Nucleon Nucleon Potential Using Dirac Constraint Dynamics

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Zusammenfassung

Es wird ein relativistisches Potentialkonzept für die Beschreibung der Nukleon-Nukleon (NN) Wechselwirkung und Streuung im Energiebereich $0 < T_{Lab} < 3$ GeV verfolgt. Paradigmatisch, NN Potentiale der Kernphysik bewähren sich für nukleare Vielteilchen Probleme wenn die experimentellen NN Daten damit bestens beschrieben werden. Mittelenergetische NN Streuung sieht, neben den nukleonischen Freiheitgraden der mittel und lang reichweitigen Wechselwirkung, die Quark-Gluon Freiheitsgrade (QCD) in Verbindung mit Mesonproduktion und Hadron Anregungen. Die genaue Zuweisung und Parametrisierung der Wechselwirkungen, zur jeweils dominierenden radial abhängigen Dynamik, ist das Thema dieser Dissertation. Es werden, um der Poincaré Invarianz Rechnung zu tragen, zwei gekoppelte Dirac Gleichungen durch *Instant Form Dynamics* eingeschränkt. Eine ausführliche Zusamenenstellung der theoretischen und mathematischen Mittel, wesentlich basierend auf den Arbeiten von Crater und Van Alstine, umfaßt einen großen Teil der Arbeit. Die Vereinfachung der gekoppelten Dirac Gleichungen in eine Art stationäre Schrödinger Gleichungen wird ausgeführt. Als Ergebnis dieser Reduktion erhält man einen Satz gekoppelter radialer Schrödinger Gleichungen mit explizite energieabhängigen Potentialen. Die Potentiale entsprechen Ausdrücken von komplexen Funktionen und deren Ableitungen. Es wurde eine umfangreiche Numerik entwickelt um die neuesten Neutron-Proton und Proton-Proton Phasenverschiebungen, GWU/SAID-2003, nach Partialwellen entwickelt, für Energien zwischen 0 und 3 GeV zu parametrisieren und berechnen. Dazu zählt auch das Deuteron. Das Wechselwirkungsmodell wird durch π, ρ, ω and σ Austausch geleitet und deren Kopplungskonstanten werden angepaßt. Dies liefert im ersten Schritt einen guten Fit der Arndtschen Phasen zwischen 0 und 300 MeV. Es zeigen die Potentiale, unabhängig vom Drehimpuls, ein repulsives Core-Potential mit Eigenschaften, das von Teilchenmassen und der relativistischen Behandlung des Problems bestimmt wird. Durch ein optisches Potential (OMP) erweitert, werden die Rechnungen von 300 MeV bis 3 GeV fortgesetzt. Damit wird die QCD dominierte kurzreichweitige Zone, mit innerer Nukleon-Anregung und Meson-Produktion, durch ein komplexes Potential im Ortsraum beschrieben. Das optische Potential, als innere Fortsetzung zum Dirac Potential des Mesonaustausches wo $r > 0.5\,\mathrm{fm}$, liefert für die Partialwellen sehr einfach eine Anpassung von Theorie und Daten. Die optischen Potentiale subsummieren die komplexe kurzlebige QCD Anregungstruktur in glatte Energieabhängigkeiten. Es zeigt sich ein konsistentes Bild wenn die Nukleonen, als separate Cluster, Anregungen und Meson-Produktion durchlaufen und die Bildung eines einzigen Dibaryon Clusters nicht dominiert. Für zukünftige Arbeiten wird vorgeschlagen den Teil des phänomenologischen OMP durch ein mikroskopisches OMP zu ersetzen und Doppelanregungen von $\Delta(3,3)$ und anderen Paaren explizite zu koppeln.

Abstract

The relativistic potential concept is fostered for the description of nucleon-nucleon (NN) interactions and scattering for energies $0 < T_{Lab} \leq 3$ GeV. It proves useful to confirm and predict nuclear properties and reactions with the implicit knowledge having the best possible agreement with experimental NN data. Medium energy NN scattering, as is accepted for low energy nuclear physics in general, is determined from proton, nucleon and meson degrees of freedom in the long range soft interaction sector, the quark gluon degrees of freedom govern the short distance hard processes. The identification and parameterization, of the combined long and short range NN domains, is the topic of this thesis. The formalism for two coupled Dirac equations, within constraint instant form dynamics, is used to study the NN interaction. The comprehensive review, of the important theoretical tools and associated mathematics, rests essentially on the work of Crater and Van Alstine. The reduction of the coupled Dirac equations into Schrödinger type equations is given. Explicitly energy dependent coupled channel potentials, for use in partial wave Schrödinger like equations, with nonlinear and complicated derivative terms, result. We developed the necessary numerics and study *np* and *pp* scattering phase shifts for energies 0 to 3 GeV and the deuteron bound state. The interactions are inspired by meson exchange of π, η, ρ, ω and σ mesons for which we adjust coupling constants. This yields, in the first instant, high quality fits to the Arndt phase shifts 0 to 300 MeV. Second, the potentials show a universal, independent from angular momentum, core potential which is generated with the relativistic meson exchange dynamics. Extrapolations towards higher energies, up to T_{Lab} equal 3 GeV, allow to separate a QCD dominated short range zone as well as inelastic nucleon excitation mechanism contributing to meson production. A local or nonlocal optical model, in addition to the meson exchange Dirac potential, produces agreement between theoretical and phase shifts data. The optical model potentials reflect a short lived complex multi hadronic intermediate structure formation of which the optical model parameters give a consistent picture. For future work, the here presented phenomenological access encourages a more microscopic and detailed use of QCD, including explicit $\Delta(3,3)$ pair formation and some obviously predominant other pair mechanism.

Contents

Chapter 1

Introduction

One of the fundamental problems in nuclear theory is to describe the properties of atomic nuclei in terms of interactions between individual particles, such as quarks, gluons, mesons, baryons, bosons and nucleons. Quarks were initially introduced as constituent particles so as to introduce a semblance of order into the *particle zoo* [273, 361]. Mesons are in this picture made up of a quark and an antiquark and baryons of three quarks. The meson exchange theory may be viewed as an effective representation of the strong nuclear interaction at low and intermediate energies the typical energies for nuclear physics in the range of $\sim$ MeV. It is reasonable to expect the quark degrees of freedom to become important only at very short distances and high energies $\sim$ GeV. The hadrons can then be fitted into multiplets and their spectrum understood. In this way new particles were even predicted [272].

Nucleons are composite objects with a rich excitation structure, whose scales are in GeV, which can be attributed to constituent quarks, these are effective particles with a typical scale of 300 MeV interacting strongly by gluon exchange. It is a paradigm that quantum chromodynamics (QCD) is the fundamental theory for the strong interaction and that the interaction between nucleons is sampling the strong interactions. Moreover, QCD at the scale of 1 GeV becomes very complicated, and hence it is rather difficult to say in this case a *priori* what kind of physics emerges. In this energy region characteristics of QCD become important such as the confinement of quarks and gluons and the spontaneous breaking of chiral symmetry. In principle, it should be determined directly from QCD. However, we do not yet have a good understanding of how such objects can emerge from QCD [564].

At low and intermediate energies, QCD coupling constants are generally large and prohibitive in order to use perturbative approaches. The existing models, based on QCD, only account for a part of the hadron properties and one is still far from describing the NN interaction in terms of QCD. This is essential before facing more complex many nucleon systems such as nuclei and it is necessary to assume simplified models. Only finite size particle degrees of freedom are assumed to be relevant. The quarks are confined inside the hadrons by the strong interactions and outside the NN interaction arises from exchange of various mesons.

The interaction description for the nuclear few and many body systems uses potential theory successfully. Hamiltonian mechanics recovers potentials as interaction description. With this background knowledge in mind, it is of interest to review cornerstones of nuclear physics which have a direct relation to NN interactions.

The first theoretical calculations for nuclear matter was made by Heisenberg and his student Euler in 1937 [215]. They calculated its properties to second order in perturbation theory assuming a two body interaction of Gaussian type for the nucleons. The work of Yukawa in 1937 [584] established the field theoretical fundamental meson exchange approach whose roots are still valid. Nuclear physics and its fundamental approaches become the forefront topic in physics for several decades [585, 191, 135, 151]. The NN interaction as well as nuclear many body problems determined the scene. When the strong short range repulsion in the NN interaction was identified, it became apparent that conventional perturbation methods were inadequate for calculation of nuclear systems with an infinitely strong hard core since two body potential matrix elements diverge. Brueckner [101], Bethe [70] and Goldstone [290] (BBG) developed the famous perturbation method based on g-matrix elements and eliminated thus divergencies. An alternative approach to handle hard core was suggested by Jastrow [338]. He introduced a variational approach by the use of trial wave functions to treat the two nucleon correlations. In addition to this, the recent approach was made by Pandharipande [448].

In the 1960's Brueckner theory brought substantial advances in nuclear physics study [101]. The first high precision NN potential was found by Hamada and Johnson in 1963 [316]. They used a hard core potential with a radius $r_c \sim 0.5$ fm. This potential accounted well for all data of this time between T_{lab} 0-300 MeV. The infinite core remained questionable and incompatible up to this date. Soon after Hamada Johnson, Reid [474] found another high precision potential with a soft core. With these potentials nuclear physics made great quantitative progress in nuclear structure and nuclear reactions [233].

Despite the efforts, high precision NN potentials do not account for the saturation properties of nuclear matter, and do not agree with experiment. The binding depends on the potential and the saturation points lie along a band, the Coester Band [132, 134], which does not contain the empirical values. The cause of this failure was linked to lack of relativity in the theory.

A relativistic approach to nuclear structure was developed by Miller and Green [419]. Using a Dirac-Hartree [322, 323] approach they were able to reproduce the binding energies and *root mean square* radii of nuclei and the spin orbit splitting of nuclear shell model orbits. A relativistic approach was applied to proton-nucleus scattering by Clark and others [127]. In the late 1980's relativistic calculations for nuclear matter were performed using realistic potentials [92].

Here, the term *realistic* refers to nuclear potentials which reproduce the two nucleon phase shift data accurately $\chi^2 \sim 1$. Representative examples are the Nijmegen [523], Paris NN potential [372] and the models developed by the Bonn group [394] and quantum inversion developed in Hamburg [277, 489].

All existing potential models are to some degree based on meson exchange. They all include the one pion exchange contribution which essentially determines the long range part of the interaction. The intermediate range has been well described by inclusion of two pion exchange processes, σ-exchange and nucleon resonances, $\Delta(3,3)$, may be excited in intermediate states [394].

The NN interaction at large distances cannot be described perturbatively. In the region where the two bags of the nucleons are not overlapping only colorless objects should be exchanged. In this region we describe the NN interaction by meson exchange. To guarantee asymptotic freedom, the mesons should interact with the quarks in the nucleons only at their surface. In order to include quark degrees of freedom in the NN interaction we must clearly consider two different regions. The first one corresponds to the region in which the two nucleon cores (quark bags) overlap. In this region quarks can be exchanged between the two quark bags and the two nucleons interact through the one gluon exchange mechanism (Fig. 1.1a). The additional quark interchange is a necessary ingredient in this basic mechanism since the nucleons are color singlets while the gluon is a color triplet. Whatever color is transferred by the interchanged gluon it is balanced by the additional quark exchange. This kind of interaction leads to hard core NN phase shifts [439, 217]. When the two bags separate, nucleons are represented as color singlets and they interact via forces arising from the exchange of color singlet objects which are identified with physical mesons (Fig. 1.1b).

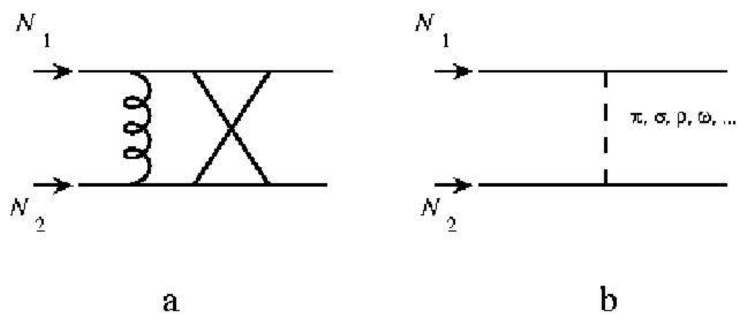


Figure 1.1: Diagrams which describe the NN interaction within the different radii. a) Overlapping two nucleons (short distances): quark gluon exchange, b) Non-overlapping two nucleons (longe distances): meson exchange.

All the above models, theories and equations, such as Bonn-CD potential [394], Bethe-Salpeter equations [71], Blankenbecler-Suger [79] and Gross equation [304, 305] are normally formulated with a kernel, which may be motivated by field theory, are phenomenological and weakly relativistic or nonrelativistic. They are simple enough to be solved accurately for realistic low energy nuclear physics. Nonrelativistic quantum mechanics, with phenomenological NN interactions, provide theories which are valid for energy momentum transfers below pion production [352]. Relativity must be included for higher energy momentum transfers. For many problems of current interest in nuclear physics these models must be consistent with the principle of special relativity. Relativity is needed to model reactions where particles are produced, reactions involving energy and momentum transfers that are comparable to the mass scales of the problem, bound systems where the binding energies are comparable to the masses of the constituent particles, and coordinate system independent treatments of problems in lepton hadron scattering [352].

If we want to describe the NN interaction simultaneously at large and short distances, we need a comprehensive theory, certainly it must be relativistic. Chapt. 2 is devoted to a discussion of the constraint Hamiltonian forms of the relativistic dynamics.

S-matrix element, bound state mass, magnetic moment, and or any other physical observable must be invariant under Poincaré transformations. The challenge of practical calculations, in the framework of Hamiltonian dynamics, is to produce invariant results for observables. An important application of Hamiltonian dynamics is nuclear physics. Traditionally, nonrelativistic model Hamiltonians were used in this field, but since the advent of powerful accelerators and technological need to far exceeding energies of involved masses, it has become clear that the implementation of a relativistic framework is unavoidable. In addition, the common practice of leaning on field theory, to construct the so called realistic nuclear forces, made it clear that also in nuclear physics one needs to take the requirements of special relativity into account.

Wigner [572] analyzed the mathematical formulation of the physical requirement of special relativity in quantum mechanics. He showed that a necessary and sufficient condition for quantum mechanical probabilities to have values that are independent of the choice of inertial coordinate system, is the existence of a unitary ray representation of the inhomogeneous Lorentz group (Poincaré group) on the quantum mechanical Hilbert space. Note that the Schrödinger equation and the existence of a Hamiltonian are consequences of applying Wigners analysis to obtain invariance under time translations.

First, Dirac formulated the problem of including interactions in relativistic classical mechanics [190]. This was done in Hamiltonian form, which has a natural canonical quantization. In the Hamiltonian formulation of classical mechanics, the goal is to construct a representation of the Poincaré group as a group of canonical transformations on phase space. Dirac analyzed these requirements infinitesimally, and showed that the problem is equivalent to the construction of a representation of the Lie algebra of the Poincaré group in terms of Poisson brackets that includes interactions in a consistent way. He gave three different types of solutions to this problem called the *point, light front* and *instant* forms of dynamics. Bakamjian and Thomas [27] constructed the first relativistic quantum mechanical model of two interacting particles in Dirac *instant* form dynamics. Foldy [239] recognized the importance of macroscopic locality as an additional constraint on these models. Coester [133] extended the work of Bakamjian and Thomas to systems of three particles, Sokolov [515] provided the general construction for N particles with a scattering operator consistent with the principle of macroscopic locality. An extensive overview of the whole subject is given in Ref. [352].

In the usual instant form, with the equal time hypersurface $x_0 = const$, translations and rotations transform quantum states in a very simple manner, because they do not change the quantization surface. These kinds of static Poincaré generators are called *kinematical* . The other generators, which change the quantization surface, are called *dynamical*. For example, the Lorentz boost mixes time and space coordinates and thus changes the hypersurface. The Hamiltonian itself, of course, is a dynamical operator. Therefore, an eigenstate in a rest frame is no longer an eigenstate in the boosted frame. Even though we know eigenstates in the rest frame, an attempt to find a new eigenstate in the boosted frame requires, in principle, requires as much effort as solving the entire problem.

Thinking of applications for nuclear physics one considers interacting fields of nucleons and mesons. To arrive at generators which act in the space of a fixed number of N nucleons one has to eliminate the mesonic degrees of freedom as well as the ones for antiparticles. A way to do this has been proposed in [285] and worked out in lowest order in the coupling constant for a field theory of scalar nucleons interacting with a scalar meson field. Relativistic direct interaction theories of particle lie between local field theory and nonrelativistic quantum mechanics [352]. When a quantum mechanics of interacting relativistic particles exists, the correspondence principle suggests in turn the existence of a classical relativistic mechanics that would reproduce the quantum version by canonical quantization [152]. A first study by Currie, Jordan and Sudarchan [169] concludes in what is known as the *non interaction theorem*, that the combined requirements of the Poincaré algebra and the *world line* condition for two particles cannot be satisfied simultaneously unless there is no direct interaction between them. Specifically, they showed that this combined set of commutation relations is unitarily equivalent to the commutation relations for operators of non-interacting particles.

The traditional treatment of interacting pairs of spinning particles began with Breit's suggestion for the spin dependent forces given in his 1929 paper [88]. Breit formed a single effective Hamiltonian by adding together two free Dirac Hamiltonians plus an interaction patterned after that in the Darwin Lagrangian for two spinless particles [171]. In the mid 1970's several investigators found a *relativistic constraint mechanics* [191, 192, 194, 542, 543, 342, 545, 366] to the problem of interacting particles. Some authors use a singular Lagrangian [342, 533, 196] as a starting point. Other bypass the Lagrangian and directly work with constrained Hamiltonian systems [190, 27, 239, 366, 542]. Constraint dynamics can be considered as a relativistic extension of classical mechanics problems with holonomic constraints. Rather than reduce the problem immediately to one with the minimum number of degrees of freedom, the problem is cast with extra degrees of freedom, plus additional constraint equations.

A physically natural way of defining a constrained dynamical system in classical mechanics is through a limiting procedure where a strong attractive potential forces the system at any fixed energy to live closer and closer to the constraint surface. The effective classical dynamics obtained in this limit turns out to depend only on the intrinsic properties of the constraint surface (e.g. the curvature) and not on the details of the constraining potential chosen on the embedding space. We use in Sec.2.5 an analogous procedure in quantum mechanics and consider the quantization of a constrained classical system as the limit of quantizations of classical systems, with extra potentials away from the surface of constraint.

When the equations are written down in covariant form it is clear that the outcome will satisfy relativistic invariance and we refer to such methods as manifestly covariant. Covariant interactions are discussed in Chapt. 5. The Dirac equation [188] for a single spin 1/2 particle utilizes a matrix algebra to construct a linear operator relationship between the energy and the momentum. Such a linear dispersion relationship is particularly useful for constructing manifestly cluster decomposable non perturbative scattering formalisms. Expectation values of the matrices can be related to physical fluxes, but the matrices themselves commute with space time translation generators. The two body Dirac equations of constraint dynamics successfully extend this one body minimal coupling form to the interacting two body system.

Crater van Alstine, et al. developed Dirac's constraint mechanics using supersymmetry for two interacting spinning particles by external potentials [152]-[167]. By combining constraint dynamics with particle supersymmetries, they extended those works to pairs of spin one half particles to obtain two body quantum bound state equations that correct not only defects in the Breit equation but those in the ladder approximation to the Bethe-Salpeter equation as well [546, 152]. These *two body dirac equations* (see Sec. 5.1.1) of constraint dynamics possess a number of important features which provide an alternative formulation of fundamental field theoretic results, yielding standard perturbative spectra and correct defects in phenomenological applications that result from patchwork introduction of interactions in particular [167],

- provide a three dimensional but covariant rearrangement of the Bethe-Salpeter equation,

- yield simple three dimensional Schrödinger like forms similar to their non-relativistic counterparts,
- implement spin dependences which are determined naturally by their incorporation of Dirac's one body structures,
- implement well defined strong potential structures that pass the necessary test that they reproduce correct QED perturbative results when solved non perturbatively,
- in phenomenological applications make unnecessary the ad hoc introduction of cutoff parameters generally used to avoid singular potentials and
- have relativistic potentials which may be related directly to the interactions of perturbative quantum field theory (for QCD) or may be introduced semiphenomenologically.

These equations provide a non perturbative or strong potential framework for extrapolating perturbative field theoretic results into the highly relativistic regime of bound light particles in a quantum mechanically well defined way.

Constrained systems are widely investigated in modern physics. Gauge field theories, quantum gravity and supergravity, string and superstring theories are examples of systems with constraints. Only few models are exactly solvable, realistic physical theories require approximate methods. A perturbation theory is one of such techniques, it is usually applied to constrained field systems such as gauge theories.

In Chapt. 6, we discussed three NN interaction models, well distinguished by radial regimes, the long range meson exchange model (OBEP, OPEP) [393, 401], the short range Effective field theory (EFT) [561, 351] and Optical model (OMP) [275, 278, 489]. We did not use EFT in this work, but it is an alternative method to describe interacting particles. A very brief introduction to the main EFT ideas is presented in Chapt. 3.1. We know, that the particle dynamics on the level of relativistic quantum mechanics stands between relativistic quantum field theory and nonrelativistic quantum mechanics. This means, EFT exploits the existence of scales in interacting systems.

EFT [561, 562] is a theoretical prescription for constructing theories spanning multiple energy scales. The physics of a system may appear radically different at various energy scales, due to low energy restrictions on available degrees of freedom and symmetries. When trying to construct a theory which spans energy scales, traditional methods of physics can be difficult to apply. Rather than stumbling on this obstacle, however, EFT provides a method to use the physical difference between energy regimes to its advantage.

How to implement EFT in nuclear physics was originally proposed by Weinberg [562] and employed by Lepage in Ref. [379]. It allows to think straightforward about the connection of EFT and potential models. As discussed in Sec.3.1, the focus is on power counting at the level of the Hamiltonian in the NN piece of the hadronic Hilbert space. The effective Lagrangian consists of NN contact interactions together with the standard heavy baryon chiral perturbation theory Lagrangian

$$L = L_{HB} + L_{NN}.$$

To connect the two energy scales, the consensus of the majority of the nuclear physics community holds that in nuclei

- nucleons are nonrelativistic,
- they interact via essentially two body forces, with small contributions from many body forces,
- the two nucleon interaction generally possesses a high degree of isospin symmetry,
- external probes usually interact with mainly one nucleon at a time.

By contrast, in QCD

- the u and d quarks are relativistic,
- the interaction is manifestly multi body, involving exchange of multiple gluons,
- there is no obvious isospin symmetry,
- external probes can and often do interact with many quarks at once.

It should not be surprising that some new ideas are required to merge these two extraordinarily different bodies of theory. Of course, we expect that QCD encompasses the physics of hadronic interactions. The root of the problem must therefore lie in the difference of energy scales.

In fact, constructing a QCD-based theory of the hadronic phase is a problem which involves three separate energy scales spanning three orders of magnitude. The first is the typical energy scale of QCD,

$$M_{QCD} \sim 1 \text{ GeV}. \tag{1.1}$$

The masses of all hadrons, except the pion, fall within this scale,[1] and the scale of chiral symmetry breaking is thought to be $M_\chi = 4\pi f_\pi$, where $f_\pi \simeq 93$ MeV is the pion decay constant. The second scale,

$$M_{nuc} \sim 100 \text{ MeV}, \tag{1.2}$$

[1]We use units where $\hbar = c = 1$.

represents the typical momentum of nucleons in a nucleus: the inverse *root mean square* charge radius of light nuclei, or the Fermi momentum of equilibrium nuclear matter. It contains also the pion decay constant itself, the mass difference between the delta isobar and the nucleon, and the mass of the pion. The final energy scale is the typical energy scale of a nucleus,

$$\frac{M_{nuc}^2}{M_{QCD}} \sim 10 \text{ MeV}. \tag{1.3}$$

The binding energy per nucleon of a nucleus is typically a few MeV. For example, the binding energy of ^{4}He is 28.296 MeV, and the binding energy per nucleon of infinite nuclear matter is 16 MeV or the binding energy of the deuteron (we calculate in Sec. 7.4) is $\sim 2.2245 MeV$. Since the goal is to construct a theory valid over an energy range of three orders of magnitude, and spanning regimes in which the physics is quite disparate, the problem is somewhat daunting.

We need coordinates, corresponding to the quantum zero point energy climbing to arbitrarily high values as $k \to \infty$ or $r < r_c$. If a singular potential itself is not sufficient to determine the scattering problem, one might be tempted to classify the singular potentials as nonrenormalizable and abandon all hope. In the modern version of the renormalization paradigm a low energy system with a clear cut separation of scales can be described by an EFT involving explicitly only the long wavelength degrees of freedom, and organized as an expansion in powers of momenta [350]. The short range dynamics can always be treated as a set of local operators. In the present context, the Dirac potential represents the long distance part of the potential. Local operators in momentum space correspond to OMP interactions in coordinate space. The essential point of EFT is that the details of the short distance physics are not of importance to low energy scattering. Hence one can simulate the delta function in an infinite number of ways. The simplest choice of a *smeared out* delta function is a simple surface Gaussian square well or Woods-Saxon shape. With a single potential, representing a given long distance force and a surface Gaussian representing unknown short distance physics, an interesting question is whether one can obtain an EFT with well defined low energy scattering observable, which are to a specified degree of accuracy insensitive to the short distance physics encoded by the Gaussian. It is the purpose of the optical model (Chapt. 6) to explore this issue. Note that we did not attempt to renormalize the coupling strength of the potential itself [107]. In the physical problems of interest, the coupling strength is completely determined by the long distance physics so there is no freedom to renormalize this parameter.

In the Sec. 6.3, two existing global medium energy NN phenomenological optical model potentials (OMP) such as vector and scalar are described and compared with experiment and with each other. By an optical potential we also mean a potential that represents the interaction between a nucleon or group of nucleons and a nucleus. When inserted into the Schrödinger equation it gives the differential cross section and polarization for elastic scattering the rection cross section and some other less important observable quantities [275]. These optical potential can be obtained in several ways. Most fundamentally, they can be calculated from the NN interaction, this is difficult and has only recently been brought to the stage of quantitative success. Alternatively, these optical potential may be found phenomenological by postulation a form of potential and adjusting its parameters to optimize the fit to the experimental data. Such phenomenological analyses may be put on a firmer physical basis by using additional information from nuclear models.

Optical potentials may also be classified by the range of experimental data they are designed to fit. Global potentials give good overall fits to the scattering from many nuclei over a range of energies. More precise fits can be obtained for particular interactions either by incorporating nuclear structure and nuclear reaction information into the calculation of the potential to fit particular sets of data.

There are many ambiguities in optical model potentials. These are familiar in phenomenological analyses, where it is often found that several potentials fit the same data equally well. It is usually thought that one of these is the *physical* potential, namely the one that is given by a microscopic calculation. It is important to identify the *physical* potential, that can be used with more confidence in situations different form those from which it was obtained. It should however by noticed that the *physical* potential may have a form that is significantly different from any of the phenomenological potentials. Furthermore, a potential is of its nature a theoretical construct, so care is necessary to describe it as *physical*. It is possible that even among microscopic potentials might be ambiguities in the sense that different types of calculations could conceivably give different potentials that nevertheless give equally good fits to the data [326].

By a microscopic optical potential we mean a potential calculated from the NN interaction and some nuclear properties, usually the density distribution. Such potentials are distinguished from phenomenological potentials which are obtained by direct fitting to experimental elastic scattering data.

The Dirac equation is used in the mean field approximation by which the nucleon (meson) fields are replaced by their expectation values. Proton nucleon (np) scattering is then described using isoscalar scalar and isoscalar vector mean fields. Here these are taken, respectively, as a optical Gaussian type scalar potential corresponding to the (fictitious) σ meson field and optical Gaussian type vector potential (timelike vector) corresponding to the meson ω meson field, together with a Coulomb and full relativistic Dirac (including relativistic kinematics) potential. The determination of the energy range, energy dependence, and isospin dependence are discussed in the last chapter.

The experimental background and motivation for analysis using an optical model and full relativistic Dirac equations is given in Chapt. 7. The Arndt [18] group has been supplies the experimental analysis of elastic NN scattering phase shift data to 3 GeV in the laboratory of kinetic energy. The necessary numerics for studying *np* and *pp* scattering phase shifts for energies 0–3 GeV and the deuteron bound state are developed in Chapt. 7.

The interactions are inspired by meson exchange of π, η, ρ, ω and σ mesons for which we adjust coupling constants. This yields, in the first instant, high quality fits to the Arndt [18] phase shifts 0–300 MeV. Second, the potentials show a universal, independent from angular momentum, core potential of which is generated from the relativistic meson exchange dynamics. Extrapolations towards higher energies, up to $T_{Lab} = 3\,\mathrm{GeV}$, allow to separate a QCD dominated short range zone as well as inelastic nucleon excitation mechanism contributing to meson production. A local and/or nonlocal optical model, in addition to the meson exchange Dirac potential, produces agreement between theoretical and data phase shifts. The result of our calculation of the Dirac equations are given in Sec. 7.3. The optical potentials are complex and short ranged, typically of nucleon size, that is known from analysis of electron scattering off a nucleon. This implies that the production processes are localized at and within the confinement surface of a nucleon. In Sec. 7.6 we display a geometry of the profile function, known from high energy diffraction scattering, which remains valid at lower energies and in the resonance dominated region. It is this result that lead us to expect that meson production is a unique QCD aspect applicable from (300 MeV) threshold up to highest energies.

The coupled two body Dirac equations, combined with the meson exchange model, yield the appearance of a repulsive practically hard core potential independent of partial wave. The universal core radius has a value $r_c = 0.5 \pm 0.025\,\mathrm{fm}$. This core radius is independent of a nucleon substructure. It depends only on masses, in particular of the exchanged mesons, and the full relativistic treatment of the NN system [280]. The meson exchange *Dirac potential*, which is described by the NN Dirac instant form dynamics, should ultimately be *limited to* $r \geq r_c \sim 0.5$ fm *in its effect*. This constraint eliminates the need for regularization of the short range Dirac potential and boundary conditions are automatically generated by the $\delta(r - r_c)$ $NN \leftrightarrow BB$ transition potentials [280]. The role of the higher partial waves deserves special attention, in particular when the hard core radius is sizeable [280].

Chapter 2

The Relativistic Particle Problems

A concept of *relativistic many body Hamiltonian dynamics* is notoriously difficult. All relativistic classical two body problems face the essential problem of separating center of momentum (CM) and relative motion. They do lead to uncoupled systems of quasi particles, of reduced mass and combined mass.

Dirac's famous paper of 1949 and there after [190, 191, 193], and the work of Bakamjian and Thomas [27] are remarkable solutions to the relativistic dynamical problem for classical and quantum mechanical systems. Poincaré invariance determines the relativistic Hamiltonian dynamics. Dirac described three different forms of Hamiltonian dynamics instant form, front form (light front form) and point form. Two other forms, suggested by Leutwyler and Stern [382], did not bring any significant extension when compared with the three general Dirac forms.

The front form has already been used in various fields including both quantum mechanics and quantum field theory. It has been successful especially in deep inelastic scattering, where the kinematic variables of phenomenological approaches coincide with the specific variables of the front form. Although also the point form was at first considered for an application in quantum field theory, it was put aside after people realized that its quantization surface (a spacetime hyperboloid) entails difficulties. However, for a quantum mechanical treatment, the point form yields advantages.

The Bakamjian Thomas construction provides for any of the different forms a simple way how to include interactions in the Poincaré generators by imposing only linear conditions on the potential. In the point form, all interactions are put into the four momentum operator, whereas the Lorentz generators remain free of interactions and are kinematical. As a result, the point form is manifestly Lorentz covariant. The dynamical equations, replacing the Schrödinger equation, are then in general the eigenvalue equations for the four components of the four momentum operator. Within the Bakamjian-Thomas framework the problem is reduced to one eigenvalue equation for the invariant mass operator.

2.1 The Poincaré Transformation

From a group theoretical point of view, the transformations of the system of reference include translation, rotations and the Lorentz transformations which are forming the Poincaré group. Under the infinitesimal transformation g of the coordinate system, with the translation a^μ and with the four dimensional rotation $\varepsilon^{\nu\mu}$,

$$x^\mu \rightarrow x'^\mu = x^\mu + a^\mu + \varepsilon^{\mu\nu} x_\nu, \tag{2.1}$$

with

$$\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}.$$

The state vector ϕ is transformed as follows

$$\phi \rightarrow \phi' = U(g)\phi, \tag{2.2}$$

where

$$U(g) = 1 + iP_\mu a^\mu + \frac{i}{2} J_{\mu\nu}\varepsilon^{\mu\nu}, \tag{2.3}$$

$$J_{\mu\nu} = -J_{\nu\mu}. \tag{2.4}$$

Here we define the metric tensor $g_{\mu\nu} = (1, -1, -1, -1)$. Once the set of infinitesimal generators are known, every finite element of the group can be expressed in terms of generators by exponentiation. In principle, the choice of a set of generators is not unique, since any linear combination of generators is again a generator so that one can replace some of them in favor of others. A standard choice, according to the parameters specified above, by setting $K^j = J^{0j}$, is

- P^μ generator of space time translations,
- K^j generator of Lorentz boosts,
- J^j generator of space rotations.

Here, $\mu = 0, 1, 2, 3$ and $j = 1, 2, 3$. The generators satisfy a set of commutation relations which is called the Lie algebra of the corresponding Lie group. Every set of generators forms a Lie algebra with different structure constants. Standard commutation relations are [352]

$$\begin{aligned} &[J^j, J^k] = i\epsilon^{jkl} J^l, \quad [K^j, K^k] = -i\epsilon^{jkl} J^l, \quad [J^j, K^k] = i\epsilon^{jkl} K^l, \\ &[P^\mu, P^\nu] = 0, \qquad [K^j, P^k] = i\delta^{jk} P^0, \qquad [J^j, P^k] = i\epsilon^{jkl} P^l, \\ &[K^j, P^0] = -iP^j, \quad [J^j, P^0] = 0, \end{aligned} \tag{2.5}$$

where ϵ^{jkl} and δ^{jk} are the Levi-Civita and Kronecker symbols, respectively. The commutation relation for the components of the four momentum are already written in a covariant way. This can also be done for the rest of the relations, if one realizes that under a Lorentz transformation the K^j and J^j transform like the six independent components of an antisymmetric tensor of rank 2. By setting $J^{0j} = K^j$, and $J^{jk} = \epsilon^{jkl} J^l$, one arrives at the covariant set of commutation relations of the generators of the Poincaré group

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \\ \frac{1}{i}[P_\mu, J_{\kappa\rho}] &= g_{\mu\rho} P_\kappa - g_{\mu\kappa} P_\rho \ , \\ \frac{1}{i}[J_{\mu\nu}, J_{\rho\gamma}] &= g_{\mu\rho} J_{\nu\gamma} - g_{\nu\rho} J_{\mu\gamma} + g_{\nu\gamma} J_{\mu\rho} - g_{\mu\gamma} J_{\nu\rho}. \end{aligned} \tag{2.6}$$

An important observation here is that now the metric $g_{\mu\nu}$ provides the structure constants of the Lie-Algebra.

2.1.1 Casimir Operators of the Poincaré Group

Having the generators and their commutation relations one can start to construct operators as functions of the generators. We have already mentioned that any linear combination of group generators is again a generator of the group. Another interesting issue is to construct the Casimir operators of the group [140, 141]. These are generator polynomials that commute with all of the generators. The Casimir operators are also important, because their eigenvalues provide quantum numbers to label the representation in use. In general for physical systems one wants to have a set of commuting self-adjoint operators, whose eigenvalues characterize the state of the system. In this sense we want to determine the mass and spin of our system of interest and construct the corresponding mass and spin operators.

The Mass Operator: In analogy to the mass m of a particle in classical relativistic mechanics, one defines the square of the mass operator M as the scalar product of the four momentum operator P^μ with itself

$$M^2 = P^\mu P_\mu = H^2 - \overrightarrow{P}^2. \tag{2.7}$$

M^2 commutes with all of the Poincaré generators, so it is a Casimir operator of the Poincaré group. If M^2 satisfies the spectral condition, i.e. $M^2 \geq 0$, then one can define its square root and write

$$M = \sqrt{H^2 - \overrightarrow{P}^2}, \tag{2.8}$$

with these relations one can now express the Hamiltonian in terms of $\overrightarrow{P}$ and M

$$P^0 = H = \sqrt{M^2 + \overrightarrow{P}^2}. \tag{2.9}$$

The Pauli-Lubanski Spin Operator : The second Casimir operator is constructed from the Pauli-Lubanski pseudovector

$$W^\mu := -\frac{1}{2}\epsilon^{\mu\nu\rho\gamma} P_\nu J_{\rho\gamma}. \tag{2.10}$$

The square of W^μ is proportional to the square of the total intrinsic spin

$$W^\mu W_\mu = -M^2 \overrightarrow{S}^2. \tag{2.11}$$

Having in mind that $W^\mu W_\mu$ commutes with all of the generators of the group, one still has to derive the commutation relations for the components of W^μ with the generators and with themselves. For that purpose it is useful to write down the components of W^μ explicitly

$$W^0 = \overrightarrow{P} \cdot \overrightarrow{J} \qquad \text{and} \qquad \overrightarrow{W} = H\overrightarrow{J} - \overrightarrow{P} \times \overrightarrow{K}, \tag{2.12}$$

we also note that W^0 and $\overrightarrow{W}$ are self adjoint. The commutation relations now are

$$\begin{aligned} &[J^j, W^0] = 0, \qquad [J^j, W^k] = i\epsilon^{jkl} W^l, \qquad [K^j, W^0] = -iW^j, \\ &[K^j, W^k] = -i\delta^{jk} W^0, \quad [P^\mu, W^\nu] = 0. \end{aligned} \tag{2.13}$$

If we compare these commutation relations for W^μ to those for the components of P^μ, we see that they are the same, meaning that W^μ transforms like a four vector under Lorentz transformations. Similar to the angular momentum operator, the components of the Pauli-Lubanski operator do not commute

$$[W^\mu, W^\nu] = i\epsilon^{\mu\nu\rho\sigma} W_\rho P_\sigma.$$

A case where $P^\mu = (M, 0)$, the Pauli-Lubanski vector $W^\mu = (W^0, \overrightarrow{W})$, is given by

$$W^0 = 0,\ W_i = \frac{1}{2} M \epsilon_{ijk} J^{kj} = M S_i. \tag{2.14}$$

and $M^2 = 0$. Then since

$$P_\mu P^\mu = W_\mu W^\mu = W_\mu P^\mu = 0, \tag{2.15}$$

W_μ and P_μ must be proportional to each other,

$$W_\mu = h P_\mu,$$

where the *helicity* h is given by

$$h = \frac{\overrightarrow{P} \cdot \overrightarrow{J}}{|\overrightarrow{P}|}. \tag{2.16}$$

The Newton-Wigner Position Operator: Bakamjian and Thomas constructed another operator, namely the so called Newton-Wigner position operator. The construction of this operator is the result of the need to have a hermitian position operator in relativistic quantum mechanics. Newton and Wigner have constructed this operator from the hermiticity requirement [432]. The definition of the operator in the instant form of relativistic dynamics makes it also possible to write the operator of total angular momentum as the usual sum of orbital angular momentum defined by the cross product of $\overrightarrow{X}_c$ and $\overrightarrow{P}$ and intrinsic angular momentum (spin). The operator reads explicitly for the case of canonical spin

$$\overrightarrow{X}_c = \frac{1}{2}[\frac{1}{H}, K]_+ - \frac{\overrightarrow{P} \times (H\overrightarrow{J} - \overrightarrow{P} \times \overrightarrow{K})}{MH(H+M)}. \tag{2.17}$$

The state vector Eq. (2.2) $\phi^{J\lambda}(p)$ corresponding to a system with definite four momentum p_μ, mass M, total angular momentum J and its projection λ to the z-axis, the following system holds

$$\begin{aligned}
P_\mu \, \phi^{J\lambda}(p) &= p_\mu \, \phi^{J\lambda}(p), \\
P^2 \, \phi^{J\lambda}(p) &= M^2 \, \phi^{J\lambda}(p), \\
S^2 \, \phi^{J\lambda}(p) &= -M^2 \, J(J+1) \, \phi^{J\lambda}(p), \\
S_3 \, \phi^{J\lambda}(p) &= M \, \lambda\phi^{J\lambda}(p).
\end{aligned} \tag{2.18}$$

A particular dynamical system is determined by the explicit form of these generators, i.e., by a particular solution of the commutation relations Eq. (2.6). If these generators are expressed in terms of the particle coordinates, we get a version of relativistic quantum mechanics with fixed number of particles. If the generators are expressed through the quantum fields, we obtain a form of the quantum field theory. As soon as the generators are fixed the state vector is determined by Eq. (2.18). For an interacting system some Poincaré generators contain the interaction. Namely, the generators changing the position of the surface, where the state vector is defined, contain the interaction. The generators, which do not change the position of the surface, do not contain the interaction and coincide with the generators of free system. Using this property, one can classify different forms of dynamics.

In Sec. 2.4 we consider the three Dirac forms of the relativistic dynamics and introduce the question how to add interactions to the operators that describe the system of interest. Before we use these forms in our approach , we review highlights of the constraint formalism and it's quantization.

2.2 Dirac's Hamiltonian Approach with Constraints

2.2.1 Classical Nonrelativistic Lagrangian Formalism

Here, we only consider closed systems whose dynamics derive from an action principle. We shall assume that the corresponding action is given as an integral over *time* of a local function of the degrees of freedom, known as the Lagrangian function. By *time*, we mean a variable parameterising time evolution of the system, and not necessarily the time as measured by and observer. We shall first consider the case of a system with a finite number of degrees of freedom, given as N functions $q_n(t)$, $n = 1, 2..., N$ of the *time* variable t. With the restrictions specified above, we thus assume that the dynamics of the system is described by an action functional of the form

$$\mathcal{A} = \int_{t_1}^{t_2} dt L(q_n, \dot{q}_n, t). \tag{2.19}$$

Here, $L(q_n, \dot{q}_n, t)$ is a Lagrange function depending on generalized coordinates q_n and generalized velocities $\dot{q}_n = \frac{d}{dt} q_n$ and time t.

The time evolution of the classical system is then obtained from the Euler-Lagrange equations of motion. These equations follow from action by the variational principle, which states that the classical trajectories $q_n(t)$ of the system are those for which the action is a stationary point of $\mathcal{A}$. A necessary condition for their existence is the vanishing of the first variation of the action, $\delta\mathcal{A} = 0$, and we have for $t_1 < t < t_2$,

$$\delta\mathcal{A} = \int_{t_1}^{t_2} dt \left\{ \left[\frac{\partial L}{\partial q_n} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} \right] \delta q_n + \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}_n} \delta q_n \right] \right\}, \tag{2.20}$$

which leads to the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} - \frac{\partial L}{\partial q_n} = 0, \tag{2.21}$$

or explicitly

$$\frac{\partial^2 L}{\partial \dot{q}_{n_1} \partial \dot{q}_{n_2}} \ddot{q}_{n_2} + \frac{\partial^2 L}{\partial \dot{q}_{n_1} \partial q_{n_2}} \dot{q}_{n_2} - \frac{\partial L}{\partial q_{n_1}} = 0. \tag{2.22}$$

Hence, time evolution of the system is described by a set of N linearly independent second order differential equations, provided the Hessian of the Lagrange function has non vanishing determinant

$$Det \frac{\partial^2 L}{\partial \dot{q}_{n_1} \partial \dot{q}_{n_2}} \neq 0.$$

The Lagrangian is called singular if the Hessian vanishes, and regular otherwise.

In the case of a regular system, the general solution of Eq. (2.22) is given in terms of (2N) integration constants for the N functions $q_n(t)$.

2.2.2 Classical Nonrelativistic Hamiltonian Mechanics

In addition to the Lagrange formulation of classical mechanics there exists a formulation due to Hamilton, the Hamiltonian formalism corresponding to the Lagrangian [423, 296, 526, 527]. We will recapitulate this briefly in order to point out where the peculiarities of *constraint dynamics* arise. Consider the differential of the Lagrange function

$$dL = dq_n \frac{\partial L}{\partial q_n} + d\dot{q}_n \frac{\partial L}{\partial \dot{q}_n} = dq_n \frac{\partial L}{\partial q_n} + d\left(\dot{q}_n \frac{\partial L}{\partial \dot{q}_n}\right) - \dot{q}_n d\left(\frac{\partial L}{\partial \dot{q}_n}\right), \tag{2.23}$$

leading to the relation

$$d\left(\dot{q}_n \frac{\partial L}{\partial \dot{q}_n} - L\right) = -dq_n \frac{\partial L}{\partial q_n} + d\left(\dot{q}_n \frac{\partial L}{\partial \dot{q}_n}\right). \tag{2.24}$$

Thus, defining the generalised momentum p_n conjugate to q_n as

$$p_n := \frac{\partial L}{\partial \dot{q}_n}, \tag{2.25}$$

and the canonical Hamiltonian H by

$$H := \dot{q}_n p_n - L, \tag{2.26}$$

the relation Eq. (2.24) takes the form

$$dH = -dq_n \frac{\partial L}{\partial q_n} + \dot{q}_n dp_n, \tag{2.27}$$

or by using the Euler-Lagrange equations of motion Eq. 2.21

$$dH = -dq_n \dot{p}_n + \dot{q}_n dp_n. \tag{2.28}$$

The result Eq. (2.27) shows that, although the canonical Hamiltonian is a priori a function of $(q_n, \dot{q}_n)$, it actually depends on these variables only through the phase space variables q_n and p_n. Therefore, H is actually a function of q_n and p_n, where p_n in turn depends on q_n and $\dot{q}_n$. Note that this result applies to regular as well as singular systems.

However, only in the case of a regular system is the map from velocity space to momentum space invertible. Indeed, such a one-to-one correspondence requires

$$Det\left(\frac{\partial p_{n_1}}{\partial \dot{q}_{n_2}}\right) \neq 0.$$

Hence, only for such systems can the dependence $p_n(q_n, \dot{q}_n)$ be inverted uniquely as $\dot{q}_n(q_n, p_n)$, in which case the definition Eq. (2.26) corresponds to a Legendre transform of the Lagrange function $L(q_n, \dot{q}_n)$. From Eq. (2.28), we then read off the Hamiltonian equations of motion

$$\dot{q}_n = \frac{\partial H}{\partial p_n}, \qquad \dot{p}_n = -\frac{\partial H}{\partial q_n}. \tag{2.29}$$

These are $(2N)$ coupled first order (in time) differential equations, whose general solution is determined in terms of $(2N)$ integration constants. These integration constants can be specified in terms of $(2N)$ boundary conditions involving $q_n(t)$ and $p_n(t)$, but here again, appropriate and consistent choices are to be made depending on the system and possibly other physical considerations.

From the Hamiltonian equations of motion Eq. (2.29), we have the following equation of motion for any function of phase space $f(q_n, p_n; t)$, with a possible explicit time dependence

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_n}\frac{\partial H}{\partial p_n} - \frac{\partial f}{\partial p_n}\frac{\partial H}{\partial q_n}. \tag{2.30}$$

Defining the Poisson bracket of two functions f and g on phase space by

$$\{f, g\} := \frac{\partial f}{\partial q_n}\frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n}\frac{\partial g}{\partial q_n}, \tag{2.31}$$

Eq. (2.30) takes the form

$$\dot{f} = \frac{\partial f}{\partial t} + \{f, H\}. \tag{2.32}$$

In this notation, the Hamiltonian equation of motion are

$$\dot{q}_n = \{q_n, H\}, \qquad \dot{p} = \{p_n, H\}. \tag{2.33}$$

Although at this point, it seems that these Poisson brackets simply provide a convenient notation, they actually play a fundamental role in classical as well as quantum mechanics.

From the definition Eq. (2.31), it is easy to establish the following properties of Poisson brackets (f, g and h are functions of phase space, and c_1, c_2 and c are constants) [289]

- antisymmetry: $\{f, g\} = -\{g, f\}$,
- linearity: $\{c_1 f + c_2 g, h\} = c_1\{f, h\} + c_2\{g, h\}$,
- existence of null elements: $\{c, f\} = 0$,
- the Jacoby identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$,
- the product rule: $\{fg, h\} = f\{g, h\} + \{f, h\}g$.

Finally, we have the fundamental Poisson brackets

$$\{q_{n_1}, q_{n_2}\} = 0,\ \{p_{n_1}, p_{n_2}\} = 0,\ \{q_{n_1}, p_{n_2}\} = \delta_{n_1,n_2}, \tag{2.34}$$

from which any Poisson bracket can be obtained using the properties above.

2.2.3 Nonrelativistic Constraint Hamiltonian Dynamics

Let us now consider a dynamical system with N degrees of freedom and with a Lagrangian $L(q_1, q_2, ..., q_N, \dot{q}_1, \dot{q}_2, ..., \dot{q}_N) = L(q, \dot{q})$. The Lagrangian is singular if

$$Det \left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right) = 0. \tag{2.35}$$

It means that, when the Lagrangian equations of motion are written in suitable coordinates, the coefficient of at least one $\ddot{q}$ is zero. Also Eq. (2.35) means that the equations given by Eq. (2.25) cannot be solved uniquely for each of the velocities $\dot{q}$ in terms of q, p. The N dimensional configuration space with coordinates q_r leads automatically to two $2N$ dimensional spaces, one is the tangent bundle of Lagrangian coordinates and velocities q_r, $\dot{q}_r$, and the other is the cotangent bundle of phase space of coordinates and momenta q_r, p_r. If the p's involve only $N-M$ independent functions of the $\dot{q}$'s, there will be M independent constraining relations

$$\phi_m(q, p) \approx 0, \; m = 1, 2, ..., M. \tag{2.36}$$

The special *weak* equality sign $\approx$ is a reminder that these relations are, of course, not valid over the entire $2N$ dimensional phase space but the solution pair (q, p) [191, 194, 294, 283]. The relations Eq. (2.36) are called *primary constraints* of the Hamiltonian formalism [525, 527, 526]. Now, we consider the quantity of $p_n\dot{q}_n - L$.

If we make variations for $p_n\dot{q}_n - L$, in the variables q and $\dot{q}$, in the coordinates and the velocities, then variations will cause variations to occur in the momentum variables p. This means that

$$\begin{aligned}
\delta(p_n\dot{q}_n - L) &= p_n\delta\dot{q}_n + \dot{q}_n\delta p_n - \frac{\partial L}{\partial q_n}\delta q_n - \frac{\partial L}{\partial \dot{q}_n}\delta\dot{q}_n \\
&= \dot{q}_n\delta p_n - \frac{\partial L}{\partial q_n}\delta q_n + \delta\dot{q}_n\left(p_n - \frac{\partial L}{\partial \dot{q}_n}\right) \\
&= \dot{q}_n\delta p_n - \frac{\partial L}{\partial q_n}\delta q_n.
\end{aligned} \tag{2.37}$$

the variation is not independent of the velocities. However, we may add to it any linear combination of the ϕ's Eq. 2.36, which are zero. The Hamiltonian, we may change to an other Hamiltonian

$$\mathcal{H} = H + \lambda_m\phi_m, \tag{2.38}$$

where the λ_m are any functions of the q's and p's.

Eq. (2.37) now gives

$$\frac{\partial \mathcal{H}}{\partial q_n}\delta q_n + \frac{\partial \mathcal{H}}{\partial p_n}\delta p_n = \dot{q}_n\delta p_n - \frac{\partial L}{\partial q_n}\delta q_n. \tag{2.39}$$

for variations δq_n, δp_n. It follows that

$$\begin{aligned} \dot{q}_n &= \frac{\partial H}{\partial p_n} + \lambda_m \frac{\partial \phi_m}{\partial p_n}, \\ -\frac{\partial L}{\partial q_n} &= \frac{\partial H}{\partial q_n} + \lambda_m \frac{\partial \phi_m}{\partial q_n}, \end{aligned}$$

and

$$\dot{p}_n = -\frac{\partial H}{\partial q_n} - \lambda_m \frac{\partial \phi_m}{\partial q_n}. \quad (2.40)$$

From definition Eq. (2.31), one can rewrite the equations of motion. For any function $g(q, p)$, we have

$$\begin{aligned} \dot{g} &= \{g, \mathcal{H}\} = \{g, H + \lambda_n \phi_n\} = \{g, H\} + \{g, \lambda_n \phi_n\} \\ &= \{g, H\} + \lambda_n \{g, \phi_n\} + \{g, \lambda_n\} \phi_n \\ &= \{g, H\} + \lambda_n \{g, \phi_n\}. \end{aligned} \quad (2.41)$$

We have the constraints Eq. (2.36), but must not use one of these constraints before working out a Poisson bracket.

An important consequence of these definitions Eq. (2.36) is the following

- the constraint Eq. (2.36) is trivially satisfied $0 = 0$,
- the constraint Eq. (2.36) is independent of λ_m, $\{\phi_m, \phi_n\} \approx 0$, determine new constraint $\chi(q_n, p_n) = 0$ so called secondary constraint,
- the constraint Eq. (2.36) defines an equation to be satisfied by the function λ_m.

This defines a linear vector space (due to the linearity of the Poisson brackets) and so any linear combination of constraints is again a constraint.

It is of great importance for our purposes the distinction between *first class* and *second class* constraints. The first are defined as the constraints which *commute* (i.e. have vanishing Poisson brackets) with all the other constraints. The second ones have at least one non vanishing bracket with some other constraint. It may happen that we can take linear combinations of *second class* constraints and obtain some *first class* constraints. Dirac showed the profound difference between this two classes in Ref. [194]. Any quantity, which is weakly equal to zero is necessarily strongly equal to a linear combination of the constraints, provided the latter are regular constraints.

2.3 Relativistic Kinematics

Particle physics utilizes relativistic kinematics in many ways: to relate energies, momenta and scattering angles in different frames of reference, to deduce the masses of unstable particles from measurements on their decay products, to work out threshold energies for production of new particles, and in a variety of other applications [408]. The problem we want to discuss here is what happens when we replace Galilean spacetime with Minkowski spacetime. All observables can be expressed in terms of invariants. The most exhaustive treatment of the subject of relativistic kinematic, is given in Ref. [104, 315], considerable interest in experimental high energy physics. While the force between particles are only imperfectly known, and are certainly for from classical, so long as the particles involved in a reaction are outside the region of mutual interaction their mean motion can be described by classical mechanics. Further, the main principle involved in the transformations conservation of the four vector of momentum is valid in both classical and quantum mechanics.

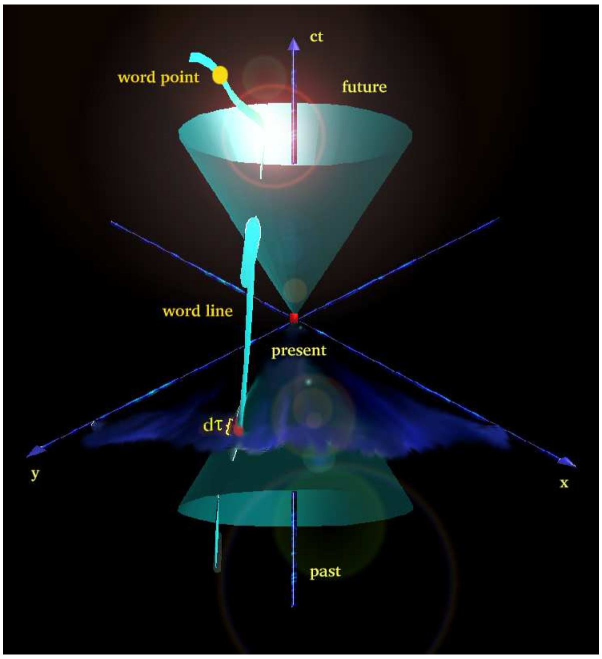


Figure 2.1: The light cone.

1. Let us consider the invariant of the four dimensional line element between two events $\overrightarrow{x}_1$ and $\overrightarrow{x}_2$ with components

$$ds^2 = cdt^2 - d\overrightarrow{x}^2, \tag{2.42}$$

where cdt timelike interval and Cartesian coordinates $d\overrightarrow{x} = \{dx, dy, dz\}$. It has become usual to talk about three different types of intervals

- $ds^2 \leq 0$, (spacelike)

- $ds^2 \geq 0$, (timelike)
- $ds^2 \equiv 0$, (lightlike).

A spacetime diagram shown in Fig. 2.1, often called a Minkowski diagram, is a geometric representation of motions in spacetime [512]. The vertical axis is usually plotted as the time axis. For timelike intervals we can further distinguish

- $s > 0$, (future)
- $s < 0$, (past)
- $s = 0$. (present or elsewhere)

The future cone represents those areas where s is positive and ct is also positive. The past cone comprises spacetime points with s positive and with ct negative. Those volumes outside the two cones represent what is sometimes called elsewhere, since they are events for which the metric s is imaginary.

Depending upon the situation one sometimes plots a single spatial dimension, or two spatial dimensions. Any point in spacetime is called a world point, and a series of world points representing the motion of some object is called a world line Fig. 2.1.

2. We have the following Lorentz transformation between the coordinates in the two systems

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}. \tag{2.43}$$

4-Vector
The prototype four vector is the position vector in spacetime, representing an event

$$\mathbf{s} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix},$$

contravariant, with components x^μ, and

$$\mathbf{s} = (-ct\, x\, y\, z),$$

covariant with components x_μ.

We are rewriting the invariant squared length Eq. (2.42)

$$d\mathbf{s}^2 = (-cdt\, dx\, dy\, dz)\begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix} = -(cdt)^2 + dx^2 + dy^2 + dz^2 = g_{\mu\nu}dx^\mu dx^\nu,$$

where $g_{\mu\nu}$ are tensor second rank with components

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.44}$$

say *metric tensor* defined as the matrix inverse of the spacetime metric $g_{\mu\nu}$, so it has components $g^{\mu\nu}$ so that

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho.$$

Along timelike directions the invariant Fig. 2.1

$$d\tau^2 = -ds^2/c^2 = dt^2 - (dx^2 + dy^2 + dz^2)/c^2 \tag{2.45}$$

is positive. Its square root is the element of proper time (chosen positive in the future pointing direction). For neighboring events on a timelike curve coincides with the *time differential* dt that is measured by a clock traveling on that curve Fig. 2.1. The proper time differential $d\tau$ and the coordinate time differential dt are related by

$$\frac{dt}{d\tau} = \gamma(v), \tag{2.46}$$

where v is the speed of the clock as measured by an observer using coordinates (ct, x, y, z).

3. The *world line* of a particle is a curve $x^\mu = x^\mu(\tau)$, parametrized by the proper time τ. The four velocity of the particle is the four vector

$$\overrightarrow{u} = \frac{dx^\mu}{d\tau}. \tag{2.47}$$

The square of any four vector is an invariant and so

$$\overrightarrow{u}^2 = V^\mu V_\nu = -c^2, \tag{2.48}$$

in any frame in *rest frame* $\overrightarrow{u}^\mu = (c, 0)$.

Associated with any particle is a quantity m_0 called its rest mass. The four momentum of a massive particle is $\overrightarrow{P} = m_0 \overrightarrow{u}$, where $\overrightarrow{u}$ is the four velocity. So

$$P^\mu = m_0 \gamma(v)(c, \overrightarrow{v}) = (mc, \overrightarrow{v}), \tag{2.49}$$

where $\overrightarrow{v}$ is usual three velocity,

$$m = \gamma(v) m_0, \qquad \text{and} \qquad \overrightarrow{p} = m\overrightarrow{v}, \tag{2.50}$$

are the relativistic inertial mass and relativistic momentum of the particle. The relativistic mass m equals the rest mass m_0 for a particle at rest and increases with speed v and $m \to \infty$ for $v \to c$.

The energy of a particle is defined as

$$E := m_0 \gamma(v) c^2 = mc^2. \tag{2.51}$$

A particle at rest has rest energy $E_0 = m_0 c^2$, so mass and energy are really equivalent! The kinetic energy of a particle is its energy due to motion,

$$W = E - E_0 = (\gamma(v) - 1) m_0 c^2. \tag{2.52}$$

The square of the four momentum is invariant,

$$P^2 = P^\mu P_\mu = -m_0^2 c^2, \tag{2.53}$$

and we can write the components of the four momentum also as

$$P^\mu = (E/c, \overrightarrow{p}) \qquad \text{and} \qquad E^2 - \overrightarrow{p}^2 c^2 = m_0^2 c^4. \tag{2.54}$$

4. In the same way that we had a transformation between spacetime coordinates, there is another Lorentz transformation between *momentum energy* coordinates, as indicated below

$$\begin{pmatrix} \frac{E'}{c} \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{E}{c} \\ p_x \\ p_y \\ p_z \end{pmatrix}. \tag{2.55}$$

We shall achieve both simultaneously and shall adopt from now on $\hbar = c = 1$.

2.3.1 Invariant Variables in the Center of Momentum

Center of Momentum (CM)
Let us first consider two particle kinematics with four momenta p_1 and p_2 and

with masses m_1 and m_2 in a Lorentz system [315]. It is possible to give the CM in terms of the three invariants

$$p_1^2 = m_1^2,\, p_2^2 = m_2^2, \qquad p_1 p_2 \qquad \text{or} \qquad (p_1 \pm p_2)^2. \tag{2.56}$$

The CM energy E is in the CM system

$$p_1^* + p_2^* = \{\epsilon_1^* + \epsilon_2^*, 0\}, \qquad \text{and} \qquad E^* = \epsilon_1^* + \epsilon_2^*. \tag{2.57}$$

Hence

$$E^{*2} = (\epsilon_1^* + \epsilon_2^*)^2 = (p_1 + p_2)^2, \tag{2.58}$$

since $(p_1 + p_2)^2$ is invariant. Now we can define the total mass M of the system by square of its total four momentum,

$$P^2 := (p_1 + p_2)^2 = M^2 = E^{*2} = (\epsilon_1 + \epsilon_2)^2 - (\overrightarrow{p}_1 + \overrightarrow{p}_2)^2 \tag{2.59}$$

or

$$\overrightarrow{P} = M\beta_{CM}\gamma_{CM}, \qquad E = M\gamma_{CM}, \tag{2.60}$$

where

$$\beta_{CM} = \frac{\overrightarrow{P}}{E} = \frac{\overrightarrow{p}_1 + \overrightarrow{p}_2}{\epsilon_1 + \epsilon_2}, \tag{2.61}$$

is the velocity of the CM in laboratory system and the corresponding γ is

$$\gamma_{CM} = \frac{E}{M}.$$

From last invariant in Eq. (2.56), we write for the Lab-system

$$p_1 p_2 = \epsilon_1 \epsilon_2 = m_1 \epsilon_2. \tag{2.62}$$

Hence

$$\epsilon_2 = E_{21}, \tag{2.63}$$

where E_{12} is the energy of the particle 2 it in the rest system of 1 and momentum

$$|\overrightarrow{p}_{21}|^2 = \frac{(p_1 p_2) - m_1^2 m_2^2}{m_1^2}, \tag{2.64}$$

and the relative velocity is

$$v_{21}^2 = \frac{|\overrightarrow{p}_{21}|^2}{E_{21}^2} = \frac{(p_1 p_2) - m_1^2 m_2^2}{(p_1 p_2)^2}. \tag{2.65}$$

Using explicitly

$$P = p_1 + p_2, \tag{2.66}$$

and

$$p_1 p_2 = \frac{1}{2}[(p_1 + p_2)^2 - p_1^2 - p_2^2] = \frac{1}{2}(M^2 - m_1^2 - m_2^2), \tag{2.67}$$

we obtain almost immediately the following expressions

$$\epsilon_1^* = \frac{M^2 + (m_1^2 - m_2^2)}{2M}, \tag{2.68a}$$

$$\epsilon_2^* = \frac{M^2 + (m_1^2 - m_2^2)}{2M}, \tag{2.68b}$$

$$\epsilon_2^* + \epsilon_1^* = M, \tag{2.68c}$$

$$|\overrightarrow{p}|^2 = |\overrightarrow{p}_1|^2 = |\overrightarrow{p}_2|^2 = \frac{M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2}{4M^2}, \tag{2.68d}$$

$$v{*}_i^2 = \left(\frac{|\overrightarrow{p}^*|}{\epsilon_i^*}\right)^2, \tag{2.68e}$$

where ϵ_i^* and v_i^* are energy and velocity, respectively, of particle i, as seen from their common CM system.

Now we shall proceed to a more complicated problem of relativistic dynamics. It would seem that only in the relativistic kinematics there are no direct experimental comparisons of physical quantities for two systems moving relative to each other, but in the relativistic dynamics everything is in order according to relativist logic the accelerators are operating. Let us try to clear up the dynamical concepts, even because the relativistic dynamics, rests upon a completely untrue relativistic kinematics.

2.4 Forms of Relativistic Dynamics

2.4.1 Including Interactions to a Hamiltonian

In relativistic classical mechanics, a problem is usually formulated in terms of a differential equation whose solution has to satisfy certain initial conditions. Causality requires only that these initial conditions have to be posed on any space like (or light like) hypersurface of Minkowski space (Sec. 2.3). In nonrelativistic mechanics initial conditions are given at a particular time $t = t_0$. The *time* evolution of the system then takes place orthogonal to the initial hypersurface chosen. In quantum mechanics the situation is similar, but one has a wave function, a solution of the dynamical equation describing the system of interest. In this case, the quantization conditions are formulated on the hypersurface chosen. In both cases are the transformation properties of the hypersurface under the elements of the Poincaré group of central importance. Dirac [190] discussed three cases of subgroups of the Poincaré group that leave hypersurfaces of Minkowski space invariant, whereas the rest of the Poincaré group does not. For each subgroup he outlines a form of relativistic dynamics. The respective subgroup is then also called the *stability group* of the corresponding form of dynamics. The generators spanning the stability group are called *kinematical*, while the others are called *dynamical* and are referred to as the *Hamiltonians* of the system. When constructing a representation of the Poincaré algebra for a system of interacting particles the kinematical generators are free of interaction terms, whereas the dynamical generators contain explicitly interactions. From this point of view the dimension of the stability group also corresponds to the number of interaction independent generators. Dirac also asked the question how interactions could be introduced in a system of free particles. He saw that the interaction terms in the Hamiltonians have to satisfy nonlinear conditions to guarantee that the interacting system is still described by a representation of the Poincaré algebra.

It is not clear from the beginning, how to add interactions to the operators. In order to describe a relativistic quantum mechanical system, one has found a set of generators of the Poincaré group that satisfies the correct commutation relations given in Eqs. (2.6). The problem can be seen best in the commutation relation

$$[P^j, K^k] = i\delta^{jk} H.$$

If interactions are added to the Hamiltonian, that means to the right hand side of this equation, then they also have to appear in one or the other part of the left hand side. This problem was noticed by Dirac [190]. In general, one can try to add interactions to H and all the other generators, even different terms for each generator, provided that the Poincaré algebra is still satisfied. Such a construction is somewhat arbitrary and not straightforward at all. Dirac restricted the examination to his three forms of dynamics, included potentials in the dynamical generators only and arrived at nonlinear conditions for the potentials, which guarantee Poincaré invariance. The differences between the three forms of dynamics are the following: including interactions only in the K's leads to the instant form, including the potential in some of the P's and some of the K's yields the front form, and including interactions only in the P's gives the point form.

So what one rather wants are only linear conditions on the potentials and a straightforward way to include given interactions in the Poincaré generators. Such a prescription was developed by Bakamjian and Thomas [27]. The Bakamjian-Thomas Construction provides a mean to add interactions to a noninteracting few body theory in a way that automatically guarantees that the resulting generators of the Poincaré group satisfy the correct commutation relations. This can be accomplished in the different forms of relativistic dynamics depending on the initial choice of the set of Poincaré generators. From each initial set, some different auxiliary operators are constructed (including the mass operator, which commutes with all generators). After the interaction potential is added to the mass operator, one gets some quite usual constraints on the potential, which ensure the validity of the commutation relations of the original generators. These can then be reconstructed from the auxiliary operators and describe the interacting theory in a Poincaré invariant manner. This yields linear constraints for the potential terms in each form of dynamics and a straightforward way for the construction of the interacting Poincaré generators.

It is quite possible to construct the Poincaré generators for one free particle. This construction can be generalized to yield operator representations for a system of two free particles. In this case the operators are constructed as a tensor product of single particle Hilbert spaces. The Poincaré generators are then sums of generators for each particle

$$P_0^\mu = P_1^\mu \otimes \mathbf{1}_2 + \mathbf{1}_1 \otimes P_2^\mu, \tag{2.69}$$

$$M_0^{\mu\nu} = M_1^{\mu\nu} \otimes \mathbf{1}_2 + \mathbf{1}_1 \otimes M_2^{\mu\nu}. \tag{2.70}$$

where the subscripts 1 and 2 denote the operators referring to the Hilbert spaces of particles 1 and 2, respectively. The subscript 0 indicates that the operator describes a system of free particles. The corresponding single particle states are combined to two particle states by Clebsch-Gordan coefficients (See Appendix.B). This construction can be done in any of the three forms of dynamics, however, the respective Clebsch-Gordan coefficients will depend on the particular form of dynamics.

The procedure to build interactions into the generators is now the following:

1. Construct the set of free two particle generators according to Eqs. (2.69) and (2.70). This set satisfies the Poincaré algebra.

2. Construct a set of auxiliary operators (depending on the form of dynamics) from the free generators with commutation relations related to the ones of the original set of generators. One of these auxiliary operators is the mass operator.

3. Add the interaction only to the mass operator without affecting the other (auxiliary) operators. This implies only linear constraints on the potential. The constraints in this case have to guarantee that the algebra of the auxiliary operators does not change.

4. Reconstruct the original set of generators from the auxiliary operators. The interaction now enters the generators via the mass operator and the Poincaré algebra is still satisfied, since the commutation relations of the auxiliary operators were not changed.

In the following subsections we will discuss Dirac's three forms of relativistic dynamics in some detail.

2.4.2 Instant Form

The laboratory observer studies the physical processes in the four dimensional space time continuum described by the coordinates $x = (t, \vec{r})$. The three dimensional space $\vec{r}$ is a hyperplane given by the equation $t = const$. An observer studies the evolution of his physical system from one plane $t = const$ to other one. The wave function $\psi(\vec{r}, t)$ of a quantum system, for a given t, is defined on this hyperplane.

This description in four dimensional space, from one fixed time plane to other one, corresponding to the different time instants is called the instant form of dynamics.

The time translations of the three dimensional plane are determined by the Hamiltonian $H = P_0$. The interaction enters also into three operators of the Lorentz transformation J_{i0}, $i = 1, 2, 3$. Indeed, two simultaneous events in one system of reference are not simultaneous ones in a moving system. Therefore, the Lorentz transformations do not leave the plane $t = const$ invariant, they change the orientation of this plane relative to the time axis. This is the reason, why the corresponding generators contain the interaction.

The other six generators, the translations and rotations inside the three dimensional space, namely, $\vec{P}$ and $\vec{J}_i = \epsilon_{ijk} J^{jk}$ coincide with the generators of the free system and are kinematical.

2.4.3 Point Form

In principle, one can define the wave function on any hyperplane in space time. Any two points of this surface cannot be connected by a light signal and, hence, an event in one of these points cannot be caused of the other one. A convenient choice is the surface of hyperboloid, $t^2 - \vec{r}^{\,2} = const$. It is invariant under Lorentz transformations. With the state vector defined on the family of these hyperboloids, we obtain the point form of dynamics.

In the point form the rotations and the Lorentz transformations do not change the hyperboloid $t^2 - \vec{r}^{\,2} = const$. Therefore all the six generators $J_{\mu\nu}$ do not contain the interaction. Whereas, the translations are much more complicated, and all the generators P_μ contain the interaction. This means that the total momentum of a system is not the sum of the particle momenta. This complicates the situation inspite of the simplification of the Lorentz boosts.

2.4.4 Front Form

The observer moving with the velocity v along z-axis describes a physical process in his coordinates (t', x', y', z'), which are related to the laboratory ones by the

Lorentz transformations:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & +v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \tag{2.71}$$

where

$$\gamma = \frac{1}{\sqrt{1-v^2}}.$$

According to Eq. (2.71), the plane $t' = const$ in the moving system corresponds to

$$(t + zv) = const,$$

in the laboratory coordinates. The evolutions are considered from one plane $t + zv = const$ to other one. Since the value of *const* is not yet specified, the factor γ can be absorbed by it. For the *null plane* we put $t' \propto t + zv = 0$. In the limiting case, when $v \to c$, we get the plane determined by the equation $t' \propto z_+ = t + z = 0$. The wave function is defined on this plane. This equation coincides with the equation for the light front $z = -t$, moving along $-z$. This is the reason, why the description in these coordinates is called the front form of dynamics, or the light front dynamics.[1]

For an arbitrary four vector a we perform the following transformation to front form coordinates

$$(a^0, a^1, a^2, a^3) \to (a^+, a^1, a^2, a^-),$$

where we have defined

$$a^+ = (a^0 + a^3)/\sqrt{2},\ a^- = (a^0 - a^3)/\sqrt{2}.$$

We also use the transverse vector part of a as

$$\mathbf{a}_\perp = (a^1, a^2).$$

In front form dynamics only three generators P_-, J_{1-}, J_{2-} do not leave the light front plane invariant and contain the interaction. Other seven generators $P_1, P_2, P_+, J_{12}, J_{-+}, J_{1+}$ and J_{2+} are the free ones.

Note also that, for a free particle, the relation between the energy and momentum

$$p_0^2 = \vec{p}^{\,2} + m^2, \tag{2.72}$$

can be rewritten in the light front coordinates as

$$2p_+p_- - \vec{p}_\perp^{\,2} = m^2, \mathrm{or} 2p_+p_- = \vec{p}_\perp^{\,2} + m^2, \tag{2.73}$$

[1]We use units with $c = 1$.

with $\vec{p}_\perp = (p_1, p_2)$. So, the light front energy p_- of a free particle is expressed by

$$p_- = \frac{\vec{p}_\perp^2 + m^2}{2p_+}, \tag{2.74}$$

where

$$p^\pm = (p^0 \pm p^3)/\sqrt{2}, \tag{2.75}$$

are the front form energy (-) and the longitudinal momentum (+). Unlike the usual instant form dispersion

$$E = \pm\sqrt{p^2 + m^2}, \tag{2.76}$$

the front form dispersion relation has a rational structure. This expression does not contain any square root, in contrast to the instant form.

The main difficulty of quantum field theory is the very complicated structure of the state vector describing the particles and even the state without any particles, the vacuum state. The state vector is usually described as a superposition of the bare quanta, corresponding to the non-interacting fields. If we *switch off* the interaction between the fields, the number of particles is conserved. As soon as we take into account the interaction, the state vector is a superposition of states with different number of particles.

If interaction is a weak, like in the case of the quantum electrodynamics, it does not change the state vector too much. Therefore, the *dressed* electron differs from the bare one only by small admixtures of photons.

The situation is drastically different when the interaction is strong. In this case, the structure of the real particle is extremely complicated. For example, the proton consists of three quarks, but these quarks are not the same quarks that appear in the initial Lagrangian of QCD. They are so called the constituent quarks which consist of bare quarks and gluons. The state vector of the proton is a complicated superposition of the bare fields. It has not yet been calculated from first principles of QCD.

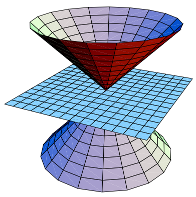
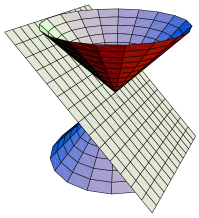
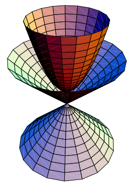
Instant Form	**Front Form**	**Point Form**
		
	Quantization Surface	
$x^0 = 0$	$x^0 + x^3 = 0$	$x^2 = a^2 > 0$, $x^0 > 0$
	Kinematical Generators	
$\mathbf{P}$ $\mathbf{J}$	P^+, $\mathbf{P}^\perp$ $E^1 = M^{+1} = \frac{K_x + J_y}{\sqrt{2}}$ $E^2 = M^{+2} = \frac{K_y - J_x}{\sqrt{2}}$ $J_z = M^{12}$ $K_z = M^{-+}$	$M^{\mu\nu}$
	Dynamical Generators	
P^0 $\mathbf{K}$	P^- $F^1 = M^{-1} = \frac{K_x - J_y}{\sqrt{2}}$ $F^2 = M^{-2} = \frac{K_y + J_x}{\sqrt{2}}$	P^μ
	Plane-wave Representation	
$\lvert\mathbf{p}\rangle$ $p^0 = \pm\sqrt{\mathbf{p}^2 + m^2}$ $p^0 > 0$ and $p^0 < 0$ not kinematically disjoint	$\lvert p^+, \mathbf{p}^\perp\rangle$ $p^- = \frac{\mathbf{p}^{\perp 2} + m^2}{2p^+}$ $p^- > 0 \leftrightarrow p^+ > 0$ $p^- < 0 \leftrightarrow p^+ < 0$	$\lvert\mathbf{u}\rangle$ $u^\mu = p^\mu/m$, $u^2 = 1$ $u^0 = \pm\sqrt{\mathbf{u}^2 + 1}$ not kinematically disjoint
	Measure	
$\int \frac{\mathrm{d}^3 p}{2p^0}$	$\int \frac{\mathrm{d}^2 p^\perp \mathrm{d}p^+}{2p^+}$	$\int \frac{\mathrm{d}^3 u}{2g^0}$

Figure 2.2: A comparison of the three dynamical forms [29].

One should emphasize that not only the proton state, but also the state without physical particles the vacuum state of the laboratory observer, is a complicated superposition of the bare particles, or, in other words, of fluctuations of the bare fields. At the same time, this description of emptiness in terms of a complicated conglomerate of particles, seems unnatural. It would be much better to work in the approach in which the vacuum is indeed nothing but emptiness. Simplifying the vacuum wave function, we simplify not only it but also the wave function of the proton and of other particles, eliminating from them, like in the vacuum wave function, the fluctuations of fields. After that on can study the real physical structure of particles.

In Fig. 2.2, we shown of the three forms. Which of the three dynamical forms should be preferred? The question is difficult to answer, in fact it is ill-posed. In principle, all three forms should yield the same physical results since physics should not depend on how one parameterizes the space time. It is dependent in it, one has made a mistake. But usually one adjusts parameters to fit the physical problem any simplify the amount of practical work. Since one knows little on the typical solutions of a field theory, it might well be worth the effort to admit also other than the conventional *instant* form. Howere, the purpose of the present work is to introduce a framework for developing NN interaction models within the context of the *instant* form of relativistic dynamics.

2.5 Quantization

We study here the canonical quantization of nonrelativistic Hamiltonian mechanics. As was discussed in Sec. 2.2.2, the basic structures that determine the Hamiltonian description of the system are the Poisson bracket structure on phase space, its values for the fundamental degrees of freedom and the classical Hamiltonian H [191].

There exist a number of different quantization schemes which have been proposed over the decades, each with its own merits and drawbacks.

1. The canonical quantization method [193, 513]: At the level of QED, the canonical quantization method is not too difficult, but the canonical quantization of more complicated theories, such as non-Abelian theories, is often prohibitively tedious.

2. The Gupta-Bleuler [307, 80] or covariant quantization method: Contrary to canonical quantization, it maintains full Lorentz symmetry, which is a great advantage. The disadvantage of this approach is that ghosts or unphysical states of negative norm are allowed to propagate in the theory.

3. The path integral [234, 235] method: This is perhaps the most elegant and powerful of all quantization programs. One advantage is that one can easily go back and forth between many of the other quantization programs to see the relationships between them. The disadvantage of the path integral approach is that functional integration is a mathematically delicate operation that may not even exist in Minkowski space (Sec. 2.3). A method to incorporate constraints in the notion of Feynman *path integrals* was formulated by Faddeev [216].

4. The Becchi-Rouet-Stora-Tyupin(BRST) [49, 50, 51, 52] quantization method: This is one of the most convenient and practical covariant approaches used for gauge theories. Like the covariant quantization program, negative norm

states or ghosts are allowed to propagate and are eliminated by applying the BRST condition onto the state vectors.

All the information is contained in a single operator, making this a very attractive formalism. We will especially discuss the canonical quantization method.

2.5.1 Canonical Quantization

On of the first methods to quantize a finite regular system is the so called canonical quantization [193, 296]. Corresponding to the fundamental degrees of freedom that define phase space, and functions on this space, we have at the quantum level linear operators acting on Hilbert space [522]. Being operators, these quantities generally do not commute among one another, so that most often ordering prescriptions are necessary in order to define quantum composite operators. Note also that quantization of a classical system always introduces a new fundamental constant, namely $\hbar$, which has the dimensions of [mass length^2 time^{-1}] [296].

The quantum Hilbert space carries an algebraic structure associated with commutation relations, in a sense that it provides a linear representation of this algebra of quantum operators. This however is not enough at the quantum level. An additional structure must be introduced on Hilbert space, namely an inner product, which will be denoted by Dirac's bracket notation $< \mid >$ [193]. This inner product has to satisfy two main requirements.

- The Hermiticity properties of the quantum operators

$$\hat{q}^\dagger = \hat{q},\ \hat{p}^\dagger = \hat{p}.$$

  In particular, the quantum Hamiltonian operator $\hat{H}$ corresponding to the classical Hamiltonian H must be defined with a choice of ordering such that $\hat{H}$ is Hermitian, self adjoint. These different orderings may lead to not equivalent quantum theories.

- The inner product must be a Hermitian inner product. This corresponds to the following property under complex conjugation

$$< \psi|\phi >^* = < \phi|\psi >,$$

  where ψ and ϕ are arbitrary quantum states.

The commutation relations of the fundamental quantum degrees of freedom are simply given by the result of the associated Poisson bracket, multiplied by $i\hbar$ (See Chapt. 4) its means that

$$\{A, B\} \longrightarrow i\hbar[\hat{A}, \hat{B}], \tag{2.77}$$

where $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$, is the commutator of the operators $\hat{A}, \hat{B}$. This prescription is in this simple form however too naive and only works for special Hamilton operators and special observables. The precise meaning o (2.77) was analyzed by Ashtekar [25].

For example, associated with the fundamental brackets given in Eq. (2.34), we have the Heisenberg commutation relations

$$[\hat{q}_{n_1}, \hat{q}_{n_2}] = 0,\ [\hat{p}_{n_1}, \hat{p}_{n_2}] = 0,\ [\hat{q}_{n_1}, \hat{p}_{n_2}] = i\hbar\delta_{n_1,n_2}, \tag{2.78}$$

where $\hat{q}_n, \hat{p}_n$ are the fundamental quantum operators corresponding to the phase space degrees of freedom [296].

The canonical quantization rules could be eventually applied in a smaller Hilbert space, which is the counterpart of the reduced phase space R. However there remains a problem because the knowledge of R is not sufficient. Wave functions depend on configuration space variables, but R although symplectic is not necessarily the cotangent space of some configuration space. And even more severe, in most cases an explicit representation of R is not available and one only knows it implicitly.

2.5.2 Quantization of Constrained Systems

The canonical quantization prescription has served us well in defining a straightforward and unambiguous way of quantizing a classical system in flat space. However, it immediately runs into problems where constrained dynamics is involved. It is easy enough to restrict the potential term to the constraint surface the difficulty arises in treating the operator ordering ambiguities in quantizing the kinetic terms.

Noticing the point that the Dirac brackets for the constrained system play the role of the Poisson brackets for the unconstrained system, we can quantize the classical constrained system. The Dirac brackets are to be replaced, in quantization, by the commutators $(\times 1/i\hbar)$. Along this canonical quantization method, therefore, it is important that we describe the system in the language of the Hamiltonian and not of Lagrangian formalism.

In case of vanishing matrix elements of constraints, it is to fix the multipliers in the classical Hamiltonian by one way or another and solve the Schrödinger equation with the corresponding Hamilton operator. Use the substitution rule Eq. (2.77) as there were no constraints. Since by this we work in a Hilbert space which is too big, define states to belong to the physical Hilbert space Γ_p if they fulfill

$$< \psi' | \hat{C}_r | \psi >= 0, \tag{2.79}$$

where $\hat{C}_r$ are the quantum analogies of all constraints in the classical theory [527].

