Chapter 2.3: $y = x + \mathrm{i}\theta\sigma\bar{\theta}$):

$$\frac{\delta}{\delta\Phi_i(y,\theta)}\Phi_j(y',\theta') = \delta_{ij}\delta(y-y')\delta(\theta-\theta') \tag{3.18}$$

by which the field variations remain chiral. For anti-chiral superfields the variation takes the form:

$$\frac{\delta}{\delta\bar{\Phi}_i(y^\dagger,\bar{\theta})}\bar{\Phi}_j(y'^\dagger,\bar{\theta}') = \delta_{ij}\delta(y^\dagger-y'^\dagger)\delta(\bar{\theta}-\bar{\theta}') \tag{3.19}$$

Expressing this in x-basis leads to (cf. [21, 22])

[17] For vector superfields, functional differentiation works exactly as in non-supersymmetric quantum field theory.

[18] This is one of the advanced Feynman rules, derived in [24].

$$
\begin{aligned}
\frac{\delta}{\delta\Phi_i(z)}\Phi_j(z') &= \delta_{ij}(-\frac{1}{4}\bar{\mathrm{D}}^2)\delta^8(z-z') \\
\frac{\delta}{\delta\bar{\Phi}_i(z)}\bar{\Phi}_j(z') &= \delta_{ij}(-\frac{1}{4}\mathrm{D}^2)\delta^8(z-z') \\
\frac{\delta}{\delta\bar{\Phi}_i(z)}\Phi_j(z') &= \frac{\delta}{\delta\Phi_i(z)}\bar{\Phi}_j(z') = 0
\end{aligned}
\tag{3.20}
$$

which is only valid under a superspace integral.

To derive propagators and Feynman rules for chiral fields, the free part of the chiral action $S_{0,\mathrm{mat}}$ (cf. (2.43)) is rewritten, using (B.27, B.28):

$$
\begin{aligned}
S_{0,\mathrm{mat}} &= \int d^8z \left\{ \bar{\Phi}_l\Phi_l + \left[\frac{1}{2}m\delta_{ij}\Phi_i\Phi_j\delta(\bar{\theta}) + \mathrm{h.c.}\right]\right\} \\
&= \int d^8z \frac{1}{2}(\Phi,\bar{\Phi})_i M_{ij} \begin{pmatrix}\Phi \\ \bar{\Phi}\end{pmatrix}_j
\end{aligned}
\tag{3.21}
$$

with

$$
M_{ij} = \delta_{ij} \begin{pmatrix} -\frac{m\mathrm{D}^2}{4\Box} & 1 \\ 1 & -\frac{m\bar{\mathrm{D}}^2}{4\Box} \end{pmatrix} . \tag{3.22}
$$

To calculate the propagator, (anti-)chiral sources are coupled in the action. Again, using (B.27) and (B.28), this is done by adding

$$
\int d^8z (\Phi,\bar{\Phi})_i \begin{pmatrix} -\frac{\mathrm{D}^2}{4\Box} & 0 \\ 0 & -\frac{\bar{\mathrm{D}}^2}{4\Box} \end{pmatrix} \begin{pmatrix}\mathrm{J} \\ \bar{\mathrm{J}}\end{pmatrix}_i . \tag{3.23}
$$

From this point on, the calculation of the propagators is straightforward, i.e. calculating the equations of motion, defining the Green's function as its "inverse" and calculating it. We refer to [21, 23]. The result is:

$$
\Delta_{ij} = \delta_{ij}\frac{1}{\Box - m^2 + \mathrm{i}\epsilon} \begin{pmatrix} \frac{m\mathrm{D}^2}{4} & \frac{\bar{\mathrm{D}}^2\mathrm{D}^2}{16} \\ \frac{\mathrm{D}^2\bar{\mathrm{D}}^2}{16} & \frac{m\bar{\mathrm{D}}^2}{4} \end{pmatrix} \delta(z-z') . \tag{3.24}
$$

The generating functional for a free chiral theory leading to this propagator is defined by

$$
Z_{0,\mathrm{mat}}[\mathrm{J},\bar{\mathrm{J}}] = \int \mathcal{D}V \exp\left\{ \mathrm{i}S_{0,\mathrm{mat}} + \mathrm{i}\int d^8z (\Phi,\bar{\Phi})_i \begin{pmatrix} -\frac{\mathrm{D}^2}{4\Box} & 0 \\ 0 & -\frac{\bar{\mathrm{D}}^2}{4\Box} \end{pmatrix} \begin{pmatrix}\mathrm{J} \\ \bar{\mathrm{J}}\end{pmatrix}_i \right\} \tag{3.25}
$$

which can be rewritten by solving the equation of motion for $(\Phi,\bar{\Phi})$ and reinserting it into the generating functional.

The result is

$$Z_{0,\mathrm{mat}}[\mathrm{J},\bar{\mathrm{J}}] = \exp\left\{-\frac{\mathrm{i}}{2}\int d^8z(\mathrm{J}(z),\bar{\mathrm{J}}(z))_i\Delta^{\mathrm{GRS}}_{ij}(z,z')\begin{pmatrix}\mathrm{J}(z')\\ \bar{\mathrm{J}}(z')\end{pmatrix}_j\right\} \tag{3.26}$$

with the Grisaru-Roček-Siegel propagator

$$\Delta^{\mathrm{GRS}}_{ij}(z,z') = \delta_{ij}\frac{1}{\Box - m^2 + \mathrm{i}\epsilon}\begin{pmatrix}\frac{m\mathrm{D}^2}{4\Box} & 1\\ 1 & \frac{m\bar{\mathrm{D}}^2}{4\Box}\end{pmatrix}\delta(z-z')\ . \tag{3.27}$$

For the ghost superfields, the same calculation applies, except that one has to take care of their odd Grassmann parity.[19] In addition the chiral ghosts are massless by construction which is protected against radiative corrections. The calculation leads to the same propagator as in (3.27) up to a sign:[20]

$$\Delta^{ab}_{\mathrm{gh}}(z,z') = \delta^{ab}\frac{1}{\Box + \mathrm{i}\epsilon}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\delta(z-z')\ . \tag{3.28}$$

The calculation of the propagator of the vector superfield is straightforward from $S_{0,\mathrm{gauge}}$ in (3.12). For simplicity, the gauge parameter γ is chosen to be $\gamma = -1$.[21] The part of the action quadratic in V then just becomes

$$S_{0,V} = \int d^8z\frac{\mathrm{Tr}}{k}\left(-V\Box V\right) \tag{3.29}$$

The propagator is therefore[22]

$$\Delta^{ab}_V(z,z') = -\frac{1}{2}\delta^{ab}\frac{1}{\Box + \mathrm{i}\epsilon}\delta^8(z-z')\ . \tag{3.30}$$

---

[19] This includes that sources for the ghosts are coupled by adding $c\left(-\frac{\mathrm{D}^2}{4\Box}\right)\eta + \bar{\eta}\left(-\frac{\mathrm{D}^2}{4\Box}\right)\bar{c} = c\left(-\frac{\mathrm{D}^2}{4\Box}\right)\eta - \bar{c}\left(-\frac{\mathrm{D}^2}{4\Box}\right)\bar{\eta}$ as this keeps the action hermitian.

[20] Note however, that $a, b, \ldots$ now label the generators in the adjoint representation which may differ from the matter representation.

[21] the so called Super-Feynman-Gauge

[22] This definition of the vector superfield propagator differs from [21] by the factor of $\frac{1}{2}$. While the convention of [21] leads to the standard normalisation of the component fields, one has to take care to read off super vertices from the superfield action, as by truncating the three-point functions, the numerical expression of the vertex becomes dependent of the normalisation. Our choice is dictated by keeping the derivation of the Feynman rules as conventional as possible.

The generating functional for free superfields is then

$$Z_{0,\mathrm{SYM}} = \exp\left(\mathrm{i}\int d^8z \frac{\mathrm{Tr}}{k}\left[\ -\frac{1}{2}(\mathrm{J},\bar{\mathrm{J}})_i \Delta^{\mathrm{GRS}}_{ij}\begin{pmatrix}\mathrm{J}\\ \bar{\mathrm{J}}\end{pmatrix}_j - \frac{1}{2}\mathscr{J}^a \Delta_V^{ab}\mathscr{J}^b \right.\right.$$
$$\left.\left. -(\eta',\bar{\eta}')_a \Delta^{ab}_{\mathrm{gh}}\begin{pmatrix}\eta\\ \bar{\eta}\end{pmatrix}_b\right]\right) \tag{3.31}$$

where $\mathscr{J}$ is the real supersource coupled to the vector superfield and $\eta', \bar{\eta}', \eta, \bar{\eta}$ are the (anti-)chiral, Grassmann-odd supersources coupled to the ghost superfields.

To calculate the Feynman rules for vertices, the generating functional can be defined in the standard way:

$$Z_{\mathrm{SYM}} = e^{\mathrm{i}S_{\mathrm{INT}}(\Phi_i \to \frac{\delta}{\mathrm{i}\delta \mathrm{J}^i}, \bar{\Phi}_j \to \frac{\delta}{\mathrm{i}\delta \bar{\mathrm{J}}^j}, V^a \to \frac{\delta}{\mathrm{i}\delta \mathscr{J}^a}, c^a \to \frac{\delta}{\mathrm{i}\delta \eta^a}, \ldots)} Z_{0,\mathrm{SYM}}\ , \tag{3.32}$$

where (from (3.13) and (2.44))

$$\begin{aligned} S_{\mathrm{INT}} &= S_{\mathrm{INT,gauge}} + S_{\mathrm{INT,mat}} \\ S_{\mathrm{INT,mat}} &= \int d^8z \left\{ \bar{\Phi}_l \left(\mathrm{e}^{2gV} - 1\right)\Phi_l + \left[\frac{1}{3!}\lambda_{ijk}\Phi_i\Phi_j\Phi_k \delta(\bar{\theta}) + \mathrm{h.c.}\right]\right\} \\ S_{\mathrm{INT,gauge}} &= \int d^8z \frac{\mathrm{Tr}}{k}\left\{\frac{1}{4}g(\bar{\mathrm{D}}^2\mathrm{D}^\alpha V)[V, \mathrm{D}_\alpha V] + g(c' + \bar{c}')\,[V, c + \bar{c}]\right. \\ &\quad -\frac{1}{8}g^2[V,(\mathrm{D}^\alpha V)]\bar{\mathrm{D}}^2[V,(\mathrm{D}_\alpha V)] - \frac{1}{6}g^2(\bar{\mathrm{D}}^2\mathrm{D}^\alpha V)[V,[V,\mathrm{D}_\alpha V]] \\ &\quad \left. +\frac{1}{3}g^2(c' + \bar{c}')\,[V,[V, c - \bar{c}]] + O(g^3)\right\}\ . \end{aligned} \tag{3.33}$$

The n-vertices are then calculated as the amputated Green's functions of the 1-particle-irreducible part of

$$\left.\frac{\delta^n Z_{\mathrm{SYM}}}{(\mathrm{i}\delta j(z_1))\ldots(\mathrm{i}\delta j(z_n))}\right|_{\mathrm{J}=\bar{\mathrm{J}}=\mathscr{J}=\eta'=\ldots=0} \tag{3.34}$$

where $j(z_i)$ is the current of the respective field. To calculate the vertices on tree-level, it is however easier to start from the effective action. It is defined by (cf. [23])

$$\begin{aligned} \Gamma_{\mathrm{SYM}}[\Phi, \bar{\Phi}, V, \ldots] &= \mathrm{i}\ln Z[\mathrm{J}, \bar{\mathrm{J}}, \mathscr{J}, \ldots] - \mathrm{J}\Phi - \bar{\mathrm{J}}\bar{\Phi} - \mathscr{J}V - \ldots \\ &= S_{\mathrm{SYM}}[\Phi, \bar{\Phi}, V, \ldots] + \text{loop corrections}\ . \end{aligned} \tag{3.35}$$

The n-vertices are then

$$\mathrm{i}\Gamma^{(F_1,\ldots,F_n)}_{0,\mathrm{SYM}}(z_1, \ldots, z_n) = \left.\frac{\mathrm{i}\delta^n S_{\mathrm{SYM}}}{\delta F(z_1)\ldots\delta F(z_n)}\right|_{\Phi=\bar{\Phi}=V=c'=\ldots=0}\ , \tag{3.36}$$

where $F(z_i)$ are the fields interacting in the vertex. To get the vertex in momentum space, (3.36) has to be Fourier transformed:[23]

$$\mathrm{i}\Gamma^{(F_1,\ldots,F_n)}_{0,\mathrm{SYM}}(p_1,\ldots,p_n) = \int d^4x_1 \ldots d^4x_n \mathrm{i}\Gamma^n_{0,\mathrm{SYM}}(z_1,\ldots,z_n) e^{-\mathrm{i}(x_1p_1+\ldots+x_np_n)} \ . \tag{3.37}$$

The vertices in coordinate space contain the derivatives D and $\bar{\mathrm{D}}$ (c.f. (3.39) and (3.41)). By Fourier transformation, they are transformed into operators $\mathrm{D}(p)$ and $\bar{\mathrm{D}}(p)$, defined in (B.30). In the following, the 3-vector-superfield-vertex and its Fourier transform are calculated. All others are similar and, knowing the first calculation, can be read off from the action. They are given in the next chapter.

For the calculation of $\Gamma^{(V^a,V^b,V^c)}_{0,\mathrm{SYM}}(z_1,z_2,z_3)$, the only relevant term of S_{SYM} is

$$\begin{aligned} S_{V^3} &= \int d^8z \frac{\mathrm{Tr}}{k}\frac{1}{4} g(\bar{\mathrm{D}}^2\mathrm{D}^\alpha V)[V,\mathrm{D}_\alpha V] \\ &= -\mathrm{i}g\frac{f^{abc}}{4}\int d^8z (\bar{\mathrm{D}}^2\mathrm{D}^\alpha V^a)(\mathrm{D}_\alpha V^b)V^c \ , \end{aligned} \tag{3.38}$$

using $V = V^aT^a$ and (2.30) for the adjoint representation.

Therefore,

$$\begin{aligned} &\mathrm{i}\Gamma^{(V^a,V^b,V^c)}_{0,\mathrm{SYM}}(z_1,z_2,z_3) = \frac{\mathrm{i}\delta^3 S_{V^3}}{\delta V^a(z_1)\delta V^b(z_2)\delta V^c(z_3)}\Big|_{V=0} \\ &= g\frac{f^{a'b'c'}}{4}\int d^8z \Big\{ (\bar{\mathrm{D}}^2\mathrm{D}^\alpha\delta^{aa'}\delta^8(z_1-z))(\mathrm{D}_\alpha\delta^{bb'}\delta^8(z_2-z))\delta^{cc'}\delta^8(z_3-z) \\ &\qquad -(\mathrm{D}_\alpha\delta^{aa'}\delta^8(z_1-z))(\bar{\mathrm{D}}^2\mathrm{D}^\alpha\delta^{bb'}\delta^8(z_2-z))\delta^{cc'}\delta^8(z_3-z) \\ &\qquad +(\bar{\mathrm{D}}^2\mathrm{D}^\alpha\delta^{aa'}\delta^8(z_1-z))\delta^{bb'}\delta^8(z_2-z)(\mathrm{D}_\alpha\delta^{cc'}\delta^8(z_3-z)) \\ &\qquad -(\mathrm{D}_\alpha\delta^{aa'}\delta^8(z_1-z))\delta^{bb'}\delta^8(z_2-z)(\bar{\mathrm{D}}^2\mathrm{D}^\alpha\delta^{cc'}\delta^8(z_3-z)) \\ &\qquad +\delta^{aa'}\delta^8(z_1-z)(\bar{\mathrm{D}}^2\mathrm{D}^\alpha\delta^{bb'}\delta^8(z_2-z))(\mathrm{D}_\alpha\delta^{cc'}\delta^8(z_3-z)) \\ &\qquad -\delta^{aa'}\delta^8(z_1-z)(\mathrm{D}_\alpha\delta^{bb'}\delta^8(z_2-z))(\bar{\mathrm{D}}^2\mathrm{D}^\alpha\delta^{cc'}\delta^8(z_3-z))\Big\} \ . \end{aligned} \tag{3.39}$$

Note that the sign of the permutation depends on the order of the D's, applied to the propagators (i.e. the contraction of the α-indices).

[23]as in Appendix B.2, the θ's are untouched by the Fourier transformation

The Fourier transformation leads to

$$
\begin{aligned}
&\mathrm{i}\Gamma^{(V^a,V^b,V^c)}_{0,\mathrm{SYM}}(p_1,p_2,p_3) = \int d^4x_1 d^4x_2 d^4x_3 \mathrm{i}\Gamma^{(V^a,V^b,V^c)}_{0,\mathrm{SYM}}(z_1,z_2,z_3)e^{-\mathrm{i}(x_1p_1+x_2p_2+x_3p_3)} \\
&= g\frac{f^{a'b'c'}}{4}\int d^4\theta d^4x \Big\{((\bar{\mathrm{D}}^2\mathrm{D}^\alpha\delta^{aa'})(p_1-p)\delta^4(\theta_1-\theta))((\mathrm{D}_\alpha)(p_2-p)\delta^{bb'}\delta^4(\theta_2-\theta)) \\
&\qquad\qquad \delta^{cc'}\delta^4(\theta_3-\theta)\delta^4(x_3-x)e^{-\mathrm{i}(x_3(p_1+p_2+p_3))} \\
&\qquad\qquad +5 \text{ other permutations}\} \\
&= g\frac{f^{a'b'c'}}{4}\int d^4\theta \Big\{((\bar{\mathrm{D}}^2\mathrm{D}^\alpha\delta^{aa'})(p_1-p)\delta^4(\theta_1-\theta))((\mathrm{D}_\alpha)(p_2-p)\delta^{bb'}\delta^4(\theta_2-\theta)) \\
&\qquad\qquad \delta^{cc'}\delta^4(\theta_3-\theta)(2\pi)^4\delta(p_1+p_2+p_3) \\
&\qquad\qquad +5 \text{ other permutations}\}
\end{aligned} \tag{3.40}
$$

where we have used the Fourier transformed D's (B.30).

Some features in this calculation are true for all vertices, namely:

- In coordinate space, every vertex is integrated over $\int d^8z$.
- In momentum space, every vertex is integrated over $\int d^4\theta$.
- Momentum is conserved in every vertex.

And, especially for this vertex:

- 3-vector superfield vertex $= g\frac{f^{abc}}{4} \times$ {6 permutations, in which $\mathrm{D}^2 D^\alpha$, D_α are applied to the three vector superfield propagators}.

For the chiral 3-vertex, a similar calculation yields

$$
\begin{aligned}
\mathrm{i}\Gamma^{(\Phi_i,\Phi_j,\Phi_k)}_{0,\mathrm{SYM}}(z_1,z_2,z_3) &= \mathrm{i}\lambda_{ijk}\int \mathrm{d}^2\theta d^4x \Bigg\{\left(-\frac{1}{4}\bar{\mathrm{D}}^2\delta^8(z-z_1)\right) \\
&\qquad \times\left(-\frac{1}{4}\bar{\mathrm{D}}^2\delta^8(z-z_2)\right)\left(-\frac{1}{4}\bar{\mathrm{D}}^2\delta^8(z-z_3)\right)\Bigg\} \\
= \mathrm{i}\lambda_{ijk}\int d^8z\,\delta^8(z-z_1)&\left(-\frac{1}{4}\bar{\mathrm{D}}^2\delta^8(z-z_3)\right)\left(-\frac{1}{4}\bar{\mathrm{D}}^2\delta^8(z-z_3)\right),
\end{aligned} \tag{3.41}
$$

so by rewriting the $\int d^2\theta d^4x$ to $\int d^8z$, one of the chiral fields is not differentiated by $-\frac{1}{4}\bar{\mathrm{D}}^2$. For external lines, however, the $-\frac{1}{4}\bar{\mathrm{D}}^2$ is needed to complete the external $\int d^2\theta d^4x$. The rules for (anti-)chiral vertices are therefore:

- In coordinate space, every vertex is integrated over $\int d^8z$.

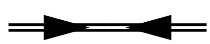
- In momentum space, every vertex is integrated over $\int d^4\theta$.
- For a chiral vertex with n internal lines, n-1 factors $-\frac{1}{4}\bar{\mathrm{D}}^2$ act on the propagators.
- For an anti-chiral vertex with n internal lines, n-1 factors $-\frac{1}{4}\mathrm{D}^2$ act on the propagators.

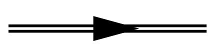
In addition to the Feynman rules, stated above, there are rules for external momenta and for internal loops, which follow from momentum conservation.

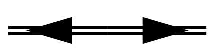
3.3 Table of Feynman Rules and D-algebra

In coordinate space, the Feynman rules are

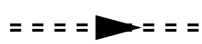
- Draw all possible topologically inequivalent graphs of the order in g which is to be considered.
- Multiply each graph with a symmetry factor $\frac{1}{S}$ where S is the number of possibilities, mapping the graph onto itself by exchanging internal lines and vertices.[24]
- For each vertex integrate over $\displaystyle\int d^8 z$.
- For each external field integrate over $\displaystyle\int \mathrm{d}^8 z_{\text{ext}}$.
- For a chiral vertex with n internal lines, $n-1$ factors $-\frac{1}{4}\bar{\mathrm{D}}^2$ act on the propagators.
- For an anti-chiral vertex with n internal lines, $n-1$ factors $-\frac{1}{4}\mathrm{D}^2$ act on the propagators.
- A ghost loop contributes a factor (-1).
- For internal lines write (propagators):

$\bar{\Phi}_i(z)$  $\bar{\Phi}_j(z') \;=\; \frac{\bar{\mathrm{D}}^2}{4}\frac{im}{\Box(\Box-m^2)+i\epsilon}\delta_{ij}\delta^8(z-z')$

$\bar{\Phi}_i(z)$  $\Phi_j(z') \;=\; \frac{\mathrm{i}}{\Box-m^2+i\epsilon}\delta_{ij}\delta^8(z-z')$

$\Phi_i(z)$  $\Phi_j(z') \;=\; \frac{\mathrm{D}^2}{4}\frac{im}{\Box(\Box-m^2)+i\epsilon}\delta_{ij}\delta^8(z-z')$

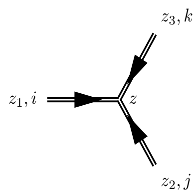
$V^a(z)$  $V^b(z') \;=\; -\frac{1}{2}\frac{\mathrm{i}}{\Box+i\epsilon}\delta^{ab}\delta^8(z-z')$

$\bar{c}'^a(z)$  $c^b(z') \;=\; -\frac{\mathrm{i}}{\Box+i\epsilon}\delta^{ab}\delta^8(z-z')$

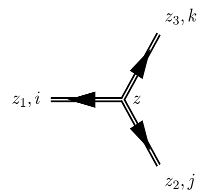
$c'^a(z)$  $\bar{c}^b(z') \;=\; \frac{\mathrm{i}}{\Box+i\epsilon}\delta^{ab}\delta^8(z-z')$

[24] the safer way of determining the symmetry factor is to derive it directly from the generating functional but for all calculations presented in this diploma thesis, this leads to the same result as the rule stated.

