Hybrid Inflation with a Local Gauge Symmetry

Jan C. Hamann

January 14, 2004

Abstract

In this thesis we review the cosmological standard model and the basics of inflation. Furthermore we analyse a hybrid inflation model derived from a supersymmetric ansatz and study the phase transition at the end of inflation in extensions of this model, in which a discrete $\mathbb{Z}_2$-symmetry of the potential will be generalised to a global $U(1)$-symmetry and a $U(1)$ gauge symmetry, respectively. These extensions avoid the problem of domain wall formation and might hence constitute a realistic description for inflation.

Hybride Inflation mit lokaler Eichsymmetrie

Jan C. Hamann

14. Januar 2004

Zusammenfassung

In dieser Arbeit geben wir einen Überblick über das kosmologische Standardmodell und die Grundlagen der Inflation. Weiterhin analysieren wir ein durch ein supersymmetrisches Szenario motiviertes Modell für hybride Inflation und betrachten den Phasenübergang am Ende der inflationären Phase in Erweiterungen dieses Modells, in denen eine diskrete $\mathbb{Z}_2$-Symmetrie des Potentials zu einer globalen $U(1)$-Symmetrie sowie zu einer $U(1)$-Eichsymmetrie verallgemeinert wird. In diesen Erweiterungen wird das Problem der Bildung von "domain walls" vermieden und somit eine realistische Beschreibung der Inflation ermöglicht.

Contents

1 Introduction **5**

2 Cosmology and Inflation **7**

2.1 Robertson-Walker Cosmology 7

2.1.1 The Friedmann Equation 8

2.1.2 Matter, Radiation and Vacuum Energy Domination 9

2.1.3 Horizons 11

2.1.4 Shortcomings of the Cosmological Standard Model 12

2.2 Inflation 14

2.2.1 The Dynamics of Scalar Fields 15

2.2.2 Slow Roll Approximation 16

2.2.3 Reheating 17

2.2.4 Solution of the Standard Model Problems 18

3 Inflationary Models **22**

3.1 Hybrid Inflation 23

3.2 A $\mathbb{Z}_2$-Symmetrical Supersymmetric Hybrid Inflation Model 24

3.2.1 A Supersymmetric Approach to Inflation 24

3.2.2 Decay of the Inflaton Field 28

3.3 Extension to a Global $U(1)$-Symmetry 31

3.3.1 Equations of Motion 32

3.3.2 Solving the Equations of Motion 35

3.4 Local $U(1)$-Symmetry and Gauge Field 36

3.4.1 Decomposition of the Gauge Field 38

3.4.2 Evolution of the Components of the Gauge Field 40

3.4.3 D-Term 42

3.4.4 Effect on the Equations of Motion 42

4 Summary and Outlook **43**

A Fluctuations of a Scalar Field **45**

B Renormalisation **47**

C Hartree Approximation **49**

Bibliography **51**

Chapter 1

Introduction

How did the universe begin? This question has troubled mankind for millenia. But only relatively recently has it become possible to attempt to answer this question on a quantitative, physical level instead of a philosophical one. The event that marks the birth of modern cosmology is Einstein's discovery of the theory of General Relativity [1], which provides the basic tool for describing the evolution of spacetime under the influence of gravity.

Initially, the universe was believed to be static. This assumption led Einstein to introduce a "cosmological constant" term to his equations. However, in the late 1920's, Hubble measured the redshift and the distance of galaxies and discovered that there is a proportionality between these two quantities. This pointed towards the conclusion that the universe is expanding. The theoretical description was given by Friedmann, using Einstein's equations and assuming homogeneity and isotropy of space. In 1965 Penzias and Wilson discovered the Cosmic Microwave Background Radiation (CMBR), another point which can only be explained by an expanding universe. This put an end to the speculations about a steady state universe.

However, Friedmann's ansatz was not entirely satisfactory. In order to match certain parameters (for instance today's spatial curvature or the spectrum of the CMBR) to observations, one would have to choose a very particular set of initial conditions. A theory that requires fine-tuning of such a degree is certainly not very elegant. In 1981, Guth [2] proposed a solution to this problem. He postulated a phase of accelerated expansion of the universe. Ironically, the acceleration is driven by the cosmological constant term, which was originally supposed to save the steady state hypothesis. The vacuum energy that leads to the cosmological constant is caused by a scalar field, usually the inflaton field, whose potential energy has a non-vanishing vacuum expectation value during inflation.

As it turns out, inflation can also explain the origin of density fluctuations, which can be interpreted as the quantum fluctuations of a scalar field (for example, the inflaton field or another scalar field, the curvaton). Recent measurements of the inhomogeneities

of the CMBR are in excellent agreement with the predictions of inflation.

Of special interest is the transition phase between inflation and the subsequent era that can be explained by standard Friedmann cosmology. The vacuum energy of the inflaton field is transferred to other fields in a non-equilibrium process, eventually leading to a thermal state, a process known as "reheating".

But the question remains: where does the inflaton field come from? Inflation cannot be accomodated within the Standard Model of particle physics, so we need to invoke an extension of the Standard Model to incorporate inflation in a more general theory. One potential candidate are supersymmetric theories. It is expected that the Large Hadron Collider (LHC) will find evidence for or against the existence of low-energy supersymmetry in nature.
Several authors have attempted to describe inflation in the context of supersymmetry [3–5]. In this thesis we will focus on a model suggested in Ref. [5]. This model leads to a hybrid inflation scenario in which reheating is initiated by a second order phase transition.

The original model proposed in Ref. [5] possesses a $\mathbb{Z}_2$-symmetrical potential. This symmetry is broken in the ground state, which leads to the formation of domain walls. However, the existence of domain walls has been ruled out by observations. In order to avoid this problem we will modify the potential to be $U(1)$-symmetrical, since broken $U(1)$-symmetries produce only one-dimensional topological defects. These defects, namely, cosmic strings, generally do not cause any trouble. Firstly, we shall analyse the case of a global $U(1)$-symmetry. Then, since global symmetries are not conserved by gravity, it appears natural to analyse the local case. This also serves a second purpose: the massless Goldstone boson that appears in the globally symmetric case can be absorbed in the gauge field if we choose the gauge condition properly. The existence of a gauge field, however, will affect the equations of motion of the other fields. We shall estimate this effect and determine whether or not it is significant.

This thesis is structured as follows: In Chapter 2 we will review the theoretical foundations of our work, namely, the Cosmological Standard Model based on Friedmann's assumptions, followed by an introduction to inflation. Chapter 3 will begin with an illustration of the mechanism of hybrid inflation models in Section 3.1. Some of the results of Refs. [5, 6] will be presented in Section 3.2. Next, we will introduce a globally $U(1)$-symmetrical variation of the basic model and analyse the dynamics of the phase transition after the critical point by numerical means. The consequences of going to a local $U(1)$-symmetry will be discussed in Section 3.4, with particular emphasis on the evolution of the gauge field. We will conclude with a summary of our results and outline prospects for further work on this topic.

Chapter 2

Cosmology and Inflation

In this chapter we shall give a brief overview of the Cosmological Standard Model and discuss its flaws and shortcomings. In Section 2.2 we will see how the concept of inflation can be used to overcome these problems. For a more detailed treatment of the cosmological standard model we recommend the books by Kolb and Turner [7] or Peacock [8]. The review by García-Bellido [9] gives a very good summary of the techniques that are used to determine cosmological parameters. For a more in-depth analysis of inflation we can refer the reader to the book by Liddle and Lyth [10] or Riotto's review [11].

2.1 Robertson-Walker Cosmology

Experimental evidence indicates that the observable universe is homogeneous and isotropic at large scales. This holds true for the distribution of galaxies as well as for the Cosmic Microwave Background Radiation. Under the assumption of isotropy and homogeneity the most general metric is the Robertson-Walker metric. In comoving spherical polar coordinates it is given by:

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \equiv \mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta + \sin^2\theta\,\mathrm{d}\phi^2)\right], \tag{2.1}$$

with the cosmic scale parameter $a(t)$ and $k = -1, 0, 1$. The comoving set of variables (r, θ, ϕ) is related to the physical spherical polars $(r_{\mathrm{ph}}, \theta_{\mathrm{ph}}, \phi_{\mathrm{ph}})$ by

$$(r, \theta, \phi) = (a^{-1}(t)\, r_{\mathrm{ph}}, \theta_{\mathrm{ph}}, \phi_{\mathrm{ph}}). \tag{2.2}$$

The different values of k correspond to different geometries of spacetime: $k = 0$ gives a flat universe, for $k = -1$ we have a hyperbolic (open) universe and $k = 1$ gives a spherical (closed) universe. Ultimately, k will determine the fate of the universe: for $k \leq 0$ it will expand forever, if $k = 1$ the expansion will stop and eventually the universe will collapse.

However, this is not necessarily true if there is a non-vanishing fraction of dark energy in the universe, for more details see [12]. The sign of the spatial curvature is determined by k:

$$R^{(3)} = \frac{6k}{a^2(t)}. \tag{2.3}$$

2.1.1 The Friedmann Equation

In order to figure out the time evolution of the scale parameter one needs to solve Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu} + \Lambda\, g_{\mu\nu}. \tag{2.4}$$

The Ricci tensor $R_{\mu\nu}$ and scalar R follow from the metric, G is Newton's constant and Λ the cosmological constant. The energy-momentum tensor $T_{\mu\nu}$ can be chosen as

$$T_{\mu\nu} = -p(t)\, g_{\mu\nu} + (p(t) + \rho(t))\, U_\mu U_\nu, \tag{2.5}$$

representing a perfect (homogeneous) fluid. The quantities $p(t)$ and $\rho(t)$ are the pressure and the energy density of the fluid and U^μ is the comoving four-velocity, which satisfies $U^\mu U_\mu = 1$. If we demand isotropy, the energy-momentum tensor becomes:

$$T_{\mu\nu} = \text{diag}\,(\rho, p, p, p). \tag{2.6}$$

Plugging these results into equation (2.4), the equation for the $(0,0)$-component reads

$$\frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} = \left(\frac{\dot{a}}{a}\right)^2 \equiv H^2, \tag{2.7}$$

known as the Friedmann equation. H characterises the expansion rate of the universe and is called Hubble parameter. We can rewrite the Friedmann equation to find the relation between the sign of the curvature and the total energy density of the universe. Using the critical density $\rho_c = \frac{3H^2}{8\pi G}$ (i.e. the energy density which leads to a flat universe in the absence of a cosmological constant) and defining the ratios $\Omega_M = \frac{\rho_M}{\rho_c}$, $\Omega_\Lambda = \frac{8\pi G\,\Lambda}{\rho_c}$ and $\Omega = \Omega_M + \Omega_\Lambda$ we obtain:

$$\Omega - 1 = \frac{k}{H^2 a^2}. \tag{2.8}$$

This tells us that for $\Omega > 1$ we have a closed universe, for $\Omega < 1$ it is open and if $\Omega = 1$ it will be flat. Observations indicate that the present value of $\Omega(\text{today}) \equiv \Omega_0$ is very close to unity (see Table (2.1)). As we will see later, inflation can explain this very nicely. The WMAP measurements [13] show that the contribution of the baryonic energy density is only about 4% of the total energy density. However, the total matter contribution has been measured as $\Omega_M \approx 27\%$. The remaining 23% are called "dark

matter". There are several candidates for dark matter, for example axions or the lightest supersymmetric particle, none of which have been experimentally detected at present though. The missing part of the energy density is attributed to "dark energy", which might be an indication towards a non-vanishing cosmological constant, or maybe the existence of the quintessence (which is essentially a scalar field which has a potential with a non-vanishing vacuum expectation value and does not interact with matter).

Parameter	Symbol	Value
Total density	Ω_0	1.02 ± 0.02
Matter density	Ω_{M}	0.27 ± 0.04
Baryon density	Ω_{B}	0.044 ± 0.004
Dark energy density	Ω_{DE}	0.73 ± 0.04
Hubble parameter today	H_0	$71\,^{+4}_{-3}$ km sec^{-1}Mpc^{-1}

Table 2.1: Cosmological parameters (from [13]). Energy densities are given in units of ρ_{c}.

The (i,i)-components of equation (2.4) give

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \Lambda = -8\pi G p. \tag{2.9}$$

Via the Bianchi identities this equation and the Friedmann equation are related to the conservation of energy:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho a^3) + p\frac{\mathrm{d}}{\mathrm{d}t}(a^3) = 0 \tag{2.10}$$

Using the Friedman equation to eliminate the first order derivative from equation (2.9) we find the equation for the acceleration of the scale factor:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{2.11}$$

If we ignore the cosmological constant for the moment, we can see that for the acceleration to acquire a positive value we need $\rho + 3p < 0$.

2.1.2 Matter, Radiation and Vacuum Energy Domination

Let us assume that the universe has an equation of state $p = w\rho$ with time-independent w. Then the energy density will evolve as $\rho \propto a^{-3(1+w)}$ and the scale factor will satisfy $a(t) \propto t^{2/3(w+1)}$. The most interesting cases are:

Matter

In the case of non-relativistic matter the pressure will vanish:

$$p = 0 \;\Rightarrow\; \rho \propto a^{-3},\; a \propto t^{2/3}. \tag{2.12}$$

As one would naively expect, matter will be diluted by the expansion of the (physical) volume, which is proportional to $a^3(t)$. Hence the dilution should be proportional to a^{-3} which is exactly what we find from the equations.

Radiation

For radiation or ultra-relativistic particles we have $p = \frac{\rho}{3}$:

$$p = \frac{\rho}{3} \;\Rightarrow\; \rho \propto a^{-4},\; a \propto t^{1/2}. \tag{2.13}$$

Just like matter, radiation will also suffer a dilution of the volume. The extra factor of a^{-1} stems from the cosmological redshift: the wavelength and thus also the energy of radiation or a relativistic particle will decrease proportional to a^{-1}.[1]

Vacuum Energy

If we consider vacuum energy, which fulfils the relation $p = -\rho$, we obtain:

$$p = -\rho \;\Rightarrow\; \rho = \text{const.} \tag{2.14}$$

Here we have an exponential time dependence of the scale factor (also known as de Sitter phase):

$$a(t) \propto \exp\left[\sqrt{\frac{\Lambda_{\text{eff}}}{3}}\, t\right] = e^{H\,t}, \tag{2.15}$$

with an effective vacuum energy density Λ_{eff} and a constant Hubble parameter. Possible sources for Λ_{eff} are a cosmological constant Λ or a scalar field with a non-vanishing vacuum expectation value (e.g. quintessence or, as we will see later, the inflaton field).

If we compare the numbers from Table 2.1 with the current radiation energy density $\Omega_R \approx 1.2 \cdot 10^{-5}$, consisting of photons and neutrinos, we can see that today's universe is dominated by matter and dark (i.e. vacuum) energy, so it seems that we are at the beginning of a de Sitter phase at the moment. Let us use the results of this section for

[1] Here we have implicitly assumed an expansion of the universe, because our universe is expanding at present. Of course, above considerations also hold true in the case of a collapsing universe, in that case we would observe a compression of matter and radiation, and a blueshift of frequencies.

an extrapolation towards earlier times. Since $\rho_{\mathrm{M}} \propto a^{-3}$ and $\rho_{\Lambda} = \text{const.}$, the contribution of matter will increase as we go back in time, thus leading to a matter dominated phase. Going back further in time, we find that eventually the radiation energy density will surpass the matter energy density while vacuum energy becomes more and more insignificant. See Ref. [14] for a more detailed dicussion.