In case of constraints as conditions on states, which occur for a first class theory with

$$H = H_0 + \lambda_i \phi_i.$$

Again solve the Schrödinger equation for certain fixed values of the multipliers and ignore the constraints. Then impose as conditions on the states

$$\hat{\phi}^i|\psi>= 0. \quad (2.80)$$

This requirement means that the state $|\psi>$ is invariant under the action of gauge transformations generated by constraints. A subspace, defined by such equations (called physical subspace) is actual the Hilbert space of a theory. This implies Eq. (2.79) for those constraints C_r which are first class. For consistency of Eq. (2.80) also

$$[\phi^i, \phi^j]|\psi>= 0,$$

has to hold. However, the quantum analogies of Poisson bracket relations

$$\{\phi^i, \phi^j\} = 0,$$

do not necessarily become

$$[\hat{\phi}^i, \hat{\phi}^j] = \hat{f}_l^{ij}\hat{\phi}^l, \quad (2.81)$$

because in general the structure coefficients are operators too and some of them may occurs on the right of the ϕ^l. This order problem might be prevented by

$$[\hat{\phi}^\mu, \hat{\phi}^\nu]|\psi>= 0.$$

In the classical theory, the constraints do not only define constraint hypersurfaces but they also generate gauge transformations. Therefore one should expect that in the quantum theory they do not restrict the full Hilbert space but also generate equivalence classes of states. Define the infinitesimal change of an operator

$$\delta\hat{H} := \epsilon_i[\hat{H}, \hat{\phi}^i], \quad (2.82)$$

and calculate the matrix element

$$< \psi'|\delta\hat{H}|\psi >= \epsilon < \psi'|\hat{H}\hat{\phi}|\psi > -\epsilon < \psi'|\hat{\phi}\hat{H}|\psi > . \quad (2.83)$$

The first term on the right hand side vanishes due to Eq. (2.80), the second would vanish in case of hermitian constraints [527].

If constraints identifies as an operator, than we use Dirac quantization rules. The difficulty arising from the nonvanishing commutator of two constraints on the physical state, the presence of second class constraints ξ_i are even more severe. If $\hat{\xi}|\psi>= 0$, there will be some

$$[\hat{\xi}_i, \hat{\xi}_j]|\psi>\neq 0, \quad (2.84)$$

since already classically

$$Det\{\xi_i, \xi_j\} \neq 0, \quad (2.85)$$

such that $\hat{C}|\psi >= 0$ must lead to contradictions unless one assumes that the constraints themselves vanish as operator $\hat{C}_r = 0$, [191, 194, 527].

So, we can quantize any singular system using Eq. (2.77), but there is no single rule for quantization of constrained systems.

In non relativistic theory all dynamical variables are bosonic and represented by ordinary commuting numbers. On the other hand, in theory of particles one is tempted, that all really fundamental particles, fermion and bosons are composite.

As it was suggested by Schwinger [507], the matrix determining the Poisson brackets and the canonical commutation relations, must be skew Hermitian, then the consistent classical and quantum dynamics may be constructed on the basis of the variation principle. Remarkably, not only a real and *skew symmetrical* matrix, but also an imaginary symmetrical one is possible. However, in the second case the canonical variables should be anticommuting. The analysis in a space of anticommuting variables (in the Grassmann algebra) as exhaustively developed by [483]. The importance of using Grassmann algebras was essential for the discovery of supersymmetries and the recent introduction of the superspace formalism.

In order to define the associated manipulations, one must introduce the notions of integration and derivation of Grassmann variables, and extend the notion Poisson brackets to Grassmann odd phase space degrees of freedom. We discuss it in the next section.

2.6 Grassmann Anticommuting Variables

The introduction of *new* variables in mathematics and other sciences has played a significant role in the development of mathematics and science. This phenomenon can be seen clearly in theoretical physics. Beginning from ordinary classical mechanics using ordinary *real* number, *complex* variables have helped the birth of quantum theory in which they are basic ingredient. Beside their use as a mathematical tool, new variables may lead to more profound physical consequences. The aim of this chapter is to show other new variables which we could use in constraint dynamics with spin one half particle.

The analysis of Grassmann algebras is known to mathematicians and has been used for a long time, but its application in physics began along with the formulation of quantum field theory [410, 64]. The formulation of bosonic field operators by using commutation relations in quantum field theory led directly to the introduction of anticommuting relations for fermionic field operators by Dirac [193],

$$\{a, a^\dagger\} = 1, \quad \{a, a\} = \{a^\dagger, a^\dagger\} = 0. \tag{2.86}$$

The form of these relations is identical to the anticommutation relations between

Grassmannian coordinates θ^i and their derivatives $\partial_i \equiv \partial/\partial\theta^i$,

$$\{\theta^i, \partial_j\} = \delta_i^j, \quad \{\theta^i, \theta^j\} = \{\partial_i, \partial_j\} = 0, \tag{2.87}$$

which leads one to conjecture the existence of fundamental anticommuting coordinates. Since then, Grassmann variables have taken an important place in theoretical physics. In particular, Grassmann variables have been playing a central role in areas such as second quantization, non Abelian gauge theories, and supersymmetry. We shall introduce a brief theory of supersymmetry.

Although the use of anticommuting variables are indispensable from almost all theoretical physics works, there has been not much attention to their use as a kind of coordinate appended to our usual space time dimension. The introduction of extra Grassmann coordinates stems from the vast amount of research on unified models of elementary particles and their interaction forces, based on the extension to *Minkowski space time.*

The first attempt to unify gravitation and electromagnetic interactions was explored by adding one bosonic extra coordinate to the the four dimensional space time in Kaluza-Klein theory [344, 358]. Commuting variable coordinates are also used in bosonic string theory [299, 502]. The introduction of extra Grassmannian coordinates has its roots in superspace formulations [181]. Supersymmetry [222] and supergravity [435] are those that use the superspace concept by introducing spinorial Grassmann variable coordinates in their efforts to unify bosons and fermions. These formulations are essential in the construction of ten dimensional superstring theories, generalized version of string theories [299, 502].

2.6.1 Grassmann Algebra

Definition

Let $\theta_i, i = 1, \ldots, n$ be a set of generators satisfying anticommutation relation

$$\{\theta_i, \theta_j\} = 0, \text{for } i, j = 1, \ldots, n \tag{2.88}$$

where $\{A, B\} \equiv AB + BA$ for arbitrary A and B. These generators will define *a Grassmann algebra G_n with n generators.* In particular, $(\theta_i)^2 = 0$. If $n \to \infty$, the corresponding algebra will be denoted by G_∞. We will limit our discussion to finite dimensional Grassmann algebra. The generalization to the infinite case is straightforward.

These generators will define a linear vector space, in which all possible polynomials of θ_i will form a basis

$$1, \ \theta_1, \ \ldots, \ \theta_2, \ \theta_1\theta_2, \ \ldots, \ \theta_{n-1}\theta_n, \ \ldots, \ \theta_1\ldots\theta_n.$$

This basis will generate a 2^n-dimensional vector space over the complex numbers, whose elements are called *supernumbers.* The monomials $\theta_{i_1}\ldots\theta_{i_p}$ will be referred to as a *monomial of degree p.*

Functions of Grassmann Variables

Any function of θ_i, $f(\theta_i)$, can be written as a combination of monomials:

$$f(\theta) = c^0 + \sum_{p=1}^{n} c^{i_1 \ldots i_p} \theta_{i_1} \ldots \theta_{i_p}. \tag{2.89}$$

We can choose without loss of generality the coefficients $c^{i_1 \ldots i_p}$ to be totally antisymmetric in their indices. These coefficients usually are ordinary complex numbers, and $f(\theta)$ is called a supernumber. A supernumber is an element of the algebra G_n. In this case, c^0 is called the *body* (which is an ordinary complex number), and the rest is called the *soul* of the supernumber $f(\theta)$. In general, every supernumber ζ can be expressed as a sum of the body ζ_B and the soul ζ_S,

$$\zeta = \zeta_B + \zeta_S.$$

The inverse of a supernumber can be derived easily. Writing

$$\zeta = \zeta_B(1 + \zeta_B^{-1}\zeta_S),$$

then

$$\zeta^{-1} = \zeta_B^{-1}(1 + \zeta_B^{-1}\zeta_S)^{-1} = \zeta_B^{-1} \sum_{n=0}^{\infty} (\zeta_B^{-1}\zeta_S)^n, \tag{2.90}$$

where a Taylor expansion is used in the last step. Then it is obvious that a supernumber has an inverse if and only if its body is not vanishing.

We can also extend any analytic function on complex numbers to a supernumber valued function on G_n, using the usual Taylor expansion:

$$f(\zeta) = \sum_{p=0}^{n} \frac{1}{p!} f^{(p)}(\zeta_B)\zeta_S^p.$$

Here $f^{(p)}(\zeta_B)$ is the p-th derivative of f at the point ζ_B, provided ζ_B is not singular.

Another useful classification of supernumber is worked out by separating its odd and even parts:

$$\zeta = \zeta_e + \zeta_o,$$

$$\zeta_e = \zeta_B + \sum_{p=1}^{n/2} c^{i_1 \ldots i_{2p}} \theta_{i_1 \ldots i_{2p}},$$

$$\zeta_o = \sum_{p=0}^{n/2-1} c^{i_1 \ldots i_{2p+1}} \theta_{i_1 \ldots i_{2p+1}},$$

for even n, or

$$\zeta_e = \zeta_B + \sum_{p=1}^{(n-1)/2} c^{i_1 \dots i_{2p}} \theta_{i_1 \dots i_{2p}},$$

$$\zeta_o = \sum_{p=0}^{(n-1)/2} c^{i_1 \dots i_{2p+1}} \theta_{i_1 \dots i_{2p+1}},$$

for odd n.

Odd supernumbers anticommute among themselves, and even supernumbers commute with everything. The set of even supernumbers will generate a commutative superalgebra of G_n, $G_n^{(+)}$. The set of odd supernumbers, $G_n^{(-)}$, do not form an algebra. The product of an even and an odd supernumber will give an odd supernumber. The product of two even supernumbers or two odd supernumbers is an even supernumber. Finally, the square of an odd supernumber vanishes.

Derivatives

Now, we define the derivative of the elements of Grassmann algebra G_n in term of its base elements

$$\begin{aligned} \frac{\partial}{\partial \theta_q} \theta_{i_1} \dots \theta_{i_p} &= \delta_{i_1 q} \theta_{i_2} \dots \theta_{i_p} - \delta_{i_2 q} \theta_{i_1} \theta_{i_3} \dots \theta_{i_p} \\ &\quad + \dots + (-1)^{p-1} \delta_{i_p q} \theta_{i_1} \dots \theta_{i_{p-1}}. \end{aligned} \tag{2.91}$$

This derivative will be denoted as *left* derivative. Its *right* partner is defined similarly

$$\begin{aligned} \theta_{i_1} \dots \theta_{i_p} \frac{\partial}{\partial \theta_q} &= \delta_{i_p q} \theta_{i_1} \dots \theta_{i_{p-1}} - \delta_{i_{p-1} q} \theta_{i_1} \dots \theta_{i_{p-2}} \theta_{i_p} \\ &\quad + \dots + (-1)^{p-1} \delta_{i_1 q} \theta_{i_2} \dots \theta_{i_p}. \end{aligned} \tag{2.92}$$

Both derivatives are linear operators in Grassmann algebra G_n. To avoid ambiguity, from now on we will only use the *left* derivative, unless indicated explicitly. The derivative operators satisfy anticommutation relation among themselves:

$$\{\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j}\} = 0. \tag{2.93}$$

Also it is easy to verify from previous result that

$$\{\theta_i, \frac{\partial}{\partial \theta_j}\} = \delta_{ij}. \tag{2.94}$$

Since any function on supernumbers can be written as superposition of the base elements, the derivative of any function on Grassmann variables can be easily calculated using these results. Other properties can be derived easily from the definition of the derivatives. For example, let $u \in G_n^{(+)}$, $v \in G_n^{(-)}$, and $w \in G_n$, then

$$\frac{\partial}{\partial\theta_i}(uw) = (\frac{\partial}{\partial\theta_i}u)w + u(\frac{\partial}{\partial\theta_i}w), \tag{2.95}$$

$$\frac{\partial}{\partial\theta_i}(vw) = (\frac{\partial}{\partial\theta_i}v)w - v(\frac{\partial}{\partial\theta_i}w). \tag{2.96}$$

Also,

$$\frac{\partial}{\partial\theta_i}(\frac{\partial}{\partial\theta_j}w) = -\frac{\partial}{\partial\theta_j}(\frac{\partial}{\partial\theta_i}w). \tag{2.97}$$

The superanalyticity is defined in the same way as analyticity for ordinary functions. If $f(v) \in G_n, v \in G_n^{(-)}$, is superanalytic, then it is simply a linear function of v

$$f(v) = a + bv, \tag{2.98}$$

where a and b are constants elements of G_n and depend on the nature of $f(v)$. For instance, if $f(v)$ is taken to be even supernumber, then a is an even supernumber and b is an odd supernumber. Let us consider an infinitesimal displacement dv in $G_n^{(-)}$ space. Assuming $f(v)$ is superanalytic, then dv will induce an infinitesimal displacement of $f(v)$, $df(v)$, in G_n space. For an arbitrary dv

$$df(v) = dv\left[\frac{d}{dv}f(v)\right], \tag{2.99}$$

from which we get the general solution above.

Integration

Integration over Grassmann variables is defined as [181]

$$\int d\theta_i = 0, \qquad \int d\theta_i\theta_i = 1. \tag{2.100}$$

The first relation follows from the requirement of invariance under any finite *translation* a for commuting variables

$$\int_{-\infty}^{+\infty} dx f(x) = \int_{-\infty}^{+\infty} dx f(x+a),$$

which is extended to anticommuting variables. The second relation fixes the normalization convention. It can be shown that $d\theta_i$ will also satisfy anticommutation relation among themselves and with θ_i

$$\begin{aligned} \{d\theta_i, d\theta_j\} &= 0, \quad \text{for } i,j = 1,\dots,n \\ \{d\theta_i, \theta_j\} &= 0. \end{aligned} \tag{2.101}$$

It is clear from their definitions that differentiation and integration over an anticommuting variable are essentially identical,

$$\int dv f(v) = \frac{d}{dv} f(v) = b, \qquad v \in G_n^{(-)}. \tag{2.102}$$

and the integral of the derivative over anticommuting variables vanishes

$$\int dv \frac{d}{dv} f(v) = \frac{d^2}{dv^2} f(v) = 0, \quad v \in G_n^{(-)}. \tag{2.103}$$

The extension to multiple integration is defined by iteration of single integral.

$$\begin{aligned} \int \theta_{i_1} \ldots \theta_{i_p} d\theta_p \ldots d\theta_1 &= \varepsilon_{i_1 \ldots i_p}, \\ \int f(\theta) d\theta_p \ldots d\theta_1 &= \frac{1}{p!} \varepsilon_{i_1 \ldots i_p} c^{i_1 \ldots i_p}, \end{aligned}$$

where $\varepsilon_{i_1 \ldots i_p}$ is the Levi-Civita tensor.

For future use we still define here a Grassmann variant of the Dirac $\delta - function$ by

$$\int f(\theta)\delta(\theta) d\theta = f(0), \tag{2.104}$$

so that

$$\delta(\theta) = 0, \quad \int \delta(\theta) d\theta = 1 \quad and \quad \int \theta\delta(\theta) d\theta = 0.$$

In case $n > 1$ Grassmann variables $\theta^1, \theta^2, ..., \theta^n$ are encountered, we define

$$\delta^n(\theta) = \delta(\theta^n)\delta(\theta^{n-1})...\delta(\theta^1) = \theta^n\theta^{n-1}...\theta^1,$$

with all the obvious properties.

Conjugation

Complex conjugation (involution) is defined as one-to-one mapping of the algebra onto itself $f(\theta) \longrightarrow \overline{f}(\theta)$, such that,

$$\begin{aligned} \overline{(\overline{f}(\theta))} &= f(\theta), \\ \overline{(f_1(\theta) + f_2(\theta))} &= \overline{f}_1(\theta) + \overline{f}_2(\theta), \\ \overline{(f_1(\theta) f_2(\theta))} &= \overline{f}_2(\theta)\, \overline{f}_1(\theta), \end{aligned}$$

for all $f, f_1, f_2 \in G_n$. A supernumber is said to be *real* if $\overline{\zeta} = \zeta$.

Until now, we have assumed that the n generators $\theta_1 \ldots \theta_n$ are *real*,

$$\overline{\theta}_i = \theta_i, \qquad i = 1, \ldots, n.$$

[Note that the base elements of the algebra need not to be real in this case.] A complex Grassmann variable, θ, can be written as summation of its real and imaginary parts $\theta = \theta_R + i\theta_I$, and the real and imaginary parts can be replaced by θ and $\overline{\theta}$ as independent generators of Grassmann algebra.

Quantization of the Grassmann Variables

Quantization sends the Grassmann algebra into a Clifford algebra. The key step is to convert the dynamical variables of the classical theory into operators by means of the prescription

$$\{A, B\} \rightarrow (i\hbar)^{-1}[\hat{A}, \hat{B}]_{\pm}, \tag{2.105}$$

where the plus sign denotes an anticommutator (to be used when A and B are odd) and the minus sign corresponds to a commutator (to be used when at least A and B is even). Applying this prescription to $\{\theta^i, \theta^j\}^* = i\delta^{ij}$ we obtain

$$[\hat{\theta}_i, \hat{\theta}_j]_+ = \hbar\delta_{ij}, \tag{2.106}$$

from which we conclude that we have a Clifford algebra. In this representation we have

$$\hat{\theta}_i = \frac{\hbar^{1/2}}{2}\sigma_i, \tag{2.107}$$

where the σ_i are the Pauli matrices.

2.7 Supersymmetry and Superfield

Sypersymmetry (susy), i.e., fermi-bose symmetry is on of the most peculiar discoveries in the history of physics.

We can also generalize the concept of supernumber to that of *superfield* [485, 227]. A superfield is defined similarly to a supernumber except that now the coefficients are *fields* instead of simply numbers. In most cases in physics they are fields over spacetime $F(x)$, where $x = (x^0, \overrightarrow{x})$. Then a superfield $F(x, \theta, \bar{\theta})$ can be written as

$$F(x, \theta, \bar{\theta}) = F^0(x) + \sum_{p=1}^{n} F^{i_1 \cdots\, i_p}(x)\theta_{i_1} \ldots \theta_{i_p}. \tag{2.108}$$

The superfield is therefore equivalent to a finite number of ordinary fields, which is a multiplet of fields. The transformation properties of superfields imply transformation properties for the multiplet components.

The method of constructing supersymmetry Hamiltonians is based on an extension of the normal space due to the introduction of additional fermion coordinates which form, together with the normal coordinates, the so called superspace. In the superspace, the superalgebra (The algebra with the Grassmann variables) can be realized linearly. Moreover, the boson and fermion fields are combined in a single superfield.

Since we want to construct supersymmetric quantum field theories, and to find representations of the susy algebra on fields. A convenient and compact way to do this is to introduce superspace and superfield, i.e. fields defined on superspace. This is particularly simple for unextended susy, so we will restrict here to $N = 1$ superspace and superfields. Then we have two extra two susy generators Q and $\bar{Q}$, as well as four generators P_μ of space time translations. The idea then is to enlarge space time labeled by the coordinates x^μ by adding two plus two anticommuting Grassmann variables θ and $\bar{\theta}$. Thus coordinates on superspace are $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$.

2.7.1 Spinors and the Poincaré Group

We begin with a review of the Lorenz and Poincaré groups and spinors in four dimensional Minkowski space. The signature is taken to be $+,-,-,-$ so that $p^2 = +m^2$ and $\mu, \nu, ...$ always are space time indices, while $i, j, ...$ are only space indices. then the metric $g_{\mu\nu}$ is diagonal with $g_{00} = 1, g_{ij} = -1$.

The Lorenz group has six generators, three rotations J_i and three boosts $K_i, i = 1, 2, 3$ with commutation relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_j. \tag{2.109}$$

To identify the mathematical structure and to construct representations of this algebra one introduces the linear combinations

$$J_j^{\pm} = \frac{1}{2}(J_j \pm iK_j), \tag{2.110}$$

in terms of which the algebra separates into two commuting $SU(2)$ algebras:

$$[J_i^{\pm}, J_j^{\pm}] = i\epsilon_{ijk}J_k^{\pm}, \quad [J_i^{\pm}, J_j^{\mp}] = 0. \tag{2.111}$$

These generators are not hermitian howere, and we see that the Lorentz group is a complexified version of $SU(2) \times SU(2)$: this group is $SL(2, C)$ [74]. To see that this group is really $SL(2, C)$ is easy: introduce the four 2×2 matrices σ_μ where σ_0 is the identity matrix and $\sigma_i, i = 1, 2, 3$ are the three Pauli matrices. Then for every four vector x^μ the 2×2 matrix $x^\mu\sigma_\mu$ is hermitian and has determinant equal to $x^\mu x_\mu$ which is a Lorentz invariant. Hence a Lorentz transformation preserves the determinant and the hermiticity of this matrix, and thus must act as $x^\mu\sigma_\mu \to Ax^\mu\sigma_\mu A^\dagger$ with $|detA| = 1$. We see that up to an irrelevant phase, A is a complex 2×2 matrix of unit determinant, i.e. an element of $SL(2, C)$. This establishes the mapping between an element of the Lorentz group and the group $SL(2, C)$.

The Poincaré group contains, in addition to the Lorentz transformations, also the translations. More precisely it is a semi direct product of the Lorentz group and the group of translation in space time. The generators of the translations are usually denoted P_μ. In addition to the commutators of the Lorentz generators J_i

(rotation) and K_i (boosts) one has the following commutation relations involving the P_μ:

$$[P_\mu, P_\nu] = 0, \quad [J_i, P_j] = i\epsilon_{ijk}P_k, \quad [J_i, P_0] = 0,$$
$$[K_i, P_j] = -iP_0, \quad [K_i, P_0] = -iP_j, \tag{2.112}$$

which state that translations commute among themselves, that the P_i are a vector and P_0 a scalar under space rotations and how P_i and P_0 mix under a boost. One defines the Lorentz generators $M_{\mu\nu} = -M_{\nu\mu}$ as $M_{0i} = K_i$ and $M_{ij} = \epsilon_{ijk}J_k$. Then the full Poincaré algebra reads

$$[P_\mu, P_\nu] = 0,$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = ig_{\nu\rho}M_{\mu\sigma} - ig_{\mu\rho}M_{\nu\sigma} - ig_{\nu\sigma}M_{\mu\rho} + ig_{\mu\sigma}M_{\nu\rho}, \tag{2.113}$$
$$[M_{\mu\nu}, P_\rho] = -ig_{\rho\mu}P_\nu + ig_{\rho\nu}P_\mu.$$

There are various equivalent ways to introduce spinors. Here we define spinors as the objects carrying the basic representation of $SL(2, C)$. Since elements of $SL(2, C)$ are complex 2×2 matrices, a spinor is a two component object $\binom{\psi=\psi_1}{\psi_2}$ transforming under an element $\mathcal{M} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, C)$ as

$$\psi_\alpha \to \psi'_\alpha = \mathcal{M}_\alpha^\beta \psi_\beta, \tag{2.114}$$

with $\alpha, \beta = 1, 2$ labeling the components. Now, unlike for $SU(2)$, for $SL(2, C)$ a representation and its complex conjugate are not equivalent. $\mathcal{M}$ and $\mathcal{M}^*$ give inequivalent representations. A two component object $\bar{\psi}$ transforming as

$$\bar{\psi}_{\dot{\alpha}} \to \bar{\psi}'_{\dot{\alpha}} = \mathcal{M}_{\dot{\alpha}}^{*\dot{\beta}} \bar{\psi}_{\dot{\beta}}, \tag{2.115}$$

is called a dotted spinor, while the above ψ is called an undotted one. Comparing the complex conjugate of Eq. (2.114) with Eq. (2.115) we see that we can identify $\bar{\psi}_{\dot{\alpha}}$ with $(\psi_\alpha)^*$.

The representation carried by the ψ_α is called $(\frac{1}{2}, 0)$ (matrices $\mathcal{M}$) and the one carried by the $\bar{\psi}_{\dot{\alpha}}$ is called $(0, \frac{1}{2})$ (matrices $\mathcal{M}^*$). They are both irreducible. Now, any $SL(2, C)$ matrix can be written as

$$\mathcal{M} = exp(a_j\sigma_j + ib_j\sigma_j), \tag{2.116}$$

$$\mathcal{M}^* = exp(a_j\sigma_j^* - ib_j\sigma_j^*). \tag{2.117}$$

This explicitly displays the generators as the spin $\frac{1}{2}$ representation of the complexified $SU(2)$, in accordance with (2.110). One introduces the Dirac matrices in the Weyl representation as

$$\gamma = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \tag{2.118}$$

A four component Dirac spinor is made from a two component undotted and a two component dotted spinor as $\binom{\psi_\alpha}{\bar{\chi}^{\dot{\alpha}}}$. Clearly it transforms as the reducible $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of Lorentz group. Then $\binom{\psi_\alpha}{0}$ and $\binom{0}{\bar{\chi}^{\dot{\alpha}}}$ are chiral Dirac (or Weyl) spinors. A Majorana spinor (see Appendix.A.1) is a Dirac spinor with $\chi \equiv \psi$, i.e. it is of the form $\binom{\psi}{\bar{\psi}^{\dot{\alpha}}}$.

The Lorentz generators are

$$\Sigma^{\mu\nu} = \frac{i}{2}\gamma^{\mu\nu}, \quad \gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{1}{2}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix}. \tag{2.119}$$

We see that indeed the undotted and dotted spinors transform separately, the generators being $i\sigma^{\mu\nu}$ for ψ_α and $i\bar{\sigma}^{\mu\nu}$ for $\bar{\psi}^{\dot{\alpha}}$ with

$$\begin{aligned} (\sigma^{\mu\nu})_\alpha^\beta &= \frac{1}{4}\left(\sigma^\mu_{\alpha\dot{\gamma}}\bar{\sigma}^{\nu\dot{\gamma}\beta} - (\mu \leftrightarrow \nu)\right), \\ (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} &= \frac{1}{4}\left(\bar{\sigma}^{\mu\dot{\alpha}\gamma}\sigma^\nu_{\gamma\dot{\beta}} - (\mu \leftrightarrow \nu)\right). \end{aligned} \tag{2.120}$$

Note that e.g. $\sigma^{12} = \bar{\sigma}^{12} = -\frac{i}{2}\sigma_3 \equiv -\frac{i}{2}\sigma_z$ so that the rotation generator $M_{12} = M^{12}$ is $\frac{1}{2}\sigma_z$ as expected.

2.7.2 The Supersymmetry Algebra

The symmetry we are looking for must connect boson and fermions. In other words, the generators Q of this symmetry must turn a bosonic state into a fermionic one, and vice versa. We want to enlarge the Poincaré algebra by generators that transform either as undotted spinors Q^I_α or as dotted spinors $\bar{Q}^I_{\cdot\alpha}$ under the Lorentz group and that commute with the translations. The extra index $I = 1, ...N$ labels the different spinorial generators in case there are more than one pair. This means that according to Eq. (2.120)

$$\begin{aligned} [P_\mu, Q^I_\alpha] &= 0, \quad [P_\mu, \bar{Q}^I_{\dot{\alpha}}] = 0, \\ [M_{\mu\nu}, Q^I_\alpha] &= i(\sigma_{\mu\nu})^\beta_\alpha Q^I_\beta, \\ [M_{\mu\nu}, \bar{Q}^{I\dot{\alpha}}] &= i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}}\bar{Q}^{I\dot{\beta}}. \end{aligned} \tag{2.121}$$

In particular, $M_{12} \equiv J_3$ and thus $[J_3, Q^I_1] = \frac{1}{2}Q^I_1$ and $[J_3, Q^I_2] = -\frac{1}{2}Q^I_2$. Since

$$\bar{Q}^{I1} = -(Q^I_2)^\dagger \quad \textit{and} \quad \bar{Q}^{I2} = (Q^I_1)^\dagger,$$

one similarly has

$$[J_3, (Q^I_2)^\dagger] = \frac{1}{2}(Q^I_2)^\dagger \quad \textit{and} \quad [J_3, (Q^I_1)^\dagger] = -\frac{1}{2}(Q^I_1)^\dagger.$$

We conclude that Q_1^I and $(Q_2^I)^\dagger$ rise the z-component of the spin by half a unit, while Q_2^I and $(Q_1^I)^\dagger$ lower it by half a unit.

Since the Q_α^I transform in the $(\frac{1}{2},0)$ representation and the $\bar{Q}_{\dot{\alpha}}^I$ in the $(0,\frac{1}{2})$, the anticommutator of Q_α^I and $\bar{Q}_{\dot{\beta}}^I$ must transform as $(\frac{1}{2},\frac{1}{2})$, i.e. as a four vector. The obvious candidate is P_μ so that we arrive at

$$\{Q_\alpha^I, \bar{Q}_{\dot{\beta}}^J\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{IJ}. \tag{2.122}$$

The δ^{IJ} can always be achieved by diagonalising an a priori arbitrary symmetric matrix and by rescaling the Q and $\bar{Q}$. furthermore, since $\bar{Q}$ is the adjoint of Q, positivity of the Hilbert space excludes zero eigenvalues of this matrix. Finally

$$\{Q_\alpha^I, \bar{Q}_\beta^J\} = \epsilon_{\alpha\beta} Z^{IJ}, \quad \{\bar{Q}_{\dot{\alpha}}^I, \bar{Q}_{\dot{\beta}}^J\} = 2\epsilon_{\dot{\alpha}\dot{\beta}} (Z^{IJ})^*. \tag{2.123}$$

The $Z^{IJ} = -Z^{JI}$ are central charges which means they commute with all generators of the full algebra. The simplest algebra has $N = 1$, i.e. they are no indices I, J and there is no possibility of central charges. this is the unextended susy algebra. In the simplest extended case, $N = 2$, there is just one central charge $Z \equiv Z^{12}$.

As over said, we restrict here to $N = 1$. The odd superspace coordinates θ_α and $\bar{\theta}_{\dot{\alpha}}$ just behave as constant (x^μ independent) spinors. Recall that as all spinors they anticommute among themselves, i.e. Eq. (2.88), and idem for the $\bar{\theta}^{\dot{\alpha}}$.

Hence an arbitrary (scalar) function Eq. (2.108) on superspace, i.e. a superfield, can always be expanded as

$$\begin{aligned} F(x,\theta,\bar{\theta}) &= f(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta} n(x) \\ &+ \theta\sigma^\mu\bar{\theta} v_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\rho(x) + \theta\theta\bar{\theta}\bar{\theta} d(x). \end{aligned} \tag{2.124}$$

Heir we have used the identities in (4.43). If F carries extra vector indices then so do the component field $f, \psi, ...$ Now, we want to realize the susy generators Q_α and their hermitian conjugates $\bar{Q}_{\dot{\alpha}} = (Q_\alpha)^\dagger$ as differential operators on superspace.[2]

We want that $i\epsilon^\alpha Q_\alpha$ generators a translation in θ^α by a constant infinitesimal spinor ϵ^α plus some translation in x^μ. The latter space time translation is determined by the susy algebra since the commutator of two such susy transformations is a translation in space time. Thus we want

$$(1 + i\epsilon Q) F(x,\theta,\bar{\theta}) = F(x + \delta x, \theta + \epsilon, \bar{\theta}), \tag{2.125}$$

[2]On this space, the Q_α operators become differential operators $Q_\alpha \to -\frac{\partial}{\partial\theta} + i\gamma^\mu\theta\partial_\mu$ in mach the same way that the P_μ and $M_{\mu\nu}$ operators are realized by $P_\mu \to -i\partial_\mu$, $M_{\mu\nu} \to -i(x_\mu\partial_\nu - x_\nu\partial_\mu + \bar{\theta}\sigma_{\mu\nu}\frac{\partial}{\partial\theta})$.

where, we arrive

$$Q_\alpha = -i\left(\frac{\partial}{\partial\theta^\alpha} - ic(\sigma^\mu\bar{\theta})_\alpha\partial_\mu\right), \tag{2.126}$$

and the hermitian conjugate is

$$\bar{Q}_{\dot{\alpha}} = i\left(\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - ic^*(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu\right), \tag{2.127}$$

and they satisfy the susy algebra, in particular

$$\{Q_\alpha, Q_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}}P_\mu = -2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu, \tag{2.128}$$

if $c = 1$, we can now give the action on the superfield F and determine δx:

$$(1 + i\epsilon Q + i\bar{\epsilon}\bar{Q})F(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\beta}}) = F(x^\mu - i\epsilon\sigma^\mu\bar{\theta} + 1\theta\sigma^\mu\bar{\epsilon}, \theta^\alpha + \epsilon^\alpha, \bar{\theta}^{\dot{\beta}} + \bar{\epsilon}^{\dot{\beta}}),$$

and the susy variation of a superfield is of course defined as

$$\delta_{\epsilon\bar{\epsilon}}F = (i\epsilon Q + i\bar{\epsilon}\bar{Q})F. \tag{2.129}$$

We find covariant derivatives D_α and $\bar{D}_{\dot{\alpha}}$ that anticommute with the susy generators Q and $\bar{Q}$. Then $\delta_{\epsilon\bar{\epsilon}}(D_\alpha F) = D_\alpha(\delta_{\epsilon\bar{\epsilon}}F)$ and idem for $\bar{D}_{\dot{\alpha}}$. It follows that

$$D_\alpha F = 0, \quad or \quad \bar{D}_{\dot{\alpha}}F = 0,$$

are susy invariant constraints one may impose to reduce the number of components in a superfield. On finds

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu,$$

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu, \tag{2.130}$$

where $\bar{D}_{\dot{\alpha}} = (D_\alpha)^\dagger$ and

$$\{D_\alpha, \bar{D}_{\dot{\beta}}\} = 2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu, \qquad \{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0,$$

$$\{D_\alpha, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0. \tag{2.131}$$

The covariant derivative plays an important role in the superfield theory. Two comments are in order here. First, any linear combination of superfields is again, a superfield. Second, the product of two superfields is also a superfield. If the first assertion is absolutely obvious, the second, perhaps, requires some explanation. It is instructive to check it explicitly, component by component.

2.7.3 Chiral Superfields

Here, we encounter a new type of superfield (chiral) and a new realization of susy in the superspace. The chiral superfields depend explicitly only on θ or only on $\bar{\theta}$.

A chiral superfield ϕ is defined by the condition

$$\bar{D}_{\dot{\alpha}}\phi = 0, \quad (left \quad chiral) \tag{2.132}$$

and an anti-chiral one $\bar{\phi}$ by

$$D_{\alpha}\bar{\phi} = 0. \quad (right \quad chiral) \tag{2.133}$$

This easily solved by observing that

$$D_{\alpha}\bar{\theta} = \bar{D}_{\dot{\alpha}}\theta = D_{\alpha}\bar{y}^{\mu} = \bar{D}_{\dot{\alpha}}y^{\mu} = 0,$$

$$y^{\mu} = x^{\mu} + i\theta\sigma^{\mu}\bar{\theta} \quad , \quad \bar{y}^{\mu} = x^{\mu} - i\theta\sigma^{\mu}\bar{\theta}. \tag{2.134}$$

Hence ϕ depends only on θ and y^{μ} and $\bar{\phi}$ only on $\bar{\theta}$ and y. Concentrating on ϕ we have component expansion

$$\phi(y,\theta) = z(y) + \sqrt{2}\theta\psi(y) - \theta\theta f(y), \tag{2.135}$$

or Taylor expanding in terms of x, θ and $\bar{\theta}$:

$$\phi(y,\theta) = z(x) + \sqrt{2}\theta\psi(x) - \theta\theta f(x)$$

$$+ i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}z(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_{\mu}\psi(x)\sigma^{\mu}\bar{\theta} - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^{2}z(x). \tag{2.136}$$

Physically, such a chiral superfield describes one complex scalar z and one Weyl fermion ψ. The field f will turn out to be an auxiliary field. For $\bar{\psi}$ we similarly have

$$\bar{\phi}(y,\bar{\theta}) = \bar{z}(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) - \bar{\theta}\bar{\theta}\bar{f}(x)$$

$$- i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\bar{z}(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^{\mu}\partial_{\mu}\bar{\psi}(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^{2}\bar{z}(x). \tag{2.137}$$

Finally, let us find the explicit susy variations of the component fields as it results form Eq. (2.129): First, for chiral superfields it is useful to change variables from $x^{\mu}, \theta, \bar{\theta}$ to $y^{\mu}, \theta, \bar{\theta}$. Then

$$Q_{\alpha} = -i\frac{\partial}{\partial\theta^{\alpha}} \quad , \quad \bar{Q}_{\dot{\alpha}} = i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + 2\theta^{\beta}\sigma^{\mu}_{\beta\dot{\alpha}}\frac{\partial}{\partial y^{\mu}}, \tag{2.138}$$

so that

$$\delta\phi(y,\theta) \equiv (i\epsilon Q + i\bar{\epsilon}\bar{Q})\phi(y,\theta) = \left(\epsilon^{\alpha}\frac{\partial}{\partial\theta^{\alpha}} + 2i\theta\sigma^{\mu}\bar{\epsilon}\frac{\partial}{\partial y^{\mu}}\right)\phi(y,\theta) =$$

$$\begin{aligned} &\sqrt{2}\epsilon\psi - 2\epsilon\theta f + 2i\theta\sigma^\mu\bar{\epsilon}(\partial_\mu z + \sqrt{2}\theta\partial_\mu\psi) \\ &= \sqrt{2}\epsilon\psi + \sqrt{2}\theta(-\sqrt{2}\epsilon f + \sqrt{2}i\sigma^\mu\bar{\epsilon}\partial_\mu z) - \theta\theta(-i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi). \end{aligned} \tag{2.139}$$

Thus we read the susy transformations of the component field:

$$\begin{aligned} \delta z &= \sqrt{2}\epsilon\psi, \quad (boson \to fermion) \\ \delta\psi &= \sqrt{2}i\sigma^\mu\bar{\epsilon}\partial_\mu z - \sqrt{2}\epsilon f, \quad (fermion \to boson) \\ \delta f &= -i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi. \quad (f \to totalderivative) \end{aligned} \tag{2.140}$$

2.7.4 Susy Invariant Action

To construct the field theoretical model we used Hamilton's method because it was not clear to us then how to formulate Lagrange's approach consistently with our invariance principle. On the contrary Hamilton's approach appeared to be very simple because the Hamiltonian of the system was among the generators of the basic algebra. It became very simple to build a representation of the algebra in terms of free field operators. Starting with this we can develop a method to determine the form of interaction [293]. At this point it was found that the basic invariance principle had to be completed with a special locality principle. Only after that did we obtain a nonlinear set of equations for the possible interaction terms.

The simplest nonsupersymmetric quantum mechanical problem is that of one degree of freedom $\phi(t)$; the corresponding action has the form

$$S = \int dt L(t), L = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - V(\phi), \tag{2.141}$$

where $V(\phi)$ is the potential energy. Now, instead of the variable $\phi(t)$, we introduce a supervariable $F(x, \theta, \bar{\theta})$ Eq. (2.108). Analogous to the fact that the space time (time in this case) linearly realizes the action of the translation generator,

$$t \to t + \tau,$$

the superspace allows one to realize linearly all susy generators. Under the susy transformations,

$$\theta \to \theta + \zeta, \quad \bar{\theta} \to \bar{\theta} + \bar{\zeta}, \quad t \to t + i\theta\bar{\zeta} - i\zeta\bar{\theta}, \tag{2.142}$$

where ζ and $\bar{\zeta}$ are the Grassmann parameters of the supertranslations.

Now all of our preparatory work has been completed, and we can finally turn to a discussion of a regular method of building supersymmetric theories. By definition, we want the action to be invariant under susy transformations:

$$\delta \int d^4x \mathcal{L}(x) = 0. \tag{2.143}$$

this is satisfied if $\mathcal{L}$ itself transforms into a total derivative. We start from the expression Eq. (2.141) for the action in ordinary quantum mechanics and can thus write the action S as

$$S_{susy} = \int dt d\theta d\bar{\theta} \left(\frac{1}{2} \bar{D}\phi D\phi - W(\phi) \right), \tag{2.144}$$

where $W(\phi)$ is an arbitrary function of ϕ, the called superpotential. This W may depend on several different ϕ_i. Using the y and θ variables one easily Taylor expands

$$W(\phi) = W(z(y)) + \sqrt{2}\frac{\partial W}{\partial z_i}\theta\psi_i(y) - \theta\theta \left(\frac{\partial W}{\partial z_i} f_i(y) + \frac{1}{2}\frac{\partial^2 W}{\partial z_i \partial z_j}\psi_i(y)\psi_j(y) \right), \tag{2.145}$$

where it is understood that $\frac{\partial W}{\partial z}$ and $\frac{\partial^2 W}{\partial z \partial z}$ are evaluated at $z(y)$. The important observation is that any Lagrangian of the form

$$\int d^2\theta d^2\bar{\theta} F(x, \theta\bar{\theta}) + \int d^2\theta W(\phi) + \int d^2\bar{\theta}[W(\phi)]^\dagger, \tag{2.146}$$

is automatically susy invariant, i.e. it transforms at most by a total derivative in space time.

The susy variation of any superfield is given by Eq. (2.129) and, since the ϵ and $\bar{\epsilon}$ are constant spinor and the Q and $\bar{Q}$ are differential operators in superspac, it is again a total derivative in all of superspace:

$$\delta F = \frac{\partial}{\partial\theta^\alpha}(-\epsilon^\alpha F) + \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}(-\bar{\epsilon}^{\dot{\alpha}} F) + \frac{\partial}{\partial x^\mu}[-i(\epsilon\sigma^\mu\bar{\theta} - \theta\sigma^\mu\bar{\epsilon})F]. \tag{2.147}$$

If now F is a chiral superfield like ϕ or $W(\phi)$ one changes variables to θ and y and one has

$$\delta\phi = \frac{\partial}{\partial\theta^\alpha}(-\epsilon^\alpha\phi(y,\theta)) + \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}(-\bar{\epsilon}^{\dot{\alpha}}\phi(y,\theta)) + \frac{\partial}{\partial y^\mu}[-i(\epsilon\sigma^\mu\bar{\theta} - \theta\sigma^\mu\bar{\epsilon})\phi(y,\theta)]. \tag{2.148}$$

The analogous result holds for an anti chiral superfield $\bar{W}(\bar{\phi}) = [W(\phi)]^\dagger$. This proves the susy of the action resulting from the space time integral of the Lagrangian Eq. (2.146).

The terms $\int d^2\theta W(\phi) + h.c.$ in the Lagrangian have the form of a potential. The kinetic terms must be provided by the term $\int d^2\theta d^2\bar{\theta} F$. The simplest choice is $F = \phi^\dagger\phi$. This is neither chiral nor anti chiral but real. To compute $\phi^\dagger\phi$ one must first expand the y^μ in terms of x^μ. We only need the terms $\sim \theta\theta\bar{\theta}\bar{\theta}$, called the D-term:

$$\begin{aligned} \phi^\dagger\phi\Big|_{\theta\theta\overline{\theta\theta}} &= -\tfrac{1}{4}z^\dagger\partial^2 z - \tfrac{1}{4}\partial^2 z^\dagger z + \tfrac{1}{2}\partial_\mu z^\dagger\partial^\mu z + f^\dagger f + \tfrac{i}{2}\partial_\mu\psi\sigma^\mu\overline{\psi} - \tfrac{i}{2}\psi\sigma^\mu\partial_\mu\overline{\psi} \\ &= \partial_\mu z^\dagger\partial^\mu z + \tfrac{i}{2}(\partial_\mu\psi\sigma^\mu\overline{\psi} - \psi\sigma^\mu\partial_\mu\overline{\psi}) + f^\dagger f + \text{total derivative}. \end{aligned} \tag{2.149}$$

Then

$$S = \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\overline{\theta}\ \ \overline{\phi}_i^\dagger\phi_i + \int \mathrm{d}^4x\mathrm{d}^2\theta\ W(\phi_i) + h.c. \tag{2.150}$$

yields

$$S = \int \mathrm{d}^4x\Big[|\partial_\mu z_i|^2 - i\psi_i\sigma^\mu\partial_\mu\overline{\psi}_i + f_i^\dagger f_i - \frac{\partial W}{\partial z_i}f_i + h.c. - \frac{1}{2}\frac{\partial^2 W}{\partial z_i\partial z_j}\psi_i\psi_j + h.c.\Big]. \tag{2.151}$$

More generally, one can replace $\phi_i^\dagger\phi_i$ by a (real) Kähler potential $K(\phi_i^\dagger, \phi_j)$. This leads to the nonlinear σ-model. In any case, the f_i have no kinetic term and hence are auxiliary fields. They should be eliminated by substituting their algebraic equations of motion

$$f_i^\dagger = \left(\frac{\partial W}{\partial z_i}\right), \tag{2.152}$$

into the action, leading to

$$S = \int \mathrm{d}^4x\Big[|\partial_\mu z_i|^2 - i\psi_i\sigma^\mu\partial_\mu\overline{\psi}_i - \left|\frac{\partial W}{\partial z_i}\right|^2 - \frac{1}{2}\frac{\partial^2 W}{\partial z_i\partial z_j}\psi_i\psi_j - \frac{1}{2}\left(\frac{\partial^2 W}{\partial z_i\partial z_j}\right)^\dagger\overline{\psi}_i\overline{\psi}_j\Big]. \tag{2.153}$$

We see that the scalar potential V is determined in terms of the superpotential W as

$$V = \sum_i \left|\frac{\partial W}{\partial z_i}\right|^2. \tag{2.154}$$

To illustrate this model, consider the simplest case of a single chiral superfield ϕ and a cubic superpotential $W(\phi) = \frac{m}{2}\phi^2 + \frac{g}{3}\phi^3$. Then $\frac{\partial W}{\partial z} = m\phi + g\phi^2$ and the action becomes

$$\begin{aligned} S_{\mathrm{WZ}} = \textstyle\int \mathrm{d}^4x \quad \Big[\quad & |\partial_\mu z|^2 - i\psi\sigma^\mu\partial_\mu\overline{\psi} - m^2|z|^2 - \tfrac{m}{2}(\psi\psi + \overline{\psi\psi}) \\ - \quad & mg(z^\dagger z^2 + (z^\dagger)^2 z) - g^2|z|^4 + g(z\psi\psi + z^\dagger\overline{\psi\psi})\Big]. \end{aligned} \tag{2.155}$$

Note that the Yukawa interactions appear with a coupling constant g that is related by susy to the bosonic coupling constants mg and g^2.

2.7.5 Vector Superfields

The $N = 1$ supermultiplet of next higher spin is the vector multiplet. The corresponding superfield $V(x, \theta, \overline{\theta})$ is real and has the expansion

$$\begin{aligned} V(x,\theta,\overline{\theta}) &= C + i\theta\chi - i\overline{\theta}\overline{\chi} + \theta\sigma^\mu\overline{\theta}v_\mu \\ &+ \tfrac{i}{2}\theta\theta(M+iN) - \tfrac{i}{2}\overline{\theta}\overline{\theta}(M-iN) \\ &+ i\theta\theta\,\overline{\theta}\left(\overline{\lambda} + \tfrac{i}{2}\overline{\sigma}^\mu\partial_\mu\chi\right) - i\overline{\theta}\overline{\theta}\,\theta\left(\lambda - \tfrac{i}{2}\sigma^\mu\partial_\mu\overline{\chi}\right) \\ &+ \tfrac{1}{2}\theta\theta\overline{\theta}\overline{\theta}\left(D - \tfrac{1}{2}\partial^2 C\right), \end{aligned} \tag{2.156}$$

where all component fields only depend on x^μ. There are 8 bosonic components (C, D, M, N, v_μ) and 8 fermionic components (χ, λ). These are too many components to describe a single supermultiplet. We want to reduce their number by making use of the supersymmetric generalization of a gauge transformation. Note that the transformation

$$V \to V + \phi + \phi^\dagger \ , \tag{2.157}$$

with ϕ a chiral superfield, implies the component transformation

$$v_\mu \to v_\mu + \partial_\mu(2\mathrm{Im}z), \tag{2.158}$$

which is an Abelian gauge transformation. We conclude that (2.157) is its desired supersymmetric generalization. If this transformation (2.157) is a symmetry (actually a gauge symmetry, as we just saw) of the theory then, by an appropriate choice of ϕ, one can transform away the components χ, C, M, N and one component of v_μ. This choice is called the Wess-Zumino gauge, and it reduces the vector superfield to

$$V_{\mathrm{WZ}} = \theta\sigma^\mu\overline{\theta}v_\mu(x) + i\theta\theta\,\overline{\theta}\overline{\lambda}(x) - i\overline{\theta}\overline{\theta}\,\theta\lambda(x) + \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}D(x). \tag{2.159}$$

Since each term contains at least one θ, the only nonvanishing power of V_{WZ} is

$$V_{\mathrm{WZ}}^2 = \theta\sigma^\mu\overline{\theta}\ \theta\sigma^\nu\overline{\theta}\ v_\mu v_\nu = \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}\ v_\mu v^\mu, \tag{2.160}$$

and $V_{\mathrm{WZ}}^n = 0$, $n \geq 3$.

To construct kinetic terms for the vector field v_μ one must act on V with the covariant derivatives D and $\overline{D}$. Define

$$W_\alpha = -\frac{1}{4}\overline{D}\overline{D}D_\alpha V \quad , \quad \overline{W}_{\dot{\alpha}} = -\frac{1}{4}DD\overline{D}_{\dot{\alpha}}V. \tag{2.161}$$

(This is appropriate for Abel Ian gauge theories and will be slightly generalized in the non-Abelian case.) Since $D^3 = \overline{D}^3 = 0$, W_α is chiral and $\overline{W}_{\dot{\alpha}}$ antichiral.

Furthermore it is clear that they behave as anticommuting Lorentz spinors. Note that they are invariant under the transformation (2.157) since

$$\begin{aligned} W_\alpha &\to W_\alpha - \tfrac{1}{4}\overline{DD}D_\alpha(\phi+\phi^\dagger) = W_\alpha + \tfrac{1}{4}\overline{D}^{\dot\beta}\overline{D}_{\dot\beta}D_\alpha\phi \\ &= W_\alpha + \tfrac{1}{4}\overline{D}^{\dot\beta}\{\overline{D}_{\dot\beta}, D_\alpha\}\phi = W_\alpha + \tfrac{i}{2}\sigma^\mu_{\alpha\dot\beta}\partial_\mu\overline{D}^{\dot\beta}\phi = W_\alpha, \end{aligned} \tag{2.162}$$

since $\overline{D}\phi = D\phi^\dagger = 0$. It is then easiest to use the WZ-gauge to compute W_α. To facilitate things further, change variables to $y^\mu, \theta^\alpha, \overline{\theta}^{\dot\alpha}$ so that

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}\frac{\partial}{\partial y^\mu} \quad , \quad \overline{D}_{\dot\alpha} = \frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}, \tag{2.163}$$

and write

$$V_{\rm WZ} = \theta\sigma^\mu\overline{\theta}v_\mu(y) + i\theta\theta\,\overline{\theta\lambda}(y) - i\overline{\theta\theta}\,\theta\lambda(y) + \frac{1}{2}\theta\theta\overline{\theta\theta}\left(D(y) - i\partial_\mu v^\mu(y)\right). \tag{2.164}$$

Then, using $\sigma^\nu\overline{\sigma}^\mu - g^{\nu\mu} = 2\sigma^{\nu\mu}$, it is straightforward to find (all arguments are y^μ)

$$\begin{aligned} D_\alpha V_{\rm WZ} &= (\sigma^\mu\overline{\theta})_\alpha v_\mu + 2i\theta_\alpha\,\overline{\theta\lambda} - i\overline{\theta\theta}\,\lambda_\alpha + \theta_\alpha\,\overline{\theta\theta}D \\ &+ 2i(\sigma^{\mu\nu}\theta)_\alpha\overline{\theta\theta}\partial_\mu v_\nu + \theta\theta\overline{\theta\theta}(\sigma^\mu\partial_\mu\overline{\lambda})_\alpha, \end{aligned} \tag{2.165}$$

and then, using $\overline{DD\theta\theta} = -4$,

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + i(\sigma^{\mu\nu}\theta)_\alpha f_{\mu\nu}(y) + \theta\theta(\sigma^\mu\partial_\mu\overline{\lambda}(y))_\alpha, \tag{2.166}$$

with

$$f_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu, \tag{2.167}$$

being the Abelian field strength associated with v_μ.

Since W_α is a chiral superfield, $\int \mathrm{d}^2\theta\ W^\alpha W_\alpha$ will be a susy invariant Lagrangian. To obtain its component expansion we need the $\theta\theta$-term (F-term) of $W^\alpha W_\alpha$:

$$W^\alpha W_\alpha\Big|_{\theta\theta} = -2i\lambda\sigma^\mu\partial_\mu\overline{\lambda} + D^2 - \frac{1}{2}(\sigma^{\mu\nu})^{\alpha\beta}(\sigma^{\rho\sigma})_{\alpha\beta}f_{\mu\nu}f_{\rho\sigma}\ , \tag{2.168}$$

where we used $(\sigma^{\mu\nu})_\alpha^{\ \beta} = \mathrm{tr}\ \sigma^{\mu\nu} = 0$. Furthermore,

$$(\sigma^{\mu\nu})^{\alpha\beta}(\sigma^{\rho\sigma})_{\alpha\beta} = \frac{1}{2}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}, \tag{2.169}$$

(with $\epsilon^{0123} = +1$) so that

$$\int \mathrm{d}^2\theta\ W^\alpha W_\alpha = -\frac{1}{2}f_{\mu\nu}f^{\mu\nu} - 2i\lambda\sigma^\mu\partial_\mu\overline{\lambda} + D^2 + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}f_{\mu\nu}f_{\rho\sigma}. \tag{2.170}$$

Note that the first three terms are real while the last one is purely imaginary.

A supersymmetric particle model consists of a collection of particle supermultiplets and a set of potentials that describe the interactions between the particles. The three potentials relevant to supersymmetry are: the superpotential W, the Kähler potential K, and the potential V for the scalar fields in the theory, derived from W and K. For $N = 1$ supersymmetry in four spacetime dimensions, the two possible types of supersymmetric particle multiplets are: the chiral multiplet, with a complex scalar field f with spin 0 and a chiral (that is, either right or left handed) fermion y with spin $\frac{1}{2}$, and the vector multiplet, composed of a real (nonchiral) fermion I with spin $\frac{1}{2}$ and a vector field A_μ with spin 1. So in the end, in a model with several generations of chiral multiplets (f_i, y_i), the action with superpotential looks like (2.151).

In quantum field theory the phenomenon of spontaneous breakdown of chiral symmetry manifest itself by the noninvariance of the vacuum under the action of the corresponding axial charge, while the equations of motion remain invariant under the chiral transformations, except for the mass terms. In quantum mechanics we do not have an explicit equivalent state of the field theory vacuum, and therefore the noninvariance of the wave equations themselves. However, one has also to guarantee that the matrix elements of the *axial vector* current divergences are still proportional to the masses of the constituent particles.

In the next chapter we want briefly to discuss some useful methods for particle physics so called effective field theory (EFT). EFT is a very powerful tool to analyze physics at low energies, without having to solve the details of dynamics at higher energy scales. One does not need to know whether there are supersymmetric particles in the 1 TeV region in order to understand the interactions of electrons and photons at energies of the order of m_e.

Chapter 3

Effective Field Theory

The particle dynamics on the level of relativistic quantum mechanics stands between relativistic quantum field theory and nonrelativistic quantum mechanics. In a previous chapters we extended nonrelativistic to relativistic quantum mechanics. The problems is to derive nuclear structure from quantum field theory. If we do, we believe that we completely understand the nuclear structure.

The goal of this chapter is to demonstrate how to use effective field theory in NN interactions.We do not use this theory but phenomenological models of nuclear physics, we give a brief review of alternative models, which describe the short range potential models from low energy quantum field theory, because EFT is useful for systems with a clear separation of scales. Its aim is to finally arrive at the complete EFT that faithfully reproduces QCD in the low energy regime relevant for nuclear physics.

What is Effective Field Theory? In brief it's a low energy approximation to arbitrary high energy physics as the nucleon energies are typically well below the complex spectrum of hadrons that exist with masses ≥ 1 GeV.

We do not know what happens in the limits $k \to \infty$ or $r \to 0$. We have a theory for the sort distance details, but the resulting system is too complicated to handle. So we encode the short distance high energy physics. This yields a Schrödinger equation with a regularized potential, of which advocates claim that its better physics and fits parameters to observables like the NN phase shift.

In particular, Weinberg [561, 562] employed power counting to the irreducible NN interaction and obtained a leading order result by iterating such type of potential in a Lippmann-Schwinger equation. This type of summation is necessary to deal with the weakly bound state (or large S-wave scattering lengths) present in the two nucleon system [414]. This is in contrast to conventional chiral perturbation theory in the meson and meson nucleon sectors, where all interactions can be treated perturbatively. Meißner [415] studied Weinberg's approach at next-to-next-to leading order (NNLO), whose results are given in Ref. [446].

A novel power counting scheme was proposed by Kaplan, Savage and Wise (KSW) [351]. They encoded the short distance NN interaction in a derivative of the high order expansions of local operators. This is in contrast with the various models of extended NN potentials with free parameters chosen to fit scattering data. Oller [442] had established a new convergent scheme to treat analytical NN interactions from a chiral effective field theory. KSW amplitudes are resumed to fulfill the unitarity or right hand cut to all orders below pion production threshold. This is achieved by matching order by order in the KSW power counting the general expression of a partial wave with resumed unitarity cut, with the inverses of the KSW amplitudes. A similar power counting with perturbative pions has been suggested by Lutz [392].

The use of Effective Field Theories in nuclear physics has grown out of attempts to link the successful phenomenology of nonrelativistic potentials to the underlying theory of strong interactions QCD. Beane and Savage [58] considered the NN potential in quenched and partially quenched QCD.

As we have discussed the CD-Bonn [393, 401] and Argonne potential interaction uses the OBEP at large distances, and the phenomenological interaction at intermediate and small distances. One can also follow the standard ideology of the quantum field theory, and model the second piece by the exchange effects for heavier mesons. Larger meson masses mean shorter distances of the interaction, this we can understand adding more mesons and using the corresponding Yukawa interactions.

At present, we are probably not at all able to say what happens with the nucleons when they are put near to one another. However, we do not really need such a complete knowledge when describing low energy NN scattering and structure of nuclei. All what we need is some kind of parameterization of the short range, high energy effects when we look at their influence on the long range, low energy observables. Such separation of scales is at the heart of the EFT. Recently, ideas of the EFT for the NN scattering were pushed further, by also adding to Lagrangian density

$$\mathcal{D}_\mu = \partial_\mu + 2i\overrightarrow{t} \circ (\overrightarrow{z} \otimes \gamma^\mu \overrightarrow{D}_\mu),$$

terms which contain six quark fields, and calculating the full energy dependence of phase shifts and mixing parameters in all partial waves [211]. The resulting effective Lagrangian density has many adjustable parameters, but the number of these parameters is comparable to that used in the parameterization of Lagrangian by exchanges. Also the description of the NN scattering data is of a comparable quality.

This shows that the ideas of the EFT really work, namely, it is not important which physical mechanism is used to model the short range effects a purely phenomenological mechanism is equally good. Our knowledge of these short range effects can be summarized in a form of optical potential model that we shall discuss below.

Another point of effective field theory is that it can easily incorporate chiral symmetry, and can be naturally extended to discuss systems with strange quarks, such as hypernuclei [491] and kaon condensation [349, 458].

3.1 Nucleon Nucleon Interactions From EFT

EFT can be used in several different ways in nuclear physics. Historically, the first one was to set the separation scale Λ around the ρ-meson mass and keep as low energy degrees of freedom the pions and the nucleons (and possibly the Δ isobar), as well as photons and leptons [561, 562, 445, 446, 362, 363]. This approach builds on and extends the success of Chiral Perturbation Theory (ChPT) in the mesonic and one baryon sectors. It shares with nuclear potential models the fact that it describes nonrelativistic nucleons interacting through a potential, but it also brings a number of ingredients of its own, such as a small expansion parameter, consistency with the chiral symmetry of QCD, and systematic and rigorous ways of including relativistic corrections and meson exchange currents.

Another way of applying EFT ideas in nuclear physics is made possible by the existence of shallow bound states, that is, binding energies much below any reasonable QCD scale [59, 364, 365, 119]. We can then set Λ around the pion mass and keep as low energy degrees of freedom only the nucleons (and photons, leptons). At least in the case of two and three body systems the bound states will be within the range of validity of this simpler theory. This *pionless* effective theory can be considered as a formalization and extension of the old effective range theory (ERT) [69] and the work on *model independent results* in three body physics [209]. The new features, besides the existence of a small parameter on which to expand, appear in a number of new short distance contributions describing exchange currents and three body forces, as well as in relativistic corrections.

3.1.1 Power Counting

A necessarily ingredient for EFT is a power counting scheme that dictates which graphs to compute in order to determine an observable to a desired order in the expansion. We wish to expand the effective potential in terms of increasing order in Λ and m_π, so let us count powers of these quantities.

The main complication in the theory of nucleons and pions is the fact that a nucleon propagator $S(q) = i/(q_0 - \overrightarrow{q}^2/2M)$ scales like $1/Q$ if q_0 scales like m_π or an external three momentum, while $S(q) \sim M/Q^2$ if q_0 scales like an external kinetic energy. Similarly, in loops $\int \mathrm{d}q_0$ can scale like Q or Q^2/M, depending on which pole is picked up. To distinguish between these two scaling properties it is convenient to define generalized em n-nucleon potentials $V^{(n)}$ comprised of those parts of connected Feynman diagrams with $2n$ external nucleon lines that have no powers of M in their scaling (except from relativistic corrections). Since there is no nucleon-antinucleon pair creation in the effective theory such a diagram always has exactly n nucleon lines running through it. $V^{(n)}$ includes diagrams which are n-nucleon irreducible and parts of diagrams which are 1-nucleon irreducible. To compute the latter contribution to $V^{(n)}$ one identifies all combinations of two or more internal nucleon lines that can be simultaneously *on shell*, and exclude their pole contributions when performing the $\int dq_0$ loop integrations. An example of the 2-pion exchange contributions to $V^{(2)}$ is shown in Fig. 3.1. A general n-nucleon Feynman diagram in the EFT can be constructed

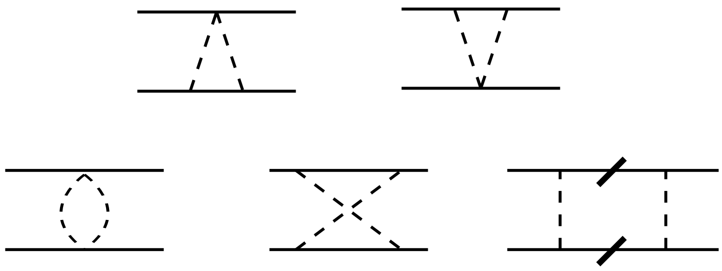


Figure 3.1: 2-pion exchange Feynman graphs contributing to the 2-nucleon potential $V^{(2)}$. The first four are 2-nucleon irreducible; the last diagram is 2-nucleon reducible, and the poles from the slashed propagators are not included in the $\int dq_0$ loop integration.

by contracting the nucleon legs of $V^{(r)}$ potentials with $r \leq n$. Treating the $V^{(r)}$'s like vertices, the $\int dq_0$ loop integrations pick up the poles of all the connecting nucleon lines. The reason for this construction is that within the $V^{(r)}$ potentials, all nucleon propagators are *off shell* and scale like $1/q_0 \sim 1/Q$. In contrast, when one picks up the pole contribution from one of the nucleon lines connecting the $V^{(r)}$ *vertices*, other nucleon lines will be almost *on shell*, and scale like M/Q^2. A contribution to the r-nucleon potential $V^{(r)}$ with ℓ loops, I_n nucleon propagators, I_π pion propagators, and V_i vertices involving n_i nucleon lines and d_i derivatives, scales like Q^μ, where

$$\begin{aligned} \mu &= 4l - I_n - 2I_\pi + \sum V_i d_i \quad , \\ \ell &= I_n + I_\pi - \sum V_i + 1 \quad , \\ I_n + r &= \frac{1}{2}\sum V_i n_i \quad . \end{aligned} \tag{3.1}$$

In this power counting we take $m_\pi \sim Q$ and treat factors of the u and d quark masses at the vertices as order Q^2. Combining these relations leads to the scaling law for the r-nucleon potential $V^{(r)}$ ($r \geq 2$):

$$\mu = 2 + 2\ell - r + \sum_i V_i (d_i + \frac{1}{2} n_i - 2) \quad . \tag{3.2}$$

Since chiral symmetry implies that the pion is derivatively coupled, it follows that $(d_i + \frac{1}{2}n_i - 2) \geq 0$, which implies that for a 2-nucleon potential, $\mu \geq 0$, and that $\mu = 0$ corresponds to tree diagrams. It is straight forward to find the scaling property for a general Feynman amplitude, by repeating the analysis that leads Eq. (3.2) treating the $V^{(r)}$ potentials as r-nucleon vertices with μ derivatives, μ given by Eq. (3.2). While Eq. (3.2) was derived assuming that $\int \mathrm{d}q_0 \sim Q$ and nucleon propagators scaled like $\sim 1/Q$, for these loop graphs they scale like Q^2/M and M/Q^2 respectively. A general Feynman diagram is constructed by stringing together r-nucleon potentials $V^{(r)}$.

3.1.2 Two Nucleon Scattering

We start by writing the Lagrangian involving only two nucleons. A system with two nucleons with zero angular momentum $L = 0$ can exist in a spin singlet ($^1S_0$) or spin triplet ($^3S_1$) state so there are two independent interactions with no derivatives,

$$\mathcal{L} = N^\dagger(i\partial_0 + \frac{\vec{\nabla}^2}{2M} + \ldots)N - C_{0t}(N^\dagger P_t N)^2 - C_{0s}(N^\dagger P_s N)^2 + \ldots, \tag{3.3}$$

where

$$\begin{aligned} P_t^i &= \frac{1}{\sqrt{8}}\sigma_2\sigma^i\tau_2, \\ P_s^A &= \frac{1}{\sqrt{8}}\tau_2\tau^A\sigma_2 \end{aligned} \tag{3.4}$$

are the projectors in the triplet and singlet *spin isospin* states (σ's act on spin space, τ's on isospin space), M is the nucleon mass and N the nucleon field.

The scattering matrix, $S = exp(2i\delta)$, is related to the NN scattering amplitude T by

$$S = 1 + i\frac{Mk}{2\pi}T.$$

Schwinger [505] has shown that the phase shift δ in the triplet *np* scattering is related to the wave number k by the relation

$$kcot\delta = -\gamma + \frac{1}{2}(\gamma^2 + k^2)r_0 + O(k^4 r_0^3),$$

where γ is related to the deuteron binding energy. We can write the T-matrix in terms of the phase shift

$$\begin{aligned} T &= \frac{4\pi}{M}\frac{e^{2i\delta} - 1}{2ik} = \frac{4\pi}{M}\frac{1}{k\cot\delta - ik} \\ &= \frac{4\pi}{M}\frac{1}{-\frac{1}{a_s} + \frac{r_{0s}}{2}k^2 + \ldots - ik}. \end{aligned} \tag{3.5}$$

It can be shown that for potentials of range $\sim R$ ($R \sim 1/m_\pi$ in this case), $k\cot\delta$ is an analytic function around $k = 0$ and that it has a cut starting at $k^2 \sim 1/R^2$, so it is well approximated by a power series as shown in the last line of Eq. (3.5). The parameter a_s (r_{0s}) is called the singlet scattering length (singlet effective range). For notational simplicity we specialize for now on the spin singlet channel.

The effective potential to order ν in the derivative expansion can be expressed in the form [561, 562, 446]

$$V^{(\nu)}(p',p) = \frac{1}{\Lambda^2}\sum_{n=0}^{\nu}\left[\frac{(p,p')}{\Lambda}\right]^n c_n, \tag{3.6}$$

where the sum here is over all possible terms extracted from (3.3) and Λ is the scale of the physics integrated out, taken to be m_π in the present context. Eq. (3.6) is intended to be symbolic; (p,p') indicates that either of these quantities may appear in the expansion, in any combination consistent with symmetry, with only their total power constrained. For instance, at $n = 4$ we have the structures $p^4 + p'^4$ and $p^2p'^2$ with coefficients c_4 and c_4', respectively. It is assumed that the coefficients c_n are natural; that is, of order unity. The fundamental assumption underlying effective field theory for the NN interaction is that this expansion in the potential, or equivalently that in the Lagrangian, may be sensibly truncated at some finite order ν.

The physical scattering amplitude is obtained by iterating the potential Eq. (3.6) using the Lippmann-Schwinger equation

$$T(p',p;E) = V(p',p) + M\int\frac{d^3q}{(2\pi)^3}\,V(p',q)\frac{1}{EM - q^2 + i\epsilon}T(q,p;E), \tag{3.7}$$

which generates the T-matrix. This procedure is illustrated in Fig. 3.2. By assumption, truncating the expansion Eq. (3.6) and retaining only its first few terms will be valid only for nucleon momenta well below Λ. It is clear that a method of regularizing the otherwise divergent integrals which occur when potentials such as Eq. (3.6) are inserted into the LS equation must be specified.

The graphs contributing to NN scattering generated by the Lagrangian in Eq. (3.3) are shown in Fig. 3.2. The L-loop graph factorizes into a power,

$$L\text{−loop graph} \sim (c\Lambda - ik)^L, \tag{3.8}$$

each one containing a linearly divergent piece and the unitarity cut ik (in the *center of mass* system with total energy k^2/M). The loop integral is linearly divergent and the coefficient c is dependent on the particular form of the regulator used, that is, the particular form the *high momentum* modes are separated from the *low momentum* ones. Using a sharp momentum cutoff, for instance, we have

$c = 2/\pi$, using dimensional regularization (DR), $c = 0$. The sum of all graphs in Fig. 3.2 is a geometrical sum giving

$$T = \frac{4\pi}{M} \frac{1}{-\frac{4\pi}{MC_{0s}} + c\Lambda - ik}. \tag{3.9}$$

We see then that terms shown explicitly in Eq. (3.3) reproduce the first term of the effective range expansion. The addition of terms with more derivatives will reproduce further terms in the effective range expansion.

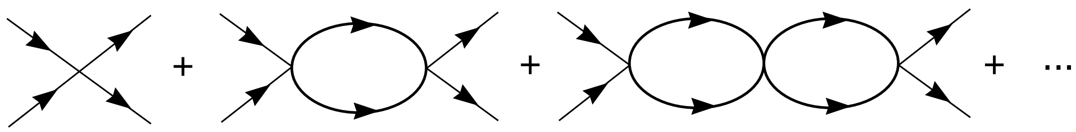


Figure 3.2: Graphs contributing to the LO NN scattering amplitude.

Let us consider two separate situations.

Natural case: For a generic potential with range R, the effective range parameters have typically similar size $a \sim r_0 \sim R$. Using DR, C_0 can be chosen to be $C_0 = 4\pi a/M$ (this choice is called minimal subtraction). The effective theory is valid for $k < 1/R$ and, in this range, T can be expanded as

$$T = \frac{4\pi}{M} \left(-a + ika^2 + \left(\frac{a^2 r_0}{2} + a^3 \right) k^2 + \dots \right). \tag{3.10}$$

Since $C_0 \sim a$, there is a one-to-one correspondence between the order in the ka expansion, the number of C_{0s} vertices and the number of loops in a graph. The leading order (LO) is given by one *tree level* diagram, the next-to-leading order (NLO) by the one-loop diagram, next-to-next-to-leading order (N^{2}LO) by the two-loop diagram involving C_0 and one *tree level* diagram with a two derivative vertex (not shown in Eq. (3.3)), and similarly for higher orders. We have then a *perturbative* expansion, even though the microscopic potential can be arbitrarily strong. If one uses a cutoff regulator the situation is slightly more complicated. Choosing $\Lambda \sim 1/R \sim 1/a$ we note that the most divergent piece of the *multi loop* graphs is as large as the *tree level* graph and must be resumed to all orders, while the energy dependent part containing powers of ikC_0 is suppressed. The pieces that need to be resumed at leading order merely renormalize the constant C_0. The one-to-one correspondence between the order in the ka expansion and the number of loops is lost in any but the DR with minimal subtraction renormalization/regularization scheme. The technical advantages arising from the use of DR and renormalization theory in this perturbative setting was used in the study of dilute gases with short range interactions in Ref. [85].

Unnatural case: In the nuclear case the scattering lengths of the two S-wave channels are much larger than the range of the potential. The physics corresponding to the large scattering lengths occurs at the QCD scale $M_{QCD} \sim 1$ GeV, what makes the discrepancy between nuclear and QCD scales even more startling. The origin of the finetuned cancellations leading to the disparity between the underlying scale and the S-wave scattering lengths (and deuteron binding energy) is presently unknown. It does not appear in any known limit of QCD like the chiral limit ($m_q \to 0$) or large number of colors ($N_c \to \infty$). We will just assume that this cancellation happens, track the dependence of observables on the new soft scale $1/a_{s,t}$ and perform our low energy expansion in powers of $kR \ll 1$ while keeping the full dependence on $ka_{s,t} \sim 1$. The singlet NN scattering amplitude, for instance, will be expanded as

$$T = -\frac{4\pi}{M}\left(\frac{a_s}{1+ika_s} + \frac{k^2 a_s^2 r_{0s}}{2}\frac{1}{(1+ika_s)^2} + \dots\right). \tag{3.11}$$

It is a little challenging to reproduce an expansion of this form in the EFT. If one uses a momentum cutoff, for instance, the constant C_{0s} has to be chosen to be $C_{0s} = (4\pi/M)(1/a_s + c\Lambda)$. The *one loop* graph is then suppressed compared to the tree level one by a factor $\sim MkC_{0s} \sim k/\Lambda$ and one would naively imagine that the *leading order* contribution is given solely by the *tree level* graph. But there are cancellations between the graphs in Fig. 3.2 and all these graphs need to be taken into account to reproduce the expansion of T above [364, 365]. In the NN scattering case considered here it is not difficult to see which graphs have to be included at each order, but in more complex situations this can be extremely tricky. A more convenient way to proceed is to use a renormalization prescription that shifts contributions from high momentum modes to the LECs in such a way as to eliminate this *accidental* cancellations between different diagrams. One can determine which diagrams contribute at each order on a diagram-by-diagram basis (manifest power counting). One way to do that is to use DR with a *power divergence subtraction* (PDS) [351].[1] In this scheme, we add and subtract to the denominator of the bubble sum in Eq. (3.9) an amount $M\mu/4\pi$, where μ is an arbitrary scale, and absorb the subtracted term in a redefinition of the constant $C_{0s}(\mu)$, that now is a function of μ. We have for the LO amplitude

$$T = -\frac{4\pi}{M}\frac{1}{\frac{4\pi}{MC_{0s}(\mu)} + ik + \mu}. \tag{3.12}$$

The constants $C_{0s}(\mu)$ is now chosen to be

$$C_{0s}(\mu) = \frac{4\pi}{M}\frac{1}{\frac{1}{a_s} - \mu}, \tag{3.13}$$

[1]Other schemes also solve this problem [413].

in order to reproduce the LO piece of the expansion in Eq. (3.11).

One can easily go to higher orders and include terms with derivatives in the Lagrangian. For instance, denoting by C_{2n} the coefficient of operators with $2n$ derivatives,

$$C_{2s} = \frac{4\pi}{M}\frac{r_{0s}}{2}\left(\frac{1}{\frac{1}{a_s}-\mu}\right)^2, \tag{3.14}$$

$$C_{4s} = \frac{4\pi}{M}\left(\frac{r_{0s}^2}{4}\left(\frac{1}{\frac{1}{a_s}-\mu}\right)^3+\frac{r_{1s}^3}{2}\left(\frac{1}{\frac{1}{a_s}-\mu}\right)^2\right), \tag{3.15}$$

where r_{1s} is the coefficient of the third term of the effective range expansion the *shape parameter*.

The β-function describing the evolution of the dimensionless coupling is

$$\mu\frac{\partial}{\partial\mu}\hat{c}_{0s}(\mu) = \hat{c}_{0s}(\mu)\left(1-\hat{c}_{0s}(\mu)\right), \text{ with } \hat{c}_{0s} \equiv -M\mu C_{0s}/4\pi. \tag{3.16}$$

The ${}^3S_1$ NN amplitude is parameterized as

$$T = \frac{4\pi}{M}\frac{1}{-\gamma+\frac{\rho(k^2+\gamma^2)}{2}+\ldots-ik}, \tag{3.17}$$

where γ^2/M is the deuteron binding energy and ρ the effective range parameter.

In the case of photons, some of these terms are just those required by gauge invariance and are determined by minimally coupling the photon to the nucleon Lagrangian. Their coefficients are thus fixed by NN scattering data and gauge invariance.

Consider some two nucleon operator of the form $X = C_{2n}^X N^\dagger N^\dagger \Gamma_X \vec{\partial}^{2n} NN$, where Γ_X is some tensor in *spin isospin* space. Its matrix element on two nucleon states is given by the diagrams involving the operator X *sandwiched* between two nucleon scattering amplitudes and by *one-loop* one body diagrams that do not involve X.

We have to make a distinction now between the cases where the operator X connects two S-wave states, two non-S-wave states, or one S-wave and one non-S-wave state. In the first case *renormalization group* invariance of the two nucleon matrix element of X implies

$$\mu\frac{\partial}{\partial\mu}C_{2n}^X(\mu)\left(\frac{T}{C_0(\mu)}\right)^2 = 0, \tag{3.18}$$

where T is the LO NN scattering matrix, which is μ independent. From that it follows that $C_{2n}^X(\mu)$ scales as $\sim(\mu-1/a)^{-2}$. Similarly, for the case where X connects one S-wave or no S-wave states $C_{2n}^X(\mu)$ scales as $\sim(\mu-1/a)^{-1}$ and

$\sim (\mu - 1/a)^0$, respectively. Using dimensional analysis to fix the powers of Λ we then have

$$C_{2n}^X(\mu) \sim \frac{1}{M(1/a-\mu)^\alpha}\frac{1}{\Lambda^{2n+1-\alpha}}, \tag{3.19}$$

where α is the number of S-wave states the operator X can connect (either $0, 1$ or 2).

In a nutshell, the power counting rules valid for the two nucleon system are [364, 365, 351, 413]:

$$\begin{aligned}
\text{fermion line} &\rightarrow M/Q^2, \\
\text{loop} &\rightarrow \frac{Q^5}{4\pi M}, \\
\vec{\partial} &\rightarrow Q, \\
\partial_0 &\rightarrow Q^2/M, \\
C_{2n} &\rightarrow \frac{4\pi}{M\Lambda^n Q^{n+1}}, \\
C_{2n}^X &\rightarrow \frac{4\pi}{M\Lambda^{2n+1-\alpha}Q^\alpha},
\end{aligned} \tag{3.20}$$

where C_{2n} is the coefficient of the two nucleon interaction with $2n$ derivatives, C_{2n}^X is the coefficient of a two nucleon operator with external current X and $2n$ derivatives, and Λ is the high energy scale $\Lambda \sim m_\pi$.

Using this rule we can determine the contributions to NN scattering at any given order. At LO, for instance, we have the series of diagrams shown in Fig. 3.2, with all the vertices containing no derivative. That is the only non perturbative resummation necessary. At NLO we have the insertion of one C_2 operator in a chain of C_0 operators. At N^{2}LO we have two insertions of C_2 and one insertion of C_4, and so on. The resulting $^3S_1$ phase shift, for example, is shown in Fig. 3.3, and compared to the Nijmegen phase shift analysis (PSA) [523]. Analytic expressions for the phase shifts can be found in Ref. [119]. They suggest convergence for momenta $k \lesssim 100$ MeV, as it is reasonable for an EFT without explicit pions. Electromagnetic effects in pp scattering were considered in the EFT approach in Ref. [368].

So, we can divide the potentials into a short range part and a long range part

$$V = V_S + V_L, \tag{3.21}$$

with V_S and V_L are the usual Gaussian type function, where $R \simeq 0.5 \pm 0.002 fm$ (we found it from the full relativistic equations, we will show later) is to be chosen such that the short range physics has on the shallow bound state to lie down.

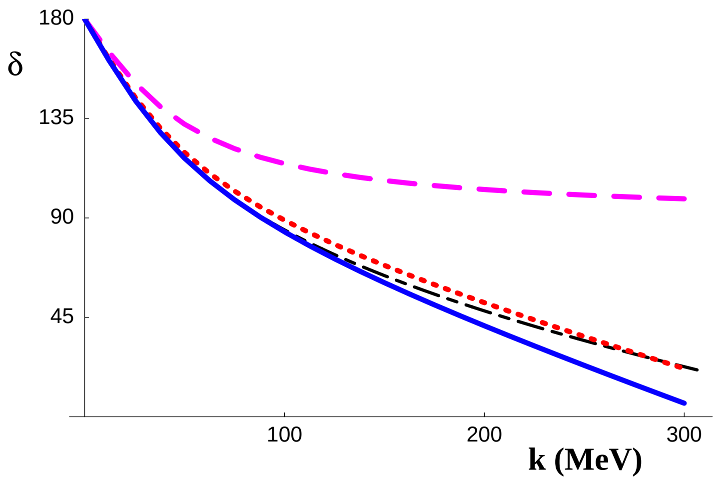


Figure 3.3: ${}^3S_1$ NN phase shift (in degrees) as function of the CM. The LO result is the dashed (purple) line, the N^{2}LO the dotted (red) line and N^{4}LO the thick (blue) solid curve. The dot-dashed (black) line is the Nijmegen PSA. From Ref. [119], courtesy of M. Savage.

3.2 Quark Gluon Plasma from EFT

The system initially composed of individual baryons and mesons whose substructure can at lower energies be studied only indirectly transforms at an energy density of roughly $1GeV/fm^3$ into a plasma of deconfined, though strongly interacting, quarks and gluons. This transition is today at the focus of considerable interest in the high energy and nuclear physics community due to the fact that quark gluon plasma (QGP), the phase of hot deconfined matter, is currently being produced in *ultra relativistic heavy ion* collisions at RHIC in Brookhaven. A similar large scale experiment is also being prepared at LHC in CERN.

The rapid progress in experimental *heavy ion* physics has set an important challenge for theorists. One needs to have a solid understanding of the processes that take place in the nuclear collisions and in particular obtain accurate numerical predictions for the different quantities measured. Whether or not the plasma produced in the present day experiments has had time to thermalize and reach an equilibrium state, it is clear that one of the most important and fundamental quantities describing the deconfined phase of QCD matter is the grand potential of quark gluon plasma. Its value is of relevance both to the study of the evolution of the heavy-ion collision products in terms of ideal hydrodynamics, and to cosmology, as the cooling rate of the very early universe depends on the energy and entropy densities of its content.

One of the fundamental properties of QCD is its asymptotic freedom [302, 457], which states that at large energies the gauge coupling constant g of the theory approaches zero. This can be seen most easily from the running of the coupling as obtained from the leading order solution to the renormalization group equation

$$g^2(\Lambda) = \frac{24\pi^2}{(11N - 2n_f)ln(\Lambda/\Lambda_{QCD})}, \tag{3.22}$$

where Λ is the renormalization scale and $\Lambda_{QCD} \sim 150$ MeV a free parameter corresponding to the characteristic energy scale of the theory. Asymptotic freedom implies that at very small distances the behavior of QCD tends to that of a free field theory making the use of perturbation theory feasible in the description of hard processes such as deep inelastic scattering. From the thermodynamic point of view this means that at least in the limit of asymptotically high temperatures or chemical potentials one might expect a perturbative approach to be fruitful in the computation of the partition function.