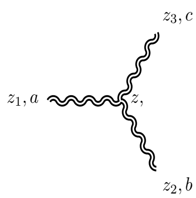
- The vertices in coordinate space are (including the integration $\int d^8z$):



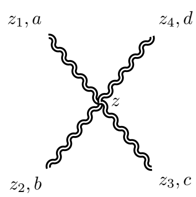
$= \mathrm{i}\lambda_{ijk} \int \mathrm{d}^8 z$



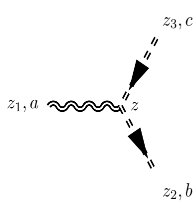
$= \mathrm{i}(\lambda_{ijk})^* \int \mathrm{d}^8 z$



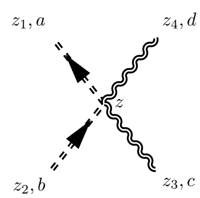
$$= g\frac{f^{a'b'c'}}{4} \int d^8 z \Big\{ (\bar{\mathrm{D}}^2 \mathrm{D}^\alpha \delta^{aa'} \delta^8(z_1 - z)) \times (\mathrm{D}_\alpha \delta^{bb'} \delta^8(z_2 - z)) \delta^{cc'} \delta^8(z_3 - z) + 5 \text{ permutations} \}$$



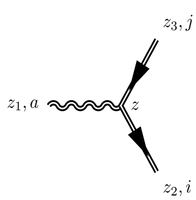
$$= -\mathrm{i}g^2 f^{a'b'e'} f^{c'd'e'} \int d^8 z \Big\{ \tfrac{1}{6} (\bar{\mathrm{D}}^2 \mathrm{D}^\alpha \delta^{a'a} \delta^8(z_1 - z)) \delta^{b'b} \delta^8(z_2 - z) \delta^{c'c} \delta^8(z_3 - z) \times (\mathrm{D}_\alpha \delta^{d'd} \delta^8(z_4 - z)) - \tfrac{1}{8} \delta_{a'a} \delta^8(z_1 - z) (\mathrm{D}^\alpha \delta^{b'b} \delta^8(z_2 - z)) \times \bar{\mathrm{D}}^2 (\delta^{c'c} \delta^8(z_3 - z) \mathrm{D}^\alpha \delta^{d'd} \delta^8(z_4 - z)) + 23 \text{ permutations} \Big\}$$



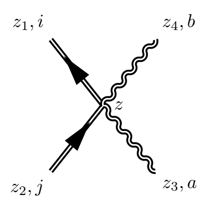
$= g f^{abc} \int d^8 z$



$= -\frac{\mathrm{i}}{3} g^2 f^{abe} f^{cde} \int d^8 z$



$= \mathrm{i}2g(\mathrm{T}^a)_{ij} \int d^8 z$

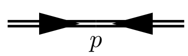
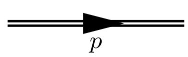
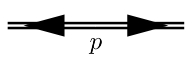
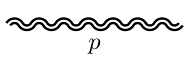
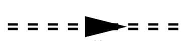
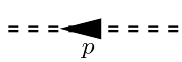


$= \mathrm{i}4g^2 (\mathrm{T}^a)_{ik} (\mathrm{T}^b)_{kj} \int d^8 z$

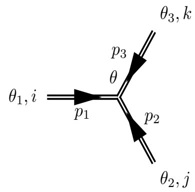
and vertices of the order $O(g^3)$ and higher.

In momentum space the Feynman rules are

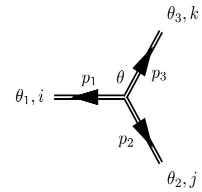
- Draw all possible topologically inequivalent graphs of the order in g which is to be considered.
- Multiply each graph with its symmetry factor $\frac{1}{S}$.
- p_i are momenta flowing along internal lines, *away* from the vertex.
- For each vertex integrate over $\int d^4\theta$.
- For each external field integrate over $\int \mathrm{d}^4\theta_{\mathrm{ext}}$.
- For the external momenta, the overall factor is $\left[\prod_{p_{\mathrm{ext}}} \int \frac{d^4 p_{\mathrm{ext}}}{(2\pi)^4}\right] (2\pi)^4 \delta^4 \left(\sum_{\mathrm{ext}} p_{\mathrm{ext}}\right)$.
- For every loop with momentum p running in it integrate over $\int \frac{d^4 p}{(2\pi)^4}$.
- For an [anti-]chiral vertex with n internal lines, $n-1$ factors $[-\frac{1}{4}\mathrm{D}^2]$ $-\frac{1}{4}\bar{\mathrm{D}}^2$ act on the propagators.
- A ghost loop contributes a factor (-1).
- The propagators in momentum space are :

$\bar\Phi_i(-p)$		$\bar\Phi_j(p)$	$=$	$\frac{\bar{\mathrm{D}}^2(p)}{4}\frac{\mathrm{i}m}{p^2(p^2+m^2)-i\epsilon}\delta_{ij}\delta^4(\theta-\theta')$
$\bar\Phi_i(-p)$		$\Phi_j(p)$	$=$	$-\frac{\mathrm{i}}{p^2+m^2-i\epsilon}\delta_{ij}\delta^4(\theta-\theta')$
$\Phi_i(-p)$		$\Phi_j(p)$	$=$	$\frac{\mathrm{D}^2(p)}{4}\frac{\mathrm{i}m}{p^2(p^2+m^2)-i\epsilon}\delta_{ij}\delta^4(\theta-\theta')$
$V^a(-p)$		$V^b(p)$	$=$	$\frac{1}{2}\frac{\mathrm{i}}{p^2-i\epsilon}\delta^{ab}\delta^4(\theta-\theta')$
$\bar c'^a(-p)$		$c^b(p)$	$=$	$\frac{\mathrm{i}}{p^2-i\epsilon}\delta^{ab}\delta^4(\theta-\theta')$
$c'^a(-p)$		$\bar c^b(p)$	$=$	$-\frac{\mathrm{i}}{p^2-i\epsilon}\delta^{ab}\delta^4(\theta-\theta')$

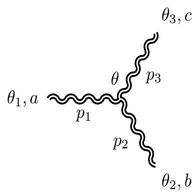
- The vertices in momentum space are:



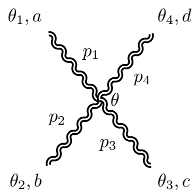
$$= \mathrm{i}\lambda_{ijk} \int \mathrm{d}^4\theta$$



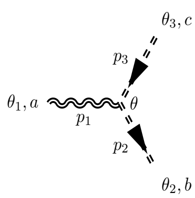
$$= \mathrm{i}(\lambda_{ijk})^* \int \mathrm{d}^4\theta$$



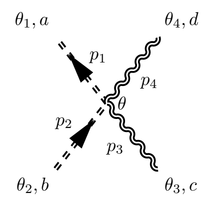
$$= g\frac{f^{a'b'c'}}{4} \int d^4\theta \Big\{ ((\bar{\mathrm{D}}^2 \mathrm{D}^\alpha \delta^{aa'})(p_1 - p)\delta^4(\theta_1 - \theta)) \\ ((\mathrm{D}_\alpha)(p_2 - p)\delta^{bb'}\delta^4(\theta_2 - \theta))\delta^{cc'}\delta^4(\theta_3 - \theta) \\ +5 \text{ permutations}\} \times (2\pi)^4\delta^4(p_1 + p_2 + p_3)$$



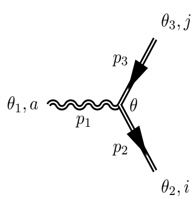
$$= -\mathrm{i}g^2 f^{a'b'e'} f^{c'd'e'} \int d^4\theta \Big\{ \tfrac{1}{6}((\bar{\mathrm{D}}^2 \mathrm{D}^\alpha(p_1 - p))\delta^{a'a}\delta^4(p_1 - p)) \\ \delta^{b'b}\delta^4(p_2 - p)\delta^{c'c}\delta^4(p_3 - p)((\mathrm{D}_\alpha)(p_4 - p)\delta^{d'd}\delta^4(p_4 - p)) \\ -\tfrac{1}{8}\delta^{a'a}\delta^4(p_1 - p)((\mathrm{D}_\alpha)(p_2 - p)\delta^{b'b}\delta^4(p_2 - p)) \\ \bar{\mathrm{D}}^2(\delta^{c'c}\delta^4(p_3 - p)(\mathrm{D}^\alpha)(p_4 - p)\delta^{d'd}\delta^4(p_4 - p)) \\ + 23 \text{ perm.}\} \times (2\pi)^4\delta^4(p_1 + p_2 + p_3 + p_4)$$



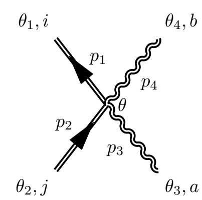
$$= g f^{abc} \int d^4\theta$$



$$= -\tfrac{\mathrm{i}}{3} g^2 f^{abe} f^{cde} \int d^4\theta$$



$$= \mathrm{i}2g(\mathrm{T}^a)_{ij} \int d^4\theta$$



$$= \mathrm{i}4g^2(\mathrm{T}^a)_{ik}(\mathrm{T}^b)_{kj} \int d^4\theta$$

and vertices of the order $O(g^3)$ and higher.

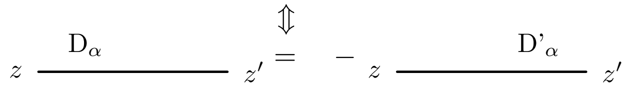
For the calculation of Feynman graphs, some of the rules in Appendix B.2 are needed. Their use is simplified by re-expressing them as graphical rules, which can be applied directly to the graphs [24, 22].

The rules which are used in the next chapter are:

- Transfer rule (coordinate space)

$$\mathrm{D}_{1\alpha}\delta^8_{12} \quad = \quad - \quad \mathrm{D}_{2\alpha}\delta^8_{12}$$

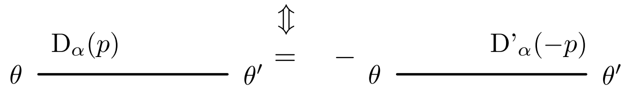
$$\Updownarrow$$



- Transfer rule (momentum space)

$$\mathrm{D}_{1\alpha}(p)\delta^4_{12} \quad = \quad - \quad \mathrm{D}_{2\alpha}(-p)\delta^4_{12}$$

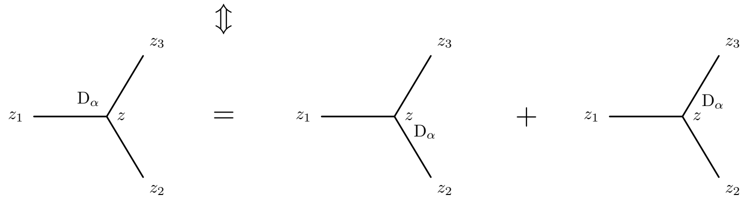
$$\Updownarrow$$



- partial integration I

$$(\mathrm{D}_{1\alpha}\delta^8_{12})\delta^8_{13}\delta^8_{14} \quad = \quad \delta^8_{12}(\mathrm{D}_{1\alpha}\delta^8_{13})\delta^8_{14} \quad + \quad \delta^8_{12}\delta^8_{13}(\mathrm{D}_{1\alpha}\delta^8_{14})$$

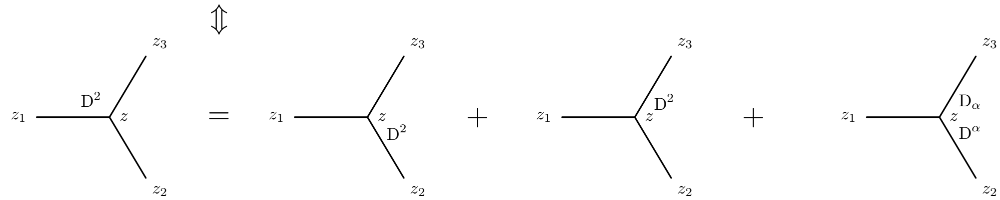
$$\Updownarrow$$



- partial integration II

$$(\mathrm{D}_1^2\delta^8_{12})\delta^8_{13}\delta^8_{14} \quad = \quad \delta^8_{12}(\mathrm{D}_1^2\delta^8_{13})\delta^8_{14} \quad + \quad \delta^8_{12}\delta^8_{13}(\mathrm{D}_1^2\delta^8_{14}) \quad + \quad \delta^8_{12}(\mathrm{D}_1^\alpha\delta^8_{13})(\mathrm{D}_{1_\alpha}\delta^8_{14})$$

$$\Updownarrow$$



The rules are for $\bar{\mathrm{D}}$ are completely analogous.

The D-algebra calculations of any Feynman graph can be performed by using these rules and the anti-commutation-relations (2.7) (in coordinate space) and (B.30) (in momentum space) to reshuffle all D's in a loop to only one propagator. There, by using the identities (B.34), only terms $O(\mathrm{D}^2\bar{\mathrm{D}}^2)$ contribute under the superspace integral and the loop is shrunk to a point in θ-space.[25] This can be done for all loops until all internal θ-integrations are performed.

The rules for partial integration are valid for a single vertex. For a full graph, however, they are only true up to a sign which depends on the number of permutations of D's in the integral expression which are done by the partial integration. This depends on the choice of writing down the integral from the graph.

A way of dealing with this ambiguity is to define an order in which the integrals are written down, then doing the manipulations, ignoring the sign of the expression and finally determining the sign by counting the number of permutations and raising/lowering of indices which need to be done to get the D's in the original order [22].

3.4 Application: Calculation of the β-Function

As an application of the Feynman rules derived in the last chapter we calculate the β-function of super Yang-Mills theory to first loop order. The result for a super Yang-Mills theory with massless chiral multiplets coupled to it is known and can be found in [23, 25].

For the calculation of the β-function, it proves convenient to rescale the vector superfield by $gV \to V$. Then, the super Yang-Mills action reads[26]

$$\begin{aligned} S'_{SYM} = \int d^8z \Bigg\{ & \frac{\mathrm{Tr}}{16kg^2}\left(W'^{\alpha}W'_{\alpha}\delta(\bar{\theta}) + \mathrm{h.c.}\right) + \frac{\mathrm{Tr}}{8\gamma kg^2}\Big((\bar{\mathrm{D}}^2V)(\mathrm{D}^2V)\Big) \\ & -\mathrm{i}\frac{\mathrm{Tr}}{kg^2}(c' + \bar{c}')(\delta(2V)) \\ & +\bar{\Phi}_l \mathrm{e}^{2V}\Phi_l + \left[\frac{1}{2}m_{ij}\Phi_i\Phi_j\delta(\bar{\theta}) + \frac{1}{3!}\lambda_{ijk}\Phi_i\Phi_j\Phi_k\delta(\bar{\theta}) + \mathrm{h.c.}\right]\Bigg\} \end{aligned} \tag{3.42}$$

where

$$W'_{\alpha} = -\frac{1}{4}\bar{\mathrm{D}}^2 e^{-2V} D_{\alpha} e^{2V} \ . \tag{3.43}$$

[25] This is possible for all superfield graphs in any N=1, D=4 supersymmetric theory. This theorem is also known a the non-renormalisation theorem of supersymmetry [24].

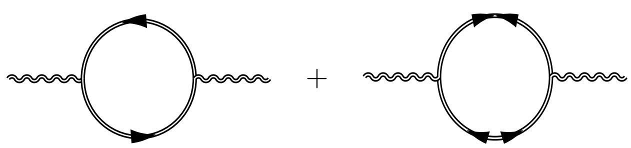
[26] cf. (3.11) and (2.42) for the gauge and matter contributions in the old normalisation

In this normalisation, the kinetic term of the chiral superfields is independant of the coupling constant g. Hence, the $\bar{\Phi}\Phi V$-vertex correction does not contribute to the renormalisation of the coupling constant. To calculate the β-function to one-loop level it is therefore sufficient to consider the radiative corrections to the vector superfield propagator.

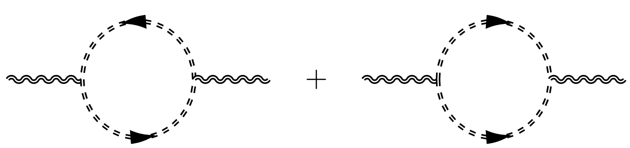
The new normalisation leads to modified Feynman rules. However, performing the calculations for the radiative corrections with these new Feynman rules is equivalent to using the Feynman rules given in the last chapter and rescaling the result by $gV \to V$.

For the radiative correction to the vector superfield propagator to first loop order several loop diagrams contribute. They are listed in (3.44). We calculate chiral loop contributions in Chapter 3.4.1. The contributions from ghost and vector superfield loops are computed in Chapter 3.4.2. With the results, we perform the rescaling and calculate the β-function in Chapter 3.4.3.

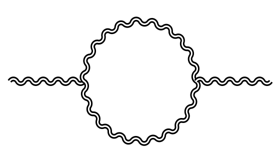
The supergraphs contributing to the the vector superfield propagator to first loop order are:



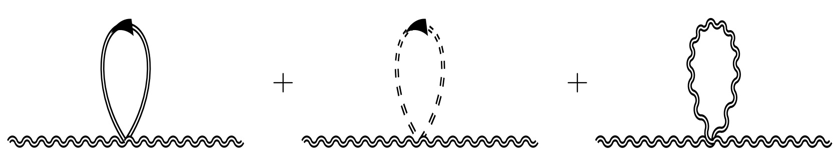
(3.44a)



(3.44b)



(3.44c)



(3.44d)

The main tools of our calculations will be the D-algebra rules in Chapter 3.3 as well as the projector algebra (B.15) and the anticommutator (B.31) and the identities (B.17) following from it to reduce expressions containing more than four D's on a propagator and the identity (B.34) to reduce the loops to points in spacetime. The result of the graphs are terms proportional to quadratic, "linear" and logarithmic divergences in the loop momentum.[27] To deal with the divergences, we have to regularise the theory. Dimensional regularisation is not appropriate for supersymmetric theories as by varying spacetime dimensions, the number of generators of the Lorentz algebra varies. For a non-supersymmetric theory, one can continue the Lorentz algebra to arbitrary dimension preserving all symmetries except for chiral symmetries (the so called γ_5-problem). However, supersymmetry is broken by dimensional regularisation as by varying the number of Lorentz generators, the number of supersymmetry generators would have to vary, too. The most convenient way to regularise in supersymmetry is therefore to analytically continue the momenta and spacetime coordinates and treat momentum integrals as in dimensional regularisation but to keep the spacetime dimension of all other tensors and spinors fixed. This is the concept of dimensional reduction [34, 35].

In general, dimensional reduction does not respect gauge symmetry. However, dealing with scalar superfields only,[28] there are no other tensors but the momenta, so there is no difference between dimensional regularisation and dimensional reduction in our case and the gauge symmetry is maintained in the regularised theory. In Appendix C we have listed all D-dimensional integrals we use in this thesis including their four- and five-dimensional limits.

3.4.1 Contributions of Chiral Superfields

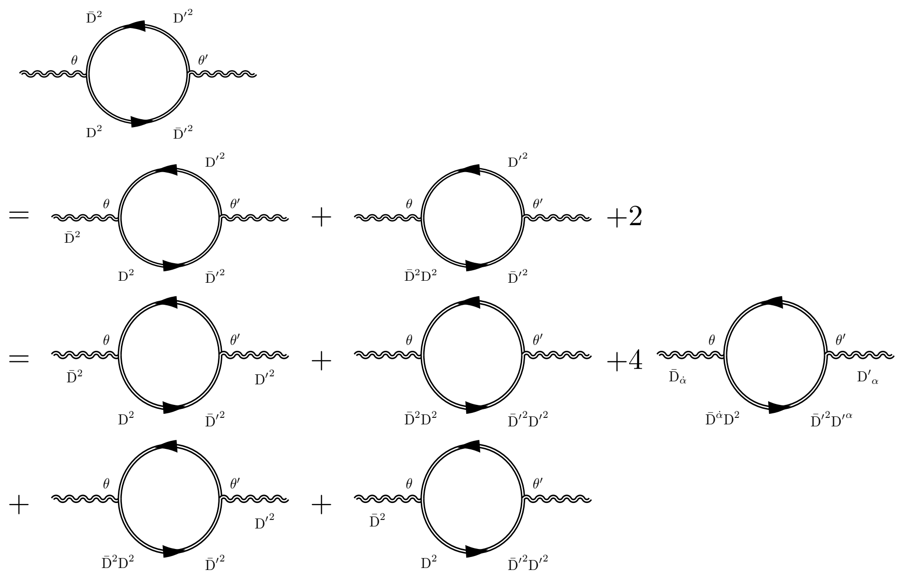
As can be seen in (3.44) there are three distinct chiral loop graphs (the graphs in (3.44a) and the left graph in (3.44d)) contributing to the one-loop correction of the vector superfield propagator.

The first step of the calculation of the left graph of (3.44a) can be done using the graphical D-algebra rules given in the last chapter. For the sake of legibility here and in what follows we write D^2 and $\bar{\mathrm{D}}^2$ on graphs to represent the $-\frac{1}{4}\mathrm{D}^2$ and $-\frac{1}{4}\bar{\mathrm{D}}^2$ factors on the chiral propagators.

[27] With "linear" divergences we mean expressions $\propto \int \frac{d^4k}{(2\pi)^4} \frac{k_m}{(k^2+m^2)((k+p)^2+m^2)}$. Due to (C.8) they do not yield linear but quadratic and logarithmic divergences.

[28] No superfield under consideration is carrying spinor or Lorentz indices. Only their components do.

The D-algebra then yields

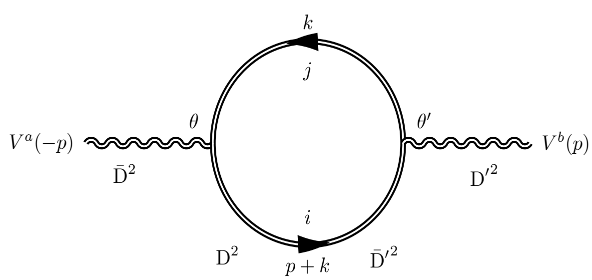


(3.45)

where we used that in the end only graphs with no D's on one internal propagator and and even number of D's on the other one[29] survive, according to (B.34).

In the last two graphs, an unequal number of D's and D̄'s act on the lower internal propagator. Therefore they do not contribute.

Using the Feynman rules, the first graph leads to[30]



$$= \int \mathrm{d}^4\theta \mathrm{d}^4\theta' \frac{\mathrm{d}^4 k}{(2\pi)^4} \left(\left(-\frac{1}{4}\bar{\mathrm{D}}^2 \right) V^a(-p,\theta) \right) \mathrm{i}2gT^a_{ij}$$
$$\left(\left(-\tfrac{1}{4}\mathrm{D}^2\right) \left(-\tfrac{1}{4}\bar{\mathrm{D}}'^2\right) \delta_{ij}\delta^4(\theta-\theta') \left(-\tfrac{\mathrm{i}}{k^2+m^2}\right) \right) \mathrm{i}2gT^b_{ji}$$
$$\delta_{ji}\delta^4(\theta'-\theta) \left(-\tfrac{\mathrm{i}}{(k+p)^2+m^2}\right) \left(-\tfrac{1}{4}\mathrm{D}'^2\right) V^b(p,\theta') \tag{3.46}$$

[29]which by anticommutation can be reduced to four D's

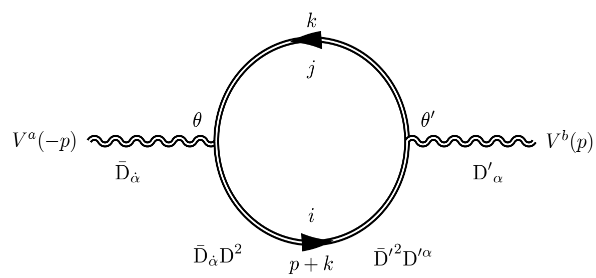
[30]We always choose the loop momentum k to run counterclockwise.

$$= 4g^2 \sum_A T_A(R) \int \mathrm{d}^4\theta \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\left(-\frac{1}{4}\mathrm{D}^2 V^a(-p)\right)\left(-\frac{1}{4}\bar{\mathrm{D}}^2 V^a(p)\right)}{(k^2+m^2)((k+p)^2+m^2)}$$

$$= 4g^2 \sum_A T_A(R) \times \left\{ \frac{1}{2}\int \mathrm{d}^4\theta \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{V^a(-p)(p^2 P_T - \frac{1}{2}\mathrm{D}^\alpha \mathrm{D}^{\dot\alpha} p_{\alpha\dot\alpha})V^a(p)}{(k^2+m^2)((k+p)^2+m^2)} \right\} \tag{3.47}$$

where $\sum_A T_A(R)\delta^{ab} = T^a_{ij}T^b_{ji}$ depends on the representation R in which the matter fields are chosen. In the last step, we did several partial integrations and used the projector property (B.16) and the $[\mathrm{D}, \bar{\mathrm{D}}^2]$ commutator from (B.32).