2.1.3 Horizons

Causality is a very important issue in cosmology. The exchange of information is limited by the speed of light, which is finite. Hence, there may be areas in the universe that are not in causal contact.
In the context of general relativity, light rays travel on null geodesics: $\mathrm{d}r = \mathrm{d}t/a(t)$. The distance a photon has traveled since the big bang (i.e. the maximum distance R_{H} a photon can have traveled) can be obtained by integrating over time:

$$R_{\mathrm{H}}(t) = a(t) \int_0^t \frac{\mathrm{d}t'}{a(t')}. \tag{2.16}$$

R_{H} is called the particle horizon. In the case of radiation or matter domination we can use the expression $a(t) \propto t^n$, $n = 2/3, 1/2$ and find:

$$R_{\mathrm{H}}(t) \propto H^{-1}(t) \propto t. \tag{2.17}$$

The particle horizon is finite and proportional to t.[2] If we compare the growth of the horizon to the growth of the scale factor ($a(t) \propto t^{1/2}$ or $a(t) \propto t^{2/3}$, respectively), we find that not only the physical scale but also the comoving scale of the particle horizon increases with time.

Let us now consider the pure de Sitter case with $w = -1$ and $\Omega_0 = 1$. Using Equation (2.15) we get:

$$R_{\mathrm{H}}(t) = a(t) \int_{-\infty}^t \frac{\mathrm{d}t'}{a(t')} = \frac{1}{H}\, e^{Ht}. \tag{2.18}$$

Note that we took $-\infty$ as the lower limit here, corresponding to $a = 0$ at the time of the big bang. In this case R_{H} is finite as well. However, if we keep in mind that $H = \text{const.}$ in a de Sitter universe and compare it to the growth of the scale factor, we can see that the comoving volume of the horizon remains constant during a vacuum dominated phase.

[2] We have assumed $\Omega_0 = 1$ in these calculations, for different values the result becomes a bit more complicated, cf. e.g. Ref. [7].

2.1.4 Shortcomings of the Cosmological Standard Model

While the Cosmological Standard Model is quite successful at describing the dynamics of the universe, there are a couple of points it fails to explain.

Flatness Problem

As we can see from Equation (2.8), if $k = 0$ then $\Omega = 1$ at all times. In the case of a positive or negative curvature however, $\Omega - 1$ will be a time-dependent quantity.

In a matter dominated period, we have

$$\Omega - 1 = \frac{a}{\zeta - a} \propto \frac{t^{1/2}}{\zeta' - t^{1/2}} \tag{2.19}$$

with $\zeta = \frac{8\pi G\rho}{3k}$. In a radiation dominated period

$$\Omega - 1 = \frac{a^2}{\zeta - a^2} \propto \frac{t^{4/3}}{\zeta' - t^{4/3}}. \tag{2.20}$$

In both cases $\Omega - 1$ increases with time and tends to zero as $t \to 0$. If we extrapolate $\Omega - 1$ from its present value of ~ 0.02 to the time of nucleosynthesis ($t_{\mathrm{N}} \approx 1\mathrm{s}$) or even to the Planck time $t_{\mathrm{Pl}} \approx 10^{-43}\mathrm{s}$, we find that

$$\begin{aligned} |\Omega(t_{\mathrm{N}}) - 1| &\lesssim \mathcal{O}(10^{-16}) \\ |\Omega(t_{\mathrm{Pl}}) - 1| &\lesssim \mathcal{O}(10^{-60}). \end{aligned} \tag{2.21}$$

So in order to get the correct value of Ω_0, the value of $\Omega - 1$ would have had to be very close to zero at t_{Pl}. What would have happened if $\Omega - 1$ had been a couple of orders of magnitude greater than this? In the case of $k = 1$ the universe would have recollapsed almost immediately and for $k = -1$ it would have cooled down to today's temperature of 3K very quickly instead of taking roughly 10^{10} years. While it does not explicitly contradict the theory, the required fine tuning of $\Omega - 1$ of an order of 10^{-60} does not make it seem very likely. Now one could invoke the anthropic principle for an explanation, but that is certainly not a very elegant way to solve this problem. In the context of inflation we will present a much more satisfactory explanation.

Entropy Problem

During a radiation dominated period the Friedmann equation can be written as

$$H^2 \approx 8\pi G\rho_R \approx \frac{T^4}{M_{\mathrm{Pl}}^2}, \tag{2.22}$$

$M_{\mathrm{Pl}} \equiv (8\pi G)^{-\frac{1}{2}} = 4.342 \cdot 10^{-9}\,\mathrm{kg} \equiv 2.436 \cdot 10^{18}\,\mathrm{GeV}$ is the Planck mass. As shown in Ref. [7], the entropy S per comoving volume is given by

$$S = g_* T^3 a^3. \tag{2.23}$$

T denotes the temperature of the radiation and g_* is the number of effectively massless degrees of freedom. Plugging these results into Equation (2.8) we arrive at

$$\Omega - 1 \approx \frac{k M_{\mathrm{Pl}}^2}{a^2 T^4} \sim \frac{k M_{\mathrm{Pl}}^2}{S^{\frac{2}{3}} T^2}. \tag{2.24}$$

Assuming that the universe expands adiabatically, i.e. $S = \mathrm{const} \equiv S_U$ and using $S_U = \frac{8\pi^3}{135} H_0^{-3} g_* T^3 \approx 10^{90}$ (cf. Ref. [11]), it follows that

$$|\Omega - 1|_{t=t_{\mathrm{Pl}}} \approx \frac{M_{\mathrm{Pl}}^2}{T_{\mathrm{Pl}}^2} \frac{1}{S_U^{2/3}} = \frac{1}{S_U^{2/3}} \approx 10^{-60}. \tag{2.25}$$

Again, we encounter a fine tuning problem. This time it is caused by the huge amount of entropy in the universe. However, we need to keep in mind that we assumed adiabaticity at all times. If we give up this assumption and allow a period of non-adiabatic expansion, this problem could be remedied.

Horizon Problem

As we have seen in Section 2.1.3, in a matter dominated universe the particle horizon grows faster than the universe expands. Hence, we are able to receive information from regions of the universe that were not in causal contact with our region before. Let us compare today's particle horizon with the physical radius of this volume at the time of the recombination of electrons and nuclei, when the CMBR was created ($t_{\mathrm{rec}} \approx 3 \cdot 10^5\,\mathrm{a}$):

$$R_{\mathrm{H}}(t_0) \frac{a(t_{\mathrm{rec}})}{a_0} \approx R_{\mathrm{H}}(t_0) \frac{T_0}{T_{\mathrm{rec}}}. \tag{2.26}$$

$T_0 \approx 2.73\,\mathrm{K}$ is the present temperature of the CMB, $T_{\mathrm{rec}} \approx 3600\,\mathrm{K} \equiv 0.3\,\mathrm{eV}$ is the temperature of the CMB at t_{rec}. Equation (2.26) assumes an adiabatic expansion, see Equation (2.23).
How big was today's causally connected region at t_{rec}? The Hubble radius H^{-1} follows from the relation

$$H^{-1} \propto \rho_M^{-\frac{1}{2}} \propto a^{\frac{3}{2}} \propto T^{-\frac{3}{2}}.$$

We get:

$$H^{-1}(t_{\mathrm{rec}}) = R_{\mathrm{H}}(t_0) \left(\frac{T_{\mathrm{rec}}}{T_0} \right)^{-\frac{3}{2}}. \tag{2.27}$$

So the volume of today's visible universe comprises

$$\left(\frac{R_{\mathrm{H}}(t_{\mathrm{rec}})}{H^{-1}(t_{\mathrm{rec}})}\right)^3 = \left(\frac{T_{\mathrm{rec}}}{T_0}\right)^{\frac{3}{2}} \approx 5 \cdot 10^5 \tag{2.28}$$

regions that were not in causal contact at the time of recombination!

After the recombination the photons essentially stopped interacting with matter, so the CMB gives us a direct image of the universe 300,000 years after the big bang. The data from COBE and WMAP show that the temperature of the CMB is homogeneous up to a factor of 10^{-4}. What is the reason for this remarkable homogeneity if those 500,000 regions were not in causal contact when the CMB originated? The Cosmological Standard Model is not able to solve this problem. A similar issue arises if we go back to the time of nucleosynthesis: here we have even more disconnected regions, and still the abundance of primordial elements seems to be homogeneous.

Relic Problem

Various extensions of the Standard Model predict that the high temperatures during the big bang lead to the creation of certain potentially very massive objects with long lifetimes. Examples include:

- Gravitinos, the supersymmetric spin-$\frac{3}{2}$ partners of the graviton with a mass of $1\,\mathrm{keV} - 100\,\mathrm{TeV}$.
- Moduli, which appear in string theory. [15]
- Topological defects [16]: domain walls (2-dimensional), cosmic strings (1-dimensional) and textures (0-dimensional), in particular magnetic monopoles: in the early universe, the energies may have been high enough to restore the symmetry of a Grand Unified Theory. The spontaneous breaking of the symmetry would lead to the production of magnetic monopoles.

Without going too much into detail here, these objects have one thing in common: their predicted abundance after the big bang is typically much higher than the upper limit imposed by experimental evidence. Essentially, if one did not find a mechanism to get rid of these relics, the universe would be dominated by their energy. Fortunately, this is not what we are observing, so we need to find an explanation why the relics are missing.

2.2 Inflation

The concept of inflation was first introduced by Guth [2] and Linde [17]. It is not supposed to replace the standard cosmology we discussed in the last section, one should think of it as an add-on that covers the evolution of our universe at very early times.

What is inflation? Basically, we postulate a phase when the universe underwent an accelerated expansion:

$$\text{Inflation} \Leftrightarrow \ddot{a}(t) > 0 \tag{2.29}$$

It would have to take place before the time of nucleosynthesis ($t_{\mathrm{N}} \approx 1\,\mathrm{s}$, $T_{\mathrm{N}} \approx 1\,\mathrm{MeV}$), since this is the earliest time we have direct experimental data from and these observations can be explained with standard cosmology.
How can this scenario be realized? As we saw in Equation (2.11), we need a form of energy with negative pressure, or, more precisely, $p < -\rho/3$. A candidate which satisfies this requirement is a scalar field, representing spin-0 particles.

2.2.1 The Dynamics of Scalar Fields

Inflation is driven by the energy of a scalar field ϕ, the inflaton field. The Lagrangean of this field is given by:

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi), \tag{2.30}$$

with a potential $V(\phi)$. In order to derive the equation of motion we will start from the action

$$\mathcal{S} = \int \mathrm{d}^4 x \, \sqrt{-g}\, \mathcal{L}, \tag{2.31}$$

where g is the determinant of the metric. For the Robertson-Walker metric we have $\sqrt{-g} = a^3$. Now we can plug this into the Euler-Lagrange equations

$$\partial^\mu \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta \partial^\mu \phi} - \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta \phi} = 0 \tag{2.32}$$

and receive:

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2(t)} \nabla^2 \phi + V'(\phi) = 0. \tag{2.33}$$

where $V'(\phi)$ denotes the derivative of the potential with respect to ϕ. If we take ϕ to be homogeneous, the gradient term will vanish and the equation of motion will represent a point mass moving in a potential V with a friction caused by the expansion of the universe.
The energy momentum tensor is given by:

$$T_{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi + V(\phi) \right). \tag{2.34}$$

In a locally orthonormal frame ($g_{\mu\nu} = \eta_{\mu\nu}$) with $\eta_{\mu\nu}$ being the Minkowski metric, we can take the momentum density $T^{0i} = \dot{\phi} \partial^i \phi$ to vanish and define the energy density ρ_ϕ and

pressure p_ϕ:

$$\rho_\phi = T_{00} = \frac{\dot{\phi}^2}{2} + V(\phi) + \frac{(\nabla\phi)^2}{2a^2} \tag{2.35}$$

$$p_\phi = T_{ii} = \frac{\dot{\phi}^2}{2} - V(\phi) + \frac{(\nabla\phi)^2}{6a^2} \tag{2.36}$$

Should the gradient term dominate, we would have $p_\phi = -\rho_\phi/3$ which is not sufficient for inflation. Let us ignore quantum fluctuations of the inflaton field for now and assume homogeneity. By imposing the condition

$$V(\phi) \gg \frac{\dot{\phi}^2}{2} \tag{2.37}$$

we end up with

$$p_\phi \simeq -\rho_\phi. \tag{2.38}$$

This is the equation of state for vacuum energy. Hence, inflation is driven by the vacuum energy of the inflaton field and all the considerations we made earlier (e.g. constant Hubble parameter, exponential growth of the scale factor) apply in this case.

2.2.2 Slow Roll Approximation

Equation (2.37) implies that the inflaton field changes its value slowly, hence the name "slow roll". In this case, the Friedmann equation reads

$$H^2 \simeq \frac{8\pi G}{3} V(\phi). \tag{2.39}$$

If we make the additional assumption that $\ddot{\phi}$ is negligible, Equation (2.33) will reduce to

$$3H\dot{\phi} \simeq -V'(\phi). \tag{2.40}$$

Another way to express these conditions is to introduce the slow roll parameters ε and η defined by

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} = 4\pi G \frac{\dot{\phi}^2}{H^2} = \frac{M_{\mathrm{Pl}}^2}{2}\left(\frac{V'}{V}\right)^2 \tag{2.41}$$

$$\eta \equiv \frac{1}{8\pi G}\frac{V''}{V} = M_{\mathrm{Pl}}^2 \frac{V''}{V} = \frac{1}{3}\frac{V''}{H^2} \tag{2.42}$$

and demand that

$$\varepsilon \ll 1 \quad \text{and} \quad \eta \ll 1. \tag{2.43}$$

Now we can calculate the amount of inflation that takes place between an initial time t_i and a final time t_f. This quantity is usually given in terms of the number of e-foldings N:

$$N \equiv \frac{a(t_f)}{a(t_i)} = \int_{t_i}^{t_f} \mathrm{d}t\, H \simeq -8\pi G \int_{\phi_i}^{\phi_f} \mathrm{d}\phi\, \frac{V}{V'}. \tag{2.44}$$

2.2.3 Reheating

Since we do not observe an inflationary universe today, obviously, inflation must have ended somehow and the universe must have gone over to a radiation dominated phase. So the question arises: How did inflation come to an end?
During the slow roll phase the inflaton field has a non-vanishing vacuum expectation value (vev), also known as the false vacuum. At the end of inflation, ϕ must have reached a stable minimum ϕ_0 with $V(\langle\phi\rangle) \equiv V(\phi_0)$ of order zero, the true vacuum. As ϕ approaches the minimum of the potential, the slope will become steeper, $\dot{\phi}$ will increase and eventually the slow roll conditions will break down.

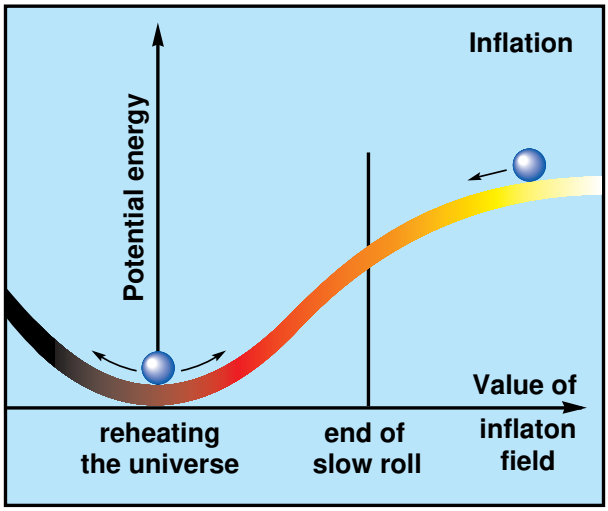

Figure 2.1: Schematic form of the inflaton potential. Slow roll takes place in the flat part. This figure is taken from Ref. [21].