EFT methods have proven to be very powerful in treating plasmas at ultra-relativistic temperatures and densities. The EFT approach has recently been summarized compactly in the form of elegant effective Lagrangian's. For many applications, it would be useful to have a unified approach that works at all temperatures and densities. One promising approach is to generalize the EFT that describes the ultrarelativistic regime. The effective Lagrangian's for ultra-relativistic quark gluon system will be reviewed below. The plasma problem of high temperature QCD was first posed by Kalashnikov and Klimov in 1980 [343] and by Gross, Pisarski, and Yaffe in 1981 [303]. The problem was that a 1-loop calculation of the gluon damping rate, which is proportional to the imaginary part of the gluon self energy, gives a gauge dependent answer. Over the next 10 years, there were about a dozen published attempts to calculate the gluon damping rate, with almost as many different answers. In 1989, Pisarski pointed out that a 1-loop calculation of the damping rate is simply incomplete [456]. A consistent calculation to leading order in the QCD coupling constant g_s must include contributions from all orders in the loop expansion. He was able to carry out the necessary resummation explicitly for the damping rate of a heavy quark. The resummation consisted of replacing the gluon propagator in the 1-loop diagram for the heavy quark self energy by an effective gluon propagator obtained by summing up the hard thermal loop corrections (the terms proportional to $g_s^2T^2$) to the gluon *self energy*. This effective propagator was first calculated by Klimov and by Weldon [359, 565], who used it to study the propagation of gluons and the screening of interactions in the high temperature limit of the QGP.

The problem of the gluon damping rate is a little more complicated. It is not enough to replace the gluon propagators in the 1-loop gluon *self energy* diagrams by effective propagators, because there are also vertex corrections that are not suppressed by any powers of g_s.

In particular, the three gluon vertex has hard thermal loop corrections proportional to $g_s^3T^2$ which contribute at the same order as the bare vertex of order g_s. Similarly, the four gluon vertex has hard thermal loop corrections proportional to $g_s^4 T^2$, which contribute at the same order as the bare vertex of order g_s^4. The three gluon vertex, and the four gluon vertex are the complete set of diagrams that need to be resumed in order to calculate the damping rate to leading order in g_s. The result of this resummation has been proven to be gauge invariant, thus solving the plasmon problem.

The resummation required to solve the plasmon problem has a simple interpretation in terms of an effective field theory. The complete damping rate to leading order in g_s is given by the imaginary part of the 1-loop gluon self energy diagrams, with the gluon propagators replaced by effective propagators and with the three gluon and four gluon vertices replaced by effective vertices obtained by adding the hard thermal loop corrections to the bare vertices. This is equivalent to calculating 1-loop diagrams in an EFT whose propagator is the effective gluon propagator of Klimov and Weldon and whose vertices are the effective three gluon and four gluon vertices that Pisarski and Braaten [84].

These propagators and vertices are related by gauge invariance, just like their counterparts in the QCD Lagrangian. The Lagrangian density which summarizes the EFT for a QGP at ultrarelativistic temperature or density has the form

$$L_{eff} = L_{QCD} + L_{gluon} + L_{quark}. \tag{3.23}$$

The first term is the usual Lagrangian density for QCD:

$$L_{QCD} = -\frac{1}{2} tr G_{\mu\nu} G^{\mu\nu} + i \sum \overline{\psi} \gamma^\mu D_\mu \psi, \tag{3.24}$$

where $G_{\mu\nu} = G^a_{\mu\nu} T^a$ is the gluon field strength contracted with generators T^a that satisfy $tr(T^a T^b) = \delta^{ab}/2$. The sum is over n_f flavors of massless quarks. The second term in Eq. (3.23) is the thermal gluon term:

$$L_{gluon} = \frac{3}{2} m_g^2 tr G_{\mu\alpha} \langle \frac{P^\alpha P^\beta}{(P \cdot D)^2} \rangle G^m u_\beta, \tag{3.25}$$

where D is the gauge covariant derivative in the adjoint representation. The angular brackets $< f(P) >$ represent the average over the spacial directions $\widehat{p}$ of the lightlike four vector $P = (p, \overrightarrow{p})$. The coefficient m_g is the thermal gluon mass

$$m_g^2 = \frac{g^2}{3} T^2 + n_f \frac{g^2}{18} \left(T^2 + \frac{3}{\pi^2} \mu^2 \right), \tag{3.26}$$

where $T$is the temperature and μ is the quark chemical potential. The quark term in Eq. (3.23) is

$$L_{quark} = i m_q^2 \sum \overline{\psi} \gamma_\mu \langle \frac{P^\mu}{P \cdot D} \rangle \psi, \tag{3.27}$$

where m_q is the thermal quark mass

$$m_q^2 = \frac{g^2}{6} \left(T^2 + \frac{1}{\pi^2} \mu^2 \right). \tag{3.28}$$

Since an EFT approach seems to provide the most efficient description of the plasma at ultrarelativistic temperatures and densities, it would be desirable to have an EFT that describes the plasma at all temperatures and densities. The propagator of this EFT should reproduce accurately the dispersion relations for transverse photons, plasmons, and the charged particle modes of the plasma. The dispersion relations should be real valued, so that damping effects can be treated as perturbations. The EFT should also describe accurately the screening effects of the plasma. Finally, the Lagrangian for this field theory should be gauge invariant.

Chapter 4

Relativistic One Body Wave Problem

I believe that such a reminder is not only useful in setting the stage for the discussion of quantum theory, but more importantly, the purpose of this section is to specify explicitly the general approach that will be used later on in the quantization of constrained systems. Some of the points discussed below are well known and can be found in different places [76].

A correct quantum theory should satisfy the requirement of relativity, laws of motion valid in one inertial system must be true in all inertial systems. Stated mathematically, relativistic quantum theory must be formulated in a Lorentz covariant forms. The theory of quantum mechanics is built upon the fundamental concepts of wave functions and operators. Linear Hermitian operators act on the wave function and correspond to the physical observables, those dynamical variables which can be measured, e.g. position, momentum and energy.

We shall first consider the transition from nonrelativistic to relativistic quantum mechanics, we shall endeavor to retain the principles underlying the nonrelativistic theory. We review them briefly [76, 193]

1. The wave function

$$\psi(q_i..., s_i..., t) \tag{4.1}$$

is a complex function of all the classical degrees of freedom, $q_1...q_n$, of the time t and of any additional degrees of freedom, such as spin s_i, which are intrinsically quantum mechanics. The wave function has no direct physical interpretation; however, $|\psi(q_1...q_n, s_1...s_n, t)|^2 \geq 0$ is interpreted as the probability of the system having values $(q_1...q_n, s_1...s_n)$ at time t.

2. Every physical observable is represented by a linear hermitian operator. The operator correspondence in a coordinate realization is

$$E_i \rightarrow i\hbar\frac{\partial}{\partial t} \quad and \quad p_i \rightarrow -i\hbar\frac{\partial}{\partial q_i}. \tag{4.2}$$

3. A physical system is in an eigenstate of the operator Ω if

$$\Omega\Phi_n = \omega_n \Phi_n, \tag{4.3}$$

where Φ_n is the n^{th} eigenstate corresponding to the eigenvalue ω_n. For a hermitian operator, ω_n is real.

4. The expansion postulate states that an arbitrary wave function, or state function, for a physical system can be expanded in a complete orthonormal set of eigenfunctions ψ_n of a complete set of commuting operators (Ω_n). We write, then,

$$\psi = \sum_n a_n \psi_n, \tag{4.4}$$

where the statement of orthonormality is

$$\sum_s \int (dq_1 ...) \psi_n^*(q_1 ..., s ..., t) \psi_m(q_1 ..., s ..., t) = \delta_{nm},$$

$|a_n|^2$ records the probability that the system is in the n-th eigenstate.

5. The result of a measurement of a physical observable is any one of its eigenvalues. The average of many measurements of the observable Ω on identically prepared systems is given by

$$< \Omega >_\psi = \sum_s \int (dq_1 ...) \psi_n^*(q_1 ..., s ..., t) \Omega \psi(q_1 ..., s ..., t). \tag{4.5}$$

6. The time development of a physical system is expressed by the Schrödinger equation

$$i\hbar \frac{\partial \psi}{t} = \hat{H}\psi, \tag{4.6}$$

where the Hamiltonian H is a linear Hermitian operator. It has no explicit time dependence for a closed physical system, that is,

$$\frac{\partial H}{\partial t} = 0,$$

in which case its eigenvalues are the possible stationary states of the system, and a superposition principle follows from the linearity of H.

4.1 The Klein-Gordon Equation

The simplest physical system is that of an isolated free particle of spin 0. Let m be it's mass and e it's charge, and suppose that it is moving in the electromagnetic potential $A^\mu := (\pi, \overrightarrow{A})$. To find the wave equation we using the correspondence principle Eq. (4.2). Putting $p^\mu := (E, \overrightarrow{p})$, this rule can be written more simply

$$p^\mu \to i\partial^\mu,$$

following this it is natural to take as the Hamiltonian of relativistic free particle

$$H = \sqrt{p^2 + m^2},$$

and to write for a relativistic quantum analogue of Schrödinger equations

$$i\hbar \frac{\partial \psi}{\partial t} = \sqrt{-\hbar^2 \nabla^2 + m^2}\psi. \tag{4.7}$$

We remove the square root operator in Eq. (4.7), writing

$$H^2 = p^2 + m^2. \tag{4.8}$$

Equivalently, iterating Eq. (4.7) and using the fact that if $[A, B] = 0$, $A\psi = B\psi$ implies $A^2\psi = B^2\psi$, we have

$$-i\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = (-\hbar^2 \nabla^2 + m^2)\psi. \tag{4.9}$$

This is recognized as the classical wave equation

$$\left[\Box + \left(\frac{m}{\hbar}\right)^2\right] \psi = 0, \tag{4.10}$$

where

$$\Box = \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x^\mu}. \tag{4.11}$$

We have two difficulty

a) in squaring the energy relation we have introduced an extraneous negative energy root

$$H = -\sqrt{p^2 + m^2}. \tag{4.12}$$

b) If we interpret $(i\frac{\hbar}{2m})\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)$ as probability density ρ. However, this is impossible, since it is not a positive definite expression.

We shall find a first order equation, it still proves impossible to retain a positive definite probability density for a single particle while at the same time providing a physical interpretation of the negative energy root of Eq. (4.8).

4.2 The Dirac Equation

Dirac formulated his relativistic wave equation under the assumption that derivatives of both time and space coordinates should occur to first order. It turns out that such an equation describes particles of spin $\frac{1}{2}$. Such an equation might assume a form

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar c}{i} \left(\alpha_1 \frac{\partial \psi}{\partial x^1} + \alpha_2 \frac{\partial \psi}{\partial x^2} + \alpha_3 \frac{\partial \psi}{\partial x^3}\right) + \beta m\psi = H\psi. \tag{4.13}$$

The wave function ψ, in analogy with the spin wave function (See below) of non-relativistic quantum mechanics, is writing as a column matric with N components

$$\psi = (\psi_1 ... \psi_N)^T,$$

and the constant coefficients α_i and β are $N \times N$ matrices. We now discuss the correct energy momentum relation for a free particle. From Eq. (4.13), each component ψ_σ of ψ must satisfy the Klein-Gordon second order equation, i.e we require that

$$[E^2 - p^2 - m^2]\psi = 0. \tag{4.14}$$

Iterating Eq. (4.13), we find

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = -\hbar^2 \sum_{i,j=1}^{3} \frac{\alpha_j \alpha_i + \alpha_i \alpha_j}{2} \frac{\partial^2 \psi}{\partial x^i \partial x^j} + \frac{\hbar m c^3}{i} \sum_{i=1}^{3} (\alpha_i \beta + \beta \alpha_i) \frac{\partial \psi}{\partial x^i} + \beta^2 m^2 \psi. \tag{4.15}$$

This wave equation and equation Klein-Gordon are identical if the four operators β, α_i anticommute and if their squares are equal to 1

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij}, \tag{4.16a}$$

$$\alpha_i \beta + \beta \alpha_i = 0, \tag{4.16b}$$

$$\alpha_i^2 = \beta^2 = 1. \tag{4.16c}$$

Since the trace is just the sum of eigenvalues, the number of positive and negative eigenvalues ± 1 must be equal, and the α_i and β must therefore be even dimension matrices. The matrices Pauli hat N=2. Now we wish to include a mass term, and therefore the smallest dimension in which the α and β can be realized is N=4. In a particular explicit representation the matrices are

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where the σ_i are the 2×2 Pauli matrices and the entries in β stand for 2×2 unit matrices. From Eq. (4.13), we make the identification of probability density

$$\rho = \psi^* \psi, \tag{4.17}$$

as a positive definite.

It is necessary that the Dirac equation and continuity equation upon which its physical interpretation rest be covariant under Lorenz transformations. In discussing covariance it is desirable to express the Dirac equation in a four dimensional notation which preserves the symmetry between ct and x^i. To this end we multiply Eq. (4.13) by β/c and introduce the notation

$$\gamma^0 = \beta, \quad \gamma^i = \beta \alpha_i, \quad i = 1, 2, 3$$

This gives

$$\left(i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x) = 0. \tag{4.18}$$

From this equation for the free electron, we pass to the Dirac equation for an electron in the electromagnetic field $(\varphi, \overrightarrow{A})$ by making the substitution

$$p^\mu \to p^\mu - eA^\mu. \tag{4.19}$$

One obtains

$$\{\gamma_\mu(i\partial_\mu - eA_\mu) - m\}\psi = 0. \tag{4.20}$$

It is known that the Dirac equation has two dynamical symmetries, spin and pseudospin symmetry. Both are approximately realized in nature: spin symmetry in *heavy light mesons* and pseudospin symmetry in nuclei.

4.2.1 The Solutions of the Dirac Equation

Solving the Dirac equation is then equivalent to finding the eigen solutions of the Hamiltonian H. The four general form of a free particle solution is

$$\psi^r(x) = u^r(\mathbf{p})e^{i\epsilon_r(p_\mu x^\mu/\hbar)}, \tag{4.21}$$

where $u(\mathbf{p})$ is a four component spinor independent of $\overrightarrow{r}$. It is determined by the eigenvalue equation

$$Hu(\mathbf{p}) = Eu(\mathbf{p}), \tag{4.22}$$

where H is the following operator

$$H = \alpha p + \beta m. \tag{4.23}$$

A simple calculation gives

$$H^2 = p^2 + m^2. \tag{4.24}$$

The only possible eigenvalues of H are therefore the two values $\pm\sqrt{p^2+m^2}$, i.e

$$E = \varepsilon E_p, \qquad (\varepsilon = \pm 1), \tag{4.25}$$

$$E_p = \sqrt{p^2+m^2}. \tag{4.26}$$

The $u^r(\mathbf{p})$ satisfy the following relations

$$(p - \epsilon_r m)u^r(\mathbf{p}) = 0, \qquad \bar{u}_r(\mathbf{p})(p - \epsilon_r m) = 0, \tag{4.27}$$

$$\bar{u}^r(\mathbf{p})u^{r'}(\mathbf{p}) = \delta_{rr'\epsilon_r}, \tag{4.28}$$

$$\sum_{r=1}^{4} \epsilon_r u^r_\alpha(\mathbf{p})\bar{u}^r_\beta(\mathbf{p}) = \delta_{\alpha\beta}. \tag{4.29}$$

We may now superpose the plane wave solutions at our disposal to construct localized packets. These packets are still solutions of the free Dirac equation, as required by the superposition principle, since the Dirac equation is linear. The solution writing of free particle with positive negative energy following form

$$\psi(x,t) = \int \frac{d^3p}{(2\pi\hbar)^{3/2}} \sqrt{\frac{m}{E}} \sum_{\pm s} [b(p,s)u(p,s)e^{-ip^\mu x_\mu/\hbar} + d^*(p,s)v(p,s)e^{+ip^\mu x_\mu/\hbar}]. \tag{4.30}$$

We now look for the eigen solutions of a Dirac particle in a static central potential $V(r)$. The Dirac Hamiltonian is then

$$H = \alpha p + \beta m + V(r). \tag{4.31}$$

where

$$\alpha p := \alpha_r \left(p_r + \frac{i}{r}(1 + \sigma \mathbf{L}) \right). \tag{4.32}$$

We introduce the radial momentum

$$p_r := -i \frac{1}{r} \frac{\partial}{\partial r} r, \tag{4.33}$$

and the *radial velocity*

$$\alpha_r := \alpha \cdot \hat{\mathbf{r}} = \rho_1 (\sigma \cdot \mathbf{r})/r. \tag{4.34}$$

From identity

$$(\sigma \cdot \mathbf{A})(\sigma \cdot \mathbf{B}) = (\mathbf{A} \cdot \mathbf{B}) + i\sigma \cdot (\mathbf{A} \times \mathbf{B}), \tag{4.35}$$

one obtains

$$(\alpha \cdot \mathbf{r})(\alpha \cdot \mathbf{p}) = (\sigma \cdot)(\sigma \cdot \mathbf{p}) = \mathbf{rp} + i\sigma \cdot \mathbf{L} = rp_r + i(1 + \sigma \mathbf{L}). \tag{4.36}$$

Whence multiplying on the left by α_r/r and using the obvious property $\alpha_r^2 = 1$, the identity

$$\alpha \cdot \mathbf{p} := \alpha_r \left(p_r + \frac{i}{r}(1 + \sigma \cdot \mathbf{L}) \right). \tag{4.37}$$

After some operations, we are leads to two coupled differential equation for the radial functions $F(r)$ and $G(r)$, namely

$$\left[-\frac{d}{dr} + \frac{\epsilon(J + \frac{1}{2})}{r} \right] G = (E - m - V)F, \tag{4.38}$$

$$\left[\frac{d}{dr} + \frac{\epsilon(J + \frac{1}{2})}{r} \right] F = (E + m - V)G. \tag{4.39}$$

These equations here play the role of equation Schrödinger in the nonrelativistic theory.

We strive to maintain these familiar principles as underpinnings of a full relativistic constraint dynamics on the language Grassmann algebra. We shall then briefly consider the extension of the approach to systems with infinite number of freedom, such as meson exchange theories, and to systems with anticommuting or Grassmann degrees of freedom.

4.3 Spin One Half Particle in Grassmann Variables

4.3.1 Spin and Grassmannian Coordinates

The most obvious application area of anticommuting variables is the description of spin. The spin operators commute with position and angular momentum operators and this shows that spin is an intrinsic property which does not have a direct relation to *space time*. In the case of angular momentum, using Dirac's quantum mechanical formalism, we have angular momentum kets $|lm\rangle$ and angular coordinates kets $|\theta\phi\rangle$, which are related by Fourier transformation. The well known spherical harmonics functions are the transformation coefficients between the two representations. Now, extending this to half integer value of angular momentum, we realize that spin is characterized by its double valuedness. At this stage, because of their similarity, we will demand the existence of a parallel analogue between angular momentum and spin concepts. At this stage, the generalization of angular momentum kets to include half integer values will produce kets for spinors with $j = 1/2$. But where is the analogue of angular coordinate kets? Since spin has nothing to do with space time, a *suggestion* was made that spin can be described in an internal space which is distinct from space time [178]. Also, since fermions are anticommute among themselves, it is natural to introduce anticommuting coordinates to describe this internal space.

Therefore, append two *real* anticommuting coordinates, θ^1 and θ^2, to space time, then

$$(\theta^1)^2 = (\theta^2)^2 = \{\theta^1, \theta^2\} = 0. \tag{4.40}$$

The wave functions $\psi(\theta)$ will have a terminating Taylor expansion

$$\psi(\theta) = a\theta^1 + b\theta^2 + c\theta^1\theta^2 + d. \tag{4.41}$$

Introducing the antisymmetric metric tensor $\eta_{12} = -\eta_{21} = 1$ will give a symplectic invariant

$$(\theta)^2 := \theta^i\theta_i = \theta^i\eta_{ij}\theta^j = 2\theta^1\theta^2.$$

Note that $\theta^k\theta_k = -\theta_k\theta^k$. The corresponding covariant coordinates are defined by

$$\theta_i = \eta_{ij}\theta^j, \qquad \eta^{ij}\eta_{jk} = \delta^i_k, \tag{4.42}$$

so

$$\begin{aligned} \theta_1 &= \theta^2, \qquad \theta_2 = -\theta^1, \\ (\theta)^2 &:= \theta_i\eta^{ji}\theta_j = 2\theta_1\theta_2. \end{aligned}$$

One can then easily prove the following useful identities:

$$\theta^\alpha\theta^\beta = -\tfrac{1}{2}\epsilon^{\alpha\beta}\theta\theta, \qquad \bar\theta^\alpha\bar\theta^\beta = \frac{1}{2}\epsilon^{\alpha\beta}\bar\theta\bar\theta, \tag{4.43a}$$

$$\theta_\alpha\theta_\beta = \tfrac{1}{2}\epsilon_{\alpha\beta}\theta\theta, \qquad \bar\theta_\alpha\bar\theta_\beta = -\frac{1}{2}\epsilon_{\alpha\beta}\bar\theta\bar\theta, \tag{4.43b}$$

$$\theta\sigma^\mu\bar\theta\theta\sigma^\nu\bar\theta = \tfrac{1}{2}\theta\theta\bar\theta\bar\theta g^{\mu\nu}, \qquad \theta\psi\theta\chi = -\frac{1}{2}\theta\theta\psi\chi. \tag{4.43c}$$

Next, define the Grassmann operators θ_i with eigenvalues θ_i,

$$\theta^i|\theta\rangle = |\theta\rangle\theta^i,$$

and construct the Grassmannian momentum operators Π_i, which are the displacement operators on wave functions in internal Grassmann space, the analogue of momentum operators in space time,

$$\Pi_i := -i\hbar\frac{\partial}{\partial\theta^i}, \tag{4.44}$$

$$\Pi_i|\pi\rangle = |\pi\rangle\pi_i. \tag{4.45}$$

The analogue of the Heisenberg commutation relations are

$$\{\theta^i,\theta^j\} = 0, \tag{4.46a}$$

$$\{\Pi_i,\Pi_j\} = 0, \tag{4.46b}$$

$$\{\theta^i,\Pi_j\} = i\hbar\delta^i_j. \tag{4.46c}$$

Now all the rest is just an imitation of the usual position and momentum operators relations in space time, except that we should apply anticommutation rules for fermionic operators and commutation rules for bosonic operators. The Grassmannian $Sp(2)$ rotation operators in the internal θ-space are defined as symmetric products of θ_i and Π_i

$$\Lambda_{ij} := \theta_i\Pi_j + \theta_j\Pi_i, \tag{4.47}$$

and are bosonic operators. Then the commutation relations for these operators are

$$[\Lambda_{11},\Lambda_{22}] = -4i\hbar\Lambda_{12}, \tag{4.48a}$$

$$[\Lambda_{12},\Lambda_{11}] = +2i\hbar\Lambda_{11}, \tag{4.48b}$$

$$[\Lambda_{12},\Lambda_{22}] = -2i\hbar\Lambda_{22}. \tag{4.48c}$$

Comparing these to the commutation relations of angular momentum operators J

$$[J_+,J_-] = +2\hbar J_3, \tag{4.49a}$$

$$[J_3,J_+] = +\hbar J_+, \tag{4.49b}$$

$$[J_3,J_-] = -\hbar J_-, \tag{4.49c}$$

we can identify spin generators $\mathbf{S}$ as

$$\Lambda_{11} = 2S_+, \quad \Lambda_{22} = 2S_-, \quad \Lambda_{12} = 2iS_3, \tag{4.50}$$

or

$$\begin{aligned} S_1 &= (\theta_1\Pi_1 + \theta_2\Pi_2)/2, & (4.51a)\\ S_2 &= (\theta_1\Pi_1 - \theta_2\Pi_2)/2i, & (4.51b)\\ S_3 &= (\theta_1\Pi_2 + \theta_2\Pi_1)/2i. & (4.51c)\end{aligned}$$

There still remains one antisymmetric product of θ and Π, i.e.

$$\Sigma = (\theta_1\Pi_2 - \theta_2\Pi_1)/i\hbar \rightarrow (\theta_1\frac{\partial}{\partial\theta_1} + \theta_2\frac{\partial}{\partial\theta_2}), \tag{4.52}$$

which acts as a scale operator on spin wave functions of θ. Therefore

$$\mathbf{S}^2 = 3\hbar^2\Sigma(\Sigma+1)/8, \tag{4.53}$$

and applying this to a linear function of θ,

$$\psi(\theta) = \langle\theta|\psi\rangle = a\theta^1 + b\theta^2, \tag{4.54}$$

will give the correct result

$$\mathbf{S}^2 = 3\hbar^2/4.$$

Since

$$\begin{aligned} S_3\theta^1 &= +\frac{1}{2}\hbar\theta^1, & (4.55a)\\ S_3\theta^2 &= -\frac{1}{2}\hbar\theta^2, & (4.55b)\end{aligned}$$

it is confirmed that we are truly dealing with spin $\frac{1}{2}$, and the wavefunction of a particle with spin $\frac{1}{2}$ can be written as a linear combination of the Grassmannian coordinates θ^1 and θ^2. Here (θ^1, θ^2) constitutes a spin-$\frac{1}{2}$ doublet.

The conjugation property of these Grassmannian coordinates are defined as

$$\begin{aligned} (\theta^i)^* &= -\theta_i = \theta^j\eta_{ji}, & (4.56)\\ (\theta^1\theta^2)^* &:= (\theta^1)^*(\theta^2)^*, & (4.57)\end{aligned}$$

from which the hermiticity of spin operators $\mathbf{S}$ is maintained, and the following requirement for normalization is satisfied,

$$\begin{aligned} \langle\psi|\psi\rangle &= \int \langle\psi|\theta\rangle d\theta^2 d\theta^1 \langle\theta|\psi\rangle \\ &= \int (a\theta^1 + b\theta^2)^* d\theta^2 d\theta^1 (a\theta^1 + b\theta^2) \\ &= |a|^2 + |b|^2 = 1. & (4.58)\end{aligned}$$

Hence we are effectively manipulating two degrees of freedom which are equivalent to two *real* Grassmannian coordinates.

The description of scalar particle states (spin 0) is given by the combinations of the two $Sp(2)$ singlets, 1 and $\theta^1\theta^2$

$$\phi(\theta) = \langle\theta|\phi\rangle = \frac{1}{\sqrt{2}}(1 \pm \theta^1\theta^2). \tag{4.59}$$

The normalization condition follows directly from conjugation properties

$$\begin{aligned} \langle\phi|\phi\rangle &= \int \langle\phi|\theta\rangle d\theta^2 d\theta^1 \langle\theta|\phi\rangle \\ &= \int \frac{1}{2}(1 \pm \theta^1\theta^2)^* d\theta^2 d\theta^1 (1 \pm \theta^1\theta^2) \\ &= 1. \end{aligned} \tag{4.60}$$

One simple example of the application of this idea is Grassmannian formalism of the electromagnetic interactions. Consider a spin $\frac{1}{2}$ particle of unit charge placed in an electromagnetic field. The interaction can be determined by encountering the standard minimal substitution,

$$\mathbf{p} \longrightarrow \mathbf{p} + \mathbf{A}. \tag{4.61}$$

In a uniform magnetic field $\mathbf{B}$, the vector potential can be identified as

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{x}, \tag{4.62}$$

and the substitution yields the familiar Zeeman coupling:

$$\mathbf{p}^2 \rightarrow (\mathbf{p} + \mathbf{A})^2 = \mathbf{p}^2 + \mathbf{B} \cdot \mathbf{L} + \frac{1}{4}(\mathbf{B} \times \mathbf{x})^2. \tag{4.63}$$

In order to account for the *spin magnetic* field interaction, the anticommuting coordinates θ are introduced. The generalization of the substitution rule is given by

$$\pi \longrightarrow \pi + g\alpha, \tag{4.64}$$

where

$$\alpha = \mathbf{B} \otimes \theta, \tag{4.65}$$

is a $Sp(2)$ spinor with components,

$$\alpha_1 = -\frac{i}{2}B_3\theta_1 + \frac{1}{2}(B_1 + iB_2)\theta_2, \tag{4.66a}$$

$$\alpha_2 = \frac{i}{2}B_3\theta_2 - \frac{1}{2}(B_1 - iB_2)\theta_1. \tag{4.66b}$$

The analogue of Eq. (4.63) is

$$\begin{aligned} \Pi_1\Pi_2 &\rightarrow \Pi_1\Pi_2 + g(\alpha_1\Pi_2 + \alpha_2\Pi_1) + g^2\alpha_1\alpha 2 \\ &= \Pi_1\Pi_2 + g\mathbf{B}\cdot\mathbf{S} + \frac{1}{4}g^2B^2\theta_1\theta_2, \end{aligned} \tag{4.67}$$

which is the contribution of kinetic energy from the anticommuting coordinates and correctly adds to the orbital interaction. The factor g introduced above is a reflection of the fact that the Lande factor is not fixed in a nonrelativistic description. Let the direction of $\mathbf{B}$ defines the z-axis, and substitute,

$$\Pi_1\Pi_2 + g\mathbf{B}\cdot\mathbf{S} \rightarrow \frac{1}{2}g\hbar B\left(\theta_1\frac{\partial}{\partial\theta_1} - \theta_2\frac{\partial}{\partial\theta_2}\right) + \hbar^2\frac{\partial^2}{\partial\theta_2\partial\theta_1}, \tag{4.68}$$

then one readily gets the eigenfunctions,

$$\psi(\theta) = \theta_1,\ (\frac{1}{2}g\hbar B + \frac{1}{4}g^2B^2\theta_1\theta_2)\phi; \qquad E = \frac{1}{2}g\hbar B, \tag{4.69a}$$

$$\psi(\theta) = \theta_2,\ (-\frac{1}{2}g\hbar B + \frac{1}{4}g^2B^2\theta_1\theta_2)\phi; \qquad E = -\frac{1}{2}g\hbar B, \tag{4.69b}$$

for spin up and down. Notice the occurrence of a scalar part ϕ with the same eigenvalues as the spinors.

4.3.2 Nonrelativistic Spin One Half Particle with Constraints

As a next simple example we outline the discussion of the nonrelativistic spinning particles [111].

We consider a nonrelativistic free particle with position coordinates $x^i(t), i = 1, 2, 3$. For the purpose of describing the spin degrees of freedom we associate with the particle three real anticommuting variables

$$\theta^i = \theta^i(t),$$

in addition to the position coordinates. We write the Lagrangian for the free nonrelativistic spin one half particle as

$$L = \frac{1}{2}m\dot{x}^2 + \frac{i}{2}\dot{\theta}\cdot\theta. \tag{4.70}$$

To pass to the Hamiltonian we define the conjugate momenta

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}_i, \tag{4.71}$$

$$\pi_k = \frac{\partial L}{\partial \dot{\theta}^k} = \frac{i}{2}\theta_k, \tag{4.72}$$

whose generalized Poisson brackets are

$$\{x^i, p_j\} = \delta^i_j, \tag{4.73a}$$

$$\{\theta^k, \pi_l\} = \delta^k_l, \tag{4.73b}$$

all other being zero. We obtain the primary constraints from Eq. (4.72)

$$\chi_k = \pi_k - \frac{i}{2}\theta_k \approx 0, \tag{4.74}$$

which are consequences of the linearity of L, in $\dot{\theta}$. To check whether there are secondary constraints we write down the total Hamiltonian

$$H_T = H_c + \lambda^i \chi_i = \frac{\mathbf{p}^2}{2m} + \lambda^i \chi_i, \tag{4.75}$$

where the λ^i are anticommuting Lagrange multipliers. The consistence conditions

$$\dot{\chi}^i = \{\chi^i, H_T\} \approx 0, \tag{4.76}$$

lead to $\lambda^i = 0$. Therefore we have secondary constraints and the χ_i are second class:

$$\{\chi_i \chi_k\} = i\delta_{ik}. \tag{4.77}$$

It is now possible to introduce Dirac brackets based on the second class constraints χ_k; after this is done one one may consider $\chi_k = 0$ as strong equations thus eliminating the π_k from the theory, which leaves the θ_i as the only Fermi variables, with modified brackets

$$\{\theta^i, \theta^j\}^* = i\delta^{ij}. \tag{4.78}$$

The equation of motion are obtained by extremizing the action under small deformations of the history of the system. The allowed deformations must obey boundary conditions he number of which is equal to the number of integration constants in the general solution of the equations of motion. Now, for the bose variables x_i, we demand as usual that x_i be fixed at the initial and final times. It is not possible however to impose a similar requirement on the θ's, as this would imply two boundary conditions for a first order differential equation.

We write the action as

$$S = \int_{t_1}^{t_2} L dt + \frac{i}{2}\theta(t_1) \cdot \theta(t_2), \tag{4.79}$$

and state that the solution of the equations of motion are those histories which yield no variation of S under the conditions

$$\delta\mathbf{x}(t_1) = 0, \quad \delta\mathbf{x}(t_2) = 0, \tag{4.80}$$

$$\delta\theta(t_1) + \delta\theta(t_2) = 0. \tag{4.81}$$

The x-dependence of the action is the usual one so it need not concern us any longer. The novelty is in the θ part.

If we vary $\theta(t)$ we find

$$\begin{aligned}\delta S = & \int_{t_1}^{t_2} dt i\dot{\theta}\cdot\delta\theta + \frac{i}{2}[\delta\theta(2)\cdot\theta(2) - \delta\theta(1)\cdot\theta(1)] + \frac{i}{2}[\delta\theta(1)\cdot\theta(2) - \theta(1)\cdot\delta\theta(2)] \\ = & \int_{t_1}^{t_2} dt i\dot{\theta}\cdot\delta\theta - \frac{i}{2}[\delta\theta(1)+\delta\theta(2)]\cdot[\theta(1)-\theta(2)], \end{aligned} \tag{4.82}$$

where we abbreviated $\delta\theta(t_1) = \delta\theta(1)$, etc. The boundary term vanishes on account of condition Eq. (4.81) and extermination of S yields just

$$\dot{\theta}_i = 0, \tag{4.83}$$

is needed.

Now, let us suppose that $\theta(1) + \theta(2)$ is given as 2χ, say. In that case there is a unique solution to Eq. (4.83) with that boundary condition, namely $\theta(t) = \chi$ for all t is fulfilled.

The action can be rewritten in Hamiltonian from as

$$S = \int_{t_1}^{t_2} dt \left(\dot{\mathbf{x}}\cdot\mathbf{p} + \frac{i}{2}\dot{\theta}\cdot\theta - \frac{\mathbf{p}^2}{2m}\right) + \frac{1}{2}\theta(1)\cdot\theta(2). \tag{4.84}$$

Having an action principle we can discuss conservation laws. The action Eq. (4.79) is invariant under translations, rotations, and Galilean transformations. Let us therefore analyze the case of rotations. Under that translation we write

$$\delta x^i = \omega^i_j x^j, \quad \delta p^i = \omega^i_j p^j, \quad \delta\theta^i = \omega^i_j \theta^j, \tag{4.85}$$

with $\omega_{ij} = -\omega_{ji}$.

The action Eq. (4.79) is clearly invariant under this transformation. On the other hand, following Noether's procedure, we can rewrite the variation of the action as

$$\begin{aligned}\delta S = & \mathbf{x}\cdot\mathbf{p}|{t_1}^{t_2} - \frac{i}{2}[\delta\theta(1)+\delta\theta(2)]\cdot[\theta(1)-\theta(2)] \\ & +(\text{term vanishing when the equations of motion hold}).\end{aligned} \tag{4.86}$$

Now, the θ term in the above equation can be rewritten in this case as

$$\frac{i}{2}\omega_{ik}[\theta^k(1)+\theta^k(2)][\theta^j(1)-\theta^j(2)] = \frac{i}{2}\omega_{ik}[\theta^j(2)\theta^k(2) - \theta^j(1)\theta^k(1)]. \tag{4.87}$$

If we now insert Eq. (4.87) into Eq. (4.86) and recall Eq. (4.85) we find that

$$J_{ik} = L_{ik} + S_{ik}, \tag{4.88}$$

is a constraint of motion, where

$$L_{ik} = x_i p_k - x_k p_i, \tag{4.89}$$

$$S_{ik} = i\theta_i\theta_k. \tag{4.90}$$

In terms of their Dirac brackets L_{ik} and S_{ik} obey the customary algebra. For example, if we define the spin vector

$$S_i = -\frac{1}{2}\epsilon_{ijk}S_{jk}, \tag{4.91}$$

we have

$$\{S_i, S_j\} = \epsilon_{ijk}S_k. \tag{4.92}$$

It is interesting to mention here that had we neglected the surface term in Eq. (4.79) and applied Noether's procedure naively to the action $S = \int Ldt$ we would have arrived at a definition for the spin of opposite sign to Eq. (4.90.)

4.3.3 Relativistic Spin One Half Particle

Let us start from the Dirac equation

$$(\hbar\gamma^\mu\partial_\mu + m)\psi = 0, \tag{4.93}$$

which implies the Klein-Gordon equation

$$(-\hbar^2\Box^2 + m^2)\psi = 0. \tag{4.94}$$

To formulate the dynamics, and additional constraint is necessary, and to take this constraint explicitly invariant introduce a new Grassmann variable θ_5. The constraint is in quantum falls

$$(p\theta) + \theta_5 = 0, \tag{4.95}$$

and Klein-Gordon equation reads

$$(\hat{p}^\mu\hat{p}_\mu + m^2) = 0. \tag{4.96}$$

The commutation relations for the quantum operators are

$$[\hat{p}_\mu, \hat{q}_\nu]_- = -i\hbar g_{\mu\nu}, \quad [\theta_\mu, \theta_\nu]_+ = \hbar g_{\mu\nu}, \quad [\theta_5, \theta_5]_+ = \hbar, \tag{4.97}$$

while the primary constraints are converted into conditions on the physical states

$$\mathcal{L} = [(\hat{p}\hat{\theta}) + m\hat{\theta}_5]\psi \approx 0, \quad \mathcal{H} = (p^2 + m^2)\psi \approx 0, \tag{4.98}$$

which obey the relations

$$\begin{aligned}
[\mathcal{L}, \mathcal{L}] &= i\mathcal{H}, \\
[\mathcal{L}, \mathcal{H}] &= 0, \\
[\mathcal{H}, \mathcal{H}] &= 0,
\end{aligned} \tag{4.99}$$

where

$$\hat{\theta}_\mu = (\hbar/2)^{1/2}\gamma_5\gamma_\mu, \quad \hat{\theta}_5 = (\hbar/2)^{1/2}\gamma_5, \tag{4.100}$$

satisfy

$$[\theta^\mu, \theta^\nu]_+ = -\hbar g^{\mu\nu}, \quad [\theta_5, \theta^\mu]_+ = 0, \quad [\theta_5, \theta_5]_+ = -\hbar. \tag{4.101}$$

Generators of the Lorenz group $J_{\mu\nu}$ are constructed along the conventional lines

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}. \tag{4.102}$$

The quantum spin vector is then given by

$$\hat{S}_i = -\frac{1}{2}\epsilon_{ijk}\hat{\theta}_j\hat{\theta}_k = -\frac{\hbar}{4}i\epsilon_{ijk}\sigma_j\sigma_k = i\frac{\hbar}{2}\sigma_i. \tag{4.103}$$

To get the quantum operator as (anti)symmetrization is necessary

$$\hat{L}_{\mu\nu} = \frac{1}{2}(\hat{q}_\mu\hat{p}_\nu + \hat{p}_\nu\hat{q}_\mu - \hat{q}_\nu\hat{p}_\mu - \hat{p}_\mu\hat{q}_\nu), \tag{4.104}$$

$$\hat{S}_{\mu\nu} = -\frac{i}{2}(\hat{\theta}_\mu\hat{\theta}_\nu - \hat{\theta}_\nu\hat{\theta}_\mu) := \frac{i}{4}\hbar(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \tag{4.105}$$

Grassmann coordinates schemes offer a natural framework for explaining the occurrence of particle generations and the application of duality constraints turns out to be a powerful tool for pruning the overabundance of states [179]. The nature of the interaction between different fields is well determined by the anticommuting properties and integration rules of Grassmann variables. The assignment of the correct component fields in the superfields and the construction of the appropriate forms of the interaction between superfields are the main problems in this formulation as well as the choice of the superfields itself. The dimension of the Grassmann manifold determines wholly the results that follow and must therefore be chosen carefully. There is no explicit rule in determining the number of Grassmann coordinates in one model of the theory, but the symmetry of the model can offer guidance as to how many anticommuting variables are needed to incorporate the corresponding internal symmetry. In other word, the internal space is the Grassmannian space. The group structures in the Grassmannian manifold are represented by generators on the Grassmann coordinates which reshuffle the monomials. This is in effect similar to the reshufflings of different states associated with each monomials. The imposition of the duality constraints will reduce the size of the residual group symmetry.

Chapter 5

Relativistic Two Body Wave Equations

To formulate the quantum theory of a particle, Dirac [189] used the Hamiltonian description of classical mechanics. The standard rules for constructing the momenta and the Hamiltonian function, however, cannot be applied when the Lagrangian is singular. In such a case it is not possible to extract the functional dependence of all the velocities on the momenta in order to obtain a Hamiltonian function of coordinates and momenta only. Dirac's method concerns the study of classical systems using the Hamiltonian method when the usual procedure fails due to the singularity of the Lagrangian [194]. Dirac gave very general rules to construct the Hamiltonian and calculate sensible brackets that can be used to describe the classical and, by the canonical quantization procedure, the quantum dynamics, which we studied in Sec. 2.2.3.

In this chapter, we are using techniques developed by Dirac to handle constraints in quantum mechanics and the method developed by Crater and Van Alstine. They derive the two body Dirac equations for eight nonderivative Lorentz invariant interactions acting separately or together [158, 388]. These include world scalar, four vector and pseudoscalar interactions among others. We can also reduce the two body Dirac equations to coupled Schrödinger like equations even with all these interactions acting together. Before we test this method in nuclear physics, in the phase shift analysis of the NN scattering problems, we review the constraint formalism and the form of the two body Dirac equations.

5.1 Hamiltonian Formulation of the Two Body Problem with Constraint Dynamics

We know from Sec. 2.2.3, that constraints of the form

$$\phi_j(q,p) \approx 0,\; j = 1...M, \tag{5.1}$$

which must be satisfied *weakly* along the physical trajectory. The constraint Eq. (5.1) say in effect that we cannot roam over the full plane space (q, p), we are confined to a particular lower dimensional hyper surface of the full space. First, we can with impunity have been carried out, need the constraints $\phi_j(q, p) = 0$, imposed [289, 417, 536]. The weakly sign $\approx$ signal this hold procedure. In this way we obtain M number of constraints which Dirac called *primary* because of their direct derivation from the Lagrangian. Notice that a Hamiltonian is required to be independent of the velocities. If we are not able to erase the $\dot{q}$ dependence, then the straightforward application of the Hamiltonian method is impossible. To solve this problem we proceed as follows. We add to H all *primary* constraints multiplied by arbitrary functions of time λ_j , to obtain the total Hamiltonian $\mathcal{H}$. The Hamiltonian is obtained by a Legendre transformation of the *velocities*

$$\mathcal{H} = H + \sum_{j=1}^{M} \lambda_j \phi_j(q, p). \tag{5.2}$$

For consistency, the constraints must not change under the temporal evolution of our system establishing the consistency equations

$$\dot{\phi}_j = [\phi_j, \mathcal{H}] \approx 0, \quad j = 1, ..., M. \tag{5.3}$$

If these equations are consistent, three cases are possible, an equation can give an identity, it can give a linear equation for the λ_j, it can give an equation containing only p's and q's, in which case it must be considered as another constraint. The constraints that arise from this procedure will be called *secondary*, for obvious reasons. Also we emphasize again that any linear combination of constraints is again a constraint.

Todorov studied the relativistic N - particle dynamics as a problem with constraints of the type

$$2\phi_i(q, p) := m_i^2 - p_i^2 + \Phi_i(q, p, s), \quad i = 1, ..., N, \tag{5.4}$$

where $\Phi_i(q, p, s)$ are Poincaré invariant functions of the particles, coordinates, momenta and spin components. We introduce p_i and position q_i of a particle, whose Poisson bracket are satisfying the usually Poincaré transformation. However, the Poisson brackets are degenerate on the mass shell for a relativistic particle

$$m^2 - p^2 = 0, \tag{5.5}$$

Crater termed it *relativistic Hamiltonian*. Caveat, Crater and et al. use a metric with the other sign[1]. He extended Todorov's idea for four dimensional space-time form in the two body system, he had two constraints

$$\phi_i(q, p) \approx 0, \qquad j = 1, 2. \tag{5.6}$$

[1] $\eta_{\mu\nu} = -g_{\mu\nu} = (-1, 1, 1, 1)$.

For spinless particles he had taken to be the generalized mass shell constraints of the two particles [153, 582], namely Eq. (5.4) rewriting

$$\mathcal{H}_1 = p_1^2 + m_1^2 + \Phi_1(x, p_1, p_2) \approx 0,$$
$$\mathcal{H}_2 = p_2^2 + m_2^2 + \Phi_2(x, p_1, p_2) \approx 0, \tag{5.7}$$

where $\mathcal{H}_1, \mathcal{H}_2$ are covariant constraints on the dynamical variables four momentum p_1, p_2. The interaction functions Φ_1, Φ_2 must be equal $\Phi_1 = \Phi_2$ a relativistic analog of Newton's third law with relative distance $x = x_1 - x_2$. The total Hamiltonian $\mathcal{H}$ from these constraints alone is

$$\mathcal{H} = \lambda_1 \mathcal{H}_1 + \lambda_2 \mathcal{H}_2, \tag{5.8}$$

with λ_i as Lagrange multipliers.

Thus the quantum forms for each individual particle constraint become Schrödinger type equations [497]

$$\mathcal{H}_i |\psi\rangle =, 0 \qquad \text{for} \qquad i = 1, 2. \tag{5.9}$$

In order that each of these constraints be conserved in time we must have

$$[\mathcal{H}_i, \mathcal{H}] |\psi\rangle = i \frac{d\mathcal{H}_i}{d\tau} |\psi\rangle = 0. \tag{5.10}$$

The CM eigentime τ is used, so that

$$[\mathcal{H}_i, \lambda_1 \mathcal{H}_1 + \lambda_2 \mathcal{H}_2] |\psi\rangle =$$
$$\{[\mathcal{H}_i, \lambda_1] \mathcal{H}_1 |\psi\rangle + \lambda_1 [\mathcal{H}_i, \mathcal{H}_1] |\psi\rangle + [\mathcal{H}_i, \lambda_2] \mathcal{H}_2 |\psi\rangle + \lambda_2 [\mathcal{H}_i, \mathcal{H}_2]\} |\psi\rangle = 0. \tag{5.11}$$

Using Eq. (5.9), the above equation leads to the compatibility condition between the two constraints

$$[\mathcal{H}_1, \mathcal{H}_2] |\psi\rangle = 0. \tag{5.12}$$

This condition guarantees that, with the Dirac Hamiltonian $\mathcal{H}$, the system evolves such that the *motion* is constrained to the surfaces on the mass shells described by the constraints $\mathcal{H}_1$ and $\mathcal{H}_2$ [582, 155, 151]. The interaction constraint becomes

$$2p_1 \dot{\Phi}_2 + 2p_2 \dot{\Phi}_1 + \{\Phi_1, \Phi_2\} \approx 0.$$

As described most recently in [582] this requires that

$$\Phi_1 = \Phi_2 = \Phi(x_\perp, p_1, p_2), \tag{5.13}$$

with the transverse coordinate defined by

$$x_{\nu\perp} = x_{12}^\mu (\eta_{\mu\nu} - P_\mu P_\nu / P^2), \tag{5.14}$$

and total momentum

$$P = p_1 + p_2. \tag{5.15}$$

Using Todorov variables [156, 539, 543],

- Relative position, $x_1 - x_2$.
- Relative momentum, $p = \frac{\epsilon_2 p_1 - \epsilon_1 p_2}{w}$.
- Total CM energy, $w = \sqrt{-P^2}$.
- Total momentum, $P = p_1 + p_2$.
- (Conserved) constituent CM energies,
  $\epsilon_1 = \frac{w^2+m_1^2-m_2^2}{2w}$, $\epsilon_2 = \frac{w^2+m_2^2-m_1^2}{2w}$.
  (In terms of these, $p_1 = \epsilon_1 \hat{P} + p, \quad p_2 = \epsilon_2 \hat{P} - p$ where $\hat{P} = \frac{P}{w}$.)
- Relativistic reduced mass and energy of a fictitious particle of relative motion, $m_w = \frac{m_1 m_2}{w}, \quad \epsilon_w = \frac{w^2-m_1^2-m_2^2}{2w}$.
- On-shell value of the relative momentum squared,
  $b^2(w) = \epsilon_w^2 - m_w^2 = \epsilon_1^2 - m_1^2 = \epsilon_2^2 - m_2^2 =$
  $\frac{1}{4w^2}(w^4 - 2w^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2)$.

CM momentum Eq. (5.15) and total energy

$$w^2 = -P^2,$$

to define the relativistic momentum, we require that the difference

$$\mathcal{H}_1 - \mathcal{H}_2 = 0,$$

is independent of the interaction, Φ, namely Eq. (5.13) forms the differences of constraints into a purely *kinematical constraint*

$$\mathcal{H}_1 - \mathcal{H}_2 = m_1^2 + p_1^2 + \Phi_1 - m_2^2 - p_2^2 - \Phi_2 \approx 0 \Rightarrow$$

$$2P \cdot p + (\epsilon_2 - \epsilon_1)w + m_1^2 - m_2^2 \approx 0. \tag{5.16}$$

In CM frame

$$P \cdot p = 0, \tag{5.17}$$

$P = (-P^0, \overrightarrow{P} = 0)$, $p = (0, \overrightarrow{p})$ and the relative momentum has a vanishing time like component

$$(-w^2, \underbrace{\overrightarrow{p}_1 + \overrightarrow{p}_2}_{0}) = P^2, \tag{5.18}$$

$$p_1 = (+\epsilon_1, \overrightarrow{p}), \quad p_1^1 = -\epsilon_1^2 + \overrightarrow{p}^2 = -m_1^2,$$

$$p_2 = (+\epsilon_2, \overrightarrow{p}), \quad p_1^2 = -\epsilon_2^2 + \overrightarrow{p}^2 = -m_2^2,$$

$$P = p_1 + p_2 = -(\epsilon_1 + \epsilon_2) = -\epsilon_1^2 - \epsilon_2^2 - 2\epsilon_1\epsilon_2 = -w^2. \tag{5.19}$$

With

$$p = \frac{\varepsilon_2}{w}p_1 - \frac{\varepsilon_1}{w}p_2 = 0, \tag{5.20}$$

the time-like components of the particle momenta in the CM system are

$$\varepsilon_1 = \frac{w^2 + m_1^2 - m_2^2}{2w},$$
$$\varepsilon_2 = \frac{w^2 + m_2^2 - m_1^2}{2w}. \tag{5.21}$$

Since

$$\mathcal{H}_1 = \overrightarrow{p}^2 - \epsilon_1^2 + m_1^2 + \Phi,$$
$$\mathcal{H}_2 = \overrightarrow{p}^2 - \epsilon_2^2 + m_2^2 + \Phi, \tag{5.22}$$

the combination of the constraints give an explicit relativistic Schrödinger equation for, one effective particle of *relative motion*

$$\mathcal{H} = (\epsilon_2 \mathcal{H}_1 + \epsilon_1 \mathcal{H}_2)/w = \overrightarrow{p}^2 - b^2 + \Phi \approx 0. \tag{5.23}$$

Thus quantum mechanically,

$$\left\{ \overrightarrow{p}^2 + \Phi(x_\perp) - b^2(w^2, m_1^2, m_2^2) \right\} |\psi\rangle = 0, \tag{5.24}$$

where

$$b^2(w^2, m_1^2, m_2^2) = \varepsilon_1^2 - m_1^2 = \varepsilon_2^2 - m_2^2$$
$$= \frac{1}{4w^2} \left\{ w^4 - 2w^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 \right\}. \tag{5.25}$$

This equation maintains the exact relativistic two body kinematics, i.e. classically

$$p^2 - b^2 = 0,$$

would imply

$$w = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2}.$$

Note that both of the constituent invariant CM energies ε_1 and ε_2 are positive for positive total CM energy w greater than the square root of $|m_1^2 - m_2^2|$. This is a direct consequence of the Eq. (5.17) which in turn depends on the "third law" condition necessary for compatibility.

In the CM system, $p = p_\perp = (0, \overrightarrow{p})$, $x_\perp = (0, \overrightarrow{r})$ and the relative energy and time are removed from the problem. The equation for the relative motion is then

$$\left\{ \overrightarrow{p}^2 + \Phi(\mathbf{r}) - b^2 \right\} |\psi\rangle = 0, \tag{5.26}$$

which has the form of a stationary non-relativistic Schrödinger equation, with $2mV \to \Phi$, $2mE_{NR} \to b^2$. Thus the relativistic treatment of the two body problem for spinless particles gives a form that has the simplicity of the ordinary stationary nonrelativistic one body Schrödinger equation and yet maintains relativistic covariance. Spin and different types of interactions can be included in a more complete framework [388, 166, 158, 421] and will be reviewed later.

The potential Φ may have a complicated dependence,

$$\Phi = \Phi(r^2, r \cdot p, P^2, p^2, P \cdot p). \tag{5.27}$$

It is a general feature of relativistic mechanics that the potentials do not appear only as functions of the relative coordinates x and relative momentum p. Ignoring explicit dependences of Φ on total and relative momenta, one is still left with an implicit dependence upon P through the vector r. It is only in the CM frame that r reduces to the momentum independent vector $(0, \mathbf{x})$ The potential Φ can also exhibit, an explicit dependence on P^2. This can arise either from a dependence of the coupling constants on P^2, or from dimensional requirements. The dependence of Φ upon $P \cdot p$ can be ignored altogether, since the latter can be eliminated of the constraints Eq. (5.17)

$$P \cdot p = 0.$$

Finally the dependences upon $r \cdot p$ and p^2 arise when nonlocal effects are approximated by local functions or from the tensor structure of the interaction. For two spinless particles the potential Φ is a function of r, p, and P.A Φ dependence on the relative momentum p has to be symmetrized in such a way that the eigenvalues of p^2 come out to be real, if no other reason forbids this result [546].

For a scalar interaction the momenta p_1 and p_2 do not appear at the vertices and therefore we have

$$\Phi = \Phi(r^2, p^2).$$

For a vector interaction p_1 and p_2 appear linearly at the vertices and to lowest order of the coupling constants we have

$$\Phi = [p_1^{\nu}, [p_2^{\mu}, C_{\mu\nu}(r, P)]_+]_+, \tag{5.28}$$

where $C_{\mu\nu}$ represents the relativistic instantaneous approximation of a vector field propagator and satisfies the properties

$$C_{\mu\nu}(r, p) = C_{\nu\mu}(r, p) = C_{\mu\nu}(-r, p).$$

To first order in C the term Eq. (5.28) could also arise from a minimal substitution in the free equations of the type

$$\begin{aligned} p_{1\mu} \to p'_{1\mu} &= p_{1\mu} - A_{1\mu} := p_{1\mu} - [p_2^{\nu}, C_{\mu\nu}]_+, \\ p_{2\mu} \to p'_{2\mu} &= p_{2\mu} - A_{2\mu} := p_{2\mu} - [p_1^{\nu}, C_{\mu\nu}]_+. \end{aligned} \tag{5.29}$$

Pseudoscalar type interactions correspond to potentials Φ which are proportional to the matrices $\gamma_5 \eta_5$,

$$\Phi = \gamma_5 \eta_5 W(r^2, p^2),$$

where the γ_5 and η_5 are the Dirac matrices acting on the fermion and antifermion spinor, $W(r^2, p^2)$ is the Pauli-Lubanski operator. If $W_{1s\alpha}$ and $W_{2s\alpha}$ are Pauli-Lubanski spin operators of particles 1 and 2,

$$\begin{aligned}
W_{1s\alpha} &= \frac{\hbar}{4}\epsilon_{\alpha\beta\mu}P^{\beta}\sigma^{\mu\nu}, \\
W_{2s\alpha} &= \frac{\hbar}{4}\epsilon_{\alpha\beta\mu}P^{\beta}\xi^{\mu\nu}, \\
W_{1s}^2 &= W_{2s}^2 = -\frac{3}{4}\hbar^2 P^2, \\
W_S &= W_{1s} + W_{2s}, \textit{(total spin)} \\
\sigma^{\mu\nu} &= \frac{1}{2i}[\gamma^{\mu}, \gamma^{\nu}],\ \eta^{\mu\nu} = \frac{1}{2i}[\eta^{\mu}, \eta^{\nu}],
\end{aligned}$$

which commute with all longitudinal variables and matrices. The pseudoscalar interactions play, also an important role in the representation of confining interactions and spontaneous breakdown of chiral symmetry [495].

An axial vector interaction is of the type

$$\Phi = -\tilde{\gamma}\cdot W_L A(r^2, p^2) = -\frac{2}{p^2}\gamma\cdot p W_L \cdot W_{1s} A(r^2, p^2),$$

where $\tilde{\gamma}^{\mu} = \gamma^{\mu}\gamma^{5}$, W_L is the relative orbital angular momentum. Such potentials cannot arise in the ladder approximation of parity conserving interactions in renormalizable field theories. However they can arise from a local approximation of fourth order irreducible diagrams in vector interactions in the Bethe-Salpeter kernel [496]. They correspond to the exchange between the fermion and the boson of two vector particles. The vector particles couple to the fermion line at two different vertices with matrices γ^{μ} and γ^{ν}, respectively. Furthermore the fermion propagator joining the two vertices is proportional to $(\gamma\cdot p_1' + m_1')$ and one finds, among other terms, the product of three γ matrices which involve the term $i\epsilon_{\mu\nu\alpha\beta}\tilde{\gamma}^{\alpha}p_1'^{\beta}$. On the boson line the vertices of the vector particle involve the momenta $p_{2\mu}'$ and $p_{2\mu}''$.

In Eq. (5.26) to relativistic kinematical and dynamical corrections are included. The corrections include dependences on the CM energy w and on the nature of the interaction. For spinless particles, interacting by way of a world scalar interaction S, one finds [151, 153, 159, 160]

$$\Phi = 2m_w S + S^2, \tag{5.30}$$

where

$$m_w = \frac{m_1 m_2}{w}. \tag{5.31}$$

For timelike vector interactions, described by $\mathcal{A}$, one finds [151, 539, 540, 159, 162]

$$\Phi = 2\varepsilon_w\mathcal{A} - \mathcal{A}^2, \tag{5.32}$$

where

$$\varepsilon_w = \frac{w^2 - m_1^2 - m_2^2}{2w}. \tag{5.33}$$

For combined spacelike and timelike vector interactions, that reproduce the correct energy spectrum for scalar QED [154],

$$\Phi = 2m_w S + S^2 + 2\varepsilon_w \mathcal{A} - \mathcal{A}^2 + \frac{1}{2}\nabla^2 \log(1 - 2\mathcal{A}/w) + \frac{1}{4}[\nabla \log(1 - 2\mathcal{A}/w)]^2. \tag{5.34}$$

The variables m_w and ε_w have been introduced by Todorov in his quasipotential approach [539, 540]. In the nonrelativistic limit, Φ approaches $2\mu(S + \mathcal{A}), \quad \mu = m_1 m_2/(m1 + m_2)$ for combined interactions. In the relativistic case, the dynamical corrections include both quadratic additions to S and $\mathcal{A}$ and CM energy dependences through m_w and ε_w. The two logarithm terms at the end of Eq. (5.34) are due to the transverse or spacelike part of the potential. Without those terms, spectral results would not agree with the standard spinless Breit and Darwin approaches.

Eq. (5.26) provides a useful way to obtain the solution of the relativistic two body problem for spinless particles with scalar and vector interactions. Below we shall include spin.

These ways of putting the invariant potential functions, for scalar S and vector $\mathcal{A}$ interactions, into Φ is used by Crater and Van Alstine for the case of two spin one half particles. These forms are not unique but are motivated by classical field theory, Crater and Van Alstine, and in quantum field theory, Sazdjian [162, 336].

5.1.1 Constraint Mechanics for Spinless Particles

Crater and Van Alstine used Dirac's constraint mechanics and supersymmetry (see Sec. 2.7) to derive a pair of coupled but compatible relativistic wave equations that generalize Dirac's equation for a single spin one half particle in an external field to a system of two spinning particles interacting through world scalar and vector potentials.

To see how one introduces relativistic dynamics through a constraint approach [156, 542, 545, 366], consider first two spinless particle Klein-Gordon equations. In constraint quantum mechanics, these equations are treated as mass shell conditions on the wave function

$$\mathcal{H}_1|\psi >= (p_1^2 + m_1^2 + \Phi_1(x, p_1, p_2))|\psi >= 0, \tag{5.35a}$$

$$\mathcal{H}_2|\psi >= (p_2^2 + m_2^2 + \Phi_2(x, p_1, p_2))|\psi >= 0. \tag{5.35b}$$

One further finds that the interaction functions, referred to as quasipotentials, must be equal

$$\Phi_1 = \Phi_2 = \Phi_w(x_\perp, p_1, p_2). \tag{5.36}$$

This is a relativistic analog of Newton's third law. The invariant variable

$$r = \sqrt{x_\perp^2}, \tag{5.37}$$

is the spatial interparticle separation (only) in the CM system. The fact that x may only appear as $x_\perp$ means that constraint mechanics controls the relative time in a covariant way although the quasipotential Φ_w may depend on other invariant combinations of $x_\perp, p_1, p_2$ i.e.

$$\ell := \sqrt{(x_\perp \times p)^2},$$

where

$$(a \times b)^\mu = \epsilon^{\nu k \lambda \mu} \hat{P}_\nu a_k b_\lambda.$$

The two quantum constraints Eq. (5.35b) can be recombined in two linear independent ways. The difference yields a wave equation that controls the relative energy

$$(\mathcal{H}_1 - \mathcal{H}_2)|\psi >= 2P \cdot p|\psi >= 0. \tag{5.38}$$

The other independent combination

$$\mathcal{H} = (\epsilon_2 \mathcal{H}_1 + \epsilon_1 \mathcal{H}_2)/w, \tag{5.39}$$

leads to the stationary Schrödinger like form

$$\mathcal{H}|\psi >= 0 \rightarrow (p^2 + \Phi_w)|\psi >= b^2(w)|\psi >= (\epsilon_w^2 - m_w^2)|\psi > . \tag{5.40}$$

In the CM system, Eqs. (5.38) implies the relation

$$p^2|\psi >= p_\perp^2|\psi >= \mathbf{p}^2|\psi >,$$

thus Eqs. (5.40) has a three dimensional Schrödinger like form. However, Eqs. (5.40) is a fully covariant equation, with a CM energy dependent potential.

We imply that each particle Φ_i is constructed from constituent scalar and vector potentials which are produced by other particle. We introduce vector and scalar interactions through the minimal momentum and mass substitutions

$$p_1^\mu = p^\mu + \epsilon_1 \hat{P}^\mu \rightarrow p_1^\mu - A_1^\mu = G_1(r,\ell)p^\mu + E_1(r,\ell)\hat{P}^\mu := \pi_1^\mu,$$

$$p_2^\mu = -p^\mu + \epsilon_2 \hat{P}^\mu \rightarrow p_2^\mu - A_2^\mu = -G_2(r,\ell)p^\mu + E_2(r,\ell)\hat{P}^\mu := \pi_2^\mu, \tag{5.41}$$

$$m_1 \rightarrow m_1 + S_1 := M_1(r,\ell), \tag{5.42a}$$

$$m_2 \rightarrow m_2 + S_2 := M_2(r,\ell), \tag{5.42b}$$

in

$$\mathcal{H}_i^0 = p_i^2 + m_i^2.$$

These substitutions are straightforward two body extensions of those generated by standard covariant coupling of a single particle to vector or scalar external fields. Thus the original free mass shell forms

$$\mathcal{H}_i^0 = p_i^2 + m_i^2, \quad i = 1, 2, \tag{5.43}$$

become

$$\mathcal{H}_i = \pi_i^2 + M_i^2, \quad i = 1, 2. \tag{5.44}$$

The procedure determines the Φ_i, in Eqs. (5.35b), in terms of constituent vector and scalar potentials.

The decomposition of Eqs. (5.41-5.42b) associates the invariant functions G_1, G_2 with spacelike vector potentials, E_1, E_2 with timelike vector potentials, and M_1, M_2 with scalar potentials. These six scalar functions are not independent. In fact, the *third law* condition on scalar, timelike and spacelike vector parts yield

$$\begin{aligned} \Phi_{1S} &= \Phi_{2S}, \\ \Phi_{1A} &= \Phi_{2A}, \text{ timelike part} \\ \Phi_{1A} &= \Phi_{2A}, \text{ spacelike part.} \end{aligned} \tag{5.45}$$

It implies

$$M_1^2 - M_2^2 = m_1^2 - m_2^2, \quad E_1^2 - E_2^2 = \epsilon_1^2 - \epsilon_2^2, \text{ and } G_1^2 = G_2^2 = G^2.$$

There are only two invariant functions for the vector and one for the scalar interaction. When vector interactions are generated by coupling to QCD fields there will be further relations among the potentials. In that case, both E_i and G_i become functions of an underlying generalized Coulombic potential $\mathcal{A}$. With this technique one allows for the presence of a short distance electromagneticlike or gauge vector containing both timelike and spacelike parts, and a long distance timelike vector, by parameterizing E_i in terms of two invariant functions $\mathcal{A}$ and $\mathcal{V}$, and G in terms of $\mathcal{A}$ alone. Thus

$$A_i^\mu = A_i^\mu(\mathcal{A}(r, \ell), \mathcal{V}(r, \ell)), \tag{5.46}$$

and

$$S_i = S_i(S(r, \ell), \mathcal{V}(r, \ell)). \tag{5.47}$$

5.1.2 Constraint Mechanics for Spin One Half Particles

Crater and van Alstine [156] used supersymmetry to find compatible Dirac operators for two spinning particles. The particles are interacting through a system of relativistic scalar and vector interactions. For two spin one half particles, we start from two compatible free Dirac equations, in terms of Todorov variables, in the forms

$$\mathcal{L}_{10}|\psi >= (\theta_1 \cdot p_1 + m_1\theta_{51})|\psi >= 0, \tag{5.48a}$$

$$\mathcal{L}_{20}|\psi >= (\theta_2 \cdot p_2 + m_2\theta_{52})|\psi >= 0, \tag{5.48b}$$

using

$$p_1 = \epsilon_1 \hat{P} + p,\ p_2 = \epsilon_2 \hat{P} - p, \tag{5.49}$$

yields

$$\mathcal{L}_{10}|\psi >= (+\theta_1 \cdot p + \epsilon_1\theta_1 \cdot \hat{P} + m_1\theta_{51})|\psi >= 0, \tag{5.50a}$$

$$\mathcal{L}_{20}|\psi >= (-\theta_2 \cdot p + \epsilon_2\theta_2 \cdot \hat{P} + m_2\theta_{52})|\psi >= 0, \tag{5.50b}$$

in the effective one body variables of Todorov [539]. Crater and Van Alstine have written the matrix coefficients of these Dirac equations not in terms of gamma matrices but in terms of products of gamma matrices whose algebraic properties permit more efficient calculation of the commutation relations appropriate to two spinning bodies. These *theta* matrices

$$\begin{aligned} \theta_i^\mu &:= i\sqrt{\frac{1}{2}}\gamma_{5i}\gamma_i^\mu, \\ \theta_{5i} &:= i\sqrt{\frac{1}{2}}\gamma_{5i}, \qquad \mu = 0,1,2,3, \qquad i = 1,2 \end{aligned} \tag{5.51}$$

satisfy the fundamental anticommutation relations

$$\begin{aligned} [\theta_i^\mu, \theta_i^\nu]_+ &= -\eta^{\mu\nu}, \\ [\theta_{5i}, \theta_i^\mu]_+ &= 0, \\ [\theta_{5i}, \theta_{5i}]_+ &= -1, \end{aligned} \tag{5.52}$$

where $[,]_+$ is anticommutator. Projected theta matrices then satisfy

$$\begin{aligned} \left[\theta_i \cdot \hat{P}, \theta_i \cdot \hat{P}\right]_+ &= 1, \\ \left[\theta_i \cdot \hat{P}, \theta_{i\perp}^\mu\right]_+ &= 0, \end{aligned} \tag{5.53}$$

where

$$\theta_\perp^\mu = \theta_{i\nu}(\eta^{\mu\nu} + \hat{P}^\mu \hat{P}^\nu). \tag{5.54}$$

The algebraic significance of the theta matrices, in the dynamical description provided by Eqs. (5.48), is that the Dirac operators $\mathcal{L}_{10}$ and $\mathcal{L}_{20}$ are exact operator square roots of the corresponding mass shelf forms $-\frac{1}{2}(p_1^2 + m_1^2)$ and $-\frac{1}{2}(p_2^2 + m_2^2)$. Additional use of covariant CM projected versions of the Dirac α and β matrices are

$$\beta_i^\mu = -\gamma_i \cdot \hat{P} = 2\theta_{5i}\theta_i \cdot \hat{P}, \tag{5.55}$$

$$\alpha_i^\mu = 2\theta_{i\perp}^\mu \theta_i \cdot \hat{P}, \tag{5.56}$$

and

$$\sigma_i^{\mu} = \gamma_{5i}\alpha_i^{\mu} = i2\sqrt{2}\theta_{5i}\theta_i \cdot \hat{P}\theta_{\perp i}.$$

Crater and Van Alstine claim to have introduced the important but unfamiliar theta matrices in order to take advantage of their remarkable algebraic properties to simplify the otherwise complicated consequences of compatibility ($[\mathcal{L}_1, \mathcal{L}_2]_- |\psi >= 0$) when interactions are present. The fundamental anticommutation relations, Eqs. (5.53), of the theta matrices lead directly through a *pseudo-classical* correspondence limit, to a graded symplectic structure in which the theta's become two commuting sets of Grassmann variables (See Sec. 2.6.1). The space possesses a graded Poisson bracket that takes the differential form of Berezin and Marinov [65, 66]. When quantized, this bracket becomes a generalized quantum bracket that is sometimes a commutator, sometimes an anticommutator depending on the nature of its operator arguments.

In terms of this bracket, all necessary commutation relations involving the quantum theta's can be carried out through operations that are isomorphic to those involving the classical brackets. For dynamical variables A_α and A_β that have well defined character (odd or even) with respect to each spin or Grassmann space, the generalized quantum bracket takes the form

$$[A_\alpha, A_\beta]_{-\eta_{\alpha\beta}} = A_\alpha A_\beta - \eta_{\alpha\beta} A_\beta A_\alpha, \tag{5.57}$$

where [152]

$$\eta_{\alpha\beta} = (-1)^{\epsilon_{\alpha 1}\epsilon_{\beta 1} + \epsilon_{\alpha 2}\epsilon_{\beta 2}}.$$

The variable

$$\epsilon_{\alpha 1} = \begin{cases} 0 & \text{if } A_\alpha \text{ is even in space one (like } \quad p, x, \theta_{51}, \theta_1 \cdot \hat{P}) \\ 1 & \text{if } A_\alpha \text{ is odd in space one (like } \quad \theta_1 \cdot x, \theta_{51}\theta_2 \cdot \hat{P}). \end{cases} \tag{5.58}$$

Similarly, $\epsilon_{\alpha 2}$ keeps track of parity in space two. Note that the last variable is then odd in both spaces doubly odd. This sorts the variables into those that are even in both spaces, odd in both, even in space one while odd in space two, and odd in space one while even in space two. In addition, there are additive combinations that do not have well defined character e.g.,

$$\theta_1 \cdot x + x \cdot p, \quad or \quad \theta_1 \cdot x + \theta_2 \cdot p.$$

For expressions that contain only one set of spin variables, when inserted as pairs of arguments of the quantum bracket, for two even variables, or one odd and one even, $\eta_{\alpha\beta} = -1$ and the bracket is a commutator. For two odd variables, $\eta_{\alpha\beta} = +1$ and the bracket is an anticommutator. The product quantum bracket, such that bracket of $A_\alpha A_\beta$ with A_γ, is

$$[A_\alpha A_\beta, A_\gamma]_{-\eta_{\alpha\gamma}\eta_{\beta\gamma}}.$$

This implies that within the Grassmann space of a single particle, the product of an odd with an odd is an even, the product of an even with an odd is an odd, and that the product of an even with an even is an even. Using the definition, in Eqs. (5.57), one finds that

$$[A_\alpha A_\beta, A_\gamma]_{-\eta_{\alpha\gamma}\eta_{\beta\gamma}} = A_\alpha[A_\beta, A_\gamma]_{-\eta_{\beta\gamma}} + \eta_{\beta\gamma}[A_\alpha, A_\gamma]_{-\eta_{\alpha\gamma}} A_\beta. \tag{5.59}$$

Next, use is made of this bracket to construct pairs of compatible Dirac equations for interacting particles.

Consider what happens when we attempt to introduce scalar interactions into the two free particle equations Eqs. (5.48). If we make the minimal substitutions Eqs. (5.41) of the spinless case, we do not obtain compatible two body Dirac equations. That is, the brackets Eqs. (5.59),

$$\mathcal{L}_1|\psi> = (+\theta_1 \cdot p + \epsilon_1\theta_1 \cdot \hat{P} + M_1\theta_{51})|\psi> = 0, \tag{5.60a}$$

$$\mathcal{L}_2|\psi> = (-\theta_2 \cdot p + \epsilon_2\theta_2 \cdot \hat{P} + M_2\theta_{52})|\psi> = 0, \tag{5.60b}$$

produce an operator that does not vanish on $|\psi>$

$$\begin{aligned}[\mathcal{L}_1, \mathcal{L}_2]_-|\psi> &= [\theta_1 \cdot p, M_2\theta_{52}] + [M_1\theta_{51}, -\theta_2 \cdot p]_-|\psi> \\ &= -i(\partial M_1 \cdot \theta_1\theta_{52} + \partial M_2 \cdot \theta_2\theta_{51})|\psi> \neq 0. \end{aligned} \tag{5.61}$$

Crater and Van Alstine used supersymmetry arguments to extend the naive $\mathcal{L}_1$ and $\mathcal{L}_2$ to forms that are compatible. The procedure has four steps.

(a) They found supersymmetry of the pseudo-classical limit of an ordinary free one body Dirac equation.

(b) They introduced interactions for a single Dirac particle with external potentials that preserved these supersymmetries. For scalar interactions this required the coordinate replacement

$$x^\mu \to \tilde{x}^\mu := x^\mu + i\frac{\theta^\mu\theta_5}{m + S(\tilde{x})}.$$

Since the Grassmann variables satisfy $\theta^2 = 0$, this self referent relation has a terminating Taylor expansion.

(c) They maintained the one body supersymmetries, for each spinning particle, through the replacement

$$x_\perp^\mu = (x_1 - x_2)_\perp^\mu \to (\tilde{x}_1 - \tilde{x}_2)_\perp^\mu,$$

in the relativistic potentials S_i, and thus obtained compatible classical constrains.

(d) Finally, they canonically quantized these constraints to obtain compatible two body Dirac equations of the form

$$\mathcal{L}_1|\psi> = (+\theta_1 \cdot p + \epsilon_1\theta_1 \cdot \hat{P} + M_1\theta_{51} - i\partial L_1 \cdot \theta_2\theta_{52}\theta_{51})|\psi> = 0, \tag{5.62a}$$
$$\mathcal{L}_2|\psi> = (-\theta_2 \cdot p + \epsilon_2\theta_2 \cdot \hat{P} + M_2\theta_{52} + i\partial L_2 \cdot \theta_1\theta_{51}\theta_{52})|\psi> = 0. \tag{5.62b}$$

Note, the requirement of compatibility leads to terms with a relatively simple structure, like recoil terms which are functions of $x_\perp$ that add to the naive forms of Eqs. (5.60). The only additional nontrivial commutators needed to check compatibility, $[\mathcal{L}_1, \mathcal{L}_2]_-|\psi>=0$, are

$$[-i\partial L_1 \cdot \theta_2\theta_{51}\theta_{52}, -\theta_2 \cdot p]_- = (i\partial L_1 \cdot p + \frac{1}{2}\partial^2 L_1)\theta_{51}\theta_{52},$$

$$[\theta_1 \cdot p, i\partial L_2 \cdot \theta_1\theta_{51}\theta_{52}]_- = (-i\partial L_2 \cdot p - \frac{1}{2}\partial^2 L_2)\theta_{52}\theta_{51},$$

$$[-i\partial L_1 \cdot \theta_2\theta_{51}\theta_{52}, M_2\theta_{52}]_- = iM_2\theta_{51}\partial L_1 \cdot \theta_2,$$

$$[M_1\theta_{51}, +i\partial L_2 \cdot \theta_1\theta_{52}\theta_{51}]_- = iM_1\theta_{52}\partial L_2 \cdot \theta_1,$$

$$[\epsilon_1\theta_1 \cdot \hat{P}, +i\partial L_2 \cdot \theta_1\theta_{52}\theta_{51}]_- = [\epsilon_1\theta_1 \cdot \hat{P}, +i\partial L_2 \cdot \theta_1]_+\theta_{51}\theta_{52} = -i\epsilon_1\hat{P} \cdot \partial L_2\theta_{51}\theta_{52} = 0,$$

$$[-i\partial L_1 \cdot \theta_2\theta_{51}\theta_{52}, \epsilon_2\theta_2 \cdot \hat{P}]_- = -[-i\partial L_2 \cdot \theta_2, \epsilon_2\theta_2 \cdot \hat{P}]_+\theta_{51}\theta_{52} = -i\epsilon_2\hat{P} \cdot \partial L_1\theta_{51}\theta_{52} = 0.$$

The last two commutators vanish since $L_i = L_i(x_\perp)$ and $\hat{P} \cdot x_\perp = 0$. Note that we have used product rule Eqs. (5.59) to determine whether to compute commutators or anticommutators. Collecting coefficients of independent matrices, we find the simple differential equations

$$\partial M_1 = M_2 \partial L_1, \tag{5.63a}$$

$$\partial M_2 = M_1 \partial L_2, \tag{5.63b}$$

$$\partial L_1 = \partial L_2. \tag{5.63c}$$

In the static limit $\partial L \to 0$, each of our equations reduces to the standard one body equation for interaction with an external scalar potential. If we solve these, while identifying the free particle rest masses $M_i(L=0) = m_i$, we obtain

$$M_1 = m_1 \cosh L + m_2 \sinh L, \tag{5.64a}$$

$$M_2 = m_2 \cosh L + m_1 \sinh L. \tag{5.64b}$$

The recoil terms at the end of Eqs. (5.62) appeared as the quantum remnants of the classical Grassmann-Taylor expansion of the mass potential generated by its argument, the supersymmetric position variable $\tilde{x}_\perp$. Note that this solution of the compatibility condition implies

$$M_1^2 - M_2^2 = m_1^2 - m_2^2, \tag{5.65}$$

the third law condition. As with the treatment of the scalar interaction using supersymmetries, such methods reduce the problem of compatibility for spinning particles to those conditions that are already needed for compatibility of spinless particles. Elsewhere [156], they have extended their supersymmetric treatment to the case of timelike vector interactions. Just as in the scalar case, the naive replacement in Eqs. (5.50)

$$\epsilon_i \rightarrow E_i(r,\ell),$$

does not lead to compatible two body Dirac equations. That is, when

$$\mathcal{L}_1|\psi>=(+\theta_1\cdot p+E_1\theta_1\cdot\hat{P}+m_1\theta_{51})|\psi>=0, \tag{5.66a}$$

$$\mathcal{L}_2|\psi>=(-\theta_2\cdot p+E_2\theta_2\cdot\hat{P}+m_2\theta_{52})|\psi>=0, \tag{5.66b}$$

they found

$$\begin{aligned}[\mathcal{L}_1,\mathcal{L}_2]_-|\psi> &= [\theta_1\partial_p, E_2\theta_2\cdot\hat{P}]_- + [E_1\theta_1\cdot\hat{P}, -\theta_2\partial_p]_-|\psi> \\ &= -i(\partial E_1\cdot\theta_1\theta_2\cdot\hat{P}+\partial E_2\cdot\theta_1\theta_1\cdot\hat{P})|\psi>\neq 0.\end{aligned}$$

This enforcement of supersymmetries, for each spinning particle, leads to the recoil corrected forms

$$\mathcal{L}_1|\psi>=(+\theta_1\cdot p+E_1\theta_1\cdot\hat{P}+m_1\theta_{51}+i\partial J_1\cdot\theta_2\theta_2\cdot\hat{P}\theta_1\cdot\hat{P})|\psi>=0, \tag{5.67a}$$

$$\mathcal{L}_2|\psi>=(-\theta_2\cdot p+E_2\theta_2\cdot\hat{P}+m_2\theta_{52}-i\partial J_2\cdot\theta_1\theta_1\cdot\hat{P}\theta_2\cdot\hat{P})|\psi>=0, \tag{5.67b}$$

The requirement of compatibility, $[\mathcal{L}_1,\mathcal{L}_2]_-\,|\psi>=0$, then yields the simple differential equations

$$\partial E_1 = E_2\partial J_1, \tag{5.68a}$$

$$\partial E_2 = E_1\partial J_2, \tag{5.68b}$$

$$\partial J_1 = \partial J_2. \tag{5.68c}$$

Note, the static limit $\partial J \rightarrow 0$, so that each of the equations reduces to the standard one body equation for a spinning particle in an external timelike vector potential. Solution of these equations with identification of the usual free particle energies $E_i(J=0)=\epsilon_i$ then gives

$$E_1 = \epsilon_1\cosh J + \epsilon_2\sinh J, \tag{5.69a}$$

$$E_2 = \epsilon_2\cosh J + \epsilon_1\sinh J. \tag{5.69b}$$

Just as for the scalar interaction, the recoil terms at the end of Eqs. (5.67) appear in the supersymmetric treatment as the quantum remnants of the Grassmann-Taylor expansion of the energy potential generated by its argument, the supersymmetric position variable $\tilde{x}_\perp$. Note, this solution of the compatibility condition yields

$$E_1^2 - E_2^2 = \epsilon_1^2 - \epsilon_2^2,$$

the third law condition for spinless particles interacting through a timelike vector potential.