The third graph yields

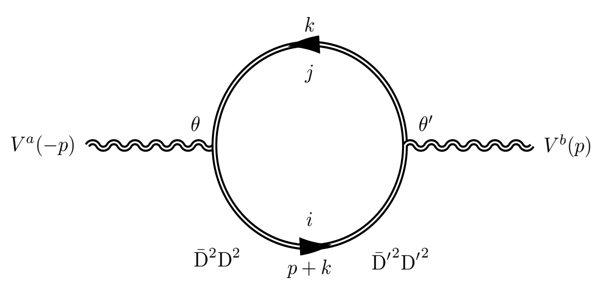


$$\begin{aligned} = 4 \times \int \mathrm{d}^4\theta \mathrm{d}^4\theta' \frac{\mathrm{d}^4 k}{(2\pi)^4} &\left(\left(-\frac{1}{4}\bar{\mathrm{D}}_{\dot\alpha}\right) V^a(-p,\theta)\right) \mathrm{i}2gT^a_{ij} \\ &\left(\left(-\tfrac{1}{4}\bar{\mathrm{D}}^{\dot\alpha}\mathrm{D}^2\right)\left(-\tfrac{1}{4}\bar{\mathrm{D}}'^2\mathrm{D}'^\alpha\right)\delta_{ij}\delta^4(\theta-\theta')\left(-\tfrac{\mathrm{i}}{k^2+m^2}\right)\right) \mathrm{i}2gT^b_{ji} \\ &\delta_{ji}\delta^4(\theta'-\theta)\left(-\tfrac{\mathrm{i}}{(k+p)^2+m^2}\right)\left(-\tfrac{1}{4}\mathrm{D}'_\alpha\right) V^b(p,\theta') \end{aligned}$$

$$= 4g^2 \sum_A T_A(R) \times \left\{ \tfrac{1}{2}\int \mathrm{d}^4\theta \tfrac{\mathrm{d}^4 k}{(2\pi)^4} \tfrac{V^a(-p)\mathrm{D}^\alpha \mathrm{D}^{\dot\alpha}(k+p)_{\alpha\dot\alpha}V^a(p)}{(k^2+m^2)((k+p)^2+m^2)} \right\} , \tag{3.48}$$

where we again used (B.32). Using (C.9), this graph therefore cancels the contribution proportional to p of the first graph.

Using the projector property (B.15), the second graph in (3.45) leads to



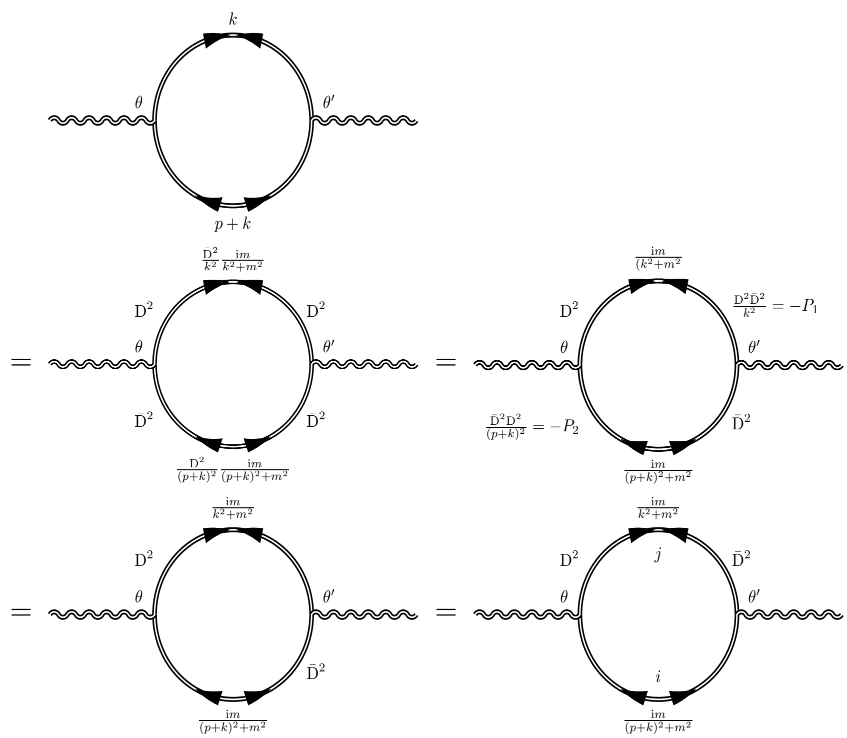
(3.49)

$$= -8g^2 \sum_A T_A(R) \times \left\{ \tfrac{1}{2}\int \mathrm{d}^4\theta \tfrac{\mathrm{d}^4 k}{(2\pi)^4} \tfrac{V^a(-p)(k+p)^2 V^a(p)}{(k^2+m^2)((k+p)^2+m^2)} \right\}$$

For the graph with the chiral superfields running anticlockwise, we get the same result. However, this is not a topologically new graph as it is just the graph with the chiral fields running clockwise and both internal lines exchanged. Therefore (3.46) and (3.49) contibute just once.

The right graph of (3.44a) only contributes in a theory where *massive* chiral supermultiplets are coupled to the gauge supermultiplet as for the massless case, the $\Phi\Phi$ and $\bar{\Phi}\bar{\Phi}$ propagators vanish.[31]

The calculation of the graph again includes some D-algebra. Using the fact that the propagators are now (anti-)chiral and the projection operators defined in (B.15), the calculation yields[32]

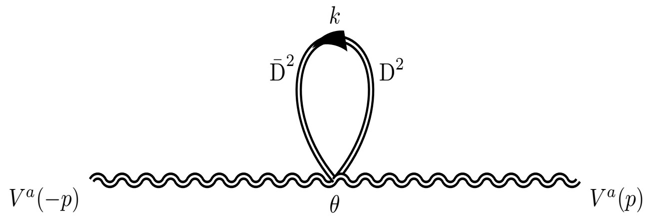


$$= -8g^2 \sum_A T_A(R) \times \left\{ \frac{1}{2} \int \mathrm{d}^4\theta \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{V^a(-p) m^2 V^a(p)}{(k^2+m^2)((k+p)^2+m^2)} \right\} \tag{3.50}$$

[31] We point out that this is not the case in the $D=5$ theory where $\Phi\Phi$ and $\bar{\Phi}\bar{\Phi}$ propagators are always present due to the derivative in 5-direction appearing in the propagator (cf. 4.45).

[32] We only consider graphs contributing to the divergence.

From the left graph of (3.44d) we get a contribution



(3.51)

$$= 8g^2 \sum_A T_A(R) \times \left\{ \tfrac{1}{2} \int \mathrm{d}^4\theta \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{V^a(-p)V^a(p)}{k^2+m^2} \right\} .$$

There is a slight subtlety in reading off the integral expression from the Feynman graph as the term $\int \mathrm{d}^4\theta \frac{1}{16}\bar{\mathrm{D}}^2\mathrm{D}^2\delta^4(\theta-\theta)$ occurs in the θ integration. Deriving the loop from the generating functional, it can be shown that this term has to be read as $\int \mathrm{d}^4\theta \mathrm{d}^4\theta' \frac{1}{16}\delta^4(\theta-\theta')\bar{\mathrm{D}}^2\mathrm{D}'^2\delta^4(\theta-\theta')$ as one might have expected.

Collecting our results from (3.46), (3.49) and (3.51), we get

$$\begin{aligned}
\Pi_\Phi(p^2) &= 4g^2 \sum_A T_A(R) \int \mathrm{d}^4\theta \frac{\mathrm{d}^4k}{(2\pi)^4} \\
&\quad \left\{ \frac{1}{2} V^a(-p) \frac{p^2 P_T + 2((k+p)^2+m^2) - 2(k+p)^2 - 2m^2}{(k^2+m^2)((k+p)^2+m^2)} V^a(p) \right\} \\
&= 4g^2 \sum_A T_A(R) \times \left\{ \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{k^2+m^2} \frac{1}{(p+k)^2+m^2} \right\} \\
&\qquad \times \left\{ \int \mathrm{d}^4\theta \, \frac{1}{2} V^a(-p) p^2 P_T V^a(p) \right\}
\end{aligned} \tag{3.52}$$

Note that the quadratic divergences in the three graphs cancel due to (C.9) and we are left with a logarithmic divergence only. This is a necessary condition for the renormalisability of the theory as a quadratic divergence would yield a term proportional to $m^2 \int d^4\theta V^a(-p,\theta)V^a(p,\theta)$ due to (C.10b) and (C.10d) for which no counterterm exists (cf. the analysis of counterterms in [23]).[33]

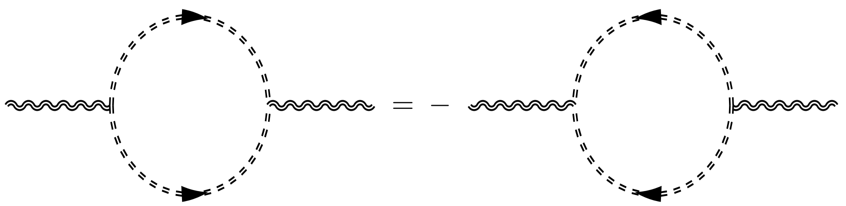
3.4.2 Contributions of Ghost and Vector Superfields

The contributions to the vector superfield propagator coming from the pure super Yang-Mills theory are the supergraphs in (3.44b), (3.44c) and the middle and right graph in (3.44d).

[33] The contributions from the ghost and vector superfield loops cannot contribute a term proportional to $m^2 \int d^4\theta V^a(-p,\theta)V^a(p,\theta)$ either as only massless propagator occur.

The tadpole graphs in (3.44d) lead to quadratic divergences which vanish due to (C.10a).

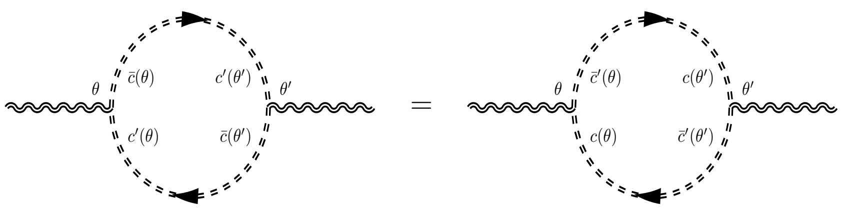
For ghost loops we have to consider the graphs in (3.44b) as well these two graphs with the vertices exchanged. Taking the statistics of the ghosts into account yields



(3.53)

These graphs therefore cancel.

For the left ghost loop in (3.44b) note, that there are two distinct ways of assigning ghost fields to the graph which contribute the same as can be seen from the Feynman rules.



(3.54)

As each of the graphs is symmetric under exchange of its vertices, both contribute once.

The calculation of (3.54) is the same as in (3.46) except that for the ghost loop we get an additional factor (-1) for the loop, and a factor of $(-\frac{1}{2})^2$ as each vertex contributes gf^{abc} instead of $2\mathrm{i}g(T^a)_{ij}$. Furthermore, for the adjoint representation, $\sum_A T_A(R)$ becomes $\mathrm{Tr}(T^aT^b) = k$ which, according to (2.30) is just the definition of the 2nd Casimir $C_2(G) \equiv k$.

The "linear" divergences of the graph (3.44b) cancel in the same way as for the chiral fields. The quadratic divergence of the graph does not need to be taken into account because the ghosts are massless and vanish in dimensional reduction because of (C.10a). We nevertheless give the expression as we will need it for the calculation in five dimensions:

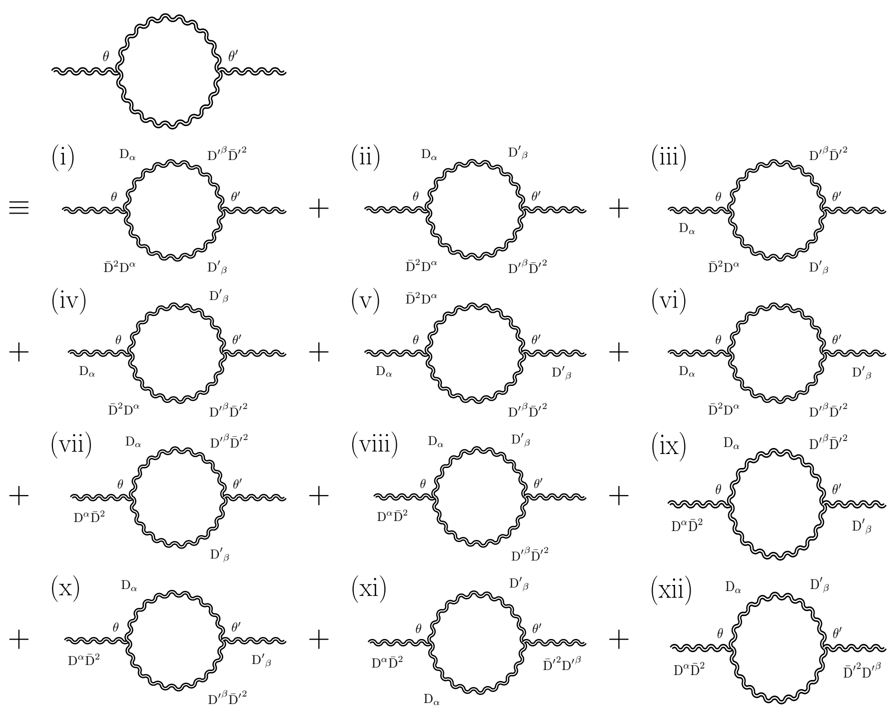
$$
\begin{aligned}
&= 4g^2 C_2(G) \times \left\{ \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{(k+p)^2}{k^2(k+p)^2} \right\} \\
&\qquad \times \left\{ \frac{1}{2} \int \mathrm{d}^4\theta V^a(-p) V^a(p) \right\}
\end{aligned}
\tag{3.55}
$$

The tadpole ghost graph in (3.44d) leads to a quadratic diverce of the same type. It does not cancel (3.55) but vanishes due to (C.10a) as well.

Having no quadratic and linear divergences, the ghost loop contribution to the vector superfield propagator is therefore

$$
\begin{aligned}
\Pi_c(p^2) &= 2 \times (-1) \times (-\tfrac{1}{2})^2 \times ((3.46) \text{ in adjoint representation}) \\
&= -2g^2 C_2(G) \times \left\{ \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 (k+p)^2} \right\} \\
&\qquad \times \left\{ \frac{1}{2} \int \mathrm{d}^4\theta V^a(-p) p^2 P_T V^a(p) \right\}
\end{aligned}
\tag{3.56}
$$

The last and most involved calculation is the one of the vector superfield loop (3.44c). As every vertex contains six permutations in which the $\bar{\mathrm{D}}^2\mathrm{D}^\alpha$ and the D_α act on the different propagators, there are 36 graphs contributing to this loop. The topologically inequivalent are[34]



(3.57)

The graphs (xi) and (xii) vanish as even after partial integration there are no four D's acting on one propagator when the other one

[34]In this calculation, there are no factors $-\frac{1}{4}$ attached to the D's acting on the propagator.

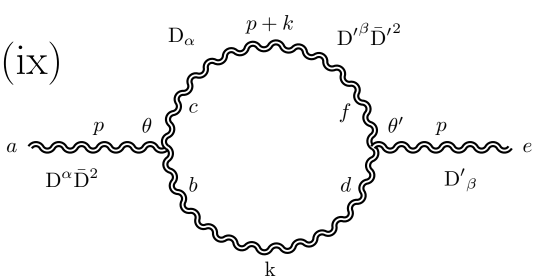
is free of D's. The graphs (iv) and (vi) vanish because after using the identity $\bar{\mathrm{D}}^2(p_2)\mathrm{D}^2(p_1)\bar{\mathrm{D}}^2(p_2) = -4(p_1+p_2)^2\bar{\mathrm{D}}^2(p_2)\delta^4(\theta_1-\theta_2)$ (cf. (B.32)), there is only $\bar{\mathrm{D}}^2$ acting on one propagator and no derivative on the other one. Quadratic divergences can only occur in the graphs (i) and (ii). Using the graphical D-algebra rules it can easily be seen that they cancel.[35] "Linear" divergences only occur in (i) and (iii) as in all other graphs as these are the only graphs in which exactly three D's and three $\bar{\mathrm{D}}$'s can end up on the lower propagator. They cancel by the same argument as the quadratic divergences.[36]

Furthermore, just considering the logarithmically divergent parts of the graphs, we get (iii)=(v)=(vii)=(viii)=(x)=(ix) while (i)=-(ix).

The vector tadpole graph (3.44d) again leads to a quadratic divergence which is canceled in dimensional reduction.

The one-loop contribution of the vector superfield loop to the vector superfield propagator is therefore $5 \times$ (ix), where (ix) is

(ix)



$$
\begin{aligned}
&= \int \mathrm{d}^4\theta \mathrm{d}^4\theta' \frac{\mathrm{d}^4k}{(2\pi)^4} \left\{ \left(\bar{\mathrm{D}}^2\mathrm{D}^\alpha V^a(-p,\theta)\right) \left(\frac{1}{4} g f^{abc}\right) \left(-\frac{1}{2}\frac{\mathrm{i}}{k^2}\delta^{bd}\delta(\theta-\theta')\right) \right. \\
&\quad \left. \left(\tfrac{1}{4} g f^{def}\right) \left(-\tfrac{1}{2}\tfrac{\mathrm{i}}{(p+k)^2}\delta^{ec}\left[\mathrm{D}_\alpha\bar{\mathrm{D}}^2\mathrm{D}^\beta\delta(\theta'-\theta)\right]\right) \mathrm{D}_\beta V^f(-p,\theta') \right\} \\
&= \tfrac{1}{64} g^2 f^{abc} f^{bcf} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \mathrm{d}^4\theta \left(\bar{\mathrm{D}}^2\mathrm{D}^\alpha V^a(-p)\right) \frac{8}{k^2}\frac{1}{(p+k)^2} \mathrm{D}_\alpha V^f(p) \\
&= -2g^2 C_2(G) \times \left\{ \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2}\frac{1}{(p+k)^2} \right\} \times \left\{ \tfrac{1}{2} \int \mathrm{d}^4\theta V^a(-p) p^2 P_T V^a(p) \right\}
\end{aligned}
\tag{3.58}
$$

where we used the definition of $C_2(G)$.

[35] After partial integration of all D's to the lower propagator, the $\bar{\mathrm{D}}'$ are contracted once top-bottom and once bottom-top yielding a relative factor of -1.

[36] Even if the quadratic and "linear" divergences would not cancel, in 4D, they would not contribute due to (C.9) and (C.10a). However as the terms manifestly cancel without using dimensional reduction, this will hold for this graph in the 5D theory as well.

Adding 5×(ix) to the ghost loop term, we get the overall divergence

$$\Pi_{V,c}(p^2) = -4g^2\left(\frac{1}{2}+\frac{5}{2}\right)C_2(G)\times\left\{\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2(p+k)^2}\right\}\times\left\{\frac{1}{2}\int\mathrm{d}^4\theta V^a(-p)p^2P_TV^a(p)\right\} \tag{3.59}$$

3.4.3 The result for the β-Function

As discussed in the beginning of Chapter 3.4, we have to rescale the theory by $gV \to V$ in order to calculate the β-function from the radiative corrections to the vector superfield propagator only. The the effective tree level action has been given in (3.42). Fourier transforming it to momentum space, the kinetic part of the vector superfield reads

$$S'^{(0)}_{V,\mathrm{kin}} = \frac{1}{g_0^2}\int\mathrm{d}^4\theta\frac{\mathrm{d}^4p}{(2\pi)^4}\left\{\frac{1}{2}V^a(-p)p^2P_TV^a(p)\right\} \tag{3.60}$$

where we write g_0 to indicate that this is the bare coupling.

Using the results (3.52) and (3.59) from the last chapter and rescaling it by $gV \to V$, the effective action to first loop order reads[37]

$$S'^{(1)}_{V,\mathrm{kin}} = S'^{(0)}_{V,\mathrm{kin}} + \frac{4}{16\pi^2}\int\mathrm{d}^4\theta\,\frac{\mathrm{d}^4p}{(2\pi)^4}\left\{\frac{1}{2}V^a(-p)p^2P_TV^a(p)\times\left[\left(\sum_A T_A(R)\right)B_0(p^2,m,m) - 3C_2(G)B_0(p^2,0,0)\right]\right\} \tag{3.61}$$

where we used the definition (C.3) of the scalar two-point integral $B(p^2, m_0, m_1)$ in dimensional regularisation.

To renormalise the theory we use the momentum subtraction scheme (cf. [33]). We define the renormalised coupling $g(\mu,\mathcal{M})$ by

$$\frac{1}{g_0^2} = \frac{1}{g^2(\mu,\mathcal{M})} - \frac{4}{16\pi^2}\left[\left(\sum_A T_A(R)\right) - 3C_2(G)\right]B_0(-\mathcal{M}^2,0,0) \tag{3.62}$$

where $\mathcal{M}$ is the renormalisation point.

[37] We only give the part relevant for the calculation of the β-function.

Rewriting $S'^{(1)}_{V,\mathrm{kin}}$ as a function of $g(\mu,\mathcal{M})$ reads

$$\begin{aligned}
S'^{(1)}_{V,\mathrm{kin}}(g(\mu,\mathcal{M}) = & \frac{1}{g^2(\mu,\mathcal{M})}\int \mathrm{d}^4\theta \frac{\mathrm{d}^4 p}{(2\pi)^4}\left\{\frac{1}{2}V^a(-p)p^2 P_T V^a(p)\right\} \\
& + \frac{4}{16\pi^2}\int \mathrm{d}^4\theta\, \frac{\mathrm{d}^4 p}{(2\pi)^4}\Bigg\{\frac{1}{2}V^a(-p)p^2 P_T V^a(p) \\
& \times \Bigg[\left(\sum_A T_A(R)\right)\big(B_0(p^2,m,m) - B_0(-\mathcal{M}^2,0,0)\big) \\
& - 3C_2(G)\big(B_0(p^2,0,0) - B_0(-\mathcal{M}^2,0,0)\big)\Bigg]\Bigg\}
\end{aligned} \tag{3.63}$$

which is finite as $\big(B_0(p^2,m_0,m_1) - B_0(-\mathcal{M}^2,0,0)\big)$ is finite (cf. Appendix C).

From (3.62), we calculate the β-function

$$\begin{aligned}
\beta(g(\mu,\mathcal{M})) &\equiv \mathcal{M}\frac{\partial}{\partial\mathcal{M}}g(\mu,\mathcal{M}) \\
&= \frac{4}{16\pi^2}\left[\left(\sum_A T_A(R)\right) - 3C_2(G)\right]g^3(\mu,\mathcal{M})\ .
\end{aligned} \tag{3.64}$$

Our result agrees with the result in [25].[38]

4 D=5 Super Yang-Mills Theory

In this chapter we will derive the Feynman rules for N=1 super Yang-Mills theory in five dimensions, following the calculation in four dimensions in the last chapter as far as possible.

In five dimensions, several new issues have to be taken into account:

- 5D Lorentz algebra leads to different spinor representations. A short review is given in Appendix A.2. We refer to [36] for a review and summary on this topic which is sufficient for our purposes. A more detailed discussion can be found in [37].
- There is no known 5D superfield formulation.

Therefore a straightforward generalisation from D=4 to D=5 is not possible. However, it is possible to express higher dimensional supersymmetric theories in terms of 4D superfields as has already been

[38] In comparing the results, note that our coupling constant is rescaled by a factor of 2 compared to [25].

shown in 1983 [17]. A few years ago in the course of the search for tools for supersymmetric orbifold GUTs this question has been readressed [18, 19, 20].