In the most basic model the inflaton field will start oscillating coherently around the minimum of its potential. During this time the energy density will no longer be dominated by the potential energy of ϕ, but by the kinetic term. These oscillations are can be interpreted as zero-momentum ϕ-particles, which can decay to lighter fields to which the inflaton couples. We can express this by adding a decay term $\Gamma_\phi\dot{\phi}$ with a decay width Γ_ϕ to Equation (2.33):

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma_\phi\dot{\phi} + V'(\phi) = 0. \tag{2.45}$$

This decay will dampen the oscillations until ϕ finally reaches the true vacuum.

It turns out that this model is not very accurate and only valid for small oscillations at the end of the reheating process. In chaotic inflation models the inflaton field will

decay very rapidly via parametric excitations (also known as 'preheating', see Ref. [18] and [19] for decay into bosons and Ref. [20] for fermionic decay). A similar effect can be observed in hybrid inflation models, as we will see later.

The decay products will thermalize and the universe which underwent an exponential cooling ($T \propto \exp(-Ht)$) during inflation will be reheated. We can define the reheating temperature T_{RH} at the time t_{RH}, when $H(t_{\mathrm{RH}}) = \Gamma_\phi$, i.e. when the lifetime of the ϕ-particles is equal to H^{-1} and their decay rate starts becoming significant. For times greater than t_{RH} the energy contribution of the inflaton field will become negligible compared to the energy of the (relativistic) decay products. So t_{RH} marks the end of the inflationary and the beginning of the radiation dominated phase.
Assuming instantaneous conversion of energy at $t = t_{\mathrm{RH}}$ we can equate the energy density of the inflaton field $\rho_\phi = \rho_i(a(t_i)/a(t_{\mathrm{RH}}))^3$ with the radiation energy density $\rho_R = (\pi^2/30) g_* T_{\mathrm{RH}}^4$. The subscript i refers to the value at the beginning of the oscillatory phase and g_* is the effective number of relativistic degrees of freedom at T_{RH}. We obtain:

$$T_{\mathrm{RH}} = \left(\frac{90}{8\pi^3 g_*}\right)^{\frac{1}{4}} \sqrt{\Gamma_\phi M_{\mathrm{Pl}}}. \tag{2.46}$$

2.2.4 Solution of the Standard Model Problems

Let us take a look at how inflation can solve the aforementioned problems and estimate a lower bound for the number of e-foldings required.

Flatness Problem

What happens to $|\Omega - 1|$ during inflation? If we take H to be constant and $a \propto e^{Ht}$, Equation (2.8) becomes

$$|\Omega - 1| \propto |k|\, e^{-2Ht}. \tag{2.47}$$

So, provided inflation lasted long enough, it is only natural that Ω should be very close to zero. This becomes even more obvious if we look at the behaviour of the spatial curvature

$$R^{(3)} \propto k\, e^{-2Ht}. \tag{2.48}$$

The curvature decreases exponentially and goes to zero, i.e. space becomes flat.

We can attempt a rough estimation of a lower bound on N now: Assuming an initial value $|\Omega - 1|_{t_i}$ of order unity at the beginning of inflation and requiring $|\Omega - 1|_{t_f} \approx 10^{-60}$ at the end of inflation, we can determine N from the relation

$$\frac{|\Omega - 1|_{t_i}}{|\Omega - 1|_{t_f}} = \left(\frac{a(t_f)}{a(t_i)}\right)^2 = e^{2N}, \tag{2.49}$$

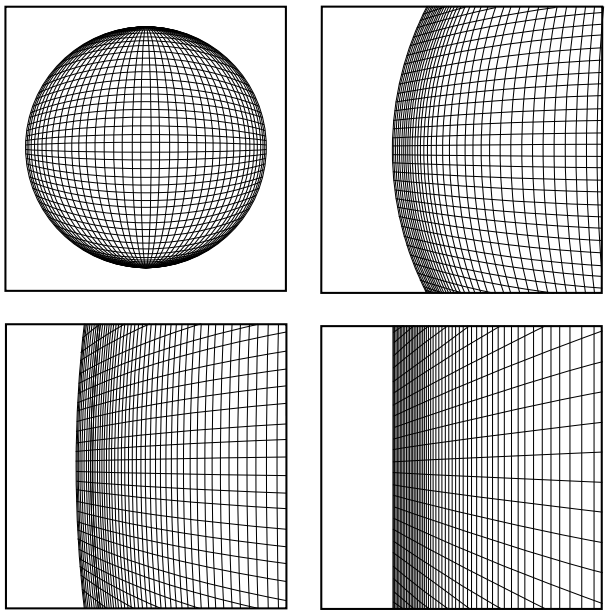

Figure 2.2: Inflation flattens the universe, just like a balloon which is inflated to enormous sizes. This figure is taken from Ref. [21].

leading to

$$N \gtrsim \frac{1}{2} \ln \left(\frac{|\Omega - 1|_{t_i}}{|\Omega - 1|_{t_f}} \right) \approx 70. \tag{2.50}$$

Entropy Problem

As we saw earlier, we need to find a way to explain the huge amount of entropy we can observe in today's universe. An adiabatic expansion leads to $S \sim a^3 T^3$. It makes sense to assume adiabaticity during the slow roll phase since the dilution due to the expansion essentially reduces the interaction rate to zero. Hence, the slow roll phase is not a suitable candidate for the production of entropy.
During the reheating phase, however, the expansion was clearly anything but adiabatic. The decay of particles and the subsequent thermalization provide an excellent source of entropy.

Again, we can estimate a lower bound on the number of e-foldings. Let us assume an entropy S_i of order one at the beginning of inflation. At the end of reheating, S will have grown by a numerical factor of C, i.e. $S_f = CS_i$. Presuming adiabatic expansion after reheating we get $S_f = S_U \approx 10^{90} S_i$ and it follows that

$$\frac{S_f}{S_i} = \left(\frac{a_f T_f}{a_i T_i} \right)^3 = e^{3N} \left(\frac{T_f}{T_i} \right)^3 = C \approx 10^{90}, \tag{2.51}$$

leading to

$$N \approx 70 + \ln\left(\frac{T_i}{T_f}\right). \tag{2.52}$$

We can only guess the temperature T_i, since we don't have any information about the situation prior to inflation, but this term is logarithmically supressed anyway, so it is safe to say that the lower bound we get here is of the same order as the result we obtained above.

Horizon Problem

Let us take a small causally connected patch of radius r_i before inflation. During inflation r_i will be inflated to a radius $r_f = e^N r_i$, while the horizon radius H^{-1} remains constant. If inflation lasts long enough, r will become greater than the horizon H^{-1}, it will "leave" the horizon. After inflation ends, during a matter or radiation dominated phase, we have $H^{-1} \propto a^{3/2}$ in the matter dominated case and $H^{-1} \propto a^2$ for radiation domination. Since $r \propto a$, the Hubble radius will grow faster than the physical scale r and scales which have left the horizon during inflation will enter it again.
In other words, during inflation space underwent a superluminal expansion and the universe grew faster than the horizon. After inflation the expansion becomes slower than the growth of the horizon, allowing the horizon to eventually catch up to any physical scale.

This way we can solve the horizon problem. Areas that were seemingly causally disconnected at the time of recombination or nucleosynthesis had actually been in causal contact before inflation, which explains the homogeneity we observe.

To calculate the minimum number of e-foldings required to solve the horizon problem, we need to demand that the largest scale we can observe today, i.e. the Hubble radius H_0^{-1} was reduced to a value $\lambda_{H_0}(t_i)$ at the beginning of inflation, which was smaller than the size of the horizon during inflation H_i^{-1}:

$$\lambda_{H_0}(t_i) = H_0^{-1}\frac{a(t_i)}{a_0} = H_0^{-1}\left(\frac{a(t_i)}{a(t_f)}\right)\left(\frac{a(t_f)}{a_0}\right) \approx H_0^{-1}e^{-N}\left(\frac{T_0}{T_f}\right) \lesssim H_i^{-1}. \tag{2.53}$$

Solving for N, we get

$$N \gtrsim \ln\left(\frac{T_0}{H_0}\right) + \ln\left(\frac{H_i}{T_f}\right) \approx 67 + \ln\left(\frac{H_i}{T_f}\right), \tag{2.54}$$

a number that is of the same order as the lower limits derived above.

Relic Problem

Assuming N=70, the density of pre-inflation relics was reduced by a factor of $e^{-3N} \approx 6 \cdot 10^{-92}$. Hence, virtually all particles that existed before inflation started were

diluted away and consequently, the matter we observe today must have been formed after inflation.

So in order to avoid problems with heavy particles dominating the universe, we need to put some constraints on the reheating temperature. For magnetic monopoles it would be sufficient to have T_{RH} lower than the GUT scale, which is about 10^{16} GeV. Gravitinos will put this bound even lower, but this is of course model dependent. In Ref. [22] for instance, the authors derive an upper bound of $T_{\mathrm{RH}} \lesssim 10^9$ GeV.

Similarly, we can put a lower bound on T_{RH}. We know that nucleosynthesis must have taken place after inflation, so $T_{\mathrm{RH}} \gtrsim 1$ MeV, which is a rather conservative value. Obviously, baryo- and leptogenesis must have taken place after inflation as well. However, we have a very strong dependence on the model again. In Affleck-Dine and electroweak baryogenesis for instance one needs $T_{\mathrm{RH}} \gtrsim 100$ GeV, but there are also models which could in principle explain baryo- and leptogenesis with a reheating temperature as low as 1 MeV via a non-thermal state after reheating.

Chapter 3

Inflationary Models

Initially, inflation was introduced as a way to explain the conceptual problems that arose from the Cosmological Standard Model. The main reason why inflation has been such a successful theory lies elsewhere though.
One of the conjectures of FRW cosmology is the homogeneity and isotropy of the universe. However, the world we live in is neither very homogeneous nor isotropic, unless we go to scales just below the Hubble radius. How could the universe deviate from the homogeneity of its initial conditions?

The temperature distribution of the CMBR provides us with an excellent picture of the inhomogeneities at the time of decoupling. The fluctuations of the CMBR correspond directly to fluctuations of the energy density and hence of the metric as well. The results of COBE [23] and WMAP [13] show that the temperature fluctuations $\delta T/T$ are of the order 10^{-5}. In the context of inflation, these fluctuations originate from quantum fluctuations of the inflaton field. During inflation the length scale of small fluctuations will be blown up exponentially and it can be shown [11] that soon after they leave the horizon they will basically 'freeze in', i.e., they will become constant and can be regarded as a classical perturbation of the homogeneous part of the inflaton field. This way, inflation can be used to explain today's large scale structure and the anisotropies of the microwave background.

However, as successful as inflation may be, it is a paradigm without a distinguished model. Consequently, there has been a very large number of suggestions on how inflation may have worked (see e.g., Ref. [10] or [24] for a selection of models). In order to find a realistic model one needs to compare its predictions with the experimental observations. Since most of these models may seem somewhat arbitrary, it would be nice if inflation could be explained in the framework of a more general theory.

In the following, we will introduce the class of hybrid inflation models, and analyse one particular model in more detail. We will also look into two extensions to the basic model and study their behaviour at the end of the inflationary phase.

3.1 Hybrid Inflation

When it comes to comparing the predictions of an inflationary model with the predictions of other theories of particle physics and observational data, one can often run into problems. One such issue is the axion problem: one finds that for large values of the Hubble parameter at the end of inflation ($H \gg 10^9$ GeV) one obtains isothermal perturbations due to quantum fluctuations of the axion field that are unacceptably large. However, most single field inflation models predict a fairly large Hubble constant, for instance $H \sim 10^{13}$ GeV for the simplest models of chaotic inflation [25]. Also, chaotic inflation models often require a very large field value of the inflaton field ($\phi \geq M_{\rm Pl}$) to sustain inflation long enough and generate the necessary number of e-foldings. For such large values of the field it is not possible to embed inflation in a low energy effective theory such as Supergravity.

To alleviate these problems Linde [25, 26] proposed a new ansatz, 'hybrid inflation', which combines two important virtues: it yields small values of H and ϕ and it does not significantly depend on initial conditions. In hybrid inflation we have an interaction between the inflaton field ϕ and one (or more) scalar fields $(\psi, ...)$. During the slow roll phase ψ will remain constant. The end of slow roll and the transition to reheating will be initiated when the inflaton field reaches a critical value $\phi_{\rm c}$ and ψ becomes unstable. This can happen either via a first order phase transition (i.e. quantum tunneling through a potential barrier) or a second order phase transition which does not involve tunneling.

Let us take a look at a simple two field hybrid inflation model with a scalar field ψ and the inflaton field ϕ. Consider the potential

$$V(\phi, \psi) = \frac{1}{2} m_\phi^2 \phi^2 + \frac{1}{4}\lambda(\psi^2 - v^2)^2 + \frac{1}{2}\, g^2 \psi^2 \phi^2, \tag{3.1}$$

where m_ϕ and v are mass parameters and λ, g are dimensionless constants. The first term is a mass term of the inflaton field, the second one can be regarded as a symmetry breaking term and the last one is a coupling term between the two fields. The form of the potential is sketched in Figure 3.1.

The effective mass of ψ is given by

$$m_{\psi,\rm eff}^2 = g^2\phi^2 - m_\psi^2, \tag{3.2}$$

with $m_\psi^2 = \lambda v^2$. Now if ϕ is greater than the critical value $\phi_{\rm c} \equiv m_\psi / g$, the effective mass squared of ψ will be positive, forcing ψ to the bottom of the potential valley. So for $\phi > \phi_{\rm c}$ and $\psi = 0$ we essentially recover the single field case with a quadratic inflaton potential. When ϕ passes the critical point and becomes smaller than $\phi_{\rm c}$, the mass squared of ψ will change sign and ψ becomes unstable. Obviously, there is no potential wall the field needs to overcome and we have a second order phase transition. The backreaction from the

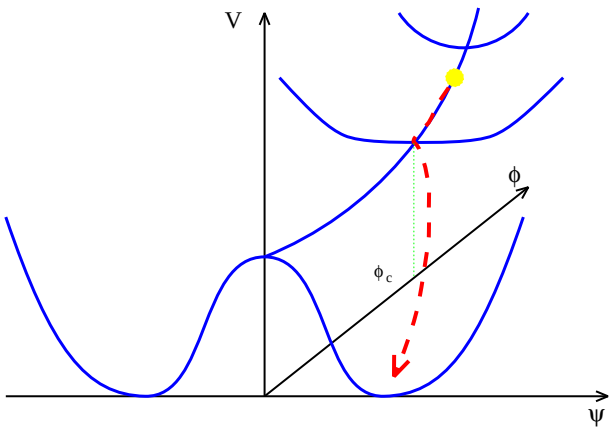

Figure 3.1: Form of the hybrid inflation potential, from Ref. [27]

tachyonic growth of ψ will terminate the slow roll of the inflaton field and eventually the fields will settle in the true vacuum at $\psi = \pm v$ and $\phi = 0$. Note that in the true vacuum state the $\mathbb{Z}_2$-symmetry $\phi \to \pm\phi$ is spontaneously broken. A very detailed analysis of this model and its cosmological implications can be found in Refs. [28] and [29].

3.2 A $\mathbb{Z}_2$-Symmetrical Supersymmetric Hybrid Inflation Model

If we want to understand inflation and the inflaton field in a greater scheme, we will have to go beyond the Standard Model of particle physics. Very promising candidates for an extension of the Standard Model are supersymmetric theories (Susy) and their locally gauge-invariant generalisations which are known under the name of Supergravity (Sugra). Scalar fields are ubiquitous in Susy theories and their potentials often have flat directions. Since we need a sufficiently flat potential for the inflaton field to allow slow roll, it does not seem unreasonable to try and identify one of these fields with the inflaton field. Furthermore, one requires a stable minimum of the potential to represent the true vacuum state.
In this section we will present a model proposed in Refs. [5, 6].