Finally, a spacelike interaction, the naive choice

$$\mathcal{L}_1|\psi>=(+G_1\theta_1\cdot p+\epsilon_1\theta_1\cdot\hat{P}+m_1\theta_{51})|\psi>=0, \tag{5.70a}$$
$$\mathcal{L}_2|\psi>=(-G_2\theta_2\cdot p+\epsilon_2\theta_2\cdot\hat{P}+m_2\theta_{52})|\psi>=0, \tag{5.70b}$$

proves incompatible

$$\begin{aligned}[\mathcal{L}_1,\mathcal{L}_2]|\psi> &= [G_1\theta_1\cdot p,-G_2\theta_2\cdot p]_-|\psi>\\ &= i(G_2\partial G_1\cdot\theta_1\theta_2\cdot p-G_1\partial G_2\cdot\theta_2\theta_1\cdot p)|\psi>\neq 0.\end{aligned} \tag{5.71}$$

However,

$$\mathcal{L}_1|\psi>=(+G_1\theta_1\cdot p+\epsilon_1\theta_1\cdot\hat{P}+m_1\theta_{51}+i\theta_2\cdot\partial G_1\theta_{1\perp}\cdot\theta_{2\perp})|\psi>=0, \tag{5.72a}$$
$$\mathcal{L}_2|\psi>=(-G_2\theta_2\cdot p+\epsilon_2\theta_2\cdot\hat{P}+m_2\theta_{52}-i\theta_1\cdot\partial G_2\theta_{2\perp}\cdot\theta_{1\perp})|\psi>=0, \tag{5.72b}$$

are compatible provided that

$$G_1\partial G_2=G_2\partial G_1. \tag{5.73}$$

Thus G_1 and G_2 differ by at most a multiplicative constant. If the corresponding Dirac equations are to become the usual free particle Dirac equations when the interaction vanishes, G_1 and G_2 must be unity in this limit. Hence, the constant must be one, and

$$G_1=G_2:=G. \tag{5.74}$$

When both, scalar and timelike four vector interactions, are present [156], the compatible two body Dirac Eqs. (5.50) turn out to be

$$\begin{aligned}\mathcal{L}_1|\psi> &= (+\theta_1\cdot p+E_1\theta_1\cdot\hat{P}+M_1\theta_{51}\\ &\quad +i\partial J\cdot\theta_2\theta_1\cdot\hat{P}\theta_2\cdot\hat{P}-i\partial L\cdot\theta_2\theta_{52}\theta_{51})|\psi>=0,\end{aligned} \tag{5.75a}$$
$$\begin{aligned}\mathcal{L}_2|\psi> &= (-\theta_2\cdot p+E_2\theta_2\cdot\hat{P}+M_2\theta_{52}\\ &\quad -i\partial J\cdot\theta_1\theta_2\cdot\hat{P}\theta_2\cdot\hat{P}+i\partial L\cdot\theta_1\theta_{51}\theta_{52})|\psi>=0,\end{aligned} \tag{5.75b}$$

in which M_1, M_2, E_1, E_2, L and J are related by Eqs. (5.64) and (5.69). When all three interactions are *turned on* at once, the solutions (5.63)-(5.64), (5.68)-(5.69), and (5.73)-(5.74) yield the compatible two body Dirac Eqs. (5.50).

$$\begin{aligned}\mathcal{L}_1|\psi> &= (+G\theta_1\cdot p+E_1\theta_1\cdot\hat{P}+M_1\theta_{51}+iG(\theta_2\cdot\partial\ln G\theta_{1\perp}\cdot\theta_{2\perp}\\ &\quad +\theta_2\cdot\partial J\theta_1\cdot\hat{P}\theta_2\cdot\hat{P}-\theta_2\cdot\partial L\theta_{52}\theta_{51}))|\psi>=0,\end{aligned} \tag{5.76a}$$
$$\begin{aligned}\mathcal{L}_2|\psi> &= (-G\theta_2\cdot p+E_2\theta_2\cdot\hat{P}+M_2\theta_{52}-iG(\theta_1\cdot\partial\ln G\theta_{1\perp}\cdot\theta_{2\perp}\\ &\quad +\theta_1\cdot\partial J\theta_1\cdot\hat{P}\theta_2\cdot\hat{P}-\theta_1\cdot\partial L\theta_{51}\theta_{52}))|\psi>=0\end{aligned} \tag{5.76b}$$

Note, the requirement of compatibility generates three spin-dependent recoil terms at the end of each Dirac equation, which can be written compactly as

$$\theta_2 \cdot \partial \begin{pmatrix} -L\theta_{51}\theta_{52} \\ J\theta_1 \cdot \hat{P}\theta_2 \cdot \hat{P} \\ \ln G\theta_{1\perp} \cdot \theta_{2\perp} \end{pmatrix} \tag{5.77a}$$

$$\theta_1 \cdot \partial \begin{pmatrix} L\theta_{51}\theta_{52}, \\ -J\theta_1 \cdot \hat{P}\theta_2 \cdot \hat{P} \\ lnG\theta_{1\perp} \cdot \theta_{2\perp} \end{pmatrix}. \tag{5.77b}$$

The physically important case of electromagnetic like interactions, related time-like and spacelike component interactions, deserves special mention. In that case, our compatible two body Dirac equations reduce to

$$\mathcal{L}_1|\psi >= (+G\theta_1 \cdot p + E_1\theta_1 \cdot \hat{P} + m_1\theta_{51} + i\theta_2 \cdot \partial G\theta_1 \cdot \theta_2)|\psi >= 0, \tag{5.78a}$$

$$\mathcal{L}_2|\psi >= (-G\theta_2 \cdot p + E_2\theta_2 \cdot \hat{P} + m_2\theta_{52} - i\theta_1\partial G\theta_1 \cdot \theta_2)|\psi >= 0, \tag{5.78b}$$

in which the compatibility restrictions of Eqs. (5.69) and Eqs. (5.74) lead to

$$E_1 = \frac{G}{2}(\epsilon_1 - \epsilon_2) + \frac{w}{2G}, \tag{5.79a}$$

$$E_2 = \frac{G}{2}(\epsilon_2 - \epsilon_1) + \frac{w}{2G}, \tag{5.79b}$$

In Eqs. (5.78), the recoil terms are combined to yield the characteristic factor

$$\theta_1 \cdot \theta_2 = \theta_{1\perp} \cdot \theta_{2\perp} - \theta_1 \cdot \hat{P}\theta_2 \cdot \hat{P}.$$

5.2 Covariant Interactions

So far, we have been able to determine appropriate modifications of free Dirac equations that lead to compatible two body Dirac equations in the presence of scalar, timelike, and spacelike vector interactions. The resulting dynamical forms of the two body Dirac equations are identical to their one body counterparts in corresponding external fields except for the presence of recoil terms (see Eqs. (5.77) that vanish when either of the particles becomes very heavy. But, how can we determine the corresponding corrections needed for the construction of compatible Dirac equations containing pseudoscalar, timelike pseudovector, spacelike pseudovector, and tensor interactions? The scalar and vector interactions employed alter classical relativistic properties, minimal mass and they are four momentum substitutions which are not available for pseudovector and pseudoscalar interactions.

However, regardless of the details of origin of the interaction terms, for each of the cases treated so far (supersymmetry, minimal substitution) they share a common algebraic hyperbolic structure. In each case, the interactions generated by hyperbolic functions of the potential whose gradient determines the magnitude of the corresponding recoil term. As we saw in our derivation of Eqs. (5.63)-(5.65) and Eqs. (5.68)-(5.71), these structures arise from the solution of the compatibility problem and enforce generalized third law conditions on the interactions. As we shall see, if we use the hyperbolic structure to rewrite our solutions for $\mathcal{L}_i$ for the three interactions introduced so far in a compact form, we find that such hyperbolic structures can be readily generalized to incorporate their axial counterparts as well as the tensor interactions. The part presented here and the next section are reviewing the pervious work done by Crater and Van Alstine [158, 75]

5.2.1 Hyperbolic Description of Potentials

Using Eqs. (5.63)-(5.64), the scalar Eqs. (5.62) can be written in the form

$$\mathcal{L}_1|\psi>= (+\theta_1\cdot p+\epsilon_1\theta_1\cdot\hat{P}+(m_1\cosh L+m_2\sinh L)\theta_{51}-i\theta_2\cdot\partial L\theta_{52}\theta_{51})|\psi>, \tag{5.80a}$$

$$\mathcal{L}_2|\psi>= (-\theta_2\cdot p+\epsilon_2\theta_2\cdot\hat{P}+(m_2\cosh L+m_1\sinh L)\theta_{52}+i\theta_1\cdot\partial L\theta_{51}\theta_{52})|\psi>. \tag{5.80b}$$

These two Dirac equations, can be brought into a more general form through the matrix

$$\mathcal{O}_1 = 2\theta_{51}\theta_{52}, \tag{5.81}$$

which is a root of unity, $\mathcal{O}_1^2 = 1$, and is odd in each theta space. We then rewrite Eq. (5.80) as

$$\mathcal{L}_1|\psi>= (\mathcal{L}_{10}+m_1(\cosh(2\Delta)-1)\theta_{51}+m_2\sinh(2\Delta)\theta_{52}+i\theta_2\cdot\partial\Delta)|\psi>, \tag{5.82a}$$

$$\mathcal{L}_2|\psi>= (\mathcal{L}_{20}+m_2(\cosh(2\Delta)-1)\theta_{52}+m_1\sinh(2\Delta)\theta_{51}+i\theta_1\cdot\partial\Delta)|\psi>, \tag{5.82b}$$

where

$$\Delta = -\mathcal{O}_1 L/2. \tag{5.83}$$

If we rearrange these equations, we find that the combinations

$$\mathbf{S}_1|\psi>= (\cosh(\Delta)\mathcal{L}_1-\sinh(\Delta)\mathcal{L}_2)|\psi>= 0, \tag{5.84a}$$

$$\mathbf{S}_2|\psi>= (\cosh(\Delta)\mathcal{L}_2-\sinh(\Delta)\mathcal{L}_1)|\psi>= 0, \tag{5.84b}$$

take the general forms

$$\mathbf{S}_1|\psi>= (\mathcal{L}_{10}\cosh(\Delta)+\mathcal{L}_{20}\sinh(\Delta))|\psi>= 0, \tag{5.85a}$$

$$\mathbf{S}_2|\psi>= (\mathcal{L}_{20}\cosh(\Delta)+\mathcal{L}_{10}\sinh(\Delta))|\psi>= 0, \tag{5.85b}$$

after we have used simple hyperbolic identities and brought the matrices on the left of each $\mathcal{L}_i$ to the right. Since the new constraints Eqs. (5.84) are nothing but algebraic rearrangements of linear combinations of the old compatible constraints $\mathcal{L}_i$, they must themselves be compatible. However, we shall verify the compatibility explicitly. We already know that the constraints $\mathcal{L}_1$ and $\mathcal{L}_2$ are compatible

$$[\mathcal{L}_1, \mathcal{L}_2]_-|\psi >= 0.$$

The commutator $[\mathbf{S}_1, \mathbf{S}_2]_-$ is a sum of four commutators. The first,

$$\begin{aligned}[\cosh(\Delta)\mathcal{L}_1, \quad \cosh(\Delta)\mathcal{L}_2]_- = \cosh(\Delta)(\cosh(\Delta)[\mathcal{L}_1, \mathcal{L}_2]_- + \\ [\mathcal{L}_1, \cosh(\Delta)]_-\mathcal{L}_2 + [\cosh(\Delta), \mathcal{L}_2]_-\mathcal{L}_1) \approx 0,\end{aligned} \tag{5.86}$$

vanishes weakly (we need to use the constraints $\mathcal{L}_i \approx 0$.) Likewise,

$$[\sinh(\Delta)\mathcal{L}_2, \sinh(\Delta)\mathcal{L}_1]_- \approx 0. \tag{5.87}$$

We are left with

$$\begin{aligned}-[\cosh(\Delta)\mathcal{L}_1, \sinh(\Delta)\mathcal{L}_1]_- - [\sinh(\Delta)\mathcal{L}_2, \cosh(\Delta)\mathcal{L}_2]_- = \\ -(\cosh(\Delta)(\sinh(\Delta)[\mathcal{L}_1, \mathcal{L}_1]_- + [\mathcal{L}_1, \sinh(\Delta)]_-\mathcal{L}_1) - \\ \sinh(\Delta)[\cosh(\Delta), \mathcal{L}_1]_-\mathcal{L}_1) + (1 \rightarrow 2) \approx 0.\end{aligned} \tag{5.88}$$

Thus

$$[\mathbf{S}_1, \mathbf{S}_2]_-|\psi >\approx 0.$$

We now conjecture that the constraints in the general forms Eqs. (5.85) are the proper forms of the introduction of relativistic interactions in the sense that all interactions known appear simply as choices for the invariant form Δ. In order to find compatible constraints of the *external potential* form $\mathcal{L}_1$ and $\mathcal{L}_2$ for more general interactions from the $\mathbf{S}_1$ and $\mathbf{S}_2$ constraints, we must first show that the new forms in Eqs. (5.85) are compatible for arbitrary $\Delta(x_\perp)$. Then, the constraints $\mathcal{L}_1$ and $\mathcal{L}_2$ which we uncover from the $\mathbf{S}_1$ and $\mathbf{S}_2$ constraints are compatible, this is isomorphic to the proof given in Eq. (5.86) but with the roles of $\mathcal{L}_i$ and $\mathbf{S}_i$ interchanged.

The new forms Eqs. (5.85) can be related to forms which were recently proposed by Sazdjian

$$\mathbf{S}_1|\Psi >= (\mathcal{L}_{10} + \mathcal{L}_{20}\mathcal{W})|\Psi >, \tag{5.89a}$$

$$\mathbf{S}_2|\Psi >= (\mathcal{L}_{20} + \mathcal{L}_{10}\mathcal{W})|\Psi > . \tag{5.89b}$$

If we identify

$$|\Psi >= \cosh(\Delta)\psi \quad \textit{and} \quad \mathcal{W} = \tanh(\Delta),$$

then they are in equivalent forms. Note, in Sazdjian equations, for a given interaction say vector, the potential $\mathcal{W}$ has a simple matrix structure (i.e., $\theta_1 \cdot \theta_2$). On the other hand, when the equations are written in this form with a Δ that has the same matrix structure, the $\mathcal{W}$ may contain additional matrix structure since the hyperbolic tangent is a nonlinear function of Δ. These additional terms will not appear for interactions whose matrix structures are roots of unity since, for such interactions (e.g., scalar, pseudoscalar, timelike vector and pseudovector), the matrix structures of Δ and $tanh(\Delta)$ are the same. But, for those interactions whose Δ's are not multiples of root unity (i.e., those for spacelike interactions), the new equations are not equivalent to Sazdjian's. Hence, in general, Sazdjian's form of the two body Dirac equations is a weak potential version (small Δ) of Eqs. (5.85). Now, Sazdjian's forms or the two body Dirac equations (and our generalized version Eqs. (5.89) are compatible for arbitrary $\mathcal{W}$ provided that $\mathcal{W} = \mathcal{W}(x_\perp)$. We have slightly altered Sazdjian's proof of compatibility of Eqs. (5.89). First, note that the relative energy constraints

$$P \cdot p|\Psi >= 0,$$

follows from

$$\begin{aligned}(\mathcal{L}_{10}\mathbf{S}_1 - \mathcal{L}_{20}\mathbf{S}_2)|\Psi > &= (\mathcal{L}_{10}^2 - \mathcal{L}_{20}^2)|\Psi > \\ &= -\frac{1}{2}(p_1^2 + m_1^2 - p_2^2 - m_2^2)|\Psi > \\ &= -P \cdot p|\Psi >= 0. \end{aligned} \quad (5.90)$$

In order to demonstrate the (weak) compatibility of the two constraints, one must calculate

$$\begin{aligned}[S_1, S_2]|\Psi > &= [\mathcal{L}_{10}, \mathcal{L}_{20}]_-|\Psi > +[\mathcal{L}_{10}, \mathcal{L}_{10}\mathcal{W}]_-|\Psi > + \\ & [\mathcal{L}_{20}\mathcal{W}, \mathcal{L}_{20}]_-|\Psi > +[\mathcal{L}_{20}\mathcal{W}, \mathcal{L}_{10}\mathcal{W}]_-|\Psi > \\ &= (\mathcal{L}_{10}^2 - \mathcal{L}_{20}^2)\mathcal{W}|\Psi > -\mathcal{L}_{10}\mathcal{W}\mathcal{L}_{10}|\Psi > -\mathcal{L}_{10}\mathcal{W}\mathcal{L}_{20}\mathcal{W}|\Psi > + \\ & \mathcal{L}_{20}\mathcal{W}\mathcal{L}_{20}|\Psi > +\mathcal{L}_{20}\mathcal{W}\mathcal{L}_{10}\mathcal{W}|\Psi >= \\ &- P \cdot p\mathcal{W}(x_\perp)|\Psi > -\mathcal{L}_{10}\mathcal{W}\mathcal{L}_1|\Psi > +\mathcal{L}_{20}\mathcal{W}\mathcal{L}_2|\Psi > . \end{aligned} \quad (5.91)$$

Using Eq. (5.89) and Eq. (5.90), and

$$[P \cdot p, \mathcal{W}(x_\perp)]_-|\Psi >= 0,$$

one then finds that each of the terms vanishes. Thus S_1 and S_2 are weakly compatible. Next we show that compatibility of Sazdjian's constraints

$$[S_1, S_2]_-|\Psi >= 0,$$

plus the constraints themselves $S_i|\Psi >= 0$ imply the compatibility of the forms Eqs. (5.85)

$$[\mathbf{S}_1, \mathbf{S}_2]|\Psi >= 0.$$

First, we observe that

$$\mathbf{S}_i|\Psi>=S_i\cosh(\Delta)|\Psi>=S_i|\Psi> .$$

Therefore,

$$\begin{aligned}
[\mathbf{S}_1,\mathbf{S}_2]|\Psi> &= (S_1\cosh(\Delta)S_2-S_2\cosh(\Delta)S_1)|\Psi> \\
&= [S_1\cosh(\Delta)]S_2|\Psi> -[S_2\cosh(\Delta)]S_1|\Psi> \\
&+ \cosh(\Delta)[S_1,S_2]|\Psi>=0. \qquad (5.92)
\end{aligned}$$

Hence, above forms Eqs. (5.85) of the two body Dirac equations are compatible for arbitrary $\Delta(x_\perp)$.

Now, from Eqs. (5.84), we see that the new constraints $\mathbf{S_i}$ are related to the original *external potential* ones by

$$\mathcal{L}_1|\psi>=(\cosh(\Delta)\mathbf{S}_1+\sinh(\Delta)\mathbf{S}_2)|\psi>, \qquad (5.93a)$$

$$\mathcal{L}_2|\psi>=(\cosh(\Delta)\mathbf{S}_2+\sinh(\Delta)\mathbf{S}_1)|\psi> . \qquad (5.93b)$$

Even though we used the scalar interaction to carry out the compatibility check in Eqs. (5.86)-(5.88), the proof that the *external potential* $\mathcal{L}_i$ constraints are compatible for arbitrary $\Delta(x_\perp)$, given the compatibility of the $\mathbf{S}_i$ for arbitrary Δ, is virtually identical to Eqs. (5.86)-(5.88). As we shall show, for eight invariant forms for $\Delta(x_\perp)$, the corresponding *external potential* form $\mathcal{L}_i$ constraints can actually be written in a form that looks like that of a one body Dirac equation, that is,

$$\begin{aligned}
\mathcal{L}_1|\psi> &= (\mathcal{L}_{10}+Z_1(x_\perp,p))|\psi> \\
&= (\theta_1\cdot p+\epsilon_1\theta_1\cdot\hat{P}+m_1\theta_{51}+Z_1(x_\perp,p))|\psi> \\
&:= (\mathbb{H}(x_\perp)\theta_1\cdot p+\mathbb{L}_1(x_\perp))|\psi>=0, \qquad (5.94a) \\
\mathcal{L}_2|\psi> &= (\mathcal{L}_{20}+Z_2(x_\perp,p))|\psi> \\
&= (-\theta_2\cdot p+\epsilon_2\theta_2\cdot\hat{P}+m_2\theta_{52}+Z_2(x_\perp,p))|\psi> \\
&:= (-\mathbb{H}(x_\perp)\theta_2\cdot p+\mathbb{L}_2(x_\perp))|\psi>=0. \qquad (5.94b)
\end{aligned}$$

Unlike the S_i or $\mathbf{S}_i$ forms of the two body Dirac equation, these external potential forms have no gross kinetic terms depending on $\theta_j\cdot p$, $i\neq j$. This property simplifies the reduction to a Schrödinger like form. Even at this stage, we see the importance of the hyperbolic structure of the equations (which the reader will recall emerged automatically for the scalar in our supersymmetry approach) in bringing Eqs. (5.93) to the *external potential* form through the identity

$$\cosh^2(\Delta)-\sinh^2(\Delta)=1. \qquad (5.95)$$

Recall also the importance of this structure in guaranteeing a physical principle, the third law.

Such properties of the hyperbolic structure are classical in that they are necessary to guarantee consistency even at the relativistic classical level. But, the hyperbolic structure has a relativistic quantum mechanical consequence as well. In two resent papers, [496, 498] Sazdjian has shown how to construct scalar products that accompany his form of the two body Dirac equations given in Eqs. (5.89). The result he obtains is (rewritten here in the notation of Crater paper)

$$< \Psi_{P',a}, \Psi_{P,b} >= (2\pi)^3\delta^3(\overrightarrow{P}' - \overrightarrow{P}) \int d^3 \left[\Psi_a^\dagger(x) \left(1 - \mathcal{W}^2 - 4\omega^2\gamma_{10}\gamma_{20}\frac{\partial \mathcal{W}}{\partial P^2}\right) \Psi_b(x)\right]$$
$$= (2\pi)^3\omega^3\delta^3(\vec{P}' - \vec{P})\delta_{ab} f_a(\omega).$$

Note that (as pointed out by Sazdjian) this scalar product is of the same kind potential dependent even if $\mathcal{W}$ is energy independent. However, using the transformation Eqs. (5.2.1) and a simple hyperbolic identity we find that the scalar product that accompanies Crater's form Eqs. (5.85) of the two body Dirac equations is given by

$$< \psi_{P',a}, \psi_{P,b} >= (2\pi)^3\delta^3(\overrightarrow{P}' - \overrightarrow{P}) \int d^3 \left[\psi_a^\dagger(x) \left(1 - \mathcal{W}^2 - 4\omega^2\gamma_{10}\gamma_{20}\frac{\partial \mathcal{W}}{\partial P^2}\right) \psi_b(x)\right]$$
$$= (2\pi)^3\omega^3\delta^3(\vec{P}' - \vec{P})\delta_{ab} f_a(\omega).$$

Note that for energy independent potentials, this scalar product is of the same potential independent form as that for the one body Dirac equation with energy independent potentials. Perhaps, the hyperbolic structure of the two body Dirac equations will turn out to be a consequence of the requirement that the scalar product take the simple $\psi^\dagger\psi$ form for energy independent potentials.

We now investigate the constraints of external potential form S_i generated by eight choices for $\Delta(x_\perp)$. In each case, we first construct the new general hyperbolic constraints Eqs. (5.85) and then pass to the corresponding *external potential* constraints through Eq. (5.93).

For scalar interactions, we shall verify that the choice

$$\Delta = -\mathcal{O}_1 L(x_\perp)/2 = -[1_1 1_2 L(x_\perp)/2]\mathcal{O}_1,$$

where

$$\mathcal{O}_1 = 2\theta_{51}\theta_{52},$$

leads to the result given in Eqs. (5.62)-(5.64).

For timelike vector interactions, the choice

$$\Delta = \mathcal{O}_2 J(x_\perp)/2 = [\gamma_1 \cdot \hat{P}\gamma_1 \cdot \hat{P} J(x_\perp/2)]\mathcal{O}_1, \tag{5.96}$$

where

$$\mathcal{O}_2 = 2\theta_1 \cdot \hat{P}\theta_2 \cdot \hat{P},$$

will lead to the result Eqs. (5.67)-(5.69).

For spacelike vector interaction, the choice

$$\Delta = \mathcal{O}_3 \mathcal{J}(x_\perp)/2 = [(\gamma_{1\perp} \cdot \gamma_{2\perp} \mathcal{J}(x_\perp))/2]\mathcal{O}_1, \tag{5.97}$$

where

$$\mathcal{O}_3 = 2\theta_{1\perp} \cdot \theta_{2\perp},$$

will lead to the result Eqs. (5.72)-(5.74). The matrices $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ are *doubly odd* (odd in each spin space) and symmetric in the labels of the two spinning particles. Here, $\mathcal{O}_1$ and $\mathcal{O}_1$ are roots of unity, $\mathcal{O}_1^2 = \mathcal{O}_2^2 = 1$. However, since

$$\mathcal{O}_3 = -\beta_1 \beta_2 \sigma_1 \cdot \sigma_2,$$

and

$$(\sigma_1 \cdot \sigma_2)^2 = 3 - 2\sigma_1 \cdot \sigma_2,$$

one finds that

$$\mathcal{O}_3^2 = 3\mathcal{E}_1 + 2\mathcal{E}_2 \mathcal{O}_3,$$

where

$$\mathcal{E}_1 = 1, \qquad \mathcal{E}_2 = 4\theta_{51}\theta_{52}\theta_1 \cdot \hat{P}\theta_2 \cdot \hat{P} = \beta_1 \beta_2.$$

There exists a fourth doubly odd matrix combination

$$\mathcal{O}_4 = \mathcal{E}_2 \mathcal{O}_3,$$

which, like $\mathcal{O}_3$ is not a root of unity ($\mathcal{O}_4^2 = \mathcal{O}_3^2 \neq 1$). So, we uncover a fourth odd interaction, the covariant *polar* part of the full tensor interaction

$$\Delta = \mathcal{F}(x_\perp)\mathcal{O}_4/2 = [\alpha_1 \cdot \alpha_2 \mathcal{F}(x_\perp)/2]\mathcal{O}_1. \tag{5.98}$$

We now construct the two external potential constraints $\mathcal{L}_1$ and $\mathcal{L}_2$ corresponding to each of these four interactions. In this construction, the theta combinations of the Dirac gamma matrices again prove useful. Their characters (even or odd) in the general brackets Eq. (5.57) and Eq. (5.59) dictate whether one should employ commutators or anticommutators to obtain the external potential forms from Eqs. (5.93) and Eqs. (5.85). First note that for each of these four interactions, $\sinh(\Delta)$ is a double odd function of the doubly odd variable Δ. Using Eq. (5.57) to guide us to the proper bracket we obtain

$$\begin{aligned}
\mathcal{L}_1|\psi> &= \cosh(\Delta)\mathbf{S}_1 + \sinh(\Delta)\mathbf{S}_2 \\
&= \cosh(\Delta)\mathcal{L}_{10}\cosh(\Delta) + \cosh(\Delta)\mathcal{L}_{20}\sinh(\Delta) \\
&\quad + \sinh(\Delta)\mathcal{L}_{20}\cosh(\Delta) + \sinh(\Delta)\mathcal{L}_{10}\sinh(\Delta) \\
&= \cosh^2(\Delta)\mathcal{L}_{10} + \cosh(\Delta)[\mathcal{L}_{10}, \cosh(\Delta)]_- \\
&\quad + \cosh(\Delta)[\mathcal{L}_{20}, \sinh(\Delta)]_+ + \sinh(\Delta)[\mathcal{L}_{20}, \cosh(\Delta)]_- \\
&\quad + \sinh(\Delta)[\mathcal{L}_{10}, \sinh(\Delta)]_+ - \sinh^2(\Delta)\mathcal{L}_{10},
\end{aligned} \tag{5.99}$$

with a similar expression for $\mathcal{L}_2$. Note how the plus sign in conjunction with the odd-odd nature of $\sinh(\Delta)$ combine to give a negative coefficient for $\sinh^2(\Delta)$ which in turn allows one to use the simple hyperbolic identity Eq. (5.95) in the construction of the external potential form Eqs. (5.94). We need to compute the four quantum brackets $[\mathcal{L}_{i0}, \cosh(\Delta)]_-$ and $[\mathcal{L}_{i0}, \sinh(\Delta)]_+$ for $i = 1, 2$.

First, we isolate the derivative parts of the constraints by using the product rule Eq. (5.59) to decompose the following parts of these four quantum brackets

$$[\theta_1 \cdot p, \cosh(\Delta)]_- = -i\theta_1 \cdot \partial(\Delta)\sinh(\Delta) + [\theta_1^\mu, \cosh(\Delta)]_- \, p_\mu, \tag{5.100a}$$

$$[-\theta_2 \cdot p, \cosh(\Delta)]_- = i\theta_2 \cdot \partial(\Delta)\sinh(\Delta) - [\theta_2^\mu, \cosh(\Delta)]_- \, p_\mu, \tag{5.100b}$$

$$[\theta_1 \cdot p, \sinh(\Delta)]_+ = -i\theta_1 \cdot \partial(\Delta)\cosh(\Delta) + [\theta_1^\mu, \sinh(\Delta)]_+ \, p_\mu, \tag{5.100c}$$

$$[-\theta_2 \cdot p, \sinh(\Delta)]_+ = i\theta_2 \cdot \partial(\Delta)\cosh(\Delta) - [\theta_2^\mu, \sinh(\Delta)]_+ \, p_\mu. \tag{5.100d}$$

Thus the derivative parts of Eq. (5.99) are

$$\begin{aligned}
\cosh(\Delta)(\; & - \; i\theta_1 \cdot \partial(\Delta)\sinh(\Delta) + i\theta_2 \cdot \partial\cosh(\Delta)) \\
& + \; \sinh(\Delta)(i\theta_2 \cdot \partial(\Delta)\sinh(\Delta) - i\theta_1 \cdot \partial\cosh(\Delta)) \\
& = \; i\theta_2 \cdot \partial(\Delta) - i([\cosh(\Delta), \theta_1^\mu]_- \sinh(\Delta) \\
& - \; [\cosh(\Delta), \theta_2^\mu]_- \cosh(\Delta) - [\sinh(\Delta), \theta_2^\mu]_+ \sinh(\Delta) \\
& + \; [\sinh(\Delta), \theta_1^\mu]_+ \cosh(\Delta))\partial_\mu(\Delta).
\end{aligned}$$

Note that the choice of commutators versus anticommutators is dictated by the facts that $\triangle$ is odd in both particles theta matrices and that the hyperbolic sine is an odd function (while the hyperbolic cosine is an even function). (Note also that $[\partial(\Delta), \Delta]_- = 0$.) As a result,

$$\begin{aligned}
\mathcal{L}_1|\psi> \; = \; & \mathcal{L}_{10} + i\theta_2 \cdot \partial(\Delta) - i([\cosh(\Delta), \theta_1^\mu]_- \sinh(\Delta) - [\cosh(\Delta), \theta_2^\mu]_- \cosh(\Delta) \\
& -[\sinh(\Delta), \theta_2^\mu]_+ \sinh(\Delta) + [\sinh(\Delta), \theta_1^\mu]_+ \cosh(\Delta))\partial_\mu(\Delta) \\
& + \cosh(\Delta)([\theta_1^\mu, \cosh(\Delta)]_- p_\mu + [\epsilon_1\theta_1 \cdot \hat{P}, \cosh(\Delta)]_- + [m_1\theta_{51}, \cosh(\Delta)]_- \\
& -[\theta_2^\mu, \sinh(\Delta)]_+ p_\mu + [\epsilon_2\theta_2 \cdot \hat{P}, \sinh(\Delta)]_+ + [m_2\theta_{52}, \sinh(\Delta)]_+) \\
& + \sinh(\Delta)(-[\theta_2^\mu, \cosh(\Delta)]_- p_\mu + [\epsilon_2\theta_2 \cdot \hat{P}, \cosh(\Delta)]_- + [m_2\theta_{52}, \cosh(\Delta)]_- \\
& +[\theta_1^\mu, \sinh(\Delta)]_+ p_\mu + [\epsilon_1\theta_1 \cdot \hat{P}, \sinh(\Delta)]_+ + [m_1\theta_{51}, \sinh(\Delta)]_+,
\end{aligned} \tag{5.101}$$

along with a similar expression for $\mathcal{L}_2$.

In each of the brackets of Eq. (5.101) which contain θ_i^μ, that matrix may be replaced by $\theta_{i\perp}$ since it is contracted with either $\partial_\mu f(x_\perp)$ or p_μ (which satisfies $P \cdot p \approx 0$).

5.2.2 The Important Eight Interactions

Crater and van Alstine considered four *polar* and four *axial* interactions [158]. The four polar interactions (or tensors of rank 0,1,2) are

1.) scalar

$$\Delta_L = -L\theta_{51}\theta_{52} = -\frac{L}{2}\mathcal{O}_1, \; \mathcal{O}_1 := -\gamma_{51}\gamma_{52}, \tag{5.102}$$

2.) timelike vector

$$\Delta_J = J\theta_1 \cdot \hat{P}\theta_2 \cdot \hat{P} := \mathcal{O}_2\frac{J}{2} = \beta_1\beta_2\frac{J}{2}\mathcal{O}_1, \tag{5.103}$$

3.) spacelike vector

$$\Delta_{\mathcal{G}} = \mathcal{G}\theta_{1\perp} \cdot \theta_{2\perp} := \mathcal{O}_3\frac{\mathcal{G}}{2} = \gamma_{1\perp} \cdot \gamma_{2\perp}\frac{\mathcal{G}}{2}\mathcal{O}_1, \tag{5.104}$$

4.) tensor (polar)

$$\Delta_{\mathcal{F}} = 4\mathcal{F}\theta_{1\perp} \cdot \theta_{2\perp}\theta_{52}\theta_{51}\theta_1 \cdot \hat{P}\theta_2 \cdot \hat{P} := \mathcal{O}_4\frac{\mathcal{F}}{2} = \alpha_1 \cdot \alpha_2\frac{\mathcal{F}}{2}\mathcal{O}_1. \tag{5.105}$$

The four *axial* interactions, or pseudotensors of rank 0,1,2, are
5.) pseudoscalar

$$\Delta_C = \frac{C}{2} := \mathcal{E}_1\frac{C}{2} = -\gamma_{51}\gamma_{52}\frac{C}{2}\mathcal{O}_1, \tag{5.106}$$

6.) timelike pseudovector

$$\Delta_H = -2H\theta_1 \cdot \hat{P}\theta_2 \cdot \hat{P}\theta_{51}\theta_{52} := -\mathcal{E}_2\frac{H}{2} = \beta_1\gamma_{51}\beta_2\gamma_{52}\frac{H}{2}\mathcal{O}_1, \tag{5.107}$$

7.) spacelike pseudovector

$$\Delta_I = -2I\theta_{1\perp} \cdot \theta_{2\perp}\theta_{51}\theta_{52} := -\mathcal{E}_3\frac{I}{2} = -\gamma_{51}\gamma_{1\perp} \cdot \gamma_{52}\gamma_{2\perp}\frac{I}{2}\mathcal{O}_1, \tag{5.108}$$

8.) and tensor(axial)

$$\Delta_Y = -2Y\theta_{1\perp} \cdot \theta_{2\perp}\theta_1 \cdot \hat{P}\theta_2 \cdot \hat{P} := -\mathcal{E}_4\frac{Y}{2} = -\sigma_1 \cdot \sigma_2\frac{Y}{2}\mathcal{O}_1. \tag{5.109}$$

L, J, $\mathcal{G}$, $\mathcal{F}$, C, H, I, Y are invariant functions of $x_\perp$. Too each of the eight relativistic interactions correspond coupled compatible Dirac equations.

1.) scalar: $\Delta = -\frac{1}{2}\mathcal{O}_1 L(x_\perp)$.
   Consequently,

$$\cosh(\Delta) = \mathcal{E}_1 \cosh\left(\frac{L}{2}\right), \; \sinh(\Delta) = -\mathcal{O}_1 \sinh\left(\frac{L}{2}\right). \tag{5.110}$$

To construct the $\mathcal{L}_1$, one needs to know the elementary brackets

$$[\mathcal{E}_1, \theta^\mu_{i\perp}]_- = [\mathcal{E}_1, \theta_i \cdot \hat{P}]_- = [\mathcal{E}_1, \theta_{51}]_- = 0, \tag{5.111}$$

$$[\mathcal{O}_1, \theta^{\mu}_{i\perp}]_+ = [\mathcal{O}_1, \theta_i \cdot \hat{P}]_+ = 0, \tag{5.112}$$

$$[\mathcal{O}_1, \theta_{5i}]_+ = -2\theta_{5j},\ i \neq j. \tag{5.113}$$

These imply that

$$[\cosh(\Delta), \theta^{\mu}_{i\perp}]_- = [\cosh(\Delta), \theta_i \cdot \hat{P}]_- = [\cosh(\Delta), \theta_{5i}]_- = 0, \tag{5.114}$$

$$[\sinh(\Delta), \theta^{\mu}_{i\perp}]_+ = [\sinh(\Delta), \theta_i \cdot \hat{P}]_+ = [\sinh(\Delta), \theta_{5i}]_+ = 2\sinh(\Delta)\theta_{5i}. \tag{5.115}$$

To perform the remaining multiplications, we use

$$\mathcal{O}_1\theta_{51} = -\theta_{52},\ \mathcal{O}_1\theta_{52} = -\theta_{51},$$

along with hyperbolic identities, to obtain

$$\begin{aligned} \mathcal{L}_1|\psi> &= (\theta_1 \cdot p + \epsilon_1\theta_1 \cdot \hat{P} + m_1\cosh(L\mathcal{O}_1)\theta_{51} \\ &\quad -m_2\sinh(L\mathcal{O}_1)\theta_{52} - i\theta_2 \cdot \frac{\partial L}{2}\mathcal{O}_1)|\psi>, \end{aligned} \tag{5.116a}$$

$$\begin{aligned} \mathcal{L}_2|\psi> &= (-\theta_1 \cdot p + \epsilon_2\theta_2 \cdot \hat{P} + m_2\cosh(L\mathcal{O}_1)\theta_{52} \\ &\quad -m_1\sinh(L\mathcal{O}_1)\theta_{51} + i\theta_1 \cdot \frac{\partial L}{2}\mathcal{O}_1)|\psi>. \end{aligned} \tag{5.116b}$$

Since $\mathcal{O}_1$ is a root of unity, $\cosh(\mathcal{O}_1 L) = \cosh(L)$, $\sinh(\mathcal{O}_1 L) = \sinh(L)$. Thus these equations are just the scalar equations Eqs. (5.62)-(5.64) that we originally derived through supersymmetric techniques.[2]

2.) timelike four vector: $\Delta = \frac{1}{2}\mathcal{O}_2 J$.
   Consequently,

$$\cosh(\Delta) = \mathcal{E}_1\cosh(J/2),\ \sinh(\Delta) = \mathcal{O}_2\sinh(J/2). \tag{5.117}$$

Carrying out steps similar to those given above for the scalar interaction, we obtain

$$\begin{aligned} \mathcal{L}_1|\psi> &= (\theta_1 \cdot p + \epsilon_1\cosh(J\mathcal{O}_2)\theta_1 \cdot \hat{P} + \epsilon_2\sinh(J\mathcal{O}_2)\theta_2 \cdot \hat{P} \\ &\quad +m_1\theta_{51} + i\theta_2 \cdot \frac{\partial J}{2}\mathcal{O}_2)|\psi>, \end{aligned} \tag{5.118a}$$

$$\begin{aligned} \mathcal{L}_2|\psi> &= (-\theta_2 \cdot p + \epsilon_2\cosh(J\mathcal{O}_2)\theta_2 \cdot \hat{P} + \epsilon_1\sinh(J\mathcal{O}_2)\theta_1 \cdot \hat{P} \\ &\quad +m_2\theta_{52} - i\theta_2 \cdot \frac{\partial J}{2}\mathcal{O}_2)|\psi>. \end{aligned} \tag{5.118b}$$

Since $\mathcal{O}_2$ is a root of unity, these equations are just those generated by supersymmetric techniques Eqs. (5.67)-(5.69).

[2]See on the Sec. 2.7.

3.) spacelike vector: $\Delta = \frac{1}{2}\mathcal{O}_3\mathcal{J}(x_\perp)$.
This case is more complex algebraically since $\mathcal{O}_3^2 \neq 1$. However, we can write

$$\mathcal{O}_3 = \mathcal{E}_2(\mathcal{E}_1 - 2\mathcal{R}), \tag{5.119}$$

where

$$\mathcal{R} = \frac{1}{2}(\mathcal{E}_1 - \mathcal{O}_4) = \frac{1}{2}(1 + \sigma_1 \cdot \sigma_2), \tag{5.120}$$

is a root of unity $\mathcal{R}^2 = 1$. Thus

$$\begin{aligned}
\cosh(\Delta) &= \cosh\left(\frac{\mathcal{E}_1\mathcal{J}}{2} - \mathcal{R}\mathcal{J}\right) \\
&= \cosh\left(\frac{\mathcal{J}}{2}\right)\cosh(\mathcal{J}) - \mathcal{R}\sinh\left(\frac{\mathcal{J}}{2}\right)\sinh(\mathcal{J}) \\
&= \cosh^3\left(\frac{\mathcal{J}}{2}\right) + \frac{1}{2}\mathcal{O}_4\sinh\left(\frac{\mathcal{J}}{2}\right)\sinh(\mathcal{J}), \qquad (5.121) \\
\sinh(\Delta) &= \mathcal{E}_2\sinh\left(\mathcal{E}_1\frac{\mathcal{J}}{2} - \mathcal{R}\mathcal{J}\right) \\
&= \mathcal{E}_2\left(\sinh\left(\frac{\mathcal{J}}{2}\right)\cosh(\mathcal{J}) - \mathcal{R}\cosh\left(\frac{\mathcal{J}}{2}\right)\sinh\left(\mathcal{J}\right)\right) \\
&= \mathcal{E}_2\sinh^3\left(\frac{\mathcal{J}}{2}\right) + \frac{1}{2}\mathcal{O}_3\cosh\left(\frac{\mathcal{J}}{2}\right)\sinh(\mathcal{J}). \qquad (5.122)
\end{aligned}$$

One needs to know the elementary brackets

$$[\mathcal{E}_2, \theta^\mu_{2\perp}]_+ = 2\theta_{2\perp}\mathcal{E}_2 = 2\mathcal{E}_2\theta^\mu_{2\perp}, \tag{5.123}$$

$$[\mathcal{E}_2, \theta_i \cdot \hat{P}]_+ = [\mathcal{E}_2, \theta_{5i}]_+ = 0. \tag{5.124}$$

Since $\mathcal{E}_2\mathcal{O}_4 = \mathcal{O}_3$, one also needs to know

$$[\mathcal{O}_3, \theta^\mu_{i\perp}]_+ = -2\theta^\mu_{j\perp},\ i \neq j, \tag{5.125}$$

$$[\mathcal{O}_3, \theta_\perp \cdot \hat{P}]_+ = [\mathcal{O}_3, \theta_{5i}]_+ = 0. \tag{5.126}$$

In addition, one must use

$$\begin{aligned}
[\mathcal{O}_4, \theta^\mu_{i\perp}]_- &= -2\theta^\mu_{i\perp}\mathcal{O}_4 + [\mathcal{O}_3, \theta^\mu_{i\perp}]_+ \\
&= -2\theta^\mu_{i\perp}\mathcal{O}_4 - 2\theta^\mu_{j\perp}\mathcal{E}_2 \\
&= 2\mathcal{O}_4, \theta^\mu_{i\perp} + 2\mathcal{E}_2\theta^\mu_{j\perp},\ i \neq j, \qquad (5.127)
\end{aligned}$$

$$[\theta_i \cdot \hat{P}, \mathcal{O}_4]_- = [\theta_{5i}, \mathcal{O}_4]_- = 0. \tag{5.128}$$

Consequently,

$$\begin{aligned}
[\cosh(\Delta), \theta^\mu_{i\perp}]_- &= -\sinh\left(\frac{\mathcal{J}}{2}\right)\sinh(\mathcal{J})(\theta^\mu_{i\perp}\mathcal{O}_4 + \theta^\mu_{j\perp}\mathcal{E}_2) \\
&= \sinh\left(\frac{\mathcal{J}}{2}\right)\sinh(\mathcal{J})(\mathcal{O}_4\theta^\mu_{i\perp} + \mathcal{E}_2\theta^\mu_{j\perp}), \qquad (5.129)
\end{aligned}$$

$$[\cosh(\Delta), \theta_i \cdot \hat{P}]_- = [\cosh(\Delta), \theta_{5i}]_- = 0, \tag{5.130}$$

while

$$\begin{aligned}
[\sinh(\Delta), \theta^\mu_{i\perp}]_+ &= 2\sinh^3\left(\frac{\mathcal{J}}{2}\right)\theta^\mu_{i\perp}\mathcal{E}_2 - \cosh\left(\frac{\mathcal{J}}{2}\right)\sinh(\mathcal{J})\theta^\mu_{j\perp} \\
&= 2\sinh^3\left(\frac{\mathcal{J}}{2}\right)\mathcal{E}_2\theta^\mu_{i\perp} - \cosh\left(\frac{\mathcal{J}}{2}\right)\sinh(\mathcal{J})\theta^\mu_{j\perp},
\end{aligned}$$

$$[\sinh(\Delta), \theta_i \cdot \hat{P}]_+ = [\sinh(\Delta), \theta_{5i}]_+ = 0. \tag{5.131}$$

One then uses

$$\mathcal{O}_4\mathcal{E}_2 = \mathcal{O}_3,\ \mathcal{O}_4\mathcal{O}_3 = \mathcal{E}_2\mathcal{O}_3^2 = 3\mathcal{E}_2 + 2\mathcal{O}_3, \mathcal{O}_4^2 = 3 + 2\mathcal{O}_4, \tag{5.132}$$

along with the identities

$$\theta^\mu_{1\perp}\mathcal{O}_3 + \theta^\mu_{1\perp}\mathcal{E}_2 + \theta^\mu_{2\perp}\mathcal{O}_4 + \theta^\mu_{2\perp} = 0, \tag{5.133}$$

$$\mathcal{O}_4\theta^\mu_{1\perp} + \mathcal{E}_2\theta^\mu_{2\perp} + \mathcal{O}_3\theta^\mu_{2\perp} + \theta^\mu_{1\perp} = 0, \tag{5.134}$$

to perform the remaining multiplications. After using numerous hyperbolic identities one finds

$$-[\cosh(\Delta), \theta^\mu_{1\perp}]_- \sinh(\Delta) - [\sinh(\Delta), \theta^\mu_{1\perp}]_+ \cosh(\Delta)$$
$$= 2\sinh\left(\frac{\mathcal{J}}{2}\right)\cosh\left(\frac{\mathcal{J}}{2}\right)\theta^\mu_{2\perp}, \tag{5.135}$$

$$[\cosh(\Delta), \theta^\mu_{2\perp}]_- \cosh(\Delta) + [\sinh(\Delta), \theta^\mu_{2\perp}]_+ \sinh(\Delta)$$
$$= 2\sinh^2\left(\frac{\mathcal{J}}{2}\right)\theta^\mu_{2\perp}, \tag{5.136}$$

$$\cosh(\Delta)[\theta^\mu_{1\perp}, \cosh(\Delta)]_- + \sinh(\Delta)[\theta^\mu_{2\perp}, \sinh(\Delta)]_+$$
$$= 2\sinh^2\left(\frac{\mathcal{J}}{2}\right)\theta^\mu_{1\perp}, \tag{5.137}$$

$$-\cosh(\Delta)[\theta^\mu_{2\perp}, \sinh(\Delta)]_+ - \sinh(\Delta)[\theta^\mu_{2\perp}, \cosh(\Delta)]_-$$
$$= 2\sinh\left(\frac{\mathcal{J}}{2}\right)\cosh\left(\frac{\mathcal{J}}{2}\right)\theta^\mu_{1\perp}, \tag{5.138}$$

so that

$$\mathcal{L}_1|\psi >= (+e^{\mathcal{J}}\theta_1 \cdot p + \epsilon_1\theta_1 \cdot \hat{P} + m_1\theta_{51} - i\exp(\mathcal{J})\theta_2 \cdot \frac{\partial\mathcal{J}}{2}\mathcal{O}_3)|\psi >, \tag{5.139a}$$

$$\mathcal{L}_2|\psi >= (-e^{\mathcal{J}}\theta_2 \cdot p + \epsilon_2\theta_2 \cdot \hat{P} + m_2\theta_{52} - i\exp(\mathcal{J})\theta_1 \cdot \frac{\partial\mathcal{J}}{2}\mathcal{O}_3)|\psi > . \tag{5.139b}$$

The external potential forms Eqs. (5.139) [with $\mathcal{J} = \ln(\mathcal{J})$], are just the constraints Eqs. (5.72)-(5.74) that we had derived earlier.

4.) tensor (Polar):$\Delta = \frac{1}{2}\mathcal{O}_4\mathcal{F}$.
Now since $\mathcal{O}_4 = \mathcal{E}_1 - 2\mathcal{R}$, we find that

$$\begin{aligned}\cosh(\Delta) &= \cosh\left(\frac{\mathcal{E}_1\mathcal{F}}{2} - \mathcal{R}\mathcal{F}\right)\\ &= \cosh\left(\frac{\mathcal{F}}{2}\right)\cosh(\mathcal{F}) - \mathcal{R}\sinh\left(\frac{\mathcal{F}}{2}\right)\sinh(\mathcal{F})\\ &= \cosh^3\left(\frac{\mathcal{F}}{2}\right) + \frac{1}{2}\mathcal{O}_4\sinh\left(\frac{\mathcal{F}}{2}\right)\sinh(\mathcal{F}), \qquad (5.140)\\ \sinh(\Delta) &= \sinh\left(\mathcal{E}_1\frac{\mathcal{F}}{2} - \mathcal{R}\mathcal{F}\right)\\ &= \sinh\left(\frac{\mathcal{F}}{2}\right)\cosh(\mathcal{F}) - \mathcal{R}\cosh\left(\frac{\mathcal{F}}{2}\right)\sinh\left(\mathcal{F}\right)\\ &= \sinh^3\left(\frac{\mathcal{F}}{2}\right) + \frac{1}{2}\mathcal{O}_4\cosh\left(\frac{\mathcal{F}}{2}\right)\sinh(\mathcal{F}). \qquad (5.141)\end{aligned}$$

In addition to the commutators given in Eq. (5.127) and Eq. (5.128) for the spacelike vector, we need the anticommutators

$$[\mathcal{O}_4, \theta^{\mu}_{i\perp}]_+ = -2\theta^{\mu}_{j\perp}\mathcal{E}_2 = -\mathcal{E}_2\theta^{\mu}_{j\perp}, \qquad (5.142)$$

$$[\mathcal{O}_4, \theta_i\cdot\hat{P}]_+ = 2\mathcal{O}_4\theta_i\cdot\hat{P},\ [\mathcal{O}_4, \theta_{5i}]_+ = 2\mathcal{O}_4\theta_{5i}. \qquad (5.143)$$

With their aid, we find that

$$\begin{aligned}[\cosh(\Delta), \theta^{\mu}_{i\perp}]_- &= -\sinh\left(\frac{\mathcal{F}}{2}\right)\sinh(\mathcal{F})(\theta^{\mu}_{i\perp}\mathcal{O}_4 + \theta^{\mu}_{j\perp}\mathcal{E}_2)\\ &= \sinh\left(\frac{\mathcal{F}}{2}\right)\sinh(\mathcal{F})(\mathcal{O}_4\theta^{\mu}_{i\perp} + \mathcal{E}_2\theta^{\mu}_{j\perp}), \qquad (5.144)\end{aligned}$$

$$[\cosh(\Delta), \theta_i\cdot\hat{P}]_- = [\cosh(\Delta/2), \theta_{5i}]_- = 0, \qquad (5.145)$$

$$\begin{aligned}[\sinh(\Delta), \theta^{\mu}_{i\perp}]_+ &= 2\sinh^3\left(\frac{\mathcal{F}}{2}\right)\theta^{\mu}_{i\perp} - \cosh\left(\frac{\mathcal{F}}{2}\right)\sinh(\mathcal{F})\theta^{\mu}_{j\perp}\mathcal{E}_2\\ &= 2\sinh^3\left(\frac{\mathcal{F}}{2}\right)\theta^{\mu}_{i\perp} - \cosh\left(\frac{\mathcal{F}}{2}\right)\sinh(\mathcal{F})\mathcal{E}_2\theta^{\mu}_{j\perp},\end{aligned} \qquad (5.146)$$

$$[\sinh(\Delta), \theta_i\cdot\hat{P}]_+ = 2\sinh(\Delta)\theta_i\cdot\hat{P},\ [\sinh(\Delta), \theta_{5i}]_+ = 2\sinh(\Delta)\theta_{5i}. \quad (5.147)$$

After using these brackets to evaluate Eq. (5.101) and performing the indicated multiplications by using Eqs. (5.129)- (5.135) and

$$\mathcal{O}_3\theta^{\mu}_{1\perp} + \mathcal{E}_2\theta^{\mu}_{1\perp} + \mathcal{O}_4\theta^{\mu}_{2\perp} + \theta^{\mu}_{2\perp} = 0, \qquad (5.148)$$

$$\theta^{\mu}_{1\perp}\mathcal{O}_3 + \theta^{\mu}_{2\perp}\mathcal{E}_2 + \theta^{\mu}_{2\perp}\mathcal{O}_4 + \theta^{\mu}_{1\perp} = 0, \qquad (5.149)$$

we obtain

$$
\begin{aligned}
&-[\cosh(\Delta), \theta^\mu_{1\perp}]_- \sinh(\Delta) - [\sinh(\Delta), \theta^\mu_{1\perp}]_+ \cosh(\Delta) \\
&= 2 \sinh\left(\frac{\mathcal{F}\mathcal{E}_2}{2}\right) \cosh\left(\frac{\mathcal{F}\mathcal{E}_2}{2}\right) \theta^\mu_{2\perp}, && (5.150) \\
&[\cosh(\Delta), \theta^\mu_{2\perp}]_- \cosh(\Delta) + [\sinh(\Delta), \theta^\mu_{2\perp}]_+ \sinh(\Delta) \\
&= 2 \sinh^2\left(\frac{\mathcal{F}\mathcal{E}_2}{2}\right) \theta^\mu_{2\perp}, && (5.151) \\
&\cosh(\Delta)[\theta^\mu_{1\perp}, \cosh(\Delta)]_- + \sinh(\Delta)[\theta^\mu_{2\perp}, \sinh(\Delta)]_+ \\
&= 2 \sinh^2\left(\frac{\mathcal{F}\mathcal{E}_2}{2}\right) \theta^\mu_{1\perp}, && (5.152) \\
&-\cosh(\Delta)[\theta^\mu_{2\perp}, \sinh(\Delta)]_+ - \sinh(\Delta)[\theta^\mu_{2\perp}, \cosh(\Delta)]_- \\
&= 2 \sinh\left(\frac{\mathcal{F}\mathcal{E}_2}{2}\right) \cosh\left(\frac{\mathcal{F}\mathcal{E}_2}{2}\right) \theta^\mu_{1\perp}. && (5.153)
\end{aligned}
$$

We find that for polar tensor interactions

$$
\begin{aligned}
\mathcal{L}_1|\psi> \;&=\; (\exp(\mathcal{F}\mathcal{E}_2)\theta_1 \cdot p + \epsilon_1 \cosh(\mathcal{F}\mathcal{O}_4)\theta_1 \cdot \hat{P} \\
&\quad + \epsilon_2 \sinh(\mathcal{F}\mathcal{O}_4)\theta_2 \cdot \hat{P} \\
&\quad + m_1 \cosh(\mathcal{F}\mathcal{O}_4)\theta_{51} + m_2 \sinh(\mathcal{F}\mathcal{O}_4)\theta_{52} \\
&\quad + i \exp(\mathcal{F}\mathcal{E}_2)\theta_2 \cdot \frac{\partial \mathcal{F}}{2} \mathcal{O}_4)|\psi>, && (5.154a) \\
\mathcal{L}_2|\psi> \;&=\; (-\exp(\mathcal{F}\mathcal{E}_2)\theta_2 \cdot p + \epsilon_2 \cosh(\mathcal{F}\mathcal{O}_4)\theta_1 \cdot \hat{P} \\
&\quad + \epsilon_1 \sinh(\mathcal{F}\mathcal{O}_4)\theta_1 \cdot \hat{P} \\
&\quad + m_2 \cosh(\mathcal{F}\mathcal{O}_4)\theta_{52} + m_1 \sinh(\mathcal{F}\mathcal{O}_4)\theta_{51} \\
&\quad - i \exp(\mathcal{F}\mathcal{E}_2)\theta_1 \cdot \frac{\partial \mathcal{F}}{2} \mathcal{O}_4)|\psi> . && (5.154b)
\end{aligned}
$$

and the four pairs of Dirac equations that we shall derive below for axial interactions are new forms which accompany the three pairs of Dirac equations that we had found previously through quantization of supersymmetric pseudo-classical forms.

The axial counterparts to the constraints Eqs. (5.93) in the case of polar interactions are

$$
\mathcal{L}_1|\psi> = \cosh(\Delta)\mathbf{S}_1|\psi> - \sinh(\Delta)\mathbf{S}_2|\psi>, \qquad (5.155a)
$$

$$
\mathcal{L}_2|\psi> = \cosh(\Delta)\mathbf{S}_2|\psi> - \sinh(\Delta)\mathbf{S}_1|\psi> . \qquad (5.155b)
$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are still given by Eqs. (5.85). Just as for polar Eqs. (5.93), the compatibility of these two constraints follows from that of the $\mathbf{S}_i$. Note that the minus sign combines with the fact Δ is even-even for the axial interactions to give a minus sign coefficient for $\sinh(\Delta)$.

We have chosen a minus sign in Eqs. (5.155) because $\sinh(\Delta)$ is even in the number of theta matrices. Since this quantity will appear in commutators instead of anticommutators, we find [in contrast to Eqs. (5.99) for polar interactions] that

$$\begin{aligned}\mathcal{L}_1|\psi> &= \cosh^2(\Delta)\mathcal{L}_{10} + \cosh(\Delta)[\mathcal{L}_{10}, \cosh(\Delta)]_- \\ &+ \cosh(\Delta)[\mathcal{L}_{20}, \sinh(\Delta)]_- - \sinh(\Delta)[\mathcal{L}_{20}, \cosh(\Delta)]_- \\ &- \sinh(\Delta)[\mathcal{L}_{10}, \sinh(\Delta)]_- - \sinh^2(\Delta)\mathcal{L}_{10}. \end{aligned} \tag{5.156}$$

Steps analogous to those below Eqs. (5.99)(with commutators appearing instead of anticommutators at appropriate places) show that the general form of the Dirac operator for the axial interaction analogous to Eqs. (5.101) for the polar is

$$\begin{aligned}\mathcal{L}_1|\psi> &= \mathcal{L}_{10} + i\theta_2\cdot\partial(\Delta) - i([\cosh(\Delta), \theta_1^\mu]_- \sinh(\Delta) - [\cosh(\Delta), \theta_2^\mu]_- \cosh(\Delta) \\ &+[\sinh(\Delta), \theta_2^\mu]_- \sinh(\Delta) - [\sinh(\Delta), \theta_1^\mu]_- \cosh(\Delta))\partial_\mu(\Delta) \\ &+ \cosh(\Delta)([\theta_1^\mu, \cosh(\Delta)]_- p_\mu + [\epsilon_1\theta_1\cdot\hat{P}, \cosh(\Delta)]_- + [m_1\theta_{51}, \cosh(\Delta)]_- \\ &-[\theta_2^\mu, \sinh(\Delta)]_- p_\mu + [\epsilon_2\theta_2\cdot\hat{P}, \sinh(\Delta)]_- + [m_2\theta_{52}, \sinh(\Delta)]_-) \\ &+ \sinh(\Delta)([\theta_2^\mu, \cosh(\Delta)]_- p_\mu - [\epsilon_2\theta_2\cdot\hat{P}, \cosh(\Delta)]_- - [m_2\theta_{52}, \cosh(\Delta)]_- \\ &-[\theta_1^\mu, \sinh(\Delta)]_- p_\mu - [\epsilon_1\theta_1\cdot\hat{P}, \sinh(\Delta)]_- - [m_1\theta_{51}, \sinh(\Delta)]_-, \end{aligned} \tag{5.157}$$

along with a similar expression for $\mathcal{L}_2$..

5.) pseudoscalar: $\Delta = \frac{1}{2}C$.
Consequently,

$$\cosh(\Delta) = \cosh(C/2),\ \sinh(\Delta) = \sinh(C/2). \tag{5.158}$$

As a result,

$$[\cosh(\Delta), \theta_{i\perp}^\mu]_- = [\cosh(\Delta), \theta_i\cdot\hat{P}]_- = [\cosh(\Delta), \theta_{5i}]_- = 0, \tag{5.159}$$

$$[\sinh(\Delta), \theta_{i\perp}^\mu]_- = [\sinh(\Delta), \theta_i\cdot\hat{P}]_- = [\sinh(\Delta), \theta_{5i}]_- = 0, \tag{5.160}$$

so that

$$\begin{aligned}\mathcal{L}_1|\psi> &= (+\theta_1\cdot p + \epsilon_1\theta_1\cdot\hat{P} + m_1\theta_{51} + i\theta_2\cdot\frac{\partial L}{2})|\psi>, \\ \mathcal{L}_2|\psi> &= (-\theta_2\cdot p + \epsilon_2\theta_2\cdot\hat{P} + m_2\theta_{52} - i\theta_1\cdot\frac{\partial L}{2})|\psi>. \end{aligned} \tag{5.161}$$

6.) timelike pseudovector: $\Delta = \frac{1}{2}\mathcal{E}_2 H$.
Then

$$\cosh(\Delta) = \cosh(H),\ \sinh(\Delta) = \mathcal{E}_2\sinh(H). \tag{5.162}$$

Thus

$$[\cosh(\Delta), \theta^{\mu}_{i\perp}]_- = [\cosh(\Delta), \theta_i \cdot \hat{P}]_- = [\cosh(\Delta), \theta_{5i}]_- = [\sinh(\Delta), \theta^{\mu}_{i\perp}]_- = 0,$$

$$[\sinh(\Delta), \theta_1 \cdot \hat{P}]_- = 2\sinh(\Delta)\theta_i \cdot \hat{P}, \quad [\sinh(\Delta), \theta_{5i}]_- = 2\sinh(\Delta)\theta_{5i}. \tag{5.163}$$

In addition,

$$\mathcal{E}_2, \theta_i{\cdot}\hat{P}[\mathcal{E}_2, \theta^{\mu}_{i\perp}]_- = 0, \quad [\mathcal{E}_2, \theta_i{\cdot}\hat{P}]_- = 2\mathcal{E}_2\theta_i{\cdot}\hat{P}, \quad [\mathcal{E}_2, \theta_{5i}]_- = 2\mathcal{E}_2\theta_{5i} \tag{5.164}$$

implies that

$$\begin{aligned}&[\sinh(\Delta), \theta^{\mu}_{i\perp}]_- = 0,\\ &[\sinh(\Delta), \theta_i \cdot \hat{P}]_- = 2\sinh(\Delta)\theta_i \cdot \hat{P},\\ &[\sinh(\Delta), \theta_{5i}]_- = 2\sinh(\Delta)\theta_{5i}.\end{aligned} \tag{5.165}$$

When we substitute these brackets into Eq. (5.157), we find

$$\begin{aligned}\mathcal{L}_1|\psi> \;&=\; (\theta_1 \cdot p + \epsilon_1 \cosh(H\mathcal{E}_2)\theta_1 \cdot \hat{P} + \epsilon_2 \sinh(H\mathcal{O}_2)\theta_2 \cdot \hat{P}\\ &\quad +m_1 \cosh(H\mathcal{E}_2)\theta_{51} + m_2 \sinh(H\mathcal{E}_2)\theta_{52} - i\theta_2 \cdot \frac{\partial H}{2}\mathcal{E}_2)|\psi>,\end{aligned} \tag{5.166a}$$

$$\begin{aligned}\mathcal{L}_2|\psi> \;&=\; (-\theta_2 \cdot p + \epsilon_2 \cosh(H\mathcal{E}_2)\theta_2 \cdot \hat{P} + \epsilon_1 \sinh(H\mathcal{O}_2)\theta_1 \cdot \hat{P}\\ &\quad +m_2 \cosh(H\mathcal{E}_2)\theta_{52} + m_1 \sinh(H\mathcal{E}_2)\theta_{51} + i\theta_1 \cdot \frac{\partial H}{2}\mathcal{E}_2)|\psi> .\end{aligned} \tag{5.166b}$$

7.) spacelike pseudovector: $\Delta = -\frac{1}{2}I(x_\perp)\mathcal{E}_3$.
Using the identity

$$\mathcal{E}_3 = \mathcal{O}_1\mathcal{O}_3 = \mathcal{O}_2(\mathcal{E}_1 - 2\mathcal{R}),$$

we find that steps similar to those given for the spacelike vector interaction and the polar part of the tensor interaction given in case 4.) yield

$$\begin{aligned}\mathcal{L}_1|\psi> \;&= (\exp(\mathcal{O}_1 I)\theta_1 \cdot p + \epsilon_1\theta_1 \cdot \hat{P} + m_1 \cosh(I\mathcal{E}_3)\theta_{51}\\ &\quad +m_2 \sinh(I\mathcal{E}_3)\theta_{52} - i\exp(\mathcal{O}_1 I)\theta_2 \cdot \tfrac{\partial I}{2}\mathcal{E}_3)|\psi>,\end{aligned} \tag{5.167a}$$

$$\begin{aligned}\mathcal{L}_2|\psi> \;&= (-\exp(\mathcal{O}_1 I)\theta_2 \cdot p + \epsilon_2\theta_2 \cdot \hat{P} + m_2 \cosh(I\mathcal{E}_3)\theta_{52}\\ &\quad +m_1 \sinh(I\mathcal{E}_3)\theta_{51} + i\exp(\mathcal{O}_1 I)\theta_1 \cdot \tfrac{\partial I}{2}\mathcal{E}_3)|\psi> .\end{aligned} \tag{5.167b}$$

8.) tensor-axial: $\Delta = \frac{1}{2}Y(x_\perp)\mathcal{E}_4$.
Using the identity

$$\mathcal{E}_4 = \mathcal{O}_2\mathcal{O}_3 = \mathcal{O}_1(\mathcal{E}_1 - 2\mathcal{R}),$$

we find that steps similar to those given for the spacelike vector interaction and the polar part of the tensor interaction given in case 4.) yield

$$\begin{aligned}
\mathcal{L}_1|\psi> &= (\exp(\mathcal{O}_2 Y)\theta_1\cdot p+\epsilon_1\cosh(Y\mathcal{E}_4)\theta_1\cdot\hat{P}+\epsilon_2\sinh(Y\mathcal{E}_4)\theta_2\cdot\hat{P} \\
&+ m_1\theta_{51}-i\exp(\mathcal{O}_2 Y)\theta_2\cdot\frac{\partial Y}{2}\mathcal{E}_4)|\psi>, \quad (5.168a)\\
\mathcal{L}_2|\psi> &= (-\exp(\mathcal{O}_2 Y)\theta_2\cdot p+\epsilon_2\cosh(Y\mathcal{E}_4)\theta_2\cdot\hat{P}+\epsilon_1\sinh(Y\mathcal{E}_4)\theta_1\cdot\hat{P} \\
&+ m_2\theta_{52}+i\exp(\mathcal{O}_2 Y)\theta_1\cdot\frac{\partial Y}{2}\mathcal{E}_4)|\psi>. \quad (5.168b)
\end{aligned}$$

5.2.3 Combination of Interactions

In physical applications two or more of these eight interactions occur in combination. We may use each of the eight interaction in Eqs. (5.85) and (5.93) separately or as a sum

$$\Delta_p=\Delta_L+\Delta_J+\Delta_{\mathcal{G}}+\Delta_{\mathcal{F}}, \quad (5.169)$$

to generate sets of two body Dirac equations with corresponding interactions. A particularly important combination occurs for electromagnetic interactions. While time- and space- like vector interactions are characterized by the respective matrices $\beta_1\beta_2$ and $\gamma_{1\perp}\cdot\gamma_{2\perp}$, a potential proportional to $\gamma_1\cdot\gamma_2$ would correspond to an electromagnetic like interaction and would require that $J=-\mathcal{G}$.

$$\Delta_{\mathcal{EM}}=\frac{(\mathcal{O}_3-\mathcal{O}_2)\mathcal{G}(x_\perp)}{2}=\frac{\gamma_1\cdot\gamma_2\mathcal{G}(x_\perp)}{2}\mathcal{O}_1. \quad (5.170)$$

Crater and Van Alstine found [166] that these and their sum

$$\Delta_a=\Delta_C+\Delta_H+\Delta_I+\Delta_Y, \quad (5.171)$$

would be used in Eqs. (5.85) but with the $\sinh(\Delta_a)$ terms in Eqs. (5.84) appearing with a negative sign instead of the plus sign as is the case polar interactions. There is no sign change in Eqs. (5.85) for Δ_a.

For systems with both polar and axial interactions [158], one uses $\Delta_p-\Delta_a$ to replace Δ in Eqs. (5.84), and $\Delta_p+\Delta_a$ to replace the Δ in Eqs. (5.85).

We examine the case of additive scalar and timelike vector interactions

$$\Delta:=\Delta_L+\Delta_J=\frac{1}{2}(\mathcal{O}_2 J(x_\perp)-\mathcal{O}_1 L(x_\perp)). \quad (5.172)$$

Since both $\mathcal{O}_1$ and $\mathcal{O}_2$ are doubly odd matrices, the general form given in Eq. (5.99) for $\mathcal{L}_1$, $\mathcal{L}_2$ constraint is still valid. We use Eqs. (5.110), Eqs. (5.117) and $\mathcal{O}_1\mathcal{O}_2=\mathcal{E}_2$ to obtain

$$\begin{aligned}
\cosh(\Delta) &= \cosh(\Delta_J)\cosh(\Delta_L)+\sinh(\Delta_J)\sinh(\Delta_L) \\
&= \cosh(\frac{J}{2})\cosh(\frac{L}{2})-\mathcal{E}_2\sinh(\frac{J}{2})\sinh(\frac{L}{2}), \quad (5.173)
\end{aligned}$$

$$\begin{aligned}\sinh(\Delta) &= \sinh(\Delta_J)\cosh(\Delta_L) + \cosh(\Delta_J)\sinh(\Delta_L) \\ &= \mathcal{O}_2 \sinh(\frac{J}{2})\cosh(\frac{L}{2}) - \mathcal{O}_1 \cosh(\frac{J}{2})\sinh(\frac{L}{2}). \qquad (5.174)\end{aligned}$$

We then make use of Eqs. (5.114)-(5.115) and similar relations for the timelike vector interactions to obtain

$$\begin{aligned}&[\cosh(\Delta), \theta^{\mu}_{i\perp}]_- = 0 = [\sinh(\Delta), \theta^{\mu}_{i\perp}]_+, \\ &[\cosh(\Delta), \theta_i \cdot \hat{P}]_- = 2\sinh(\Delta_J)\sinh(\Delta_L)\theta_i \cdot \hat{P}, \\ &[\cosh(\Delta), \theta_{5i}]_- = 2\sinh(\Delta_J)\sinh(\Delta_L)\theta_{5i}, \\ &[\sinh(\Delta), \theta_i \cdot \hat{P}]_+ = 2\sinh(\Delta_J)\cosh(\Delta_L)\theta_i \cdot \hat{P}, \\ &[\sinh(\Delta), \theta_{5i}]_- = 2\cosh(\Delta_J)\sinh(\Delta_L)\theta_{5i}.\end{aligned}$$

Substitution of these results into Eq. (5.101) then yields

$$\begin{aligned}\mathcal{L}_1|\psi> &= (\theta_1 \cdot p + \epsilon_1 \cosh(\mathcal{O}_2 J)\theta_1 \cdot \hat{P} + \epsilon_2 \sinh(\mathcal{O}_2 J)\theta_2 \cdot \hat{P} + m_1 \cosh(\mathcal{O}_1 L)\theta_{51} \\ &\quad - m_2 \sinh(\mathcal{O}_1)\theta_{52} + i\theta_2 \cdot \frac{\partial}{2}(J\mathcal{O}_2 - L\mathcal{O}_1))|\psi>, \qquad (5.175a) \\ \mathcal{L}_2|\psi> &= (-\theta_2 \cdot p + \epsilon_2 \cosh(\mathcal{O}_2 J)\theta_2 \cdot \hat{P} + \epsilon_1 \sinh(\mathcal{O}_2 J)\theta_1 \cdot \hat{P} + m_2 \cosh(\mathcal{O}_1 L)\theta_{52} \\ &\quad - m_1 \sinh(\mathcal{O}_1)\theta_{51} - i\theta_1 \cdot \frac{\partial}{2}(J\mathcal{O}_2 - L\mathcal{O}_1))|\psi> . \qquad (5.175b)\end{aligned}$$

When we use the fact that $\mathcal{O}_i$'s are roots of unity in each hyperbolic function, we are similar results Eqs. (5.75). Next, we treat the more complicated case of additive timelike and spacelike interaction for which

$$\Delta := \Delta_J + \Delta_{\mathcal{G}} = \frac{1}{2}(\mathcal{O}_2 J(x_\perp) + \mathcal{O}_3 \mathcal{G}(x_\perp)). \qquad (5.176)$$

When we use

$$\begin{aligned}\cosh(\Delta) &= \cosh(\Delta_{\mathcal{G}})\cosh(\Delta_J) + \sinh(\Delta_{\mathcal{G}})\sinh(\Delta_J), \\ \sinh(\Delta) &= \sinh(\Delta_{\mathcal{G}})\cosh(\Delta_J) + \cosh(\Delta_{\mathcal{G}})\sinh(\Delta_J),\end{aligned}$$

Eqs. (5.114), (5.129) and (5.131), we find that

$$\begin{aligned}&[\cosh(\Delta), \theta^{\mu}_{i\perp}]_- = [\cosh(\Delta_{\mathcal{G}}), \theta^{\mu}_{i\perp}]_- \cosh(\Delta_J) - [\sinh(\Delta_{\mathcal{G}}), \theta^{\mu}_{i\perp}]_+ \sinh(\Delta_J), \\ &[\sinh(\Delta), \theta_i \cdot \hat{P}]_- = \sinh(\Delta_{\mathcal{G}})[\sinh(\Delta_J), \theta_i \cdot \hat{P}]_+, \\ &[\sinh(\Delta), \theta_{5i}]_+ = 0, \\ &[\cosh(\Delta), \theta_i \cdot \hat{P}]_- = \sinh(\Delta_{\mathcal{G}})[\sinh(\Delta_J), \theta_i \cdot \hat{P}]_+, \\ &[\cosh(\Delta), \theta_{5i}]_- = 0, \\ &[\sinh(\Delta), \theta^{\mu}_{i\perp}]_+ = [\sinh(\Delta_{\mathcal{G}}), \theta^{\mu}_{i\perp}]_+ \cosh(\Delta_J) - [\cosh(\Delta_{\mathcal{G}}), \theta^{\mu}_{i\perp}]_- \sinh(\Delta_J),\end{aligned}$$

So that

$$\begin{aligned}\mathcal{L}_1|\psi> &= (\exp(\mathcal{G})\theta_1\cdot p + \cosh(\mathcal{O}_2 J)\epsilon_1\theta_1\cdot\hat{P} + \sinh(\mathcal{O}_2 J)\epsilon_2\theta_2\cdot\hat{P} \\ &\quad + m_1\theta_{51} + i\exp(\mathcal{G})\theta_2\cdot\frac{\partial}{2}(\mathcal{G}\mathcal{O}_3 + J\mathcal{O}_2))|\psi>, \qquad (5.177a)\\ \mathcal{L}_2|\psi> &= (-\exp(\mathcal{G})\theta_2\cdot p + \cosh(\mathcal{O}_2 J)\epsilon_2\theta_2\cdot\hat{P} + \sinh(\mathcal{O}_2 J)\epsilon_1\theta_1\cdot\hat{P} \\ &\quad + m_2\theta_{52} - i\exp(\mathcal{G})\theta_1\cdot\frac{\partial}{2}(\mathcal{G}\mathcal{O}_3 + J\mathcal{O}_2))|\psi>. \qquad (5.177b)\end{aligned}$$

For electromagnetic like interactions the potentials are related through

$$J = -\mathcal{G},$$

and the combination

$$\Delta = (\mathcal{G}\mathcal{O}_3 + J\mathcal{O}_2)/2 = (\mathcal{G}\theta_1\cdot\theta_2)/2,$$

then

$$\begin{aligned}\mathcal{L}_1|\psi> &= (\exp(\mathcal{G})\theta_1\cdot p + \cosh(\mathcal{O}_2 J)\epsilon_1\theta_1\cdot\hat{P} - \sinh(\mathcal{O}_2 J)\epsilon_2\theta_2\cdot\hat{P} \\ &\quad + m_1\theta_{51} + i\exp(\mathcal{G})\theta_2\cdot\frac{\partial}{2}(\mathcal{G}\theta_1\cdot\theta_2))|\psi>, \qquad (5.178a)\\ \mathcal{L}_2|\psi> &= (\exp(\mathcal{G})\theta_2\cdot p + \cosh(\mathcal{O}_2 J)\epsilon_2\theta_2\cdot\hat{P} - \sinh(\mathcal{O}_2 J)\epsilon_1\theta_1\cdot\hat{P} \\ &\quad + m_2\theta_{52} - i\exp(\mathcal{G})\theta_1\cdot\frac{\partial}{2}(\mathcal{G}\theta_1\cdot\theta_2))|\psi>. \qquad (5.178b)\end{aligned}$$

If we identify

$$G = exp(\mathcal{G}),$$

we reproduce Eqs. (5.78).

Next, we examine the still more complex structure generated by addition of polar and axial interactions, such as produced by electromagnetics when the Fierz transformated annihilation channel is included.[3] We will construct $\mathcal{L}_i$ constraints from combinations of $\mathbf{S_i}$ that yield the simple external potential forms. For the polar interactions, Δ is an odd-odd matrix and $\mathcal{L}_i$ is is given by Eq. (5.101), there as for the axial interactions, Δ is an even-even interaction and $\mathcal{L}_i$ is given Eq. (5.157). We still start from general constraints Eqs. (5.85) but with $\Delta = \Delta_{\mathcal{O}} + \Delta_{\mathcal{E}} := \Delta_+$ and $\Delta_- = \Delta_{\mathcal{O}} - \Delta_{\mathcal{E}}$, taking

$$\mathcal{L}_1|\psi>= \cosh(\Delta_-)\mathbf{S}_1|\psi> + \sinh(\Delta_-)\mathbf{S}_2|\psi>, \qquad (5.179a)$$
$$\mathcal{L}_2|\psi>= \cosh(\Delta_-)\mathbf{S}_2|\psi> + \sinh(\Delta_-)\mathbf{S}_1|\psi>. \qquad (5.179b)$$

permit to use the simple hyperbolic identity Eq. (5.95). That is, the plus sign coefficient of $\Delta_{\mathcal{O}}$ and the minus sign coefficient $\Delta_{\mathcal{E}}$ in conjunction with the odd-odd nature of $\Delta_{\mathcal{O}}$ and $\sinh(\Delta_{\mathcal{E}})$ for the polar interactions and the even-even

[3]See Appendix A.2.

nature of $\Delta_{\mathcal{E}}$ and $\sinh(\Delta_{\mathcal{E}})$ for the axial interactions combine to give a minus sign coefficient for $\sinh^2(\Delta_-)$, which, in turn, allow us to use the simple hyperbolic identity Eq. (5.95) in the construction of the external potential form Eq. (5.94). Note that Eqs. (5.179) generalize our two earlier forms Eqs. (5.93) and Eqs. (5.155) reducing to them when either $\Delta_{\mathcal{O}} = 0$ or $\Delta_{\mathcal{E}} = 0$. The compatibility of these two constraints follows from those of the $\mathbf{S_i}$ just as did that of Eq. (5.93). Next, we consider how to generalize Eq. (5.101) and Eq. (5.157), the equations for the external potential forms of the constraints. We begin with the identity

$$\cosh(\Delta_+) = \cosh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}}) + \sinh(\Delta_{\mathcal{O}})\sinh(\Delta_{\mathcal{E}}),$$
$$\sinh(\Delta_+) = \sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}}) + \cosh(\Delta_{\mathcal{O}})\sinh(\Delta_{\mathcal{E}}).$$

The even or odd character of the functions and their respective arguments dictate whether the $\mathcal{L}_{i0}$ form commutators or anticommutators as they pass through $\cosh(\Delta_+)$ and $\sinh(\Delta_+)$ in Eqs. (5.85). To evaluate it, we use

$$\begin{aligned} \mathcal{L}_{10}\cosh(\Delta_+) &= \cosh(\Delta_-)\mathcal{L}_{10} + \\ &\quad [\mathcal{L}_{10}, \cosh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_- + [\mathcal{L}_{10}, \sinh(\Delta_{\mathcal{O}})\sinh(\Delta_{\mathcal{E}})]_+ \qquad (5.180) \\ \mathcal{L}_{20}\sinh(\Delta_+) &= -\sinh(\Delta_-)\mathcal{L}_{20} + \\ &\quad [\mathcal{L}_{20}, \sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_+ + [\mathcal{L}_{20}, \cosh(\Delta_{\mathcal{O}})\sinh(\Delta_{\mathcal{E}})]_-. \qquad (5.181) \end{aligned}$$

We find

$$\begin{aligned} \mathcal{L}_1|\psi> \;=\;& \cosh^2(\Delta_-)\mathcal{L}_{10} + \cosh(\Delta_-)([\mathcal{L}_{10}, \cosh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_- \\ & +[\mathcal{L}_{10}, \sinh(\Delta_{\mathcal{O}})\sinh(\Delta_{\mathcal{E}})]_+ + [\mathcal{L}_{20}, \sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_+ \\ & +[\mathcal{L}_{20}, \cosh(\Delta_{\mathcal{O}})\sinh(\Delta_{\mathcal{E}})]_-) \\ & +\sinh(\Delta_-)([\mathcal{L}_{20}, \cosh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_- \\ & +[\mathcal{L}_{20}, \sinh(\Delta_{\mathcal{O}})\sinh(\Delta_{\mathcal{E}})]_+ + [\mathcal{L}_{10}, \sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_+ \\ & +[\mathcal{L}_{10}, \cosh(\Delta_{\mathcal{O}})\sinh(\Delta_{\mathcal{E}})]_- - \sinh^2(\Delta_-)\mathcal{L}_{10}), \qquad (5.182) \end{aligned}$$

accompanied by a similar expression for $\mathcal{L}_2$. Just as we did for polar and axial interactions alone, we isolate the derivative part of the interaction for this combination. We find

$$\begin{aligned} & [\theta_1 \cdot p, \cosh\Delta_{\mathcal{O}}\cosh\Delta_{\mathcal{E}}]_- + [\theta_1 \cdot p, \sinh\Delta_{\mathcal{O}}\sinh\Delta_{\mathcal{E}}]_+ = \\ & \quad -i\theta_1\partial(\Delta_+)\sinh(\Delta_+) + [\theta^\mu_{1\perp}, \cosh\Delta_{\mathcal{O}}\cosh\Delta_{\mathcal{E}}]_- p_\mu \\ & \quad +[\theta^\mu_{1\perp}, \sinh\Delta_{\mathcal{O}}\sinh\Delta_{\mathcal{E}}]_+ p_\mu, \qquad (5.183) \end{aligned}$$

$$\begin{aligned} & [-\theta_2 \cdot p, \sinh\Delta_{\mathcal{O}}\cosh\Delta_{\mathcal{E}}]_+ + [-\theta_2 \cdot p, \sinh\Delta_{\mathcal{O}}\sinh\Delta_{\mathcal{E}}]_- = \\ & \quad +i\theta_2\partial(\Delta_+)\cosh(\Delta_+) - [\theta^\mu_{2\perp}, \sinh\Delta_{\mathcal{O}}\cosh\Delta_{\mathcal{E}}]_+ p_\mu \\ & \quad -[\theta^\mu_{2\perp}, \cosh\Delta_{\mathcal{O}}\sinh\Delta_{\mathcal{E}}]_- p_\mu, \qquad (5.184) \end{aligned}$$

$$\begin{aligned}&[-\theta_2 \cdot p, \cosh\Delta_{\mathcal{O}} \cosh\Delta_{\mathcal{E}}]_- + [\theta_2 \cdot p, \sinh\Delta_{\mathcal{O}} \sinh\Delta_{\mathcal{E}}]_+ = \\ &\quad +i\theta_2\partial(\Delta_+)\sinh(\Delta_+) - [\theta^\mu_{2\perp}, \cosh\Delta_{\mathcal{O}} \sinh\Delta_{\mathcal{E}}]_- p_\mu \\ &\quad -[\theta^\mu_{2\perp}, \sinh\Delta_{\mathcal{O}} \sinh\Delta_{\mathcal{E}}]_+ p_\mu, \end{aligned} \tag{5.185}$$

$$\begin{aligned}&[\theta_1 \cdot p, \sinh\Delta_{\mathcal{O}} \cosh\Delta_{\mathcal{E}}]_+ + [\theta_1 \cdot p, \cosh\Delta_{\mathcal{O}} \sinh\Delta_{\mathcal{E}}]_- = \\ &\quad -i\theta_1\partial(\Delta_+)\sinh(\Delta_+) + [\theta^\mu_{1\perp}, \cosh\Delta_{\mathcal{O}} \sinh\Delta_{\mathcal{E}}]_- p_\mu \\ &\quad +[\theta^\mu_{1\perp}, \sinh\Delta_{\mathcal{O}} \cosh\Delta_{\mathcal{E}}]_+ p_\mu, \end{aligned} \tag{5.186}$$

The derivative part of Eq. (5.182) is

$$\begin{aligned}&\cosh(\Delta_-)(-i\theta_1\partial(\Delta_+)\sinh(\Delta_+) + i\theta_2\partial(\Delta_+)\cosh(\Delta_+)) + \\ &\sinh(\Delta_-)(i\theta_2\partial(\Delta_+)\sinh(\Delta_+) - i\theta_1\partial(\Delta_+)\cosh(\Delta_+)) \\ &= i\theta_2\partial(\Delta_+) - i([\cosh\Delta_{\mathcal{O}}\cosh\Delta_{\mathcal{E}}, \theta^\mu_{1\perp}]_- \sinh(\Delta_+) \\ &\quad -[\sinh\Delta_{\mathcal{O}}\cosh\Delta_{\mathcal{E}}, \theta^\mu_{1\perp}]_+ \sinh(\Delta_+) \\ &\quad -[\cosh\Delta_{\mathcal{O}}\cosh\Delta_{\mathcal{E}}, \theta^\mu_{2\perp}]_- \cosh(\Delta_+) \\ &\quad +[\sinh\Delta_{\mathcal{O}}\sinh\Delta_{\mathcal{E}}, \theta^\mu_{2\perp}]_+ \cosh(\Delta_+) \\ &\quad -[\sinh\Delta_{\mathcal{O}}\cosh\Delta_{\mathcal{E}}, \theta^\mu_{2\perp}]_- \sinh(\Delta_+) \\ &\quad +[\sinh\Delta_{\mathcal{O}}\cosh\Delta_{\mathcal{E}}, \theta^\mu_{2\perp}]_- \sinh(\Delta_+) \\ &\quad +[\sinh\Delta_{\mathcal{O}}\cosh\Delta_{\mathcal{E}}, \theta^\mu_{1\perp}]_+ \cosh(\Delta_+) \\ &\quad -[\cosh\Delta_{\mathcal{O}}\sinh\Delta_{\mathcal{E}}, \theta^\mu_{2\perp}]_- \cosh(\Delta_+))\partial_\mu(\Delta_+).\end{aligned}$$

When we collect all terms, we find

$$\begin{aligned}\mathcal{L}_1|\psi> \;=\; & \mathcal{L}_{10}|\psi> + i\theta_2\partial(\Delta_+) - i([\cosh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}}), \theta^\mu_{1\perp}]_- \sinh(\Delta_+) \\ & -[\sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}}), \theta^\mu_{1\perp}]_+ \sinh(\Delta_+) \\ & -[\cosh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}}), \theta^\mu_{2\perp}]_- \cosh(\Delta_+) \\ & +[\sinh(\Delta_{\mathcal{O}})\sinh(\Delta_{\mathcal{E}}), \theta^\mu_{2\perp}]_+ \cosh(\Delta_+) \\ & -[\sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}}), \theta^\mu_{2\perp}]_- \sinh(\Delta_+) \\ & +[\sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}}), \theta^\mu_{2\perp}]_- \sinh(\Delta_+) \\ & +[\sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}}), \theta^\mu_{1\perp}]_+ \cosh(\Delta_+) \\ & -[\cosh(\Delta_{\mathcal{O}})\sinh(\Delta_{\mathcal{E}}), \theta^\mu_{2\perp}]_- \cosh(\Delta_+))\partial_\mu(\Delta_+) \\ & +\cos(\Delta_-)([\theta^\mu_{1\perp}, \cosh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_- p_\mu \\ & +[\theta^\mu_{1\perp}, \sinh(\Delta_{\mathcal{O}})\sinh(\Delta_{\mathcal{E}})]_+ p_\mu \\ & +[\epsilon_1\theta_1 \cdot \hat{P}, \cosh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_- + [\epsilon_1\theta_1 \cdot \hat{P}, \sinh(\Delta_{\mathcal{O}})\sinh(\Delta_{\mathcal{E}})]_+ \\ & +[m_1\theta_{51}, \cosh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_- + [m_1\theta_{51}, \sinh(\Delta_{\mathcal{O}})\sinh(\Delta_{\mathcal{E}})]_+ \\ & -[\theta^\mu_{2\perp}, \sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_+ p_\mu \\ & +[\theta^\mu_{2\perp}, \cosh(\Delta_{\mathcal{O}})\sinh(\Delta_{\mathcal{E}})]_- p_\mu \\ & +[\epsilon_2\theta_2 \cdot \hat{P}, \sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_+ + [\epsilon_2\theta_2 \cdot \hat{P}, \cosh(\Delta_{\mathcal{O}})\sinh(\Delta_{\mathcal{E}})]_- \\ & +[m_2\theta_{52}, \sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_+ + [m_2\theta_{52}, \cosh(\Delta_{\mathcal{O}})\sinh(\Delta_{\mathcal{E}})]_-) +\end{aligned}$$

$$
\begin{aligned}
&+\sinh(\Delta_-)([\theta^\mu_{2\perp}, \cosh(\Delta_{\mathcal{O}})\sinh(\Delta_{\mathcal{E}})]_- p_\mu \\
&+[\theta^\mu_{2\perp}, \sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_+ p_\mu \\
&+[\epsilon_2\theta_2\cdot\hat{P}, \cosh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_- + [\epsilon_2\theta_2\cdot\hat{P}, \sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_+ \\
&+[m_2\theta_{52}, \cosh(\Delta_{\mathcal{O}})\sinh(\Delta_{\mathcal{E}})]_- + [m_2\theta_{52}, \sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_+ \\
&+[\theta^\mu_{1\perp}, \sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_+ p_\mu \\
&+[\theta^\mu_{1\perp}, \sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_- p_\mu \\
&+[\epsilon_1\theta_1\cdot\hat{P}, \sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_+ + [\epsilon_1\theta_1\cdot\hat{P}, \sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_- \\
&+[m_1\theta_{51}, \sinh(\Delta_{\mathcal{O}})\cosh(\Delta_{\mathcal{E}})]_+ \\
&+[m_1\theta_{51}, \sinh(\Delta_{\mathcal{E}})\cosh(\Delta_{\mathcal{O}})]_-)|\psi>, \qquad (5.187)
\end{aligned}
$$

With a similar expression for $\mathcal{L}_2$. This complicated expression simplifies to either Eq. (5.101) or Eq. (5.157) if either $\Delta_{\mathcal{E}}$ or $\Delta_{\mathcal{O}}$ vanishes.