In Chapter 4.1, we follow [20] in expressing an N=1, D=5 super Yang-Mills theory in manifestly gauge invariant 4D superfields and fix the gauge, adopting the gauge fixing function in [17][39] to derive the Feynman rules of the D=5 theory in 4D superfields. The Feynman rules are then summarised in Chapter 4.2. Finally, as an application we use the Feynman rules to calculate the β-function for N=1, D=5 Super Yang-Mills Theory along the lines of the 4D case of Chapter 3.4.

4.1 D=5 Supersymmetry in 4D Superfields

Supercharges are Lorentz spinors. Having an 8-dimensional spinor representation in D=5 (cf. Table 1 in Appendix A.2), there are eight supercharges in N=1, D=5 supersymmetry. As shown in Appendix A.2, the D=5 spinors can be decomposed into two D=4 spinors. This implies for the supercharges that N=1, D=5 supersymmetry can be expressed by an N=2, D=4 supersymmetry with an $SU(2)_R$ symmetry which stems from the $USp(2)$ automorphism group of the D=5 spinors (cf. Appendix A.2 and [37]).

To discuss D=5 supersymmetry in 4D superfields we need an off-shell representation of the D=5 vector and chiral supermultiplet which is described in the following two chapters. Due to the spinor representation this is analogous to finding N=2, D=4 off-shell representations (cf. [25]).

In the following, the space-time coodinates will be denoted by $(x_0, \ldots, x_3, x_5 \equiv y) = x_M$.

4.1.1 The D=5 Vector Supermultiplet

The field content of an N=1, D=5 vector supermultiplet is a vectorfield v^M, a real scalar Σ and an $SU(2)_R$ gaugino doublet λ^i_α, to which, following [25], an $SU(2)_R$ triplet of real auxiliary fields is added for the off-shell multiplet.

The off-shell action of an N=1, D=5 Super Yang-Mills theory is

[39] There the gauge fixing is done for a D=10 theory but it can be applied to D=5 in a straightforward manner.

given by (cf. [20])

$$S = \int \mathrm{d}^4x\, \mathrm{d}y \frac{1}{2kg^2}\Big\{ -\frac{1}{2}\mathrm{Tr}(F_{MN})^2 - \mathrm{Tr}(D_M\Sigma)^2 - \mathrm{Tr}(\bar{\lambda}_i \mathrm{i}\Gamma^M D_M \lambda^i) + \mathrm{Tr}(X^a)^2 + \mathrm{Tr}(\bar{\lambda}_i[\Sigma, \lambda^i])\Big\} \tag{4.1}$$

where $D_M = \partial_M + \mathrm{i}A_M$ is the convariant derivative and $F_{MN} = -\frac{\mathrm{i}}{g}[D_M, D_N]$ is the field strength tensor. The supersymmetry transformations expressed in terms of symplectic 4-spinors λ^i are given in [20] as well as in terms of 4D Weyl spinors λ_L and λ_R i.e. after using the decomposition given in (A.18). We list them for completeness.

$$\begin{aligned}
\delta_{\xi_L} v^m =& \mathrm{i}\bar{\xi}_L\bar{\sigma}^m\lambda_L + \mathrm{i}\xi_L\sigma^m\bar{\lambda}_L \\
\delta_{\xi_L} v^5 =& -\bar{\xi}_L\bar{\lambda_R} - \xi_L\lambda_R \\
\delta_{\xi_L}\Sigma =& \mathrm{i}\bar{\xi}_L\bar{\lambda_R} - \mathrm{i}\xi_L\lambda_R \\
\delta_{\xi_L}\lambda_L =& \sigma^{mn}F_{mn}\xi_L - \mathrm{i}D_5\Sigma\xi_L + \mathrm{i}X^3\xi_L \\
\delta_{\xi_L}\lambda_R =& \mathrm{i}\sigma^m F_{5m}\bar{\xi}_L - \sigma^m D_m\Sigma\bar{\xi}_L + \mathrm{i}(X^1 + \mathrm{i}X^2)\xi_L \\
\delta_{\xi_L}(X^1 + \mathrm{i}X^2) =& 2\bar{\xi}_L\bar{\sigma}^m D_m\lambda_R - 2\mathrm{i}\bar{\xi}_L D_5\bar{\lambda}_L + \mathrm{i}[\Sigma, 2\bar{\xi}_L\bar{\lambda}_L] \\
\delta_{\xi_L}X^3 =& \bar{\xi}_L\bar{\sigma}^m D_m\lambda_L + \mathrm{i}\bar{\xi}_L D_5\bar{\lambda}_R - \xi_L\sigma^m D_m\bar{\lambda}_L \\
& - \mathrm{i}\xi_L D_5\lambda_R + \mathrm{i}[\Sigma, (\bar{\xi}_L\bar{\lambda}_R + \xi_L\lambda_R)]
\end{aligned} \tag{4.2}$$

where $\sigma^{mn} = \frac{1}{4}(\sigma^m\bar{\sigma}^n - \sigma^n\bar{\sigma}^m)$.

We only give the ξ_L transformations though the action under consideration has an N=2 supersymmetry. Compactification on an orbifold breaks half of the supersymmetry, such that in 4D only an N=1 supersymmetry[40] survives. For the non-compactified theory, the D=4, N=2 supersymmetry is not broken. We will see that the superfield formulation reproduces the D=5 component field action. Therefore the superfield action possesses the full D=4, N=2 supersymmetry though only an N=1 supersymmetry is manifest.

The first crutial observation from the supersymmetry transformation is that v_m, λ_l and $(X^3 - D_5\Sigma)$ transform like the components of a vector superfield in Wess-Zumino gauge

$$V = -\theta\sigma^m\bar{\theta}v_m + \mathrm{i}\theta^2\bar{\theta}\bar{\lambda}_L - \mathrm{i}\bar{\theta}^2\theta\lambda_L + \frac{1}{2}\theta^2\bar{\theta}^2(X^3 - D_5\Sigma). \tag{4.3}$$

The gauge transformation can be defined as in D=4 by[41]

$$e^{2gV} \to e^{-\mathrm{i}\bar{\Lambda}}e^{2gV}e^{\mathrm{i}\Lambda}. \tag{4.4}$$

[40] with four supersymmetry generators

[41] This is in slight deviation to [20] in order to match with our conventions for the 4D theory.

The remaining components can be grouped such that their transformation is just the one of a chiral superfield[42]

$$\Phi = (\Sigma + \mathrm{i}A_5) + \sqrt{2}\theta(-\mathrm{i}\sqrt{2}\lambda_R) + \theta^2(X^1 + \mathrm{i}X^2) \tag{4.5}$$

if Φ transforms as

$$g\Phi \rightarrow e^{-\mathrm{i}\Lambda}(\partial_5 + g\Phi)e^{\mathrm{i}\Lambda}. \tag{4.6}$$

From this transformation, it was realised in [20] that $\nabla_y \equiv \nabla_5 \equiv \partial_5 + g\Phi$ is a gauge covariant derivative in the 5-direction. This is one of the main progresses compared to [19].

In our conventions for the vector superfield

$$\nabla_y e^{2gV} = \partial_5 e^{2gV} - g\bar{\Phi}e^{2gV} - e^{2gV}g\Phi \tag{4.7}$$

the transformation is

$$\nabla_y e^{2gV} \rightarrow e^{-\mathrm{i}\bar{\Lambda}}(\nabla_y e^{2gV})e^{\mathrm{i}\Lambda}. \tag{4.8}$$

Using the covariant derivative, the D=5 action can be rewritten in terms of 4D superfields only[43]:

$$S = \int \mathrm{d}^8 z\, \mathrm{d}y \left\{ \frac{1}{4kg^2}\mathrm{Tr}\left[\frac{1}{4}\left(W^\alpha W_\alpha \delta^2(\bar{\theta}) + \text{h.c.}\right) + \left(e^{-2gV}\nabla_y e^{2gV}\right)^2\right]\right\} \tag{4.9}$$

where

$$W_\alpha = -\frac{1}{4}\bar{\mathrm{D}}^2 e^{-2gV} D_\alpha e^{2gV} \tag{4.10}$$

like in the 4D case.

It is remarkable that given the action (4.9), the component field decomposition into V and Φ can be derived by demanding D=5 Lorentz invariance of the Lagrangian and D=5 Lorentz covariance of the component fields only.

From the field content, it is clear that the fields have to be split into a vector and a chiral supermultiplet. v^m is the only spin 1 field and therefore has to form the $\theta\bar{\theta}$-component of V. From ξ_L-supersymmetry, the gaugino is also fixed to be λ_L, $\bar{\lambda}_L$, so the only freedom in the vector superfield components (4.3) is in the real auxiliary field D. We demand it to be a D=5 Lorentz scalar. For the chiral field, the spinor component has to be proportional to λ_R and the relative factor to λ_L is fixed by D=5 Lorentz covariance. The linear combinations of the scalars in (4.5) remain to be fixed. For now we call them A and F as done in (2.15) for a chiral superfield.

[42] in the y-basis

[43] Compared to [20] we rescale $2V \rightarrow 2gV$ and $W_\alpha \rightarrow \frac{1}{2}W_\alpha$.

Now, inserting the component fields into (4.9), the equation of motion for the D field is

$$D^a = -\left\{\partial_5 \delta^{ac} + \frac{\mathrm{i}}{2} g f^{abc} (A - A^\dagger)^b\right\} (A + A^\dagger)^c \ . \tag{4.11}$$

By demanding D=5 Lorentz covariance, the imaginary part of A is identified to be the gauge field component in 5-direction, v_5. We call the real component Σ. Furthermore it is obvious that D^a is not a D=5 Lorentz scalar, but $D' = X - D_5\Sigma$ is a scalar. Therefore the only freedom left is the assignment of the SU(2)-triplet to the scalar degree of freedom in D and the (complex) field F. Similar arguments will be used to derive the component expansion of the hypermultiplet in Chapter 4.1.2.

After this short interlude, we continue with the derivation of the Feynman rules. Knowing the action (4.9), we can now follow the path of the last chapter, meaning, expanding the action in powers of the coupling constant, fixing the gauge and calculating the related ghost action, introducing sources for the gauge field as well as for the ghosts to derive the propagator from the path integral to finally read off the vertices of the theory.

The super Yang-Mills action (4.9), expanded to second order in the coupling constant reads

$$\begin{aligned}
S = \int \mathrm{d}^8 z\, \mathrm{d}y \frac{\mathrm{Tr}}{k} \Bigg\{ & \frac{1}{8} V \mathrm{D}^\alpha \bar{\mathrm{D}}^2 \mathrm{D}_\alpha V + \frac{1}{4} g (\bar{\mathrm{D}}^2 \mathrm{D}^\alpha V)[V, \mathrm{D}_\alpha V] \\
& - \frac{1}{8} g^2 [V, (\mathrm{D}^\alpha V)] \bar{\mathrm{D}}^2 [V, (\mathrm{D}_\alpha V)] - \frac{1}{6} g^2 (\bar{\mathrm{D}}^2 \mathrm{D}^\alpha V)[V, [V, \mathrm{D}_\alpha V]] \\
& + \frac{1}{2} \bar{\Phi}\Phi - (\Phi + \bar{\Phi})(\partial_5 V) + (\partial_5 V)^2 \\
& + \mathrm{i} g f^{abc} \left\{ (\partial_5 V)^a V^b (\Phi + \bar{\Phi})^c - (\Phi + \bar{\Phi})^a V^b (\Phi + \bar{\Phi})^c \right\} \\
& - 4 g^2 f^{ade} f^{bce} \Bigg\{ \frac{1}{3} (\partial_5 V)^a V^b (\partial_5 V)^c V^d - \frac{2}{3} (\partial_5 V)^a V^b (\Phi + \bar{\Phi})^c V^d \\
& \qquad + \frac{1}{2} (\Phi + \bar{\Phi})^a V^b (\Phi + \bar{\Phi})^c V^d \Bigg\} + O(g^3) \Bigg\}
\end{aligned} \tag{4.12}$$

where the first two lines are just the expansion known from the 4D action while the last four lines stem from the ∇_y-Term in (4.9).

In the quadratic part of the action a $(\Phi + \bar{\Phi})(\partial_5 V)$-term appears which mixes the V and Φ fields. To calculate the propagators, we choose a gauge fixing function which compensates this term[44]

$$S_{GF} = \frac{\mathrm{Tr}}{kg^2} \int \mathrm{d}^8 z\, \mathrm{d}y \Bigg\{ \left(\frac{1}{\sqrt{\xi}} \sqrt{2} \frac{\bar{\mathrm{D}}^2}{4} gV + \sqrt{\frac{\xi}{2}} \frac{\bar{\mathrm{D}}^2}{4\Box} \partial_5 g \bar{\Phi} \right) \\ \times \left(\frac{1}{\sqrt{\xi}} (\sqrt{2} \frac{\mathrm{D}^2}{4} gV + \sqrt{\frac{\xi}{2}} \frac{\mathrm{D}^2}{4\Box} \partial_5 g \Phi \right) \Bigg\} \tag{4.13}$$

leading to the free action

$$\begin{aligned} S_0 + S_{GF} = \frac{\mathrm{Tr}}{k} \int \mathrm{d}^8 z\, \mathrm{d}y \Bigg\{ & - V \Box \left(P_T - \frac{1}{\xi}(P_1 + P_2) \right) V - V \partial_5 \partial_5 V \\ & - \frac{\xi}{2} \bar{\Phi} \frac{\partial_5 \partial_5}{\Box} \Phi + \frac{1}{2} \bar{\Phi} \Phi \Bigg\} \\ + \frac{\mathrm{Tr}}{k} \int \mathrm{d}^8 z & \left((\Phi + \bar{\Phi}) V + V(\partial_5 V) - \frac{1}{2} \bar{\Phi} \frac{\partial_5}{\Box} \Phi \right) \Bigg|_y \ . \end{aligned} \tag{4.14}$$

The y-surface terms result from partial intergrations. If the theory is not compactified, they can be neglected. However closer inspection shows that even for a compactified theory with a Z_2 symmetry, the surface terms vanish as Φ and V have opposite Z_2 parity as well as any field and its derivative in 5-direction.

The ghost action can be derived from the gauge fixing function

$$F = \frac{\mathrm{Tr}}{\sqrt{2}k} \frac{1}{\sqrt{\xi}} \left(-\frac{\bar{\mathrm{D}}^2}{4} \right) \left(2gV + \xi \frac{\partial_5}{\Box} g \bar{\Phi} \right) = 0 \tag{4.15}$$

exactly as has been done in Chapter 3, i.e. defining the group measure in the path integral, calculating it like in (3.7) and then replacing the ordinary gauge transformation superfields by anticommuting ghost superfields. The result is

$$\begin{aligned} S_{FP} &= \frac{\mathrm{Tr}}{k} \int \mathrm{d}^8 z\, \mathrm{d}y \int \mathrm{d}^8 z\, \mathrm{d}y \left\{ -\mathrm{i}(c' + \bar{c}')(\delta(2gV) - \xi \frac{\partial_5^2}{\Box}(c - \bar{c})) \right\} \\ &= \frac{\mathrm{Tr}}{k} \int \mathrm{d}^8 z\, \mathrm{d}y \Bigg\{ \bar{c}' c - c' \bar{c} - \xi \bar{c}' \frac{\partial_5^2}{\Box} c + \xi c' \frac{\partial_5^2}{\Box} \bar{c} + g(c' + \bar{c}')\, [V, c + \bar{c}] \\ & \qquad + \frac{1}{3} g^2 (c' + \bar{c}')\, [V, [V, c - \bar{c}]] + O(g^3) \Bigg\} \ . \end{aligned} \tag{4.16}$$

[44] In [17] the same type of gauge fixing is used for a D=10 super Yang-Mills theory. It is pointed out that this fixing is similar to the R_ξ-gauge in a spontaneously broken (non-supersymmetric) Yang-Mills theory (cf. eg. [38]).

To calculate the propagators, we again choose the super Feynman gauge $\xi{=}-1$. The generating functional for the vector superfield is then given by

$$\begin{aligned} Z_{0,V}(\mathscr{J}) &= \frac{\mathrm{Tr}}{k}\int \mathcal{D}V \exp\left\{\mathrm{i}\int \mathrm{d}^8z\,\mathrm{d}y - V\Box V - V\partial_5^2 V + \mathscr{J}V\right\} \\ &= \exp\left\{\frac{\mathrm{i}}{2}\int \mathrm{d}^8z\,\mathrm{d}y\,\mathscr{J}^a(z,y)\Delta^{ab}_{5,V}(z-z',y-y')\mathscr{J}^b(z',y')\right\} \end{aligned} \tag{4.17}$$

where the propagator $\Delta_{5,V}$ in coordinate space is defined by

$$(\partial_5^2+\Box)\Delta^{ab}_{5,V}(z-z',y-y') = -\frac{1}{2}\delta^{ab}\delta^8(z-z')\delta(y-y') \tag{4.18}$$

in the usual way.[45]

The propagator is thus

$$\Delta^{ab}_{5,V}(z-z',y-y') = -\frac{1}{2}\delta^{ab}\mathcal{A}_V(z-z',y-y') \tag{4.19}$$

where $\mathcal{A}_V$ is a solution to

$$(\partial_5^2+\Box)\mathcal{A}_V(z-z',y-y') = \delta^8(z-z')\delta(y-y')\ . \tag{4.20}$$

If we do not compactify the y-direction or impose any boundary conditions, the solution is just

$$\Delta^{ab}_{5,V}(z-z',y-y') = -\frac{1}{2}\frac{1}{\partial_5^2+\Box}\delta^{ab}\delta^8(z-z')\delta(y-y')\ . \tag{4.21}$$

In 4-momentum space[46] this leads to the propagator

$$\Delta^{ab}_{5,V}(y-y') = -\frac{1}{2}\frac{1}{\partial_5^2-p^2}\delta^{ab}\delta^4(\theta-\theta')\delta(y-y')\ . \tag{4.22}$$

The propagator for the Φ superfield can be calculated in analogy to the chiral propagator in four dimensions. Rewriting the free action, the free generating functional reads

[45] The factor $\frac{1}{2}$ stems from the fact that in the free action, we have the kinetic term $-V\Box V$ instead of $-\frac{1}{2}V\Box V$ as in Chapter 3.

[46] i.e. we Fourier transform in $x_0,\ldots,x_3$ and regard y as a parameter which is not being transformed

$$Z_{0,\Phi}(J,\bar{J}) = \int \mathcal{D}\Phi\mathcal{D}\bar{\Phi}\exp\Bigg\{\mathrm{i}\int \mathrm{d}^8z\,\mathrm{d}y\frac{1}{2}(\Phi,\bar{\Phi})^a M_\Phi \begin{pmatrix}\Phi\\ \bar{\Phi}\end{pmatrix}^a$$

$$+ (\Phi,\bar{\Phi})^a \begin{pmatrix} -\frac{\mathrm{D}^2}{4\Box} & 0 \\ 0 & -\frac{\bar{\mathrm{D}}^2}{4\Box}\end{pmatrix}\begin{pmatrix}\mathrm{J}\\ \bar{\mathrm{J}}\end{pmatrix}^a\Bigg\}$$

$$= \exp\left\{-\frac{\mathrm{i}}{2}\int \mathrm{d}^8z\,\mathrm{d}y(J(z,y),\bar{\mathrm{J}}(z,y))^a\Delta^{ab}_{5,\Phi}(z-z',y-y')\begin{pmatrix}\mathrm{J}(z',y')\\ \bar{\mathrm{J}}(z',y')\end{pmatrix}^b\right\} \tag{4.23}$$

where M_Φ is read off from the action (4.14) to be[47]

$$M_\Phi = \frac{1}{2}\begin{pmatrix} 0 & 1+\frac{\partial_5^2}{\Box} \\ 1+\frac{\partial_5^2}{\Box} & 0\end{pmatrix} \tag{4.24}$$

and the chiral propagator $\Delta^{ab}_{5,\Phi}(z-z',y-y')$ is defined by

$$\begin{aligned}&\frac{1}{4}\begin{pmatrix}\bar{\mathrm{D}}^2 & 0\\ 0 & \mathrm{D}^2\end{pmatrix} M_\Phi\Delta^{ab}_{5,\Phi}(z-z',y-y')\\ &= -\begin{pmatrix} -\frac{1}{4}\bar{\mathrm{D}}^2 & 0\\ 0 & -\frac{1}{4}\mathrm{D}^2\end{pmatrix}\delta^{ab}\delta^8(z-z')\delta(y-y')\ .\end{aligned} \tag{4.25}$$

To solve for the propagator we once again use the method of projection operators outlined in [21]. The result for the propagator is

$$\Delta^{ab}_{5,\Phi}(z-z',y-y') = 2\delta^{ab}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\mathcal{A}_\Phi(z-z',y-y') \tag{4.26}$$

where $\mathcal{A}_\Phi$ is (analogous to the Φ propagator) a solution to

$$(\partial_5^2+\Box)\mathcal{A}_\Phi(z-z',y-y') = \delta^8(z-z')\delta(y-y')\ . \tag{4.27}$$

Without compactification of the y-direction the solution is

$$\mathcal{A}_\Phi(z-z',y-y') = \frac{1}{\partial_5^2+\Box}\delta^8(z-z')\delta(y-y') \tag{4.28}$$

hence

$$\Delta^{ab}_{5,\Phi}(z-z',y-y') = \frac{2}{\partial_5^2+\Box}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\delta^{ab}\delta^8(z-z')\delta(y-y')\ . \tag{4.29}$$

[47] in super Feynman gauge

The propagator in momentum space reads

$$\Delta^{ab}_{5,\Phi}(y-y') = -\frac{2}{p^2-\partial_5^2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\delta^{ab}\delta^4(\theta-\theta')\delta(y-y') \ . \qquad (4.30)$$

For the ghost propagator the calculation is analogous. As in Chapter 3, the ghost terms in the action (4.16) have a factor of 2, compared to the Φ terms in the action (4.9), leading to a factor of $\frac{1}{2}$ in the propagator and one has to take care for the differing statistics of the ghosts. The ghost propagator in coordinate space is

$$\Delta^{ab}_{5,c}(z-z', y-y') = \delta^{ab}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathcal{A}_c(z-z', y-y') \qquad (4.31)$$

with $\mathcal{A}_c$ defined in the same way as $\mathcal{A}_\Phi$ in (4.27) and (4.20) and the same solution for non-compactified 5-direction.[48]

The vertices of the theory can be read off from the expanded action (4.12). Most of them are an obvious extension of the vertices of the 4D theory, given in Chapter 3.3. Note however that in the 5D theory there is also a $(\partial_5 V)V(\Phi+\bar{\Phi})$-vertex which is missing in the 4D theory. In Chapter 4.3 it will become obvious that this vertex is crucial to cancel quadratic divergences in the vector superfield propagator one loop correction arising from the Φ loop and the ghost loop.

4.1.2 The D=5 Chiral Supermultiplet and its Coupling to the Vector Supermultiplet

To find an off-shell formulation for the N=1, D=5 chiral multiplet, again, the equivalence to the N=2, D=4 theory with a $USp(2)$ symmetry described in the beginning of Chapter 4 can be used. Following [25], the field content is given by a complex scalar $SU(2)_R$ doublet H^i and an $SU(2)_R$ singlet Dirac spinor Ψ which can be decomposed into two Weyl spinors $\Psi = (\psi, \bar{\psi}^c)^T$ to which a complex scalar $SU(2)_R$ doublet $\tilde{F}_i$ of auxiliary fields is added for the off-shell multiplet. The supersymmetry transformations are (cf.[18])

$$\begin{aligned} \delta_\xi H^i =& -\sqrt{2}\epsilon^{ij}\bar{\xi}_j\Psi \\ \delta_\xi \Psi =& \sqrt{2}\mathrm{i}\Gamma^M\partial_M H^i\epsilon_{ij}\xi^j + \sqrt{2}\tilde{F}_i\xi^i \\ \delta_\xi \tilde{F}_i =& \sqrt{2}\mathrm{i}\bar{\xi}_i\Gamma^M\partial_M\Psi \ , \end{aligned} \qquad (4.32)$$

[48]Note however that if different boundary conditions are imposed to the superfields V, Φ and the ghosts, their propagators might differ even though their implicit definition (4.27) is identical.

for which the invariant action is

$$S=\int \mathrm{d}^8z\,\mathrm{d}y\left(-(\partial_M H)^\dagger_i(\partial^M H^i)-\bar\Psi(\mathrm{i}\Gamma^M\partial_M+m)\Psi\right) \tag{4.33}$$

Note that no trilinear couplings occur. Ψ carries spin $\frac{1}{2}$ and $SU(2)_R$-spin 0 while H^i carries spin 0 and $SU(2)_R$-spin $\frac{1}{2}$. Due to the additional $SU(2)_R$ symmetry, all Yukawa coupling terms allowed by the other symmetries[49] are forbidden.