3.2.1 A Supersymmetric Approach to Inflation

A comprehensive introduction to Susy and Sugra would be far beyond the scope of this thesis. We will refer the interested reader to the books of Wess and Bagger [30], Buchbinder and Kuzenko [31] or the review by Dragon, Ellwanger and Schmidt [32].

Essentially, SUSY is a symmetry between bosons and fermions, i.e. the supersymmetry transformation turns fermions into bosons and vice versa. If the symmetry is not broken, particles and their supersymmetric partners will have the same masses.
Since the supersymmetric partners of the Standard Model particles have not been observed, they cannot have the same masses and hence, if SUSY exists, it must be broken. There are various ways of breaking supersymmetry, we are going to look at a superpotential which uses a Polonyi term [33] to achieve this.

Consider the chiral superfields S, T and Σ and the superpotential given by

$$W = \lambda\, T(M_{\mathrm{G}}^2 - \Sigma^2) + M_{\mathrm{S}}^2(\beta + S). \tag{3.3}$$

M_{S} and M_{G} are mass scales and λ is a coupling constant. The constant $\beta \equiv (2-\sqrt{3})\, M_{\mathrm{Pl}}$ has been chosen such that the cosmological constant vanishes. This superpotential is invariant under the $\mathbb{Z}_2$-transformation $\Sigma \to -\Sigma$. The first term breaks the global $(\mathbb{Z}_2)$-symmetry in the ground state while preserving supersymmetry. The second term is a Polonyi term and responsible for SUSY breaking. M_{S} is the scale of supersymmetry breaking and leads to a gravitino mass of $m_{3/2} = (M_{\mathrm{S}}^2/M_{\mathrm{Pl}})\exp(2-\sqrt{3})$. In the following we will assume $m_{3/2} \simeq 130$ GeV corresponding to a supersymmetry breaking scale of $M_{\mathrm{S}} \simeq 1.6 \cdot 10^{10}$ GeV.

If we rotate the fields S and T by an angle $\xi \equiv M_{\mathrm{S}}^2/(\lambda M_{\mathrm{G}}^2)$:

$$\begin{aligned} \Phi &= \frac{\xi S}{\sqrt{1+\xi^2}} + \frac{T}{\sqrt{1+\xi^2}} \\ \Psi &= \frac{S}{\sqrt{1+\xi^2}} - \frac{\xi T}{\sqrt{1+\xi^2}} \end{aligned} \tag{3.4}$$

and define

$$\begin{aligned} \lambda_1 &= \frac{\lambda}{\sqrt{1+\xi^2}} \\ \lambda_2 &= \frac{\xi\lambda}{\sqrt{1+\xi^2}} \\ \text{and} \qquad M &= M_G\sqrt{1+\xi^2}, \end{aligned} \tag{3.5}$$

the superpotential can be written as

$$W_{\mathrm{O}} = \lambda_1\Phi(M^2 - \Sigma^2) + \lambda_2\Psi\Sigma^2 + M_{\mathrm{S}}^2\beta. \tag{3.6}$$

In this form one can recognize the superpotential as an O'Raifeartaigh model [34], which represents the simplest model of F-term SUSY breaking. In Ref. [5] it is shown that successful inflation requires $\xi \ll 1$, i.e. $\lambda M_{\mathrm{G}}^2 \gg M_{\mathrm{S}}^2$.

Let us calculate the scalar potential. In the following we will ignore the spinor components of the superfields and limit our analysis to the scalar components, which will be denoted by the same symbols as the corresponding superfields. In the globally supersymmetric case one would use

$$V_0 = \sum_i \left| \frac{\partial W}{\partial z_i} \right|^2 \tag{3.7}$$

with $z_i = T, S, \Sigma$; $(i = 1, 2, 3)$, resulting in

$$V_0 = \lambda^2 |M_{\rm G}^2 - \Sigma^2|^2 + 4\lambda^2 |T|^2 |\Sigma|^2 + M_{\rm S}^4. \tag{3.8}$$

This potential yields vacuum expectation values of $\langle T \rangle = 0$ and $\langle \Sigma \rangle = \pm M_{\rm G}$ while S is a flat direction and hence undetermined in the ground state. We are, however, interested in the locally supersymmetric case, so we need to take supergravity corrections into account. The SUGRA scalar potential is given by

$$V = e^{K/M_{\rm Pl}^2} \left[\left| \frac{\partial W}{\partial z_i} + \frac{z_i^* W}{M_{\rm Pl}^2} \right|^2 - 3 \frac{|W|^2}{M_{\rm Pl}^2} \right], \tag{3.9}$$

where $K = \sum_i |z_i|^2$ is the canonical Kähler potential. We can expand V in powers of $M_{\rm Pl}^{-1}$ and receive, up to higher order terms,

$$\begin{aligned} V &= V_0 + \lambda \frac{M_{\rm G}^2}{M_{\rm Pl}^2} \beta M_{\rm S}^2 (T + T^*) + \mathcal{O}(|T|^2/M_{\rm Pl}^2) \\ &= V_0 + 2(2 - \sqrt{3}) \lambda \frac{M_{\rm S}^2 M_{\rm G}^2}{M_{\rm Pl}} \, {\rm Re}\,(T) + \mathcal{O}(|T|^2/M_{\rm Pl}^2) \\ &\equiv V_0 + \tilde{\kappa} \phi + \mathcal{O}(|T|^2/M_{\rm Pl}^2), \end{aligned} \tag{3.10}$$

where we have defined $\tilde{\kappa} \equiv 2^{3/2}(2 - \sqrt{3}) \lambda \frac{M_{\rm S}^2 M_{\rm G}^2}{M_{\rm Pl}}$ and $T \equiv (\phi + i\chi)/\sqrt{2}$ with ϕ and χ real. Likewise, we can split Σ into a real and an imaginary part: $\Sigma \equiv (\sigma + i\rho)/\sqrt{2}$.
S will acquire a small mass from the higher order corrections, giving it a vacuum expectation value. The vacuum expectation values of the fields in the true vacuum are

$$\langle T \rangle = 0, \qquad \langle \Sigma \rangle = \pm M_{\rm G}, \qquad \langle S \rangle = (2 - \sqrt{3}) \, M_{\rm Pl}. \tag{3.11}$$

Let us identify ϕ with the inflaton field: For $\Sigma \approx 0$ the slope of ϕ, which is needed to sustain the slow roll, will be caused by the linear supergravity correction term.

Indeed, for $|T|^2 \gg M_{\rm G}^2/2$ the effective mass of Σ will be positive, pushing Σ to zero and thereby allowing slow roll inflation in this regime. Also, we will have minima of the potential for $S = 0$ and $\chi = 0$, the corresponding masses are small, however. Provided that slow roll lasts long enough though, it is reasonable to assume that at the end of this phase these fields will not be excited.

Similar to the model we treated in the previous section there is a critical value $\phi_c = M_G$, at which the effective mass of σ will change sign. The effective mass of ρ however, will not change sign and continue to grow as ϕ approaches its minimum.

Therefore, we shall limit our analysis to the evolution of the fields ϕ and σ, whose scalar potential reads

$$V(\phi,\sigma) = \lambda^2\left(M_G^2 - \frac{\sigma^2}{2}\right)^2 + \lambda^2\phi^2\sigma^2 + \tilde{\kappa}\phi + \mathcal{O}(|T|^2/M_{Pl}^2). \tag{3.12}$$

We also omitted the constant term M_S^4, since it does not affect the equations of motion.

In order to analyse this model quantitatively, we need to fix some parameters. In Ref. [5] it was shown that $M_G = 10^{16}$ GeV and $\lambda = 1.3 \cdot 10^{-5}$ is a sensible choice of parameters, which ensures we are in the linear regime of hybrid inflation (i.e. we can ignore higher order and loop effects) and agrees with the COBE normalisation. Together with our choice of M_S we can deduce numerical values for the Hubble constant H and the SUGRA correction parameter $\tilde{\kappa}$.

Parameter	Value
M_S	$1.6 \cdot 10^{10}$ GeV
$m_{3/2}$	130 GeV
M_G	10^{16} GeV
λ	$1.3 \cdot 10^{-5}$
H	$3.1 \cdot 10^8$ GeV
$\tilde{\kappa}$	$(4.6 \cdot 10^9 \text{ GeV})^3$

Table 3.1: Representative values for the parameters of this model.

The energy density during the slow roll phase is $V \simeq \lambda^2 M_G^4$ with a Hubble parameter $H \simeq \lambda M_G^2/(\sqrt{3}M_{Pl})$. As we expected, we have $V \ll M_{Pl}$ and $H < 10^9$ GeV, so the axion problem is avoided.

Let us double-check if this potential actually allows slow roll by verifying that the slow roll conditions are fulfilled:
Ignoring the contributions of order σ^2 the slow roll parameters are given by

$$\begin{aligned} \varepsilon &= \frac{M_{Pl}^2}{2}\left(\frac{V'}{V}\right)^2 \\ &= \frac{4\xi^2\beta^2}{M_{Pl}^2}\left(1 - \frac{\phi^3}{2\sqrt{2}\,\xi\beta M_{Pl}^2} + \ldots\right) \qquad (3.13) \\ \eta &= M_{Pl}^2\frac{V''}{V} = \frac{3}{2}\frac{\phi^2}{M_{Pl}^2} + \ldots \qquad (3.14) \end{aligned}$$

Obviously, the conditions (2.43) are satisfied for $\phi \ll M_{Pl}$.

The equations of motion can be calculated with Equation (2.33) and are given by

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2(t)}\nabla^2\phi + \frac{\partial V}{\partial \phi} = 0 \tag{3.15}$$

$$\ddot{\sigma} + 3H\dot{\sigma} - \frac{1}{a^2(t)}\nabla^2\sigma + \frac{\partial V}{\partial \sigma} = 0. \tag{3.16}$$

Let us split up ϕ into a classical, homogeneous part ϕ_{cl} and its quantum fluctuations $\delta\phi$,

$$\phi(x,t) = \phi_{\mathrm{cl}}(t) + \delta\phi(x,t). \tag{3.17}$$

Around the critical point we have $\phi \approx \phi_{\mathrm{cl}} \approx M_{\mathrm{G}} \gg \delta\phi$ and $\tilde{\kappa} \gg 2\lambda^2\sigma^2\phi$. This leads to an equation of motion of ϕ_{cl} which reads

$$\ddot{\phi}_{\mathrm{cl}} + 3H\dot{\phi}_{\mathrm{cl}} + \tilde{\kappa} = 0. \tag{3.18}$$

Defining $t = 0$ at the critical point, i.e. $\phi(t=0) = \phi_{\mathrm{c}} = M_{\mathrm{G}}$, the solution for ϕ_{cl} is given by

$$\phi_{\mathrm{cl}} \simeq M_{\mathrm{G}} - \frac{\tilde{\kappa}}{3H}t. \tag{3.19}$$

We will be using this as an approximation for ϕ near the critical point. In order to get a rough idea of how the fields behave when the mass squared of σ turns negative, we will expand the potential in t up to linear order and disregard terms that are of an order greater than two in the fields. This will lead to a linearised potential

$$V_{\mathrm{lin}} = -\frac{1}{2}D^3\, t\, \sigma^2, \tag{3.20}$$

with $D = (4\lambda^2\tilde{\kappa}M_{\mathrm{G}}/(3H))^{1/3} \simeq 3.4H$. Note that this approximation neglects the interaction between the fields and the backreaction, that will eventually terminate the growth of the fluctuations of σ. We will discuss the effect of the higher order terms in more detail for a similar case in Section 3.3.

3.2.2 Decay of the Inflaton Field

Our primary interest lies in the energy transfer from the inflaton field to the fluctuations of the σ-field near the critical point. To analyse the fluctuation, we need to perform a Fourier transformation of the fields and their corresponding equations of motion.

Since the critical point marks the end of the slow roll phase, we can assume that the amount of inflation that has taken place up to this point is sufficient to regard space as flat. Furthermore, we assume that the energy of the universe is entirely dominated by vacuum energy, i.e. we have a flat de Sitter space.

The Fourier transformation of the fields is then given by

$$\phi(x,t) = \phi_{\mathrm{cl}}(t) + e^{-\frac{3}{2}Ht}\,(2\pi)^{-\frac{3}{2}} \int d^3k\,(a_\phi(k)\,\phi_k(t)\,e^{ikx} + a_\phi^\dagger(k)\,\phi_k^*(t)\,e^{-ikx}) \tag{3.21}$$

$$\sigma(x,t) = e^{-\frac{3}{2}Ht}\,(2\pi)^{-\frac{3}{2}} \int d^3k\,(a_\sigma(k)\,\sigma_k(t)\,e^{ikx} + a_\sigma^\dagger(k)\,\sigma_k^*(t)\,e^{-ikx}), \tag{3.22}$$

with creation and annihilation operators $a^\dagger(k), a(k)$ and the Fourier coefficients ϕ_k and σ_k. Since $\langle\sigma\rangle = 0$, there will be no homogeneous part in the expansion of σ.

Combining these with Equations (3.15), (3.16) and (3.20), we obtain the equations of motion for the Fourier coefficients

$$\ddot{\phi_k} + \left(k^2 e^{-2Ht} - \frac{9}{4}H^2\right)\phi_k = 0 \tag{3.23}$$

$$\ddot{\sigma_k} + \left(k^2 e^{-2Ht} - \frac{9}{4}H^2 - D^3 t\right)\sigma_k = 0. \tag{3.24}$$

For numerical reasons it is convenient to reparametrise these equations by defining $\tau \equiv Ht$, $\kappa \equiv k/H$ and $\Xi \equiv D^3/H^3$, resulting in

$$\phi_\kappa'' + \left(\kappa^2 e^{-2\tau} - \frac{9}{4}\right)\phi_\kappa = 0 \tag{3.25}$$

$$\sigma_\kappa'' + \left(\kappa^2 e^{-2\tau} - \frac{9}{4} - \Xi\tau\right)\sigma_\kappa = 0. \tag{3.26}$$

Here the prime denotes a differentiation with respect to the new time variable τ. In Appendix A we show that Equation (3.25) is solved by a Hankel function

$$\begin{aligned}\phi_\kappa &= -\frac{1}{2}\sqrt{\frac{\pi}{H}}\,H^{(1)}_{3/2}(\kappa e^{-\tau}) \\ &= -\frac{1}{\sqrt{2H}}\,\kappa^{-\frac{3}{2}}\,e^{\frac{3}{2}\tau}\,(\kappa e^{-\tau} + i)\,\exp\left[i\,\kappa e^{-\tau}\right],\end{aligned} \tag{3.27}$$

which is the solution for a massless field in flat Robertson-Walker spacetime.

Equation (3.26) cannot be solved analytically. An approximate analytical solution was constructed in Ref. [6]: One can define $\tau_t(\kappa)$

$$\kappa^2 e^{-2\tau_t} - \frac{9}{4} = |\Xi\tau_t|, \tag{3.28}$$

which marks the time at which a given mode κ will start growing tachyonically. For $\tau \ll \tau_t$ the solution can be approximated by a Hankel function, for $\tau \gg \tau_t$ it is given by Airy functions [35].