We specialize this result to the additive scalar and pseudoscalar interaction for which

$$\Delta_{\mathcal{O}} = -\frac{L(x_\perp)\mathcal{O}_1}{2},\ \Delta_{\mathcal{E}} = \frac{C(x_\perp)}{2}\mathcal{E}_1 = \frac{C(x_\perp)}{2}. \qquad (5.188)$$

This combination is important not only as part of the Fierz transformed annihilation structure of electrodynamics but also for phenomenological studies of the two nucleon problem. This particular case is especially simple since virtually all of the commutators and anticommutators in Eq. (5.187) vanish with the exception of the anticommutators that involve the m_i factors. These combine to give

$$
\begin{aligned}
&\cosh(\Delta_-)(2\sinh(\Delta_L)\sinh(\Delta_C)m_1\theta_{51} + 2\sinh(\Delta_L)\cosh(\Delta_{\mathcal{E}})m_2\theta_{52}) \\
&\qquad \times(2\sinh(\Delta_L)\sinh(\Delta_C)m_2\theta_{52} + 2\sinh(\Delta_L)\cosh(\Delta_C)m_1\theta_{51}) \\
&= (2\sinh^2(\Delta_L)m_1\theta_{51} + 2\sinh(\Delta_{\mathcal{L}})\cosh(\Delta_L)m_2\theta_{52}.
\end{aligned}
$$

Thus, in this case,

$$
\begin{aligned}
\mathcal{L}_1|\psi> &= (\theta_1\cdot p + \epsilon_1\theta_1\cdot\hat{P} + m_1\cosh(L\mathcal{O}_1)\theta_{51} \\
&\quad -m_2\sinh(L\mathcal{O}_1)\theta_{52} + i\theta_2\cdot\frac{\partial}{2}(C - L\mathcal{O}_1))|\psi>, \qquad (5.189a) \\
\mathcal{L}_2|\psi> &= (-\theta_1\cdot p + \epsilon_2\theta_2\cdot\hat{P} + m_2\cosh(L\mathcal{O}_1)\theta_{52} \\
&\quad -m_1\sinh(L\mathcal{O}_1)\theta_{51} - i\theta_1\cdot\frac{\partial}{2}(C - L\mathcal{O}_1))|\psi> . \qquad (5.189b)
\end{aligned}
$$

We can now use the above expression Eqs. (5.187) with its analog for $\mathcal{L}_2$ to calculate the general hyperbolic constraint, $\mathcal{L}_1$ and $\mathcal{L}_2$ for all eight interactions acting in concert, i.e., for the general invariant matrix,

$$\Delta = D_+ = \Delta_p + \Delta_a, \qquad (5.190)$$

where

$$\Delta_p = \Delta_L + \Delta_J + \Delta_{\mathcal{G}} + \Delta_{\mathcal{F}},$$
$$\Delta_a = \Delta_C + \Delta_H + \Delta_I + \Delta_Y.$$

The complete hyperbolic two body Dirac equations for all eight interaction acting together are

$$\begin{aligned}\mathcal{L}_1|\psi>= \{&\exp(\mathcal{G}+\mathcal{F}\mathcal{E}_2+I\mathcal{O}_1+Y\mathcal{O}_2) \\ &\times[\theta_1\cdot p-\frac{i}{2}\theta_2\cdot\partial(L\mathcal{O}_1-J\mathcal{O}_2-\mathcal{G}\mathcal{O}_3-\mathcal{F}\mathcal{O}_4-C\mathcal{E}_1+H\mathcal{E}_2+I\mathcal{E}_3+Y\mathcal{E}_4)] \\ &+\epsilon_1\cosh(J\mathcal{O}_2+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+Y\mathcal{E}_4)\theta_1\cdot\hat{P} \\ &+\epsilon_2\sinh(J\mathcal{O}_2+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+Y\mathcal{E}_4)\theta_2\cdot\hat{P} \\ &+m_1\cosh(-L\mathcal{O}_1+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+I\mathcal{E}_3)\theta_{51} \\ &+m_2\sinh(-L\mathcal{O}_1+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+I\mathcal{E}_3)\theta_{52}\}|\psi>=0,\end{aligned} \tag{5.191a}$$

$$\begin{aligned}\mathcal{L}_2|\psi>= \{&-\exp(\mathcal{G}+\mathcal{F}\mathcal{E}_2+I\mathcal{O}_1+Y\mathcal{O}_2) \\ &\times[\theta_2\cdot p-\frac{i}{2}\theta_1\cdot\partial(L\mathcal{O}_1-J\mathcal{O}_2-\mathcal{G}\mathcal{O}_3-\mathcal{F}\mathcal{O}_4-C\mathcal{E}_1+H\mathcal{E}_2+I\mathcal{E}_3+Y\mathcal{E}_4)] \\ &+\epsilon_1\sinh(J\mathcal{O}_2+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+Y\mathcal{E}_4)\theta_1\cdot\hat{P} \\ &+\epsilon_2\cosh(J\mathcal{O}_2+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+Y\mathcal{E}_4)\theta_2\cdot\hat{P} \\ &+m_1\sinh(-L\mathcal{O}_1+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+I\mathcal{E}_3)\theta_{51} \\ &+m_2\cosh(-L\mathcal{O}_1+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+I\mathcal{E}_3)\theta_{52}\}|\psi>=0.\end{aligned} \tag{5.191b}$$

What is remarkable is that the above hyperbolic and exponential structures account for all of the *interference* terms between the various interactions. The interactions acting separately or in subgroupings are simple reductions of the above. For example, in the case of the combined scalar, timelike, spacelike and pseudoscalar interactions,

$$\Delta = \Delta_J + \Delta_L + \Delta_{\mathcal{G}} + \Delta_C, \tag{5.192}$$

and the two body Dirac Eqs. (5.191b) reduce to

$$\begin{aligned}\mathcal{L}_1|\psi> = (&\exp(\mathcal{G})\theta_1\cdot p+E_1\theta_1\cdot\hat{P}+M_1\theta_{51} \\ &+i\frac{\exp(\mathcal{G})}{2}\theta_2\cdot\partial(\mathcal{G}\mathcal{O}_3+J\mathcal{O}_2-L\mathcal{O}_1+C\mathcal{E}_1))|\psi>=0,\end{aligned} \tag{5.193a}$$

$$\begin{aligned}\mathcal{L}_2|\psi> = (&-\exp(\mathcal{G})\theta_2\cdot p+E_2\theta_2\cdot\hat{P}+M_2\theta_{52} \\ &-i\frac{\exp(\mathcal{G})}{2}\theta_1\cdot\partial(\mathcal{G}\mathcal{O}_3+J\mathcal{O}_2-L\mathcal{O}_1+C\mathcal{E}_1))|\psi>=0.\end{aligned} \tag{5.193b}$$

where

$$M_1 = m_1 \cosh(L) + m_2 \sinh(L), \tag{5.194a}$$

$$M_2 = m_2 \cosh(L) + m_1 \sinh(L), \tag{5.194b}$$

$$E_1 = \epsilon_1 \cosh(J) + \epsilon_2 \sinh(J), \tag{5.195a}$$

$$E_2 = \epsilon_2 \cosh(J) + \epsilon_1 \sinh(J), \tag{5.195b}$$

$$G = e^{\mathcal{G}}. \tag{5.196}$$

The scalar generator produces the mass or scalar potential M_i terms, the timelike vector generator produces the energy or timelike potential E_i terms, and the spacelike vector generator produces the transverse or spacelike momentum G terms, while the pseudoscalar generator produces only spin dependent terms. The vector and scalar interactions also have additional spin dependent recoil terms essential for compatibility.

The polar tensor interaction defined by the function $\mathcal{F}(x_\perp)$ likewise contributes both to the mass potential M_i terms as well as the energy or timelike potential E_i terms. The spacelike vector generator $\mathcal{G}$, $\mathcal{F}$ produces spacelike vector dependent potential terms that are momentum dependent.

In the limit $m_1 \to \infty$ (or $m_2 \to \infty$), (when one of the particles become infinitely massive), the extra terms $\partial\mathcal{G}$, $\partial J, \partial L$ and ∂C in Eqs. (5.193) vanish, and one recovers the one body Dirac equation in an external potential.

5.3 Reduction of the Coupled Two Body Dirac Equations

Now one can use the complete hyperbolic constraint two body Dirac equations Eqs.(5.191), to derive the Schrödinger like eigenvalue equation for the combined interactions: $L(x_\perp), J(x_\perp), H(x_\perp), C(x_\perp), \mathcal{G}(x_\perp), \mathcal{F}(x_\perp), I(x_\perp), Y(x_\perp)$ [388]. We only with the final forms stationary Schrödinger equations was include extra optical potentials. The basic method we use here has some similarities to the reduction of the single particle Dirac equation to a Schrödinger like form (the Pauli reduction) and to related work by Sazdjian [498, 421].

The state vector $|\psi\rangle$ appearing in the two body Dirac equations Eq. (5.191b) is a Dirac spinor written as

$$|\psi\rangle = \begin{bmatrix} |\psi\rangle_1 \\ |\psi\rangle_2 \\ |\psi\rangle_3 \\ |\psi\rangle_4 \end{bmatrix} \tag{5.197}$$

where each $|\psi\rangle_i$ is itself a four component spinor. $|\psi\rangle$ has a total of sixteen components and the matrices $\mathcal{O}_i$'s, $\mathcal{E}_i$'s are all sixteen by sixteen.

We use the block forms of the gamma matrices given by in [388]

$$\beta_2 = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbf{1}_4 & 0 \\ 0 & -\mathbf{1}_4 \end{pmatrix},$$

$$\gamma_{52} = \begin{pmatrix} \gamma_5 & 0 \\ 0 & \gamma_5 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & \mathbf{1}_4 \\ \mathbf{1}_4 & 0 \end{pmatrix},$$

$$\beta_2\gamma_{52} = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}, \quad \rho = \begin{pmatrix} 0 & \mathbf{1}_4 \\ -\mathbf{1}_4 & 0 \end{pmatrix},$$

$$\beta_1\gamma_{51}\gamma_{52} = \begin{pmatrix} 0 & \gamma_5 \\ -\gamma_5 & 0 \end{pmatrix}, \quad \gamma_{51}\gamma_{52} = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}, \quad \beta_2\gamma_{52}\gamma_{51} = \begin{pmatrix} 0 & \rho \\ \rho & 0 \end{pmatrix}$$

and

$$\Sigma_i^\mu = \gamma_{5i}\beta_i\gamma_{\perp i}^\mu, \ i = 1, 2. \tag{5.198}$$

The Σ_i^μ are four vector generalizations of the Pauli matrices of particle one and two. In the CM frame, the time component is zero and the spatial components are the usual Pauli matrices for each particle.

We rewrite Eqs. (5.193) by multiplying the first by $\sqrt{2}i\beta_1$ and the second by $\sqrt{2}i\beta_2$ yielding [388]

$$[T_1(\beta_1\beta_2) + U_1(\beta_1\beta_2)\gamma_{51}\gamma_{52}]|\psi\rangle = (E_1 + M_1\beta_1)\gamma_{51}|\psi\rangle, \tag{5.199}$$

$$[T_2(\beta_1\beta_2) + U_2(\beta_1\beta_2)\gamma_{51}\gamma_{52}]|\psi\rangle = (E_2 + M_2\beta_2)\gamma_{52}|\psi\rangle, \tag{5.200}$$

in which the kinetic and recoil terms are

$$T_1(\beta_1\beta_2) = \exp(\mathcal{G})[\mathbf{\Sigma}_1 \cdot p - \frac{i}{2}\beta_1\beta_2(\mathbf{\Sigma}_2 \cdot \partial(-C + \mathcal{G}\beta_1\beta_2\mathbf{\Sigma}_1 \cdot \mathbf{\Sigma}_2)], \tag{5.201}$$

$$T_2(\beta_1\beta_2) = \exp(\mathcal{G})[\mathbf{\Sigma}_2 \cdot p - \frac{i}{2}\beta_1\beta_2(\mathbf{\Sigma}_1 \cdot \partial(-C + \mathcal{G}\beta_1\beta_2\mathbf{\Sigma}_1 \cdot \mathbf{\Sigma}_2)], \tag{5.202}$$

$$U_1(\beta_1\beta_2) = \exp(\mathcal{G})[-\frac{i}{2}\beta_1\beta_2\mathbf{\Sigma}_2 \cdot \partial(J\beta_1\beta_2 - L)], \tag{5.203}$$

$$U_2(\beta_1\beta_2) = \exp(\mathcal{G})[-\frac{i}{2}\beta_1\beta_2\mathbf{\Sigma}_1 \cdot \partial(J\beta_1\beta_2 - L)], \tag{5.204}$$

while the timelike and scalar potentials E_i, M_i are given above in Eqs. (5.194) and (5.195)

The final result of the matrix multiplication in Eqs. (5.200) is a set of eight simultaneous equations for the Dirac spinors $|\psi\rangle_1, |\psi\rangle_2, |\psi\rangle_3, |\psi\rangle_4$. In an arbitrary frame, the result of the matrix calculation produces the eight simultaneous equations ($\sigma_i^\mu|\psi\rangle \to \Sigma_i^\mu|\psi\rangle_{1,2,3,4}$) [388].

$$
\begin{aligned}
& T_1(+1)|\psi_1\rangle + U_1(+1)|\psi_4\rangle = (E_1 + M_1)|\psi_3\rangle, && (5.205)\\
& T_1(-1)|\psi_2\rangle + U_1(-1)|\psi_3\rangle = (E_1 + M_1)|\psi_4\rangle, && (5.206)\\
& T_1(-1)|\psi_3\rangle + U_1(-1)|\psi_2\rangle = (E_1 - M_1)|\psi_1\rangle, && (5.207)\\
& T_1(+1)|\psi_4\rangle + U_1(+1)|\psi_1\rangle = (E_1 - M_1)|\psi_2\rangle, && (5.208)\\
- & T_2(+1)|\psi_1\rangle - U_2(+1)|\psi_4\rangle = (E_2 + M_2)|\psi_2\rangle, && (5.209)\\
- & T_2(-1)|\psi_2\rangle - U_2(-1)|\psi_3\rangle = (E_2 - M_2)|\psi_1\rangle, && (5.210)\\
- & T_2(-1)|\psi_3\rangle - U_2(-1)|\psi_2\rangle = (E_2 + M_2)|\psi_4\rangle, && (5.211)\\
- & T_2(+1)|\psi_4\rangle - U_2(+1)|\psi_1\rangle = (E_2 - M_2)|\psi_3\rangle. && (5.212)
\end{aligned}
$$

We reduce the above set of eight equations to a second order Schrödinger like equation by a process of substitution and elimination using the combination of the four Dirac spinors given below [388]:

$$|\phi_\pm\rangle := |\psi_1\rangle \pm |\psi_4\rangle, \quad (5.213)$$

$$|\chi_\pm\rangle := |\psi_2\rangle \pm |\psi_3\rangle. \quad (5.214)$$

Eq. (5.205)+Eq. (5.208) yields

$$D_1^{++}|\phi_+\rangle = E_1|\chi_+\rangle - M_1|\chi_-\rangle, \quad (5.215)$$

Eq. (5.209)+Eq. (5.212) yields

$$-D_2^{++}|\phi_+\rangle = E_2|\chi_+\rangle + M_2|\chi_-\rangle, \quad (5.216)$$

Eq. (5.206)+Eq. (5.207) yields

$$D_1^{-+}|\chi_+\rangle = E_1|\phi_+\rangle - M_1|\phi_-\rangle, \quad (5.217)$$

Eq. (5.206)-Eq. (5.207) yields

$$D_1^{--}|\chi_-\rangle = -E_1|\phi_-\rangle + M_1|\phi_+\rangle, \quad (5.218)$$

in which the *kinetic recoil* terms appear through the combinations

$$
\begin{aligned}
D_1^{++} &:= T_1(+1) + U_1(+1) = \\
&\quad \exp\mathcal{G}\Big[\sigma_1\cdot p + \frac{i}{2}\sigma_2\cdot\partial\big[L + \mathcal{G}(1-\sigma_1\cdot\sigma_2)\big]\Big], && (5.219)\\
D_2^{++} &:= T_2(+1) + U_2(+1) = \\
&\quad \exp\mathcal{G}\Big[\sigma_2\cdot p + \frac{i}{2}\sigma_1\cdot\partial\big[L + \mathcal{G}(1-\sigma_1\cdot\sigma_2)\big]\Big], && (5.220)\\
D_1^{-+} &:= T_1(-1) + U_1(-1) = \\
&\quad \exp\mathcal{G}\Big[\sigma_1\cdot p + \frac{i}{2}\sigma_2\cdot\partial\big[-L + \mathcal{G}(1-\sigma_1\cdot\sigma_2)\big]\Big], && (5.221)\\
D_1^{--} &:= T_1(-1) - U_1(-1) = \\
&\quad \exp\mathcal{G}\Big[\sigma_1\cdot p + \frac{i}{2}\sigma_2\cdot\partial\big[L - \mathcal{G}(1+\sigma_1\cdot\sigma_2)\big]\Big]. && (5.222)
\end{aligned}
$$

Solve Eq. (5.215) and Eq. (5.216) for $|\chi_+\rangle$ and $|\chi_-\rangle$ we obtain

$$|\chi_+\rangle = \frac{1}{\mathcal{D}}(M_2 D_1^{++} - M_1 D_2^{++})|\phi_+\rangle, \tag{5.223}$$

$$|\chi_-\rangle = -\frac{1}{\mathcal{D}}(E_2 D_1^{++} + E_1 D_2^{++})|\phi_+\rangle, \tag{5.224}$$

$$\tag{5.225}$$

in which

$$\mathcal{D} := E_1 M_2 + E_2 M_1. \tag{5.226}$$

Solve Eq. (5.217) and Eq. (5.218) for $|\phi_+\rangle$

$$E_1 D_1^{-+}\chi_+ - M_1 D_1^{--}\chi_- = \mathcal{B}^2\phi_+, \tag{5.227}$$

in which

$$\mathcal{B}^2 := E_1^2 - M_1^2. \tag{5.228}$$

We combined Eq. (5.223) and Eq. (5.224) in Eq. (5.227) that yields the following equation (simplified here for electromagnetic like interactions ($\partial J := \frac{\partial E_1}{E_2} = -\partial G$) and scalar interactions alone)

$$\begin{aligned}
&[E_1 D_1^{-+}\frac{1}{E_1 M_2 + E_2 M_1}(M_2 D_1^{++} - M_1 D_2^{++}) \\
&+M_1 D_1^{--}\frac{1}{E_1 M_2 + E_2 M_1}(E_2 D_1^{++} + E_1 D_2^{++})]|\phi_+\rangle \\
&= (E_1^2 - M_1^2)|\phi_+\rangle.
\end{aligned} \tag{5.229}$$

We display all the general spin dependent structures in $\Phi(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, w)$ explicitly, very similar to what appears in nonrelativistic formalisms. We do this by expressing it explicitly in terms of its matrix $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$, and operator $\mathbf{p}$ structure in the CM system ($\hat{P} = (1, \mathbf{0})$). We are working in the CM frame (i.e. $x_\perp = (\mathbf{r}, 0)$), so all the interaction functions $(L(x_\perp), J(x_\perp), C(x_\perp), \mathcal{G}(x_\perp))$ are functions of $r = \sqrt{x_\perp^2} = |\mathbf{r}|$, $F = F(r)$

We derive explicitly the reduction of the following equations [388]

$$\begin{aligned}
&hE_1[\boldsymbol{\sigma}_1\cdot\mathbf{p} - i\boldsymbol{\sigma}_2\cdot(\mathbf{d} + \mathbf{k}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]hF_1[\boldsymbol{\sigma}_1\cdot\mathbf{p} - i\boldsymbol{\sigma}_2\cdot(\mathbf{z} + \mathbf{k}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]|\phi_+\rangle \qquad (a)\\
&+hM_1[\boldsymbol{\sigma}_1\cdot\mathbf{p} - i\boldsymbol{\sigma}_2\cdot(\mathbf{o} + \mathbf{k}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]hF_3[\boldsymbol{\sigma}_1\cdot\mathbf{p} - i\boldsymbol{\sigma}_2\cdot(\mathbf{z} + \mathbf{k}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]|\phi_+\rangle \qquad (b)\\
&-hE_1[\boldsymbol{\sigma}_1\cdot\mathbf{p} - i\boldsymbol{\sigma}_2\cdot(\mathbf{d} + \mathbf{k}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]hF_2[\boldsymbol{\sigma}_2\cdot\mathbf{p} - i\boldsymbol{\sigma}_1\cdot(\mathbf{z} + \mathbf{k}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]|\phi_+\rangle \qquad (c)\\
&+hM_1[\boldsymbol{\sigma}_1\cdot\mathbf{p} - i\boldsymbol{\sigma}_2\cdot(\mathbf{o} + \mathbf{k}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]hF_4[\boldsymbol{\sigma}_2\cdot\mathbf{p} - i\boldsymbol{\sigma}_1\cdot(\mathbf{z} + \mathbf{k}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]|\phi_+\rangle \qquad (d)\\
&= \mathcal{B}^2|\phi_+\rangle.
\end{aligned} \tag{5.230}$$

in which

$$\begin{aligned}\mathcal{B}^2 &= {E_1}^2 - {M_1}^2 = {E_2}^2 - {M_2}^2 \\ &= b^2(w) + (\epsilon_1^2 + \epsilon_2^2)\sinh^2(J) + 2\epsilon_1\epsilon_2 \sinh(J)\cosh(J) \\ &\quad -(m_1^2 + m_2^2)\sinh^2(L) - 2m_1m_2\sinh(L)\cosh(L). \end{aligned} \tag{5.231}$$

and

$$\begin{aligned} h &:= \exp(\mathcal{G}), \\ \mathbf{k} &:= \frac{1}{2}\nabla\log(h), \\ \mathbf{z} &:= \frac{1}{2}\nabla(-C + J - L), \\ \mathbf{d} &:= \frac{1}{2}\nabla(C + J + L), \\ \mathbf{o} &:= \frac{1}{2}\nabla(C - J - L), \end{aligned}$$

with

$$\begin{aligned} F_1 &:= \frac{M_2}{\mathcal{D}}, \\ F_2 &:= \frac{M_1}{\mathcal{D}}, \\ F_3 &:= \frac{E_2}{\mathcal{D}}, \\ F_4 &:= \frac{E_1}{\mathcal{D}}, \\ \mathcal{D} &:= E_1M_2 + E_2M_1. \end{aligned}$$

Eq. (5.230) is a second order Schrödinger like eigenvalue equation for the newly defined wavefunction $|\phi_+\rangle$ in the form.

$$(p_\perp^2 + \Phi(r, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, w))|\phi_+\rangle = b^2(w)|\phi_+\rangle. \tag{5.232}$$

Eq. (5.258) for $\mathcal{B}^2$, that provide us with the primary spin independent part of Φ, the quasipotential [539, 543]. Note that in the CM system $p_\perp^2 = \mathbf{p}^2$, $\sigma = (0, \boldsymbol{\sigma})$. For future reference we will refer to the four sets of terms on the left hand side as the Eq. (5.230) (a),(b),(c),(d) term.

Now we proceed with a different derivation than Long and Crater's derivation [388]. The aim is to produce a Schrödinger like form like in Eq. (5.232) involving the Pauli matrices for both particles.

Substitute $\mathbf{d}$, h, F_1, $\mathbf{z}$, $\mathbf{k}$'s expressions to (a) term of Eq. (5.230), we obtain

$$\text{(a) term} = \exp(\mathcal{G})E_1\{[\boldsymbol{\sigma}_1 \cdot \mathbf{p} - \frac{i}{2}\boldsymbol{\sigma}_2 \cdot \nabla(C + J + L) - \frac{i}{2}\nabla\mathcal{G} \cdot (\boldsymbol{\sigma}_1 + i\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)]$$

$$\times\exp(\mathcal{G})\frac{M_2}{\mathcal{D}}[\boldsymbol{\sigma}_1\cdot\mathbf{p}-\frac{i}{2}\boldsymbol{\sigma}_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)]\},\quad(5.233)$$

working out the commutation relation of $\boldsymbol{\sigma}_1\cdot\mathbf{p}$ in above expression, we can find

(a) term= $\exp(\mathcal{G})E_1\times$

$$\begin{aligned}&\{\exp(\mathcal{G})\frac{M_2}{\mathcal{D}}[\mathbf{p}^2-\frac{i}{2}\boldsymbol{\sigma}_2\cdot\nabla(-C+J-L)(\boldsymbol{\sigma}_1\cdot\mathbf{p})\\&-\frac{i}{2}\nabla\mathcal{G}\cdot[(\mathbf{p}+i(\boldsymbol{\sigma}_1\times\mathbf{p})-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\mathbf{p}+\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_2\cdot\mathbf{p})-i(\boldsymbol{\sigma}_2\times\mathbf{p})]]\\&+\frac{1}{i}\boldsymbol{\sigma}_1\cdot\partial[\exp(\mathcal{G})\frac{M_2}{\mathcal{D}}[\boldsymbol{\sigma}_1\cdot\mathbf{p}-\frac{i}{2}\boldsymbol{\sigma}_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)]]\\&-\frac{i}{2}[\boldsymbol{\sigma}_2\cdot\nabla(C+J+L)+\nabla\mathcal{G}\cdot(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)]\exp(\mathcal{G})\frac{M_2}{\mathcal{D}}[\boldsymbol{\sigma}_1\cdot\mathbf{p}\\&-\frac{i}{2}\boldsymbol{\sigma}_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)]\}.\end{aligned}\quad(5.234)$$

Likewise we can the find (b),(c),(d) terms.

(b) term= $\exp(\mathcal{G})M_1\times$

$$\begin{aligned}&\{\exp(\mathcal{G})\frac{E_2}{\mathcal{D}}[\mathbf{p}^2-\frac{i}{2}\boldsymbol{\sigma}_2\cdot\nabla(-C+J-L)(\boldsymbol{\sigma}_1\cdot\mathbf{p})\\&-\frac{i}{2}\nabla\mathcal{G}\cdot[(\mathbf{p}+i(\boldsymbol{\sigma}_1\times\mathbf{p})-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\mathbf{p}+\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_2\cdot\mathbf{p})-i(\boldsymbol{\sigma}_2\times\mathbf{p})]]\\&+\frac{1}{i}\boldsymbol{\sigma}_1\cdot\partial[\exp(\mathcal{G})\frac{E_2}{\mathcal{D}}[\boldsymbol{\sigma}_1\cdot\mathbf{p}-\frac{i}{2}\boldsymbol{\sigma}_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)]]\\&-\frac{i}{2}[\boldsymbol{\sigma}_2\cdot\nabla(C-J-L)+\nabla\mathcal{G}\cdot(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)]\exp(\mathcal{G})\frac{E_2}{\mathcal{D}}[\boldsymbol{\sigma}_1\cdot\mathbf{p}\\&-\frac{i}{2}\boldsymbol{\sigma}_2\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)]\},\end{aligned}\quad(5.235)$$

(c) term= $-\exp(\mathcal{G})E_1\times$

$$\begin{aligned}&\{\exp(\mathcal{G})\frac{M_1}{\mathcal{D}}[(\boldsymbol{\sigma}_2\cdot\mathbf{p})(\boldsymbol{\sigma}_1\cdot\mathbf{p})-\frac{i}{2}\boldsymbol{\sigma}_1\cdot\nabla(-C+J-L)(\boldsymbol{\sigma}_1\cdot\mathbf{p})\\&-\frac{i}{2}\nabla\mathcal{G}\cdot[(\boldsymbol{\sigma}_2(\boldsymbol{\sigma}_1\cdot\mathbf{p})-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\mathbf{p}+\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_2\cdot\mathbf{p})+i(\boldsymbol{\sigma}_2\times\mathbf{p})]]\\&+\frac{1}{i}\boldsymbol{\sigma}_1\cdot\partial[\exp(\mathcal{G})\frac{M_1}{\mathcal{D}}[\boldsymbol{\sigma}_2\cdot\mathbf{p}-\frac{i}{2}\boldsymbol{\sigma}_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\boldsymbol{\sigma}_2+i\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_1)]]\\&-\frac{i}{2}[\boldsymbol{\sigma}_2\cdot\nabla(C+J+L)+\nabla\mathcal{G}\cdot(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)]\exp(\mathcal{G})\frac{M_1}{\mathcal{D}}[\boldsymbol{\sigma}_2\cdot\mathbf{p}\\&-\frac{i}{2}\boldsymbol{\sigma}_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\boldsymbol{\sigma}_2+i\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_1)]\},\end{aligned}\quad(5.236)$$

(d) term= $\exp(\mathcal{G})M_1\times$

$$
\begin{aligned}
&\{\exp(\mathcal{G})\frac{E_1}{\mathcal{D}}[(\boldsymbol{\sigma}_2\cdot\mathbf{p})(\boldsymbol{\sigma}_1\cdot\mathbf{p})-\frac{i}{2}\boldsymbol{\sigma}_1\cdot\nabla(-C+J-L)(\boldsymbol{\sigma}_1\cdot\mathbf{p})\\
&-\frac{i}{2}\nabla\mathcal{G}\cdot[(\boldsymbol{\sigma}_2(\boldsymbol{\sigma}_1\cdot\mathbf{p})-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\mathbf{p}+\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_2\cdot\mathbf{p})+i(\boldsymbol{\sigma}_2\times\mathbf{p})]]\\
&+\frac{1}{i}\boldsymbol{\sigma}_1\cdot\partial[\exp(\mathcal{G})\frac{E_1}{\mathcal{D}}[\boldsymbol{\sigma}_2\cdot\mathbf{p}-\frac{i}{2}\boldsymbol{\sigma}_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\boldsymbol{\sigma}_2+i\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_1)]]\\
&-\frac{i}{2}[\boldsymbol{\sigma}_2\cdot\nabla(C-J-L)+\nabla\mathcal{G}\cdot(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)]\exp(\mathcal{G})\frac{E_1}{\mathcal{D}}[\boldsymbol{\sigma}_2\cdot\mathbf{p}\\
&-\frac{i}{2}\boldsymbol{\sigma}_1\cdot\nabla(-C+J-L)-\frac{i}{2}\nabla\mathcal{G}\cdot(\boldsymbol{\sigma}_2+i\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_1)]\}. \qquad (5.237)
\end{aligned}
$$

5.3.1 Pauli Reduction

The multiplication properties of a matrix form for the wave functions with the Pauli matrices σ_1 and σ_2 Eq. (5.232), instead of a spinor form, make them a particularly suitable choice. For this reason we transform the four component spinor ϕ_+ into 2×2 matrix wave function composed of a scalar spherical harmonic part (spin zero) and a vector spherical harmonic part (spin one) by the redefinition (in the CM system)

$$\phi_+ \to \phi_+\sigma_y := \Phi := \phi + \phi\cdot\sigma.$$

The above definition includes a transpose and left multiplication by σ_y. This leads to the properties

$$\sigma_1\phi_+ \to \sigma\Phi,\ \sigma_2\phi_+ \to -\Phi\sigma. \qquad (5.238)$$

This in turn leads to

$$\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}\phi_+ \to -\sigma\cdot\Phi\sigma = -3\phi+\phi\cdot\sigma, \qquad (5.239)$$

which shows the singlet and triplet nature of the two parts of the matrix wave function. In general, since an arbitrary function $F(\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2})$ can be expanded into a form

$$F(\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}) = A + B\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}, \qquad (5.240)$$

we have

$$
\begin{aligned}
F(\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2})\phi_+ &= A+B\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}\phi_+ \to [A\Phi - B\sigma\cdot\Phi\sigma]\\
&= (A-3B)\phi + (A+B)\phi\cdot\sigma\\
&= F(-3)\phi + F(+1)\phi\cdot\sigma. \qquad (5.241)
\end{aligned}
$$

In computing the four sets of terms on left hand side of Eq. (5.230), we use the identities

$$
\begin{aligned}
&\sigma_1\cdot A\phi_+ \to \sigma\cdot A\Phi = A\cdot\phi + \sigma\cdot[A\phi + iA\times\phi],\\
&\sigma_1\cdot A\phi_+ \to -A\cdot\Phi\sigma = -A\cdot\phi - \sigma\cdot[A\phi - iA\times\phi].
\end{aligned}
$$

The calculation of Eq. (5.230) then proceeds by reducing the left hand side to a form $L + \mathbf{L} \cdot \sigma$ in which $[\pi_{i1}]$ we call L the *scalar* and $\mathbf{L}$ the *vector* term, respectively. We define the left hand side of Eq. (5.230) in these terms as

$$\begin{aligned} \Pi\Phi = \mathcal{L} &= L + \mathbf{L} \cdot \sigma := \text{left hand side} \quad (5.230) \\ &= (5.230a) + (5.230b) + (5.230c) + (5.230d). \end{aligned} \tag{5.242}$$

Having defined the left hand side in the terms $L + \mathbf{L} \cdot \sigma$ we now proceed to calculate L and $\mathbf{L}$ explicitly.

We write the left hand Eq. (5.230)

$$L + \mathbf{L} \cdot \phi := \Pi\Phi = \Pi_1\Phi + \Pi_2\Phi + \Pi_3\Phi + \Pi_4\Phi, \tag{5.243}$$

and $\Pi_1\Phi = (5.230a), \Pi_2\Phi = (5.230b), \Pi_3\Phi = (5.230c), \Pi_4\Phi = (5.230d)$. Notice that each Π_i is a product of four matrix factors

$$\Pi_i = [\pi_{i1}][\pi_{i2}][\pi_{i3}][\pi_{i4}].$$

Explicitly Π_1 is defined

$$\begin{aligned} \Pi_1 &= \underbrace{hE_1}_{\pi_{11}} \underbrace{[\boldsymbol{\sigma}_1 \cdot \mathbf{p} - i\boldsymbol{\sigma}_2 \cdot (\mathbf{d} + \mathbf{k}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)]}_{\pi_{12}} \underbrace{hF_1}_{\pi_{13}} \underbrace{[\boldsymbol{\sigma}_1 \cdot \mathbf{p} - i\boldsymbol{\sigma}_2 \cdot (\mathbf{z} + \mathbf{k}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)]}_{\pi_{14}} \\ &= [\pi_{i1}][\pi_{i2}][\pi_{i3}][\pi_{i4}]. \end{aligned} \tag{5.244}$$

Due to the properties Eq. (5.238), each of the π_{ij} terms acts on the matrix wave function F to produce yet another matrix form

$$[\pi_{ij}]\Phi = \mathcal{Z}_i = Z_i + \mathbf{Z}_i \cdot \sigma.$$

The multiplication of each of the four $\Pi_i\Phi$ terms can now proceed as

$$[\pi_{i1}][\pi_{i2}][\pi_{i3}][\pi_{i4}]\Pi = [\pi_{i1}][\pi_{i2}][\pi_{i3}]\mathcal{N}_i = [\pi_{i1}][\pi_{i2}]\mathcal{P}_i = [\pi_{i1}]\mathcal{Q}_i = \mathcal{L}_i = L_i + \mathbf{L_i} \cdot \sigma,$$

where $\mathcal{N}_i := N_i + \mathbf{N_i} \cdot \sigma$, same for $\mathcal{P}_i, \mathcal{Q}_i$. For each of the four matrix multiplications $\Pi_i\Phi,\ i = 1, 2, 3, 4$ we obtain a term of the form $L_i + \mathbf{L_i} \cdot \sigma_\mathbf{i}$. For the complete calculation of Eq. (5.243)

$$\begin{aligned} R &:= L_1 + L_2 + L_3 + L_4 \\ \mathbf{R} &:= \mathbf{L_1} + \mathbf{L_2} + \mathbf{L_3} + \mathbf{L_4}. \end{aligned}$$

Inspection of Eq. (5.230) reveals that the matrix structures of the terms Eq. (5.230a) and Eq. (5.230b) are the same. In fact Eq. (5.230a) becomes Eq. (5.230b) (and vice versa) if we make the variable exchanges

$$E_1 \Leftrightarrow M_1,\ F_1 \Leftrightarrow F_3, \mathbf{d} \Leftrightarrow \mathbf{o}. \tag{5.245}$$

This simplifies the calculation of Eq. (5.243): we can calculate the first term $\Pi_1\Phi$, which is Eq. (5.230a), and then find the second term $\Pi_2\Phi$, which is Eq. (5.230b), by the variable exchanges of Eq. (5.245). Inspection of Eq. (5.230) reveals that the matrix structures of the terms Eq. (5.230c) and Eq. (5.230d) are the same. In fact Eq. (5.230c) becomes Eq. (5.230d) (and vice versa) if we make the variable exchanges

$$E_1 \Leftrightarrow M_1,\; F_2 \Leftrightarrow F_4, bfd \Leftrightarrow o. \tag{5.246}$$

In light of the simplification produced by the variable exchanges of relations Eq. (5.245) and Eq. (5.245), we proceed to calculate Eq. (5.243): first we calculate $\Pi_1\Phi = \mathcal{L}_1 = L_1 + \mathbf{L_1}\cdot\sigma$, then use it to find $\Pi_2\Phi = \mathcal{L}_2 = L_2 + \mathbf{L_2}\cdot\sigma$, by the variable exchanges Eq. (5.245), then we calculate $\Pi_3\Phi = \mathcal{L}_3 = L_3 + \mathbf{L_3}\cdot\sigma$ and use it to find $\Pi_4\Phi = \mathcal{L}_4 = L_4 + \mathbf{L_4}\cdot\sigma$ by the variable exchanges Eq. (5.245).

Calculation $\mathcal{L}_1 = L_1 + \mathbf{L_1}\cdot\sigma$:

$$[\pi_{11}][\pi_{12}][\pi_{13}][\pi_{14}]\Pi = [\pi_{11}][\pi_{12}][\pi_{13}]\mathcal{N}_1 = [\pi_{11}][\pi_{12}]\mathcal{P}_1$$
$$= [\pi_{11}]\mathcal{Q}_1 = \mathcal{L}_1 = L_1 + \mathbf{L_1}\cdot\sigma,$$

Starting with the right most term $\mathcal{N}_1$ (it is understood that σ_1 and σ_2 act in accordance with Eq. (5.238),

$$[\pi_{14}]\Phi = [\boldsymbol{\sigma}_1\cdot\mathbf{p} - i\boldsymbol{\sigma}_2\cdot(\mathbf{z} + \mathbf{k}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]\Phi = \mathcal{N}_1 := N_1 + \mathbf{N_1}\cdot\sigma, \tag{5.247}$$

where N_1 and $\mathbf{N_1}$ are

$$\begin{aligned}
&N_1 = [\mathbf{p} + i(\mathbf{z}+\mathbf{k})]\cdot\phi,\\
&\mathbf{N}_1 = [\mathbf{p} + i(\mathbf{z} - \mathbf{3k})]\cdot\phi + i[\mathbf{p} - i(\mathbf{z}+\mathbf{k})]\times\phi,\\
&[\pi_{13}]\mathcal{N}_1 = hF_1\mathcal{N}_1 = \mathcal{P}_1 := P_1 + \mathbf{P_1}\cdot\sigma,
\end{aligned}$$

P_1 and $\mathbf{P_1}$ are

$$P_1 = hF_1N_1,\, \mathbf{P_1} = hF_1\mathbf{N_1}, \tag{5.248}$$

nonumber $$[\pi_{12}]\mathcal{P}_1 = [\sigma_\mathbf{1}\cdot\mathbf{p} - \mathbf{i}\sigma_\mathbf{2}\cdot(\mathbf{d} + \mathbf{k}\sigma_\mathbf{1}\cdot\sigma_\mathbf{2})]\mathcal{P}_1 = \mathcal{Q}_1 := Q_1 + \mathbf{Q_1}\cdot\sigma, \tag{5.249}$$

and Q_1 and $\mathbf{Q_1}$ are

$$\begin{aligned}
&Q_1 = [\mathbf{p} + i(\mathbf{d}+\mathbf{k})]\cdot\mathbf{P}_1,\\
&\mathbf{Q}_1 = [\mathbf{p} + i(\mathbf{d} - \mathbf{3k})]\cdot P_1 + i[\mathbf{p} - i(\mathbf{d}+\mathbf{k})]\times\mathbf{P}_1,\\
&[\pi_{11}]\mathcal{Q}_1 = hE_1\mathcal{Q}_1 = \mathcal{L}_1 := L_1 + \mathbf{L_1}\cdot\sigma,
\end{aligned}$$

where finally L_1 and $\mathbf{L_1}$ are

$$L_1 = hE_1\mathcal{Q}_1,\, \mathbf{L}_1 = hE_1\mathcal{Q}_\mathbf{1}, \tag{5.250}$$

Combining the multiplicative factors, the *scalar* part of L_1 is therefore

$$L_1 = hE_1[\mathbf{p} + i(\mathbf{d} + \mathbf{k})] \cdot hF_1([\mathbf{p} + i(\mathbf{z} - \mathbf{3k})]\phi + i[\mathbf{p} - i(\mathbf{z} + \mathbf{k})] \times \phi), \quad (5.251)$$

while the *vector* part of $\mathbf{L_1}$ is

$$\mathbf{L}_1 = hE_1([\mathbf{p} + i(\mathbf{d} - \mathbf{3k})] \cdot hF_1[\mathbf{p} + i(\mathbf{z} + \mathbf{k})]\phi + i[\mathbf{p} - i(\mathbf{d} + \mathbf{k})]$$
$$\times hF_1([\mathbf{p} + i(\mathbf{z} - \mathbf{3k})]\phi + i[\mathbf{p} - i(\mathbf{z} + \mathbf{k})] \times \phi)). \quad (5.252)$$

Now, we repeat our previous results for other *scalar* and *vector* parts of L_i, $i = 2, 3, 4$ and $\mathbf{L_i}$, $i = 2, 3, 4$, and combining they are

$$\begin{aligned}\sum_{i=1}^{4} L_i &= h[hE_1(F_1 + F_2) + hM_1(F_3 - F_4)]\mathbf{p} \cdot (\mathbf{p} + i(\mathbf{z} - \mathbf{3k}))\phi \\ &+i(hE_1[h(F_1 + F_2)(\mathbf{d} + \mathbf{k}) - \nabla(h(F_1 + F_2))] \\ &+hM_1[h(F_3 - F_4)(\mathbf{o} + \mathbf{k}) - \nabla(h(F_3 - F_4))]) \cdot (\mathbf{p} \\ &+i(\mathbf{z} - \mathbf{3k}))\phi ih[hE_1(F_1 - F_2) + hM_1(F_3 + F_4)]\mathbf{p} \cdot (\mathbf{p} \\ &-i(\mathbf{z} + \mathbf{k})) \times \phi - (hE_1[h(F_1 - F_2)(\mathbf{d} + \mathbf{k}) \\ &-\nabla(h(F_1 - F_2))] + hM_1[h(F_3 + F_4)(\mathbf{o} + \mathbf{k}) \\ &-\nabla(h(F_3 - F_4))]) \cdot (\mathbf{p} - i(\mathbf{z} + \mathbf{k})) \times \phi, \quad (5.253)\end{aligned}$$

for the scalar portion. We can simplify the vector portion below

$$\begin{aligned}\sum_{\mathbf{i=1}}^{4} \mathbf{L_i} &= h[hE_1(F_1 + F_2) + hM_1(F_3 - F_4)]\mathbf{p} \cdot (\mathbf{p} + i(\mathbf{z} + \mathbf{k})) \cdot \phi \\ &+i(hE_1[h(F_1 + F_2)(\mathbf{d} + \mathbf{3k}) - \nabla(h(F_1 + F_2))] \\ &+hM_1[h(F_3 - F_4)(\mathbf{o} - \mathbf{3k}) - \nabla(h(F_3 - F_4))]) \\ &\times(\mathbf{p} + i(\mathbf{z} + \mathbf{k}))\phi ih[hE_1(F_1 - F_2) + hM_1(F_3 + F_4)] \\ &\times\mathbf{p} \times (\mathbf{p} + i(\mathbf{z} - \mathbf{3k}))\phi + (hE_1[h(F_1 + F_2)(\mathbf{d} + \mathbf{k}) \\ &+\nabla(h(F_1 + F_2))] + hM_1[h(F_3 - F_4)(\mathbf{o} + \mathbf{k}) \\ &+\nabla(h(F_3 - F_4))]) \times (\mathbf{p} + i(\mathbf{z} - \mathbf{3k}))\phi - h[hE_1(F_1 \\ &-F_2) + hM_1(F_3 + F_4)]\mathbf{p} \times [(\mathbf{p} - i(\mathbf{z} + \mathbf{k})) \times \phi] \\ &+i(hE_1[h(F_1 - F_2)(\mathbf{d} + \mathbf{k}) + \nabla(h(F_1 - F_2))] \\ &+hM_1[h(F_3 + F_4)(\mathbf{o} + \mathbf{k}) + \nabla(h(F_3 + F_4))]) \cdot \\ &\times(\mathbf{p} - i(\mathbf{z} + \mathbf{k})) \times \phi. \quad (5.254)\end{aligned}$$

Using the following vector identities ($\mathbf{A} = A\hat{\mathbf{r}}$),

$$\begin{aligned}
&\mathbf{p} \times (\mathbf{p} \times \phi) = \mathbf{p}(\mathbf{p} \cdot \phi) - \mathbf{p^2}\phi, \\
&i\mathbf{p} \times (\mathbf{A} \times \phi) = \left[A' - \frac{A}{r}\right] \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \phi) + \left(i\mathbf{A} \cdot \mathbf{p} - \nabla \cdot + \frac{A}{r}\right) \phi, \\
&\hat{\mathbf{r}}(\hat{\mathbf{r}} \times \phi) = \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \phi) - \phi, \\
&i\mathbf{A} \times (\mathbf{p} \times \phi) = iA_i\mathbf{p}\phi_i - i\mathbf{A} \cdot \mathbf{p}\phi, \\
&r_i\mathbf{p}\phi_i = \mathbf{r}(\mathbf{p} \cdot \phi) + \mathbf{L} \times \phi,
\end{aligned} \tag{5.255}$$

leads to a second order Schrödinger like eigenvalue equation for the four component wave function $|\phi_+\rangle = |\psi\rangle_1 + |\psi\rangle_4$ in the general form

$$(\mathbf{p}^2 + \Phi(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, w))|\phi_+\rangle = b^2(w)|\phi_+\rangle. \tag{5.256}$$

Simplification of the final result by using identities involving $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ and grouping by the $\mathbf{p}^2$ term , Darwin term ($\hat{\mathbf{r}} \cdot \mathbf{p}$), spin orbit angular momentum term $\mathbf{L} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)$, spin orbit angular momentum difference term $\mathbf{L} \cdot (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)$, spin spin term $(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)$, tensor term $(\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}})(\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})$, additional spin dependent terms $\mathbf{L} \cdot (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)$ and $(\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}})(\boldsymbol{\sigma}_2 \cdot \mathbf{p}) + (\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})(\boldsymbol{\sigma}_1 \cdot \mathbf{p})$ and spin independent terms. Collecting all terms for the (a) + (b) + (c) + (d) terms above Eq. (5.230), we obtain

$$\begin{aligned}
&\Bigg\{\mathbf{p}^2 - i\left[2\mathcal{G}' - \frac{E_2M_2 + M_1E_1}{\mathcal{D}}(J+L)'\right](\hat{\mathbf{r}} \cdot \mathbf{p}) \\
&- \frac{i(J-L)'}{2}((\sigma_1 \cdot \hat{\mathbf{r}})(\sigma_2 \cdot \mathbf{p}) + (\sigma_2 \cdot \hat{\mathbf{r}})(\sigma_1 \cdot \mathbf{p})) \\
&-\frac{1}{2}\nabla^2\mathcal{G} - \frac{1}{4}\mathcal{G}'^2 - \frac{1}{4}(C+J-L)'(-C+J-L)' + \frac{1}{2}\frac{E_2M_2 + M_1E_1}{\mathcal{D}}\mathcal{G}'(J+L)' \\
&+(\sigma_1 \cdot \sigma_2)\left[\frac{1}{2}\nabla^2\mathcal{G} + \frac{1}{2}\mathcal{G}'^2 - \frac{1}{2}\frac{E_2M_2 + M_1E_1}{\mathcal{D}}\mathcal{G}'(J+L)'\right. \\
&\left.-\frac{1}{2}\mathcal{G}'C' - \frac{1}{2}\frac{\mathcal{G}'}{r} - \frac{1}{2}\frac{(-C+J-L)'}{r}\right] \\
&+\frac{\mathbf{L} \cdot (\sigma_1 + \sigma_2)}{r}\left[\mathcal{G}' - \frac{1}{2}\frac{E_2M_2 + M_1E_1}{\mathcal{D}}(J+L)'\right] \\
&-\frac{\mathbf{L} \cdot (\sigma_1 - \sigma_2)}{r}\frac{1}{2}\frac{E_2M_2 - M_1E_1}{\mathcal{D}}(J+L)' \\
&+\frac{\mathbf{L} \cdot (\sigma_1 \times \sigma_2)}{r}\frac{i}{2}\frac{M_2E_1 - M_1E_2}{\mathcal{D}}(J+L)' \\
&+(\sigma_1 \cdot \hat{\mathbf{r}})(\sigma_2 \cdot \hat{\mathbf{r}})\left[-\frac{1}{2}\nabla^2(-C+J-L) - \frac{1}{2}\nabla^2\mathcal{G} - \mathcal{G}'(-C+J-L)' - \mathcal{G}'^2 + \frac{3}{2r}\mathcal{G}'\right. \\
&\left.+\frac{3}{2r}(-C+J-L)' + \frac{1}{2}\frac{E_2M_2 + M_1E_1}{\mathcal{D}}(J+L)'(\mathcal{G} - C + J - L)'\right]\Bigg\}|\phi_+\rangle \\
&= e^{-2\mathcal{G}}\mathcal{B}^2|\phi_+\rangle,
\end{aligned} \tag{5.257}$$

where

$$\begin{aligned}
\mathcal{D} &:= E_1 M_2 + E_2 M_1, \\
\mathcal{B}^2 &= {E_1}^2 - {M_1}^2 = {E_2}^2 - {M_2}^2, \\
&= b^2(w) + (\epsilon_1^2 + \epsilon_2^2)\text{sinh}^2(J) + 2\epsilon_1\epsilon_2 \sinh(J)\cosh(J), \\
&- (m_1^2 + m_2^2)\text{sinh}^2(L) - 2m_1 m_2 \sinh(L)\cosh(L). \qquad (5.258)
\end{aligned}$$

$E_i, M_i, C, J, L, \mathcal{G}$ are all functions of the invariant r. We point out that Eq. (5.257) differs from the forms presented in [388]. Whereas the above equation involve four component spinor wave functions, the ones given in [388] are obtained in terms of matrix wave functions involving one component scalar and three component vector wave functions.

All of the above equations when reduced to radial form in next section having also a first derivative terms from the $\hat{\mathbf{r}}\cdot\mathbf{p}$ and $(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}})(\boldsymbol{\sigma}_2\cdot\mathbf{p})+(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})(\boldsymbol{\sigma}_1\cdot\mathbf{p})$ terms. An advantage of the above for the relativistic case is that they are Schrödinger like equations which we solve numerically.

5.3.2 Reduction to Radial Form

The general form of the eigenvalue equation given in Eq. (5.257) is

$$\begin{aligned}
&[\mathbf{p}^2 - ig'\hat{\mathbf{r}}\cdot\mathbf{p} + \frac{g'}{2r}\vec{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2) - ih'(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\mathbf{p} + \boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_1\cdot\mathbf{p}) \\
&+ k\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + n\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}} + l\vec{L}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2) + ij\vec{L}\cdot(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2) + m]|\phi_+\rangle \\
&= \mathcal{B}^2 e^{-2\mathcal{G}}|\phi_+\rangle. \qquad (5.259)
\end{aligned}$$

The m term is the spin independent part involving derivatives of the potentials. For the equal mass case, two terms drop out (see Eq. (5.257)), and the above equation becomes [75]

$$\begin{aligned}
&[\mathbf{p}^2 - ig'\hat{\mathbf{r}}\cdot\mathbf{p} + \frac{g'}{2r}\vec{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2) - ih'(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\mathbf{p} + \boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_1\cdot\mathbf{p}) \\
&+ k\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + n\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}} + m]|\phi_+\rangle = \mathcal{B}^2 e^{-2\mathcal{G}}|\phi_+\rangle. \qquad (5.260)
\end{aligned}$$

We introduce the spin dependent scale change

$$|\phi_+\rangle \equiv \exp(F + K\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})|\psi_+\rangle \equiv (A + B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})|\psi_+\rangle. \qquad (5.261)$$

with F, K, A, B to be determined. We find that

$$\begin{aligned}
\mathbf{p}|\phi_+\rangle &= (A + B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\mathbf{p}|\psi_+\rangle - i(A' + B'\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}|\psi_+\rangle \\
&- i\frac{B}{r}[(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\hat{\mathbf{r}})\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}} + (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\hat{\mathbf{r}})\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}]|\psi_+\rangle, \qquad (5.262)
\end{aligned}$$

and

$$\frac{g'}{2r}\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)|\phi_+\rangle=(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\frac{g'}{2r}\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)|\psi_+\rangle$$

$$+\frac{g'}{2r}B[2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-4ir\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\,\hat{\mathbf{r}}\cdot\mathbf{p}+2ir(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\mathbf{p}+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_1\cdot\mathbf{p})-6\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}]|\psi_+\rangle.$$

We thus find that

$$-ig'\hat{\mathbf{r}}\cdot\mathbf{p}|\phi_+\rangle=(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})(-ig'\hat{\mathbf{r}}\cdot\mathbf{p})|\psi_+\rangle+C|\psi_+\rangle$$

and

$$\begin{aligned}&-ih'(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\mathbf{p}+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_1\cdot\mathbf{p})|\phi_+\rangle\\&=(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})(-ih'[\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\mathbf{p}+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_1\cdot\mathbf{p}])|\psi_+\rangle+D|\psi_+\rangle\end{aligned}$$

and finally

$$\begin{aligned}\mathbf{p}^2|\phi_+\rangle&=(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\mathbf{p}^2|\psi_+\rangle-2i(A'+B'\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\cdot\mathbf{p}|\psi_+\rangle\\&\quad+i\frac{2B}{r}[2\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\,\hat{\mathbf{r}}\cdot\mathbf{p}-(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\mathbf{p}+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_1\cdot\mathbf{p}]|\psi_+\rangle+E|\psi_+\rangle,\end{aligned}\tag{5.263}$$

where C and D and E do not involve $\mathbf{p}$ and are given by

$$C=-g'(A'+B'\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}),\tag{5.264}$$

$$D=-2h'(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}A'+B')-2h'\frac{B}{r}[\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)+2-\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2],\tag{5.265}$$

and

$$E=-(A''+B''\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})-\frac{2}{r}(A'+B'\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})-2\frac{B}{r^2}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}).\tag{5.266}$$

The general form of the eigenvalue equation then becomes after some detail [75]

$$\begin{aligned}&(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})[\mathbf{p}^2-ig'\hat{\mathbf{r}}\cdot\mathbf{p}+\frac{g'}{2r}\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\\&-ih'(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\mathbf{p}+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_1\cdot\mathbf{p})]|\psi_+\rangle\\&+(\frac{g'}{2r}B[2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-4ir\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\,\hat{\mathbf{r}}\cdot\mathbf{p}\\&+2ir(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\mathbf{p}+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_1\cdot\mathbf{p})-6\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}]\\&-2i(A'+B'\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\cdot\mathbf{p}+i\frac{2B}{r}[2\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\,\hat{\mathbf{r}}\cdot\mathbf{p}\\&-(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\mathbf{p}+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_1\cdot\mathbf{p})]\\&+(k\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+n\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})+R+m)|\psi_+\rangle\\&=\mathcal{B}^2\exp(-2\mathcal{G})(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})|\psi_+\rangle\end{aligned}\tag{5.267}$$

in which $R = C + D + E$.

Now, to bring this equation to the desired Schrödinger like form with no linear $\mathbf{p}$ term we multiply both sides by

$$(A + B\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})^{-1} = \frac{(A - B\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})}{A^2 - B^2} \tag{5.268}$$

and find, using the exponential form above that appears in Eq. (5.261), (and some detail [75])

$$\begin{aligned}&(A + B\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})^{-1}[-2i(A' + B'\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})]\hat{\mathbf{r}} \cdot \mathbf{p} \\ &= -2i(F' + K'\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} \cdot \mathbf{p},\end{aligned} \tag{5.269}$$

and

$$\begin{aligned}&(A + B\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})^{-1} i\frac{2B}{r}[2\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p} - (\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \mathbf{p} + \boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_1 \cdot \mathbf{p})] \\ &= \frac{2i\sinh(K)\cosh(K)}{r}[2\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p} - (\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \mathbf{p} + \boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_1 \cdot \mathbf{p})] + G\end{aligned}$$

where [75]

$$G = -\frac{2\sinh^2(K)}{r^2}\mathbf{L} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2), \tag{5.270}$$

and

$$\begin{aligned}&(A + B\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})^{-1}\frac{g'}{2r}B[2\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 - 4ir\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p} \\ &+ 2ir(\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \mathbf{p} + \boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_1 \cdot \mathbf{p}) - 6\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}] \\ &= \frac{ig'\sinh(K)\cosh(K)}{2r}[-4r\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p} + 2r(\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \mathbf{p} + \boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_1 \cdot \mathbf{p}) \\ &- 2i\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + 6i\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}] + H\end{aligned} \tag{5.271}$$

where [75]

$$H = \frac{g'\sinh^2(K)}{2r}[2\mathbf{L} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) - 2\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}} + 2\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + 4].$$

Note that G and H do not contain linear $\mathbf{p}$ type of terms. Now collect the three different linear $\mathbf{p}$ type of terms in Eq. (5.267):

$$(-2iF' - ig')\hat{\mathbf{r}} \cdot \mathbf{p}, \tag{5.272}$$

$$(-2i\frac{\sinh(K)\cosh(K)}{r} - ih' + ig'\sinh(K)\cosh(K))(\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \mathbf{p} + \boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_1 \cdot \mathbf{p}), \tag{5.273}$$

$$(4i\frac{\sinh(K)\cosh(K)}{r} - 2i\sinh(K)\cosh(K)g' - 2iK')\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot \mathbf{p}. \tag{5.274}$$

If we set the first of the above equations to 0, we obtain the expected result (for the uncoupled portion of the equation)

$$F' = -g'/2. \tag{5.275}$$

If we set $h' = -K'$ and use $\mathbf{p} = \hat{\mathbf{r}}(\hat{\mathbf{r}}.\mathbf{p}) - \frac{\hat{\mathbf{r}} \times L}{r}$ then the two expressions (5.273) and (5.274) combine to

$$(2\frac{\sinh(K)\cosh(K)}{r} + h' - g'\sinh(K)\cosh(K))\frac{\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}\vec{L} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)}{r} \tag{5.276}$$

which contains no $\hat{\mathbf{r}} \cdot \mathbf{p}$. Thus the matrix scale change

$$|\phi_+\rangle = \exp(-g/2)\exp(-h\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})|\psi_+\rangle \tag{5.277}$$

eliminates the linear $\mathbf{p}$ terms.

Further note that

$$(A + B\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})^{-1}(k\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + n\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})(A + B\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})$$
$$= (k\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + n\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}),$$

$$(A + B\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})^{-1}C|\psi_+\rangle = -g'(F' + K'\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})|\psi_+\rangle, \tag{5.278}$$

and (after some algebraic detail [75])

$$(A + B\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})^{-1}D|\psi_+\rangle = -2h'(K' + F'\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})|\psi_+\rangle$$

$$-2h'\frac{\cosh(K)\sinh(K)}{r}[\mathbf{L} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) + 2 - \boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}} + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2]|\psi_+\rangle$$

$$+2h'\frac{\sinh^2(K)}{r}[\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}\mathbf{L} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) + 3\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}} - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2]|\psi_+\rangle. \tag{5.279}$$

also

$$(A + B\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})^{-1}E|\psi_+\rangle = -[F'' + F'^2 + K'^2 + (2F'K' + K'')\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}]$$

$$-\frac{2}{r}[F' + K'\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}] - 2\frac{\cosh(K)\sinh(K)}{r^2}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 - 3\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})$$

$$+2\frac{\sinh^2(K)}{r^2}(\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}} - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 - 2). \tag{5.280}$$

So combining all terms and grouping by $\mathbf{p}^2$ term , spin independent terms, spin-orbit angular momentum term $\mathbf{L} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)$, spin-spin term $(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)$, tensor term

$(\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}})(\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}})$, additional spin independent term we have our Schrödinger-like equation

$$
\begin{aligned}
&\{\mathbf{p}^2 + \frac{2g' \sinh^2(K)}{r} - g'F' - 2h'K' - 4h'\frac{\cosh(K)\sinh(K)}{r} \\
&- F'' - F'^2 - K'^2 - \frac{2}{r}F' - 4\frac{\sinh^2(K)}{r^2} \\
&+ \mathbf{L}\cdot(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)[\frac{g'}{2r} + \frac{g'\sinh^2(K)}{r} - \frac{2\sinh^2(K)}{r^2} - 2h'\frac{\cosh(K)\sinh(K)}{r}] \\
&+ \boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}\mathbf{L}\cdot(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\times \\
&(2h'\frac{\sinh^2(K)}{r} + 2\frac{\sinh(K)\cosh(K)}{r^2} + \frac{h'}{r} - \frac{g'\sinh(K)\cosh(K)}{r}) \\
&+ \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2[k + \frac{g'\cosh(K)\sinh(K)}{r} + \frac{g'\sinh^2(K)}{r} - 2h'\frac{\cosh(K)\sinh(K)}{r} \\
&- 2h'\frac{\sinh^2(K)}{r} - 2\frac{\cosh(K)\sinh(K)}{r^2} - 2\frac{\sinh^2(K)}{r^2}] \\
&+ \boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}}\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}[n - \frac{3g'\cosh(K)\sinh(K)}{r} - \frac{g'\sinh^2 K}{r} - g'K' - 2h'F' \\
&+ \frac{2h'\cosh K \sinh K}{r} + 6h'\frac{\sinh^2(K)}{r} - (2F'K' + K'') - \frac{2}{r}K' \\
&+ 6\frac{\cosh(K)\sinh(K)}{r^2} + 2\frac{\sinh^2(K)}{r^2}] + m\}|\psi_+\rangle = \mathcal{B}^2 e^{-2\mathcal{G}}|\psi_+\rangle
\end{aligned}
\tag{5.281}
$$

Comparing Eq. (5.260) with Eq. (5.257) we find

$$
\begin{aligned}
k &= \frac{1}{2}\nabla^2\mathcal{G} + \frac{1}{2}\mathcal{G}'^2 - \frac{1}{2}\mathcal{G}'\log'\mathcal{D} - \frac{1}{2}\mathcal{G}'C' - \frac{1}{2}\frac{\mathcal{G}'}{r} - \frac{1}{2}\frac{(-C+J-L)'}{r}, \\
g' &= 2\mathcal{G}' - \frac{E_2M_2 + M_1E_1}{\mathcal{D}}(J+L)' = 2\mathcal{G}' - \log'\mathcal{D} = -2F', \\
h' &= \frac{(J-L)'}{2} = -K', \\
n &= -\frac{1}{2}\nabla^2(-C+J-L) - \frac{1}{2}\nabla^2\mathcal{G} - \mathcal{G}'(-C+J-L)' - \mathcal{G}'^2 + \frac{3}{2r}\mathcal{G}' \\
&+ \frac{3}{2r}(-C+J-L)' + \frac{1}{2}\log'\mathcal{D}(\mathcal{G} - C + J - L)', \\
m &= -\frac{1}{2}\nabla^2\mathcal{G} - \frac{1}{4}\mathcal{G}'^2 - \frac{1}{4}(C+J-L)'(-C+J-L)' + \frac{1}{2}\mathcal{G}'\log'\mathcal{D}.
\end{aligned}
$$

Eq. (5.281) and it's derivation is an important part of this work which become by Bin Liu and Crater. It will provide us with a way to derive phase shift equations using work by other authors who developed methods for the nonrelativistic Schrödinger equation. We using the radial form of the coordinate space form of this equation (5.257).

5.3.3 The Radial Eigenvalue Equations

The following are radial eigenvalue equations corresponding to Eq. (5.257) after getting rid of the first derivative terms for singlet states $^1S_0$, $^1P_1$, $^1D_2$(a general singlet ^1J_j), triplet states $^3P_1$(a general let ^3J_j), a general $s=1$, $j=l+1$ ($^3P_0$,$^3S_1$ states), and a general $s=1$, $j=l+1$ ($^3D_1$ state). $^1S_0$, $^1P_1$, $^1D_2$ (a general singlet ^1J_j) $\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)=0$, $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2=-3$, $\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}=-1$.

$$\{-\frac{d^2}{dr^2}+\frac{j(j+1)}{r^2}+\frac{g'^2}{4}+h'^2+\frac{g''}{2}+\frac{g'}{r}-3k-j-g'h'-h''-\frac{2h'}{r}+m\}v=\mathcal{B}^2\epsilon^{-2\mathcal{G}}v \tag{5.282}$$

$^3P_1$(a general triplet ^3J_j) $\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)=-2$, $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2=1$, $\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}$ =1.

$$\{-\frac{d^2}{dr^2}+\frac{j(j+1)}{r^2}+\frac{g'^2}{4}+h'^2+\frac{g''}{2}+k+n+g'h'+h''+m\}v=\mathcal{B}^2\epsilon^{-2\mathcal{G}}v, \tag{5.283}$$

$s=1$, $j=l+1$ ($^3S_1$ states) $\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)=2(j-1)$, $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2=1$, $\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}=\frac{1}{2j+1}$ (diagonal term), and $\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}=\frac{2\sqrt{j(j+1)}}{2j+1}$(off diagonal term).

$$\begin{aligned}
&\{-\frac{d^2}{dr^2}+\frac{j(j-1)}{r^2}+\frac{3g'\sinh^2 h}{r}+6h'\frac{\cosh h\sinh h}{r}-\frac{6\sinh^2 h}{r^2}\\
&-\frac{g'\cosh h\sinh h}{r}-2h'\frac{\sinh^2 h}{r}+2\frac{\cosh h\sinh h}{r^2}+\frac{g'^2}{4}+h'^2+\frac{g''}{2}+\frac{g'}{r}\\
&+k+2(j-1)[\frac{g'}{2r}+\frac{g'\sinh^2 h}{r}-2\frac{\sinh^2 h}{r^2}+2h'\frac{\cosh h\sinh h}{r}]\\
&+\frac{2(j-1)}{2j+1}[2h'\frac{\sinh^2 h}{r}-2\frac{\cosh h\sinh h}{r^2}+\frac{h'}{r}+\frac{g'\cosh h\sinh h}{r}]\\
&+\frac{1}{2j+1}[\frac{3g'\cosh h\sinh h}{r}-\frac{g'\sinh^2 h}{r}-2h'\frac{\cosh h\sinh h}{r}+6h'\frac{\sinh^2 h}{r}\\
&-6\frac{\cosh h\sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}+n+g'h'+h''+\frac{2h'}{r}]+m\}u_+\\
&+\frac{2\sqrt{j(j+1)}}{2j+1}\{\frac{3g'\cosh h\sinh h}{r}-\frac{g'\sinh^2 h}{r}-2h'\frac{\cosh h\sinh h}{r}+6h'\frac{\sinh^2 h}{r}\\
&-6\frac{\cosh h\sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}+n+g'h'+h''+\frac{2h'}{r}\\
&+2(j-1)[\frac{2h'\sinh^2(h)}{r}-\frac{2\cosh(h)\sinh(h)}{r^2}+\frac{h'}{r}+\frac{g'\cosh(h)\sinh(h)}{r}]\}u_-=\mathcal{B}^2\epsilon^{-2\mathcal{G}}u_+,
\end{aligned} \tag{5.284}$$

$s = 1$, $j = l - 1$ (${}^3P_0, {}^3D_1$ states) $\mathbf{L} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) = -2(j+2)$, $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 = 1$, $\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}} \boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}} = -\frac{1}{2j+1}$(diagonal term), and $\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}} \boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}} = \frac{2\sqrt{j(j+1)}}{2j+1}$(off diagonal term).

$$
\begin{aligned}
&\{-\frac{d^2}{dr^2} + \frac{(j+1)(j+2)}{r^2} + \frac{3g' \sinh^2 h}{r} + 6h' \frac{\cosh h \sinh h}{r} - \frac{6\sinh^2 h}{r^2} \\
&-\frac{g' \cosh h \sinh h}{r} - 2h' \frac{\sinh^2 h}{r} + 2\frac{\cosh h \sinh h}{r^2} + \frac{g'^2}{4} + h'^2 + \frac{g''}{2} + \frac{g'}{r} \\
&+k + 2(j+2)[\frac{g'}{2r} + \frac{g' \sinh^2 h}{r} - 2\frac{\sinh^2 h}{r^2} + 2h' \frac{\cosh h \sinh h}{r}] \\
&+\frac{2(j-1)}{2j+1}[2h' \frac{\sinh^2 h}{r} - 2\frac{\cosh h \sinh h}{r^2} + \frac{h'}{r} + \frac{g' \cosh h \sinh h}{r}] \\
&-\frac{1}{2j+1}[\frac{3g' \cosh h \sinh h}{r} - \frac{g' \sinh^2 h}{r} - 2h' \frac{\cosh h \sinh h}{r} + 6h' \frac{\sinh^2 h}{r} \\
&-6\frac{\cosh h \sinh h}{r^2} + 2\frac{\sinh^2 h}{r^2} + n + g'h' + h'' + \frac{2h'}{r}] + m\}u_- \\
&+\frac{2\sqrt{j(j+1)}}{2j+1}\{\frac{3g' \cosh h \sinh h}{r} - \frac{g' \sinh^2 h}{r} - 2h' \frac{\cosh h \sinh h}{r} \\
&+6h' \frac{\sinh^2 h}{r} - 6\frac{\cosh h \sinh h}{r^2} + 2\frac{\sinh^2 h}{r^2} + n + g'h' + h'' + \frac{2h'}{r} \\
&-2(j+2)[\frac{2h' \sinh^2(h)}{r} - \frac{2\cosh(h)\sinh(h)}{r^2} + \frac{h'}{r} + \frac{g' \cosh(h)\sinh(h)}{r}]\}u_+ = \mathcal{B}^2 \epsilon^{-2\mathcal{G}} u_-,
\end{aligned}
\tag{5.285}
$$

Substituting for g', h', m, n, k we obtain the radial equations and potentials Φ given in the text.

$s = 0, j = l$:

$$
A\left[\left\{-\frac{d^2}{dr^2} + \frac{j(j+1)}{r^2} - B'\left(\frac{d}{dr} - \frac{1}{r}\right) + \frac{1}{2}\nabla^2 F - \frac{1}{4}K'F'\right\}u_{j0j} + N\frac{w(m_1 - m_2)}{\mathcal{D}(1)}U'\frac{\sqrt{j(j+1)}}{r}u_{j1j}\right] = \mathcal{B}^2(-3)u_{j0j}. \tag{5.286}
$$

$s = 1, j = l$:

$$
E\left[\left\{-\frac{d^2}{dr^2} + \frac{j(j+1)}{r^2} + (S-Q)'\frac{d}{dr} + 2\frac{Q'}{r} - \nabla^2 Q + S'Q'\right\}u_{j1j} + T\frac{(\epsilon_1 - \epsilon_2)(m_1 + m_2)}{\mathcal{D}(1)}U'\frac{\sqrt{j(j+1)}}{r}u_{j0j}\right] = \mathcal{B}^2(1)u_{j1j}. \tag{5.287}
$$

$s = 0, j = l + 1$:

$$\left\{\left[E + \frac{j}{2j+1}(O - E)\right]\left(-\frac{d^2}{dr^2} + \frac{j(j-1)}{r^2}\right) + \right.$$
$$\left[E(S-Q)' + \frac{j}{2j+1}(E(Q-S)' + O(Q+R)')\right] \times$$
$$\frac{d}{dr} + \frac{1}{2j+1}[Q'[E(-j^2+j+2) + O(j^2-2j)] -$$
$$ES'j(j+1) - OR'j^2]\frac{1}{r} + \frac{1}{2j+1}[\nabla^2 Q(-(j+1)E +$$
$$\left. Oj) + Q'(ES'(j+1) - OR'j)]\right\} u_- + \frac{\sqrt{j(j+1)}}{2j+1} \times$$
$$\left\{(E-O)\left(-\frac{d^2}{dr^2} - (2j+1)\frac{d}{dr} - \frac{j^2-1}{r^2}\right) + \right.$$
$$[E(S-Q)' - O(R+Q)']\frac{d}{dr} + (E[(2-j)Q + S(j+1)]' +$$
$$O[(2-j)Q - (j+1)R]')\frac{1}{r} - (E+O)\nabla^2 Q +$$
$$\left. Q'(OR' + ES')\right\} u_+ = \mathcal{B}^2(1)u_-. \tag{5.288}$$

$s = 1, j = l - 1$:

$$\left\{\left[E + \frac{j+1}{2j+1}(O - E)\right]\left(-\frac{d^2}{dr^2} + \frac{(j+1)(j+2)}{r^2}\right) + \right.$$
$$\left[E(S-Q)' + \frac{j+1}{2j+1}(E(Q-S)' + O(Q+R)')\right] \times$$
$$\frac{d}{dr} + \frac{1}{2j+1}[Q'[E(j^2+3j) - O(j^2+4j+3)] +$$
$$ES'j(j+1) + OR'(j+1)^2]\frac{1}{r} + \frac{1}{2j+1}[\nabla^2 Q(-jE +$$
$$\left. O(j+1)) + Q'(ES'j - OR'(j+1))]\right\} u_+ + \frac{\sqrt{j(j+1)}}{2j+1} \times$$
$$\left\{(E-O)\left(-\frac{d^2}{dr^2} + (2j+1)\frac{d}{dr} - \frac{j(j+2)}{r^2}\right) + \right.$$
$$[E(S-Q)' - O(R+Q)']\frac{d}{dr} + (E[(3+j)Q - Sj]' +$$

$$O[(3+j)Q+jR]')\frac{1}{r}-(E+O)\nabla^2 Q+$$
$$\left.Q'(OR'+ES')\right\}u_- = \mathcal{B}^2(1)u_+. \tag{5.289}$$

where we have

$$\begin{aligned}
\mathcal{B}^2(-3) &= \mathcal{B}^2(1)=\mathcal{B}^2=E_1^2-M_1^2=E_2^2-M_2^2= \\
&\quad b^2(w)+(\epsilon_1^2+\epsilon_2^2)sh^2(J)+2\epsilon_1\epsilon_2 sh(J)ch(J)- \\
&\quad (m_1^2+m_2^2)sh^2(L)+2m_1m_2 sh(L)ch(L), \\
\mathcal{D}^2(-3) &= \mathcal{D}^2(-1)=\mathcal{D}^2(1)=\mathcal{D}^2=E_1M_2+E_2M_1 \\
&= (\epsilon_1 m_2+\epsilon_2 m_1)ch(J+L)+(\epsilon_1 m_1+\epsilon_2 m_1)sh(J+L),
\end{aligned}$$

$$\begin{aligned}
K &= 3H+C+J-L-\mathcal{G}-5\mathcal{F}-3I-3Y+2\ln(\mathcal{D}(1)), \\
A &= e^{(2\mathcal{G}+4\mathcal{F}-2H)}\frac{\mathcal{D}(3)}{\mathcal{D}(1)}, \\
B &= 2\mathcal{G}+4\mathcal{F}-2H-\ln(\mathcal{D}(1))-J+L+3I+3Y, \\
F &= H-C+J-L-3\mathcal{G}-3\mathcal{F}-3I-3Y, \\
N &= e^{(2H-2\mathcal{F}-I-Y+J-L)}, \\
U &= J+L+I-Y, \\
O &= e^{(2\mathcal{G}-2H-4\mathcal{F})}\frac{\mathcal{D}(-1)}{\mathcal{D}(-3)}, \\
E &= e^{(2\mathcal{G}+2H)}\frac{\mathcal{D}(-1)}{\mathcal{D}(1)}, \\
Q &= \frac{1}{2}(H-C+J-L+\mathcal{G}+I+Y), \\
R &= \frac{1}{2}(3H+C+J-L-5\mathcal{G}+7\mathcal{F}+I+Y)+\ln(\mathcal{D}(-3)), \\
S &= \frac{1}{2}(3H+C+J-L+3\mathcal{G}-\mathcal{F}+I+Y)+\ln(\mathcal{D}(1)), \\
T &= e^{(-2H+2\mathcal{F}+L-J+I+Y)}\frac{\mathcal{D}(-1)}{\mathcal{D}(1)}.
\end{aligned}$$

5.4 Numerical Procedures

The standard form of Eq. (5.257) for singlet and coupled channels is

$$A(r)f''+B(r)f'+C(r)f=0, \tag{5.290}$$

where $A(r), B(r), C(r)$ are matrix functions. For a *normal form* we multiply with A^{-1} viz.

$$f''+A^{-1}(r)B(r)f'+A^{-1}(r)C(r)f=0. \tag{5.291}$$

Such equation have been the subject for numerical analysis in the past. In particular nuclear physics problems this was solved by an iteration scheme developed by Raynal, see ECIS [471]. The singlet channel can most easily be solved by closed form factorization, but in practical terms one does not gain much in performance when compared with the interactive scheme. To solve above equation, we can use different numerical methods.