Defining the superfields[50]

$$\begin{aligned}\tilde H =&H^1+\sqrt{2}\theta\psi_L+\theta^2\tilde F_1\\ \tilde H^c =&H_2^\dagger+\sqrt{2}\theta\psi_R+\theta^2\tilde F^{\dagger 2}\end{aligned} \tag{4.34}$$

the action can be re-written as [19]

$$S=\int \mathrm{d}^8z\,\mathrm{d}y\left\{\bar{\tilde H}\tilde H+\tilde H^c\bar{\tilde H}^c+\left(\tilde H^c(\partial_5+m)\tilde H\delta(\bar\theta)+h.c.\right)\right\} \tag{4.35}$$

The next step is to couple the chiral supermultiplet to the vector supermultiplet. Let $\tilde H$ be in a representation of the gauge group. Then the chiral multiplets transform according to

$$\tilde H\to e^{-\mathrm{i}\Lambda}\tilde H \qquad , \qquad \tilde H^c\to\tilde H^c e^{\mathrm{i}\bar\Lambda} \tag{4.36}$$

As in 4D, to keep the kinetic terms gauge invariant, they have to be modified. Furthermore, the ∂_5 derivative has to be replaced by the covariant derivative in y-direction ∇_y. The action for the hypermultiplet including the coupling to the vector supermultiplet is

$$S=\int \mathrm{d}^8z\mathrm{d}y\left\{\bar{\tilde H}e^{2gV}\tilde H+\tilde H^c e^{-2gV}\bar{\tilde H}^c+\left(\tilde H^c(\nabla_y+m)\tilde H\delta(\bar\theta)+\mathrm{h.c.}\right)\right\}\ . \tag{4.37}$$

Before proceeding with the derivation of the super Feynman rules, we have to point out one subtlety which has not been adessed in [19] but is resolved in the discussion of the D=5 hypermultiplet in [20]. For the vector supermultiplet, we have seen that it was neccessary to align the physical fields appropriately in the component fields in order to keep D=5 Lorentz covariance for the component fields. It turns out that after gauging the hypermultiplet, this issue has be taken into account there as well.

[49] and therefore in D=4

[50] in y-basis

Taking (4.37) as a starting point and calculating the component field action using the superfields (4.34) leads to the equations of motion for the auxiliary fields which are

$$\begin{aligned}\tilde{F}_1 &= -(\partial_5 + \mathrm{i}A_5)H^2 + \Sigma H^2 \\ \tilde{F}^{\dagger 2} &= -(\partial_5 + \mathrm{i}A_5)H_2 - H_1^\dagger \Sigma\end{aligned} \tag{4.38}$$

and their hermitian conjugates.

They are obviously not D=5 Lorentz covariant, which can be cured by redefining the auxiliary fields by

$$\begin{aligned}F_1 &= \tilde{F}_1 + (D_5 - \Sigma)H^2 \\ F^{\dagger 2} &= -\tilde{F}^{\dagger 2} - (D_5 + \Sigma)H_1^\dagger\end{aligned} \tag{4.39}$$

leading to the the gauged chiral superfields (cf. [20])

$$\begin{aligned}H =& H^1 + \sqrt{2}\theta\psi_L + \theta^2(F_1 + D_5 H^2 - \Sigma H^2) \\ H^c =& H_2^\dagger + \sqrt{2}\theta\psi_R + \theta^2(-F^{\dagger 2} - D_5 H_1^\dagger - H_1^\dagger \Sigma)\end{aligned} \tag{4.40}$$

As for the vector supermultiplet, the asignment of the component fields to H and H^c is therefore uniquely fixed if we demand 5D Lorentz invariance.

Replacing the superfields $\tilde{H}$, $\tilde{H}^c$ by H, H^c in the action (4.37) it is straightforward to calculate the action in component fields. The supersymmetry transformations of the component fields can be computed, applying $\xi_L \mathrm{Q}_L$ to the superfields. Therefore the 5D hypermultiplet action in component fields can be rederived from the superfield formulation.[51]

We will now proceed deriving the Feynman rules for the hypermultiplet starting from the action (4.37). Its expansion reads

$$\begin{aligned}S = \frac{\mathrm{Tr}}{k}\int \mathrm{d}^8 z\, \mathrm{d}y\, \big\{ &\bar{H}H + H^c\bar{H}^c \\ &+H^c(\partial_5 + m)H\delta(\bar{\theta}) + \bar{H}(-\partial_5 + m)\bar{H}^c\delta(\theta) \\ &+g\left(2\bar{H}VH - 2H^cV\bar{H}^c + H^c\Phi H\delta(\bar{\theta}) + \bar{H}\bar{\Phi}\bar{H}^c\delta(\theta)\right) \\ &+2g^2\left(\bar{H}V^2H - H^cV^2\bar{H}^c\right) + O(g^3)\big\}\end{aligned} \tag{4.41}$$

[51] The ξ_R supersymmetry is a symmetry of the action. However it appears rather accidential, starting from the superfield formulation.

The vertices can be read off from the action so we are left with the calculation of the chiral propagator. From the free part of (4.41), we define the generating functional[52]

$$
\begin{aligned}
Z_{0,H}(\mathrm{J}_H,\bar{\mathrm{J}}_H,\mathrm{J}_{H^c},\bar{\mathrm{J}}_{H^c}) = &\int \mathcal{D}H\mathcal{D}\bar{H}\mathcal{D}H^c\mathcal{D}\bar{H}^c \\
&\exp\left\{\mathrm{i}\int \mathrm{d}^8 z\,\mathrm{d}y\left[(H^c,\bar{H})M_H\begin{pmatrix} H \\ \bar{H}^c\end{pmatrix}\right.\right. \\
&+(H^c,\bar{H})\begin{pmatrix} -\frac{\mathrm{D}^2}{4\Box} & 0 \\ 0 & -\frac{\bar{\mathrm{D}}^2}{4\Box}\end{pmatrix}\begin{pmatrix}\mathrm{J}_{H^c} \\ \bar{\mathrm{J}}_H\end{pmatrix} \\
&\left.\left.+(H,\bar{H}^c)\begin{pmatrix} -\frac{\mathrm{D}^2}{4\Box} & 0 \\ 0 & -\frac{\bar{\mathrm{D}}^2}{4\Box}\end{pmatrix}\begin{pmatrix}\mathrm{J}_{H} \\ \bar{\mathrm{J}}_{H^c}\end{pmatrix}\right]\right\} \\
= \exp&\left\{-\mathrm{i}\int \mathrm{d}^8\ z\,\mathrm{d}y(\mathrm{J}_{H^c},\bar{\mathrm{J}}_H)_i\Delta_{5,H_{ij}}(z-z',y-y')\begin{pmatrix}\mathrm{J}'_{H^c} \\ \bar{J}'_H\end{pmatrix}\right\}
\end{aligned}
\tag{4.42}
$$

where M_H is

$$
(M_H)_{ij} = \delta_{ij}\begin{pmatrix} -\frac{\mathrm{D}^2}{4\Box}(\partial_5+m_i) & 1 \\ 1 & -\frac{\bar{\mathrm{D}}^2}{4\Box}(-\partial_5+m_i)\end{pmatrix} \tag{4.43}
$$

and the chiral propagator $(\Delta_{5,H})_{ij}$ is defined by

$$
\begin{pmatrix} \frac{\mathrm{D}^2}{4\Box} & 0 \\ 0 & \frac{\bar{\mathrm{D}}^2}{4\Box}\end{pmatrix}(M_H)_{ij}(\Delta_{5,H})_{ij} \equiv -\begin{pmatrix} -\frac{\mathrm{D}^2}{4\Box} & 0 \\ 0 & -\frac{\bar{\mathrm{D}}^2}{4\Box}\end{pmatrix}\delta^8(z-z')\delta(y-y') \tag{4.44}
$$

As for the Φ propagator, using the method of projectors outlined in [21], the propagator reads

$$
(\Delta_{5,H})_{ij} = \delta_{ij}\begin{pmatrix} (m-\partial_5)\frac{\mathrm{D}^2}{4\Box} & 1 \\ 1 & (m+\partial_5)\frac{\bar{\mathrm{D}}^2}{4\Box}\end{pmatrix}\mathcal{A}_H \tag{4.45}
$$

where $\mathcal{A}_H$ is a solution to

$$
(\partial_5^2+\Box-m^2)\mathcal{A}_H(z-z',y-y') = \delta^8(z-z')\delta(y-y') \tag{4.46}
$$

The solution for a non-compactified theory is

$$
\mathcal{A}_H(z-z',y-y') = \frac{1}{\partial_5^2+\Box-m^2}\delta^8(z-z')\delta(y-y')\ . \tag{4.47}
$$

Note that in contrast to the 4D theory even for a massless hypermultiplet there are chiral H^cH and anti-chiral $\bar{H}^c\bar{H}$ propagators containing the 5-derivative.

[52]We denote $\mathrm{J}(z,y)\equiv\mathrm{J}$ and $\mathrm{J}(z',y')\equiv\mathrm{J}'$.

4.2 Feynman Rules for D=5, N=1 Super Yang-Mills-Theory

In this chapter, we summarise the Feynman rules derived in Chapter 4.1.1 and 4.1.2. The action is given in 4D superfields. Therefore all rules of 4D D-algebra apply. Furthermore, all vertices of the 4D action apart from the Yukawa couplings are contained in the D=5 action.

Differences in the Feynman rules are due to

- 5D Lorentz covariance, leading to propagators having an additional ∂_5-term
- the covariant derivative ∇_y in y-direction which introduces the chiral field Φ as a part of the D=5 vector supermultiplet which lead to
- new coupling terms of Φ with the vector superfield and the chiral matter superfields appearing in the theory.

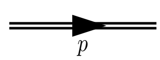
We once give the complete action for D=5, N=1 super Yang-Mills theory in superfields, collecting the terms for the gauge fixed action from (4.14) and (4.16), the super Yang-Mills interaction terms from (4.12)and the chiral action terms from (4.41).

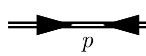
$$
\begin{aligned}
S = \frac{\mathrm{Tr}}{k} \int \mathrm{d}^8 z\, \mathrm{d}y \Bigg\{ & - V\Box V - V\partial_5^2 V + \frac{1}{2}\bar{\Phi}\frac{\partial_5^2}{\Box}\Phi + \frac{1}{2}\bar{\Phi}\Phi \\
& \qquad\qquad + \left(\bar{c}' c - c'\bar{c} + \bar{c}'\frac{\partial_5^2}{\Box}c - c'\frac{\partial_5^2}{\Box}\bar{c}\right) \\
& + \frac{1}{4} g(\bar{\mathrm{D}}^2 \mathrm{D}^\alpha V)[V, \mathrm{D}_\alpha V] + g(c' + \bar{c}')\,[V, c + \bar{c}] \\
& + \mathrm{i} g f^{abc}\left[(\partial_5 V)^a V^b (\Phi + \bar{\Phi})^c - \bar{\Phi}^a V^b \Phi^c\right] \\
& - \frac{1}{8} g^2 [V, (\mathrm{D}^\alpha V)]\bar{\mathrm{D}}^2 [V, (\mathrm{D}_\alpha V)] - \frac{1}{6} g^2 (\bar{\mathrm{D}}^2 \mathrm{D}^\alpha V)[V, [V, \mathrm{D}_\alpha V]] \\
& - 4 g^2 f^{ade} f^{bce}\bigg\{\frac{1}{3}(\partial_5 V)^a V^b (\partial_5 V)^c V^d - \frac{2}{3}(\partial_5 V)^a V^b (\Phi + \bar{\Phi})^c V^d \\
& \qquad + \frac{1}{2}\bar{\Phi}^a V^b \Phi^c V^d\bigg\} + \frac{1}{6} g^2 (c' + \bar{c}')\,[V, [V, c - \bar{c}]] \\
& + \bar{H}H + H^c \bar{H}^c + H^c(\partial_5 + m)H\delta(\bar{\theta}) + \bar{H}(-\partial_5 + m)\bar{H}^c\delta(\theta) \\
& + g\left(2\bar{H}VH - 2H^c V\bar{H}^c + H^c\Phi H\delta(\bar{\theta}) + \bar{H}\bar{\Phi}\bar{H}^c\delta(\theta)\right) \\
& + 2g^2\left(\bar{H}V^2 H - H^c V^2 \bar{H}^c\right) + O(g^3)\Bigg\}
\end{aligned}
\tag{4.48}
$$

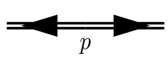
The vertices can be read off from it in the same way as it has been done in the 4D theory. In the following summary, we only give the vertex graphs.

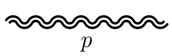
The D=5 Feynman rules in momentum space with y in coordinate space, following from (4.48) are:

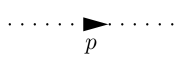
- Draw all possible topologically inequivalent graphs of the order in g which is to be considered.
- Multiply each graph with its symmetry factor.
- p_i are momenta flowing along internal lines, away from the vertex.
- For each vertex integrate over $\int \mathrm{d}^4\theta\, \mathrm{d}y$.
- For each external field integrate over $\int \mathrm{d}^4\theta\, \mathrm{d}y_{\mathrm{ext}}$.
- For the external momenta, the overall factor is $\left[\prod_{p_{\mathrm{ext}}} \int \frac{\mathrm{d}^4 p_{\mathrm{ext}}}{(2\pi)^4}\right] (2\pi)^4 \delta^4 \left(\sum_{\mathrm{ext}} p_{\mathrm{ext}}\right)$.
- For every loop with momentum p running in it integrate over $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$
- For an [anti-]chiral vertex with n internal lines, $n-1$ factors $[-\frac{1}{4}\mathrm{D}^2]$ $-\frac{1}{4}\bar{\mathrm{D}}^2$ act on the propagators.
- A ghost loop contributes a factor (-1).
- The propagators are

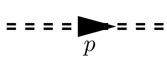
$\bar{H}_i(-p)$, $\bar{H}^c_i(-p)$  $H_j(p)$, $H^c_j(p)$ $= -\frac{\mathrm{i}}{p^2-\partial_5^2+m^2}\delta_{ij}\delta^4(\theta-\theta')\delta(y-y')$

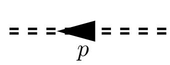
$\bar{H}_i(-p)$  $\bar{H}^c_j(p)$ $= \frac{\bar{\mathrm{D}}^2(p)}{4}\frac{\mathrm{i}(m+\partial_5)}{p^2(p^2-\partial_5^2+m^2)}\delta_{ij}\delta^4(\theta-\theta')\delta(y-y')$

$H^c_i(-p)$  $H_j(p)$ $= \frac{\mathrm{D}^2(p)}{4}\frac{\mathrm{i}(m-\partial_5)}{p^2(p^2-\partial_5^2+m^2)}\delta_{ij}\delta^4(\theta-\theta')\delta(y-y')$

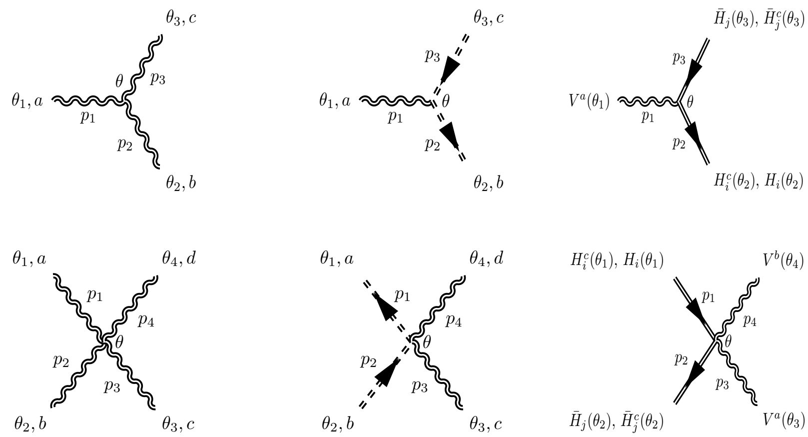
$V^a(-p)$  $V^b(p)$ $= \frac{1}{2}\frac{\mathrm{i}}{p^2-\partial_5^2}\delta^{ab}\delta^4(\theta-\theta')\delta(y-y')$

$\bar{\Phi}^a(-p)$  $\Phi^b(p)$ $= 2\frac{\mathrm{i}}{p^2-\partial_5^2}\delta^{ab}\delta^4(\theta-\theta')\delta(y-y')$

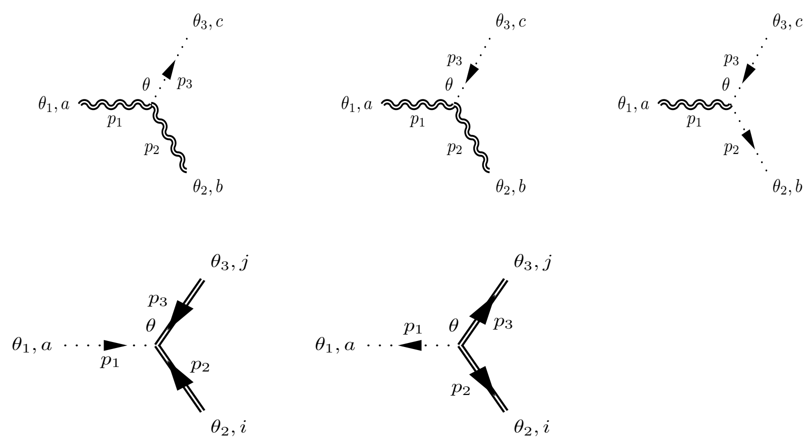
$\bar{c}'^a(-p)$  $c^b(p)$ $= \frac{\mathrm{i}}{p^2-\partial_5^2}\delta^{ab}\delta^4(\theta-\theta')\delta(y-y')$

$c'^a(-p)$  $\bar{c}^b(p)$ $= -\frac{\mathrm{i}}{p^2-\partial_5^2}\delta^{ab}\delta^4(\theta-\theta')\delta(y-y')$

- Vertices also present in the 4D theory



- Vertices not present in the 4D theory



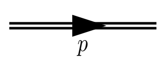
- Note that no $\Phi\Phi\Phi$ and $\bar{\Phi}\bar{\Phi}\bar{\Phi}$ vertices are present. These (anti)chiral Yukawa couplings are forbidden by $SU(2)_R$ symmetry (cf. Chapter 4.1.2).

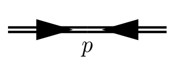
In the Feynman rules given above, y is kept in coordinate space while x_m is Fourier transformed which proves useful in compactified theories.[53] For our calculation in the next chapter for a non-compactified D=5 theory it is of advantage to Fourier transform y as well.

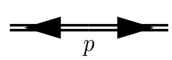
[53]By compactification in y-direction, D=5 Lorentz symmetry is broken and y is treated as a parameter rather than a spacetime coordinate. For an example of this approach cf. [10].

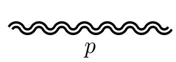
The Feynman rules in full momentum space (p_m, p_5) are:

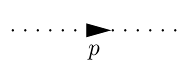
- Draw all possible topologically inequivalent graphs of the order in g which is to be considered.
- Multiply each graph with its symmetry factor.
- p_i are momenta flowing along internal lines, away from the vertex.
- For each vertex integrate over $\displaystyle\int \mathrm{d}^4\theta$.
- For each external field integrate over $\displaystyle\int \mathrm{d}^4\theta$.
- For the external momenta, the overall factor is $\displaystyle\left[\prod_{p_{\text{ext}}} \int \frac{\mathrm{d}^4 p_{\text{ext}}\, \mathrm{d}p_{5,\text{ext}}}{(2\pi)^5}\right] (2\pi)^5 \delta^5\left(\sum_{\text{ext}} (p_{\text{ext}}, p_{5,\text{ext}})\right)$.
- For every loop with momentum p running in it integrate over $\displaystyle\int \frac{\mathrm{d}^4 p\, \mathrm{d}p_5}{(2\pi)^5}$
- For an [anti-]chiral vertex with n internal lines, $n-1$ factors $[-\frac{1}{4}\mathrm{D}^2]$ $-\frac{1}{4}\bar{\mathrm{D}}^2$ act on the propagators.
- A ghost loop contributes a factor (-1).
- The propagators in full momentum space are

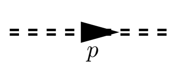
$\bar{H}_i(-p)$, $\bar{H}^c_i(-p)$  $H_j(p)$, $H^c_j(p)$ $\quad = \quad -\frac{\mathrm{i}}{p^2+p_5^2+m^2}\delta_{ij}\delta^4(\theta-\theta')$

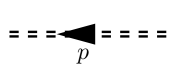
$\bar{H}_i(-p)$  $\bar{H}^c_j(p)$ $\quad = \quad \frac{\bar{\mathrm{D}}^2(p)}{4}\frac{\mathrm{i}(m+\mathrm{i}p_5)}{p^2(p^2+p_5^2+m^2)}\delta_{ij}\delta^4(\theta-\theta')$

$H^c_i(-p)$  $H_j(p)$ $\quad = \quad \frac{\mathrm{D}^2(p)}{4}\frac{\mathrm{i}(m-\mathrm{i}p_5)}{p^2(p^2+p_5^2+m^2)}\delta_{ij}\delta^4(\theta-\theta')$

$V^a(-p)$  $V^b(p)$ $\quad = \quad \frac{1}{2}\frac{\mathrm{i}}{p^2+p_5^2}\delta^{ab}\delta^4(\theta-\theta')$

$\bar{\Phi}^a(-p)$  $\Phi^b(p)$ $\quad = \quad 2\frac{\mathrm{i}}{p^2+p_5^2}\delta^{ab}\delta^4(\theta-\theta')$

$\bar{c}'^a(-p)$  $c^b(p)$ $\quad = \quad \frac{\mathrm{i}}{p^2+p_5^2}\delta^{ab}\delta^4(\theta-\theta')$

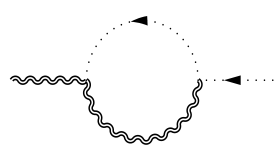
$c'^a(-p)$  $\bar{c}^b(p)$ $\quad = \quad -\frac{\mathrm{i}}{p^2+p_5^2}\delta^{ab}\delta^4(\theta-\theta')$

- The vertex graphs are the same as for non-Fourier transformed y.

4.3 Application: Calculation of the 5D β-Function

As an application in $N{=}1$, $D{=}5$ super Yang-Mills theory, we calculate the β-function on one-loop level. Analogous to the the 4D calculation, it is sufficient to consider the corrections to the vector propagator and its superpartner to renormalise the coupling constant and therefore to calculate the β-function if we rescale the gauge fields by a factor of $1/g$. In the 4D theory, this just meant rescaling $gV \to V$.

In the 5D theory, we have to rescale $g\Phi \to \Phi$ as well, because the 5D vectorfield multiplet is given by linear combinations of the component fields of V and Φ. Therefore the radiative corrections relevant for the renormalisation of the coupling constant at one-loop are corrections to the V *and* the Φ superfield propagator. Furthermore, as has been shown in chapter 3.4 the V and Φ superfields a priory mix due to a $\Phi(\partial_5 V)$-term. The mixing term vanishes on tree level due to a special choice for the gauge-fixing function which in turn takes the 4D superfields V and Φ as well as the ghosts into a 5D Lorentz covariant form.[54] However, on one-loop level, there is a non-vanishing contribution to the mixing term from the supergraph



(4.49)

which has to be taken into account.[55] This term on its own is not 5D Lorentz covariant as can easily be seen from the Feynman rules. The same is true for the radiative correction to the V and the Φ propagator.