To measure the growth of the fluctuations we can calculate $\langle\sigma^2(\tau)\rangle$, which can be expressed in terms of the spectral function f_κ:

$$\langle\sigma^2(\tau)\rangle = H^3 \int \frac{d^3\kappa}{(2\pi)^3}\, e^{-3\tau}|\sigma(\tau)|^2 \equiv \int d\kappa\, f_\kappa(\tau). \tag{3.29}$$

In general, however, $\langle\sigma^2(\tau)\rangle$ will be a divergent quantity. Therefore, we shall have to renormalise it by introducing a (possibly time-dependent) cutoff scale Λ_κ, see also Appendix B for a more detailed discussion. We can then define the renormalised variance

$$\langle\sigma^2(\tau)\rangle_{\mathrm{r}} = H^3 \int^{\Lambda_\kappa} \frac{d^3\kappa}{(2\pi)^3}\, \left(e^{-3\tau}|\sigma(\tau)|^2 - |\sigma(0)|^2\right). \tag{3.30}$$

In order to solve Equation (3.26) we need to fix the initial conditions. In the simplest approximation we can assume free field initial conditions $\sigma_\kappa(0) = 1/\sqrt{2\kappa}$ and $\sigma'_\kappa(0) = -i\sqrt{\kappa/2}$. If we want to be more precise we can use a Hankel function at negative times. For $\kappa^2 e^{-2\tau} \gg -\Xi\tau$, Equation (3.26) will reduce to Equation (3.25) with a solution given by Equation (3.27). For $\tau = -10$ for instance, one would expect the Hankel function to be a good initial condition for $\kappa \gg 10^{-3}$, i.e. $\kappa \gtrsim 10^{-2}$. In Figures 3.2 and 3.3 we have compared the spectral functions we can obtain from the free field approximation and from the Hankel functions.

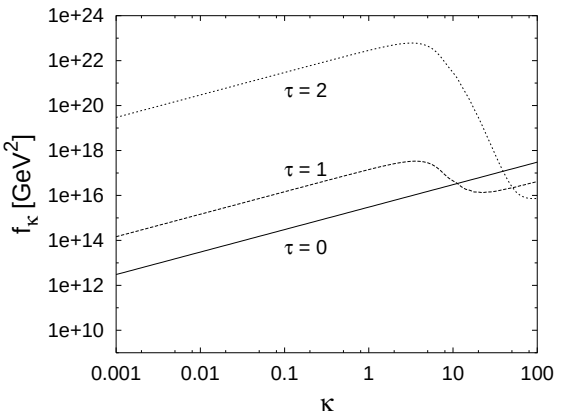

Figure 3.2: Spectral function f_κ of the variance $\langle\sigma^2(\tau)\rangle$ with free field initial conditions at $\tau = 0$

We find that the small momentum modes start growing rapidly after $\tau = 0$. The higher momentum modes ($\kappa \gtrsim 10^2$) are suppressed due to the fact that their mass is still positive and by the redshift from the expansion of the universe. The spectral functions have a maximum at $\kappa \sim \mathcal{O}(1)$, which suggests that the main contribution to the growth of the variance lies in this area. Qualitatively, the free field initial conditions give a decent approximation to the solution we get from the Hankel initial conditions, regarding the

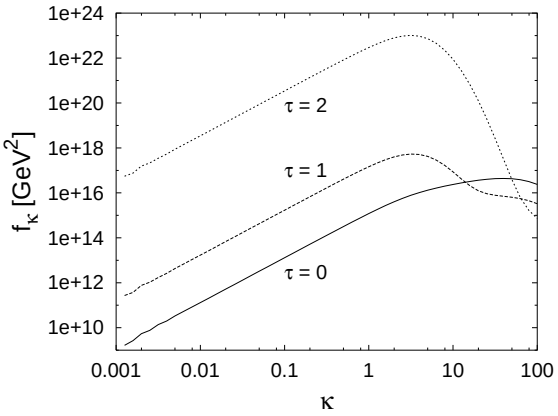

Figure 3.3: Spectral function f_κ of the variance $\langle\sigma^2(\tau)\rangle$ with Hankel function as initial condition at $\tau = -10$.

position and absolute value of the maximum and the suppression at large κ. Since the free field initial conditions are considerably easier to handle numerically, we shall continue to use them for the extensions of this model.

Based on these results we can choose a momentum cutoff

$$\Lambda_\kappa(\tau) = \kappa_0 e^\tau, \tag{3.31}$$

which avoids a time-dependence of the quadratic divergence of $\langle\sigma^2\rangle$. Note that even though the momentum cutoff is time dependent in comoving coordinates, in physical coordinates it is actually constant (see Appendix B for more details on the choice of the cutoff). We have calculated the variance of σ using $\kappa_0 = 4$, corresponding to $\Lambda_\kappa(0) = 4$ and $\Lambda_\kappa(3) \approx 80$.

We notice a very rapid growth of the fluctuations, which is characteristical for hybrid inflation models and known under the name of 'tachyonic preheating'. The fluctuations will reach the true vacuum expectation value of σ, which is given by $\langle\sigma\rangle = \sqrt{2}M_\mathrm{G}$, at $\tau \approx 3$. So we can expect the non-linear terms, which will end the tachyonic growth, to take effect around this time.

3.3 Extension to a Global $U(1)$-Symmetry

We have seen that the $\mathbb{Z}_2$-symmetrical model has two equivalent true vacuum ground states, one with $\langle\sigma\rangle = +M_\mathrm{G}$ and one with $\langle\sigma\rangle = -M_\mathrm{G}$. When the inflaton field reaches the critical point and the mass of σ becomes negative, the field will start rolling down one side of the potential. Whether it rolls down on the positive or on the negative side of the potential is completely random though. So, depending on the position in space, there

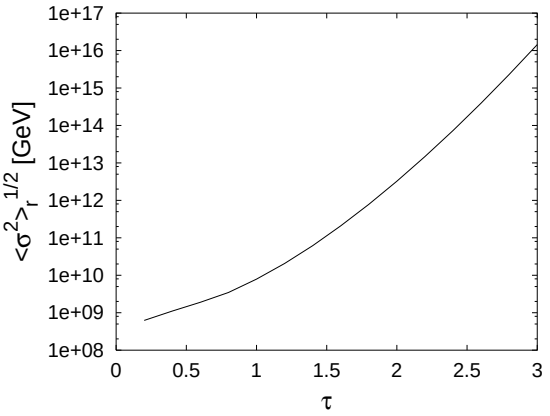

Figure 3.4: Growth of the standard deviation $\langle\sigma^2\rangle_{\mathrm{r}}^{1/2}$ with time. The plot is based on a Hankel function as initial condition at $\tau = -10$ with a momentum cutoff $\Lambda_\kappa = 4e^\tau$

will be regions ("domains") with a positive vacuum expectation value and ones with a negative vacuum expectation value. At the border of two such regions we will have two-dimensional topological defects, called domain walls. The domain walls are stable objects that can be interpreted as potential barrier between the two vacua and hence they contribute to the energy density of the universe. However, cosmologically, domain walls are ruled out, since their contribution to the total energy density of the universe would be too high [37]. One could remedy this problem by breaking the symmetry explicitly and giving one of the vacua a slightly higher energy density, which would destabilize the domain walls.

We have taken a different approach here. In order to avoid the domain wall problem, we modified the superpotential such that the $\mathbb{Z}_2$-symmetry of the potential is replaced by a $U(1)$-symmetry. In a $U(1)$-symmetric potential domain walls do not exist at all. We do still have topological defects in the form of cosmic strings, but as long as their energy density per string length is not too high ($\lesssim 10^{16}$ GeV) the total energy density will not be too large and they will not pose any cosmological problems [10].

3.3.1 Equations of Motion

Based on the superpotential in Equation (3.3) we will analyse the superpotential

$$W = \lambda T(M_{\mathrm{G}}^2 - \Sigma_+\Sigma_-) + M_{\mathrm{S}}^2(\beta + S). \tag{3.32}$$

Σ_+ and Σ_- are chiral superfields with opposite charges $\pm g$ under a $U(1)$-transformation:

$$\Sigma_+ \;\rightarrow\; e^{ig\varphi}\,\Sigma_+ \tag{3.33}$$

$$\Sigma_- \;\rightarrow\; e^{-ig\varphi}\,\Sigma_- \tag{3.34}$$

while S and T are uncharged. The scalar potential including SUGRA corrections can be determined in the same fashion as in the previous section. We obtain

$$\begin{aligned} V &= V_0 + \tilde{\kappa}\phi + \mathcal{O}(M_{\mathrm{Pl}}^{-2}) \simeq \sum_i \left|\frac{\partial W}{\partial z_i}\right| + \tilde{\kappa}\phi \\ &= \lambda^2 |M_{\mathrm{G}}^2 - \Sigma_+\Sigma_-|^2 + M_{\mathrm{S}}^4 + \lambda^2|T|^2\left(|\Sigma_+|^2 + |\Sigma_-|^2\right) + \tilde{\kappa}\phi \end{aligned} \tag{3.35}$$

At this point it is convenient to perform the field transformation

$$\Theta_\pm = \frac{1}{\sqrt{2}}(\Sigma_+ \pm \Sigma_-^*) \tag{3.36}$$

$$\Leftrightarrow \quad \Sigma_+ = \frac{1}{\sqrt{2}}(\Theta_+ + \Theta_-) \tag{3.37}$$

$$\text{and} \quad \Sigma_- = \frac{1}{\sqrt{2}}(\Theta_+ - \Theta_-)^* \tag{3.38}$$

to get rid of the term linear in Σ_+ and Σ_-. Plugging this into Equation (3.35) we obtain the scalar potential

$$\begin{aligned} V = M_{\mathrm{S}}^4 + \lambda^2 \Bigg[& M_{\mathrm{G}}^4 - \frac{1}{2}M_{\mathrm{G}}^2(|\Theta_+|^2 - |\Theta_-|^2) \\ & + \frac{1}{4}\left((|\Theta_+|^2 - |\Theta_-|^2)^2 - (\Theta_+^*\Theta_- - \Theta_+\Theta_-^*)^2\right) \\ & + |T|^2(|\Theta_+|^2 + |\Theta_-|^2)\Bigg] + \tilde{\kappa}\phi. \end{aligned} \tag{3.39}$$

In the slow roll regime $\phi > M_{\mathrm{G}}$ the effective masses of Θ_+ and Θ_- are positive, pushing the fields to their minimum at $\langle\Theta_+\rangle = \langle\Theta_-\rangle = 0$. After ϕ passes the critical point $\phi_{\mathrm{c}} = M_{\mathrm{G}}$, the effective mass of Θ_+ become negative and Θ_+ will acquire a non-vanishing expectation value, while the mass of Θ_- will remain positive, keeping the field in the minimum. Similar to what we did in the previous section, we will ignore the imaginary part of T and forget about the contributions of Θ_- for now. Furthermore, we can express Θ_+ in polar form[1]

$$\Theta_+ \equiv \theta \, \exp\left[i\,\frac{\gamma}{M_{\mathrm{G}}}\right]. \tag{3.40}$$

Plugging this into the potential (3.39), omitting the constant terms and setting $\Theta_- = 0$ we obtain

$$V(\phi,\theta,\gamma) \simeq \lambda^2\left[M_{\mathrm{G}}^4 + \frac{1}{2}(\phi^2 - M_{\mathrm{G}}^2)\,\theta^2 + \frac{1}{4}\theta^4\right] + \tilde{\kappa}\phi. \tag{3.41}$$

[1]This ansatz is somewhat problematic for negative times since $\langle\theta\rangle = 0$, hence θ can be zero, but for positive times, θ will start growing and we will not run into problems.

This is, up to a factor 2, the same potential as in the $\mathbb{Z}_2$-symmetrical case (cf. Equation (3.12)). There is, however, a subtle difference between the two cases.

Unlike σ, which is the real component of a complex scalar field, θ is the absolute value of a complex field. This means we cannot use Equation (2.33) to determine the equations of motion, there will be an interaction with the phase γ, even though it does not appear in the potential. In this case, the equations of motion are given by

$$\ddot{\theta} + 3H\dot{\theta} - \frac{1}{a^2}\nabla^2\theta + \frac{1}{2}\frac{\partial}{\partial\theta}V - \frac{1}{M_{\mathrm{G}}^2}\,\partial^\mu\gamma\partial_\mu\gamma\;\theta = 0 \tag{3.42}$$

$$\ddot{\gamma} + 3H\dot{\gamma} - \frac{1}{a^2}\nabla^2\gamma + 2\frac{1}{\theta}\,\partial^\mu\theta\partial_\mu\gamma = 0 \tag{3.43}$$

The phase γ does not appear in the potential and in particular it does not have a mass term. Hence, γ represents a flat direction of the potential and can be interpreted as the (massless) Goldstone boson of the symmetry.

The interaction term $\frac{1}{M_{\mathrm{G}}^2}\,\partial^\mu\gamma\partial_\mu\gamma\;\theta \equiv I_\gamma$ in the equation of motion for θ corresponds to a correction of the effective mass of θ. So any initial value of I_γ could be absorbed by a redefinition of the origin of time. In the Hartree approximation without expansion of the universe we have $\partial^\mu\gamma\partial_\mu\gamma = 0$ at all times, so for the sake of simplicity we shall ignore I_γ in the following.

With the derivatives of the potential

$$\frac{\partial V}{\partial\theta} = \lambda^2(\phi^2 - M_{\mathrm{G}}^2)\theta + \lambda^2\theta^3 \tag{3.44}$$

$$\frac{\partial V}{\partial\phi} = \theta^2\phi + \tilde{\kappa} \tag{3.45}$$

we obtain the equations of motion for θ and the inflaton field:

$$\ddot{\theta} + 3H\dot{\theta} - \frac{1}{a^2}\nabla^2\,\theta + \frac{\lambda^2}{2}\left(\phi^2 - M_{\mathrm{G}}^2\right)\theta + \frac{\lambda^2}{2}\,\theta^3 = 0 \tag{3.46}$$

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + \theta^2\phi + \tilde{\kappa} = 0. \tag{3.47}$$

Applying the Hartree approximation (see Appendix C) and reparametrising with $\kappa = k/H$ and $\tau = Ht$ leads to the following equations of motion for the Fourier components of the fields

$$\phi_\kappa'' + \left(\kappa^2 e^{2\tau} + \frac{1}{H^2}\langle\theta^2\rangle - \frac{9}{4}\right)\phi_\kappa = 0 \tag{3.48}$$

$$\theta_\kappa'' + \left(\kappa^2 e^{-2\tau} + \frac{\lambda^2}{2H^2}(\langle\phi^2\rangle - M_{\mathrm{G}}^2) + \frac{3\lambda^2}{2H^2}\langle\theta^2\rangle - \frac{9}{4}\right)\theta_\kappa = 0. \tag{3.49}$$

3.3.2 Solving the Equations of Motion

Again, our primary interest lies in the fluctuations of θ. The form of Equation (3.49) suggests an iterative procedure to end up with a self-consistent solution. Like in the previous section we will approximate $\langle\phi^2\rangle \approx \phi_{\mathrm{cl}}^2$ and assume ϕ_{cl} to be linear (cf. Equation (3.19)), thus decoupling the two equations of motion.

In order to start the iteration we need to guess a reasonable form for $\langle\theta^2_{(0)}\rangle$. For $\tau \lesssim 2$ we certainly have $\frac{3\lambda^2}{2H^2}\langle\theta^2\rangle \ll 1$, hence our first step will be to solve Equation (3.49) for $\langle\theta^2_{(0)}\rangle = 0$. We have already solved this case in Equation (3.26), just with a different value for Ξ. Now we can determine $\langle\theta^2_{(1)}\rangle$. We have done this numerically by fitting a quadratic function of the form $\langle\theta^2_{(1)}\rangle \simeq a + b\cdot 10^{\tau} + c\cdot(10^{\tau})^2$ to the plot in Figure 3.5.