We used next notations, $x_0 < x_1 < ...$, $f(x_i) = f_i, f'(x_i) = f'_i, f''(x_i) = f''_i, ...$ Starting with some arbitrary values x_n, $0 \leq n \leq N$, then with $f_0 = 0$, we first need to find solutions to the homogeneous equation, the above yields recursively

$$f''_1 + D_1 f_1 = 0, \tag{5.292}$$

where $D_1 = A^{-1}(r)C(r)$, and $D_2 = -A^{-1}(r)B(r)$ are matrix functions. With the solution f'_1

$$f''_2 + D_1 f_2 = D_2 f'_1 := W, \tag{5.293}$$

where W is the inhomogeneous term. The existence of a regular solution f_2 of the above differential equations yields

$$\begin{aligned}
&f''_3 + D_1 f_3 = D_2 f'_2, \\
&f''_4 + D_1 f_4 = D_2 f'_3, \\
&... \\
&f''_{n-1} + D_1 f_{n-1} = D_2 f'_{n-2}, \\
&f''_n + D_1 f_n = D_2 f'_{n-1}.
\end{aligned}$$

It is clear that these equations for f''_n are formally similar to those which determine the first derivative f'_{n-1} using any numerical method, after this we can solve the equations numerical very well. One such method is a Lagrange interpolation

$$f'_{n-1} = \sum_{k=0}^{N} \ell'_k f_k + R'_N, \tag{5.294}$$

where

$$\begin{aligned}
&R'_N(r) = \frac{f^{(N+1)}}{(N+1)!}(\xi)\pi'_N(r) + \frac{\pi_N(x)}{(N+1)!}\frac{d}{dx}f^{(N+1)}(\xi), \\
&\ell' = \sum_{j=0, j\neq k}^{N} \frac{\pi_N(r)}{(r-r_k)(r-r_j)\pi'_N(r_k)}, \\
&\xi = \xi(r),\ (r_0 < \xi < r_N), \\
&\pi_N(r) = (r-r_0)(r-r_1)...(r-r_N),
\end{aligned}$$

and $\pi'_n(r)$ is its derivative. For four points Eq. (5.294) is

$$\begin{aligned}
f'_p = f(r_0 + ph) &= \frac{1}{h}\{-\frac{3p^2 - 6p + 2}{6}f_{-1} + \frac{3p^2 - 4p - 1}{2}f_0 \\
&- \frac{3p^2 - 2p - 2}{2}f_1 + \frac{3p^2 - 1}{6}f_2 + R'_3\}.
\end{aligned} \tag{5.295}$$

Similarly, in Eq. (5.293) we apply Milne's method,

$$f'_{n+1} = f'_{n-3} + \frac{4h}{3}(2f''_{n-2} - f''_{n-1} + 2f''_n + O(h^5)), \tag{5.296}$$

$$f'_{n+1} = f'_{n-1} + \frac{h}{3}(f''_{n-1} + 4f''_n + f''_{n+1} + O(h^5)), \tag{5.297}$$

and any other numerical differentiation method [529].

Another powerful method is the Numerov algorithm [278], which we use only after elimination of the first derivative terms. In this case with the ansatz

$$f = gh,$$

in Eq. (5.291), we find

$$g''h + gh'' + 2g'h' + A^{-1}B(g'h + gh') + A^{-1}Cgh = 0. \tag{5.298}$$

To eliminate the first derivative terms h', we solve the differential equation

$$2g'h' + A^{-1}Bgh' = (2g' + A^{-1}Bg)h' = 0, \tag{5.299}$$

for any $h' \neq 0$ and boundary condition

$$\lim_{r\to\infty} g(r) = 1 = \text{unit matrix}.$$

The solution of this first order differential equation

$$g' = -\frac{A^{-1}B}{2}g, \tag{5.300}$$

is readily found for singlet channel equation by using

$$g(r) = g(\infty)\exp\left[-\frac{1}{2}\int_\infty^r A^{-1}(x)B(x)\,dx\right], \tag{5.301}$$

or more generally numerical integration of the differential equation starting with the boundary conditions

$$\lim_{r\to\infty} g(r) = 1.$$

Therewith emerges a modified differential equation without first derivatives for h when the solution for g is inserted and new potentials are defined. The new equation for

$$h'' = U(r)h, \tag{5.302}$$

is readily solved with the Numerov algorithm [278], and matched asymptotically to known free solutions and thus yields the S-matrix.

5.4.1 Numerov Algorithm

The solution of radial Schrödinger equations is certainly not new and generally deserves no mention. Here, we dwell upon the details since we found the specified elements to have a *normal form* of related problems in other fields of physics and engineering which were tested with parallel computing facilities. The Numerov algorithm has been widely used for singlet and coupled channels Schrödinger equations since it gives sufficient numerical accuracy with minimal operations [472]. The standard form of linear homogeneous or inhomogeneous Schrödinger equations which we have to solve is

$$f_i''(r) = \sum_j V_{ij}(r) f_j(r) + W_i(r), \tag{5.303}$$

where $W_i(r) = 0$ for homogeneous equations.

In case of Schrödinger equations which include a first derivative

$$f_i''(r) = \sum V_{ij}^0(0) f_j(r) + \sum V_{ij}^{(1)}(r) f_j'(r), \tag{5.304}$$

we identify

$$W_i(r) := \sum V_{ij}^{(1)}(r) f_j'(r), \tag{5.305}$$

in which the derivative solution $f_j'(r)$ is obtained iteratively. To determine the scattering matrix S, we need to compute only the regular solution

$$f_i(r) \rightarrow f_i(\alpha, r),$$

with the iteration counter $\alpha = -1, 0, 1, ..., N$.

$$f_i''(-1, r) = \sum V_{ij}^{(0)}(r) f_i(-1, r), \tag{5.306}$$

with

$$f_i(-1, r)|_{r\rightarrow 0} = 0.$$

$$f_i(r) \rightarrow f_i(\alpha, r),$$

with the iteration counter $\alpha = 0, 1, ..., N$. We start the iteration with

$$\begin{aligned} f_i''(0, r) &= \sum V_{ij}^{(0)}(r) f_i(0, r), \\ W_i(0, r) &= 0, \end{aligned}$$

for a regular solution

$$f_i(0, r)|_{r\rightarrow 0} = 0.$$

Thus

$$f_i''(\alpha, r) = \sum V_{ij}^{(0)}(r) f_i(\alpha, r) + W_i(\alpha, r), \tag{5.307}$$

with

$$W_i(\alpha, r) = \sum V_{ij}^{(1)}(r) f_j'(\alpha - 1, r),$$

and the derivatives are computed by a three or five point Lagrange interpolation formula. The potential $V_{ij}^{(0)}(r)$ and $V_{ij}^{(1)}(r)$ are properly regularized near the origin to guarantee a stable numerical solution and convergence of the iterative scheme. This is well satisfied for our NN optical model. For singlet channels the algorithm is

$$f_{n+1} = 2f_n - f_{n-1} + \frac{h^2}{12}\left(u_{n+1} + 10u_n + u_{n-1}\right), \tag{5.308}$$

or

$$\begin{aligned}\left(1 - \frac{h^2}{12}V_{n+1}\right) f_{n+1} &= \left(2 + \frac{10h^2}{12}V_n\right) f_n - \left(1 - \frac{h^2}{12}V_{n-1}\right) f_{n-1} \\ &+ \frac{h^2}{12}\left(W_{n+1} + 10\, W_n + W_{n-1}\right). \end{aligned} \tag{5.309}$$

These expressions generalize for coupled channels using standard vector and matrix algebra.

A significant reduction of operations is found by using the substitution

$$\xi_n = \left(1 - \frac{h^2}{12}V_n\right) f_n, \tag{5.310}$$

in Eq. (5.309). It gives

$$\xi_{n+1} = 2\xi_n - \xi_{n-1} + \mathcal{U}_n, \tag{5.311}$$

and the inhomogeneous equation

$$\xi_{n+1} = 2\xi_n - \xi_{n-1} + \mathcal{U}_n + \frac{h^2}{12}\left(W_{n+1} + 10W_n + W_{n-1}\right), \tag{5.312}$$

with

$$\mathcal{U}_n = \frac{h^2 V_n}{1 - \frac{h^2}{12}V_n}\xi_n.$$

Back-transformations from $\xi_i \to f_i$ use either of the two possibilities

$$f_i = \xi_i + \frac{1}{12}\mathcal{U}_i, \text{ or } \quad f_i = \frac{\xi_{i+1} + 10\xi_i + \xi_{i-1}}{12}. \tag{5.313}$$

5.4.2 Calculation of Phase Shift

We evaluate the equation Eq. (5.257) numerically for uncoupled and coupled channels using a Numerov method (Sec. 5.4). The physical solutions are matched asymptotically, $\lim_{r\to\infty}$ to Riccati-Hankel functions

$$u_\alpha^+(r, k) \sim \frac{1}{2i}\left[-h_\alpha^-(rk) + h_\alpha^+(rk) S_\alpha(k)\right]. \tag{5.314}$$

The irregular outgoing wave Jost solutions are

$$\mathcal{J}_\alpha^+(r,k) \sim h_\alpha^+(rk), \tag{5.315}$$

and the regular solutions are asymptotically

$$\lim_{r\to\infty} \psi_\ell^{(\pm,0)}(r,k,q)\mathcal{N}_\ell = j_\ell(rq) + h_\ell^{(\pm,0)}(rk)\frac{q}{k}T_\ell^{(\pm,0)}(k^2,k,q), \tag{5.316}$$

to determine the half off-shell t-matrix $T_\ell^{(\pm,0)}(k^2,k,q)$ and the normalization $\mathcal{N}_\ell$. Spherical Riccati functions are symbolized by $j_\ell(x)$, $h_\ell^\pm(x)$ and $h_\ell^0(x) = n_\ell(x)$.

The on-shell t-matrix gives the S-matrix by the relation

$$S(k) = 1 + 2i\,T^{(+)}(k^2,k,k). \tag{5.317}$$

To solve for coupled channels $^3SD_1$, $^3PF_2$, *etc.*, two linear independent regular solutions are calculated [256, 278].

The VPI/GWU solutions [11] are parameterizations of the elastic channel NN S-matrix. They consider

$$S_1 = (1 + iK_4)(1 - iK_4)^{-1}, \tag{5.318}$$

which inverts to give

$$K_4 = i(1 - S_1)(1 + S_1)^{-1} = Re\,K_4 + iIm\,K_4. \tag{5.319}$$

The real part of this K-matrix is related to a unitary S-matrix (S_6) and therewith phase shifts $\delta^\pm$ and ϵ are defined by

$$S_6 = \frac{(1 + i\,Re\,K_4)}{(1 - i\,Re\,K_4)} = \left\{ \begin{array}{cc} \cos 2\varepsilon \exp 2i\delta^- & i\sin 2\varepsilon \exp i(\delta^- + \delta^+) \\ i\sin 2\varepsilon \exp i(\delta^- + \delta^+) & \cos 2\varepsilon \exp 2i\delta^+ \end{array} \right\}. \tag{5.320}$$

The absorption parameters $\rho^\pm$ and μ relate to the imaginary part of that K-matrix by

$$Im\,K_4 = \left\{ \begin{array}{cc} \tan^2\rho^- & \tan\rho^- \tan\rho^+ \cos\mu \\ \tan\rho^- \tan\rho^+ \cos\mu & \tan^2\rho^+ \end{array} \right\}. \tag{5.321}$$

These relations simplify to $K = \tan\delta + i\tan^2\rho$ for uncoupled channels.

Chapter 6

Application to NN Interactions

6.1 Boson Exchange Models

All existing potential models are to some degree based on meson exchange. They all include the one pion exchange contribution which essentially determines the long range part of the interaction. As such, the meson exchange model has been very successful in describing the empirical features of the NN force. Both the tensor and spin orbit forces are easily accounted for in the meson exchange model [393, 394].

The starting point for meson exchange theory is a phenomenological Lagrangian which describes the interaction between baryons and meson fields. For the NN interaction at low and intermediate energies the three relevant meson fields show sysmmeties of scalar (s),(σ, δ), pseudoscalar (ps), (π, η) and vector (v), (ρ, ω) fields.

The interaction Lagrangian that couples these fields to the nucleons in lowest order are

$$\mathcal{L}_s = g_s \overline{\psi}\psi\phi^{(s)}, \tag{6.1}$$

$$\mathcal{L}_{ps} = -ig_{ps}\overline{\psi}\gamma^5\psi\phi^{(ps)}, \quad or \quad \mathcal{L}_{pv} = -\frac{f_{ps}}{m_{ps}}\overline{\psi}i\gamma^5\psi\partial_\mu\phi^{(ps)}, \tag{6.2}$$

$$\mathcal{L}_v = -ig_v\overline{\psi}\gamma^\mu\psi\partial\phi^{(v)}_\mu - \frac{f_v}{4M}\overline{\psi}i\sigma^{\mu\nu}\left(\partial_\mu\phi^{(v)}_\nu - \partial_\nu\phi^{(v)}_\mu\right)\psi, \tag{6.3}$$

where M is the nucleon mass and ψ is the nucleon Dirac field with its adjoint defined by $\overline{\psi} = \psi\gamma^0$, while $\phi^{(s)}$, $\phi^{(ps)}$ and $\phi^{(v)}$ are the scalar, pseudoscalar and vector meson fields respectively. Correspondingly, g_s, g_{ps} and g_v are the coupling constants. There is also a tensor coupling f_v between nucleons and the vector mesons. Actually, there are two alternative ways to couple pseudoscalar fields to nucleons. The one given above is the pseudoscalar coupling *ps*. The so called gradient coupling or pseudovector *pv* coupling is an effective coupling derived from chiral symmetry [394, 560]. The *ps* and the *pv* couplings are equivalent for on-shell nucleons when the coupling constants are related by $f_{ps} = g_{ps}(m_{ps}/2M)$. The coupling constants are constrained by NN scattering data. Empirical information also constrains the ratio of g_v to f_v.

From the Lagrangian, the Hamiltonian can be derived, and using time dependent perturbation theory the NN interaction can be visualized in terms of Feynman diagrams involving meson baryon coupling. This field theory was first developed for quantum electrodynamics. When electromagnetic interactions are involved, it is necessary to apply perturbation theory, which appears quite reasonable for a coupling constant $\alpha \approx 1/137$. Meson baryon coupling constants originate from the strong interaction and are generally large of the order 1-10, thus, the perturbation expansion becomes increasingly divergent at shorter distances. For intermediate and long ranges the diagrammatic expansion is assumed to converge. However, due to the quark structure and finite size of the nucleons the meson exchange model for the NN interaction generates little confidence at short range. This problem has been never overcome and a saving argument is the strongly repulsive core which keeps the nucleons apart and hides the genuine quark gluon processes between the nucleons. Traditionally, the short range part of the NN interaction is treated phenomenologically by introducing vertex form factors, which are effectively to the extended by adjusting several parameters in fits to NN data. The form factors suppress the meson exchange at small distances. Using this short range regularization, the meson exchange theory yield quantative results in the framework of time ordered perturbation theory [401] for NN energies $T_{Lab} < 300$MeV. Above 1 GeV and above, this meson exchange mechanism becomes relatively small in comparison with the kinetic energy and the hard core domain starts to display violently the quark gluon dynamics. From the NN phase shift for $T_{Lab} > 1$GeV we get the impression that this violent core dynamics can be regularized with a few complex boundary conditions whose energy dependence reflects our ignorance about low energy QCD dynamics. In summary, the lowest order contribution to the NN scattering establishes one boson exchange, which comprise the exchange of the six nonstrange mesons

$$V^{OBE} = \sum_{\alpha=\pi,\eta,\rho,\omega,\delta,\sigma} V_\alpha^{OBE}. \tag{6.4}$$

The heavy vector mesons ρ and ω are important at short distances where the NN interaction becomes repulsive, a feature which is enhanced by relativistic kinematics. Nevertheless, the intermediate distance attraction is generated by σ meson which accounts for correlated pairs of pions with total spin $J = 0$ and isospin $T = 0$ [327]. In the one boson approximation the contribution from these processes is effectively included by exchange of a fictitious scalar meson usually denoted by *sigma* [394]. The long range part of the NN interaction is well established both theoretically and experimentally to arise from Coulomb and one pion exchange.

Most of the uncertainty in nuclear processes comes from the short distance interactions ($r < 0.75 fm$) between two or more nucleons. Even when one is interested only in low energy phenomena, the short distance contributions are important. In perturbation theory, for instance, the influence of short distance physics on low energy observable appears in the existence of ultraviolet divergent integrals, that is, in the dominance of high momentum modes over the small momentum ones. Sensitivity of large distance observables on short distance physics is not an unusual situation in physics, it is in fact pervasive in many fields. One way of dealing with it, as we shall do, is to model the short distance physics and solve the problem within a semimicroscopic approach. In the case of nuclear systems this would lead either to a calculation of nuclear processes directly from QCD (which is currently impossible and would be, even if possible, a highly inefficient way of approaching the problem) or to the use of meson exchange/quark/skyrmion/... models.

6.2 Instant Form Potentials

We first consider how to model $\mathcal{G}$ and L, corresponding to vector and scalar interactions.

We may rewrite the *external potential form* of the covariant two body Dirac equations for two relativistic spin one half particles interacting through scalar and vector potentials as, see Eqs. (5.193) without the pseudoscalar interaction,

$$\mathcal{S}_1|\psi\rangle \equiv \gamma_{51}(\gamma_1\cdot(p_1-A_1)+m_1+S_1)|\psi\rangle=0, \tag{6.5a}$$

$$\mathcal{S}_2|\psi\rangle \equiv \gamma_{52}(\gamma_2\cdot(p_2-A_2)+m_2+S_2)|\psi\rangle=0. \tag{6.5b}$$

A_i^μ and S_i introduce the interactions that the i^{th} particle experience due to the presence of the other particle, both are spin dependent [546, 152, 156, 153, 154, 155, 157]. In order to identify these potentials we use Eqs. (5.193), and (5.194,5.195). Then we find that the momentum dependent vector potentials A_i^μ are given in terms of three invariant functions [156, 157] G, E_1, E_2

$$A_1^\mu = ((\epsilon_1-E_1)-i\frac{G}{2}\gamma_2\cdot\frac{\partial E_1}{E_2}\gamma_2\cdot\hat{P})\hat{P}+(1-G)p^\mu-\frac{i}{2}\partial G\cdot\gamma_{2\perp}\gamma_{2\perp}^\mu, \tag{6.6}$$

$$A_2^\mu = ((\epsilon_2-E_2)-i\frac{G}{2}\gamma_1\cdot\frac{\partial E_2}{E1}\gamma_1\cdot\hat{P})\hat{P}+(1-G)p^\mu-\frac{i}{2}\partial G\cdot\gamma_{1\perp}\gamma_{1\perp}^\mu, \tag{6.7}$$

where

$$G=\exp(\mathcal{G}), \tag{6.8}$$

(with $\hat{P}^2=-1$, where $\hat{P}\equiv P/w$) while the scalar potentials S_i are given in terms of three invariant functions [152, 156, 157] G, M_1, M_2

$$S_1 = M_1-m_1-\frac{i}{2}G\gamma_2\cdot\frac{\partial M_1}{M_2}, \tag{6.9}$$

$$S_2 = M_2 - m_2 - \frac{i}{2} G\gamma_1 \cdot \frac{\partial M_2}{M_1}. \tag{6.10}$$

In QCD, the scalar potentials S_i are semiphenomenological long range interactions. The vector potentials A_i^μ are semiphenomenological in the long range while in the short range are closely related to perturbative quantum field theory. Of course this rewrite does not change the fact that $\mathcal{S}_1$ and $\mathcal{S}_2$ still satisfy the compatibility condition

$$[\mathcal{S}_1, \mathcal{S}_2]|\psi >= 0. \tag{6.11}$$

In order that Eq. (6.5a) and Eq. (6.5b) satisfy Eq. (6.11), it is necessary that the invariant functions G, E_1, E_2, M_1 and M_2 depend on the relative separation,

$$x = x_1 - x_2,$$

only through the spacelike coordinate four vector

$$x_\perp^\mu = x^\mu + \hat{P}^\mu(\hat{P} \cdot x),$$

perpendicular to the total four momentum P. For QCD and QED applications, G, E_1, E_2 are functions [153, 157] of an invariant $\mathcal{A}$. The explicit forms for functions E_1, E_2, G are

$$E_1 = G(\epsilon_1 - \mathcal{A}), \tag{6.12}$$

$$E_2 = G(\epsilon_2 - \mathcal{A}), \tag{6.13}$$

and

$$G^2 = \frac{1}{(1 - \frac{2\mathcal{A}}{w})}. \tag{6.14}$$

The function $\mathcal{A}(r)$ is responsible for the covariant electromagnetic like A_i^μ. Even though the dependencies of E_1, E_2, G on $\mathcal{A}$ is not unique, they are constrained by the requirement that they yield an effective Hamiltonian with the correct nonrelativistic and semirelativistic limits (classical and quantum mechanical [162, 336]). For QCD and QED application , M_1 and M_2 are functions of two invariant functions [152, 157], $\mathcal{A}(r)$ and $S(r)$

$$M_1^2(\mathcal{A}, S) = m_1^2 + G^2(2m_w S + S^2), \tag{6.15a}$$

$$M_2^2(\mathcal{A}, S) = m_2^2 + G^2(2m_w S + S^2). \tag{6.15b}$$

The invariant function $S(r)$ is responsible for the scalar potential since $S_i = 0$, if $S(r) = 0$, while $\mathcal{A}(r)$ contributes to the S_i (if $S(r) \neq 0$) as well as to the vector potential A_i^μ. So, finally, the five invariant functions G, E_1, E_2, M_1 and M_2 (or $\mathcal{G} = -J, L$) depend on two independent invariant potential functions S and $\mathcal{A}$. (Compare also the spin independent portions to Eqs. (5.30,5.32) through calculation of $E_i^2 - M_i^2 - b^2$.)

Expressing G, E_1, E_2, M_1 and M_2 in terms of S and $\mathcal{A}$ is important for semiphenomenological and other applications that emphasize the relationship of the interactions to effective external potentials of the two associated one body problems. However, the five invariants G, E_1, E_2, M_1 and M_2 can also be expressed in the hyperbolic representation [158] in terms of the three invariants L, J and $\mathcal{G}$ (see Eqs. (5.194), (5.195) and (5.196)). L, J and $\mathcal{G}$ generate scalar, timelike vector and spacelike vector interactions respectively and enter into our Dirac equations via the sum $\Delta_L + \Delta_J + \Delta_{\mathcal{G}}$ where Eqs. (5.102,5.103,5.104) define $\Delta_L, \Delta_J, \Delta_{\mathcal{G}}$.

We may use Eq. (5.93) to relate the matrix potentials Δ to a given field theoretical or semiphenomenological Feynman amplitude. As mentioned earlier, a matrix amplitude proportional to $\gamma_1^\mu \cdot \gamma_{2\mu}$ corresponding to an electromagnetic like interaction would require [160] $J = -\mathcal{G}$. Matrix amplitude proportional to either $I_1 I_2$ or $\gamma_1 \cdot \hat{P} \gamma_2 \cdot \hat{P}$ would correspond to semiphenomenological scalar or timelike vector interactions. The two body Dirac equations in the hyperbolic form of Eq. (5.93) give a simple version [158] for the norm of the sixteen component Dirac spinor. The two body Dirac equations in *external potential* form, Eq. (6.5a) and Eq. (6.5b), (or more generally (5.193) are simpler to reduce to the Schrödinger like form and are useful for numerical calculations (see Sazdjian [421] for a related reduction). We describe the parameterization of the pseudoscalar interaction C below in Eq. (6.17).

6.2.1 Modeling the Invariant Interaction Functions

Bin Liu and Crater had used the following scalar interactions in two body Dirac equations (see Eqs. (6.9, 6.10), 6.15b)):

$$S = -g_\sigma^2 \frac{e^{-m_\sigma r}}{r} - (\tau_1 \cdot \tau_2) g_{a_0}^2 \frac{e^{-m_{a_0} r}}{r} - g_{f_0}^2 \frac{e^{-m_{f_0} r}}{r}, \tag{6.16}$$

where g_σ^2, $g_{a_0}^2$, $g_{f_0}^2$ are coupling constants for the σ, a_0 and f_0 mesons and m_σ, m_{a_0} and m_{f_0} the corresponding masses. $(\tau_1 \cdot \tau_2)$ is 1 or -3 for isospin triplet or singlet states.

Pseudoscalar interactions are assumed to enter into two body Dirac equations in the form (see Eq. (5.257))

$$C = (\tau_1 \cdot \tau_2) \frac{g_\pi^2}{w} \frac{e^{-m_\pi r}}{r} + \frac{g_\eta^2}{w} \frac{e^{-m_\eta r}}{r} + \frac{g_{\eta'}^2}{w} \frac{e^{-m_{\eta'} r}}{r}, \tag{6.17}$$

where $w = \epsilon_1 + \epsilon_2$ is the total energy of the two nucleon system. g_π^2, g_η^2, $g_{\eta'}^2$ are coupling constants for mesons π, η and η' respectively and m_π, m_η and $m_{\eta'}$ the corresponding masses. This form for C yields the correct limit at low energy.

Vector interactions enter into two body Dirac equations in the form (see Eqs. (6.12) and (6.14))

$$A = (\tau_1 \cdot \tau_2) g_\rho^2 \frac{e^{-m_\rho r}}{r} + g_w^2 \frac{e^{-m_w r}}{r} + g_\phi^2 \frac{e^{-m_\phi r}}{r}, \tag{6.18}$$

where g_ρ^2, g_ω^2, g_ϕ^2 are coupling constants for mesons ρ, ω and ϕ and m_ρ, m_ω and m_ϕ are the corresponding masses.

In the constraint equations, A and S are relativistic invariant functions of the invariant separation $r = \sqrt{x_\perp^2}$ (see below for the distinction between $\mathcal{A}$ and A). Since it is possible that A and S, as identified from the nonrelativistic limit, can take on large positive and negative values, it is necessary to modify G, E_1, E_2, M_1 and M_2 so that the interaction functions remain real when A become large and repulsive [167]. These modifications are not unique but must maintain correct limits.

Model 1 Crater and van Alstine [167] used the Adler-Piran potential in order to compute meson spectra

$$\mathcal{A} = exp(-\beta r)[V_{AP} - \frac{c_4}{r}] + \frac{c_4}{r} + \frac{e_1 e_2}{r},$$
$$S = V_{AP} + \frac{e_1 e_2}{r} - \mathcal{A} = (V_{AP} - \frac{c_4}{r})(1 - exp(-\beta r)). \tag{6.19}$$

For $E_i = G(\epsilon_i - \mathcal{A})$ to be real we need only ensure that G be real which requires that $\mathcal{A} \leq w/2$. This restriction on $\mathcal{A}$ is enough to ensure that

$$M_i = G\sqrt{m_i^2(1 - 2\mathcal{A}/w) + 2m_w S + S^2},$$

be real as well (so long as $S \geq 0$). (As we shall show below in our discussion on the static limit, the case of $S < 0$ does not require any further restrictions.) In order that $\mathcal{A}$ satisfy this inequality, we must modify it and S so that

$$S(r) + \mathcal{A}(r) = V_{AP}(r) + e_1 e_2/r \equiv A + \bar{S},$$

with A and $\bar{S}$ given by the right hand sides of Eq. (6.19) respectively. Then we reidentify $\mathcal{A}$ and S such that

$$\mathcal{A} = A, \;\; A \leq 0, \tag{6.20}$$
$$\mathcal{A} = \frac{AA_0}{\sqrt{A^2 + A_0^2}}, \quad A > 0 \tag{6.21}$$
$$S = \bar{S} + A - \mathcal{A}, \tag{6.22}$$

where $A_0 = w/2$. This parameterization gives $\mathcal{A}$ and S that are continuous through their first derivatives. We next consider problems that may arise in the limit that one of the masses becomes very large. We must modify the M_i so that they have the correct static limits (when say $m_2 \to \infty$).

In the spinless case, the potential forms Eq. (6.19) are determined by requiring the desired nonrelativistic limit and compatibility of the covariant generalized mass shell constraints for the constituent particles,

$$\mathcal{H}_i = (p_i - A_i)^2 + (m_i + S_i)^2 \approx 0,$$

in which constituent vector potentials are given by

$$A_i^\mu = \alpha_i p_1^\mu + \beta_i p_2^\mu,$$

introduced by minimal substitutions. The classical compatibility condition on the two constraints,

$$\{\mathcal{H}_1, \mathcal{H}_2\} \approx 0,$$

implies that only two of the four scalars α_i, β_i and one of the two scalars S_i are independent. These three independent system scalars, $\mathcal{A}, \mathcal{V}$ and S, must be functions of $x_\perp$ and are related to constituent potentials A_i^μ, S_i by

$$M_1^2 = (m_1 + S_1(S, \mathcal{A}))^2 = m_1^2 + G(2m_w S + S^2), \tag{6.23}$$

$$M_2^2 = (m_2 + S_2(S, \mathcal{A}))^2 = m_2^2 + G(2m_w S + S^2),\ S > 0. \tag{6.24}$$

At first sight, Eq. (6.23) does appear to give $M_1 \to m_1 + S$. (Note $G \to 1$ in this heavy mass limit). However, this limit is only true if $m_1 + S \geq 0$.

We now take advantage of the hyperbolic parameterization given in Eqs. (5.194). Let us assume that L is a monotonic function of S for $S \geq 0$ given by

$$\exp(L) = \frac{M_{10} + M_{20}}{m_1 + m_2} \equiv \exp(L_0(S)),\ S \geq 0, \tag{6.25}$$

in which M_{i0} are defined as the forms of M_i given in Eqs. (6.23). We choose this form since only in the region where S is large and negative do we expect problems. We desire a form for M_i which goes over to $m_i + S$ when S becomes large and negative and one of the constituent rest masses is large. At the same time we require an expression that has the correct weak potential behavior. The original forms in Eq. (6.23) for M_i do have the correct weak potential form. So, for weak potentials, using Eq. (5.194) and Eq. (6.23) we solve for $\sinh(L)$ and obtain

$$\sinh(L) \approx \frac{S}{w}\left(1 + \frac{\epsilon_w S}{w m_w}\right). \tag{6.26}$$

We then need a modification of this, valid in regions of large negative S that at $S = 0$ is continuous through second derivatives with that obtained above, that yields correct strong coupling static limit spectral results, and yields correct weak coupling equal mass results *obtained non perturbatively (i.e. numerically).* Although these restrictions are not sufficient to uniquely define an extrapolated $\sinh(L)$ they are severe enough to narrow the choices significantly. For the present treatment, we choose

$$\sinh(L) = \frac{S}{w}\left(1 + \frac{\epsilon_w S}{\sqrt{w^2 + S^2} m_w}\right). \tag{6.27}$$

This satisfies the continuity condition and gives numerical results that satisfy the other two restrictions. Consequently, for $S < 0$ and large m_2 we obtain a spectrum (both ground and excited states) that for this extreme case agrees with that of the exact solution obtained from the one body Dirac equation

$$(\gamma_1 \cdot p_1 + m_1 + S)\psi = 0. \tag{6.28}$$

Furthermore, Van Alstine obtained a spectrum for equal and unequal mass cases that agrees with the standard perturbative results as given in [160].

Another choice that works just as well is

$$\sinh(L) = \frac{S}{w}\left(1 + \frac{S}{\sqrt{(wm_w/\epsilon_w)^2 + S^2 w}}\right), \tag{6.29}$$

while one that satisfies the continuity condition, works well numerically for the extreme limit above, but fails the perturbative test

$$\sinh(L) = \frac{S}{w}\left(1 + \frac{\epsilon_w S}{\sqrt{m_w^2 + S^2 w}}\right).$$

Since this applications in QCD combine both scalar and electromagnetic like vector interactions we must impose similar conditions for the case of Eq. (6.23) with $\mathcal{A} \neq 0$. Following the same procedure, combining Eqs. (5.194,6.23) for weak S but arbitrary $\mathcal{A}$, yields

$$\sinh(L) \approx \frac{S}{wD}\left(1 + \frac{(\epsilon_w - A)S}{wDm_w}\right), \tag{6.30}$$

in which $D = 1/G^2 = 1 - 2\mathcal{A}/w$. We then test our assumption for the case $S = \mathcal{A} = -\alpha/r$, as before for the extreme cases of unequal mass, large coupling and equal mass, small coupling. We find an extrapolation that works reasonably well

$$\sinh(L) = \frac{S}{wD}\left(1 + \frac{(\epsilon_w - \mathcal{A})S}{\sqrt{w^2 + S^2 D m_w}}\right), \; S < 0. \tag{6.31}$$

We note that these requirements rule out the plausible choice

$$\sinh(L) = \frac{S}{wD}\left(1 + \frac{S}{\sqrt{m_w D w/(\epsilon_w - \mathcal{A}))^2 + S^2}}\right),$$

whose $\mathcal{A} = 0$ counterpart worked well above.

We emphasize that a crucial feature of the $\sinh(L)$ extrapolations is that for fixed S, in the static limit (e.g. $m_2 >> m_1$) $\sinh(L) \to S/w$ which leads to $M_1 \to m_1 + S$. Note that as opposed to what happens for scalar potentials, strong $\mathcal{A}$ potentials have no problem in the static limit where the restriction $\mathcal{A} < w/2 \to \infty$ on $\mathcal{A}$ is automatically satisfied.

Model 2 This model comes from the work of H. Sazdjian [336]. Using a special techniques of amplitude summation, he is able to sum an infinite number of Feynman diagrams (of the ladder and cross ladder variety). For the vector interactions, he obtained results that correspond to Eq. (5.32) to Eq. (5.34) and Eq. (6.12) to Eq. (6.14)(modified here in Eq. (6.21) for $A \geq 0$). For scalar interactions ($L(S, A)$) he obtained two results. One again agrees with Eq. (5.30) and Eq. (6.15b). As we have seen above this must be modified (see Eq. (6.31)) for $S \leq 0$. His second result is the one we use here for our second model for ($L(S, A)$). That replaces Eq. (6.31)and Eq. (6.23) with the model:

$$S + A > 0,$$

then

$$S \longrightarrow -\mathcal{A} + \frac{(S + A)w}{\sqrt{4(S + A)^2 + w^2}}, \tag{6.32}$$

while if

$$S + A < 0,$$

we let

$$S \longrightarrow -\mathcal{A} + S + A. \tag{6.33}$$

In both case we let

$$\sinh L = \sinh(-\frac{1}{2}\ln(1 - \frac{2(S + \mathcal{A})}{w}) - \mathcal{G}). \tag{6.34}$$

6.3 The Nucleon Nucleon Optical Model

Von Geramb et al. [256] analyzed partial wave amplitudes for NN scattering to 2.5 GeV, in which resonance and meson production effects are evident for energies above pion production threshold. This analyses was based upon boson exchange or quantum inversion potentials. They also added short range Gaussian potentials to their real reference model potentials, Nijmegen or inversion potentials. The energy dependences of these Gaussian was very smooth save for precise effects caused by the known Δ and $N^\star$ resonances.

Here we will analyze the SP03, and SM00 partial wave amplitudes for NN scattering to 3 GeV with complex short range Gaussian potential on the radius $r \sim 0.5 fm$, in addition to the real Dirac reference potential. Before that, we will discuss shortly the effective Hamiltonian formalism of Feshbach [232].

First, Feshbach in his work Ref. [232] used the generalized optical potentials with projection operators. His approach, the total many body wave function Ψ is partitioned into an *open channel* segment $P\Psi$, a part that is of interest for particular phenomenon, and a *closed channel* segment, the remaining part of the wave function $Q\Psi$

$$\Psi = P\Psi + Q\Psi. \tag{6.35}$$

P space is include all real longe range interactions, Q contains NN reactions.

The components $P\Psi$ and $Q\Psi$ will in general be coupled by interactions. Let ψ_i be the wave functions of the target in its different states, $i = 0$ will denote the ground state, $i = 1, 2, ...$, the 1^{st}, 2^{nd},..., excited states. For total system

$$\Psi = \sum_{i=0}^{\infty} u_i(r)\psi_i(r_1, r_2, ..., r_A), \tag{6.36}$$

where $(r_1, r_2, ..., r_A)$ stand for positional as well as intrinsic coordinates of the A particles in the target, while $u_i(r)$ are functions of the position and intrinsic coordinates r of the projectile. The projection operator P is such that acting on the function Ψ of Eq. (6.35) it projects out a number of terms $u_i\psi_i$ corresponding to a given set of channels of interest

$$P\Psi = \sum_{i=0}^{n} u_i\psi_i, \tag{6.37}$$

it means that include the open channels. We define Q and P projection operators that satisfy the relations

$$P + Q = 1, \qquad PQ = 0, \qquad P^2 = P, \qquad Q^2 = Q. \tag{6.38}$$

It is a matter of convenience to choose $Q\Psi$ to be orthogonal to $P\Psi$, as is insured by Eq. (6.38).

By eliminating the Q channels, a Schrödinger equation is obtained for the P channels, and an effective Hamiltonian can be derived, which in turn can be used to analyze various aspects of the nuclear many body problem [256]. The resulting equations depend only on the existence of an projection operator, not on explicit realization thereof. To find the effective Hamiltonian, we are starting with Schrödinger equation

$$(E - H)\Psi = 0, \tag{6.39}$$

where Ψ is given by Eq. (6.35) so that

$$(E - H)(P\Psi + Q\Psi) = 0. \tag{6.40}$$

Acting on the left by P or Q and on the right by P and using Eq. (6.38), we get the two equations

$$(E - H_{PP})(P\Psi) = H_{PQ}(Q\Psi), \tag{6.41a}$$
$$(E - H_{QQ})(Q\Psi) = H_{QP}(P\Psi), \tag{6.41b}$$

where $H_{PP} = PHP$, $H_{QQ} = QHQ$ and similarly for H_{QP} and H_{PQ}. Solving the second equation, with outgoing wave boundary conditions

$$Q\Psi = \frac{1}{E - H_{QQ} + i\epsilon} H_{QP}(P\Psi). \tag{6.42}$$

Substitution into the first equation yields

$$(E - H_{eff})(P\Psi) = 0, \tag{6.43}$$

where the energy dependent effective Hamiltonian is

$$H_{eff} = H_{PP} + H_{PQ}\frac{1}{E - H_{QQ} + i\epsilon}H_{QP}. \tag{6.44}$$

From this Eq. (6.44), we see that H_{eff} is a complex operator, the first term on the right side is a divergent free interaction and the second term is singular, which describe a coupling between the $P\Psi$ and $Q\Psi$. The operator $(E - H_{QQ} + i\epsilon)^{-1}$ provides then a propagation within the $Q\Psi$ part only. This equation, the Lippman-Schwinger equation, is an integral equation for Ψ with a Green's function kernel. The operator H_{QQ} consist of a discrete

$$H_{QQ}\Phi_s = \mathcal{E}_s\Phi_s,$$

and a continuum spectrum

$$H_{QQ}\Phi(\mathcal{E}, \alpha) = \mathcal{E}\Phi(\mathcal{E}, \alpha),$$

where α is an extra index which completes the classification of the continuum states [380]. Finally, if we use

$$\lim_{\epsilon\to 0^+}\int_a^b \frac{f(x)}{x - x_0 + i\epsilon}dx = \mathcal{P}\int_a^b \frac{f(x)}{x - x_0}dx - i\pi f(x_0), \tag{6.45}$$

we have

$$\begin{aligned} \text{Re } H_{eff}(E) &= H_{PP} + \sum_s \frac{H_{PQ}|\Phi_s >< \Phi_s|H_{QP}}{E - \mathcal{E}_s} \\ &\quad + \mathcal{P}\int d\alpha \int d\mathcal{E}\frac{H_{PQ}|\Phi(\mathcal{E}, \alpha) >< \Phi(\mathcal{E}, \alpha)|H_{QP}}{E - \mathcal{E}}, \end{aligned} \tag{6.46}$$

$$\text{Im } H_{eff}(E) = -\pi \int d\alpha H_{PQ}|\Phi(E, \alpha) >< \Phi(E, \alpha)|H_{QP}. \tag{6.47}$$

The optical model is highly nonlocal, but local equivalent potentials are commonly used. Substituting the second into the first equation, we get the dispersion type relation

$$\text{Re } H_{eff}(E) = H_{PP} + \sum_s \frac{H_{PQ}|\Phi_s >< \Phi_s|H_{QP}}{E - \mathcal{E}_s} - \frac{1}{\pi}\mathcal{P}\int \frac{\text{Im } H_{eff}}{E - \mathcal{E}}d\mathcal{E}. \tag{6.48}$$

Von Geramb et al. for their optical potential together with meson exchange model in Refs. [278, 256], described the Q space functions as doorway states, describing the QCD entrance sector from finite size nucleons, mesons and possible other particles.

Here, in the NN interaction towards higher energies, the Dirac potential is the background real reference potential V_{ref}. For a two body problem with scalar and vector interaction, a relativistic reduced mass and energy of a fictitious particle of relative motion

$$m_w = \frac{m_1 m_2}{w},$$

and

$$\epsilon_w = \frac{w^2 - m_1^2 - m_2^2}{2w},$$

the invariants is then

$$\epsilon_\omega^2 - \overrightarrow{p}^2 = m_\omega^2. \tag{6.49}$$

We can apply the techniques already developed for the *no spin* potentials to the above Eq. (6.49) by the substitutions $\epsilon_\omega \mapsto \epsilon_\omega - A$, and $m_\omega \mapsto m_\omega + S$. It yields

$$\overrightarrow{p}^2 + 2m_\omega S + S^2 + 2\epsilon_\omega A - A^2 - \epsilon_\omega^2 + m_\omega^2 = 0. \tag{6.50}$$

The effective single particle Eq. (6.50) is often referred to as a Schrödinger equivalent equation since it has the same radial form as Schrödinger equation with relativistic kinematics. Thus, given a Dirac optical potential, one can use it in a Schrödinger analysis, with relativistic kinematics simply by converting to the second order form. This relation holds for pairs of particles in form of coupled systems

$$\begin{aligned}
\{\overrightarrow{p}^2(1) + 2m_\omega(1)S(1) &+ S^2(1) + 2\epsilon_\omega(1)A - (1)A^2(1) - \\
&\epsilon_\omega^2(1) + m_\omega^2(1)\}|\psi_1\psi_2> = \mathsf{C}(1,2)|\psi_1\psi_2>, \\
\{\overrightarrow{p}^2(2) + 2m_\omega(2)S(2) &+ S^2(2) + 2\epsilon_\omega(2)A - (2)A^2(2) - \\
&\epsilon_\omega^2(2) + m_\omega^2(2)\}|\psi_1\psi_2> = \mathsf{C}(1,2)|\psi_1\psi_2>. \quad (6.51)
\end{aligned}$$

with product wave function and orthogonality of their intrinsic parts yields

$$\begin{aligned}
&<\psi_2|\{\overrightarrow{p}^2(1) + 2m_\omega(1)S(1) + S^2(1) + 2\epsilon_\omega(1)A - (1)A^2(1) - \epsilon_\omega^2(1) + m_\omega^2(1)\}|\psi_1\psi_2> \\
&\qquad = -<\psi_2|\mathsf{C}(1,2)|\psi_1>|\psi_2> = -V_{21}|\psi_2>, \\
&<\psi_1|\{\underbrace{\overrightarrow{p}^2(2)}_{H_2} + \underbrace{2m_\omega(2)S(2) + S^2(2) + 2\epsilon_\omega(2)A - (2)A^2(2)}_{V_{22}} - \underbrace{\epsilon_\omega^2(2) + m_\omega^2(2)}_{k_2^2}\}|\psi_1\psi_2> \\
&\qquad = -<\psi_1|\mathsf{C}(1,2)|\psi_2>|\psi_1> = -V_{12}|\psi_1>.
\end{aligned}$$

The right hand sides define coupling matrix elements between channel 1 and 2. Rewrite the two equations as

$$(H_1^0 + V_{11} - k_1^2)|\psi_1> = -V_{12}|\psi_2>, \tag{6.52}$$
$$(H_2^0 + V_{22} - k_2^2)|\psi_2> = -V_{21}|\psi_1>, \tag{6.53}$$

with incoming flux restricted to channel 1, we invert formally and substitute the second equation

$$|\psi_2 >= -(H_2^0 + V_{22} - k_2^2)^{-1} V_{21} |\psi_1 > . \tag{6.54}$$

into the first one

$$(H_1^0 + V_{11} - k_1^2)|\psi_1 >= -V_{12}(H_2^0 + V_{22} - k_2^2)^{-1} V_{21}|\psi_1 > . \tag{6.55}$$

We may evaluate the singular integral in terms of its principle value and pole contribution (outgoing waves)

$$\{V_{12}(H_2^0 + V_{22} - k_2^2)^{-1} V_{21}|\psi_1 >\} = P \int dE\, V_{12}(H_2^0 + V_{22} - k_2^2)^{-1} V_{21}|\psi_1 > -$$

$$i\pi \int \delta(E^2 - k_2^2)\, dE\, V_{12}(H_2^0 + V_{22} - k_2^2)^{-1} V_{21}|\psi_1 >, \tag{6.56}$$

from which follows that the elimination of the second channel introduces effectively a complex interaction. The imaginary part comes from the loss of flux, in case the second channel is energetically open. Coupling closed channels does not generate an imaginary potential. This is the case for NN scattering below the meson production threshold $T_{lab} < 300$ MeV, or excitation of NA at very low energy. The optical potential comprises a complicated substructure of channel coupling but in summary appears to be quite simple and distinguished energy dependent real and imaginary path

$$V_{_{OMP}}(r, E) + i\, W_{_{OMP}}(r, E).$$

In the center-of-momentum system, $p = p_\perp = (0, \mathbf{p})$, $x_\perp = (0, \mathbf{r})$ and the relative energy and time are removed from the problem. The equation for the relative motion is then

$$\{\overrightarrow{p}^2 + V_{ref} + V_{_{OMP}} + i\, W_{_{OMP}} - k^2\}|\psi >= 0, \tag{6.57}$$

where V_{ref} is the Dirac reference potential from Eq. (5.257).

The optical potential considered here may be taken as Woods-Saxon, surface Gaussian or any linear combination of them, it consists of two part. Consider one part is a potential S which transforms as a scalar

$$S = \frac{S_0(k^2)}{1 + exp(\frac{r - R_0}{a})}, \tag{6.58}$$

the other part , a potential $\mathcal{A}$, which transforms as a vector

$$\mathcal{A} = \frac{\mathcal{A}_0(k^2)}{1 + exp(\frac{r - R_0}{a})}. \tag{6.59}$$

The new OMP model is dependent on the fitting of data to determine the parameters in the assumed potential. The resulting Dirac equation (6.57) is suitable for simultaneous analyses of np and nn scattering data up to several GeV.

6.3.1 Refined Optical Model Potential

A fundamental difficulty in an EFT description of nuclear forces is that they are necessarily non perturbative, so that an infinite series of Feynman diagrams must be summed. Which diagrams must be summed may be known, and summing them is equivalent to solving a Schrödinger equation. However, an EFT yields graphs which require renormalization, giving rise to a Schrödinger potential which is singular to solve conventionally. In effective field theories, the potential will in general have a singular behavior, such as $1/r^2$, $1/r^3$, $\delta^3(r)$, and worse. Such potentials do not allow a conventional solution of the Schrödinger equation, or equivalently, lead to divergent diagrams in the field theory. In field theory it is well known how to deal with divergences one merely regularizes the integrals and then renormalize the coupling of the theory, absorbing terms that diverge as the cutoff is removes them into the definition of the renormalized coupling. When this is done, there is no cutoff dependence in the theory.

OMP-1: In this model, we solve the Schrödinger equation (6.57) for NN scattering with a radial weight function applied to the Dirac potential. In practical terms, this reduces the hard core Dirac potential with a Fermi distribution.

We assume the interaction between two nucleons is described by a potential, $V = V_S + V_L$, consisting of a short and long range part. This separation is justified by the weak and small long range interaction where changes are small on the scale of the nucleon diameter ~ 1 fm. The short range interaction is strong and its scale is 0.1 fm. The short range interaction acts directly on quarks and causes quark spin flip or quark exchange. Only the short range interaction is able to transform a nucleon into a $\Delta(3,3)$ or N^* and generates directly or indirectly mesons. We simplify things in this model by taking the long range potential V_L to be V_{ref} and a weighted V_{ref} as V_{OMP} in the short range domain.
The real potential is identified with the Dirac potential

$$\mathrm{Re}\, V = V_{Dirac} + V_{OMP} = V_{Dirac} - V_0 \frac{V_{Dirac}}{(1 + e^{(r-R_0)/a})}, \tag{6.60}$$

where the real optical potential is fixed with

$$V_{OMP} = -V_0 \frac{V_{Dirac}}{(1 + e^{(r-R_0)/a})}. \tag{6.61}$$

It vanishes rapidly outside the short range domain $r > 0.5$ fm. The imaginary optical potential is a short range surface peaked normalized Gaussian

$$\mathrm{Im}\, V = W_{OMP} = W_0 N_G \, e^{-\frac{(r-R_0)^2}{a^2}}, \tag{6.62}$$

where N_G is the normalization. The total potential V is

$$V = V_{Dirac} + V_{OMP} + i\, W_{OMP}, \tag{6.63}$$

Eq. (6.57) yields

$$\{\overrightarrow{p}^2 + V - k^2\}|\psi >= 0. \tag{6.64}$$

As pointed out by Beane et al. [57] and Lepage [379], EFT uses a similar model, but specifies a high momentum cut-off as regularization. In EFT the cut-off has a physical meaning like in renormalizable theories [379]. In fact the strategy of effective field theories is such that one should not pick either too low a cut-off or too high a cut-off : if one chooses too low a cut-off , one risks the danger of throwing away relevant degrees of freedom and hence correct physics while if one chooses too high a cut-off, one introduces irrelevant degrees of freedom and hence makes the theory unnecessarily complicated. The astute in doing EFT is in choosing the proper cut-off. In subsection 7.6.1 are shown numerical results of Eq. (6.64).

OMP-2: The short range dynamics can always be treated as a set of local operators. In this model, the Dirac potential represents the long range part of the potential. Local operators in momentum space correspond to OMP interactions in coordinate space. We assume that the short range physics is represented by the OMP plus the Dirac's reference potential.

The short range physics is represented by a real optical potential

$$V_{OMP} = \frac{V_0}{(1 + e^{(r-R_0)/a})}, \tag{6.65}$$

of Woods-Saxon type and the imaginary part

$$W_{OMP} = W_0 N_G \, e^{-\frac{(r-R_0)^2}{a^2}}, \tag{6.66}$$

of a normalized Gaussian type, where N_G is the normalization. The total potential V is given by

$$V = V_{Dirac} + V_{OMP} + i\, W_{OMP}, \tag{6.67}$$

with adjustable V_0 and W_0 as function of energy and partial wave, R_0 and a are fixed. Eq. (6.57) yields

$$\{\overrightarrow{p}^2 + V - k^2\}|\psi >= 0. \tag{6.68}$$

In subsection 7.6.1 we show results for OMP-2 calculations. We expect results which are qualitatively similar to OMP-1.

OMP-3: We now consider the more interesting case of an interaction with both long and short range dynamics well separated.

The long range interaction is identified with the Dirac potential with a sharp cut-off somewhat outside the hard core radius $r_c = 0.5$ fm, *viz.*

$$V_L = V_{Dirac}\left(1 - \frac{1}{(1 + e^{(r-R_s)/a_s})}\right), \tag{6.69}$$

with $R_s = 0.514$ fm and a_s=0.02 fm.

The short range interaction, $V_S(r)$ for $r < r_c$ or $r \sim r_c$, is dominated by the intrinsic dynamics of strongly overlapping (> 75%) two nucleons. The notion of nucleons, as individual entities, is not predominant for the dynamics of this domain. The highly non perturbative quark gluon dynamics becomes relevant for the (strongly overlapping NN) six quark system. The nucleon spin looses its immediate relevance in the interaction of quarks in favor of quark gluon flavors. At the end, from this complicated microscopic structure remains only a smooth short range scalar and vector potential. We apply the techniques already developed for the *no spin* potentials to

$$\epsilon_\omega^2 - \overrightarrow{p}^2 = m_\omega^2,$$

by the substitutions $\epsilon_\omega \mapsto \epsilon_\omega - A$, and $m_\omega \mapsto m_\omega + S$. It yields

$$\{\overrightarrow{p}^2 + 2m_\omega S + S^2 + 2\epsilon_\omega A - A^2 - \epsilon_\omega^2 + m_\omega^2\}|\phi >= 0. \tag{6.70}$$

The effective single particle Eq. (6.70) is a Schrödinger type equation with relativistic kinematics.

The combined interactions include both quadratic additions to S and $\mathcal{A}$ as well as CM energy dependences through m_w and ε_w. A recoil term must also be added. This is an expression containing logarithmic terms, Eq. (6.72f), which are due to the transverse or spacelike part of the potential. Without those terms, spectral results would have not agreed with the standard (but more complex) spinless Breit and Darwin approaches.

The OMP-3 model wave equation is

$$\{\overrightarrow{p}^2 - k^2 + V_{scalar} + V_{vector} + V_{recoil} + V_{_{OMP}} + i\, W_{_{OMP}} + V_{ref}\}|\psi >= 0, \tag{6.71}$$

where

$$V_{ref} = V_{Dirac}\left(1 - \frac{1}{(1 + e^{(r-R_s)/a_s})}\right), \tag{6.72a}$$

$$V_{scalar} = 2m_w S + S^2, \tag{6.72b}$$

where

$$S = S(r) = \frac{S_0}{(1 + e^{\frac{r-R_s}{a_s}})}, \tag{6.72c}$$

$$V_{vector} = 2\epsilon_w \mathcal{A} - \mathcal{A}^2, \tag{6.72d}$$

where

$$\mathcal{A} = \mathcal{A}(r) = \frac{\mathcal{A}_0}{(1 + e^{\frac{r-R_s}{a_s}})}, \tag{6.72e}$$

and

$$V_{recoil} = \frac{1}{2}\nabla^2 \log(1 - 2\mathcal{A}/w) + \frac{1}{4}[\nabla \log(1 - 2\mathcal{A}/w)]^2, \tag{6.72f}$$

with $R_s = 0.514$ fm and $a_s = 0.02$ fm. The potential strengths S_0 and $\mathcal{A}_0$ are fixed by the condition

$$2m_\omega(k^2)S + S^2 + 2\epsilon_\omega(k^2)A - A^2 \sim k^2, \tag{6.72g}$$

which implies that the two interacting nucleons have exhausted and transfered all NN kinetic energy of relative motion into intrinsic excitations of the multiquark system. This condition is *approximately* satisfied with $S_0 = -480$ MeV and $\mathcal{A}_0 = 600$ MeV and the energy dependence of m_ω and ϵ_ω, see Todorov variables in Sec. 5.1. We use these parameters for all energies T_{Lab} and in all partial waves. The surface OMP uses normalized Gaussians for its
real

$$V_{OMP} = V_0(T_{Lab})N_G\, e^{-\frac{(r-R_O)^2}{a_O^2}}, \tag{6.72h}$$

and imaginary

$$W_{OMP} = W_0(T_{Lab})N_G\, e^{-\frac{(r-R_O)^2}{a_O^2}} \tag{6.72i}$$

parts, where $R_0 = 0.514$ fm and $a_0 = 0.01$ fm, N_G is the normalization coefficient. The OMP surface potential parameters V_0 and W_0 are adjusted freely for all the partial waves and at any energy. This independent adjustments are justified as it accounts for low density of possible intermediate states and in particular the strong channel dependence of $\Delta(3,3)$ and other $N^\star$ resonances and excitations.

Let us recall a classical nuclear physics reaction, the ^{3}He $+^3$He induced reaction with meson production ^{3}He$+^3$He $\rightarrow$ ^{3}He$+^3$He + meson. In this example, the smooth and long range ion-ion potential is unable to produce a meson. Rather, we associate the meson production with a pair of nucleons, one from each ion, as a hard process.

On the QCD level we expect, as predominant effect, spin-flip and exchange of one active valence quark in each of the two nucleons. We suppose a grinding impact which is favored in the partial wave channels $^1D_2$, $^3D_2$, $^3PF_2$, $^1F_3$ and $^3F_3$. This limitation is a consequence of QCD confinement. Contrary to the ^{3}He example of complex nuclear scattering there are no quarks in the outer surface and tail region for NN. This situation is described by the peaked Gaussian optical model with proper parameters in geometry and strengths V_0 and W_0. We envisage the time development in a scheme shown in Fig. 6.1. This view must first be supported by more numerical work and secondly use of the microscopic theory of optical models. To define a g-matrix, as effective medium and density dependent quark-quark interaction, shall be an obvious goal for the future [6, 278]. Such microscopic approach would permit to calculate the real and imaginary strengths V_0 and W_0 without fitting procedure. Before we reach this goal a good understanding of what is needed form QCD is required. Tables (7.7-7.12) contain V_0 and W_0 values. A comprehensive discussion and graphical representation of potential details to the OMP-3 are given in subsection (7.6.2).

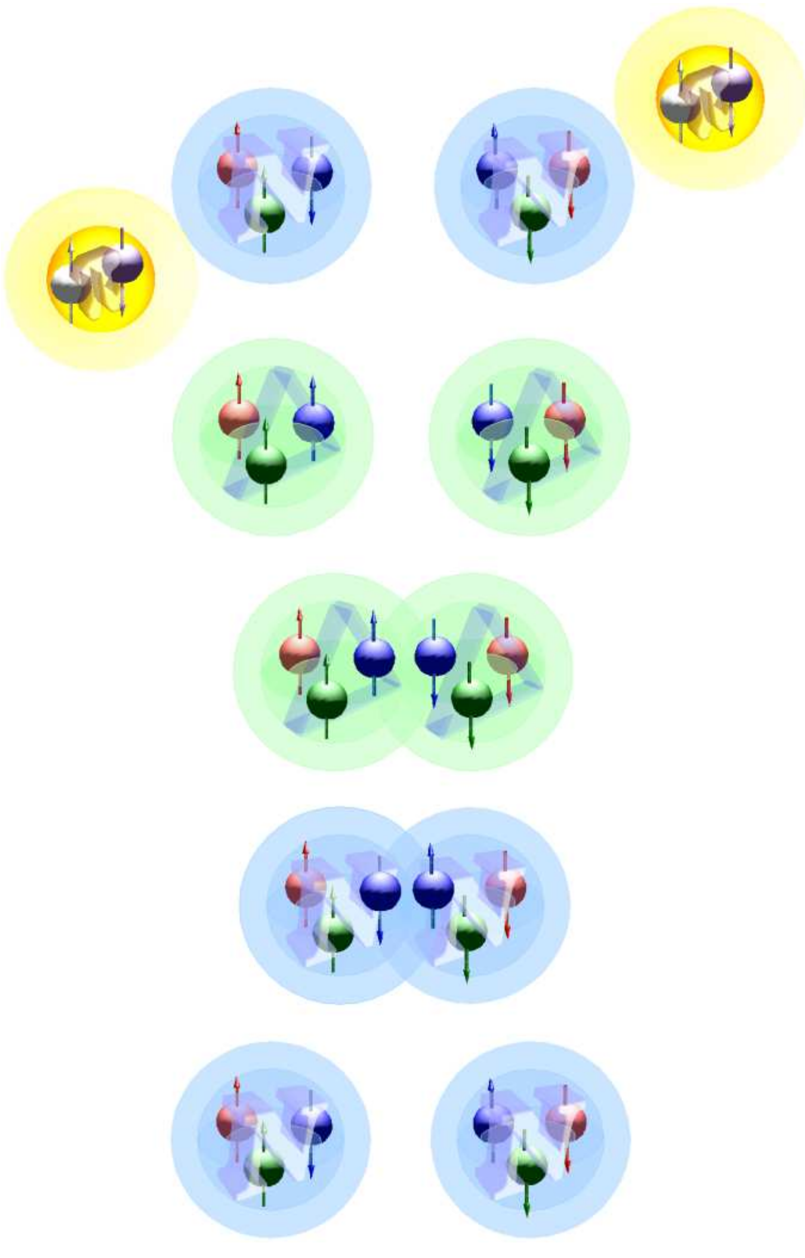


Figure 6.1: Quark exchange mechanism and spin-flip. A reaction scheme for $p+p \rightarrow p+p+2\pi^0$ in which the meson production is mediated by two intermediate Δs and their decay. The spin-flip or quark exchange is caused by contact or short range qq interactions in the overlapping/touching confinement zone of the two nucleons.

Chapter 7

Numerical Results

7.1 Implicit Knowledge

The history of particles and constituents of atomic nuclei is as long as 13 billion years. The standard character of this history are quarks and leptons. Immediately following the *big bang*, these characters were the ingredients of a hot (10^{15}K) and extremely dense particle soup. Today the universe has cooled down considerably and the sparkling stars, during a fright bright night, are living signs of all beginning. As the universe aged and cooled the matter within it went through a variety of phases as it adapted to the different conditions. One of the most significant of these phases occurred when the free deconfined quarks became bound (confined) to form matter and were no longer able to exist in isolation. The process by which quarks changed from a deconfined to a confined state is not well understand. In recent years we have gained a great deal of insurgent into quarks confinement, however we still lack a full theoretical understanding derived from *first principles.* To appreciate the significance of confinement and chiral symmetry breaking (CSB), or dynamical chiral symmetry breaking (DCSB), it is important to understand our current model of how all matter in the universe, from the galaxies to subatomic particles behave. The interaction of all matter that governs our daily lives can be explained in terms of fundamental particles (quarks and leptons) and the gauge or interaction force particles (gravitons, photons, gluons, W and Z).

In order to visualize how these particles and their interactions fit into our current understanding of the universe, imagine that we are looking at the universe through a magnifying glass. At a first glance, we see galaxies, the stars and planets. We can describe the motion of these objects by the *gravitational interaction* and *classical mechanics.* Gravitational interaction, from a submicroscopic point of view, involves the exchange of gravitons. These particles have not yet been observed.

As we look more closely at a star, we see charged particles like protons and electrons. Charged particles obey the laws of the electromagnetic interaction, of which we describe the visible phenomena in terms of classical electrodynamics and make essential use of the special theory of relativity. The interaction of charged particles requires quantum theory and exchange of photons. Not satisfied, we stare more closely into the atomic nuclei and discover protons, neutrons and nucleons as the smallest constituents of matter. After a careful investigation, we discover quarks which cannot escape from protons and neutrons. The force between quarks is called the strong interaction. It involves the exchange of gluons and it is responsible for keeping matter together. Finally, we discover that some particles are not stable, they can transform into other particles, they decay. These decays are described by the *weak interaction* and involve the exchange of W and Z mesons.

The paradigm for describing the strong interaction is called quantum chromodynamics (QCD). The basic ingredient of QCD are quarks and gluons. Quarks carry a fractional electric charge and feel the electromagnetic force, but they carry also another form of charge called *color*. QCD evolved from this idea and is closely modeled on the theory of quantum electrodynamics (QED) which explains electromagnetic interactions between electrically charged particles through the exchange of photons. Gluons transmit the color force between one quark and another, in the same way the photons transmit the electromagnetic force between electrically charged particles. However, there is one crucial difference between photons and gluons. Photons are electrically neutral, uncharged, so they do not interact with other photons. Gluons on the other hand have color charge and they interact strongly with each other as well as with quarks. Gluons are massless and appear to be free only within ~ 1 fm which is also the typical size of nucleons and hadrons in general. Hadrons are strongly interacting particles and every hadron is either a baryon (3 quarks state) or a meson (quark antiquark pair). Gluons are confined to a hadron, much as quarks are, gluons as well as quarks make themselves behave as particles like only indirectly by generating jets of hadrons (baryon and meson) in high energy collisions.

The nature of the strong interaction between quarks and gluons in QCD depends on the relevant energy scale. Coupling constants are energy dependent. Asymptotic freedom is the property that the coupling between quarks and gluons decreases as the energy scale increases (high energy is the values of > 50 GeV). At high energies, quarks are free and have a very small mass in this high energy weak coupling limit where perturbative QCD calculations are valid. Perturbative QCD can be compared with experimental results from high energy accelerators. Indeed, the success of asymptotic freedom and perturbative QCD has been part of the whole story of the standard model. Away from the high energy region perturbative theory fails in many regards.

In the low energy infrared (IR) region where quarks and gluons are confined inside hadrons, the coupling is so large that the perturbative methods must be replaced by new theoretical concepts. At this points it is appropriate to discuss briefly some of the primary concept such as confinement and CSB. Confinement has proven to be difficult to understand quantitatively. When a high energy proton is colliding with another proton, this may be visualized as a destructive process after which the quarks should steam apart. We do not see individual quarks emerging, instead, the quarks regroup quark antiquark pairs are created and ultimately hadron jets carry away momentum and energy conserving the overall electric charge. In the IR region, the strength of the quark and gluon interaction becomes increasingly large. This implies that the quarks bind more tightly together giving rise to confinement, for low energy nuclear physics generating individual nucleons and mesons. The quark gluon content of the hadrons disappear in favour of point like hadrons.

One can obtain an intuitive idea about the nature of confinement by picturing quarks as being bound by strings or flux tubes as first proposed by Nambu [430] and which have recently been studied on the lattice. When the quark antiquark pair inside a meson is close together it exchanges gluons and creates a very strong color free field that binds the quark together. At short distances, much smaller than the size of a meson, the quarks move as if they were free. When quarks are further apart the color interaction becomes stronger through the interaction of gluons with one another. The color field lines of force, between the quarks and antiquarks, are squeezed into a tubelike region. The further the quarks are pulled apart the higher the energies that must be added to the system. Eventually a limit is reached where it is energetically cheaper for the color force field to snap into a new quark antiquark pair. Energy is conserved in this process because the energy of color force fields is converted into the mass of new quarks, the color force field can relax back to an unstretched state. Therefore, in QCD the observed hadrons are composites of quarks with a surface where quark antiquark pairs assemble to mesons.

A complementary picture of confinement was proposed by Wilson. The Wilson criteria for confinement requires that the potential between two (infinitely) heavy quarks rises linearly with distance, exactly how the potential is defined in terms of Wilson loops is as a technical point which is not relevant here. Nevertheless, it is important to note that the strong coupling approximation of lattice gauge theory demonstrates that quarks are indeed confined. While we have a qualitative description of how confinement works, we do not have a quantitative one.

The other low energy aspect of quark gluon systems is chiral symmetry breaking (CSB) or dynamical chiral symmetry breaking (DCSB). We visualize CSB as follows: At high energy, where the interaction between quarks is weak and chiral symmetry preserved the quarks are free, are moving (speed of light) with almost zero mass. If these quarks move apart the color interaction between them becomes stronger and they start to slow down. At the point where the strength of the interaction reaches a critical limit, the quarks gain dynamic mass by breaking the chiral symmetry. This process is referred to as DCSB. How the structure of the QCD vacuum generates this DCSB is one of the mysteries of strong interaction physics. The process of DCSB generates 98% of the mass of the proton, neutron, hadron in general and thus 98% of the mass of a human body and and all other matter is generated by this remarkable non perturbative phenomenon.

In low energy scattering the in-state comprises two well separated protons (nucleons) each containing three constituent quarks which are confined. During their approach they are marginally deflected by the long range Coulomb interaction $r > 10$ fm, some more effects are guaranteed from π and σ exchange for relative distances $1 < r < 10$ fm. For separation distances $r < 1$ fm we expect ultimately an overlap of the confined six quark system.

One of the clearest signals of the quark gluon degrees of freedom that one may expect is a resonance whose properties are related to those degrees of freedom [387]. It is customary that six quark resonances in two baryon reactions are called dibaryons. In terms of a bag model, the originally separated bags are fusing into a single intermediate bag with six quark content, forming a dibaryon. Dibaryon formation, in slow motion, is the topic of medium energy proton-proton, more general NN scattering with a long list of questions:

- How is the relative kinetic energy of the two nucleons transformed into two nucleons and one or more mesons?
- Do metastable dibaryon systems exist?
- What is the livetime of metastable dibaryons?
- How do constituent quarks behave during dibaryon formation and meson nucleon decay?
- What are the volume and surface QCD physical quantities?
- What geometric changes are due during dibaryonic fusion and fission?

Theoretically, all the QCD models, including lattice QCD calculations, predict that there should be quasistable dibaryon resonance, but in contrast, experimentally, no quasistable dibaryon has been observed, except the molecular deuteron state [555]. The existing models are quite successful for the meson and baryon sectors, but not enough for hadronic interactions. Microscopic models, such as dynamical quark model, bag models, Skyrme models, and QCD sum rules, relate the internal baryon structure to the strong interaction of the confined constituents.

According to the quark model, mesons are composed of a pair of quark and antiquark, while baryons are composed of three quarks. Both mesons and baryons are color singlets. Most of the experimentally observed hadrons can be easily accommodated in the quark model. Any state with quark content other than $q\overline{q}$ or qqq is beyond the quark model, which is termed as nonconventional or exotic. However, besides conventional mesons and baryons, QCD itself does not exclude the existence of nonconventional states such as glueballs (gg; ggg; . . .), hybrid mesons ($q\overline{q}$g), and other multiquark states ($qq\overline{qq}, qqqq\overline{q}, qqq\overline{qqq}, qqqqqq; ...$).

In the early days of QCD, Jaffe proposed the H particle [333] with the MIT bag model, which was a six quark state. Unfortunately, it was not found experimentally. For two years there has accumulated some experimental evidence of possible existence of glueballs and hybrid mesons with exotic quantum numbers such as $J^{PC} = 1^{-+}$ [314].

In the original bag model each baryon is a cavity with quarks and gluons. First, consider a six quark system with three quarks in one baryon and three in another baryon. The distribution of quarks in the two baryon system depends upon the relative separation between the two baryons. When two bags approach each other they can overlap and consequently form a new bag which contains six quarks. The shape and size of this bag depends on the configuration of the six quark state. There are two individual bags for the long range part of the interaction. If only quarks and gluons are considered there will not be any interaction between these two bags and the quarks are confined in two separate regions. So in the absence of a new mechanism the original bag model can not describe the long range interaction between baryons. Such a new mechanism is provided by chiral symmetry, i.e., by requiring the underlying dynamics to be chiral symmetric.

For nuclear studies, we need to combine the resulting short range quark picture and the well studied meson exchange mechanisms to construct a model which can quantatively describe the NN data. Wang [554, 552] and other have developed some progressive model to obtain the full NN interaction from QCD. Cahill et al. [105] have developed EFT which takes spontaneous CSB into account. Constituent quarks and Goldstone bosons appear here as the effective degrees of freedom for low energy QCD physics. This model has been applied both to π and σ meson internal structures and to meson interactions, but not yet to NN interactions [532]. Glozman, Riska and Brown [284, 477] propose a phenomenological model, with consistent quarks and Goldstone bosons as the effective degrees of freedom for describing baryon spectroscopy and a π, σ quark coupling model has been tried for NN interactions [521]. Models with constituent quarks and effective one gluon exchange [54] have been developed for the description of hadron spectroscopy.

7.2 NN Potentials Using Dirac Constraint Instant Form Dynamics

The formalism of coupled two body Dirac equations, within constraint instant form dynamics, is used to study the NN interaction. This particular approach for two spin 1/2 particles was developed by Crater, Van Alstine, Long and Liu [152, 156, 388, 167, 75]. They define a Poincaré invariant interaction in terms of eight, by their symmetries classified, interactions with the implication that they satisfy certain compatibility conditions [388]. This approach yields in its final form explicitly energy dependent coupled channel potentials for use in partial wave Schrödinger like equations [75]. We rederived and followed their expressions up to a certain point and developed our own numerics to study *np* and *pp* scattering phase shifts, for $0 < T_{Lab} < 3\,\text{GeV}$. The comparison with recent data makes use of GWU/VPI SAID phase shift solutions SM00 and SP03 [18].

The NN interaction is described within the paradigm of exchange mechanism involving $\pi, \rho, \omega\, \sigma$ and other mesons exchanges [401] to make up what we call the *Dirac potential*. A comparison with most recent experimental data, GWU/VPI SP03 phase shifts, requires the adjustment of coupling constants and a regularization of the short range interaction domain $r < 0.15$ fm. For energies above pion production threshold $280 < T_{Lab} < 3000$ MeV we added a phenomenological complex optical model potential to the short range region $r < 0.5$ fm and around $r \sim 0.5$ fm, downgraded the Dirac potential, which we treat as a reference potential, in the nomenclature of previous approaches. This addition brings the theoretical elastic channel S-matrix in perfect agreement with the experimental data S-matrix from GWU/VPI and, more importantly, permits the identification of the QCD predominant reaction domain along the relative distance of the nucleons.

The coupled two body Dirac equations, combined with the meson exchange model, yield as first new result the appearance of a repulsive, practically hard core *potential, independent of partial wave*. The universal core radius has a value $r_c = 0.5 \pm 0.025\,\text{fm}$. This core radius emerges independently of a nucleon substructure. It depends only on masses, in particular of the exchanged mesons, and the full relativistic treatment of the NN system. This feature is not present with equal distinctness in any of the current NN best fit potentials of *np* and *pp* data [523, 574, 401]. For purpose of comparison, we show results of the Argonne AV18 potential [574].

The fitting process of coupling constants uses data in the submeson production domain, $0 < T_{Lab} < 280\,\text{MeV}$, of *np* and *pp* partial wave phase shifts. For $T_{Lab} > 280\,\text{MeV}$, single and double intrinsic nucleon excitations, $\Delta(3,3)$ and other low excited hadrons, as well as simple and complex reactive meson productions contribute. This is well known and demands beyond NN a more complex coupled channels problem to solve. We curtail the problem to NN scattering using an optical model potential (OMP) in [256, 278] addition to the Dirac NN reference potential. Despite of complicated inelasticities, the rules of angular momentum, isospin selection and the complex energy dependences, some of the partial waves show that the *real phase shifts* $\delta(T)$ are well reproduced (extrapolated) by the Dirac potential alone. Most clearly, this is realized in the ${}^1S_0, {}^3P_0$ and ${}^3P_1$ channels and $T_{Lab} < 1100\,\text{MeV}$ [256, 279].

In Fig. 7.1 we show an intuitive and guiding scheme which distinguishes interaction domains as function of separation between the two nucleons. This scheme is in accordance with coupled channels.

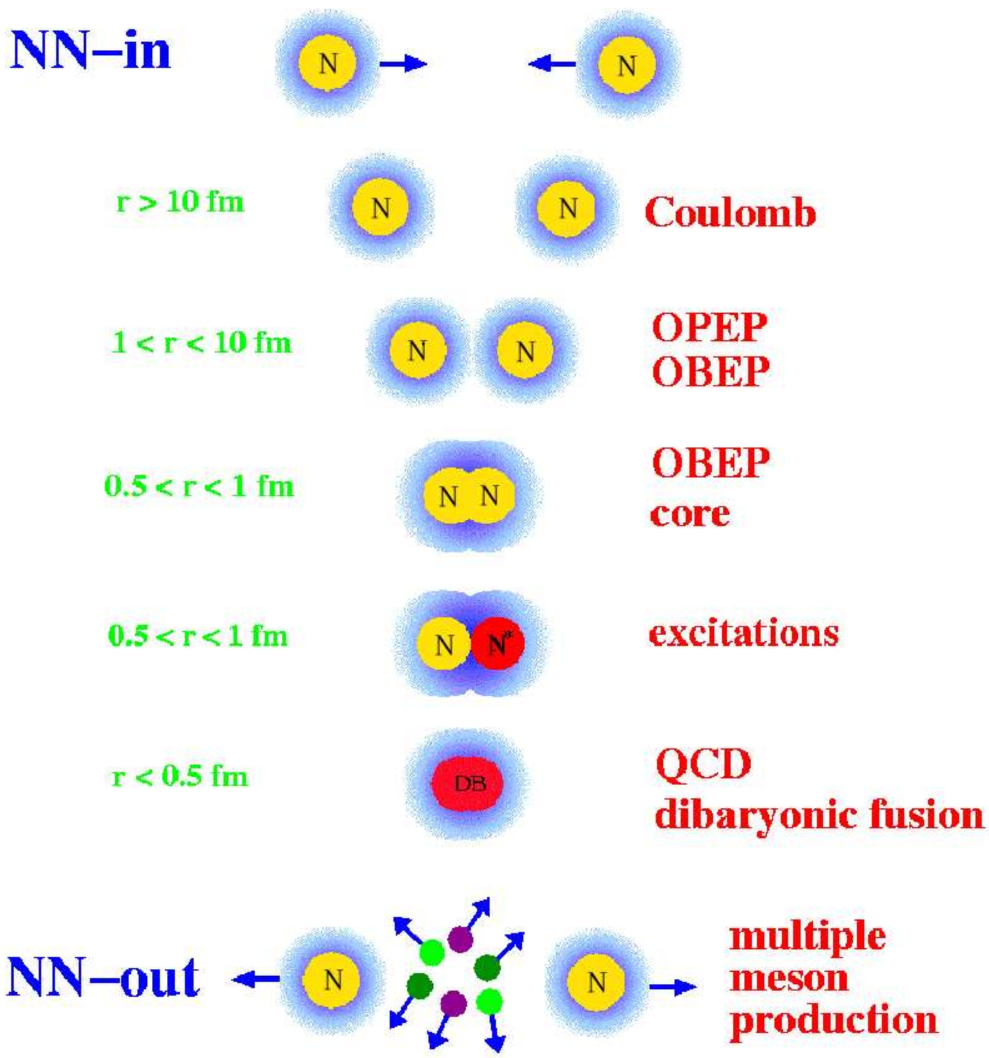


Figure 7.1: NN scattering and reaction scheme for $T_{Lab} < 3\,\mathrm{GeV}$.

7.3 Dirac Potentials and Partial Wave Phase Shifts

We use the convention of [388, 75] with the Coulomb parameter

$$\eta(k) = \frac{\epsilon_\omega\, e^2}{k}\delta_{pp},$$

for pp scattering which includes the timelike Coulomb interaction. This energy dependent $\eta(k)$ is to be compared with standard nonrelativistic Coulomb potentials [574]. Magnetic moment effect are neglected. The model specification for $L(x_\perp), J(x_\perp), C(x_\perp)$ and $\mathcal{G}(x_\perp)$ follows Liu and Crater *model I* [75] to specify:

scalar

$$S = -g_\sigma^2 \frac{e^{-m_\sigma r}}{r} - (\tau_1 \cdot \tau_2) g_{a_0}^2 \frac{e^{-m_{a_0} r}}{r} - g_{f_0}^2 \frac{e^{-m_{f_0} r}}{r},$$

pseudo scalar

$$C = (\tau_1 \cdot \tau_2) \frac{g_\pi^2}{\omega} \frac{e^{-m_\pi r}}{r} + \frac{g_\eta^2}{\omega} \frac{e^{-m_\eta r}}{r} - \frac{g_{\eta'}^2}{\omega} \frac{e^{-m_{\eta'} r}}{r},$$

and vector

$$A = (\tau_1 \cdot \tau_2) g_\rho^2 \frac{e^{-m_\rho r}}{r} + g_\omega^2 \frac{e^{-m_\omega r}}{r} + g_\phi^2 \frac{e^{-m_\phi r}}{r},$$

interactions (See Section.6.2.1). All Yukawa form factors are regularized with a normalized Gaussian

$$\frac{e^{-m\,r}}{r} \to N_G(a) \int dx^3\, \frac{e^{-m\,x}}{x}\, e^{-(\vec{r}-\vec{x})^2/a^2} \quad \text{with} \quad a = 0.14142\,\text{fm}. \tag{7.1}$$

This models π, η, ρ, ω, δ and σ exchanges. Meson masses and coupling constants are listed in Table. 7.5. In Figs. 7.6 - 7.11 are shown Dirac potentials for three values, $T_{Lab} = [0.1, 1, 2]$ GeV (red lines). In comparison are shown the results of the popular Argonne AV18 potential (blue line) [574]. The remarkable feature of the Dirac potentials is their universal repulsive core with $r_c \sim 0.5$ fm. The only exception is the ${}^3PF_1$ channel where the ansatz of repulsion turns, surprisingly, to attraction. Such attractions may occur also in other channels when the fitting of coupling constant is not properly limited. With the introduction of the optical model, inparticular with OMP-3, this feature is eliminated in favour of short range QCD dynamics.

In Figs. 7.12 - 7.17 are shown the phase shifts of SM00 (green), SP03 (blue) and theoretical results, real Dirac potential solutions (red), real Dirac potentials with adjusted complex OMP-1 and OMP-2 added are *coinciding* with the data of SP03 (blue lines).

The Dirac instant form dynamic yields partial wave spin, isospin and energy (α channel) dependent NN potentials $V_\alpha^D(r,T)$ to which we add a local or nonlocal optical model potential, version like OMP-1, OMP-2 and OMP-3 are used, whose strengths are fitted to data.

Since local/nonlocal potentials imply similar results, we restricted ourselves here to the local optical potential and reference the more general and nonlocal case [256].

7.4 Calculation of the Deuteron Wave Functions

Using Dirac potential, the resolution of the Schrödinger equation in the $T = 0$, $J = 1$ np channel leads to the bound state wave function. It is only loosely bound, having a binding energy E_B much less than the average value between air of nucleons in all the other stable nuclei. The tensor force requires that the deuteron wave function be a mixture of ${}^3S_1$ and ${}^3D_1$ components. Successful NN potentials must have several basic characteristics in order to satisfactorily describe the deuteron static properties and the NN scattering data. The deuteron is a very unique nucleus in many respects. We discussed here some characteristics of deuteron.

In momentum space, the deuteron wave function is given by

$$\Psi_d^M(k) = \left[\psi_0(k)\mathcal{Y}_{01}^{1M}(\hat{k}) + \psi_2(k)\mathcal{Y}_{21}^{1M}(\hat{k})\right]\zeta_0^0,$$

where

$$\mathcal{Y}_{LS}^{JM}(\hat{k}) = \sum_{m_L m_S} < J, M|L, m_L; S, m_S > \mathcal{Y}_L^M(\hat{\mathbf{k}})|S, m_S >,$$

are the normalized eigenfunctions of the two nucleon orbital angular momentum L, spin S, and total angular momentum J with projection M; $\zeta_T^{M_T}$ denotes the normalized eigenstates of the total isospin T with projection M_T of the two nucleons. The normalization is

$$\langle\Psi_d^M|\Psi_d^M\rangle = \int_0^\infty dkk^2\left[\psi_0^2(k) + \psi_2^2(k)\right] = 1.$$

The momentum space wave functions be Fourier transformed into the configuration space wave functions u and w by

$$\frac{u_L(r)}{r} = \sqrt{\frac{2}{\pi}}\int_0^\infty dkk^2 j_L(kr)\psi_L(k)\ ,$$

with $u_0(r) \equiv u(r)$, $u_2(r) \equiv w(r)$, and j_L the spherical Bessel functions. The normalization is

$$\int_0^\infty dr\left[u^2(r) + w^2(r)\right] = 1.$$

The asymptotic behavior of the wave functions for large values of r are

$$\begin{aligned} u(r) &\sim A_S e^{-\gamma r}, \\ w(r) &\sim A_D e^{-\gamma r}\left[1 + \frac{3}{(\gamma r)} + \frac{3}{(\gamma r)^2}\right], \end{aligned}$$

where A_S and A_D are known as the asymptotic S- and D-state normalizations, respectively. In addition, one defines the *D/S-state ratio*

$$\eta \equiv A_D/A_S. \tag{7.2}$$

Other deuteron parameters of interest are the quadrupole moment

$$Q_d = \frac{1}{20}\int_0^\infty drr^2 w(r)\left[\sqrt{8}u(r) - w(r)\right], \tag{7.3}$$

the *root mean square* or matter radius

$$r_d = \frac{1}{2}\left\{\int_0^\infty drr^2\left[u^2(r) + w^2(r)\right]\right\}^{1/2}, \tag{7.4}$$

and the $S-$ and D-state probability

$$P_S = \int_0^\infty dru^2(r), \qquad P_D = \int_0^\infty drw^2(r). \tag{7.5}$$

The magnetic moment of the deuteron is determined entirely by the D state probability P_D

$$\mu_d = \mu_s - \frac{3}{2}\left(\mu_s - \frac{1}{2}\right)P_D, \tag{7.6}$$

where $\mu_s = \mu_n + \mu_p$ is the isoscalar nucleon magnetic moment.