It should be possible to make the mixing contribution vanish and to restore the V and the Φ superfield propagators 5D Lorentz covariant by modifying the gauge fixing term in the one-loop effective action. This is however not nessessary for the calculation of the β-function. As the component fields of V and Φ are linear combinations of the vector supermultiplet, the relative factors of the $\bar{\Phi}\Phi$, the $V\Phi$ and the VV counterterms are fixed. We will compute the radiative correction

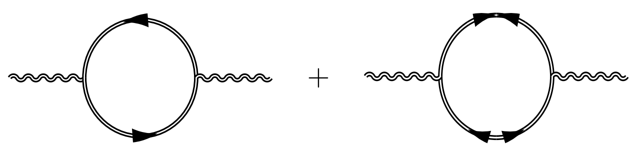
[54] Before the gauge-fixing, the 5D vector multiplet is 5D Lorentz covariant but the 4D superfields as a "collection" of component fields are not. For example, the five spacetime components of the 5D vector field are split up.

[55] This is similar to the situation in a spontaniously broken gauge theory (cf. eg. [39]).

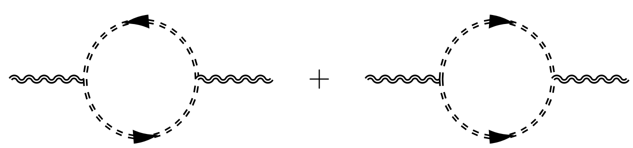
to the vector superfield propagator in Chapter 4.3.1 and assume that the other radiative corrections contribute such that the β-function is a 5D Lorentz scalar as it has to be. Using the first loop order correction, we then compute the 5D β-function in Chapter 4.3.2.

4.3.1 Radiative Corrections to the Vector Superfield Propagator

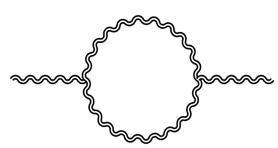
The contributing supergraphs to the correction of the vector superfield propagator to first loop order are



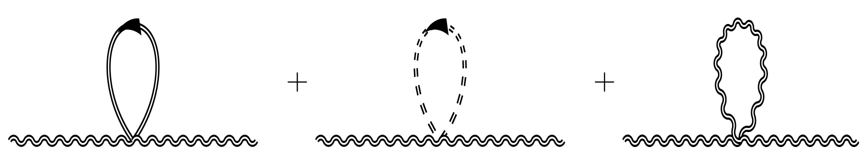
(4.50a)



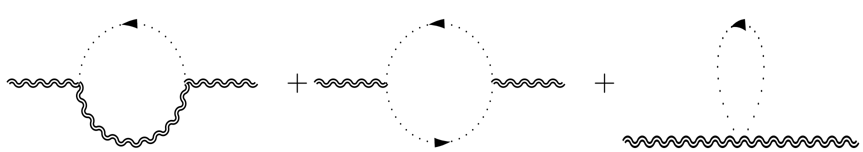
(4.50b)



(4.50c)



(4.50d)

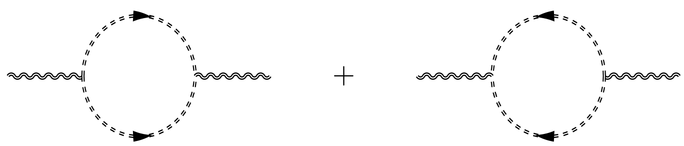


(4.50e)

We first calculate the contributions from the gauge sector (graphs (4.50b-4.50e)).

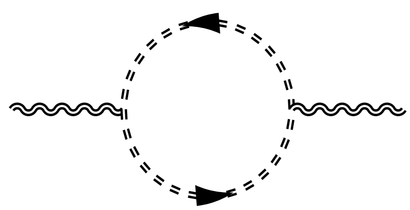
From the action (4.48) it is obvious that the VVV, $Vc'c$ and $V\Phi\Phi$ vertices do not contain any p_5. Therefore no new permutations of derivatives appear, compared to the 4D calculation. The D-algebra for the ghost-loop (4.50b) and the Φ-loop in (4.50e) in five dimensions is identical to the computation in (3.46) in four dimensions as well as the D-algebra for the vector-loop (4.50c) in 5D is identical to (3.58) in 4D. The only differences are the propagators where $\frac{1}{k^2}$ has to be replaced by $\frac{1}{k^2+k_5^2}$ and numerical factors of propagators and vertices.

The results for the ghost-loop are



$$= 0 \tag{4.51}$$

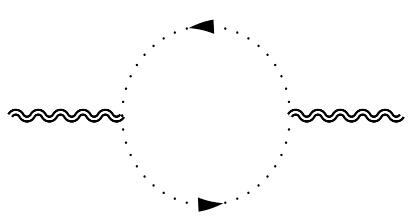
and

$2\times$ 

$$\begin{aligned}
&= 4g^2\left[-\tfrac{1}{2}C_2(G)\right] \\
&\times\left\{\frac{1}{2}\int \mathrm{d}^4\theta \frac{\mathrm{d}^4k\,\mathrm{d}k_5}{(2\pi)^5}\frac{V^a(-p,-p_5)\left[p^2P_T-2(k+p)^2\right]V^a(p,p_5)}{(k^2+k_5^2)((k+p)^2+(k+p)_5^2)}\right\}\ ,
\end{aligned} \tag{4.52}$$

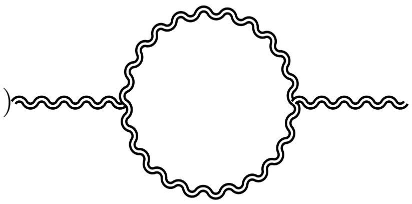
where the p^2P_T term is from (3.46), the "linear" divergences cancel analogous to (3.46) and (3.51) and the quadratic divergence stems from (3.49). The initial factor of 2 again comes from the two inequivalent possibilities to assign ghost superfields to the graph.

For the Φ-loop, the same calculation applies except for factors on vertices and propagators. This leads to



$$\begin{aligned}
&= 4g^2\left[C_2(G)\right] \\
&\times\left\{\frac{1}{2}\int \mathrm{d}^4\theta \frac{\mathrm{d}^4k\,\mathrm{d}k_5}{(2\pi)^5}\frac{V^a(-p,-p_5)\left[p^2P_T-2(k+p)^2\right]V^a(p,p_5)}{(k^2+k_5^2)((k+p)^2+(k+p)_5^2)}\right\}\ .
\end{aligned} \tag{4.53}$$

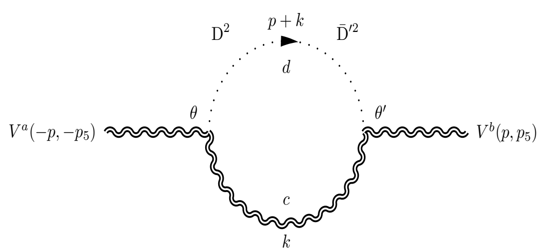
Analogous to (3.58), the vectorloop yields

(5×)


$$= 4g^2 \left[-\tfrac{5}{2}C_2(G)\right]$$
$$\times \left\{ \frac{1}{2}\int \mathrm{d}^4\theta \frac{\mathrm{d}^4k\ \mathrm{d}k_5}{(2\pi)^5} \frac{V^a(-p,-p_5)\left[p^2 P_T\right] V^a(p,p_5)}{(k^2+k_5^2)((k+p)^2+(k+p)_5^2)} \right\} \tag{4.54}$$

where the initial factor of 5 results from the number of topolgically distinct graphs with permutated D's and $\bar{\mathrm{D}}^2 D$'s (cf. Chapter 3.4). Remember that the "linear" and quadratic divergences in this graph cancel identically as has been shown in Chapter 3.4.

The "new" graph in (4.50e) from the $(p_5V)V(\bar{\Phi}+\Phi)$-vertex comes in 2×2 permutations. One of them is



$$= \int \mathrm{d}^4\theta \mathrm{d}^4\theta' \frac{\mathrm{d}^4k\ \mathrm{d}k_5}{(2\pi)^5} \left[\mathrm{i}p_5 V^a(-p,-p_5,\theta)\right] g f^{acd}$$
$$\left(\mathrm{i}k_5 \tfrac{1}{2}\tfrac{\mathrm{i}}{k^2+k_5^2}\right)\delta(\theta-\theta')(-gf^{cbd})$$
$$\left(2\tfrac{\mathrm{i}}{(k+p)^2+(k+p)_5^2)^2}\right)\left(-\tfrac{1}{4}\mathrm{D}'^2\right)\left(-\tfrac{1}{4}\bar{\mathrm{D}}^2\right)V^b(p,p_5,\theta')$$
$$= -4g^2 C_2(G) \times \left\{ \frac{1}{2}\int \mathrm{d}^4\theta \frac{\mathrm{d}^4k\ \mathrm{d}k_5}{(2\pi)^5} \frac{V^a(-p,-p_5)\left[k_5p_5\right] V^a(p,p_5)}{(k^2+k_5^2)((k+p)^2+(k+p)_5^2)} \right\} \tag{4.55}$$

where the factor of (-1) in front of the second vertex comes from the permutation chosen. The other three terms have factors of p_5k_5, p_5^2 and k_5^2 instead of k_5p_5 in the numerator. The graph with $\bar{\Phi}$ propagating in the loop instead of Φ has to be taken into account for hermiticity and leads to the same result.

The total contribution of the graph is therefore

$$\Pi_{V,\Phi} = -8g^2 C_2(G) \times \left\{ \frac{1}{2}\int \mathrm{d}^4\theta \frac{\mathrm{d}^4k\ \mathrm{d}k_5}{(2\pi)^5} \frac{V^a(-p,-p_5)\left[(p+k)_5^2\right] V^a(p,p_5)}{(k^2+k_5^2)((k+p)^2+(k+p)_5^2)} \right\} \tag{4.56}$$

A few remarks on the divergences are in order. Speaking of logarithmic, "linear" and quadratic divergences in this context, we mean terms proportional to $\int \frac{\mathrm{d}^4k\,\mathrm{d}_5^k}{(2\pi)^5}\frac{X}{(k^2+k_5^2)((k+p)^2+(k+p)_5^2)}$ with $X=1$, k_m and k^2, respectively, in analogy to the four dimensional theory. In five dimensions, these terms are not 5D Lorentz covariant as the k_5 component expressions are missing in the numerator. If the apropriate k_5-terms are added, it is obvious from power counting arguments that these expressions lead to divergences of linear, quadratic and qubic order. However, using dimensional reduction in five dimensions, these terms lead to *finite* contributions as can be seen from the identities (C.11a) and (C.11b) in the appendix.

The quadratic divergences arising from the ghost and Φ-loop are not Lorentz covariant. However the divergence added by the $V\Phi$-loop contributes exactly the missing $(k+p)_5^2$ term. Using (C.11a) with $m=0$ shows that

$$\begin{aligned}&\Pi_{\Phi,\mathrm{quadr}}+\Pi_{c,\mathrm{quadr}}+\Pi_{V\Phi}=-8g^2C_2(G)\\&\times\left\{\frac{1}{2}\int \mathrm{d}^4\theta\frac{\mathrm{d}^4k\,\mathrm{d}k_5}{(2\pi)^5}\frac{V^a(-p,-p_5)\left[(p+k)^2+(p+k)_5^2\right]V^a(p,p_5)}{(k^2+k_5^2)((k+p)^2+(k+p)_5^2)}\right\}=0\ .\end{aligned}\tag{4.57}$$

The V, Φ and ghost tadpole graphs all lead to contributions proportional to

$$\Pi_{(V,\Phi,c)\ \mathrm{tadpoles}}\propto\int \mathrm{d}^4\theta\frac{\mathrm{d}^4k\,\mathrm{d}k_5}{(2\pi)^5}\frac{V^a(-p,-p_5)V^a(p,p_5)}{k^2}\tag{4.58}$$

which vanish in dimensional reduction.

Adding up all remaining contributions from (4.52), (4.54), (4.53), (4.55) yields the one-loop correction to a D=5 Super Yang-Mills theory without matter coupled to it which is

$$\begin{aligned}\Pi^{5\mathrm{D}}_{V,c,\Phi}&=4g^2C_2(G)\left[1-\frac{1}{2}-\frac{5}{2}\right]\\&\times\left\{\frac{1}{2}\int \mathrm{d}^4\theta\frac{\mathrm{d}^4k\,\mathrm{d}k_5}{(2\pi)^5}\frac{V^a(-p,-p_5)p^2P_TV^a(p,p_5)}{(k^2+k_5^2)((k+p)^2+(k+p)_5^2)}\right\}\ .\end{aligned}\tag{4.59}$$

Comparing (4.59) to the pure gauge contribution of 4D super Yang-Mills theory in (3.59) which reads

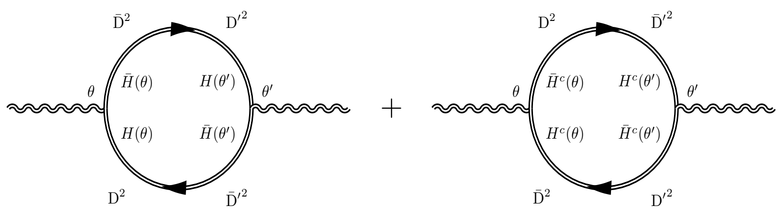
$$\Pi^{4\mathrm{D}}_{V,c}=4g^2C_2(G)\left[-\frac{1}{2}-\frac{5}{2}\right]\left\{\frac{1}{2}\int \mathrm{d}^4\theta\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{V^a(-p)p^2P_TV^a(p)}{k^2(k+p)^2}\right\}$$

we note that the contributions from the 5D ghost- and V-loop are what one would have guessed knowing the 4D result. The Φ superfield reduces the 5D contribution from $3C_2(G)\times\ldots$ to $2C_2(G)\times\ldots$ as one would expect for a chiral field in the adjoint representation in four dimensions.

Note that this expression is not 5D Lorentz covariant.

The one-loop hyperfield contribution to the vector superfield propagator is given in the two graphs in (4.50a) and the left tadpole graph in (4.50d). The calculation of both graphs can directly be taken from the 4D theory. The cancelation of the "linear" divergences works in the same way. Note however that from the hypermultiplet propagators and the $\bar{H}VH$ and $H^cV\bar{H}^c$ vertices there are two distinct assignments of chiral fields to the left graph which from the Feynman rules can be seen to contribute the same.

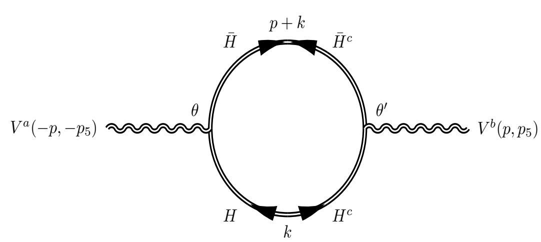
Thus the contribution from the left graph in (4.50a) is



$$
\begin{aligned}
&= 2\times 4g^2\left[\textstyle\sum_A T_A(R)\right] \\
&\quad\times\left\{\frac{1}{2}\int \mathrm{d}^4\theta \frac{\mathrm{d}^4k\,\mathrm{d}k_5}{(2\pi)^5}\frac{V^a(-p,-p_5)(p^2P_T-2(p+k)^2)V^a(p,p_5)}{(k^2+k_5^2+m^2)((k+p)^2+(k+p)_5^2+m^2)}\right\}
\end{aligned}
\tag{4.60}
$$

For the right graph in (4.50a) it seems that there is just one way to assign the fields H and H^c to the graph. However, as we will show, it is not hermitian. Like for the $V\Phi$-loop, the mirror graph is therefore not equivalent and has to be taken into account as well.

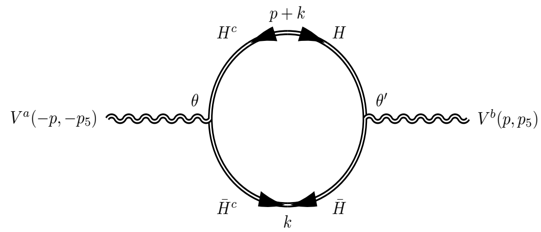
The contribution of one of the graphs is



$$
\begin{aligned}
&= -8g^2\left[\textstyle\sum_A T_A(R)\right] \\
&\quad\times\left\{\frac{1}{2}\int \mathrm{d}^4\theta \frac{\mathrm{d}^4k\,\mathrm{d}k_5}{(2\pi)^5}\frac{V^a(-p,-p_5)\left[(m+\mathrm{i}(p+k)_5^2)(m-\mathrm{i}k_5^2)\right]V^a(p,p_5)}{(k^2+k_5^2+m^2)((k+p)^2+(k+p)_5^2+m^2)}\right\}
\end{aligned}
\tag{4.61}
$$

where the numerator is the product of the numerators of the propagators. It yields $(m+\mathrm{i}(p+k)_5^2)(m-\mathrm{i}k_5^2)=m^2+(p+k)_5k_5+\mathrm{i}mp_5$.

The contribution of the other graph is



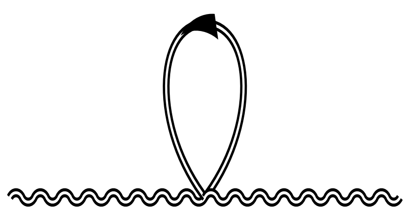
$$-8g^2\left[\textstyle\sum_A T_A(R)\right]$$
$$\times\left\{\frac{1}{2}\int \mathrm{d}^4\theta \frac{\mathrm{d}^4k\ \mathrm{d}k_5}{(2\pi)^5}\frac{V^a(-p,-p_5)\left[(m-\mathrm{i}(p+k)_5^2)(m+\mathrm{i}k_5^2)\right]V^a(p,p_5)}{(k^2+k_5^2+m^2)((k+p)^2+(k+p)_5^2+m^2)}\right\} \tag{4.62}$$

where the numerator is $(m-\mathrm{i}(p+k)_5^2)(m+\mathrm{i}k_5^2)= m^2+(p+k)_5k_5-\mathrm{i}mp_5$.

The total contribution of the massive hypermultiplet-loop is thus given by

$$\Pi_{H,\text{mass}} = 2\times(-8)g^2\left[\sum_A T_A(R)\right]$$
$$\times\left\{\frac{1}{2}\int \mathrm{d}^4\theta \frac{\mathrm{d}^4k\ \mathrm{d}k_5}{(2\pi)^5}\frac{V^a(-p,-p_5)\left[m^2+k_5^2+p_5k_5\right]V^a(p,p_5)}{(k^2+k_5^2+m^2)((k+p)^2+(k+p)_5^2+m^2)}\right\} \tag{4.63}$$

The hypermultiplet tadpole graph in (4.50d) has the result

$2\times$  (4.64)

$$= 2\times 8g^2\left[\textstyle\sum_A T_A(R)\right]$$
$$\times\left\{\frac{1}{2}\int \mathrm{d}^4\theta \frac{\mathrm{d}^4k\ \mathrm{d}k_5}{(2\pi)^5}\frac{V^a(-p,-p_5)V^a(p,p_5)}{k^2+k_5^2+m^2}\right\}\ .$$

Completing the numerator of (4.63) by $(-p_5k_5-p^2)$, this term cancels the quadratic divergent parts of (4.64) and (4.60).

The total contribution of the hypermultiplet-loops to the vector superfield propagator is therefore

$$\Pi^{5\mathrm{D}}_{\mathrm{hyp}} = 4g^2 \left[2 \sum_A T_A(R) \right] \times \left\{ \frac{1}{2} \int \mathrm{d}^4\theta \frac{\mathrm{d}^4k \, \mathrm{d}k_5}{(2\pi)^5} \frac{V^a(-p,-p_5) \left[p^2 P_T + 2p_5^2 + 2p_5 k_5 \right] V^a(p,p_5)}{(k^2+k_5^2+m^2)((k+p)^2+(k+p)_5^2+m^2)} \right\} \tag{4.65}$$

Comparing (4.65) to the matter superfield contribution of 4D super Yang-Mills theory in (3.52) which reads

$$\Pi^{4\mathrm{D}}_{chiral} = 4g^2 \left[\sum_A T_A(R) \right] \times \left\{ \frac{1}{2} \int \mathrm{d}^4\theta \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{V^a(-p) \left[p^2 P_T \right] V^a(p,p_5)}{(k^2+m^2)((k+p)^2+m^2)} \right\}$$

it is obvious that a hypermultiplet in five dimensions contributes the same as two chiral supermultiplets in four dimensions as one might expect from the fact that the 5D hypermultiplet can be expressed as an 4D $SU(2)_R$ superfield doublet.

Like the gauge-loop contribution (4.59), the hypermultiplet-loop contribution is not 5D Lorentz covariant.

As has been discussed in the beginning of Chapter 4.3, the radiative correction of the vector superfield propagator is sufficient for the calculation of the β-function, knowing, that it is a 5D Lorentz scalar. 5D Lorentz covariance of the complete radiative correction has to be restored by the corrections to the Φ superfield propagator and the $\Phi(\partial_5 V)$ mixing term. Thus, the Lorentz covariant radiative correction to the vector superfield propagator following from (4.59) and (4.65) is

$$\begin{aligned}\Pi^{5\mathrm{D}}_{V,\mathrm{LC}} = {} & 4g^2 \times \left[2 \sum_A T_A(R) \right] \\ & \times \left\{ \frac{1}{2} \int \mathrm{d}^4\theta \frac{\mathrm{d}^4k \, \mathrm{d}k_5}{(2\pi)^5} \frac{V^a(-p,-p_5) \left[p^2 + p_5^2 \right] P_T V^a(p,p_5)}{(k^2+k_5^2+m^2)((k+p)^2+(k+p)_5^2+m^2)} \right\} \\ & + 4g^2 \times [-2C_2(G)] \\ & \times \left\{ \frac{1}{2} \int \mathrm{d}^4\theta \frac{\mathrm{d}^4k \, \mathrm{d}k_5}{(2\pi)^5} \frac{V^a(-p,-p_5) \left[p^2 + p_5^2 \right] P_T V^a(p,p_5)}{(k^2+k_5^2)((k+p)^2+(k+p)_5^2)} \right\}\end{aligned} \tag{4.66}$$

By this the Lorentz invariant radiative correction to the Φ superfield propagator is fixed to be

$$
\begin{aligned}
\Pi^{\rm 5D}_{\Phi,\rm LC} = 4g^2 &\times \left[2\sum_A T_A(R)\right] \\
&\times \left\{\frac{1}{2}\int \mathrm{d}^4\theta \frac{\mathrm{d}^4k\,\mathrm{d}k_5}{(2\pi)^5}\frac{\bar\Phi^a(-p,-p_5)\Phi^a(p,p_5)}{(k^2+k_5^2+m^2)((k+p)^2+(k+p)_5^2+m^2)}\right\} \\
&+ 4g^2 \times [-2C_2(G)] \\
&\times \left\{\frac{1}{2}\int \mathrm{d}^4\theta \frac{\mathrm{d}^4k\,\mathrm{d}k_5}{(2\pi)^5}\frac{\bar\Phi^a(-p,-p_5)\Phi^a(p,p_5)}{(k^2+k_5^2)((k+p)^2+(k+p)_5^2)}\right\}
\end{aligned}
\tag{4.67}
$$

4.3.2 The Result for the 5D β-Function

In general, a five dimensional theory is non-renormalisable as the coupling constant has mass dimension $-1/2$. Nevertheless, the theory can be regarded as an effective theory and, as for the four dimensional theory, the coupling depends on the energy scale.