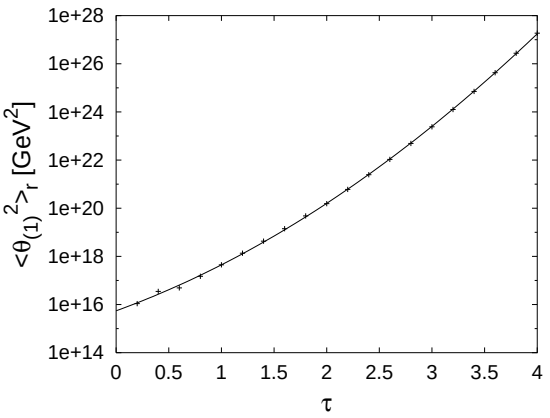

Figure 3.5: Quadratic approximation for $\langle\theta^2_{(1)}\rangle$, dots represent the values from the numerical calculation, the line is the quadratic approximation. Numerical values are $\langle\theta^2_{(1)}\rangle \simeq 0.06 + 38.53\cdot 10^{\tau} + 2.10\cdot(10^{\tau})^2$.

Now we take this result and plug it into Equation (3.49) again to find $\langle\theta^2_{(2)}\rangle$, which we would expect to show the effects of the backreaction and a transition from faster than exponential growth to oscillations as the $\langle\theta^2\rangle$-term starts to dominate the equation of motion. And indeed, as Figure 3.6 shows, at around $\tau = 4$ the backreaction will kick in and terminate the growth of the fluctuations.

In the previous section we estimated this time to be near $\tau = 3$, however, in the equations $U(1)$-symmetrical model the velocity of the inflaton field is suppressed by an additonal factor of $\frac{1}{4}$, hence the fluctuations will not grow as fast and the backreaction will set in at a later time. After the next few iterations one would reckon to see a saturation of the variance for $\tau \gtrsim 4$. For a qualitative comparison we have included the plot of a lattice simulation [38], which shows the expected behaviour, i. e. $\langle\theta^2\rangle$ converging to its vacuum expectation value.

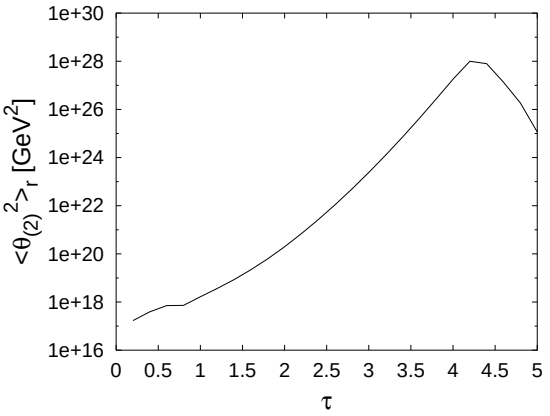

Figure 3.6: Plot of $\langle \theta^2_{(2)} \rangle$.

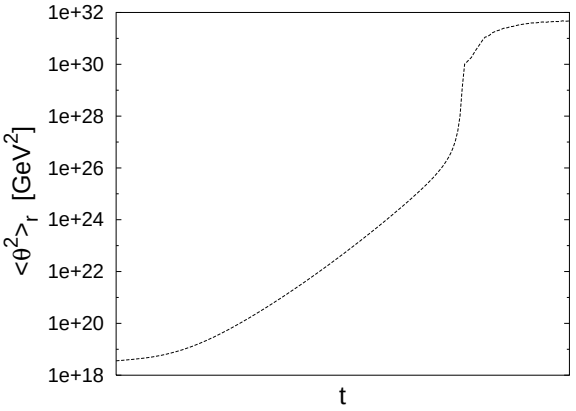

Figure 3.7: Plot of $\langle \theta^2 \rangle$ calculated with a lattice simulation [38], t is given in arbitrary units.

3.4 Local $U(1)$-Symmetry and Gauge Field

We have seen that a global $U(1)$-symmetric theory gives rise to a massless Goldstone boson, which has not been observed in nature. Therefore, it is interesting to consider the case in which the broken symmetry is a local gauge invariance. In local gauge theories the degree of freedom associated with the Goldstone boson can be absorbed by the helicity zero state of the gauge boson if one uses the unitary gauge. (see for example Ref. [39]). Moreover, a global symmetry is not conserved in supergravity, hence it appears natural to go to a local symmetry instead.
The system will be invariant under a symmetry transformation

$$\Theta_\pm \to \Theta_\pm \, e^{\pm i g \Lambda(x)}. \tag{3.50}$$

with an x-dependent real scalar field Λ. In the Lagrange density $\mathcal{L}$ we need to replace the normal derivatives ∂^μ with covariant derivatives $D^\mu \equiv \partial^\mu + igA^\mu$, where g is the gauge coupling constant and A^μ the gauge field, which transforms under the gauge transformation like

$$A^\mu(x) \to A^\mu(x) - \frac{1}{g}\partial^\mu \Lambda(x). \tag{3.51}$$

This will lead to interaction terms between the scalar fields of the theory and the gauge field. Our goal in this section is to estimate the effect of a local symmetry on the evolution of our system after the critical point. To avoid unnecessarily complicated calculations, we will therefore focus on the interactions between the tachyonically growing field Θ_+ and the gauge field and neglect the other fields.
Let us set up the Lagrange density. In addition to the kinetic term $D_\mu\Theta_+ D^\mu\Theta_+$ and the potential $V(\Theta_+)$ we can add another gauge invariant term which contains the field strength tensor $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and is the kinetic term of the gauge field.

$$\mathcal{L} = \frac{1}{2}D_\mu\Theta_+ D^\mu\Theta_+ - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - V(\Theta_+) \tag{3.52}$$

Plugging in the covariant derivatives and writing Θ_+ in polar form as

$$\Theta_+ = \theta \ \exp\left[i\frac{\gamma}{M_\mathrm{G}}\right], \tag{3.53}$$

we obtain

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\partial_\mu\theta\partial^\mu\theta + \frac{1}{2M_\mathrm{G}^2}\theta^2\partial_\mu\gamma\partial^\mu\gamma \\ &+ \frac{g}{M_\mathrm{G}}\theta A_\mu\partial^\mu\gamma - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\theta,\gamma).\end{aligned} \tag{3.54}$$

A careful analysis [40] reveals that this Lagrangean has one degree of freedom more than we require.[2] This degree of freedom can be removed by imposing a gauge condition on the fields, leaving us with 4 physical fields. The natural choice in our case is the unitary gauge

$$\frac{g}{M_\mathrm{G}}\theta\gamma = 0. \tag{3.55}$$

As long as $\theta \neq 0$, which is clearly satisfied for positive times, this is equivalent to

$$\gamma = 0. \tag{3.56}$$

[2]It might seem as though there are two degrees of freedom too much since we have 6 fields (γ, θ and A^μ) and only require 4 physical degrees of freedom. We will see later that one of these fields, A^0, can be written as a linear combination of $\dot{A}^i$, leaving us with 5 linearly independent fields.

In other words, the Goldstone boson vanishes, which was the motivation for going to a local symmetry in the first place. This leads to a considerable simplification of the Lagrangean

$$\mathcal{L} = \frac{1}{2}\partial_\mu\theta\partial^\mu\theta + \frac{1}{2}g^2\theta^2 A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\theta), \tag{3.57}$$

resulting in an equation of motion for θ that is given by

$$\partial_\mu\partial^\mu\theta + 3H\dot{\theta} - g^2\theta A_\mu A^\mu + \frac{\partial V}{\partial\theta} = 0, \tag{3.58}$$

using the Hartree approximation we have

$$\partial_\mu\partial^\mu\theta + 3H\dot{\theta} - g^2\theta\langle A_\mu A^\mu\rangle + \frac{\partial V}{\partial\theta} = 0. \tag{3.59}$$

With the exception of the term containing the gauge field, this is the same equation of motion we already analysed in the globally symmetric case. So as to estimate the effect of this extra term, we shall take a closer look at the gauge field:
The equation of motion for A^μ reads

$$\partial_\nu F^{\mu\nu} + 3HF^{\mu 0} - g^2\theta^2 A^\mu = 0 \tag{3.60}$$

$$\Leftrightarrow \qquad \partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) + 3H(\partial^\mu A^0 - \partial^0 A^\mu) - g^2\theta^2 A^\mu = 0. \tag{3.61}$$

3.4.1 Decomposition of the Gauge Field

Since the equations of motion of the gauge field contain spatial derivatives of A^μ, it is convenient to perform a Fourier transformation and study the behaviour of the Fourier coefficients. With a Fourier transformation that is given by

$$A^\mu(x,t) = \frac{1}{(2\pi)^{3/2}}\int d^3k\left(a_k\hat{A}^\mu(k,t)\ e^{ikx} + a_k^\dagger\hat{A}^{\mu *}(k,t)\ e^{-ikx}\right) \tag{3.62}$$

and using the Hartree approximation, we obtain the following equations of motion

$$-i\,\partial_0(k_j\hat{A}^j) + k_j k^j\hat{A}^0 - g^2\langle\theta^2\rangle A^0 = 0 \tag{3.63}$$

$$-ik^j\dot{\hat{A}}^0 - \ddot{\hat{A}}^j + ik_l(ik^j\hat{A}^l - ik^l\hat{A}^j) + 3H(-ik^j\hat{A}^0 - \dot{\hat{A}}^j) - g^2\langle\theta^2\rangle\hat{A}^j = 0. \tag{3.64}$$

Equation (3.63) does not have any time derivatives of $\hat{A}^0$, so we can solve for $\hat{A}^0$ and express it in terms of $\vec{k}\vec{\hat{A}} \equiv -a^2 k_j\hat{A}^j$:

$$\hat{A}^0 = i\frac{\vec{k}\dot{\vec{\hat{A}}} - 2H\vec{k}\vec{\hat{A}}}{k^2 + g^2e^{2Ht}\langle\theta^2\rangle}, \tag{3.65}$$

where we have used $k^2 \equiv -a^2 k_j k^j \simeq -k_j k^j e^{2Ht}$ and assumed flat de Sitter space. Now we can plug this into Equation (3.64), leading to

$$\ddot{\vec{\hat{A}}}^j + 3H\dot{\hat{A}}^j + (k^2 e^{-2Ht} + g^2\langle\theta^2\rangle)\hat{A}^j - k^j\vec{k}\Bigg[e^{-2Ht}\vec{\hat{A}} + \frac{1}{k^2 + g^2 e^{2Ht}\langle\theta^2\rangle}\left(\ddot{\vec{\hat{A}}} + H\dot{\vec{\hat{A}}} - 6H^2\vec{\hat{A}}\right) + \frac{2g^2 e^{2Ht}(H\langle\theta^2\rangle + \langle\theta\dot{\theta}\rangle)}{(k^2 + g^2 e^{2Ht}\langle\theta^2\rangle)^2}\left(2H\vec{\hat{A}} - \dot{\vec{\hat{A}}}\right)\Bigg] = 0. \tag{3.66}$$

Again, it will prove helpful to reparametrise these equations with $\kappa = k/H$ and $\tau = Ht$, leading to

$$\hat{A}^{j\prime\prime} + 3\hat{A}^{j\prime} + (\kappa^2 e^{-2\tau} + h^2\langle\theta^2\rangle)\hat{A}^j - \kappa^j\vec{\kappa}\Bigg[e^{-2\tau}\vec{\hat{A}} + \frac{1}{\kappa^2 + h^2 e^{2\tau}\langle\theta^2\rangle}\left(\vec{\hat{A}}'' + \vec{\hat{A}}' - 6\vec{\hat{A}}\right) + \frac{2h^2 e^{2\tau}(\langle\theta^2\rangle + \langle\theta\theta'\rangle)}{(\kappa^2 + h^2 e^{2\tau}\langle\theta^2\rangle)^2}\left(2\vec{\hat{A}} - \vec{\hat{A}}'\right)\Bigg] = 0, \tag{3.67}$$

with $h^2 = g^2/H^2$.

We can further simplify these equations by defining the longitudinal and the transversal parts of $\hat{A}^j$

$$\hat{A}_{\shortparallel} \equiv \frac{\kappa_j}{|\vec{\kappa}|}\hat{A}^j \tag{3.68}$$

$$\hat{A}_{\perp} \equiv \frac{1}{|\vec{\kappa}|}\epsilon_{lmn}\kappa^m\hat{A}^n, \tag{3.69}$$

where ϵ_{lmn} is the 3-dimensional totally antisymmetric tensor. A multiplication of Equation (3.67) with $\kappa_j/|\vec{\kappa}|$ will yield the equation of motion for $\hat{A}_{||}$:

$$\tilde{A}_{\shortparallel}''\left(1 - \frac{\kappa^2}{\kappa^2 + h^2 e^{2\tau}\langle\theta^2\rangle}\right) + \tilde{A}_{\shortparallel}'\left(3 - \frac{\kappa^2}{\kappa^2 + h^2 e^{2\tau}\langle\theta^2\rangle} + 2h^2 e^{2\tau}\kappa^2\frac{\langle\theta^2\rangle + \langle\theta\theta'\rangle}{(\kappa^2 + h^2 e^{2\tau}\langle\theta^2\rangle)^2}\right) + \tilde{A}_{\shortparallel}\left(h^2\langle\theta^2\rangle + 6\frac{\kappa^2}{\kappa^2 + h^2 e^{2\tau}\langle\theta^2\rangle} - 4h^2 e^{2\tau}\kappa^2\frac{\langle\theta^2\rangle + \langle\theta\theta'\rangle}{(\kappa^2 + h^2 e^{2\tau}\langle\theta^2\rangle)^2}\right) = 0. \tag{3.70}$$

Lastly, we define $\tilde{A}_{\shortparallel} \equiv e^{\frac{3}{2}\tau}\hat{A}_{\shortparallel}$ and obtain

$$\tilde{A}_{\shortparallel}''\left(1 - \frac{\kappa^2}{\kappa^2 + h^2 e^{2\tau}\langle\theta^2\rangle}\right) + \tilde{A}_{\shortparallel}'\left(2\,\frac{\kappa^2}{\kappa^2 + h^2 e^{2\tau}\langle\theta^2\rangle} + 2h^2 e^{2\tau}\kappa^2\frac{\langle\theta^2\rangle + \langle\theta\theta'\rangle}{(\kappa^2 + h^2 e^{2\tau}\langle\theta^2\rangle)^2}\right) + \tilde{A}_{\shortparallel}\left(h^2\langle\theta^2\rangle + \frac{33}{4}\,\frac{\kappa^2}{\kappa^2 + h^2 e^{2\tau}\langle\theta^2\rangle} - 4h^2 e^{2\tau}\kappa^2\frac{\langle\theta^2\rangle + \langle\theta\theta'\rangle}{(\kappa^2 + h^2 e^{2\tau}\langle\theta^2\rangle)^2} - \frac{9}{4}\right) = 0. \tag{3.71}$$

The equation for the component perpendicular to $\vec{\kappa}$ is a lot simpler. We can obtain it by multiplying Equation (3.67) with $\epsilon_{lmn}\kappa^m/|\vec{\kappa}|$.

$$\ddot{\hat{A}}_\perp + 3\dot{\hat{A}}_\perp + (\kappa^2 e^{-2\tau} + h^2\langle\theta^2\rangle)\hat{A}_\perp = 0 \tag{3.72}$$

And with $\tilde{A}_\perp \equiv e^{\frac{3}{2}\tau}\hat{A}_\perp$ we get

$$\tilde{A}_\perp'' + \left(\kappa^2 e^{-2\tau} + h^2\langle\theta^2\rangle - \frac{9}{4}\right)\tilde{A}_\perp = 0. \tag{3.73}$$

Equation (3.73) is the equation of motion of a massive field with a time-dependent mass. Unlike Equation (3.26) however, the mass squared will remain positive and we do not expect an exponential growth.