Table 7.1: Parameters for the deuteron properties

Parameter	Bonn-b	Argonne	Nijmegen-II	Nijmegen-I
$-\epsilon_d(MeV)$	2.2245	2.2258	2.2259	2.22589
$P_D(\%)$	4.85	5.759	5.652	5.677
A_D/A_S	0.0256	0.02215	0.02232	0.02242
κ (1/fm)	0.23153	0.2316	0.2316	0.2315

Fuda Front	Fuda Instant	Dirac instant form	Experiment
2.225	2.224	2.2245	2.224575
4.64(5.05)	4.41(4.99)	5.2621	—
0.0256	0.0257	0.02529	0.0256
0.2317	0.2315	0.2341	-

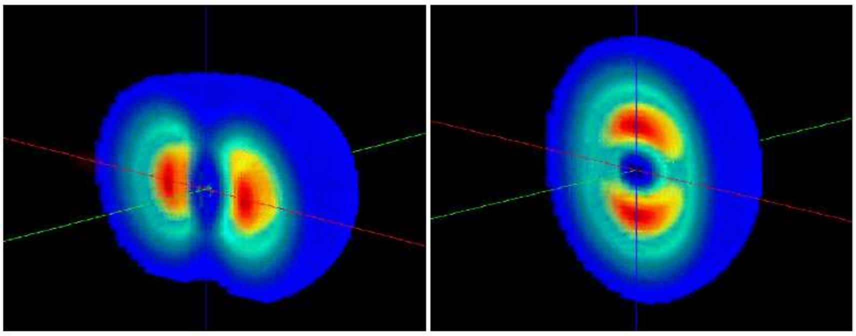


Figure 7.2: Deuteron densities $|\psi^0(x)|^2$ and $|\psi^1(x)|^2$ which depends upon the projection of the total deuteron angular momentum $M = 0$ (left, therefore one may regard as the "static" state of the deuteron) and $M = 1$ (right, as a spinning harmonic oscillator state), the size and shape of the deuteron. The figure has been adopted from [261, 242]

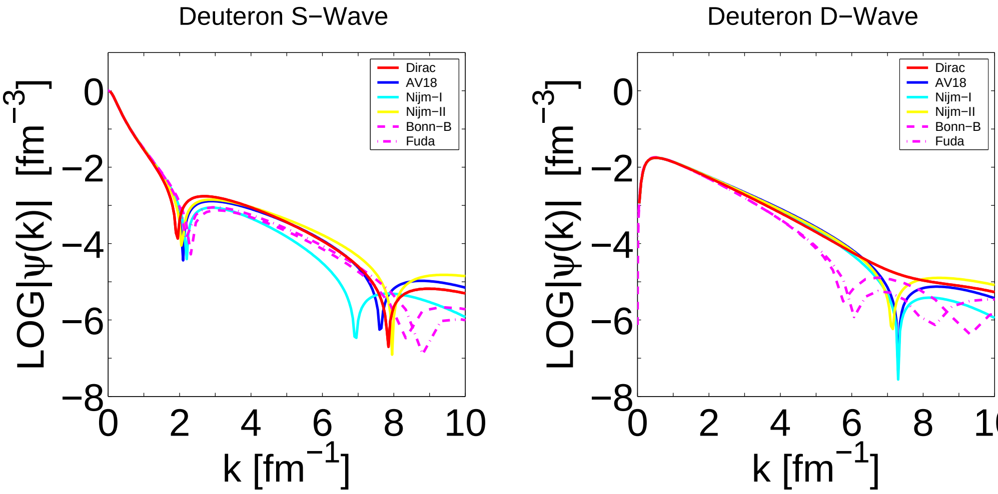


Figure 7.3: The S- and D-wave deuteron wave functions in the momentum space calculated from the Dirac potential (red) which in Table. 7.2 listed the numerical data, Argonne AV18 (blue), Nijmegen-I (cobalt), Nijmegen-II (yellow), Bonn-B (dashed magenta), Fuda [252, 253] (dot-dashed magenta).

Table 7.2: The values of deuteron wave from Dirac potential.

R (fm)	$k_S(fm^{-3})$	$k_D(fm^{-3})$
0.0	12.19866	-0.01421
0.3	3.61446	-0.20387
0.6	1.11872	-0.21497
0.9	0.41844	-0.17809
1.2	0.16213	-0.13799
1.5	0.05454	-0.10417
1.8	0.00714	-0.07769
2.1	-0.01313	-0.05754
2.4	-0.02053	-0.04239
2.7	-0.02176	-0.03108
3.0	-0.02016	-0.02268
3.3	-0.01743	-0.01646
3.6	-0.01444	-0.01187
3.9	-0.01160	-0.00851
4.2	-0.00910	-0.00606
4.5	-0.00699	-0.00428
4.8	-0.00527	-0.00301
5.1	-0.00390	-0.00210
5.4	-0.00283	-0.00146
5.7	-0.00200	-0.00101
6.0	-0.00138	-0.00070
6.3	-0.00092	-0.00049
6.6	-0.00058	-0.00035
6.9	-0.00034	-0.00026
7.2	-0.00018	-0.00020
7.5	-0.00007	-0.00017
7.8	0.00000	-0.00014
8.1	0.00004	-0.00012
8.4	0.00007	-0.00011
8.7	0.00008	-0.00010
9.0	0.00008	-0.00009
9.3	0.00007	-0.00008
9.6	0.00007	-0.00007
9.9	0.00006	-0.00006

7.5 The Nucleon Nucleon Optical Potential

The notion of an optical model is useful in cases when the S-matrix is not unitary and flux disappears into open inelastic or reaction channels. The optical model is often expressed in terms of a complex and energy dependent local potential where the imaginary part effectively describes the loss of flux without specification of the inelastic channels. A less popular alternative to a complex optical model potential is the introduction of pseudo channels. Here we follow the optical potential approach [256] with the underlying reaction scheme shown in Fig. 7.4.

The source of the BB channel (dibaryon formation), in the NN core domain, is mediated by a delta-function or a narrow Gaussian function. Intrinsic nucleon excitations are also mediated by a narrow Gaussian, also in the core domain. Inelasticities are either generated by coupling BB (dibaryon) to asymptotic many body final states, composed of two nucleons and mesons, or decay of XY, composed of one or two intrinsic nucleon excitations, into asymptotic many body final states. Within the inner core region, $r < r_c$, NN and XY wave function components are strongly attenuated either by a real potential which leaves no room for a remaining CM kinetic energy, or a strong absorption.

The optical model, or coupled channel treatment, suggest strongly a limitation of the meson exchange mechanism to $r > r_c$. The meson exchange *Dirac potential*, which is described by the NN Dirac instant form dynamics, should ultimately be *limited to* $r \geq r_c \sim 0.5\,fm$ *in its effect.* This constraint eliminates the need for regularization of the short range Dirac potential and boundary conditions are automatically generated by the localized $\delta(r-r_c)\ NN \leftrightarrow BB$ transition potentials. This proposal is demonstrated in Fig. 7.5. The strengths and location of δ-function interactions establish boundary conditions which are to be determined by BB and XY models. Herein lies the essential point of our new model. Dirac potentials play only the role of a weakly distorting shield which prevents us from seeing the naked refinement surface of hadronic QCD dynamics- it recalls the P-matrix formalism. A realization of the full coupled channels problem which replaces a particular optical model by explicitly treated $\Delta(3,3)$ channels, is in progress.

Without repeating the specification of details, the coupling scheme has the structure shown in Fig. 7.4 Below pion production threshold, $T_{Lab} \sim 280\,\mathrm{MeV}$, the NN S-matrix is unitary for all practical purposes. Above this energy, excitation of Δ(3,3), most likely two Δ(3,3) resonances are the predominant mechanism. It is *obviously present* in the NN $^1D_2$, $^3F_3$ and $^3PF_2$ channels. Isospin conservation suppresses a coupling to NΔ in the *np* $T = 0$ channels. NN scattering, for energies below 3 GeV in general show, compared with nucleon-nucleus scattering, a weak and smoothly energy dependent coupling to inelastic channels. A perturbative treatment of inelastic and reaction channels with DWBA methods is thus certainly justified.

A key issue for all secondary applications, of NN scattering, is a high quality

$$\mathcal{H}_{BB} + \mathcal{V}_{BB}^{NN}\delta(r - r_c) + \mathcal{V}_{BB} \qquad \text{Dibaryon}$$

$$\updownarrow$$

$$\mathcal{V}_{NN}^{BB}\delta(r - r_c) + \mathcal{V}_{NN}^{XY}g(r - r_c) + H_{NN} + V_{NN}^{\text{Dirac}} \quad \text{NN elastic}$$

$$\updownarrow$$

$$\mathcal{V}_{XY}^{NN}g(r - r_c) + \mathcal{H}_{XY} + \mathcal{V}_{XY} \quad \text{N}^\star\text{N}^\star\text{decay.}$$

Figure 7.4: Coupled channels reaction scheme.

reproduction of the elastic NN scattering channel. Inverse scattering methods are useful for this purpose. These methods use the experimental data in form of partial wave phase shifts as input and determine the optical model potential as a correction to a theoretically defined and numerically realized reference potentials.

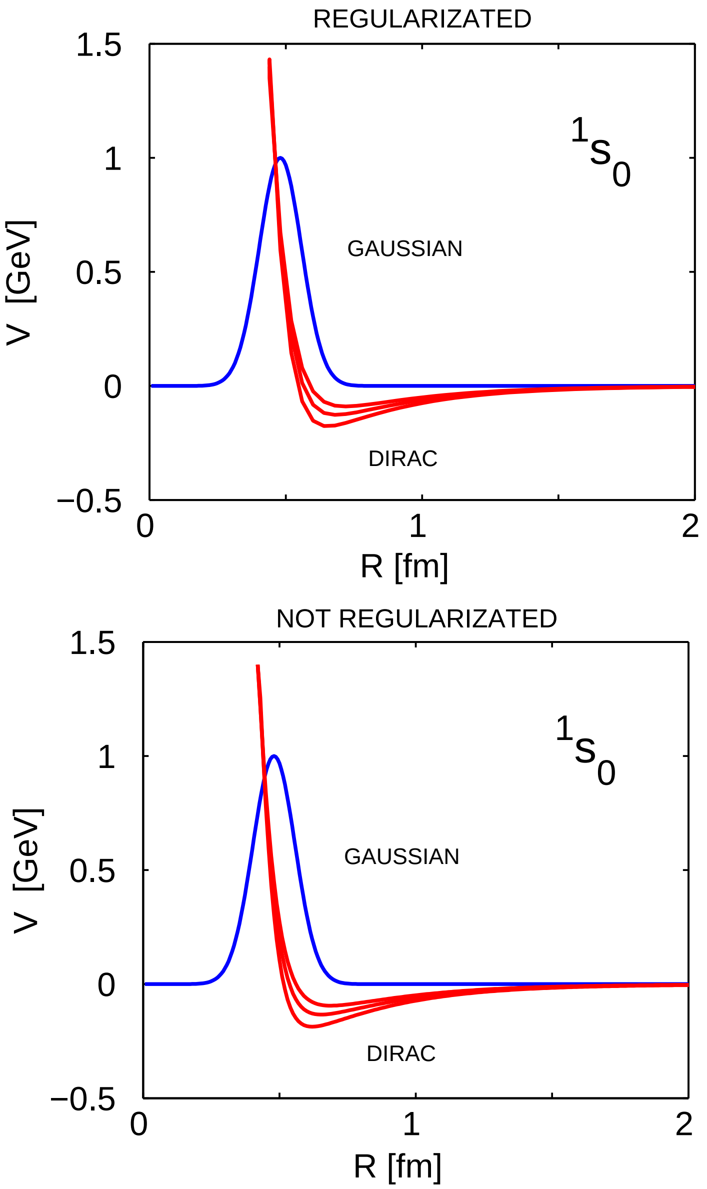


Figure 7.5: Dirac potential (red), upper figure regularized near origine and lower figure, not regularized near origine, for the np $^1S_0$ channel, T_{Lab} 0.1, 1 and 2 GeV, showing a long range OPEP tail, an attractive pocket $\sim 0.75\,\mathrm{fm}$ and a core repulsion with $r_c = 0.5 \pm 0.025\,\mathrm{fm}$ (blue). The regularized and not regularized potentials show, in all channels, only small differences. Regularization does not affect the conclusion drawn about the core geometry but helps to keep numbers reasonable near the origin. Also inserted are Gaussian form factors $g(r - r_0) \sim \exp(-(r - r_0)^2/a^2)$ which are used with the optical model (blue line curve). In this figure $r_0 = r_c = 0.5\,\mathrm{fm}$ and $a = 0.2\,\mathrm{fm}$.

Table 7.3: The meson parameters for Fuda instant form potential [251, 252, 253].

Mesons	π	η	σ	ρ	ω	δ
Channel	138.03	548.8	516.77 (674.72)	769	782.6	983.0
	13.83	2.83	6.95(13.01)	0.88	23.55	3.10

Table 7.4: The meson parameters for CD-BONN potential.

Mesons	π	η	σ	ρ	ω	δ
Channel	138.03	547.45	400 -	768.5	782.6	983.0
$^1S_0$	13.6	-	3.96451(452)	0.84	20	
$^3P_0$	13.6	-	7.866(560)	0.84	20	
$^3P_1$	13.6	-	2.346(424)	0.84	20	
$^1D_2$	13.6	-	2.236(400)	0.84	20	
$^3F_3$	13.6	-	1.53(-)	0.84	20	

Table 7.5: The meson parameters for Dirac potential.

Mesons	π	η	σ	ρ	ω
Channel	138.03	547.45	280-780	768.5	782.6
$^1S_0, ^1D_2, ^1G_4, ^1I_6$	13.84	3.00	9.95(577.44)	-	0.93
$^3P_0$	13.84	3.00	1.4(500.00)	-	20.0
$^3P_1$	13.84	3.00	2.72(437.00)	-	7.93
$^3F_3, ^3H_5$	13.84	3.00	2.48(500.00)	-	6.73
$^1P_1, ^1F_3, ^1H_5,$	13.84	3.00	6.52(500.00)	-	-
$^3D_2, ^3G_4, ^3I_6$	13.84	3.00	9.7(500.00)	-	20.0
$^3PF_2$	13.84	3.00	3.28(500.00)	-	3.66
$^3FG_4$	13.84	-	1.4(500.00) 26.6(780.00)	-	14.28
$^3HJ_6$	13.84	3.00	6.66(550.00)	-	14.63
$^3SD_1$	13.84	3.00	9.6(500.00)	-	-10.51
$^3DG_3$	13.84	3.00	5.25(500.00) 0.63(280.00)	-	
$^3GI_5$	13.84	3.00	2.43(383.00)	-	

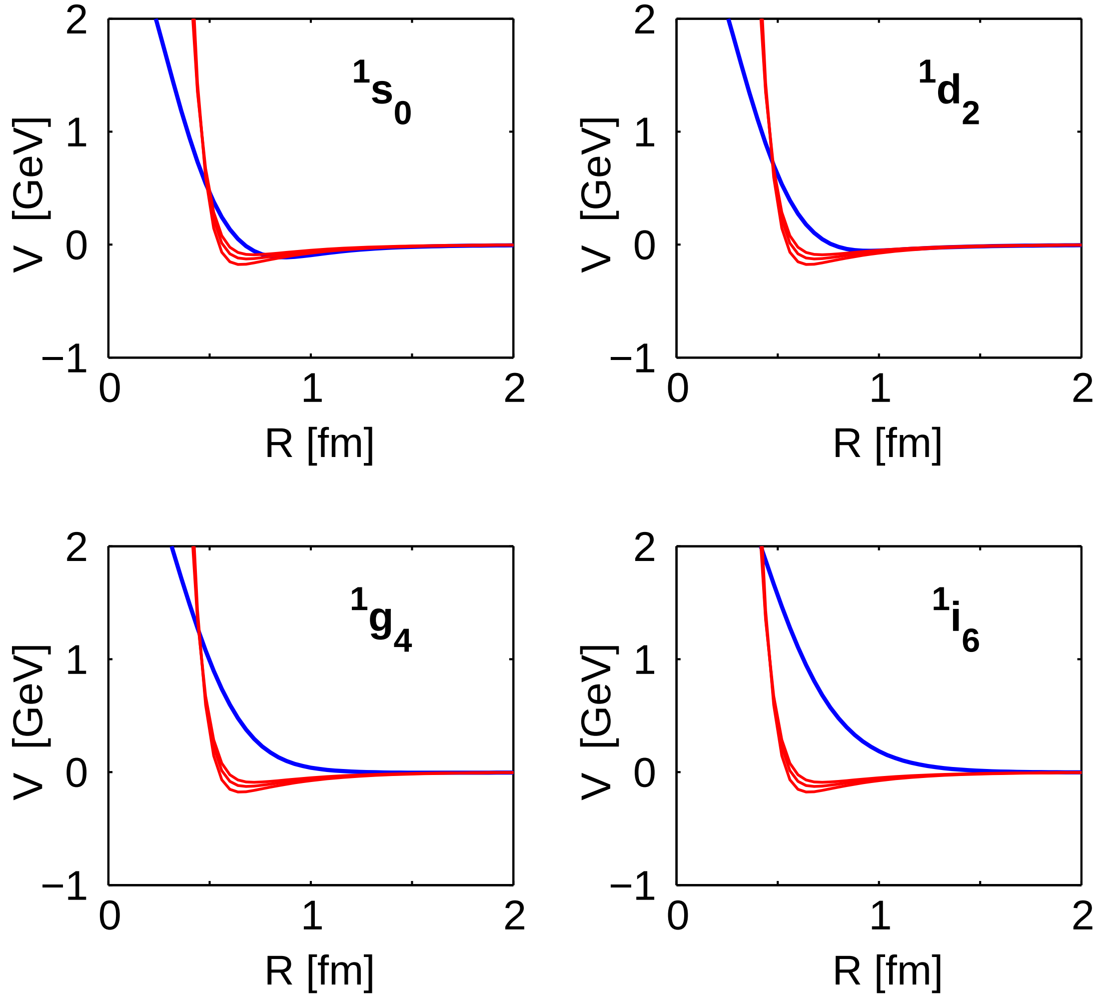


Figure 7.6: Energy dependent Dirac potentials (red) for singlet $S = 0$, $T = 1$ channels with T_{Lab} 0.1, 1 and 2 GeV. The AV18 channel potential is shown as blue line.

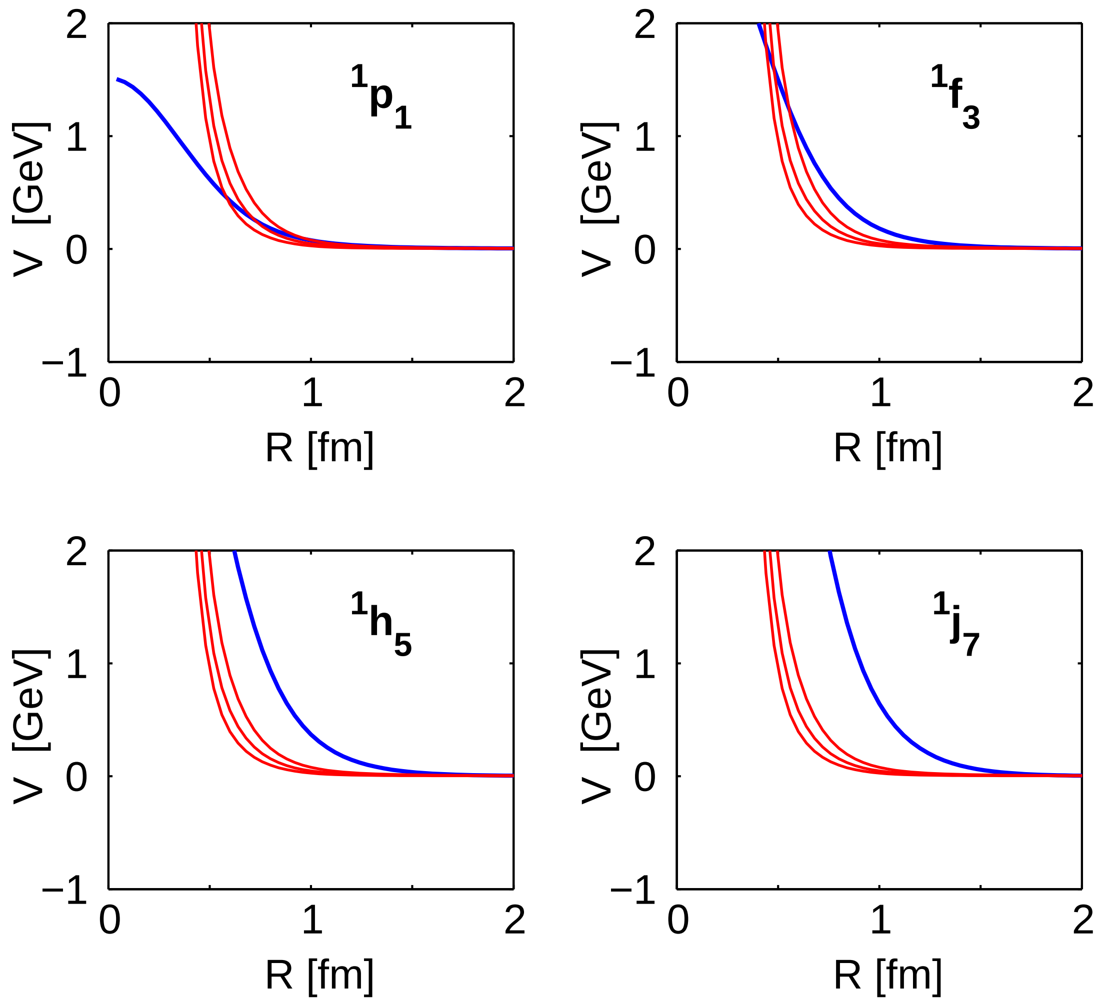


Figure 7.7: Energy dependent Dirac potentials (red) for singlet $S = 0$, $T = 0$ channels with T_{Lab} 0.1, 1 and 2 GeV. The AV18 channel potential is shown as blue line.

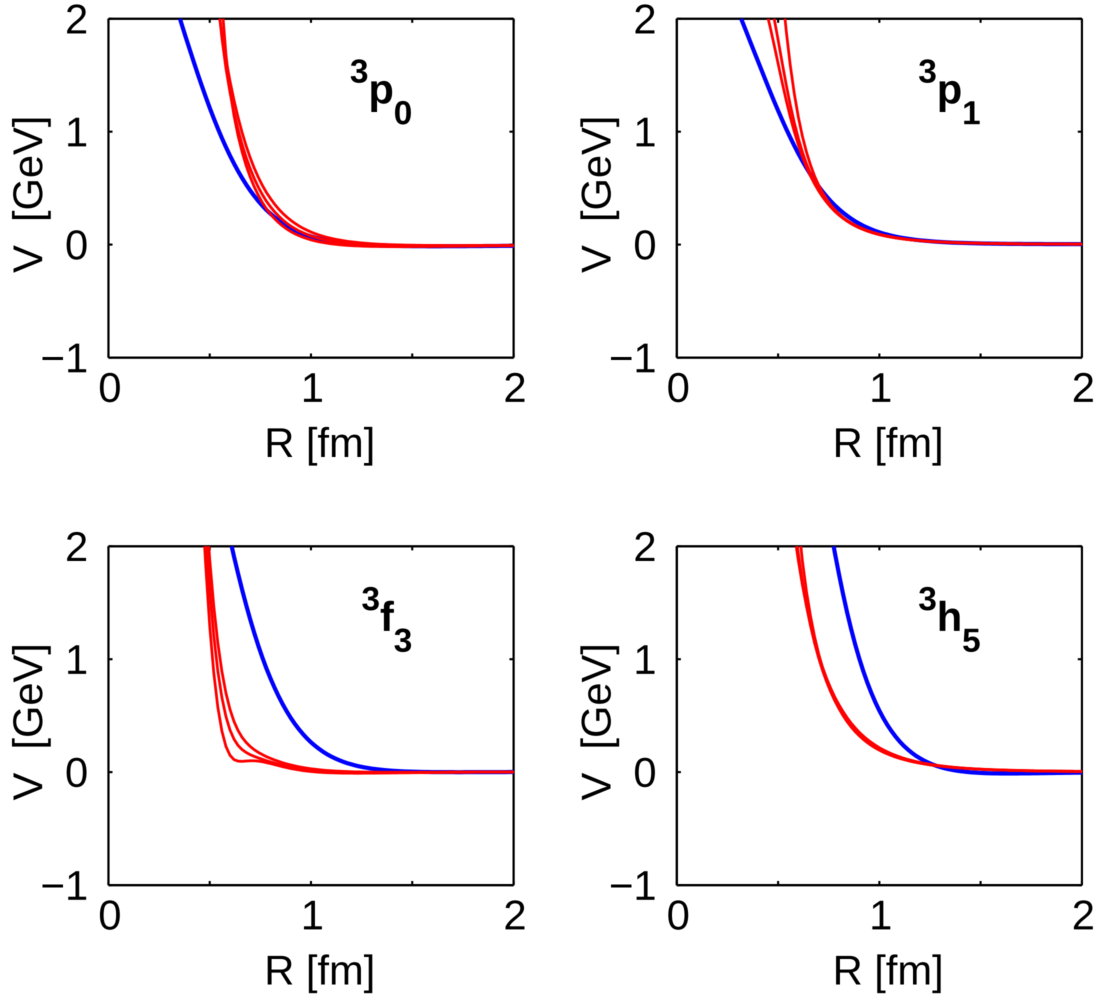


Figure 7.8: Energy dependent Dirac potentials (red) for triplet $S = 1$, $T = 1$ uncoupled channels with T_{Lab} 0.1, 1 and 2 GeV. The AV18 channel potential is shown as blue line.

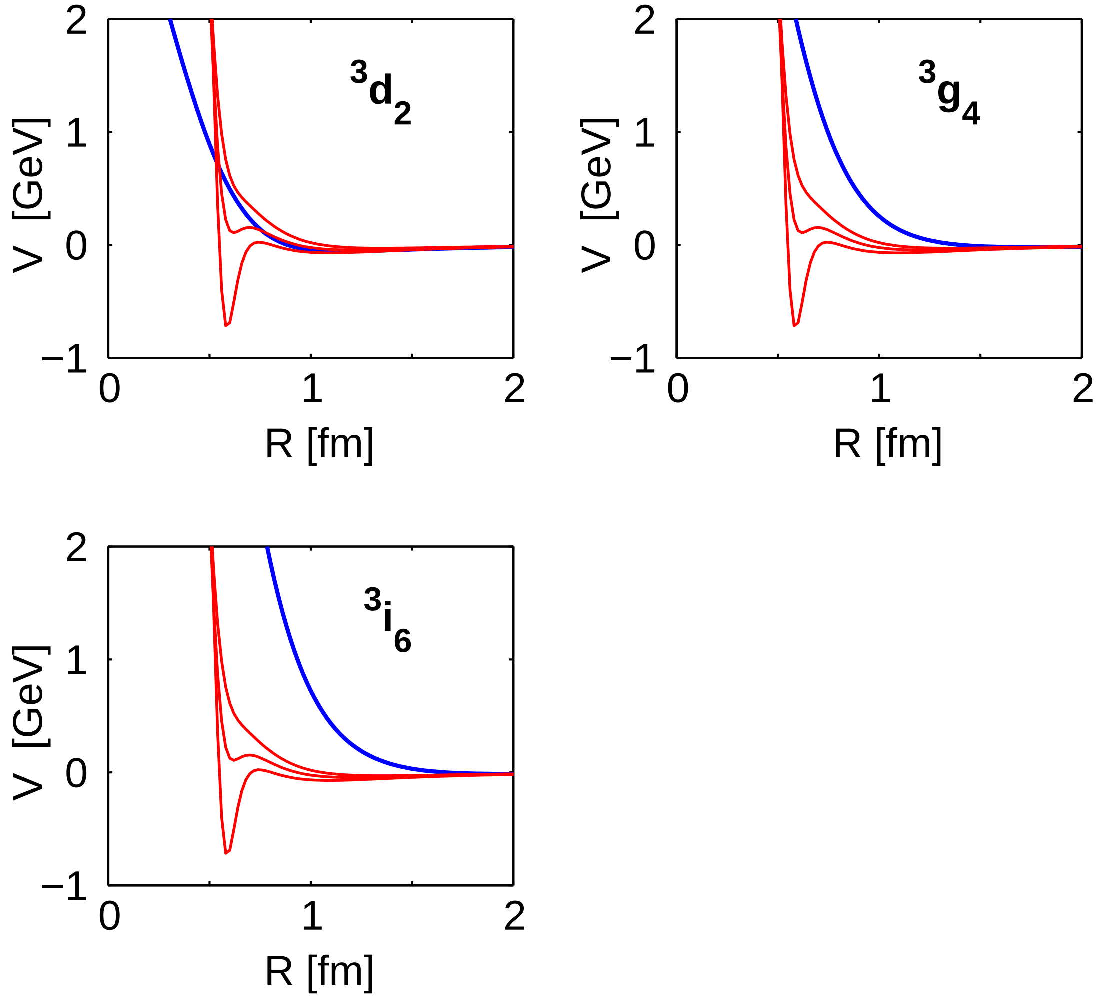


Figure 7.9: Energy dependent Dirac potentials (red) for triplet $S = 1$, $T = 0$ uncoupled channels with T_{Lab} 0.1, 1 and 2 GeV. The AV18 channel potential is shown as blue line. A comparison of these Dirac potentials shows surprisingly an attractive peak outside the hard core radius. This feature is not visible in AV18. Its narrow geometry scale is 0.2 fm, when compared with the nucleon of 1 fm. We suggest to associate it with a QCD mechanism for which the OMP-3 gives further details, see Subsec. 7.6.2, and Fig. 7.27-7.28, but no complete understanding.

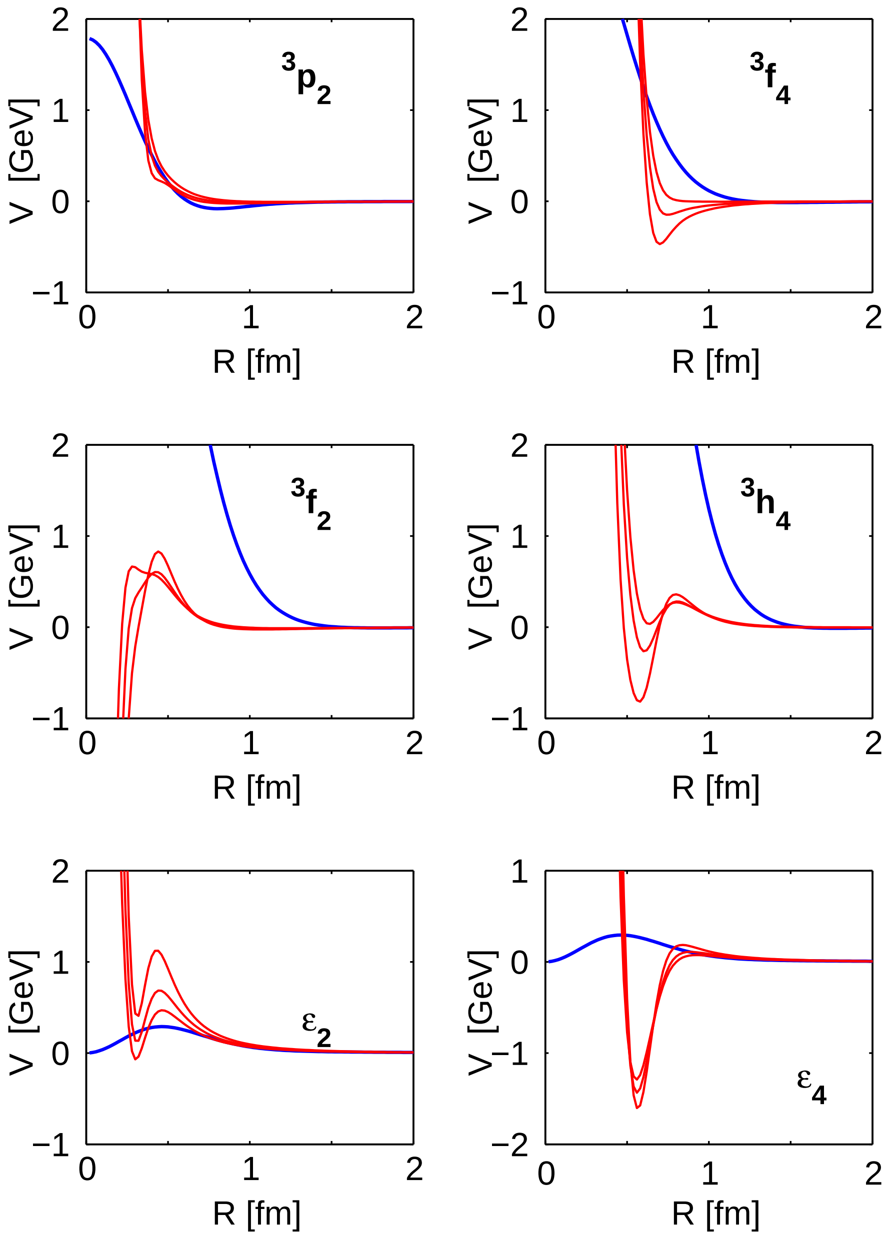


Figure 7.10: Energy dependent Dirac potentials (red) for triplet $S = 1$, $T = 1$ coupled channels with T_{Lab} 0.1, 1 and 2 GeV. The AV18 channel potential is shown as blue line. These channels show a large variation in the Dirac potentials. The pocket around the hard core radius changes dramatically and in the partial wave $^3F_2$ we get a short range attraction. The fit to the $^3PF_2$ data was notoriously difficult and not satisfying when compared with the other channel results. Together with $^1D_2$ and $^3F_3$, the $^3PF_2$ channel shows the strongest and divers effect in the QCD transition surface. This feature is not a question of adding and fitting heavier mesons into OBEP.

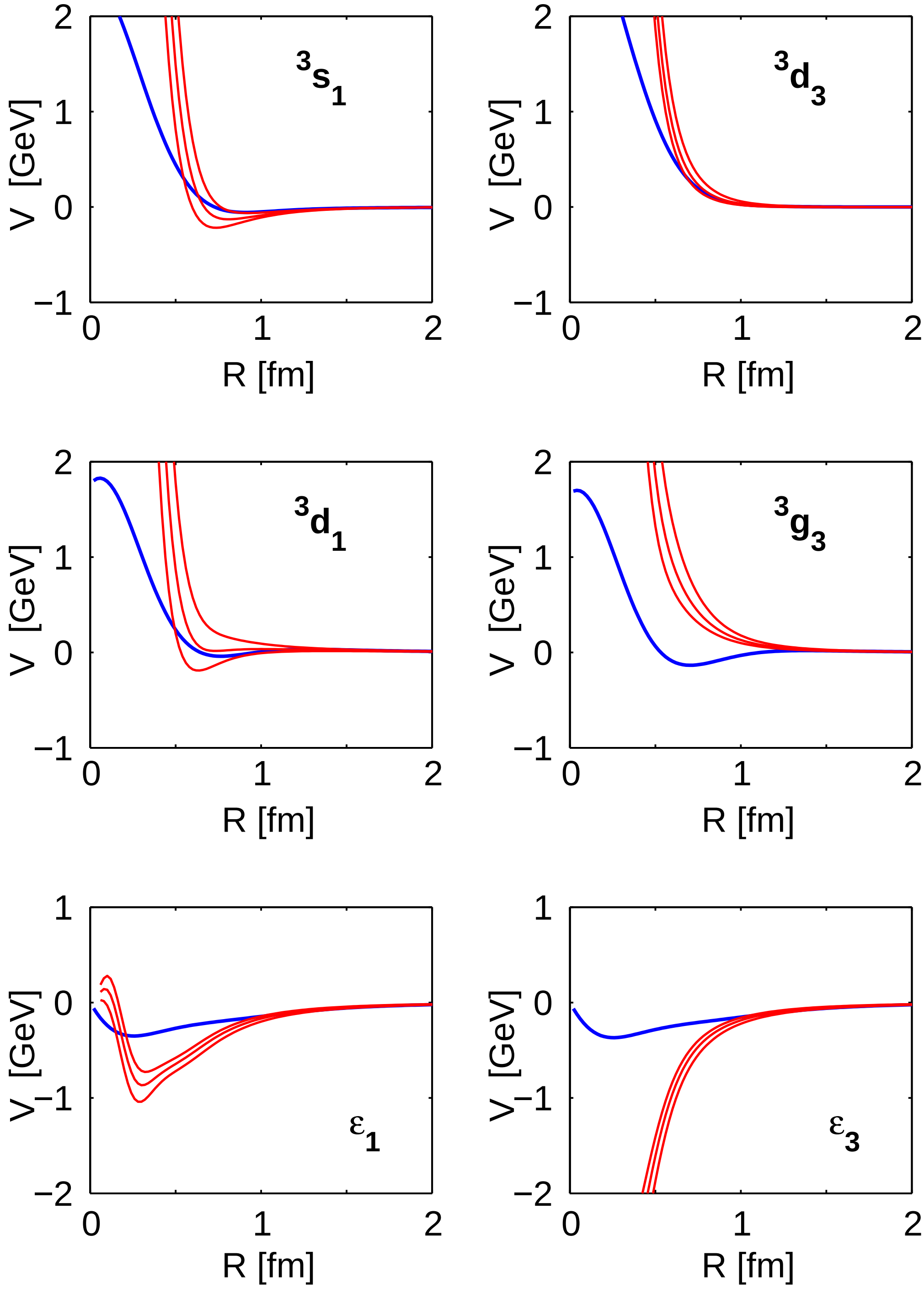


Figure 7.11: Energy dependent Dirac potentials (red) for triplet $S = 1$, $T = 0$ coupled channels with T_{Lab} 0.1, 1 and 2 GeV. The AV18 channel potential is shown as blue line.

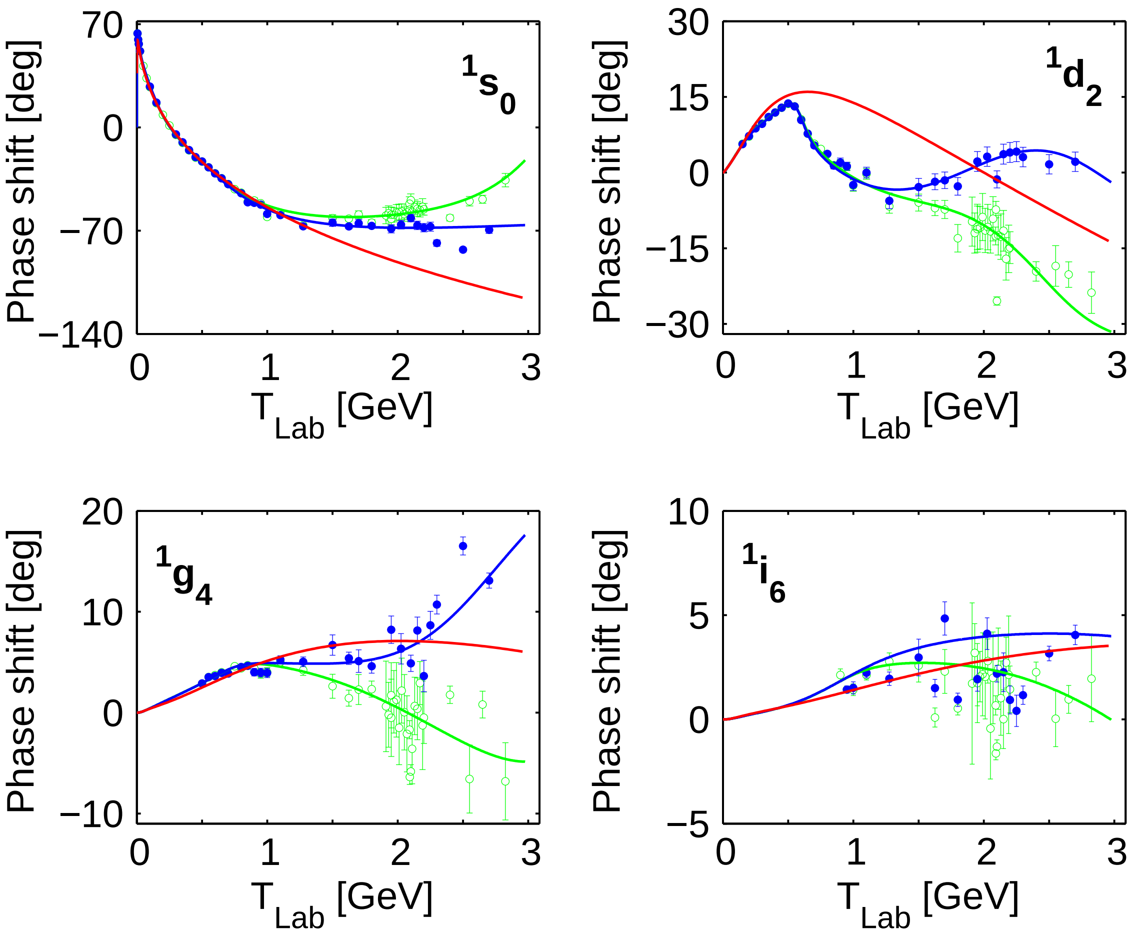


Figure 7.12: Singlet channels $S = 0$, $T = 1$, np [0,3] GeV, SM00; continuous solution (green), single energy solutions, open green circles, with error bars, SP03 continuous solution (blue), single energy solutions, full blue circles, with error bars; real phase shift data $\delta(T)$ and theoretical results; real Dirac potential solutions, full red line, real Dirac potentials are *coinciding* with the data of SP03 blue line.

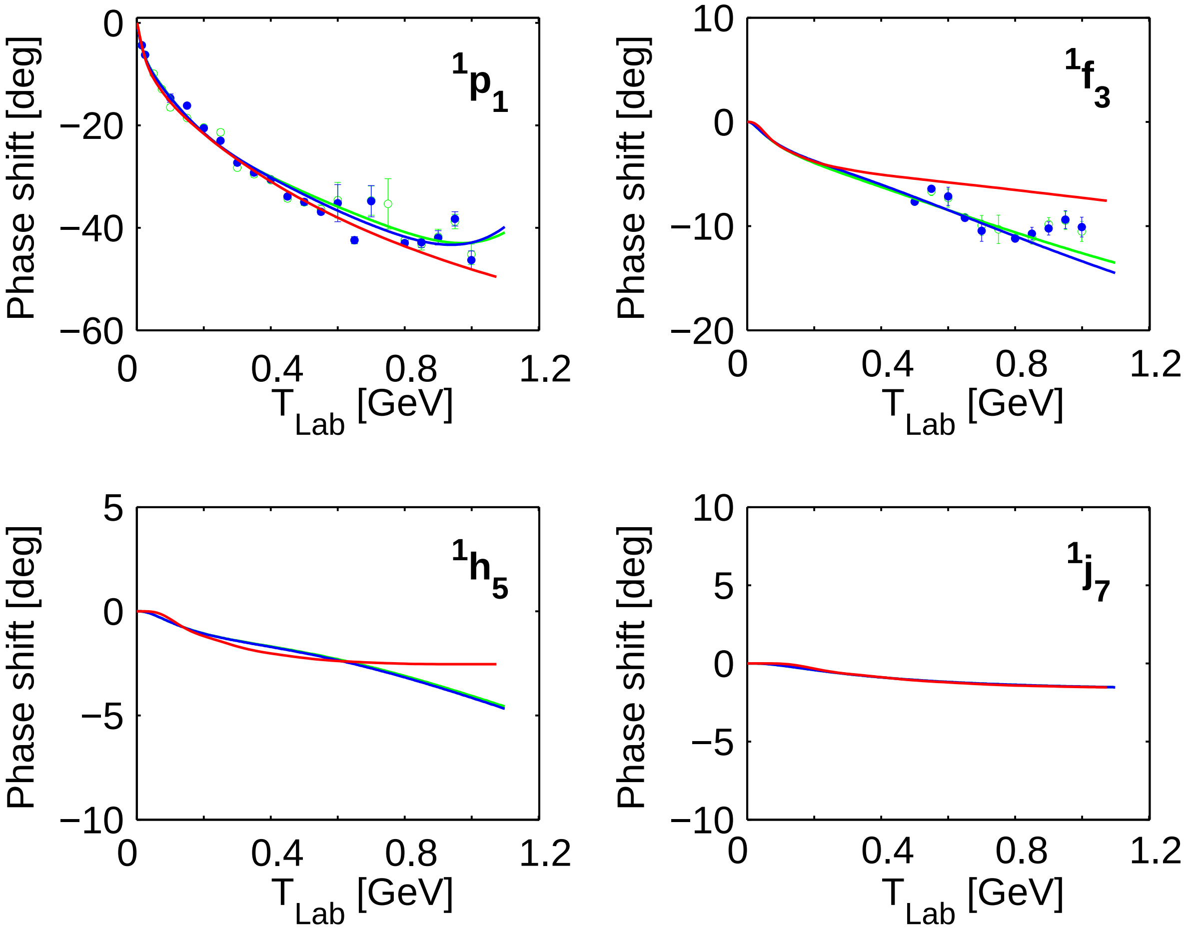


Figure 7.13: Singlet channels $S = 0$, $T = 0$, np [0,1.2] GeV, SM00; continuous solution (green), single energy solutions, open green circles, with error bars, SP03 continuous solution (blue), single energy solutions, full blue circles, with error bars; real phase shift data $\delta(T)$ and theoretical results; real Dirac potential solutions, full red line, real Dirac potentials are *coinciding* with the data of SP03 blue line.

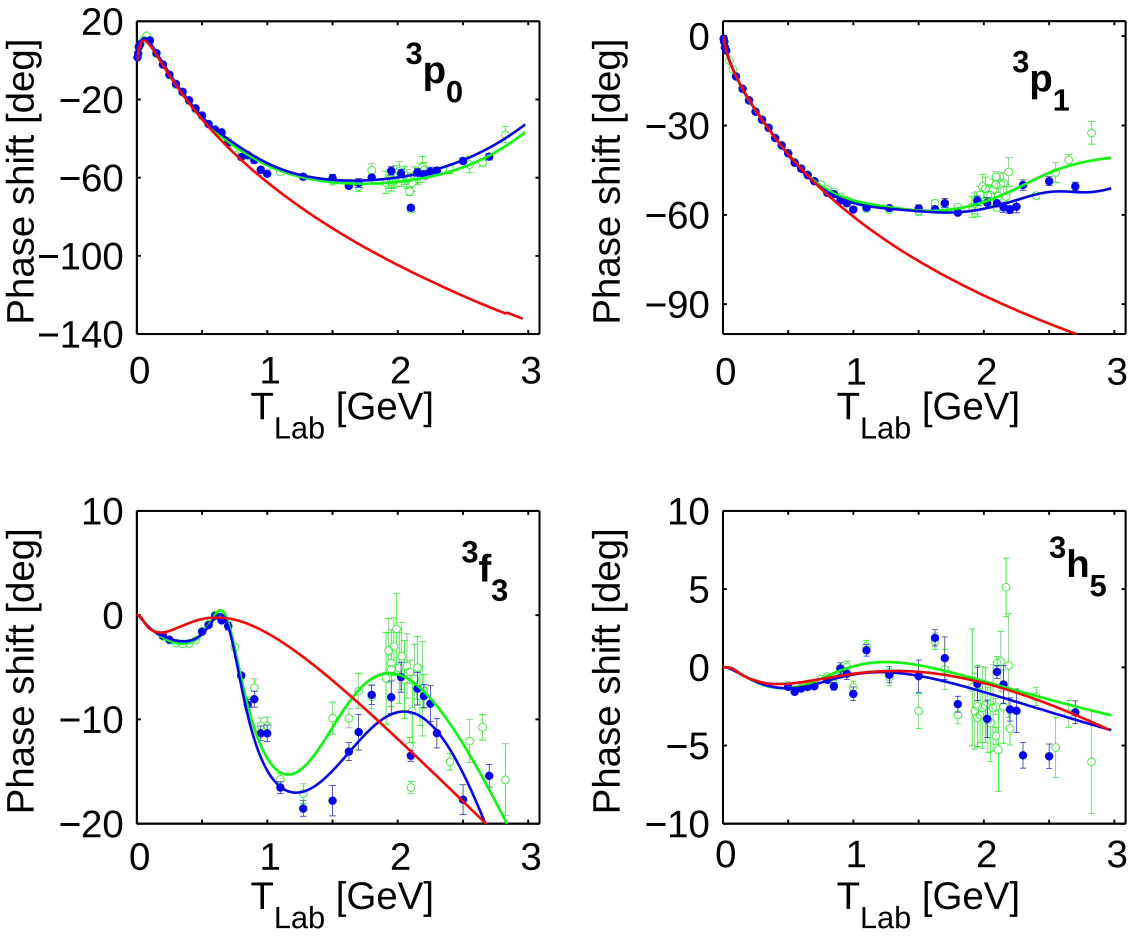


Figure 7.14: Triplet channels $S = 1$, $T = 1$, np [0,3] GeV, SM00; continuous solution (green), single energy solutions, open green circles, with error bars, SP03 continuous solution (blue), single energy solutions, full blue circles, with error bars; real phase shift data $\delta(T)$ and theoretical results; real Dirac potential solutions, full red line, real Dirac potentials are *coinciding* with the data of SP03 blue line.

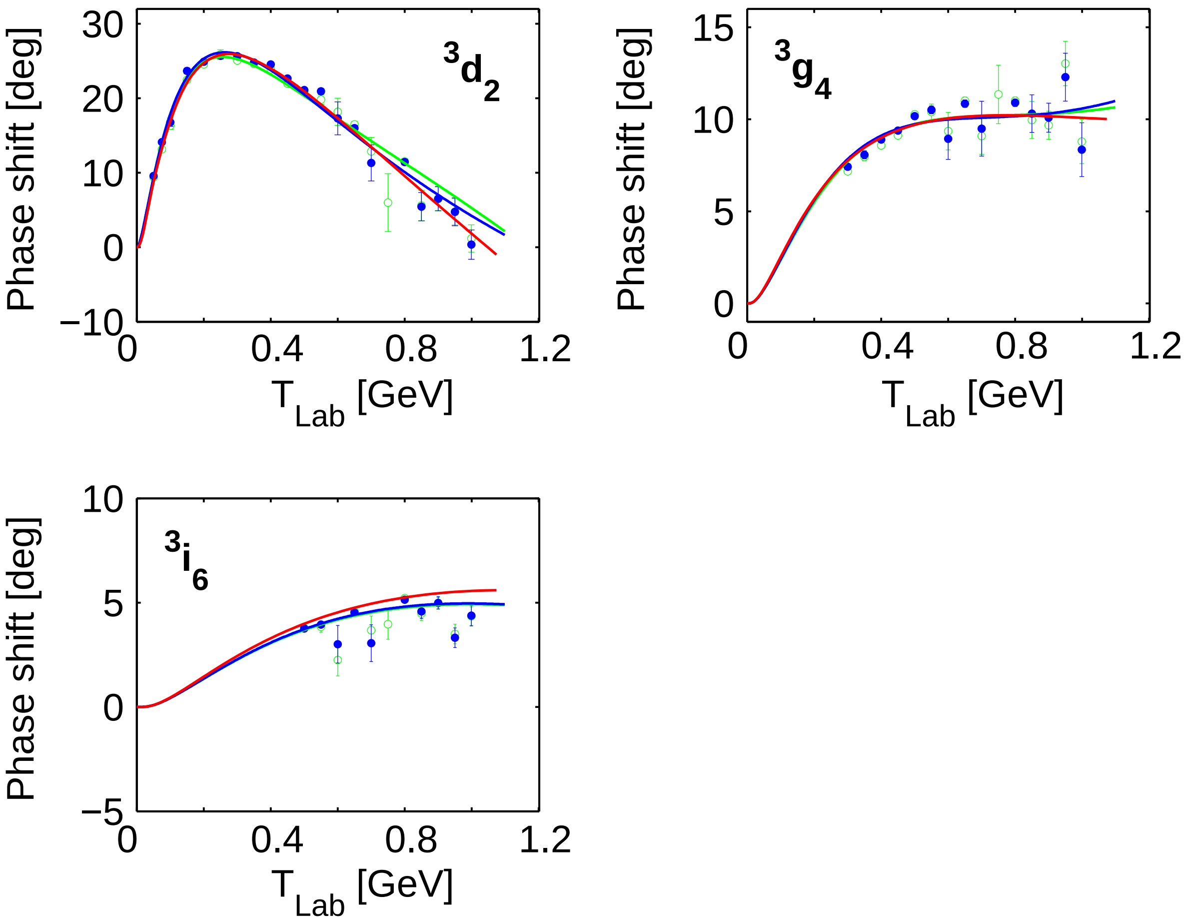


Figure 7.15: Triplet channels $S = 1$, $T = 1$, np [0,1.2] GeV, SM00; continuous solution (green), single energy solutions, open green circles, with error bars, SP03 continuous solution (blue), single energy solutions, full blue circles, with error bars; real phase shift data $\delta(T)$ and theoretical results; real Dirac potential solutions, full red line, real Dirac potentials are *coinciding* with the data of SP03 blue line.

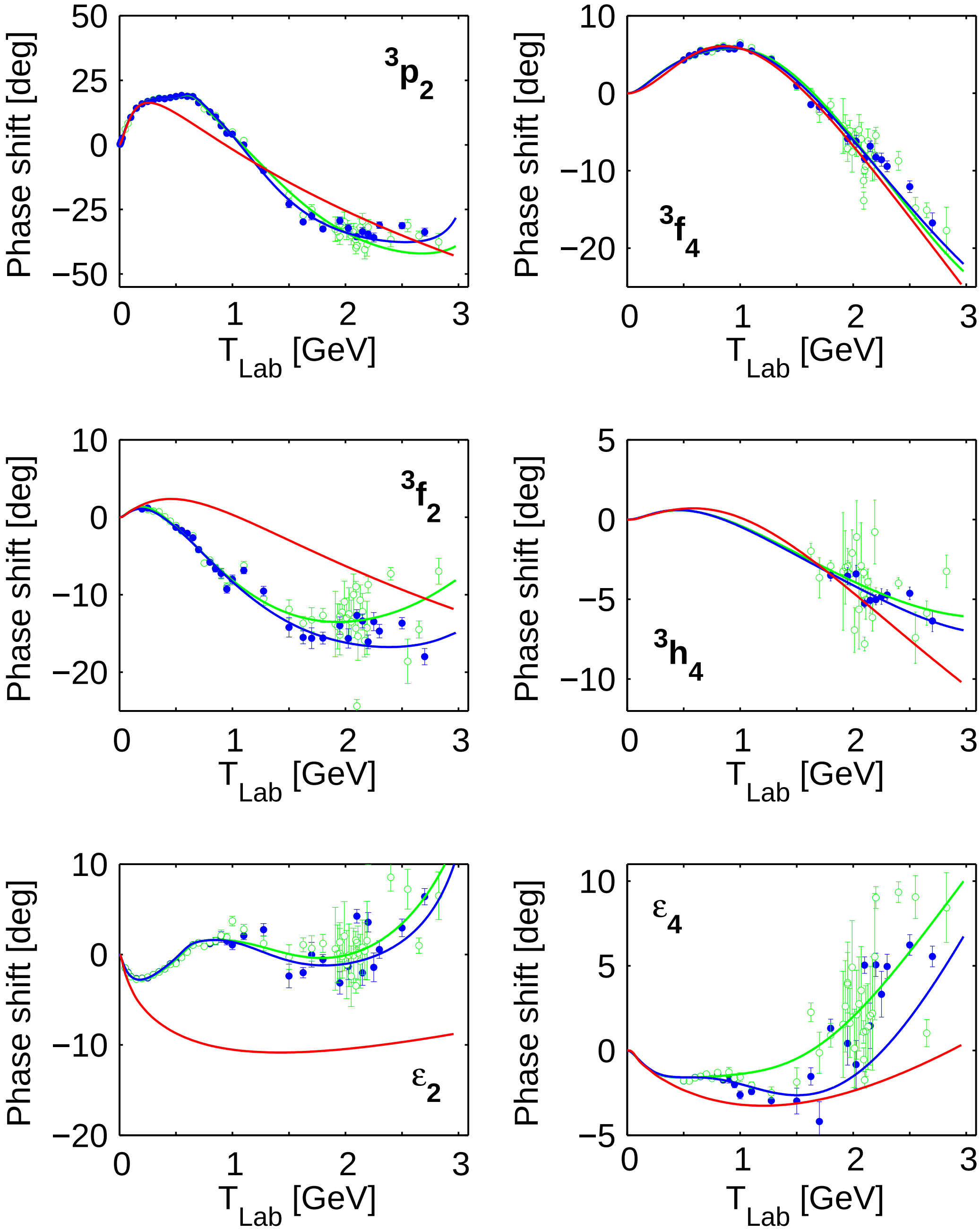


Figure 7.16: Coupled channels $S = 1$, $T = 1$, np [0,3] GeV, SM00; continuous solution (green), single energy solutions, open green circles, with error bars, SP03 continuous solution (blue), single energy solutions, full blue circles, with error bars; real phase shift data $\delta(T)$ and theoretical results; real Dirac potential solutions, full red line, real Dirac potentials are *coinciding* with the data of SP03 blue line.

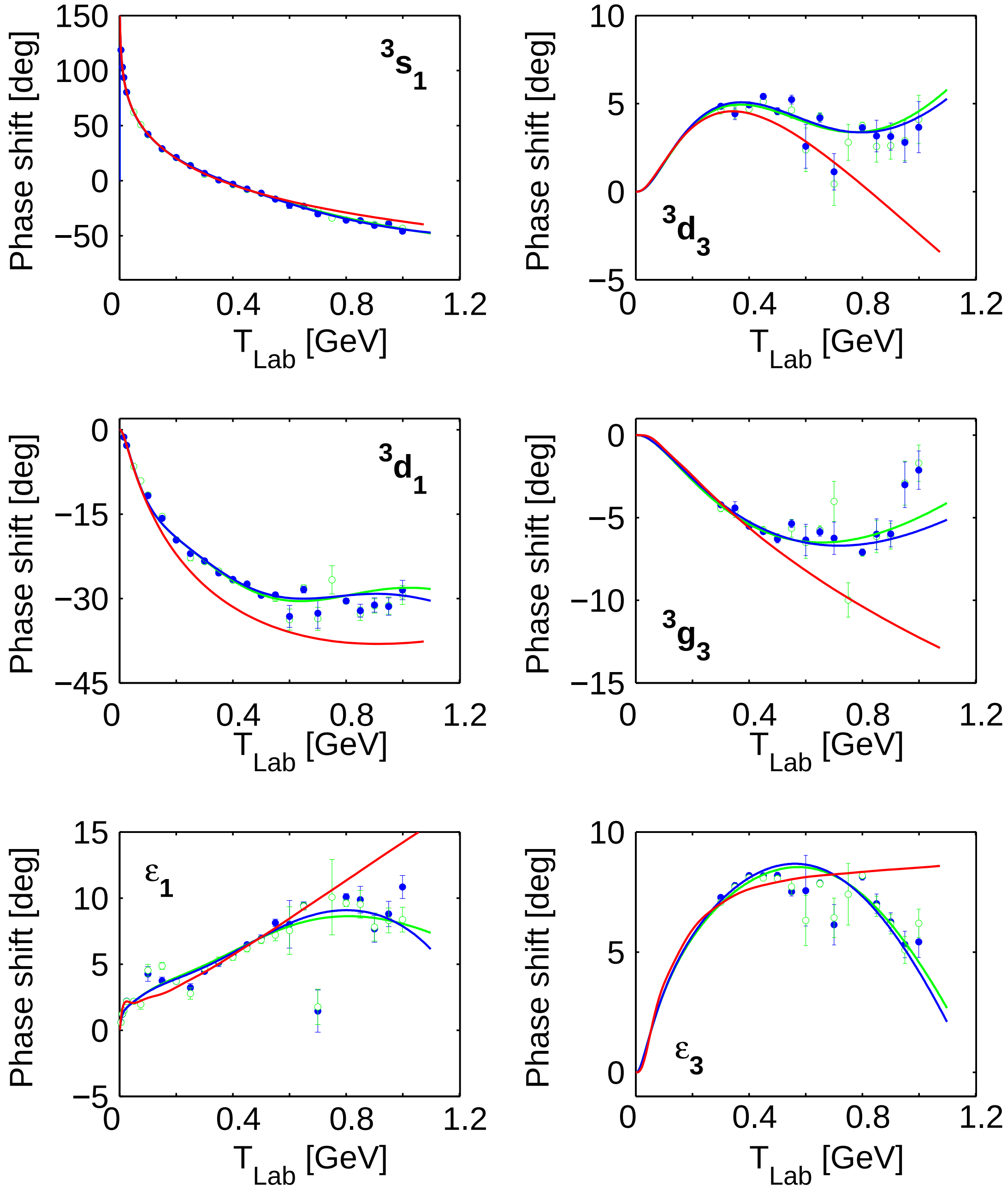


Figure 7.17: Coupled channels $S = 1$, $T = 1$, np [0,1.2] GeV, SM00; continuous solution (green), single energy solutions, open green circles, with error bars, SP03 continuous solution (blue), single energy solutions, full blue circles, with error bars; real phase shift data $\delta(T)$ and theoretical results; real Dirac potential solutions, full red line, real Dirac potentials are *coinciding* with the data of SP03 blue line.

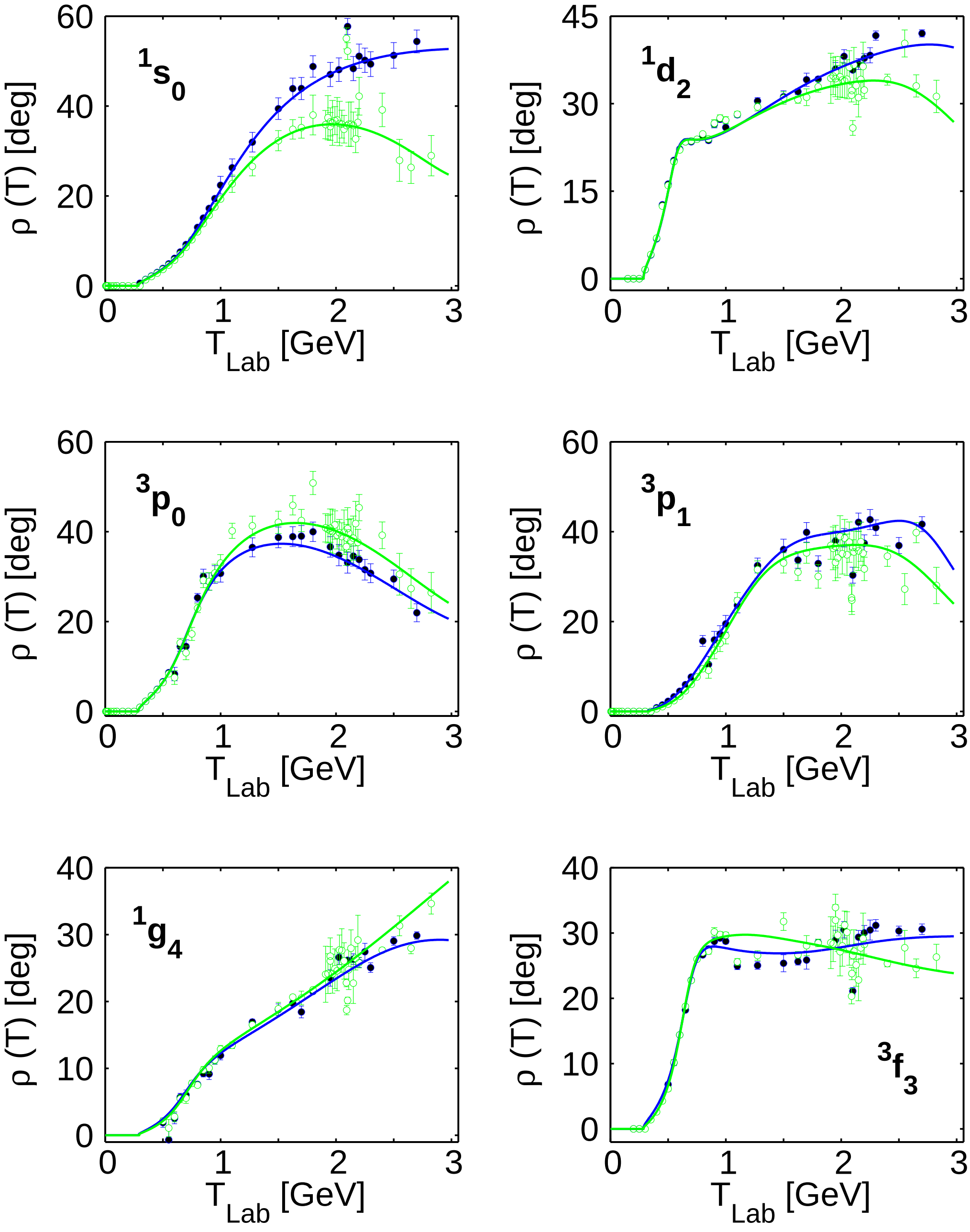


Figure 7.18: Available absorption phase shifts $\rho(T)$. Dirac potentials generate no absorption and OMP-1, OMP-2 and OMP-3 parameters are adjusted to reproduce the continuous energy solutions of $\rho(T)$. SM00, continuous solution (green) single energy solutions (open green circles with error bars) and SP03 continuous solution (blue) single energy solutions (full blue circles with error bars).

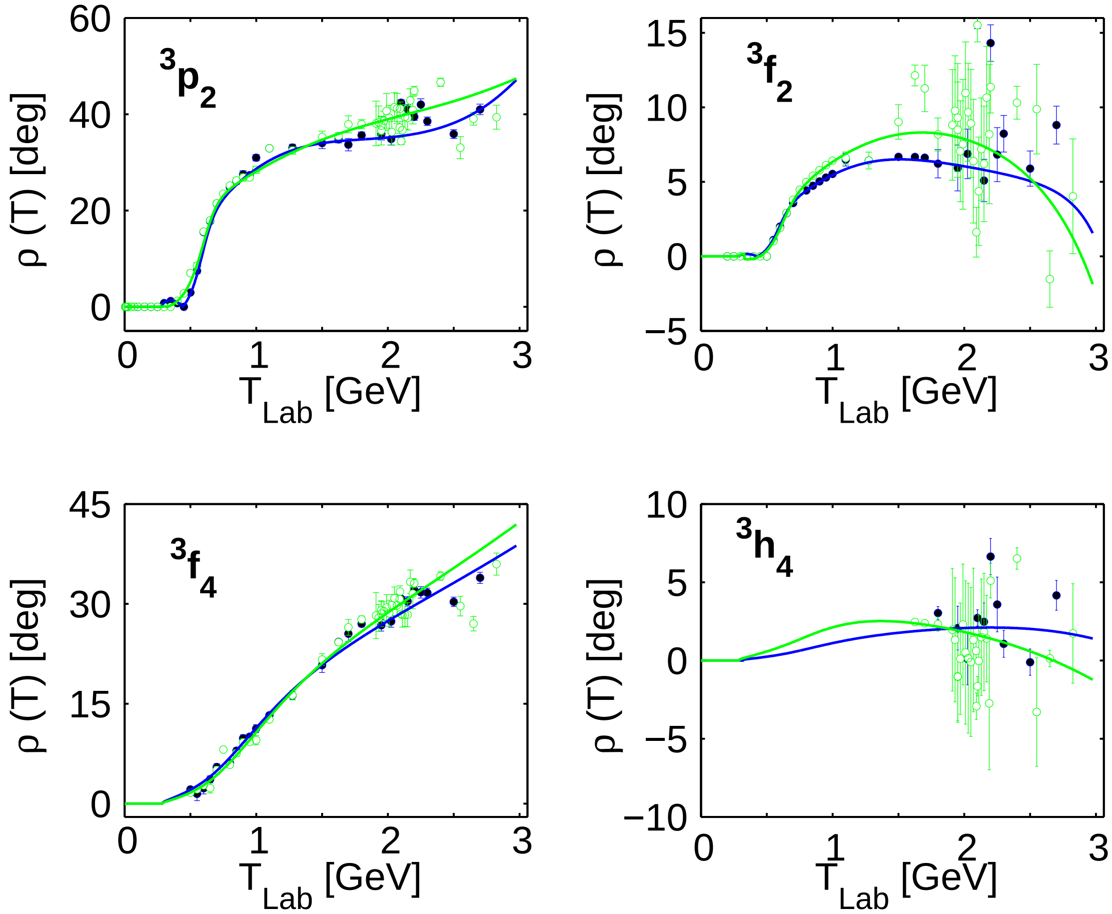


Figure 7.19: Continuation of Fig. 7.18.

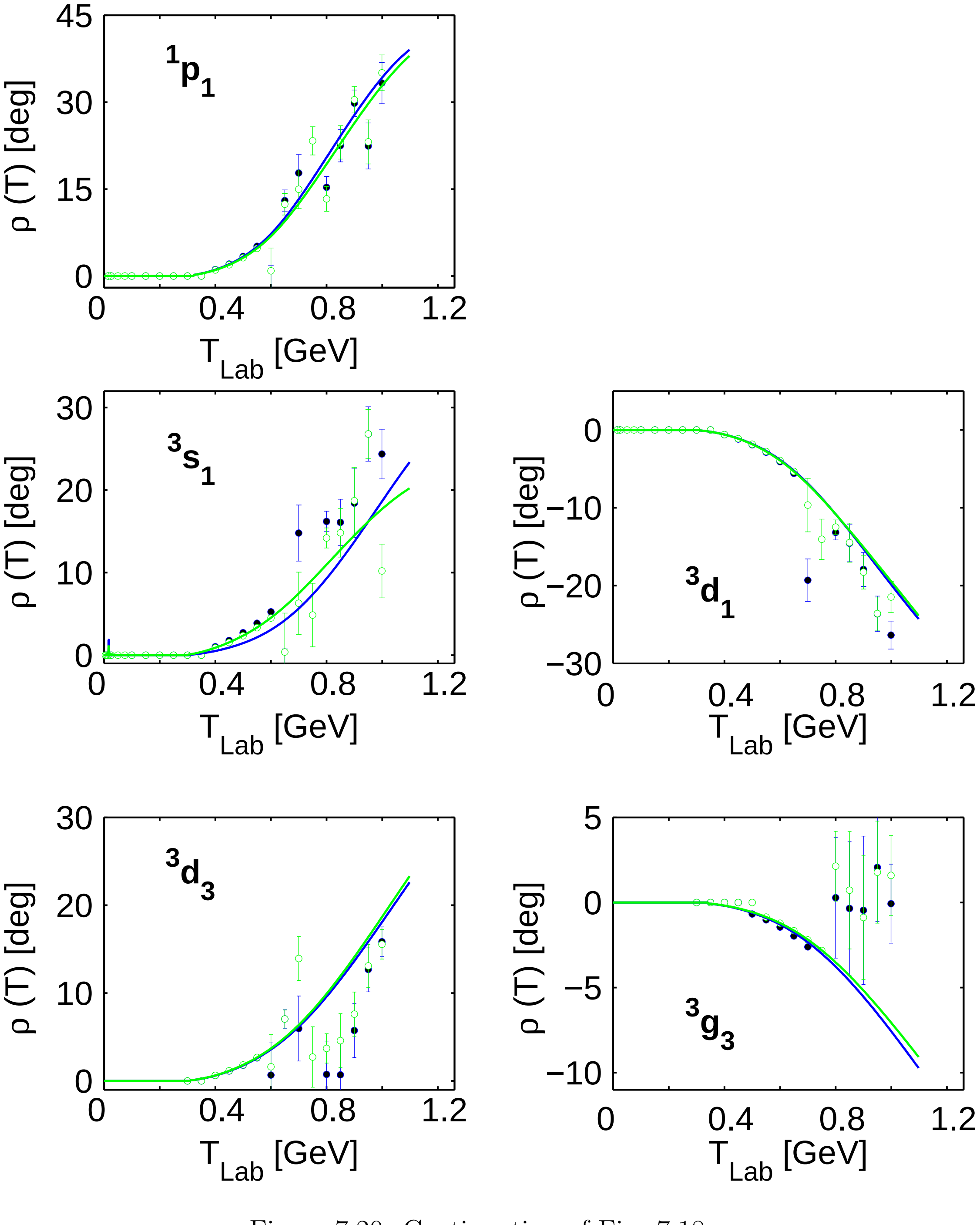


Figure 7.20: Continuation of Fig. 7.18.

7.5.1 $^1S_0$, $^3P_0$ and $^3P_1$ Channels

NN phase shift data, see Figs. 7.12-7.17 and absorption 7.18, show in almost all channels and for $T_{Lab} > 280\,\text{MeV}$ a complicated energy dependence and deviations from the Dirac potential predictions. Exceptional cases are the $^1S_0$, $^3P_0$ and $^3P_1$ channels, shown in Figs. 7.12, 7.14. They show a *practical perfect reproduction*

$$\delta_{Data} = \delta_{Dirac}, \quad \text{for} \quad 0 < T_{Lab} < 1100\,\text{MeV} \tag{7.7}$$

by the Dirac potential alone. However, the absorption

$$\rho_{Data} \neq 0, \quad \text{whereas} \quad \rho_{Dirac} = 0 \quad \text{for} \quad 280 < T_{Lab} < 1100\,\text{MeV}. \tag{7.8}$$

This demands an *optical potential* $V_{_{OMP}} + iW_{_{OMP}}$ that leaves the real phase shifts $\delta(k)$ unchanged but generates an absorption $\rho(T) > 0$ for $280 < T_{Lab} < 1100\,\text{MeV}$. As optical model interaction, we used OMP-1, OMP2 and OMP-3/or a narrow normalized Gaussian, see Fig. 7.5,

$$W(r) = \begin{cases} W_0\, N_G(r_0, a_0)\exp(-(r-r_0)^2/a_0^2), \\ W_\delta\, \delta(r-r_0), \end{cases} \tag{7.9}$$

with $V_{_{OMP}} = 0$. The following conclusions are drawn: The phase shifts $\delta(T)$, $\rho(T)$ for $280 < T_{Lab} < 1100$ MeV imply an optical potential at the surface of the repulsive core for $^1S_0$, $^3P_0$ and $^3P_1$ partial waves. Intermediate dibaryons are practically not formed, the BB channel is realized by a dibaryon fusion/scission picture as shown in Fig. 7.21. The BB dibaryon quark dynamic is reduced to energy dependent complex boundary conditions at the core radius permitting meson production. The meson exchange mechanism Fig. 7.22 is not valid inside the core radius. Caveat, at this stage of our work, we integrated from the origin through the core region realizing a small real wave function at the core radius [280]. In Fig. 7.23, 7.25 are shown the *np* OMP-1 strengths values. The crucial center of baryon NN and BB transition radius is $r_0 = 0.5 \pm 0.025\,\text{fm}$. At higher energies $T_{Lab} > 1.1\,\text{GeV}$, the transition surface becomes more and more faded, washed out and translucent when the energy of dibaryon states matches the total energy of the NN system. Intermediate short lived dibaryons $J^P_{BB} = 0^+,\ 0^-,\ 1^1$ are formed, see Fig. 7.21.

We estimate, from the phase behavior in these three channels, the total energy (lowest mass) of a dibaryon system $m_{BB} = 2400 \pm 150\,\text{MeV}$ and a width $\Gamma > 150\,\text{MeV}$.

Coupling to dibaryons is realized for $T_{Lab} > 1100\,\text{MeV}$ and the fusion/scission picture may change gradually into a fusion/fission picture as shown in Fig. 7.21. The hadronic pair excitation, however, is likely to dominate the dynamics as also for higher energies, Fig. 6.1.

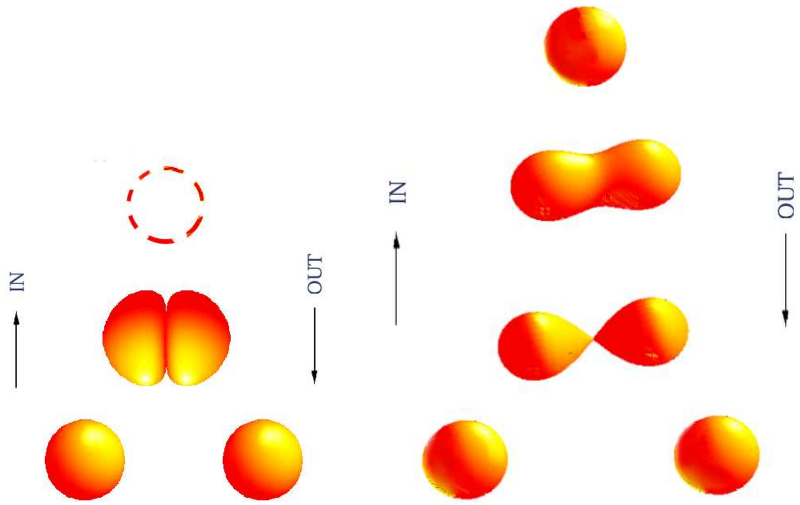


Figure 7.21: Left, caused by the Pauli exclusion principle for a six quark dibaryon and $T_{Lab,NN} < 1100\,\mathrm{MeV}$ suggests a futile ringing of the nucleons and a suppression of dibaryon formation. One or half nucleons may experience a transition, $N \mapsto \Delta(3,3)$ XY hadrons. It gives the impression of a fusion/scission mechanism. Right, the formation of dibaryons, at sufficient high energy, is governed by medium to longer ranged quark gluon flux tubes with fusion into a six quark hadron sized dibaryon with sequential decay. This inspires a fusion/fission mechanism.

7.6 First Geometrical Interpretation

One important aspect of using effective NN interactions is related to regularization at short distances. It is known that the meson theory breaks down at short range, ($r \sim 0.5$) fm, due to the intrinsic structure of the nucleons [394]. For that reason, in most meson models, the one boson exchange potentials are usually regularized to remove the singularities at the origin by introducing a form factor in a phenomenological way. Usually monopole, dipole, exponential, Woods-Saxon or other form factors [524, 574] have been used accounting for the finite size of the nucleons and pions. It has been established [523] that the most important feature is the principal range of the form factors, while their detailed analytical structure does not have a large impact on NN processes and the overall results are insensitive to the details of how the form factors have been chosen. The effective NN interactions in the medium, which are usually parameterized in terms of *meson exchange type* propagators of Yukawa shape, are clearly missing meson nucleon vertex form factors. In folding calculations for optical model and transition potentials, an averaging over the intrinsic momentum distribution of the

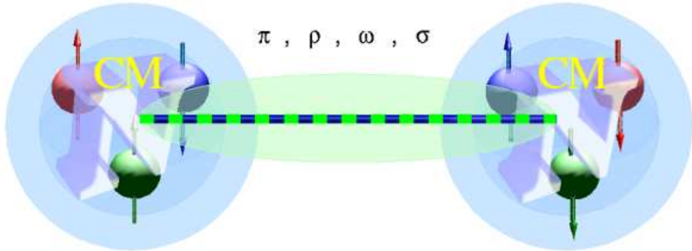


Figure 7.22: Meson exchange mechanism.

target nucleus is carried out. In these conditions the use of *on-shell* meson exchange propagators might lead to spurious effects since contributions from large off-shell momenta can be overestimated. Such uncertainties are avoided by introducing nucleonic form factors depending on the off-shellness of the nucleon and suppressing large off-shell energies and momenta [306]. Thus one may exclude contributions from small distances where the composite structure of mesons and nucleons would become visible. As phenomenological approach we choose the folding model, see Eq. (6.57).

In the previous section, we found a universal repulsive core radius with $r_c \sim$ 0.5 fm for all channels. It means that the meson exchange mechanism generates potential energies which are small in comparison with nucleon masses. However, the repulsive core reaches within a fraction of the nucleon radius several GeV and more. Such core formation implies hard momentum mechanism which cannot be taken from the nucleon as a whole but rather from its constituents.

This implies that the meson exchange mechanism should be limited to $r > r_c$ where OBEP effects remain small when compared with the nucleon mass or kinetic energies $T_{Lab} > 300$ MeV. The repulsive core, has for $T_{Lab} < 300$ MeV, no other effect but to reduce the relative wave function to extremely small values. This has no sizeable effect for low energy nuclear phenomena and high precision NN potentials [278] which give excellent fits to phase shifts data. Use of such potentials in few and many body nuclear physics produced no essential worries. This view has changed with NN models [433, 520, 401].

In this section we are studying the geometrical picture of NN scattering around the region $\sim$ 0.5 fm and the strengths and local OMP-3 in which the particle absorption processes are the relevant mechanisms where ultimately QCD effects dominate.

Our calculation is a solution of the energy dependent full relativistic equations in coordinate space.

7.6.1 Numerical Results of OMP-1 and OMP-2

In Sec. 6.3, we discussed three different OMP models, which can give us perfect fits to the experimental phase shift data. The results of OMP-1 and OMP-2 models are plotted in Figs. (7.23-7.26) with $T_{Lab} > 300$ MeV and three values of the cut-off radius R_0. The OMP-1 and OMP-2 models are very simple approaches to introduce the optical model. Its real part is proportional to the short range Dirac potential, thus V_0 controls only the strength of the short range Dirac potentials. What part of radial range acts as *short range* is controlled by R_0. The imaginary OMP is a normalized Gaussian. It can be radially shifted by R_0, the width control parameter is a. The $T = 0$ $^1P_1$ channel data are limited within $0 < T_{Lab} < 1.1$ GeV, the rest curves are made by SAID. The potential strengths V_0, W_0 vary smoothly and suggest the coupling to $\Delta(3, 3)$ to be resonance like for $T_{Lab} < 1$ GeV. OMP-1 is highly qualitative and should not be used to draw serious conclusion.

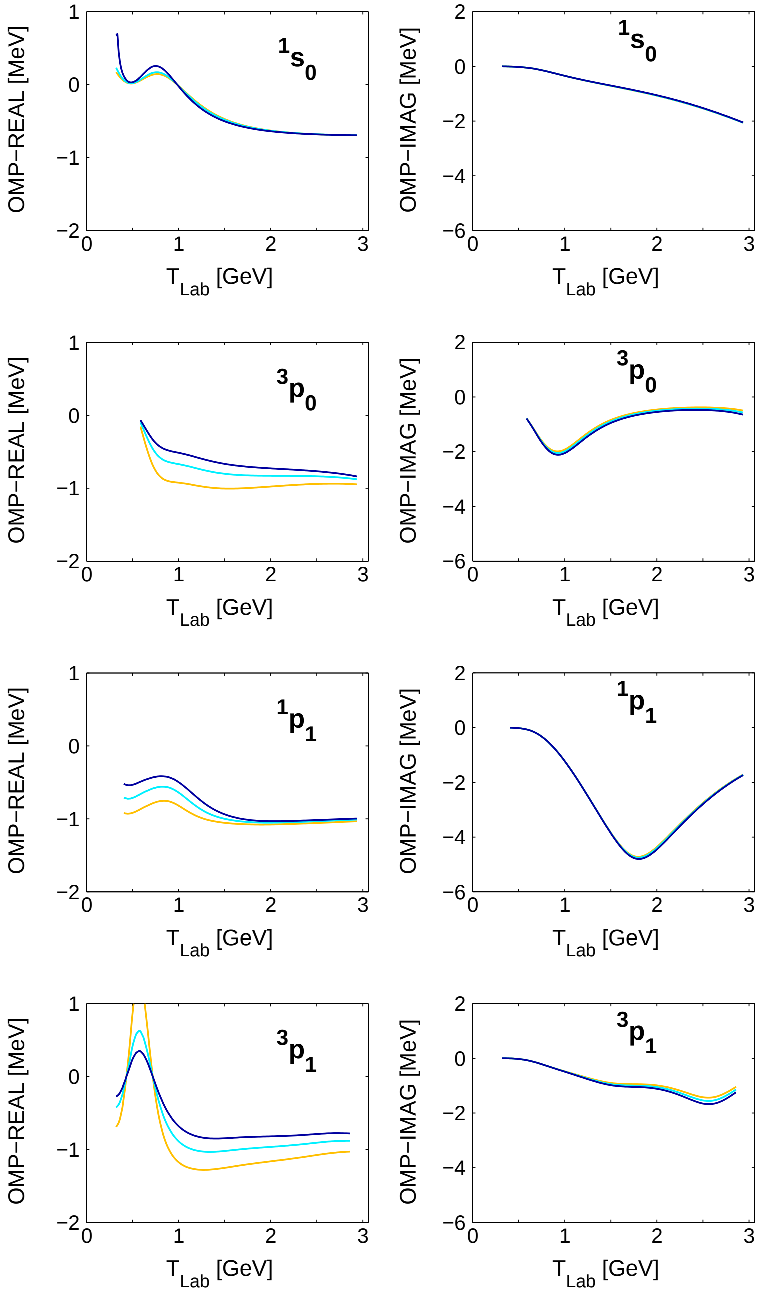


Figure 7.23: The numerical results from OMP-1. Real optical potential strengths V_0 (left) in $V_{OMP} = -V_0 V_{Dirac}(1 + exp((r - R_0)/a))^{-1}\hbar^2/m$ for different Woods-Saxon radii $R_0 = 0.5, 0.6, 1$ fm and diffuseness $a = 0.02$. Imaginary potential strengths W_0 (right) $W_{OMP} = W_0 N_G\, exp(-(r - R_0)^2/a^2)$, $a = 0.2$ of normalized Gaussian type.