In order to calculate the β-function from the radiative correction to the vector supermultiplet only, we have to rescale $gV \to V$ and $g\Phi \to \Phi$. The Fourier transformed tree level action then reads[56]

$$
\begin{aligned}
S^{\rm 5D\,(0)}_{V,\rm kin} = \frac{\rm Tr}{kg_0^2}\int \mathrm{d}^4\theta \frac{\mathrm{d}^4p\,\mathrm{d}p_5}{(2\pi)^5}\Big\{&\frac{1}{2}V^a(-p,-p_5)\left[p^2+p_5^2\right]P_T V^a(p,p_5) \\
&+\frac{1}{2}\bar\Phi(-p,-p_5)\Phi(p,p_5)\Big\}
\end{aligned}
\tag{4.68}
$$

Using our results (4.66) and (4.67) from the last chapter, the relevant part of the one-loop effective action is

$$
\begin{aligned}
S^{\rm 5D\,(1)}_{V,\rm kin} = S^{\rm 5D\,(0)}_{V,\rm kin} + \frac{4}{16\pi^2}\int \mathrm{d}^4\theta \frac{\mathrm{d}^4p\,\mathrm{d}p_5}{(2\pi)^5}\Bigg\{&\Bigg[\frac{1}{2}\bar\Phi(-p,-p_5)\Phi(p,p_5) \\
&+\frac{1}{2}V^a(-p,-p_5)\left[p^2+p_5^2\right]P_T V^a(p,p_5)\Bigg] \\
&\times\left[2\left(\sum_A T_A(R)\right)B_0(p^2,m,m) - 2C_2(G)B_0(p^2,0,0)\right]\Bigg\}
\end{aligned}
\tag{4.69}
$$

As in the four dimensional calculation, we use the momentum sub-

[56] Again, we only give the part relevant for the computation of the β-function.

traction scheme and define the coupling $g(\mu, \mathcal{M})$ by

$$\frac{1}{g^2(\mu, \mathcal{M})} \equiv \frac{1}{g_0^2} + \frac{4}{16\pi^2}\left[2\left(\sum_A T_A(R)\right) - 2C_2(G)\right] B_0(-\mathcal{M}^2, 0, 0) \ . \tag{4.70}$$

Note that in five dimensions, all divergences are absorbed by dimensional reduction and the integrals B_0 are already finite.

From the definition of the coupling (4.70) we can calculate the dependence of the coupling from the renormalisation point using the result for the scalar two-point integral in five dimensions (C.11b):

$$\mathcal{M}\frac{\partial}{\partial \mathcal{M}} g(\mu, \mathcal{M}) = \frac{1}{32\pi}\left[\left(\sum_A T_A(R)\right) - C_2(G)\right] \frac{\mathcal{M}}{\mu} g^3 \ . \tag{4.71}$$

In contrast to the four dimensional β-function, this expression explicitly depends on μ and $\mathcal{M}$. To avoid this explicit dependance, we rescale the coupling

$$\bar{g} \equiv \sqrt{\frac{\mathcal{M}}{\mu}} g \ . \tag{4.72}$$

and define the five dimensional β-function as a function of $\bar{g}$

$$\beta_{5\mathrm{D}}(\bar{g}) \equiv \mathcal{M}\frac{\partial}{\partial \mathcal{M}} \bar{g}(\mu, \mathcal{M})) \ , \tag{4.73}$$

leading to our final result for the β-function to one-loop order

$$\beta_{5\mathrm{D}}(\bar{g}) = \frac{1}{2}\bar{g} + \frac{1}{32\pi}\left[\left(\sum_A T_A(R)\right) - C_2(G)\right] \bar{g}^3 \ . \tag{4.74}$$

As the four dimensional β-function, $\beta_{5\mathrm{D}}$ contains a term cubic in the coupling constant which depends on group theoretical factors only. The change of these factors from four to five dimensions has already been discussed in the previous chapter. As a result, the cubic term of the 5D β-function vanishes if one massless hypermultiplet in the adjoint representation is coupled to the gauge supermultiplet.

Apart from group theoretical term, (4.74) contains a power-law running term proportional to $\bar{g}$, which is determined by the mass dimension of the gauge coupling constant, independant of the gauge group and of matter coupled to the gauge multiplet.

5 Summary and Outlook

In this diploma thesis we have derived the Feynman rules for a D=5, N=1 supersymmetric Yang-Mills theory in terms of four dimensional superfields and have used this formulation to derive the β-function of the theory.

As a starting point, we have used the superfield formulation of the five dimensional theory given in [20]. In the beginning of Chapter 4.1.1 we have given a brief review of the basic ideas and concepts presented in [20] and have shown that the distribution of the fields in the five dimensional vector multiplet on the component fields of the four dimensional vector superfield V and the chiral superfield Φ is determined if one demands the component fields to be 5D Lorentz covariant on tree level. We have dealt with the five dimensional massive chiral hypermultiplet in an analogous way in the beginning of Chapter 4.1.2.

The five dimensional theory has been quantised following the pattern of the known quantisation of a four dimensional N=1 supersymmetric Yang-Mills theory which has been reviewed in Chapter 3. An important new feature of the five dimensional theory is the R_ξ-like supersymmetric gauge fixing (4.15) which eliminates the $\Phi(\partial_5 V)$-mixing term in the action and at the same time makes the V, Φ and ghost superfield propagators 5D Lorentz covariant.

Having the gauge fixed action, we have defined the generating functional and from it, the superfield propagators as the two-point Greens functions of the theory. We point out that for a theory, compactified on a manifold or an orbifold, the same procedure holds. The compactification is manifest in the boundary conditions of the propagators only. In Chapter 4.3.1 we have calculated the propagators for a non-compactified 5-direction. The Feynman rules for the vertices have been read off from the action. From the derivation from the generating functional it can be seen that the vertices are independant of the boundary conditions for the propagators.

As one main result of this thesis, the Feynman rules for the D=5, N=1 supersymmetric Yang-Mills theory are given in Chapter 4.2.

As an application, we have calculated the β-function of the theory. This turns out to be technically more involved than the analogous calculation in four dimensions. Apart from additional graphs which have to be considered we find one-loop level contributions to the $\Phi(\partial_5 V)$-terms which on tree level has been gauged away. Due to the properties of the gauge fixing, these mixing terms also effect the 5D Lorentz convariance of the four dimensional superfields. We point out that this is a purely technical problem. It is convenient to work with 5D Lorentz covariant superfields on a given loop level but physically, 5D Lorentz covariance is demanded for the fields of the 5D vector multiplet and its loop corrections, not for a special distribution of its degrees of freedom on the superfields V and Φ.

We have calculated the one-loop contribution to the vector superfield propagator. Knowing that both superfields V and Φ stem from the 5D vector supermultiplet, this result is sufficient to determine the dependance of the coupling from the renormalisation scale on one-loop level. From it, we have defined and calculated the β-function to be (4.74)

$$\begin{aligned}\beta_{5\mathrm{D}}(\bar{g}) &\equiv \mathcal{M}\frac{\partial}{\partial\mathcal{M}}\bar{g}(\mu,\mathcal{M})) \\ &= \frac{1}{2}\bar{g} + \frac{1}{32\pi}\left[\left(\sum_A T_A(R)\right) - C_2(G)\right]\bar{g}^3 \ .\end{aligned}$$

Comparing the result with the four dimensional β-function, the main differences are the presence of a power-law contribution in $\beta_{5\mathrm{D}}$ resulting from the mass dimension of the gauge coupling and the numerical difference in the group theoretical factors by which the cubic part of $\beta_{5\mathrm{D}}$ cancels if one massless hypermultiplet in the adjoint representation is coupled to the gauge multiplet while in four dimensions, three massless matter multiplets are needed to make the cubic term (and therefore the whole β-function) vanish.

A special field of interest where the formalism developed in this thesis can be applied is the study of higher dimensional supersymmetric theories compactified on orbifolds as has been motivated in the introduction. For an unbroken supersymmetric five dimensional gauge theory on an orbifold the derived Feynman rules can be used if the propagators are recalculated according to the boundary conditions induced by the compactification and the parity assignment from the orbifold projection. An example for this can be found in [10].

To use the superfield formalism for more realistic models, a next step would be to include gauge and supersymmetry breaking. Gauge breaking by orbifolding can be implemented easily as its origin is the assignment of orbifold parities to the gauge fields which is again implemented by the boundary conditions to the superfield propagators. For supersymmetry breaking we are not at the stage to comment on specific ways on implementing the variaty of supersymmetry breaking mechanisms. Ref. [10] provides an example for which it is possible.

Another direction for further work is the generalisation to higher spacetime dimensions. Already in 1983 in [17], a superfield formulation for an D=10, N=1 supersymmetric gauge theory is given and the Feynman rules are derived. From [19] it can be seen that many of the concepts of the ten dimensional theory also apply to lower spacetime dimensions, including the five dimensional theories as the simplest case.

As mentioned in the introduction, there are promising setups for GUTs using $SO(10)$ on a six dimensional orbifold [14, 15, 16]. Having a superfield formulation of higher dimensional supersymmetry might simplify further studies on this.

A Conventions and Notation

A.1 Spinors in Four Dimensional Spacetime

In this diploma thesis the conventions of [21] are used. The D-dimensional spacetime metric is

$$\eta_{mn} = \mathrm{diag}(-1, 1, \ldots, 1) \tag{A.1}$$

The Γ-matrices are chosen in the Weyl representation and read in D=4

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix} \tag{A.2}$$

where $\sigma^m = (-1, \sigma^i)$ and $\bar{\sigma}^m = (-1, -\sigma^i)$ and σ^i are the Pauli matrices.

In four dimensions, there are two inequivalent spinor representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. Their generators are given by

$$\begin{aligned}
&\left(\frac{1}{2}, 0\right) \text{ generators:} \qquad (\sigma^{mn})_\alpha{}^\beta = \frac{1}{4}\left(\sigma^m_{\alpha\dot{\alpha}}\bar{\sigma}^{n\dot{\alpha}\beta} - \sigma^n_{\alpha\dot{\alpha}}\bar{\sigma}^{m\dot{\alpha}\beta}\right) \\
&\left(0, \frac{1}{2}\right) \text{ generators:} \qquad (\bar{\sigma}^{mn})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{4}\left(\bar{\sigma}^{m\dot{\alpha}\alpha}\sigma^n_{\alpha\dot{\beta}} - \bar{\sigma}^{n\dot{\alpha}\alpha}\sigma^m_{\alpha\dot{\beta}}\right)
\end{aligned} \tag{A.3}$$

Their representation spaces are the left- and righthanded Weyl fermions ψ_α and $\bar{\psi}_{\dot{\alpha}}$ which can be combined to a Dirac spinor

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}, \qquad \bar{\Psi}_D = \left(\chi^\alpha, \bar{\psi}_{\dot{\alpha}}\right) \tag{A.4}$$

The tensors $\varepsilon_{\alpha\beta}$ and $\varepsilon_{\dot{\alpha}\dot{\beta}}$ are invariant under $SL(2, \mathbb{C})$ and can be used to raise and lower indices. Choosing $\varepsilon^{12} = \varepsilon_{21} = \varepsilon^{\dot{1}\dot{2}} = \varepsilon_{\dot{2}\dot{1}} = 1$ one can define raising and lowering by

$$\begin{aligned}
\psi^\alpha &= \varepsilon^{\alpha\beta}\psi_\beta \\
\psi_\alpha &= \varepsilon_{\alpha\beta}\psi^\beta \\
\bar{\sigma}^{m\dot{\alpha}\alpha} &= \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon^{\alpha\beta}\sigma^m_{\beta\dot{\beta}} \ .
\end{aligned} \tag{A.5}$$

With these definition one finds

$$\begin{aligned}
\psi\chi &\equiv \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = \psi^\alpha\chi_\alpha \equiv \chi\psi \\
\bar{\psi}\bar{\chi} &\equiv \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \equiv \bar{\chi}\bar{\psi} = (\chi\psi)^\dagger \ .
\end{aligned} \tag{A.6}$$

A.2 Lorentz Algebra and Spinors in Higher Dimensions

In this appendix we summarise some results on higher-dimensional Lorentz algebra which are of importance for this thesis. Most of them can be found in [36] and in some more detail in [37, 40]. For the case of D=5 which is explicitly given in the end of the chapter we refer to [18, 20].

The Lorentz algebra in D dimensions is given by

$$[M_{MN}, M_{OP}] = \mathrm{i}(\eta_{NO}M_{MP} - \eta_{NP}M_{MO} - \eta_{MO}M_{NP} + \eta_{MP}M_{NO}), \tag{A.7}$$

where $M, N, O, P = 0\ldots(D-1)$.

Spinor representations of the Lorentz algebra can be found by defining the Dirac matrices $\Gamma_0, \ldots \Gamma_{D-1}$ which satisfy the Dirac algebra

$$\{\Sigma_M, \Gamma_N\} = 2\eta_{MN}\mathbb{1}\ . \tag{A.8}$$

The spinor reprensentation of the Lorentz algebra is then given by

$$M_{AB} = \frac{\mathrm{i}}{2}\Gamma_{[A}\Gamma_{B]}\ , \tag{A.9}$$

so the task is to find a representation of the D-dimensional Dirac algebra. It can be shown that the complex dimension n of the irreducible representations of the Dirac algebra in D dimensions is given by

$$n = \begin{cases} 2^{\frac{D}{2}} \text{ for } D \text{ even} \\ 2^{\frac{D-1}{2}} \text{ for } D \text{ odd} \end{cases} \tag{A.10}$$

In [**?**] it is shown how to construct the representations recursively. We only give the five-dimensional algebra in the end of the chapter.

As a next step, one has to consider whether the representation of the Dirac algebra is irreducible i.e. if there is an (anti)automorphism group which divides the representation space into invariant subspaces. For the following we refer to [36, 37] and just list the main results.

One way of reducing the representation is by chiral projectors, which in D=4 are given by

$$\mathrm{P}_{L,R} = \frac{1}{2}(\mathbb{1} \mp \mathrm{i}\Gamma_5) \tag{A.11}$$

where

$$\Gamma_5 \equiv \Gamma_0\Gamma_1\Gamma_2\Gamma_3 \tag{A.12}$$

A generalisation of Γ_5 from D=4 to arbitrary dimensions can be given by

$$\Gamma_{D+1} \equiv (\mathrm{i})^{\frac{D}{2}} \Gamma_0 \cdot \ \ldots \ \cdot \Gamma_{D-1} \tag{A.13}$$

From (A.8), it is obvious that for even D, $\{\Gamma_{D+1}, \Gamma_M\} = 0$ for any $M = 0 \ldots D-1$ while for odd D, $[\Gamma_{D+1}, \Gamma_M] = 0$. By Schur's lemma, it follows, that for odd D, $\Gamma_{D+1} \sim \mathbb{1}$. Therefore in odd dimensions, Γ_{D+1} cannot be used to define chiral projectors, while for even dimensions, chiral spinors exist.

Another way of reducing the spinor representation is to use the Dirac algebra (anti)automorphism induced by the conjugation matrix C. This leads to real, pseudoreal or complex spinor representations depending on D. For real and complex representations ($D = 0, 1, 2, 3, 4$ mod 8), a Majorana condition

$$\psi = C\bar{\psi}^T \tag{A.14}$$

can be imposed, while for pseudoreal representations ($D = 5, 6, 7$ mod 8) one can impose a symplectic Majorana condition

$$\psi_i = \Omega_{ij} C \ \bar{\psi}_j^T \tag{A.15}$$

where $i, j = 1 \ldots N$ and Ω_{ij} is a symplectic invariant metric.

All this is summarised in the following table.

Table 1: Spinor representations in various dimensions [36, 37]

Dimension	2	3	4	5	6	7	8	9	10	11
Real spinor dim.	4	4	8	8	16	16	32	32	64	64
Chiral spinors	yes	-	yes	-	yes	-	yes	-	yes	-
Reality condititon	real	real	compl.	psreal	psreal	psreal	compl.	real	real	real
Majorana spinors	yes	yes	yes	-	-	-	yes	yes	yes	yes
Ch. M. spinors	yes	-	-	-	-	-	-	-	yes	-
Min. spinor dim.	1	2	4	8	8	16	16	16	32	64

For the special case of D=5 we adopt the conventions of [20][57] in which the Dirac algebra is given by

$$\Gamma^M = \left(\begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix}, \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \right). \tag{A.16}$$

[57] In D=4 they agree with [21] as ours do.

The spinors are chosen in a pseudoreal representation with the antiautomorphism group $USp(2)$ (cf. [37]) leading to symplectic Majorana spinors ψ_i with the symplectic Majorana condition

$$\psi^i = \epsilon^{ij} C \ \left(\bar{\psi}^j\right)^T \tag{A.17}$$

where ψ_i, $i = 1, 2$ transforms under $SU(2)_R$. Here, the symplectic metric is just ϵ_{ij}. The charge conjugation C can be chosen as $C = \mathrm{diag}(\mathrm{i}\sigma^2, \mathrm{i}\sigma^2)$.

To be able to express D=5 supersymmetry in 4D superfields as it is done in [19, 20] and reviewed in Chapter 4.1, it is necessary to express D=5 symplectic Majorana spinors in terms of D=4 spinors which can be established by the the decomposition

$$\psi^1 = \begin{pmatrix} (\psi_L)_\alpha \\ (\bar{\psi_R})^{\dot{\alpha}} \end{pmatrix}, \psi^2 = \begin{pmatrix} (\psi_R)_\alpha \\ -(\bar{\psi_L})^{\dot{\alpha}} \end{pmatrix}, \bar{\psi}_1 = \begin{pmatrix} (\psi_R)^\alpha \\ (\bar{\psi_L})_{\dot{\alpha}} \end{pmatrix}^T, \bar{\psi}_2 = \begin{pmatrix} -(\psi_L)^\alpha \\ (\bar{\psi_R})_{\dot{\alpha}} \end{pmatrix}^T . \tag{A.18}$$

B Useful Identities

In the following section, some identities are listed which are used frequently throughout this diploma thesis. Most of them can be found in [21, 41] or follow by simple calculation.

B.1 4D Spinor Algebra

$$\sigma^m_{\alpha\dot{\alpha}} \bar{\sigma}^{n\dot{\alpha}\beta} = -\eta^{mn} \delta^\beta_\alpha + 2 (\sigma^{mn})_\alpha{}^\beta \tag{B.1}$$

$$\bar{\sigma}^{m\dot{\alpha}\alpha} \sigma^n_{\alpha\dot{\beta}} = -\eta^{mn} \delta^{\dot{\alpha}}_{\dot{\beta}} + 2 (\bar{\sigma}^{mn})^{\dot{\alpha}}{}_{\dot{\beta}} \tag{B.2}$$

$$\mathrm{Tr}(\sigma^m \bar{\sigma}^n) = -2\eta^{mn} \tag{B.3}$$

$$\sigma^m_{\alpha\dot{\alpha}} \bar{\sigma}^{\dot{\beta}\beta}_m = -2 \delta^\beta_\alpha \delta^{\dot{\beta}}_{\dot{\alpha}} \tag{B.4}$$

$$(\sigma^m)_{\alpha\dot{\alpha}} (\sigma_m)_{\beta\dot{\beta}} = -2 \varepsilon_{\alpha\beta} \varepsilon_{\dot{\alpha}\dot{\beta}} \tag{B.5}$$

$$(\sigma^m \bar{\sigma}^n + \sigma^n \bar{\sigma}^m)_\alpha{}^\beta = -2\eta^{mn} \delta^\beta_\alpha \tag{B.6}$$

$$(\bar{\sigma}^m \sigma^n + \bar{\sigma}^n \sigma^m)^{\dot{\alpha}}{}_{\dot{\beta}} = -2\eta^{mn} \delta^{\dot{\alpha}}_{\dot{\beta}} \tag{B.7}$$

$$(\sigma^{mn})_\alpha{}^\beta \varepsilon_{\beta\gamma} = (\sigma^{mn})_\gamma{}^\beta \varepsilon_{\beta\alpha} \tag{B.8}$$

$$\varepsilon^{mnrs} \sigma_{rs} = -2\mathrm{i} \sigma^{mn}, \quad \varepsilon^{mnrs} \bar{\sigma}_{rs} = 2\mathrm{i} \bar{\sigma}^{mn} \tag{B.9}$$

B.2 Identities involving $\boldsymbol{\theta}$'s, D's and Superspace Integrals

Let θ_α and $\bar{\theta}_{\dot{\alpha}}$ be spinorial Grassmann variables.

$$\begin{array}{rcl} \theta^\alpha\theta^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta}\theta\theta & , & \theta_\alpha\theta_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}\theta\theta \\ \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \frac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} & , & \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\frac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \end{array} \tag{B.10}$$

For convenience, we write $\sigma^n_{\alpha\dot{\alpha}}\partial_n = \partial_{\alpha\dot{\alpha}}$, $\sigma^n_{\alpha\dot{\alpha}}v_n = v_{\alpha\dot{\alpha}}, \ldots$ on which indices are raised and lowered by the ε-tensors, keeping in mind (A.5). Then, the definition of the right supersymmetry generating operators D_α and $\bar{\mathrm{D}}_{\dot{\alpha}}$(2.7) read

$$\mathrm{D}_\alpha = \partial_\alpha + \mathrm{i}\partial_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} \quad , \quad \bar{\mathrm{D}}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} - \mathrm{i}\theta^\alpha\partial_{\alpha\dot{\alpha}} \tag{B.11}$$

where $\partial_\alpha = \frac{\partial}{\partial\theta^\alpha}$ and $\partial_{\dot{\alpha}} = \frac{\partial}{\partial\theta^{\dot{\alpha}}}$ are defined by

$$\begin{array}{rcl} \partial^\alpha\theta_\beta = \delta^\alpha_\beta & , & \partial_\alpha\theta^\beta = \delta^\beta_\alpha \\ \partial^{\dot{\alpha}}\theta_{\dot{\beta}} = \delta^{\dot{\alpha}}_{\dot{\beta}} & , & \partial_{\dot{\alpha}}\theta^{\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}}\ . \end{array} \tag{B.12}$$

Therefore

$$\partial^\alpha = -\varepsilon^{\alpha\beta}\partial_\beta \quad , \quad \partial^{\dot{\alpha}} = -\varepsilon^{\dot{\alpha}\dot{\beta}}\partial_{\dot{\beta}} \tag{B.13}$$

and

$$\mathrm{D}^\alpha = -\partial^\alpha + \mathrm{i}\partial^\alpha{}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} \quad , \quad \bar{\mathrm{D}}^{\dot{\alpha}} = +\partial^{\dot{\alpha}} - \mathrm{i}\theta^\alpha\partial_\alpha{}^{\dot{\alpha}}\ . \tag{B.14}$$

From the D's and $\bar{\mathrm{D}}$'s, a group of operators including projection operators on (anti-)chiral fields can be constructed [21]:

$$\begin{array}{lll} P_1 = \dfrac{\mathrm{D}^2\bar{\mathrm{D}}^2}{16\Box}, & P_2 = \dfrac{\bar{\mathrm{D}}^2\mathrm{D}^2}{16\Box} & P_T = -\dfrac{\mathrm{D}\bar{\mathrm{D}}^2\mathrm{D}}{8\Box} = -\dfrac{\bar{\mathrm{D}}\mathrm{D}^2\bar{\mathrm{D}}}{8\Box} \\ P_+ = \dfrac{\mathrm{D}^2}{4\Box^{\frac{1}{2}}}, & P_- = \dfrac{\bar{\mathrm{D}}^2}{4\Box^{\frac{1}{2}}}\ . & \end{array} \tag{B.15}$$

Here P_1 is a projector on chiral fields, P_2 is a projector on antichiral fields and P_T is a projector on the so-called transverse superfield.[58] They fulfill

$$P_1 + P_2 + P_T = 1\ . \tag{B.16}$$

The whole multiplication table reads

[58] another irreducible superfield

	P_1	P_2	P_+	P_-	P_T
P_1	P_1	0	P_+	0	0
P_2	0	P_2	0	P_-	0
P_+	0	P_+	0	P_1	0
P_-	P_-	0	P_2	0	0
P_T	0	0	0	0	P_T

where the elements from the first column are multiplied with the elements of the first row.