3.4.2 Evolution of the Components of the Gauge Field

If we want to compute $\tilde{A}_\perp$ and $\tilde{A}_{\shortparallel}$, we need to find a good approximation for $\langle\theta^2\rangle$ first. Provided that Equation (3.59) is not dominated by the term containing the gauge field, the solution we found in the globally symmetric case should be a reasonable guess. For small times ($\tau \lesssim 3$) we can use our quadratic fit to $\langle\theta^2_{(1)}\rangle$, given by

$$\langle\theta^2\rangle \simeq 0.06 + 38.53 \cdot 10^\tau + 2.10 \cdot (10^\tau)^2. \tag{3.74}$$

This will allow us to compute $\tilde{A}_\perp$, $\tilde{A}_{\shortparallel}$ and, by plugging $\tilde{A}_{\shortparallel}$ into Equation (3.63), also $\tilde{A}^0 \equiv e^{\frac{3}{2}\tau}\hat{A}^0$. Similar to what we did in the previous section, we can calculate

$$\langle A_0 A^0\rangle = \int \mathrm{d}^3k\; e^{-3\tau}|\tilde{A}_0\tilde{A}^0| \tag{3.75}$$

and likewise for $\langle A^2_{\shortparallel}\rangle$ and $\langle A^2_\perp\rangle$. And finally, we have

$$\langle A_\mu A^\mu\rangle \simeq \langle A_0 A^0\rangle - \langle A^2_{\shortparallel}\rangle - 2\langle A^2_\perp\rangle \tag{3.76}$$

so we can estimate the effect of the local symmetry on the equation of motion of θ.

We have performed the calculation with a strong gauge coupling $g = 10^{-2}$ and taken free field initial conditions for the components of the Fourier coefficients of the gauge field.

Figure 3.8 shows the results of our computation. It turns out that $\langle A_\mu A^\mu\rangle$ is negative between $\tau = 0$ and $\tau = 4$. Starting at a value of $\sim 2 \cdot 10^{18}$ GeV^2, it decreases with time. Before we can discuss what influence this will have on the equation of motion of θ, let us first look at another effect that accompanies the move to a local gauge invariance, the D-term.

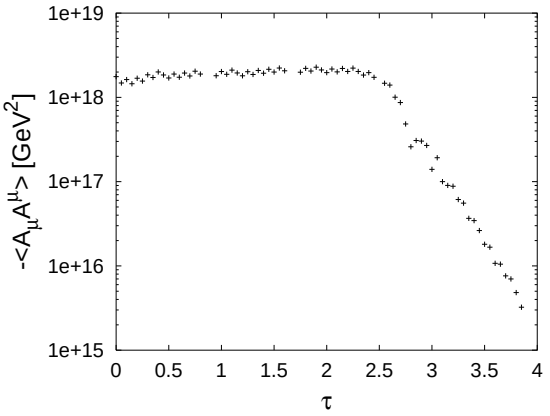

Figure 3.8: Plot of $-\langle A_\mu A^\mu \rangle$. What looks like a small oscillation is actually an artifact of the numerical integration and would not appear if one calculated a higher number of data points.

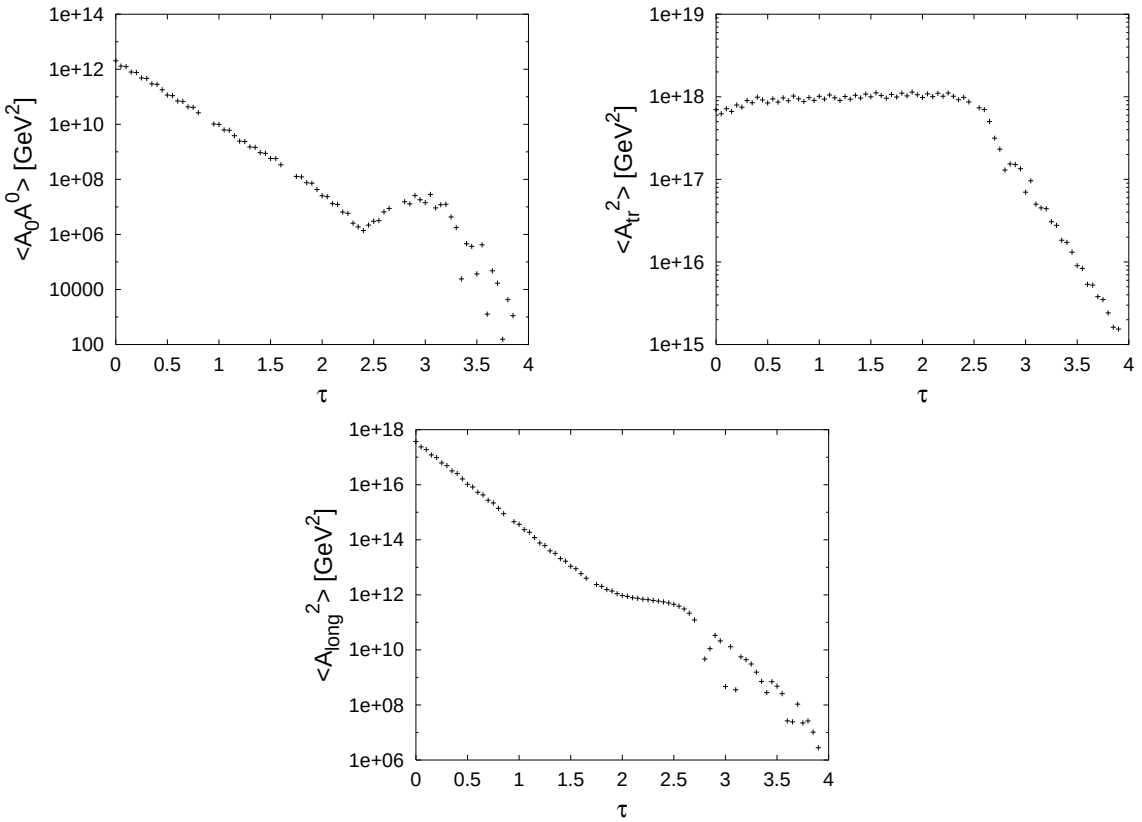

Figure 3.9: Plot of the components of the gauge field. Top left: $\langle A_0 A^0 \rangle$. Top right: $\langle A_\perp^2 \rangle$. Bottom: $\langle A_{||}^2 \rangle$.

3.4.3 D-Term

In a supersymmetric theory, the gauge field has to be interpreted as the $\theta\bar{\theta}$-component of a vector superfield $\mathcal{V}$, which satisfies $\mathcal{V} = \bar{\mathcal{V}}$. If we introduce a vector superfield, however, we also need to add an interaction term between the $\theta^2\bar{\theta}^2$-component of $\mathcal{V}$, which is usually denoted as D, and the scalar components of the scalar superfields of the theory, in our case given by Σ_+ and Σ_-. This term has the form $\pm\Sigma_\pm^* D\Sigma_\pm$. Since D does not have a kinetic term, it can be integrated out and expressed in terms of $\Sigma_\pm$. In our model, D is given by

$$D = -g^2\left(|\Sigma_+|^2 - |\Sigma_-|^2\right), \tag{3.77}$$

leading to an additional term in the scalar potential

$$V_D = \frac{g^2}{2}\left(|\Sigma_+|^2 - |\Sigma_-|^2\right)^2. \tag{3.78}$$

Or, expressed in terms of Θ_+ and Θ_-

$$V_D = \frac{g^2}{2}\left(\Theta_+^*\Theta_- + \Theta_+\Theta_-^*\right)^2. \tag{3.79}$$

Now the question arises if this term will have a significant effect on the evolution of the fields. Since V_D is of order two in $\theta = |\Theta_+|$ it represents a correction to the mass of θ. Yet, the correction is only of order Θ_-^2, so we may safely neglect it. At the same time V_D is a mass term for $|\Theta_-|$ as well, but since it is positive it will not change the fact that $|\Theta_-|$ is pushed to its minimum. Hence, we shall ignore the contribution of V_D in the following.

3.4.4 Effect on the Equations of Motion

If we write Equation (3.59) in terms of τ and κ and apply the Hartree approximation, we obtain the following equation of motion for the Fourier components of θ:

$$\theta_\kappa'' + \left(\kappa^2 e^{-2\tau} + \frac{\lambda^2}{2H^2}(\langle\phi^2\rangle - M_{\mathrm{G}}^2) + \frac{3\lambda^2}{2H^2}\langle\theta^2\rangle - h^2\langle A_\mu A^\mu\rangle - \frac{9}{4}\right)\theta_\kappa = 0. \tag{3.80}$$

Comparing this with Equation (3.49), we find that the only difference is the term $-h^2\langle A_\mu A^\mu\rangle$. With $h^2 \approx 10^{-21}\ \mathrm{GeV}^{-2}$ and $\langle A_\mu A^\mu\rangle < 2\cdot 10^{18}\ \mathrm{GeV}^2$ we have

$$\left|h^2\langle A_\mu A^\mu\rangle\right| \lesssim 2\cdot 10^{-3} \ll \frac{9}{4}. \tag{3.81}$$

Therefore, we have shown that the gauge field does not have a significant effect on the evolution of θ. Similarly, we can neglect the effect on ϕ, so the calculations we performed in the globally symmetric case remain valid in the local case as well.

Chapter 4

Summary and Outlook

The inflationary paradigm has been very successful since its introduction in 1981. It beautifully solves cosmological puzzles, such as the relic or the horizon problem. However, there are a huge number of models for inflation. Recent observations have excluded most of the simpler models, which contain only one scalar field. Therefore, models with more than one field, hybrid models, have attracted the attention of cosmologists lately. In this thesis we have analysed a particular class of hybrid inflation models that arise quite naturally from a supersymmetric theory.

Our work is based on the results of Refs. [5, 6], which we have reviewed. In order to avoid the domain wall problem that comes with a $\mathbb{Z}_2$-symmetrical potential we have extended the above model to a global $U(1)$-symmetry. The equations of motion were set up and compared to the $\mathbb{Z}_2$-symmetrical case. We found that, within our approximations, the equations remained the same, apart from a rescaling of the inflaton velocity at the critical point. We studied the reheating mechanism, i.e., the energy transfer from the inflaton field to the scalar field θ, including the effects of the backreaction, which eventually terminates the process.

Furthermore, we have expanded our model to a local $U(1)$-symmetry and analysed the behaviour of the gauge field and its effect on the dynamics of the other fields in detail. We came to the conclusion that these effects are negligible so that the considerations for the globally symmetric case remain valid for a local symmetry.

In this thesis we have laid the groundwork for further calculations which would enable us to give some quantitative predictions that can be compared with known data. Of particular interest would be an analysis of:

- the properties of the final thermal state, in particular the reheating temperature T_{RH} due to its significance for baryo- and leptogenesis (as we discussed in Section 2.2.4),
- the formation of topological defects, their contribution to the total energy density of the universe, and
- a simulation of the expected spectrum of the CMBR.

Appendix A

Fluctuations of a Scalar Field

We shall derive the equations of motion of the quantum fluctuations of a scalar field in the inflationary universe and attempt to solve them. The results can be used as initial conditions when we are trying to determine their evolution at later times.

Consider a scalar field χ. We can split this field into a classical, homogeneous part $\chi_{\rm cl}$ and its fluctuations $\delta\chi$

$$\chi(x,t) = \chi_{\rm cl}(t) + \delta\chi(x,t). \tag{A.1}$$

The fluctuations can be expanded in Fourier modes:

$$\delta\chi(x,t) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3k \, \left(a_k \chi_k(t) \, e^{ikx} + a_k^\dagger \chi_k^*(t) \, e^{-ikx} \right), \tag{A.2}$$

where the χ_k are the Fourier coefficients of the fluctuations and a_k and $a_k^\dagger$ are the annihilation and creation operators for modes with momentum k. They satisfy the canonical commutation relations

$$[a_k, a_{k'}^\dagger] = (2\pi)^3 \delta^3(k - k'). \tag{A.3}$$

And the canonically conjugate operators to χ are given by

$$\pi(x,t) = \dot{\chi}(x,t). \tag{A.4}$$

Let us examine the case of a massive field with no higher order interactions, i.e.

$$V(\chi) = \frac{1}{2} m_\chi^2 \chi^2. \tag{A.5}$$

The equation of motion for the Fourier components is given by

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a^2} + m_\chi^2 \right) \chi_k = 0. \tag{A.6}$$

It is convenient to rewrite this equation in terms of $\xi \equiv e^{\frac{3}{2}Ht}\chi$ in order to get rid of the term containing the first derivative. We obtain:

$$\ddot{\xi}_k + \left(\frac{k^2}{a^2} + m_\chi^2 - \frac{9}{4}H^2\right)\xi_k = 0. \tag{A.7}$$

Defining $\nu_\chi^2 \equiv \frac{9}{4} - \frac{m_\chi^2}{H^2}$ we have

$$\ddot{\xi}_k + \left(\frac{k^2}{a^2} - H^2\nu_\chi^2\right) = 0. \tag{A.8}$$

For ν_χ real the solution to this equation is given by Hankel functions:

$$\xi_k(t) = c_1(k)H_{\nu_\chi}^{(1)}(z) + c_2(k)H_{\nu_\chi}^{(2)}(z), \tag{A.9}$$

where $z = \frac{k}{aH}$, $H_{\nu_\chi}^{(1)}(z)$ and $H_{\nu_\chi}^{(2)}(z)$ are the Hankel functions of the first and second kind, respectively. In the ultraviolet regime ($k \gg aH$) we would expect a plane wave solution $\xi_k = e^{-iz}/\sqrt{2k}$, the factor of $1/\sqrt{2k}$ arises if we require the annihilation and creation operators to fulfil the correct commutation relations.
If we want the solution for ξ_k to coincide with the plane wave solution for small wavelengths, we can determine the parameters $c_1(k)$ and $c_2(k)$ from the asymptotic behaviour of the Hankel functions:

$$H_{\nu_\chi}^{(1)}(z \gg 1) \approx \sqrt{\frac{2}{\pi z}}\, e^{i(x - \frac{\pi}{2}\nu_\chi - \frac{\pi}{4})} \tag{A.10}$$

$$H_{\nu_\chi}^{(2)}(z \gg 1) \approx \sqrt{\frac{2}{\pi z}}\, e^{-i(x - \frac{\pi}{2}\nu_\chi - \frac{\pi}{4})}. \tag{A.11}$$

With $c_1(k) = 0$ and $c_2(k) = \frac{\sqrt{\pi}}{2}e^{i(\nu_\chi + \frac{1}{2})\frac{\pi}{2}}$ the solution becomes:

$$\xi_k = \frac{1}{2}\sqrt{\frac{\pi}{H}}\, H_{\nu_\chi}^{(1)}(z). \tag{A.12}$$

In the case of a massless field $m_\chi = 0$ we have $\nu_\chi = \frac{3}{2}$ and the solution reduces to

$$\xi_k = \frac{1}{2}\sqrt{\frac{\pi}{H}}\, H_{3/2}^{(1)}(z) \tag{A.13}$$

$$= -\frac{1}{\sqrt{2H}}\, z^{-\frac{3}{2}}\,(z+i)\, e^{iz}. \tag{A.14}$$

Appendix B

Renormalisation

We will show that the variance $\langle\phi^2\rangle$ of a scalar field ϕ is a divergent quantity and discuss a normalisation via a momentum cutoff. For simplicity we shall consider a massless field ϕ in the pure de Sitter case. The mode functions ϕ_κ are given by a Hankel function:

$$\phi_\kappa = -\frac{1}{\sqrt{2H}}\left(\kappa e^{-\tau}\right)^{3/2}\exp\left(\kappa e^{-\tau}\right)\left(\kappa e^{-\tau}+i\right). \tag{B.1}$$

We can now calculate the variance

$$\langle\phi^2\rangle = \phi_{\rm cl}^2 + H^3 e^{-3\tau}\int\frac{\mathrm{d}^3\kappa}{(2\pi)^3}\,\phi_\kappa\phi_\kappa^* \tag{B.2}$$

$$= \phi_{\rm cl}^2 + \frac{H^2}{2}\,e^{-3\tau}\int\frac{\mathrm{d}^3\kappa}{(2\pi)^3}\,\frac{1}{\kappa}\,e^{\tau}\left(1+\frac{1}{\kappa^2}e^{2\tau}\right) \tag{B.3}$$

$$= \phi_{\rm cl}^2 + \frac{H^2}{2\pi^2}\int\mathrm{d}\kappa\,\left(\kappa e^{-2\tau}+\frac{1}{\kappa}\right). \tag{B.4}$$

Introducing an ultraviolet cutoff $\Lambda_{\rm UV}$ and an infrared cutoff $\Lambda_{\rm IR}$, we can integrate and get

$$\langle\phi^2\rangle = \phi_{\rm cl}^2 + \frac{H^2}{2\pi^2}\,e^{-2\tau}\Lambda_{\rm UV}^2 + \frac{H^2}{4\pi^2}\,\ln\frac{\Lambda_{\rm UV}}{\Lambda_{\rm IR}}. \tag{B.5}$$

$$\simeq \phi_{\rm cl}^2 + \frac{H^2}{2\pi^2}\,e^{-2\tau}\Lambda_{\rm UV}^2 \tag{B.6}$$

We end up with a time-dependent quadratic ultraviolet divergence and a time-independent logarithmic divergence that depends on both the infrared and the ultraviolet cutoff. The time-dependence of the quadratic term is somewhat troublesome, but we can remedy this problem by giving the ultraviolet cutoff a time-dependence. Defining

$$\Lambda_\kappa \equiv e^{\tau}\Lambda_{\rm UV} \tag{B.7}$$

we obtain

$$\langle\phi^2\rangle = \phi_{\rm cl}^2 + \frac{H^2}{2\pi^2}\,\Lambda_\kappa^2. \tag{B.8}$$

Since $\Lambda_{\rm UV}$ is a comoving quantity, Λ_κ can be interpreted as a physical cutoff scale. Now we can renormalise $\langle\phi^2\rangle$ by subtracting $\langle\phi^2(0)\rangle$:

$$\langle\phi^2(\tau)\rangle_{\rm r} = \langle\phi^2(\tau)\rangle - \langle\phi^2(0)\rangle = \phi_{\rm cl}^2(\tau). \tag{B.9}$$

For our calculations we will apply the same procedure to massive fields σ, i.e.

$$\langle\sigma^2(\tau)\rangle_{\rm r} = H^3 e^{-3\tau}\int^{e^\tau\Lambda_{\rm UV}}\frac{{\rm d}^3\kappa}{(2\pi)^3}\,|\sigma(\tau)|^2 - \int^{\Lambda_{\rm UV}}\frac{{\rm d}^3\kappa}{(2\pi)^3}\,|\sigma(0)|^2. \tag{B.10}$$

Appendix C

Hartree Approximation

Consider a scalar field ϕ with a potential $V(\phi)$ in a FRW metric. The Fourier decomposition of ϕ is defined by

$$\phi(x,t) = \phi_{\rm cl}(t) + \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3k \, \left(a_k \phi_k(t) \, e^{ikx} + a_k^\dagger \phi_k^*(t) \, e^{-ikx} \right). \tag{C.1}$$

The equation of motion is given by

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + \frac{\partial V}{\partial \phi} = 0. \tag{C.2}$$

If we want to determine the equations of motion for the Fourier components ϕ_k, there is a direct correspondence between the two equations, as long as ϕ does not appear in higher than linear order. We have

$$\ddot{\phi} \to \ddot{\phi}_k \tag{C.3}$$

$$\dot{\phi} \to \dot{\phi}_k \tag{C.4}$$

$$\phi \to \phi_k \tag{C.5}$$

$$\nabla^2\phi \to -k^2\phi_k, \tag{C.6}$$

and so on. If however the derivative of the potential contains terms which are of quadratic or higher order in ϕ, the procedure is not so straightforward. Let us take a look at a quartic potential $V = \frac{\lambda}{4}\phi^4$, which leads to a cubic term in the equations of motion. Ignoring the classical part of ϕ and focussing on the fluctuations we have

$$\begin{aligned} \phi^3 = \frac{1}{(2\pi)^{9/2}} \iiint \mathrm{d}^3k'' \mathrm{d}^3k' \mathrm{d}^3k \, \left(a_{k''} \phi_{k''}(t) \, e^{ik''x} + a_{k''}^\dagger \phi_{k''}^*(t) \, e^{-ik''x} \right) \\ \left(a_{k'} \phi_{k'}(t) \, e^{ik'x} + a_{k'}^\dagger \phi_{k'}^*(t) \, e^{-ik'x} \right) \left(a_k \phi_k(t) \, e^{ikx} + a_k^\dagger \phi_k^*(t) \, e^{-ikx} \right). \end{aligned} \tag{C.7}$$

There is no hope of handling this expression analytically. Even numerically, the only chance of dealing with this is to perform the calculations on a finite lattice, thereby

turning the integrations into finite sums.
We will be taking a different road here. By applying the Hartree-approximation (also known as mean-field approximation) we will linearise the equation of motion and avoid the problem of the higher order terms. The idea is to remove two of the integrations by taking the vacuum expectation value.

$$\begin{aligned}\langle\phi^2\rangle - \phi_{\mathrm{cl}}^2 &= \frac{1}{(2\pi)^3} \iint \mathrm{d}^3k' \mathrm{d}^3k \, \langle 0| \left(a_{k'}\phi_{k'}(t)\, e^{ik'x} + a_{k'}^\dagger \phi_{k'}^*(t)\, e^{-ik'x}\right) \\ &\quad \left(a_k\phi_k(t)\, e^{ikx} + a_k^\dagger\phi_k^*(t)\, e^{-ikx}\right)|0\rangle \\ &= \frac{1}{(2\pi)^3} \iint \mathrm{d}^3k' \mathrm{d}^3k \, \langle 0| \left(a_{k'}a_k^\dagger\phi_{k'}(t)\phi_k^*(t)\, e^{i(k'-k)x}\right)|0\rangle \\ &= \frac{1}{(2\pi)^3} \iint \mathrm{d}^3k' \mathrm{d}^3k \, \phi_{k'}(t)\phi_k^*(t)\, (2\pi)^3\delta^{(3)}(k-k')\, e^{i(k'-k)x} \\ &= \int \mathrm{d}^3k' \, |\phi_{k'}|^2. \end{aligned} \tag{C.8}$$

The Hartree approximation is

$$\phi^3 \approx \langle\phi^2\rangle\phi. \tag{C.9}$$

We neglect the interactions between modes of different momenta, but as long as the spectrum of the ϕ_k is sufficiently smooth we can expect it to be a reasonable approximation. We can now use an iterative procedure to make the result self-consistent. In terms of the correspondence between the equations of motion we obtain

$$\phi^3 \to 3\langle\phi^2\rangle\phi_k. \tag{C.10}$$

If we have two scalar fields ϕ and ψ and an interaction term of the form $\psi^2\phi$ in the equations of motion, we can proceed similarly and get

$$\psi^2\phi \to \langle\psi^2\rangle\phi_k. \tag{C.11}$$

Bibliography

[1] A. Einstein, "Die Grundlage der allgemeinen Relativitätstheorie," Annalen der Physik 49 (1916), 769-822.

[2] A. H. Guth, "The Inflationary Universe: A Possible Solution To The Horizon And Flatness Problems," Phys. Rev. D **23** (1981) 347.

[3] E. J. Copeland, A. R. Liddle, D. H. Lyth, E. D. Stewart and D. Wands, "False vacuum inflation with Einstein gravity," Phys. Rev. D **49** (1994) 6410 [arXiv:astro-ph/9401011].

[4] G. R. Dvali, Q. Shafi and R. K. Schaefer, "Large Scale Structure And Supersymmetric Inflation Without Fine Tuning," Phys. Rev. Lett. **73** (1994) 1886 [arXiv:hep-ph/9406319].

[5] W. Buchmüller, L. Covi and D. Delepine, "Inflation and supersymmetry breaking," Phys. Lett. B **491** (2000) 183 [arXiv:hep-ph/0006168].

[6] T. Asaka, W. Buchmüller and L. Covi, "False vacuum decay after inflation," Phys. Lett. B **510** (2001) 271 [arXiv:hep-ph/0104037].

[7] E. W. Kolb and M. S. Turner, "The Early Universe," Addison Wesley (1990).

[8] J. A. Peacock, "Cosmological Physics," Cambridge University Press (1998).

[9] J. García-Bellido, "Astrophysics and cosmology," arXiv:hep-ph/0004188.

[10] A. R. Liddle and D. H. Lyth, "Cosmological Inflation and Large Scale Structure," Cambridge University Press (2000).

[11] A. Riotto, "Inflation and the theory of cosmological perturbations," arXiv:hep-ph/0210162.

[12] P. B. Pal, "Determination of cosmological parameters: An introduction for non-specialists," Pramana **54** (2000) 79 [arXiv:hep-ph/9906447].

[13] C. L. Bennett *et al.*, "First Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Preliminary Maps and Basic Results," Astrophys. J. Suppl. **148** (2003) 1 [arXiv:astro-ph/0302207].

[14] S. M. Carroll, "Why is the Universe Accelerating?," arXiv:astro-ph/0310342.

[15] B. de Carlos, J. A. Casas, F. Quevedo and E. Roulet, "Model independent properties and cosmological implications of the dilaton and moduli sectors of 4-d strings," Phys. Lett. B **318** (1993) 447 [arXiv:hep-ph/9308325].

[16] M. B. Hindmarsh and T. W. Kibble, "Cosmic strings," Rept. Prog. Phys. **58** (1995) 477 [arXiv:hep-ph/9411342].

[17] A. D. Linde, "A New Inflationary Universe Scenario: A Possible Solution Of The Horizon, Flatness, Homogeneity, Isotropy And Primordial Monopole Problems," Phys. Lett. B **108** (1982) 389.

[18] L. Kofman, A. D. Linde and A. A. Starobinsky, "Reheating after inflation," Phys. Rev. Lett. **73** (1994) 3195 [arXiv:hep-th/9405187].

[19] L. Kofman, A. D. Linde and A. A. Starobinsky, "Towards the theory of reheating after inflation," Phys. Rev. D **56** (1997) 3258 [arXiv:hep-ph/9704452].

[20] P. B. Greene and L. Kofman, "On the theory of fermionic preheating," Phys. Rev. D **62** (2000) 123516 [arXiv:hep-ph/0003018].

[21] J. Garcia-Bellido, "The origin of matter and structure in the universe," Phil. Trans. Roy. Soc. Lond. A **357** (1999) 3237 [arXiv:astro-ph/9906497].

[22] M. Kawasaki, K. Kohri and T. Moroi, "Radiative decay of a massive particle and the non-thermal process in primordial nucleosynthesis," Phys. Rev. D **63** (2001) 103502 [arXiv:hep-ph/0012279].

[23] C. L. Bennett *et al.*, "4-Year COBE DMR Cosmic Microwave Background Observations: Maps and Basic Results," Astrophys. J. **464** (1996) L1 [arXiv:astro-ph/9601067].

[24] D. H. Lyth and A. Riotto, "Particle physics models of inflation and the cosmological density perturbation," Phys. Rept. **314** (1999) 1 [arXiv:hep-ph/9807278].

[25] A. D. Linde, "Axions in inflationary cosmology," Phys. Lett. B **259** (1991) 38.

[26] A. D. Linde, "Eternal Extended Inflation And Graceful Exit From Old Inflation Without Jordan-Brans-Dicke," Phys. Lett. B **249** (1990) 18.

[27] A. R. Liddle, "Acceleration of the universe," New Astron. Rev. **45** (2001) 235 [arXiv:astro-ph/0009491].

[28] E. J. Copeland, S. Pascoli and A. Rajantie, "Dynamics of tachyonic preheating after hybrid inflation," Phys. Rev. D **65** (2002) 103517 [arXiv:hep-ph/0202031].

[29] J. Garcia-Bellido, M. Garcia Perez and A. Gonzalez-Arroyo, "Symmetry breaking and false vacuum decay after hybrid inflation," Phys. Rev. D **67** (2003) 103501 [arXiv:hep-ph/0208228].

[30] J. Wess and J. Bagger, "Supersymmetry and Supergravity," Princeton University Press (1992).

[31] I. L. Buchbinder and S. M. Kuzenko, "Ideas and Methods of Supersymmetry and Supergravity: Or a Walk through Superspace" , Institute of Physics Publishing (1998).

[32] N. Dragon, U. Ellwanger and M. G. Schmidt, "Supersymmetry And Supergravity," Prog. Part. Nucl. Phys. **18** (1987) 1.

[33] J. Polonyi, "Generalization Of The Massive Scalar Multiplet Coupling To The Supergravity,"

[34] L. O'Raifeartaigh, "Spontaneous Symmetry Breaking For Chiral Scalar Superfields," Nucl. Phys. B **96** (1975) 331.

[35] M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions", Dover Publications (1974).

[36] D. Boyanovsky, D. Cormier, H. J. de Vega, R. Holman, A. Singh and M. Srednicki, "Scalar field dynamics in Friedman Robertson Walker spacetimes," Phys. Rev. D **56** (1997) 1939 [arXiv:hep-ph/9703327].

[37] Y. B. Zeldovich, I. Y. Kobzarev and L. B. Okun, "Cosmological Consequences Of The Spontaneous Breakdown Of Discrete Symmetry," Zh. Eksp. Teor. Fiz. **67** (1974) 3 [Sov. Phys. JETP **40** (1974) 1].

[38] B. Janutta, "Entstehung Topologischer Defekte am Ende der Inflation," Diplomarbeit Universität Hamburg (2004).

[39] S. Weinberg, "The Quantum Theory of Fields, Volume 2", Cambridge University Press (1996).

[40] F. Mandl and G. Shaw. "Quantum field theory" John Wiley and Sons (1984).

Acknowledgements

I would like to extend my gratitude to Professor Wilfried Buchmüller for suggesting this interesting topic. Special thanks to Laura Covi for her supervision, advice and patience. Thanks to Professor Jan Louis for his co-supervision on behalf of the Universität Hamburg. Toffi, for making sure that this thesis is not full of nasty typos and LaTeX blunders. Benny, for plots, tea and cookies! My friendly officemates and all the members of the DESY theory group. And Y^3. ♡
And, last but not least, my parents, whose support made all this possible.

Erklärung gemäß Diplomprüfungsordnung

Hiermit versichere ich, diese Arbeit selbständing angefertigt zu haben. Die wesentlichen Quellen und Hilfsmittel sind angegeben. Ich erkläre mein Einverständnis mit der Auslage meiner Diplomarbeit in der Bibliothek des Fachbereichs Physik der Universität Hamburg.

Hamburg, den 14.01.2001

Jan C. Hamann