Further relations for the D's are:

$$\begin{aligned}
\frac{1}{2}[\mathrm{D}_\alpha, \bar{\mathrm{D}}_{\dot{\alpha}}] &= -\bar{\mathrm{D}}_{\dot{\alpha}}\mathrm{D}_\alpha - \mathrm{i}\partial_{\alpha\dot{\alpha}} \\
\frac{1}{4}[\mathrm{D}_\alpha, \bar{\mathrm{D}}^2] &= -\mathrm{i}\partial_{\alpha\dot{\alpha}}\bar{\mathrm{D}}^{\dot{\alpha}} \\
\frac{1}{4}[\bar{\mathrm{D}}_{\dot{\alpha}}, \mathrm{D}^2] &= \mathrm{i}\mathrm{D}^\alpha\partial_{\alpha\dot{\alpha}} \\
\frac{1}{8}[\mathrm{D}^2, \bar{\mathrm{D}}^2] &= -\mathrm{i}\mathrm{D}^\alpha\partial_{\alpha\dot{\alpha}}\bar{\mathrm{D}}^{\dot{\alpha}} + 2\Box \\
\mathrm{D}^2\bar{\mathrm{D}}^2\mathrm{D}^2 &= 16\Box\mathrm{D}^2 \\
\bar{\mathrm{D}}^2\mathrm{D}^2\bar{\mathrm{D}}^2 &= 16\Box\bar{\mathrm{D}}^2
\end{aligned} \tag{B.17}$$

As the D's are spinorial differential operators, there is a generalised Leibnitz rule for them, acting on superfields:

$$\mathrm{D}_\alpha(FG) = (\mathrm{D}_\alpha F)G \pm F(\mathrm{D}_\alpha G) \tag{B.18}$$

and therefore

$$\begin{aligned}
\mathrm{D}^2(FG) &= (\mathrm{D}^2F)G + F(\mathrm{D}^2G) \pm 2(\mathrm{D}F)(\mathrm{D}G) \\
\bar{\mathrm{D}}^2(FG) &= (\bar{\mathrm{D}}^2F)G + F(\bar{\mathrm{D}}^2G) \pm 2(\bar{\mathrm{D}}F)(\bar{\mathrm{D}}G)
\end{aligned} \tag{B.19}$$

with $+$ for a grassmann-even and $-$ for a grassmann-odd superfield F.

An important yet obvious property of D_α concerning superspace integration is that

$$\int \mathrm{d}^4x\mathrm{D}_\alpha F = \int \mathrm{d}^4x\partial_\alpha F + \text{surface terms} \ . \tag{B.20}$$

Using

$$\int \mathrm{d}^4\theta\partial_\alpha F = 0 \tag{B.21}$$

as a consequence of the definition of the Grassmann integral, one finds, that

$$\int \mathrm{d}^8z\mathrm{D}_\alpha F = \int \mathrm{d}^8z\bar{\mathrm{D}}_{\dot{\alpha}}F = 0 \tag{B.22}$$

up to surface terms. From this, we get a rule for partial integration and some relations following from it:

$$\begin{aligned} \int \mathrm{d}^8 z F \mathrm{D}_\alpha G &= \mp \int \mathrm{d}^8 z (\mathrm{D}_\alpha F) G \\ \int \mathrm{d}^8 z F \mathrm{D}^2 G &= \int \mathrm{d}^8 z (\mathrm{D}^2 F) G \\ \int \mathrm{d}^8 F \mathrm{D}\bar{\mathrm{D}}^2 \mathrm{D} G &= \int \mathrm{d}^8 (\mathrm{D}\bar{\mathrm{D}}^2 \mathrm{D} F) G \\ \int \mathrm{d}^8 F \mathrm{D}^2 \bar{\mathrm{D}}^2 G &= \int \mathrm{d}^8 (\bar{\mathrm{D}}^2 \mathrm{D}^2 F) G \ . \end{aligned} \tag{B.23}$$

For calculations of Feynman supergraphs the action of D's (appearing in vertices and propagators) on $\delta^8(z_1 - z_2) \equiv \delta^8_{12}$ (appearing in propagators) is important. Some identities in coordinate space are

$$\begin{aligned} \mathrm{D}_{1\alpha}\delta^8_{12} &= -\mathrm{D}_{2\alpha}\delta^8_{12} \\ \bar{\mathrm{D}}_{1\dot{\alpha}}\delta^8_{12} &= -\bar{\mathrm{D}}_{2\dot{\alpha}}\delta^8_{12} \\ \mathrm{D}^2_1\delta^8_{12} &= \mathrm{D}^2_2\delta^8_{12} \\ \bar{\mathrm{D}}^2_1\delta^8_{12} &= \bar{\mathrm{D}}^2_2\delta^8_{12} \\ \mathrm{D}^2_1\bar{\mathrm{D}}^2_1\delta^8_{12} &= \bar{\mathrm{D}}^2_2\mathrm{D}^2_2\delta^8_{12} \\ \bar{\mathrm{D}}^2_1\mathrm{D}^2_1\delta^8_{12} &= \mathrm{D}^2_1\bar{\mathrm{D}}^2_2\delta^8_{12} \\ &\vdots \end{aligned} \tag{B.24}$$

Here, properties of $\delta(x_1 - x_2)$-distributions like $\partial_{l_1}\delta(x_1 - x_2) = -\partial_{l_2}\delta(x_1 - x_2)$ have been used.

Combining this with (B.23) leads to

$$\begin{aligned} (\mathrm{D}_{1\alpha}\delta^8_{12})\delta^8_{13}\delta^8_{14} &= \delta^8_{12}(\mathrm{D}_{1\alpha}\delta^8_{13})\delta^8_{14} + \delta^8_{12}\delta^8_{13}(\mathrm{D}_{1\alpha}\delta^8_{14}) \\ (\mathrm{D}^2_1\delta^8_{12})\delta^8_{13}\delta^8_{14} &= \delta^8_{12}(\mathrm{D}^2_1\delta^8_{13})\delta^8_{14} + \delta^8_{12}\delta^8_{13}(\mathrm{D}^2_1\delta^8_{14}) + \delta^8_{12}(\mathrm{D}^\alpha_1\delta^8_{13})(\mathrm{D}_{1_\alpha}\delta^8_{14}) \\ &\vdots \end{aligned} \tag{B.25}$$

By simple calculation, one also find the following identities for D's acting on $\delta^4_{12} = \delta(\theta_1 - \theta_2)\delta(\bar{\theta}_1 - \bar{\theta}_2)$:

$$\begin{aligned} \mathrm{D}^2_1\delta^4_{12} &= -4\mathrm{e}^{-\mathrm{i}(\theta_1-\theta_2)\sigma^m\partial_m\bar{\theta}_1} \\ \bar{\mathrm{D}}^2_1\delta^4_{12} &= -4\mathrm{e}^{\mathrm{i}\theta_1\sigma^m\partial_m(\bar{\theta}_1-\bar{\theta}_2)} \\ \bar{\mathrm{D}}^2_1\mathrm{D}^2_1\delta^4_{12} &= 16\mathrm{e}^{\mathrm{i}(\theta_1\sigma^m\partial_m\bar{\theta}_1+\theta_2\sigma^m\partial_m\bar{\theta}_2-2\theta_1\sigma^m\partial_m\bar{\theta}_2)} \\ \mathrm{D}^2_1\bar{\mathrm{D}}^2_1\delta^4_{12} &= 16\mathrm{e}^{-\mathrm{i}(\theta_1\sigma^m\partial_m\bar{\theta}_1+\theta_2\sigma^m\partial_m\bar{\theta}_2-2\theta_2\sigma^m\partial_m\bar{\theta}_1)} \ . \end{aligned} \tag{B.26}$$

Two identities which will be used to rewrite the chiral action in Chapter 3.3 are derived by using the above relations as well as the projectors on chiral fields (B.15):

$$\begin{aligned}\int \mathrm{d}^8 z \frac{1}{2} m \Phi\Phi\delta(\bar\theta) &= \int \mathrm{d}^8 z \frac{1}{2} m \Phi \left(\frac{\bar{\mathrm{D}}^2 \mathrm{D}^2}{16\Box}\Phi\right)\delta(\bar\theta) \\ &= \int \mathrm{d}^8 z \frac{1}{2} m \Phi \frac{\mathrm{D}^2}{16\Box}\Phi \cdot (-4) + \text{spacetime derivatives} \\ &= \int \mathrm{d}^8 z \left(-\Phi\frac{m\mathrm{D}^2}{8\Box}\Phi\right) + \text{spacetime derivatives}\ .\end{aligned} \tag{B.27}$$

Analogously for anti-chiral fields:

$$\int \mathrm{d}^8 z \frac{1}{2} m \bar\Phi\bar\Phi\delta(\theta) = \int \mathrm{d}^8 z \left(-\bar\Phi\frac{m\bar{\mathrm{D}}^2}{8\Box}\bar\Phi\right) + \text{spacetime derivatives} \tag{B.28}$$

In order to get D-algebra rules in momentum-space, a Fourier transformation on x_i is performed, i.e. using $\delta(x_1-x_2) = \int \frac{d^4p}{(2\pi)^4}\exp[-\mathrm{i}p(x_1-x_2)]$, the identities (B.24) become

$$\begin{aligned}\mathrm{D}_{1\alpha}\delta^4_{12}\mathrm{e}^{-\mathrm{i}p(x_1-x_2)} &= -\mathrm{D}_{2\alpha}\delta_{12}\mathrm{e}^{\mathrm{i}p(x_1-x_2)} \\ \bar{\mathrm{D}}_{1\dot\alpha}\delta^4_{12}\mathrm{e}^{-\mathrm{i}p(x_1-x_2)} &= -\bar{\mathrm{D}}_{2\dot\alpha}\delta_{12}\mathrm{e}^{\mathrm{i}p(x_1-x_2)} \\ \mathrm{D}^2_1\delta^4_{12}\mathrm{e}^{-\mathrm{i}p(x_1-x_2)} &= \mathrm{D}^2_2\delta_{12}\mathrm{e}^{\mathrm{i}p(x_1-x_2)} \\ \bar{\mathrm{D}}^2_1\delta^4_{12}\mathrm{e}^{-\mathrm{i}p(x_1-x_2)} &= \bar{\mathrm{D}}^2_2\delta_{12}\mathrm{e}^{-\mathrm{i}p(x_1-x_2)} \\ \mathrm{D}^2_1\bar{\mathrm{D}}^2_1\delta^4_{12}\mathrm{e}^{-\mathrm{i}p(x_1-x_2)} &= \bar{\mathrm{D}}^2_2\mathrm{D}^2_2\delta_{12}\mathrm{e}^{-\mathrm{i}p(x_1-x_2)} \\ \bar{\mathrm{D}}^2_1\mathrm{D}^2_1\delta^4_{12}\mathrm{e}^{-\mathrm{i}p(x_1-x_2)} &= \mathrm{D}^2_1\bar{\mathrm{D}}^2_2\delta_{12}\mathrm{e}^{-\mathrm{i}p(x_1-x_2)} \\ &\vdots\end{aligned} \tag{B.29}$$

In order to do calculations in momentum space only, the D-operators can be Fourier transformed, too. Definition (2.7) in p-space read:

$$\begin{aligned}\mathrm{D}_\alpha(p) &= \mathrm{D}_\alpha(p,\theta,\bar\theta) = \partial_\alpha + p_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} \\ \bar{\mathrm{D}}_{\dot\alpha}(p) &= \bar{\mathrm{D}}_{\dot\alpha}(p,\theta,\bar\theta) = -\partial_{\dot\alpha} - \theta^\alpha p_{\alpha\dot\alpha}\end{aligned} \tag{B.30}$$

with the anticommutation relations

$$\begin{aligned}\{\mathrm{D}_\alpha(p_1), \mathrm{D}_\beta(p_2)\} &= 0 = \{\bar{\mathrm{D}}_{\dot\alpha}(p_1), \bar{\mathrm{D}}_{\dot\beta}(p_2)\} \\ \{\mathrm{D}_\alpha(p_1), \bar{\mathrm{D}}_{\dot\beta}(p_2)\} &= -2(p_1-p_2)_{\alpha\dot\alpha}\delta^4(\theta_1-\theta_2)\ .\end{aligned} \tag{B.31}$$

With the definitions (B.30), we get further relations from (B.17):

$$
\begin{aligned}
[\mathrm{D}_\alpha(p_1), \bar{\mathrm{D}}_{\dot\alpha}(p_2)] &= -2\{\bar{\mathrm{D}}_{\dot\alpha}(p_2)\mathrm{D}_\alpha(p_1) + (p_1 - p_2)_{\alpha\dot\alpha}\}\delta^4(\theta_1 - \theta_2) \\
[\mathrm{D}_\alpha(p_1), \bar{\mathrm{D}}^2(p_2)] &= -2(p_1 + p_2)_{\alpha\dot\alpha}\bar{\mathrm{D}}^{\dot\alpha}(p_2)\delta^4(\theta_1 - \theta_2) \\
[\bar{\mathrm{D}}_{\dot\alpha}(p_1), \mathrm{D}^2(p_2)] &= 2\mathrm{D}^\alpha(p_2) \cdot (p_1 + p_2)_{\alpha\dot\alpha}\delta^4(\theta_1 - \theta_2) \\
[\mathrm{D}^2(p_1), \bar{\mathrm{D}}^2(p_2)] &= 4\Big\{(p_1 + p_2)^2 \\
&\qquad + \mathrm{D}^\alpha(p_1) \cdot (p_1 + p_2)_{\alpha\dot\alpha}\bar{\mathrm{D}}^{\dot\alpha}(p_2)\Big\}\delta^4(\theta_1 - \theta_2) \\
\mathrm{D}^2(p_1)\bar{\mathrm{D}}^2(p_2)\mathrm{D}^2(p_1) &= -4(p_1 + p_2)^2\mathrm{D}^2(p_1)\delta^4(\theta_1 - \theta_2) \\
\bar{\mathrm{D}}^2(p_2)\mathrm{D}^2(p_1)\bar{\mathrm{D}}^2(p_2) &= -4(p_1 + p_2)^2\bar{\mathrm{D}}^2(p_2)\delta^4(\theta_1 - \theta_2)
\end{aligned}
\tag{B.32}
$$

as well as the action of the $\mathrm{D}(p)$'s on δ-distributions from (B.24):

$$
\begin{aligned}
\mathrm{D}_\alpha(p_1)\delta^4_{12} &= -\mathrm{D}_\alpha(-p_2)\delta^4_{12} \\
\bar{\mathrm{D}}_{\dot\alpha}(p_1)\delta^4_{12} &= -\bar{\mathrm{D}}_{\dot\alpha}(-p_2)\delta^4_{12} \\
\mathrm{D}^2(p_1)\delta^4_{12} &= \mathrm{D}^2(-p_2)\delta^4_{12} \\
\bar{\mathrm{D}}^2(p_1)\delta^4_{12} &= \bar{\mathrm{D}}^2(-p_2)\delta^4_{12} \\
\mathrm{D}^2\bar{\mathrm{D}}^2(p_1)\delta^4_{12} &= \bar{\mathrm{D}}^2\mathrm{D}^2(-p_2)\delta^8_{12} \\
\bar{\mathrm{D}}^2\mathrm{D}^2(p_1)\delta^4_{12} &= \mathrm{D}^2\bar{\mathrm{D}}^2(-p_2)\delta^8_{12} \\
&\ \vdots\ .
\end{aligned}
\tag{B.33}
$$

These relations will be used in Chapter 3.3 to shift D's from one vertex to another, leading to the so called transfer rules.

From (B.26) and (B.30) also follows that

$$
\delta^4_{12}\mathrm{D}^2_1\bar{\mathrm{D}}^2_1\delta^8_{12} = \delta^4_{12}\bar{\mathrm{D}}^2_1\mathrm{D}^2_1\delta^8_{12} = \delta^4_{12}\mathrm{D}_1\bar{\mathrm{D}}^2_1\mathrm{D}_1\delta^8_{12} = 16\delta^8_{12} \tag{B.34}
$$

which will be used to reduce loops in Feynman graphs to points in θ-space.

C Integrals for Dimensional Reduction

In this appendix we give the integrals we used for dimensional reduction in the limits D=4 and D=5. As we deal with scalar superfields only, dimensional reduction does not differ from dimensional regularisation.[59]

[59] For the original papers cf. [42, 43]. The scalar integrals we need are discussed in most of the textbooks on Quantum Field Theory (at least for D=4). We used [38, 33, 44].

The basic integral we need is

$$\mu^{4-D} \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{(l^2+M)^n} = \frac{\mathrm{i}}{16\pi}(4\pi\mu^2)^{\frac{4-D}{2}} \frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)} M^{\frac{D}{2}-n} \tag{C.1}$$

From it, one can read off the scalar one-point integral

$$\begin{aligned} A_0(m) &\equiv \frac{16\pi^2}{\mathrm{i}} \mu^{4-D} \int \frac{\mathrm{d}^D}{(2\pi)^D} \frac{1}{(l^2+M)} \\ \frac{\mathrm{i}}{16\pi^2} A_0(m) & \\ &= (4\pi\mu^2)^{\frac{4-D}{2}} \Gamma(1-\frac{D}{2}) M^{\frac{D}{2}-1} \ . \end{aligned} \tag{C.2}$$

We furthermore need the scalar two-point integral

$$B_0(p^2; m_0, m_1) \equiv \frac{16\pi^2}{\mathrm{i}} \mu^{4-D} \int \frac{\mathrm{d}^D k}{2\pi^D} \frac{1}{(k^2+m_0^2)((k+p)^2+m_1^2)} \tag{C.3}$$

It can be calculated from (C.1) with $n=2$ by using the Feynman parametrisation

$$B_0(p^2; m_0, m_1) = \frac{16\pi^2}{\mathrm{i}} \mu^{4-D} \int \frac{\mathrm{d}^D l}{2\pi^D} \int_0^1 dx \frac{1}{(l^2+M)^2} \tag{C.4}$$

with

$$\begin{aligned} l &= k + px \\ M &= (-p^2x^2 + (p^2+m_1^2)x + m_0^2(1-x)) \ . \end{aligned} \tag{C.5}$$

(C.1) yields

$$B_0(p^2; m_0, m_1) = (4\pi\mu^2)^{\frac{4-D}{2}} \frac{\Gamma(2-\frac{D}{2})}{\Gamma(2)} \int_0^1 \mathrm{d}x M^{\frac{D-4}{2}} \ . \tag{C.6}$$

Furthermore we use an identity for the two-point tensor integral

$$B_n(p^2; m_0, m_1) \equiv \frac{16\pi^2}{\mathrm{i}} \mu^{4-D} \int \frac{\mathrm{d}^D k}{2\pi^D} \frac{k_n}{(k^2+m_0^2)((k+p)^2+m_1^2)} \tag{C.7}$$

Using Lorentz decomposition it can be shown that

$$B_n(p^2; m_0, m_1) = \frac{p_m}{2p^2} \left(A_0(m_1) - A_0(m_0) - (p^2 + m_1^2 - m_0^2) B_0(p^2; m_0, m_1) \right) \tag{C.8}$$

For the special case of $m_0 = m_1$ it follows that

$$\begin{aligned} &\mu^{4-D} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{p_m + 2k_m}{(k^2+m_1^2)((k+p)^2+m^2)} \\ &= \frac{\mathrm{i}}{16\pi^2} \left[p_m B_0(p^2; m, m) + 2B_m(p^2; m, m) \right] = 0 \end{aligned} \tag{C.9}$$

For D=4, the Γ functions in the integrals A_0 and B_0 have a pole and A_0 and B_0 therefore diverge. To isolate the divergences, the expressions are expanded in $\delta = \frac{4-D}{2}$, using $\Gamma(\delta) = \frac{1}{\delta} - \gamma_E + O(\delta)$ with the Euler constant $\gamma_E = 0.577\ldots$. In what follows, $\Delta \equiv \frac{1}{\delta} - \gamma_E + \ln 4\pi$. The results are (cf. e.g. [33])[60]

$$\lim_{D\to 4} A(0) = 0 \tag{C.10a}$$

$$\lim_{D\to 4} A(m) = m^2(\Delta - \ln \frac{m^2}{\mu^2}) + O(\delta) \tag{C.10b}$$

$$\begin{aligned} \lim_{D\to 4} B_0(p^2; m_0, m_1) = \Delta + 2 - \ln \frac{m_0 m_1}{\mu^2} \\ - \frac{m_0^2 - m_1^2}{p^2} \ln \frac{m_1}{m_0} + \frac{m_0 m_1}{p^2} \left(\frac{1}{r} - r \right) \ln r \ , \end{aligned}$$

$$\text{where } r = \frac{p^2 + m_0^2 + m_1^2 \pm \sqrt{(p^2+m_0^2+m_1^2)^2 - 4m_0^2 m_1^2}}{2m_0 m_1} \tag{C.10c}$$

$$\begin{aligned} &\lim_{D\to 4} B_0(p^2; m, m) = \\ &= \Delta + 2 - \ln \frac{m^2}{\mu^2} \begin{cases} \sqrt{1 - \frac{4m^2}{p^2}} \left[\ln \frac{1+\sqrt{1+\frac{4m^2}{p^2}}}{1-\sqrt{1+\frac{4m^2}{p^2}}} - \mathrm{i}\pi \right] & \text{for } -p^2 > 4m^2 \\ 2\sqrt{-\frac{4m^2}{p^2} - 1} \ \arctan \left[\left(-\frac{4m^2}{p^2} - 1 \right)^{1/2} \right] & \text{for } -p^2 \le 4m^2 \end{cases} \end{aligned} \tag{C.10d}$$

$$\lim_{D\to 4} B_0(p^2; 0, 0) = \Delta + 2 - \ln \frac{-p^2}{\mu^2} + O(\delta) \tag{C.10e}$$

$$\lim_{D\to 4} B_0(0; m, m) = \Delta + 2 - \ln \frac{m^2}{\mu^2} + O(\delta) \ , \tag{C.10f}$$

[60]The special cases for B_0, namely $m_0 = m_1$, $m_0 = m_1 = 0$ and $(p = 0, m_0 = m_1)$ cannot be read off from the general solution in 4D but have to be calculated from (C.6).

In D=5, only $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$ and $\Gamma(-\frac{3}{2}) = \frac{4}{3}\sqrt{\pi}$ appear in the integrals we need for dimensional regularisation. Therefore no divergences occur.

The 5D integrals can directly be calculated from (C.2) and (C.6). In this thesis we only need the results for $A_0(m)$ and $B_0(-\mathcal{M}^2, 0, 0)$

$$A_0(m) = \frac{2}{3}\pi\frac{m^3}{\mu} \tag{C.11a}$$

$$B_0(-\mathcal{M}^2, 0, 0) = -\frac{\pi}{8}\frac{\mathcal{M}}{\mu} \tag{C.11b}$$

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Acknowledgements

I would like to thank my supervisor, Wilfried Buchmüller, for giving me the opportunity to write my diploma thesis at the DESY theory group, his foresight in choosing the subject, his advice, many helpful questions and remarks. Thanks to the whole theory group of DESY, its open doors and an atmosphere I have enjoyed a lot.

Arthur Hebecker helped me to learn about 5D supersymmetry and orbifolds. Thank you, as well as Arndt Brandenburg, Michael Ratz, Dominik Stöckinger and Alexander Westphal for many helpful (and sometimes late-night) discussions. Special thanks to Christoph Lüdeling for a nice year sharing the office, almost never agreeing in discussions, revising this thesis more than once and for a good friendship.

For the supervision on behalf of the Universität Hamburg I would like to thank Jan Louis, as well as for his lecture on String Theory which for me was a gate to a more fundamental motivation of my work.

Finally I would like to thank my parents who always supported me in every possible way, all of my friends and especially my girlfriend for her love, understanding and support.

Thank you all.