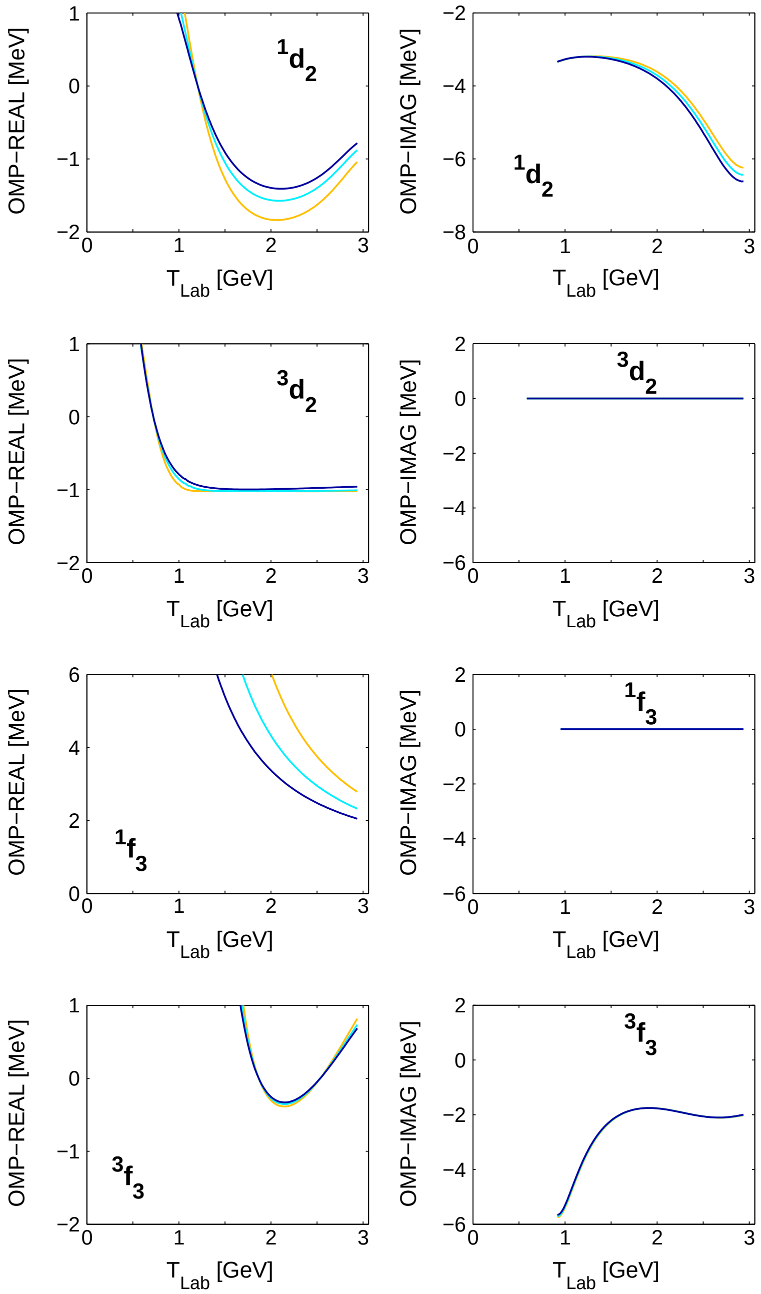


Figure 7.24: Continuation of Fig. 7.23

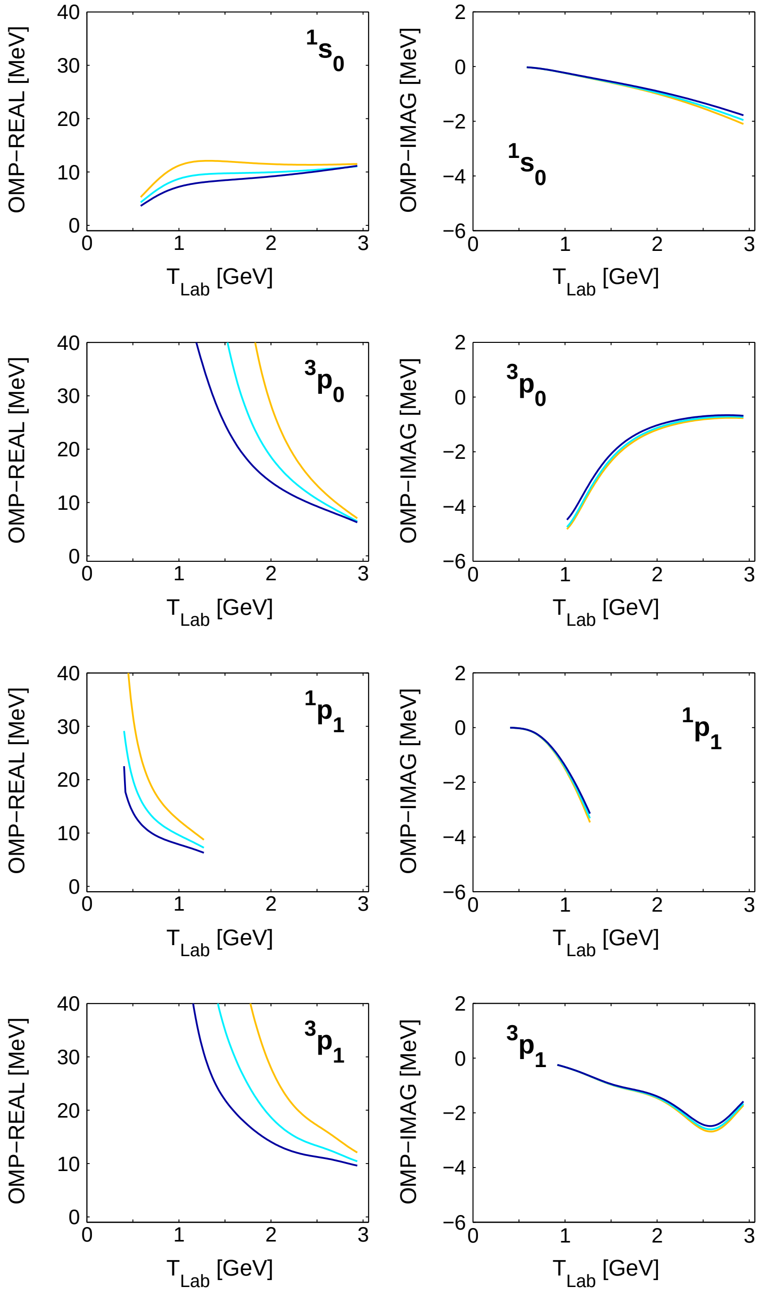


Figure 7.25: The numerical results from OMP-2. Real optical potential strengths V_0 (left) in $V_{OMP} = V_0(1 + exp((r - R_0)/a))^{-1}\hbar^2/m$ for different Woods-Saxon radii $R_0 = 0.5, 0.6, 1$ fm and diffuseness $a = 0.02$. Imaginary potential strengths W_0 (right) in $W_{OMP} = W_0 N_G\, exp(-(r-R_0)^2/a^2)$, $a = 0.2$ of normalized Gaussian type.

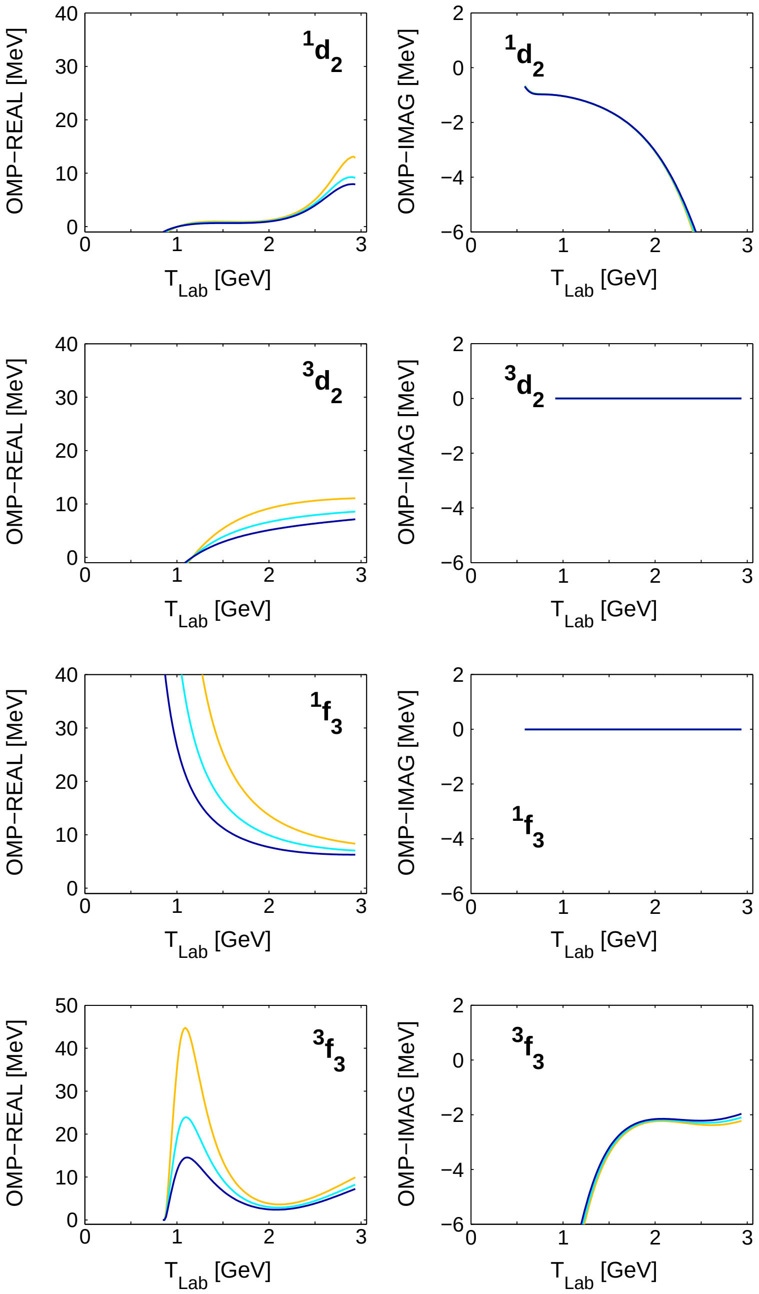


Figure 7.26: Continuation of Fig. 7.25

7.6.2 Numerical Results of OMP-3

This model combines the long range part of the Dirac potential, OBEP for $r > 0.5$ fm, and the QCD relevant region $r \sim 0.5$ fm which we describe with the OMP-3 of Subsec. 6.3.1. It is, no doubt, a phenomenological model. Its merits are a consistent picture, expressed in terms of energy dependent V_0 and W_0, for all partial waves of *pp* and *np* scattering. The Dirac potential is cut-off within 0.514 fm. This choice eliminates all sudden attractions of the Dirac potential for $r \sim 0.5$ fm. The short range scalar and vector potential was fixed with Eq. 6.72g. The real Gaussian potential may be repulsive, $V_0 > 0$, as well as attractive, $V_0 < 0$, and the imaginary Gaussian strengths are absorptive, $W_0 < 0$.

Figs. 7.27 and 7.28 show the channel and energy dependence of V_0 and W_0 for isospin $T = 1$, essentially *pp* data. In the low energy region, $T_{Lab} < 300$ MeV, we observe often an attractive pocket with $V_0 < 0$. Low partial waves show a significantly stronger repulsion as high partial waves. We interpret it as a consequence for need of angular momentum transfer to the six quark modes. Lack of NN orbital angular momentum yields repulsion. The obviously favored ${}^1D_2$ channel responds with attraction in the QCD domain.

The absorption W_0 shows only a dramatic difference in the ${}^1D_2$ channel. However, SM00 and SP03 show unexpected large changes Figs. 7.12 and 7.18. This suggest, another solution may bring $W_0({}^1D_2)$ again down. *np* channels, $T = 0$, shown in Figs. 7.29-7.32 allow a similar interpretation, as given for $T = 1$, but lack of absorption data classifies them as preliminary.

An animation of the OMP-3 results is shown in Figs. 7.33-7.50. The parameters are listed in Table. 7.6. The fitted potential strengths for GWU/VPI SAID SP03 are listed in Tables. 7.7-7.12.

Table 7.6: The parameters for OMP-3, obtained from Eq. (6.71).

Potentials	Potential strength [MeV]	Radius [fm]	Diffuseness [fm]
V_{ref}	Eq. (5.257) (Figs. 7.6-7.11)	0.514	0.02
V_{scalar}	-480	0.514	0.02
V_{vector}	600	0.514	0.02
V_{OMP}	Figs.7.27, 7.29	0.514	0.01
W_{OMP}	Figs.7.28, 7.30	0.514	0.01

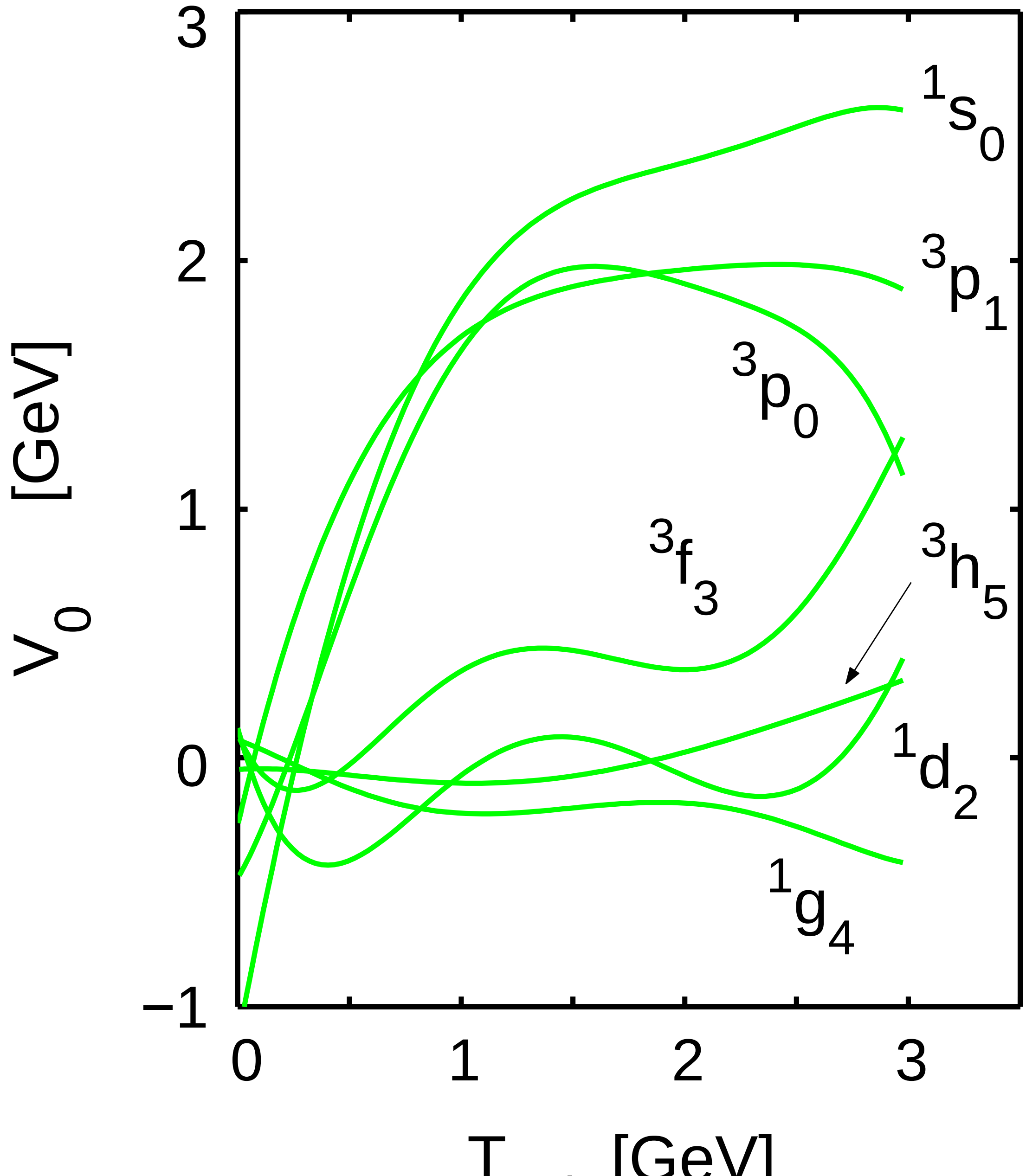


Figure 7.27: The real potential strengths OMP-3 $V_0(T_{Lab})$ of the Gaussian for isospin $T = 1$ singlet and triplet channels

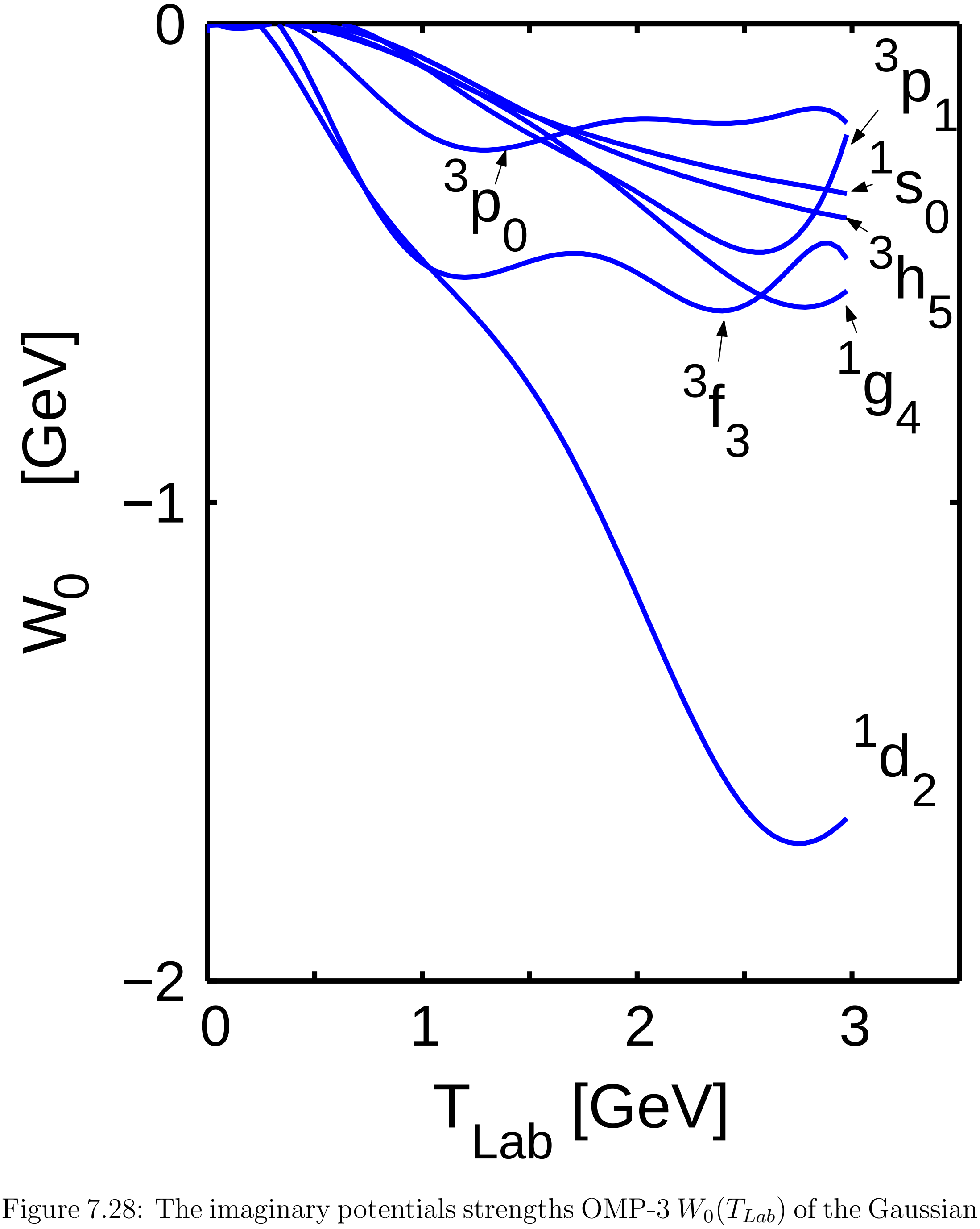


Figure 7.28: The imaginary potentials strengths OMP-3 $W_0(T_{Lab})$ of the Gaussian for isospin $T = 1$ singlet and triplet channels.

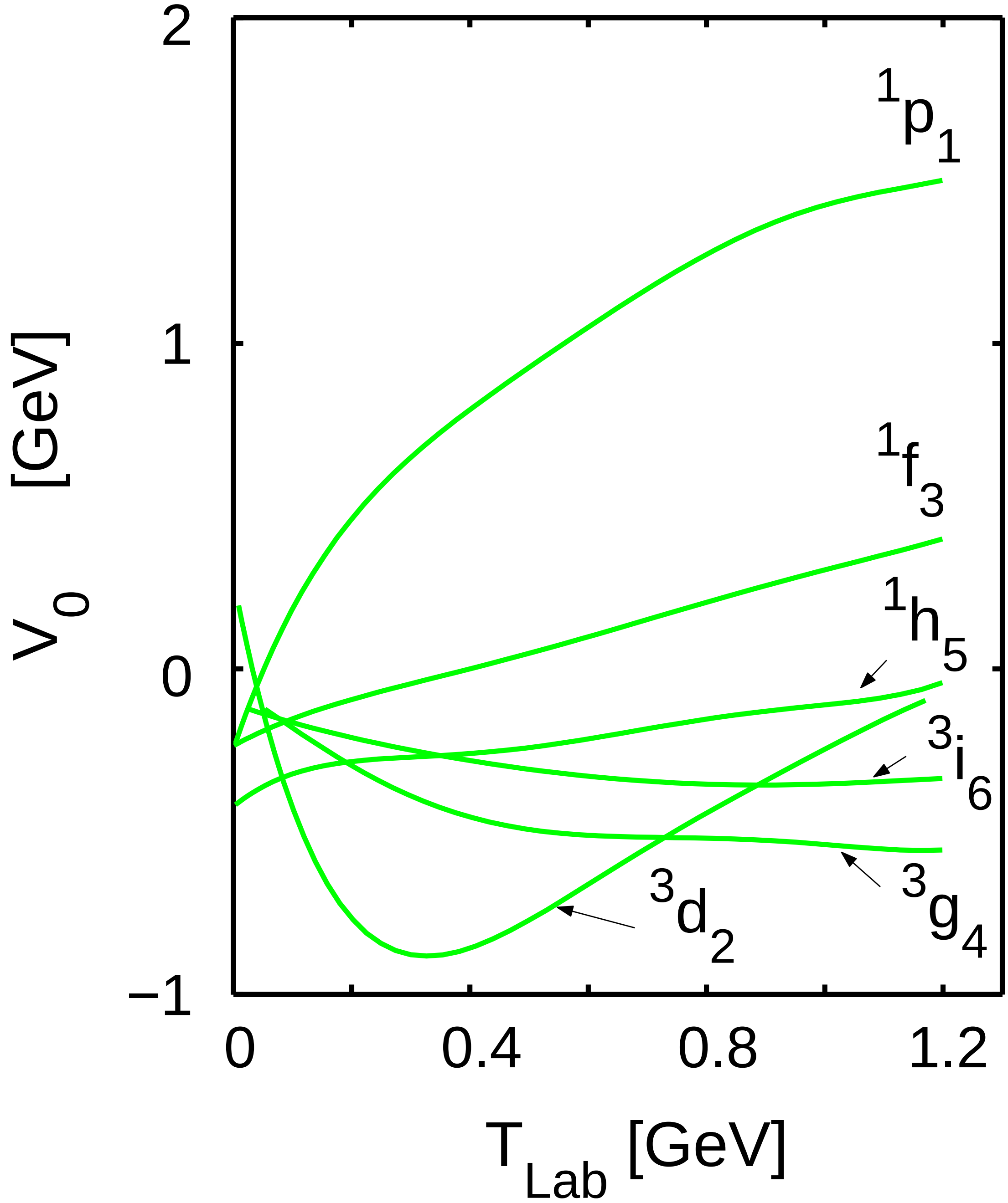


Figure 7.29: The real potential strengths OMP-3 $V_0(T_{Lab})$ of the Gaussian for isospin $T = 0$ singlet and triplet channels.

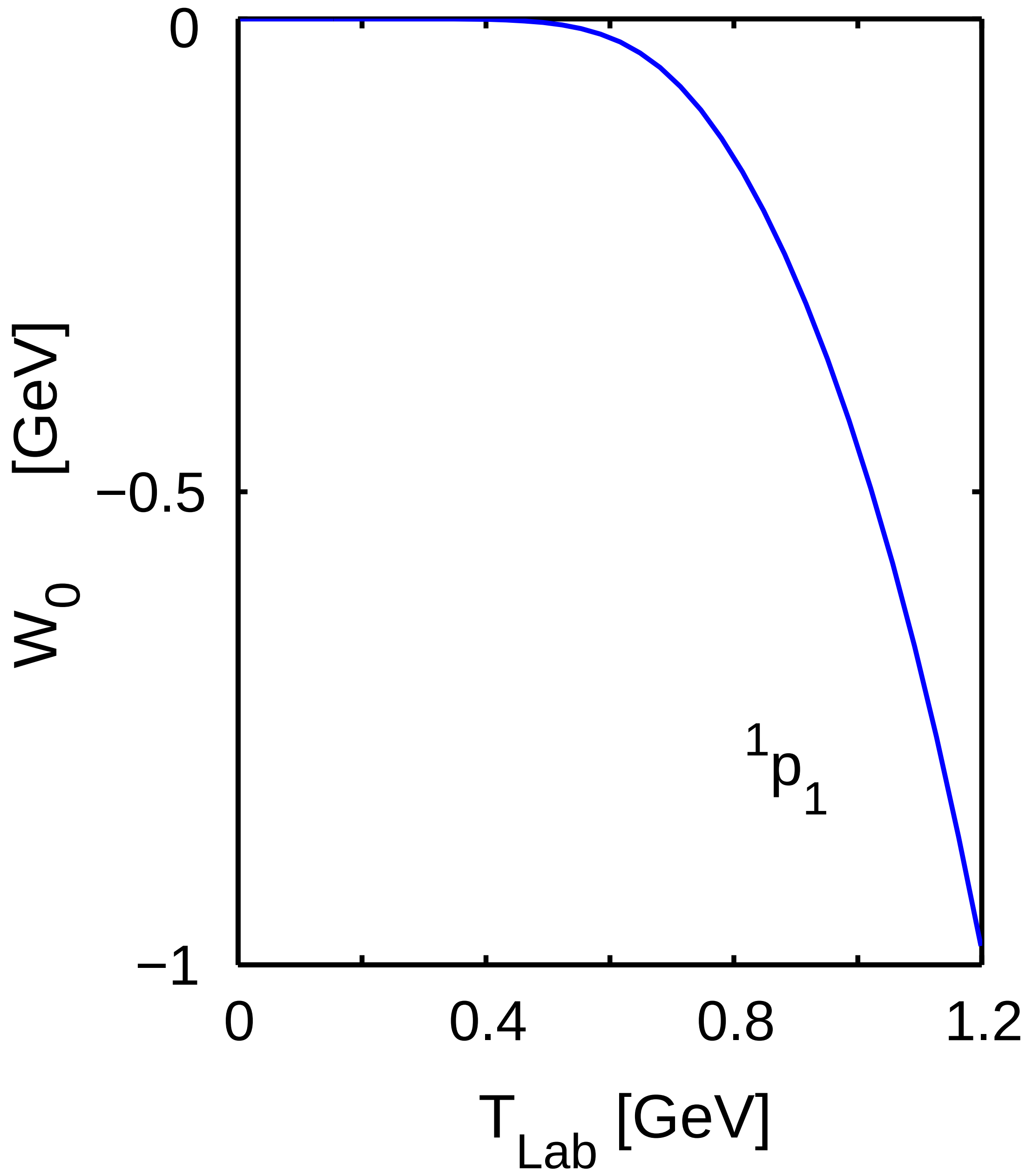


Figure 7.30: The imaginary potentials strengths OMP-3 $W_0(T_{Lab})$ of the Gaussian for isospin $T = 0$ singlet channels, the experimental data do not existent for other channels.

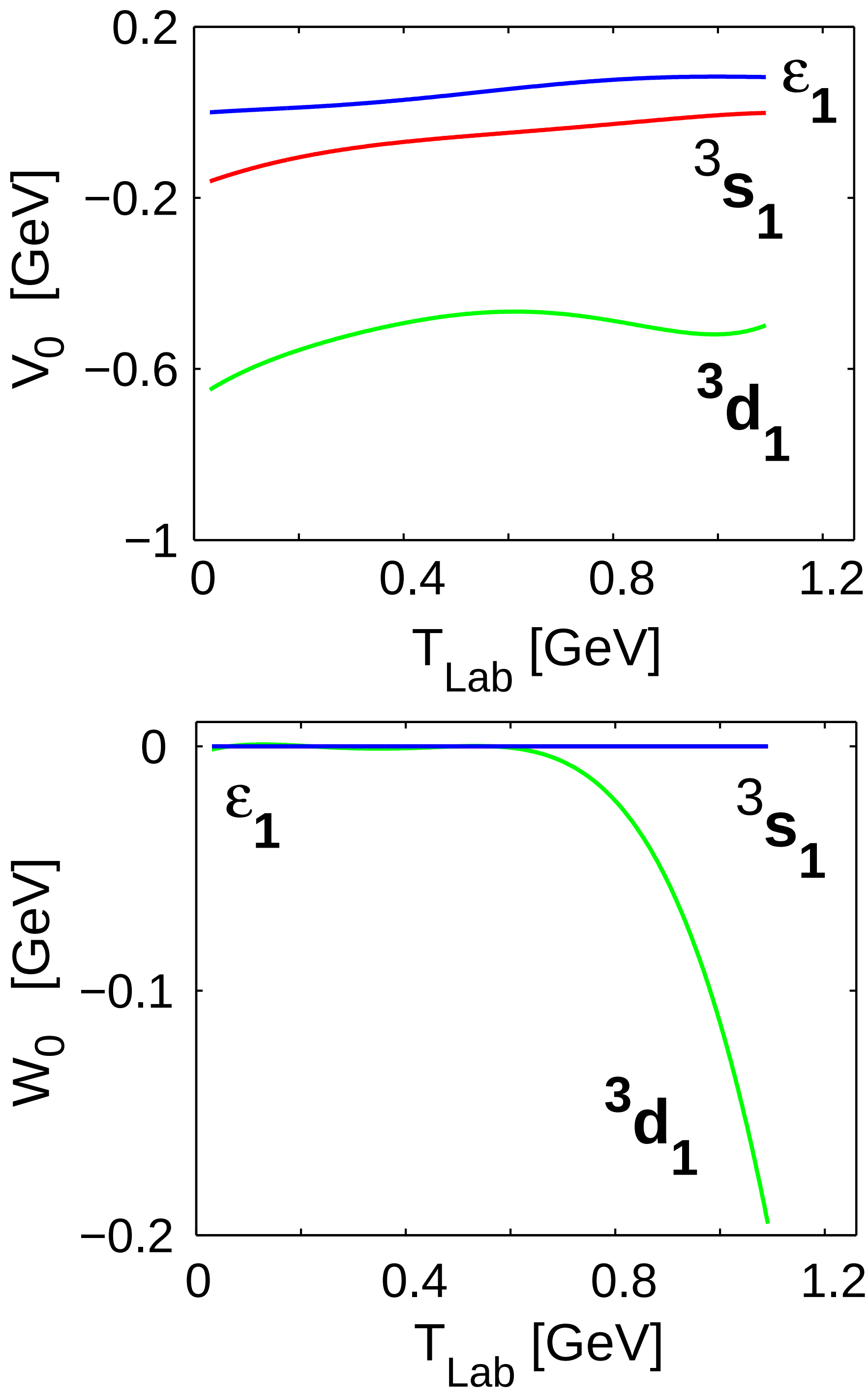


Figure 7.31: The real (upper) and imagine (below) potential strengths OMP-3 $V_0(T_{Lab})$ of the Gaussian for isospin $T = 1$ coupled channels.

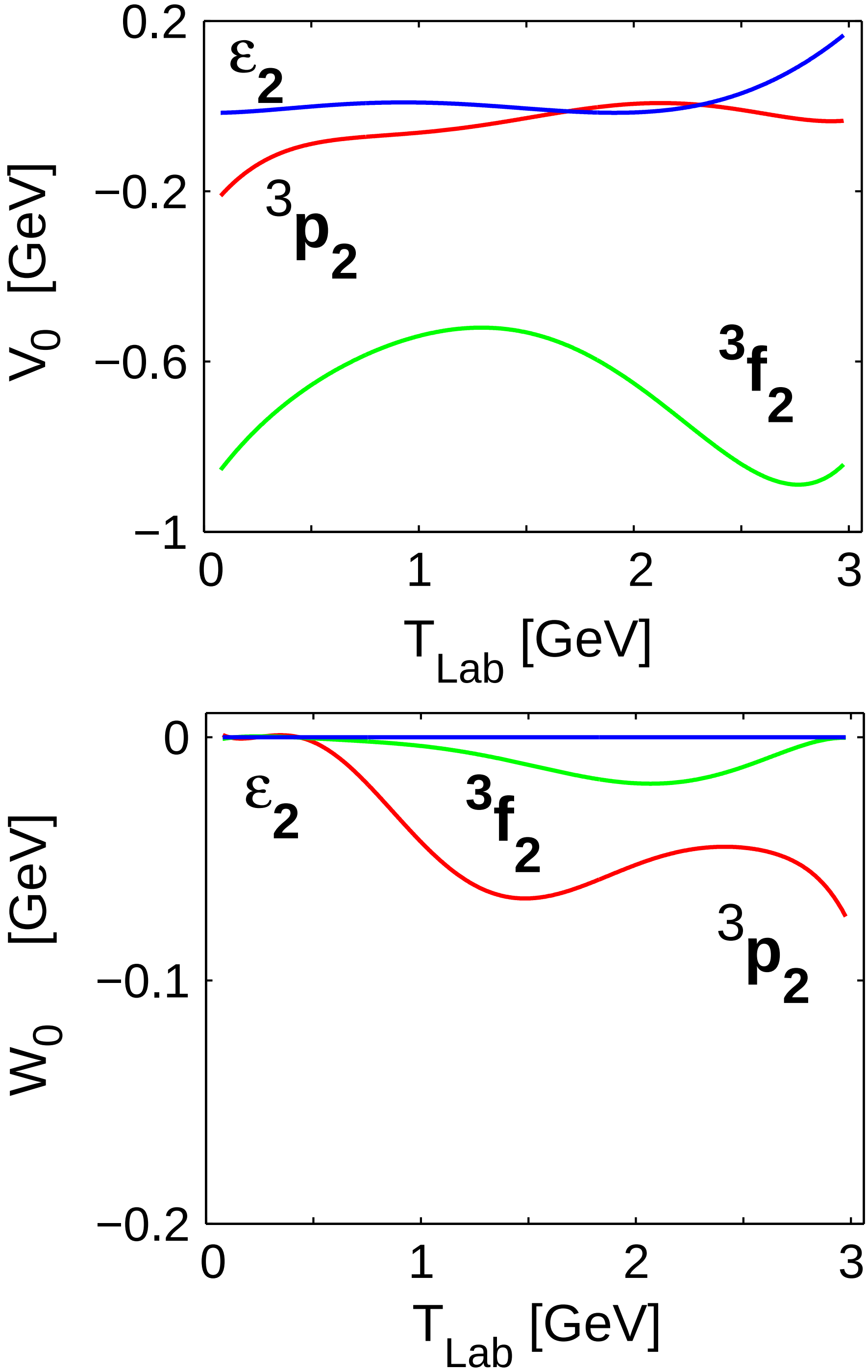
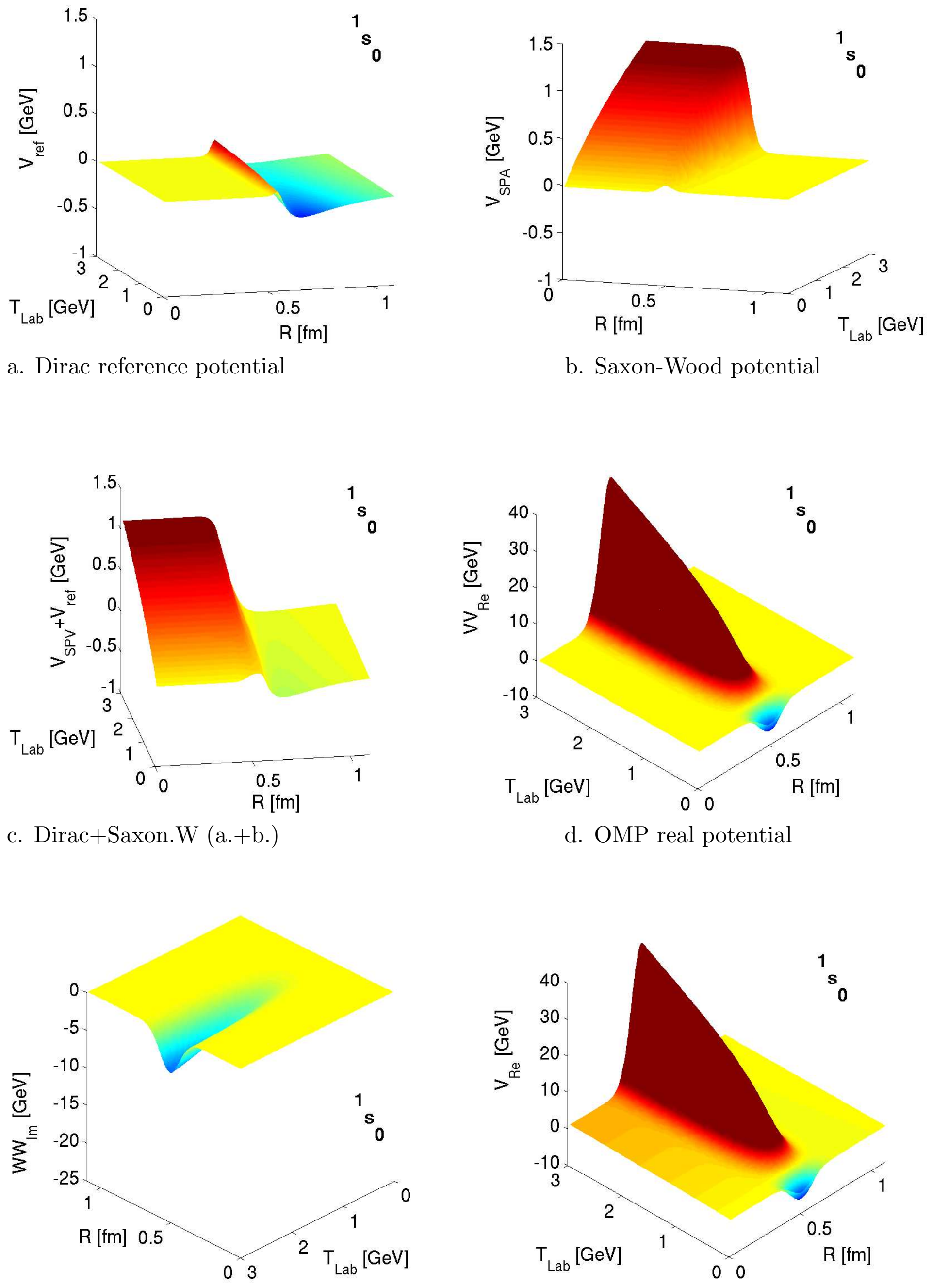
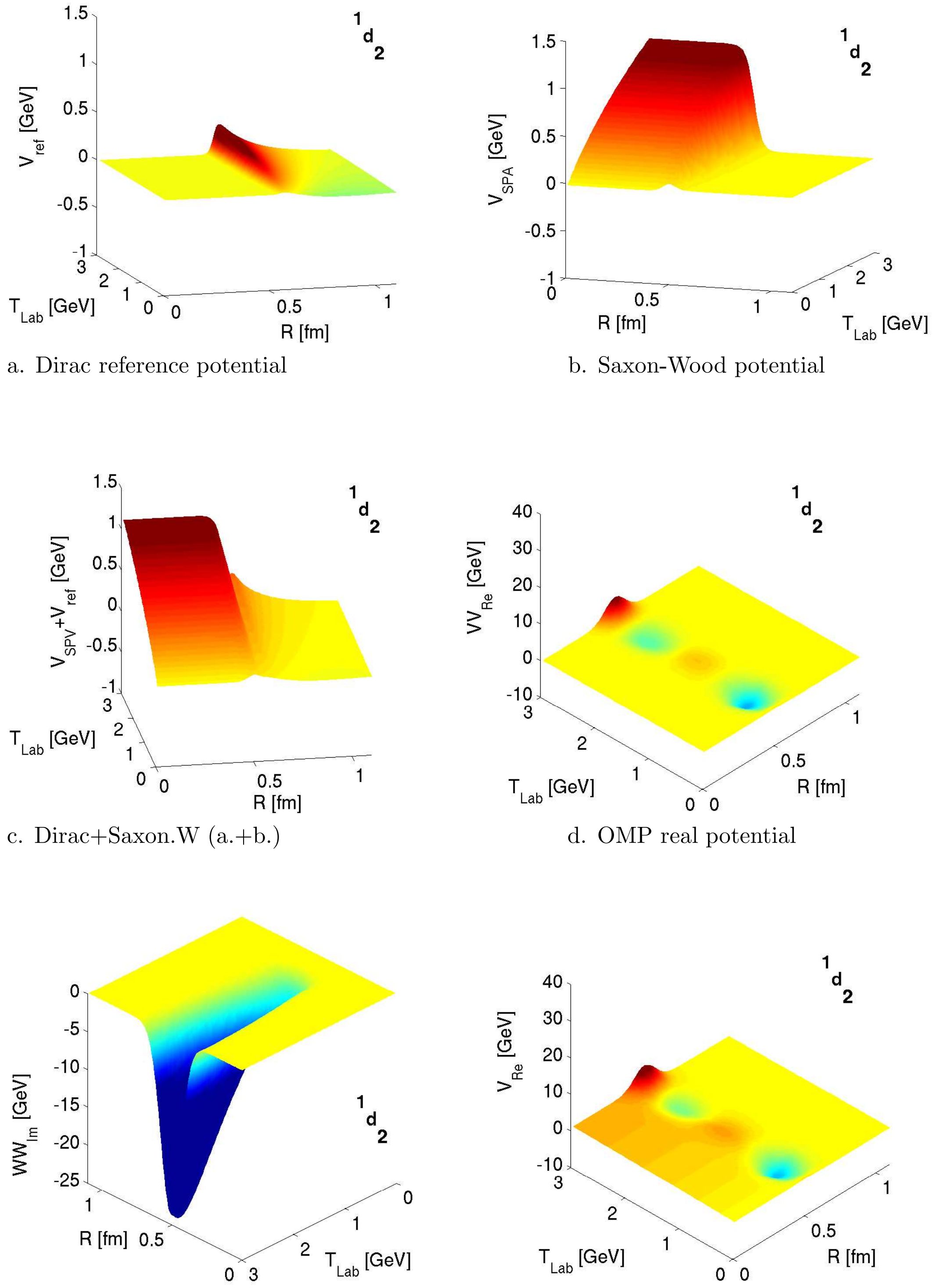


Figure 7.32: The real (upper) and imagine (below) potential strengths OMP-3 $V_0(T_{Lab})$ of the Gaussian for isospin $T = 0$ coupled channels.



a. Dirac reference potential
b. Saxon-Wood potential
c. Dirac+Saxon.W (a.+b.)
d. OMP real potential
e. OMP imaginary potential
f. full real (a.+b.+d.)

Figure 7.33: Three dimensional representation of the potential strengths for $^1S_0$, $T = 1$. a.) The energy depend Dirac reference potential for $r > 0.514$ fm using Eq.(5.257). b.) Superposition of scalar and vector potentials Eq. (6.71). c.) The sum of (a.) and (b.). d.) The OMP-3 real potential. e.) Imaginary part of OMP-3. f.) The total real potential.



a. Dirac reference potential

b. Saxon-Wood potential

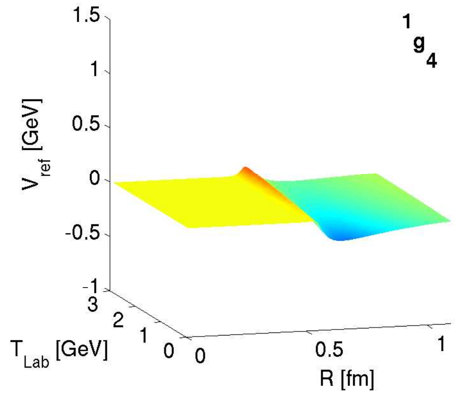
c. Dirac+Saxon.W (a.+b.)

d. OMP real potential

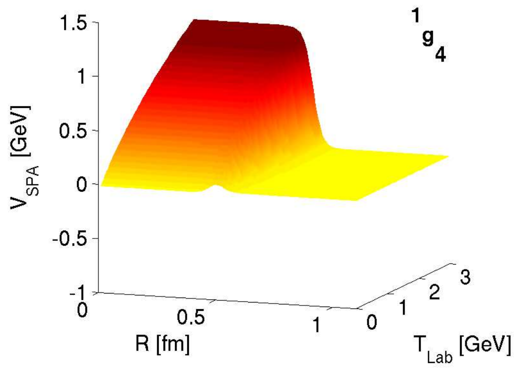
e. OMP imaginary potential

f. full real (a.+b.+d.)

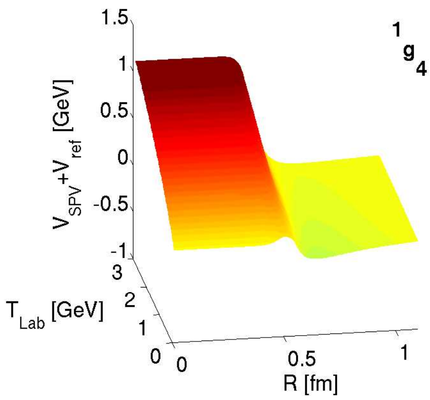
Figure 7.34: Three dimensional representation of the potential strengths for $^1D_2$, $T = 1$. (*Continuation of Fig. 7.33*).



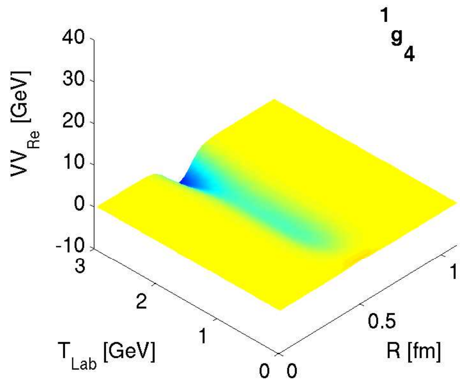
a. Dirac reference potential



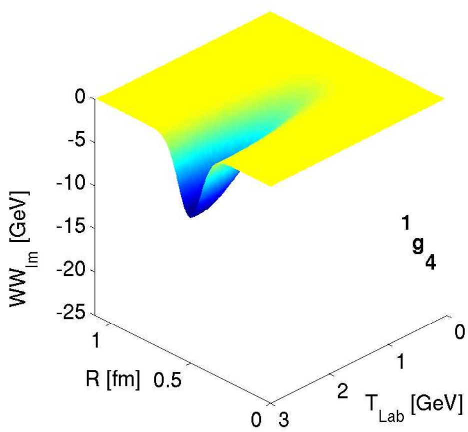
b. Saxon-Wood potential



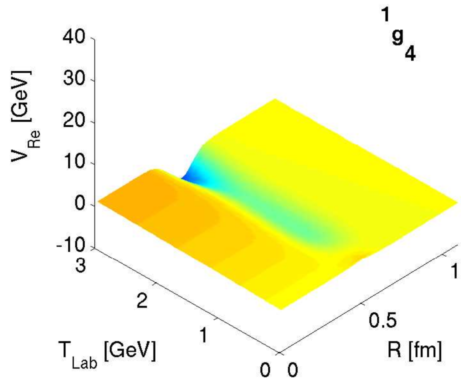
c. Dirac+Saxon.W (a.+b.)



d. OMP real potential

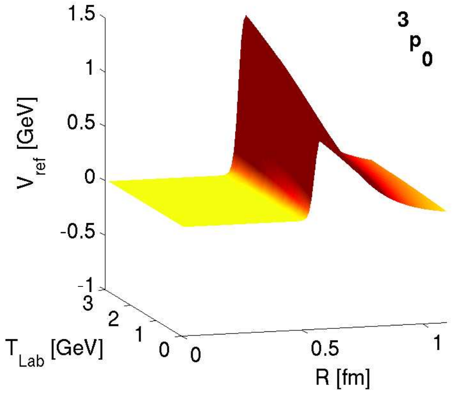


e. OMP imaginary potential

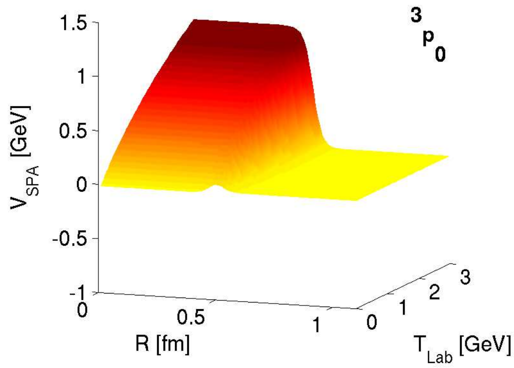


f. full real (a.+b.+d.)

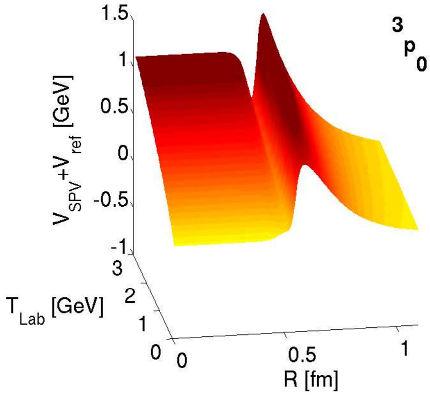
Figure 7.35: Three dimensional representation of the potential strengths for $^1G_4$, $T = 1$. (*Continuation of Fig. 7.33*).



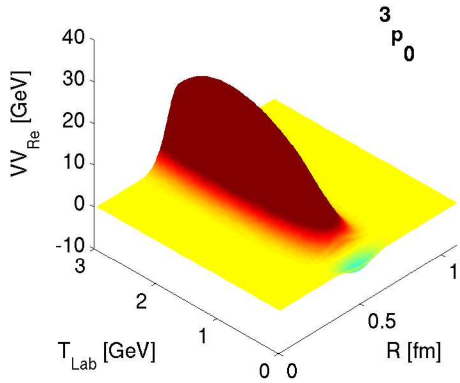
a. Dirac reference potential



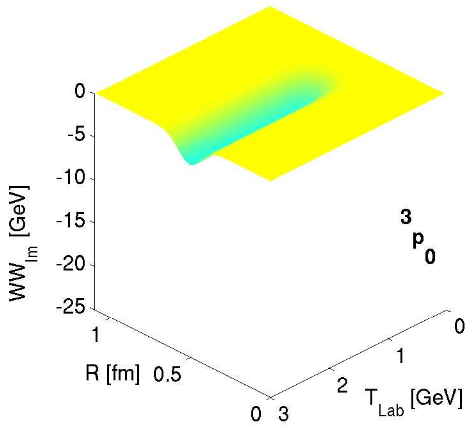
b. Saxon-Wood potential



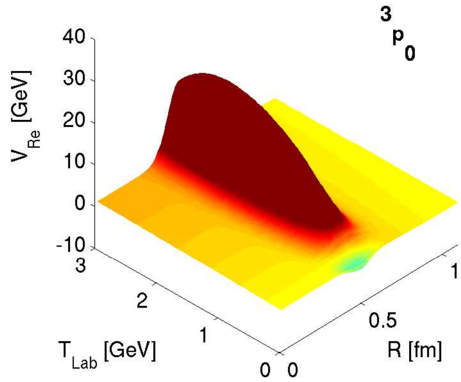
c. Dirac+Saxon.W (a.+b.)



d. OMP real potential

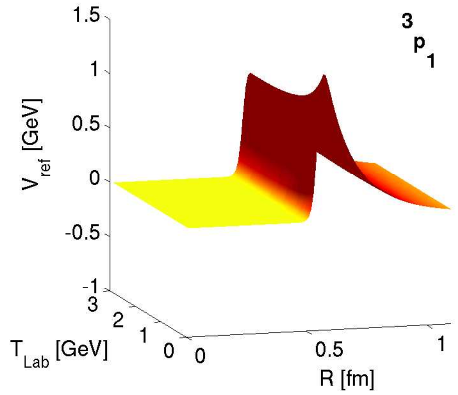


e. OMP imaginary potential

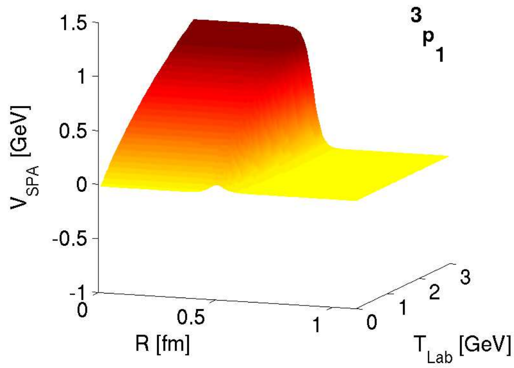


f. full real (a.+b.+d.)

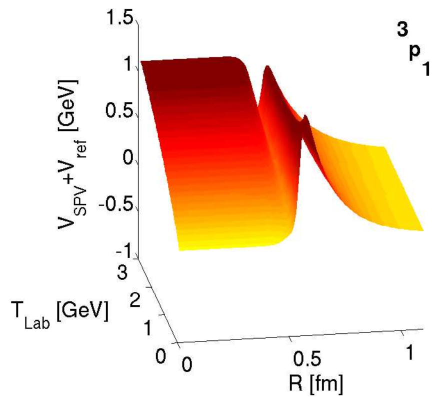
Figure 7.36: Three dimensional representation of the potential strengths for $^3P_0$, $T = 1$. (*Continuation of Fig. 7.33*).



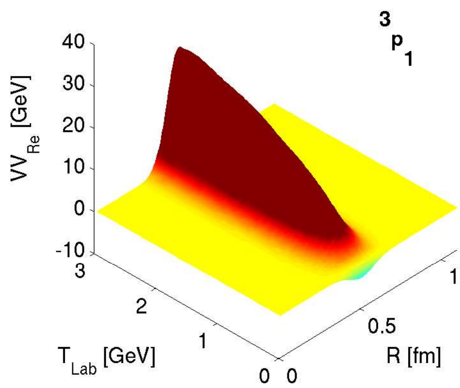
a. Dirac reference potential



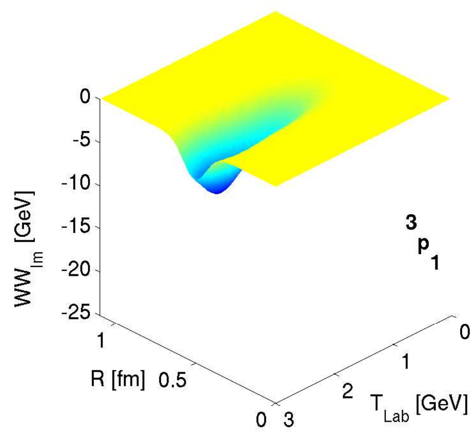
b. Saxon-Wood potential



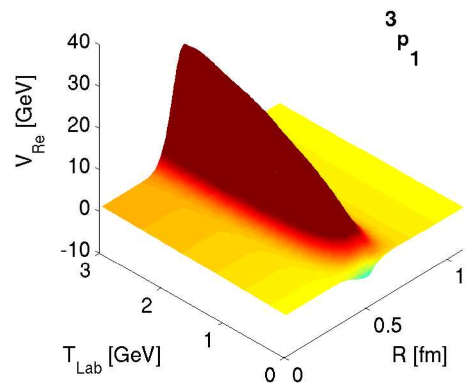
c. Dirac+Saxon.W (a.+b.)



d. OMP real potential

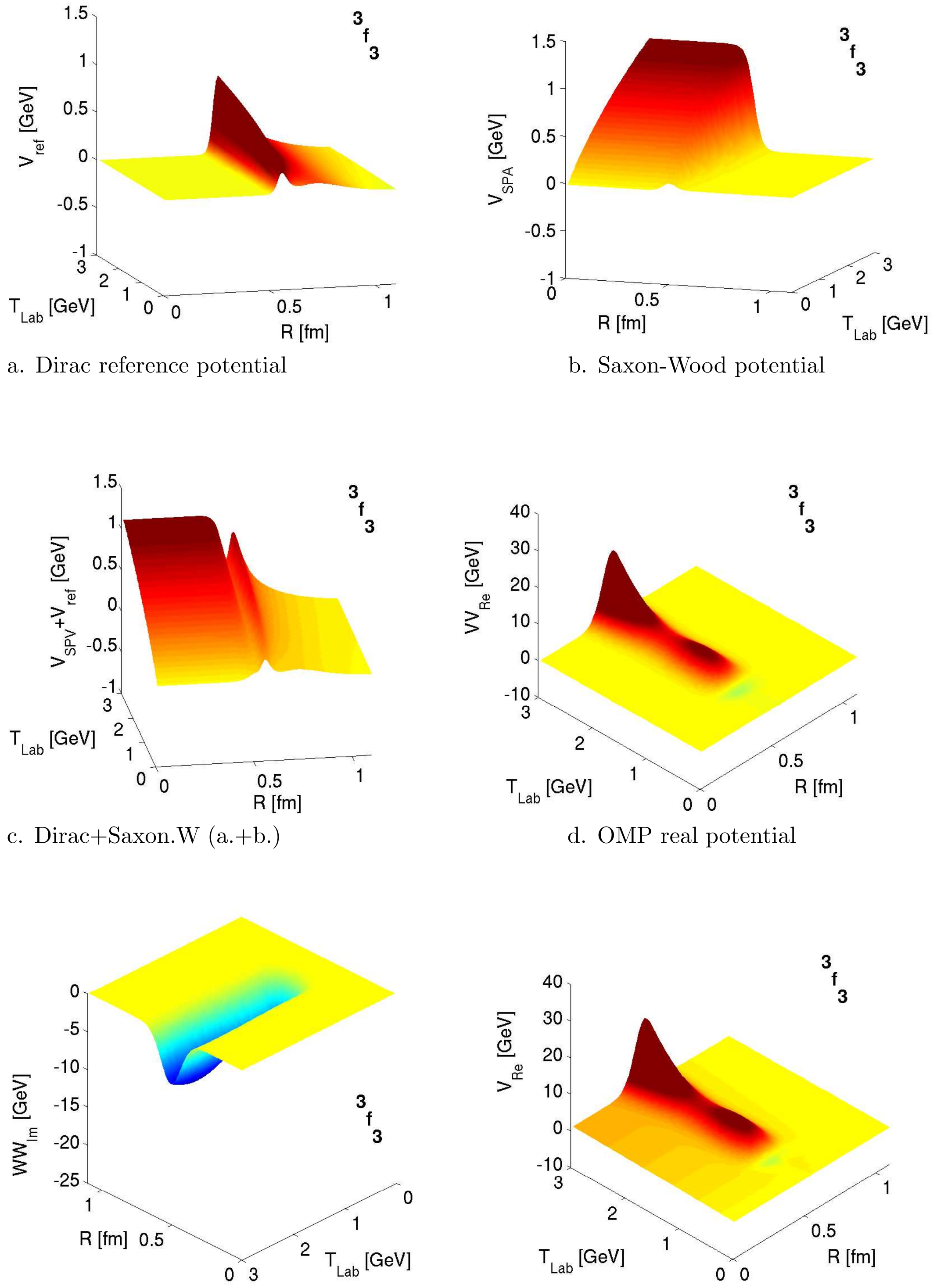


e. OMP imaginary potential



f. full real (a.+b.+d.)

Figure 7.37: Three dimensional representation of the potential strengths for $^3P_1$, $T = 1$. (*Continuation of Fig. 7.33*).



a. Dirac reference potential

b. Saxon-Wood potential

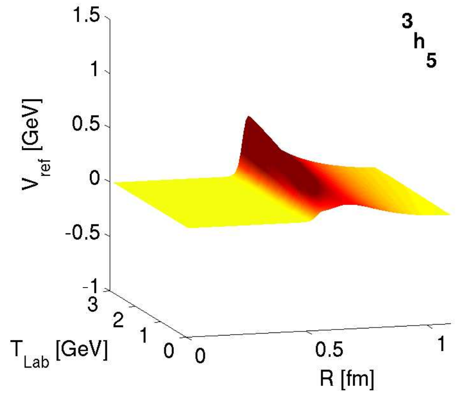
c. Dirac+Saxon.W (a.+b.)

d. OMP real potential

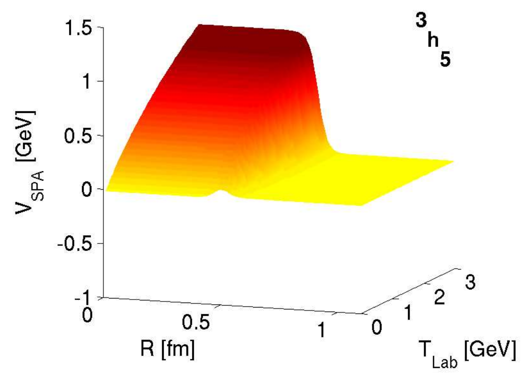
e. OMP imaginary potential

f. full real (a.+b.+d.)

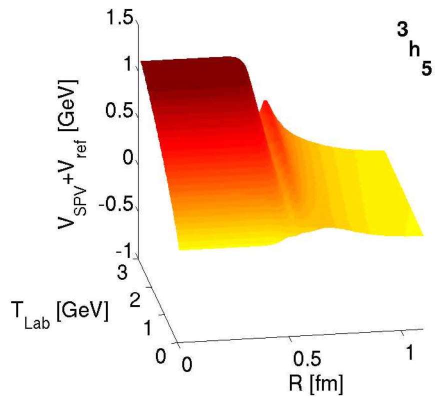
Figure 7.38: Three dimensional representation of the potential strengths for channel ${}^3f3_0$, $T = 1$. (*Continuation of Fig. 7.33*).



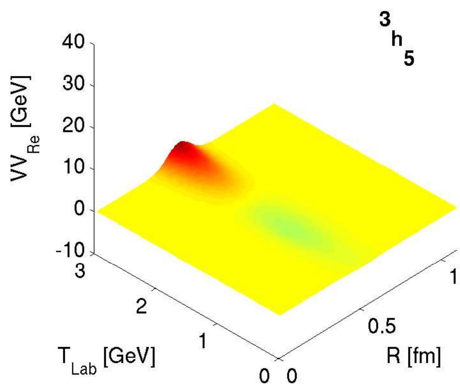
a. Dirac reference potential



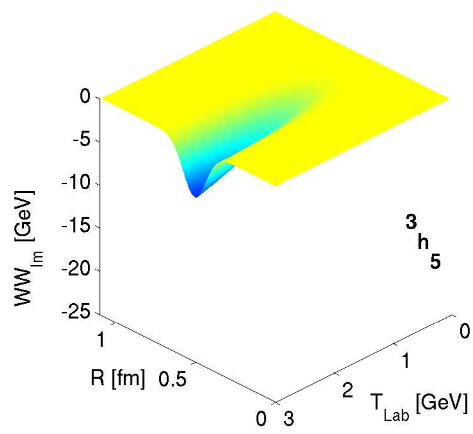
b. Saxon-Wood potential



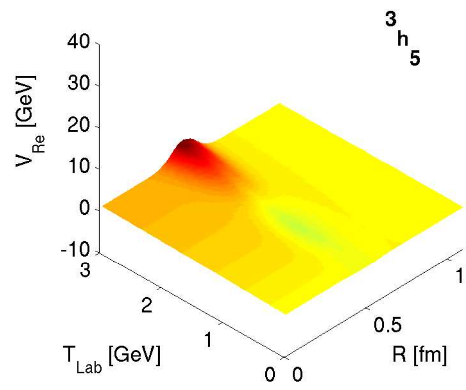
c. Dirac+Saxon.W (a.+b.)



d. OMP real potential

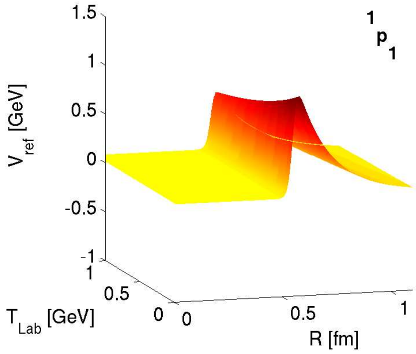


e. OMP imaginary potential

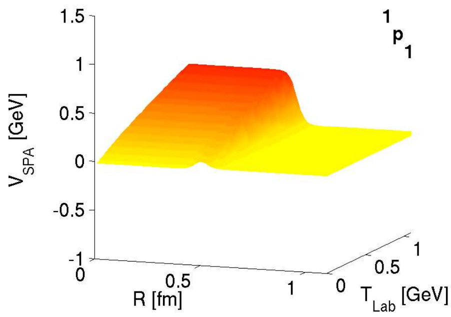


f. full real (a.+b.+d.)

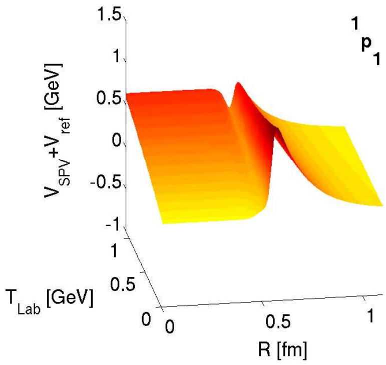
Figure 7.39: Three dimensional representation of the potential strengths for $^3H_5$, $T = 1$. (*Continuation of Fig. 7.33*).



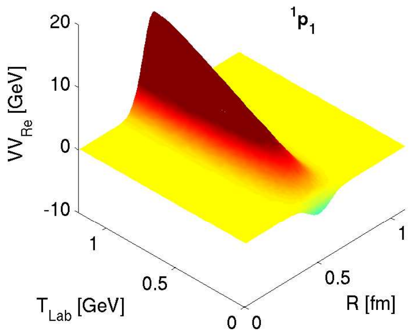
a. Dirac reference potential



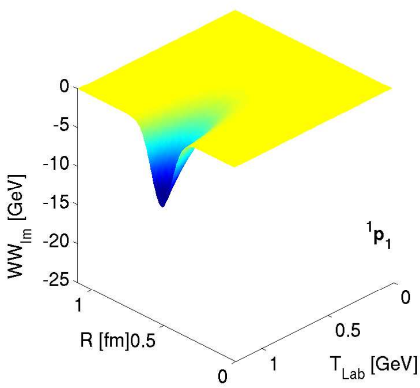
b. Saxon-Wood potential



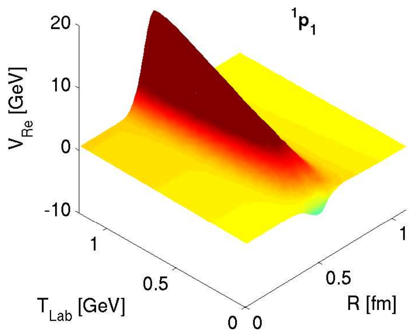
c. Dirac+Saxon.W (a.+b.)



d. OMP real potential

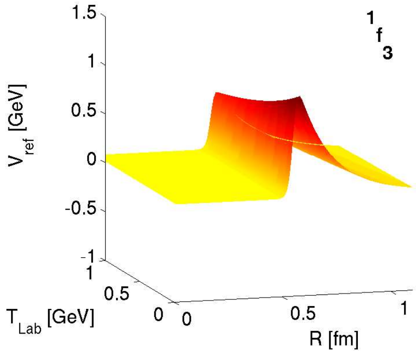


e. OMP imaginary potential

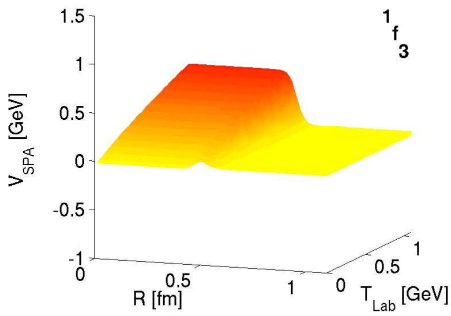


f. full real (a.+b.+d.)

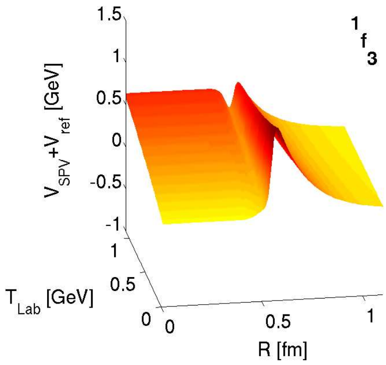
Figure 7.40: Three dimensional representation of the potential strengths for $^1P_1$, $T = 0$. (*Continuation of Fig. 7.33*).



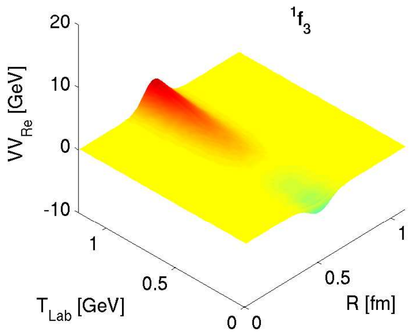
a. Dirac reference potential



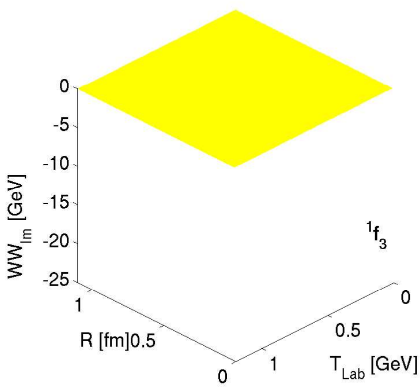
b. Saxon-Wood potential



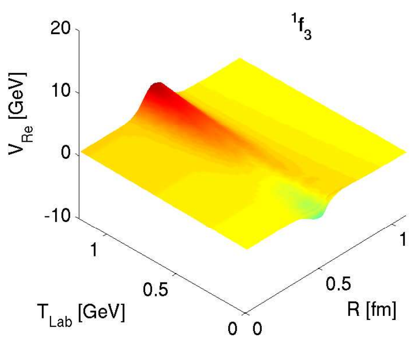
c. Dirac+Saxon.W (a.+b.)



d. OMP real potential

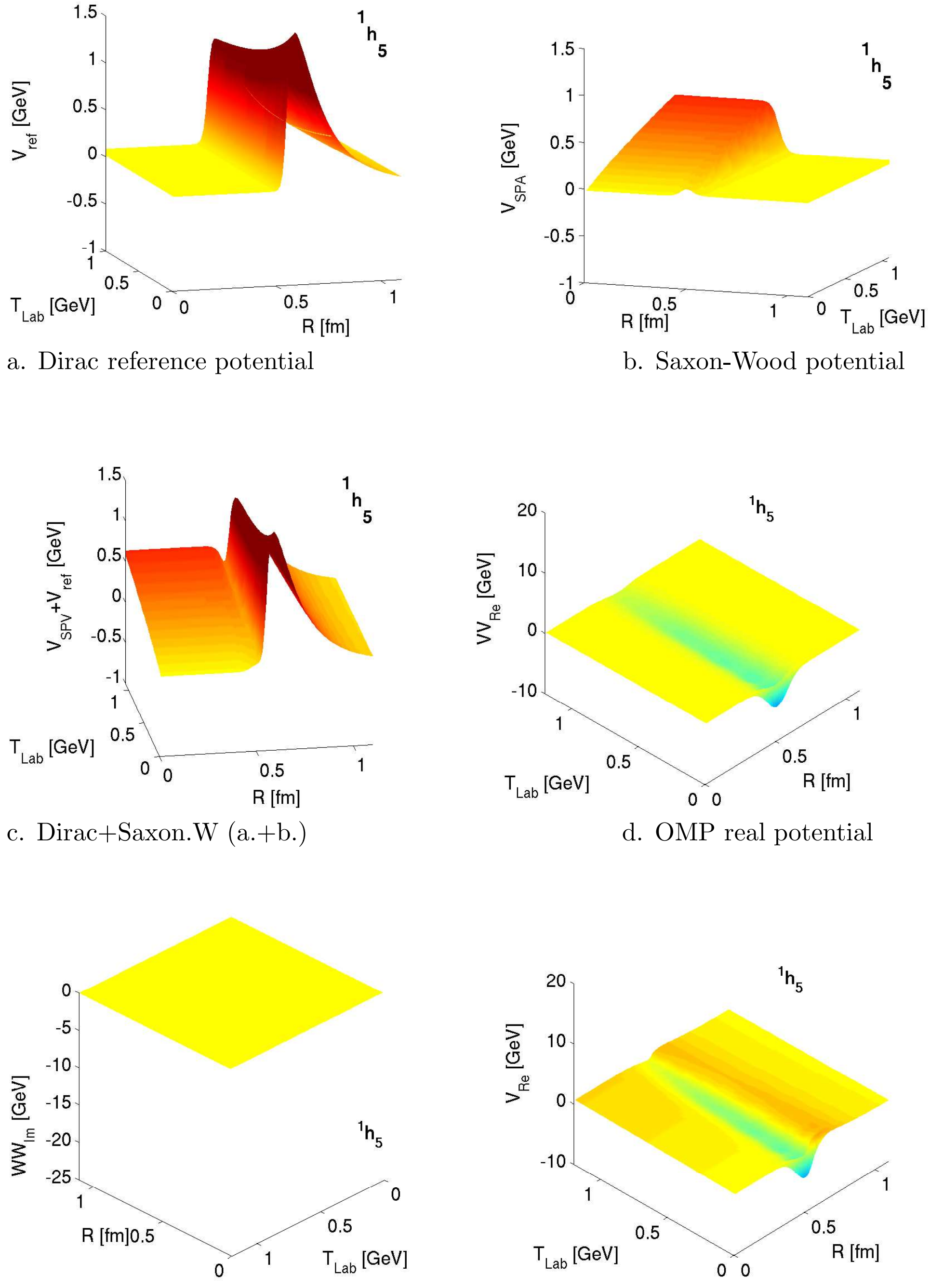


e. OMP imaginary potential



f. full real (a.+b.+d.)

Figure 7.41: Three dimensional representation of the potential strengths for ${}^1F_3$, $T = 0$. (*Continuation of Fig. 7.33*).



a. Dirac reference potential

b. Saxon-Wood potential

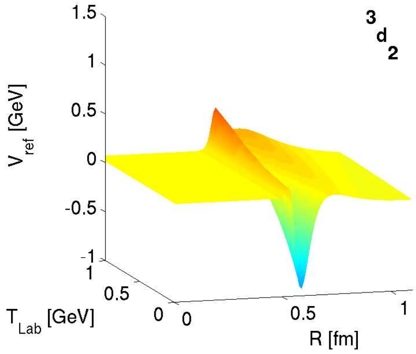
c. Dirac+Saxon.W (a.+b.)

d. OMP real potential

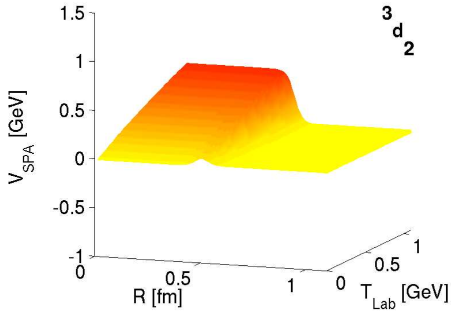
e. OMP imaginary potential

f. full real (a.+b.+d.)

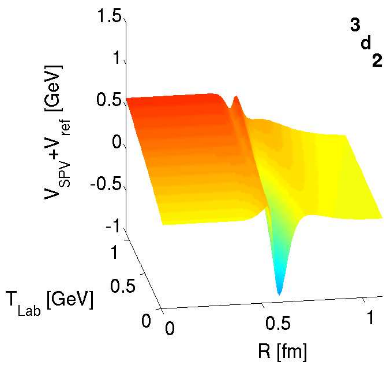
Figure 7.42: Three dimensional representation of the potential strengths for $^1H_5$, $T = 0$. (*Continuation of Fig. 7.33*).



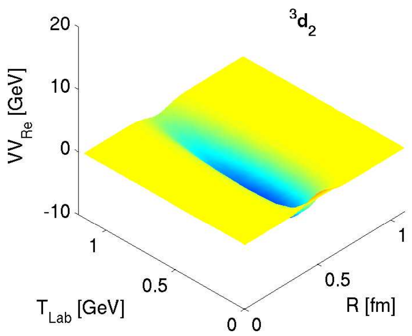
a. Dirac reference potential



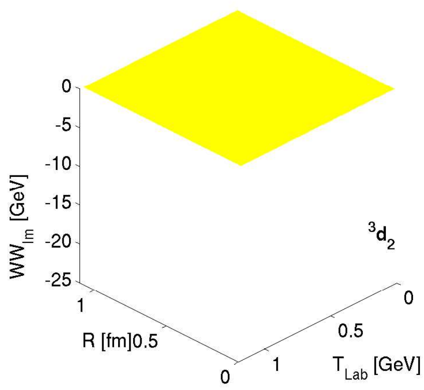
b. Saxon-Wood potential



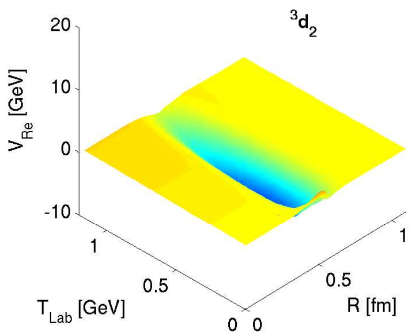
c. Dirac+Saxon.W (a.+b.)



d. OMP real potential

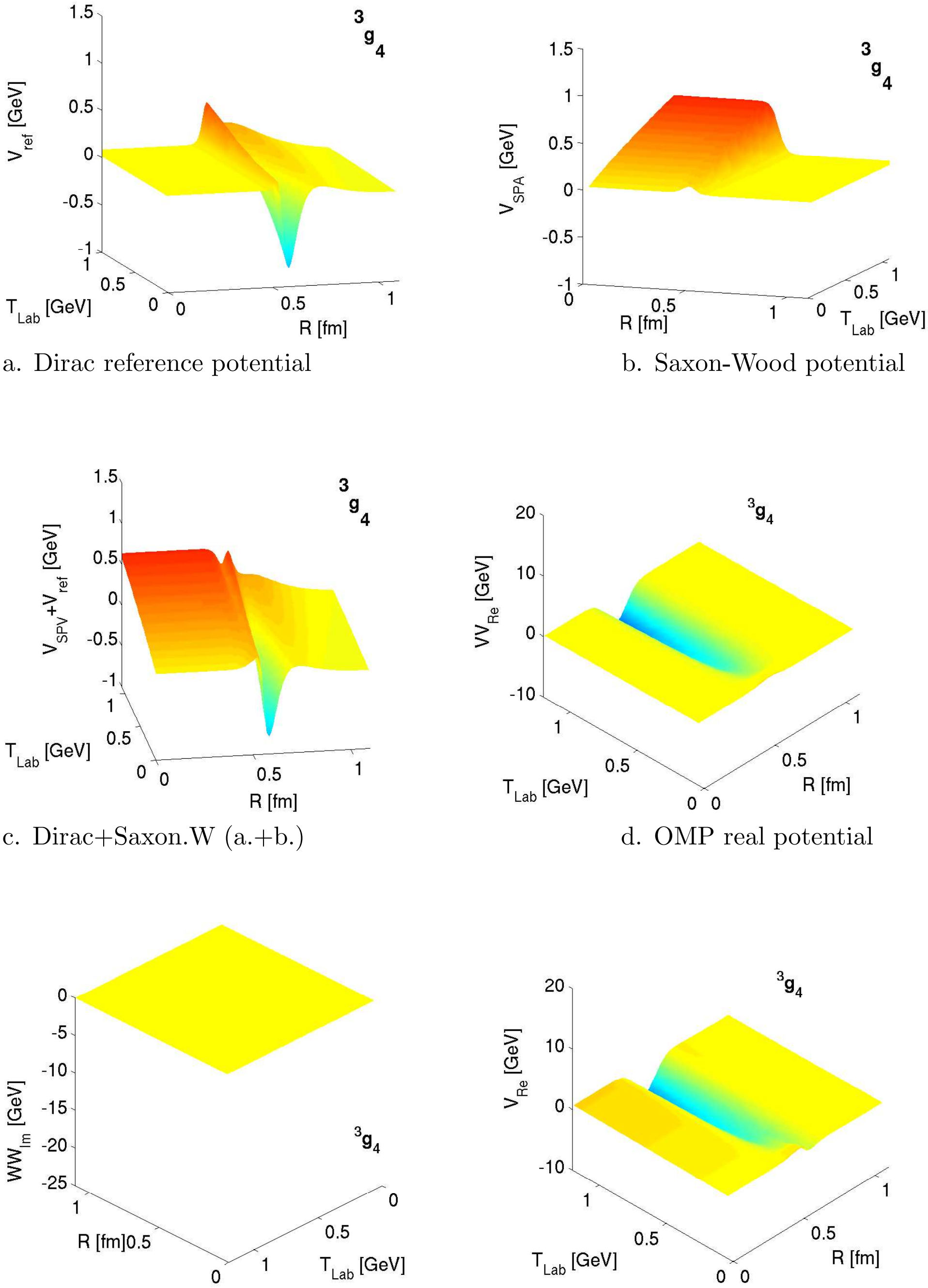


e. OMP imaginary potential



f. full real (a.+b.+d.)

Figure 7.43: Three dimensional representation of the potential strengths for $^3D_2$, $T = 0$. (*Continuation of Fig. 7.33*).



a. Dirac reference potential

b. Saxon-Wood potential

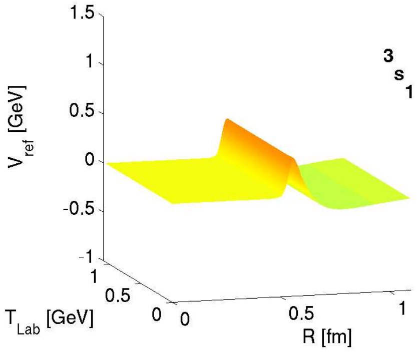
c. Dirac+Saxon.W (a.+b.)

d. OMP real potential

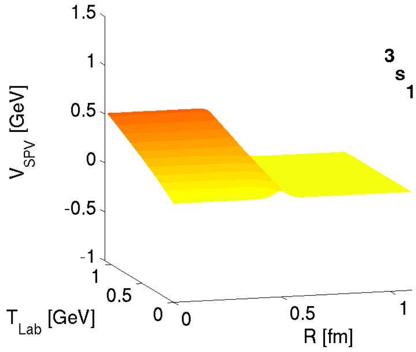
e. OMP imaginary potential

f. full real (a.+b.+d.)

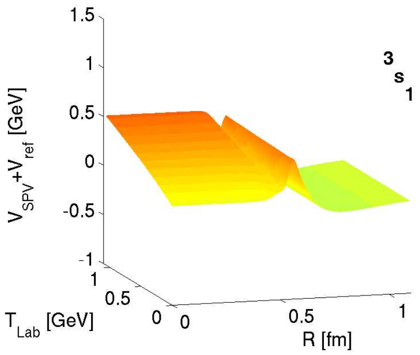
Figure 7.44: Three dimensional representation of the potential strengths for $^3G_4$, $T = 0$. (*Continuation of Fig. 7.33*).



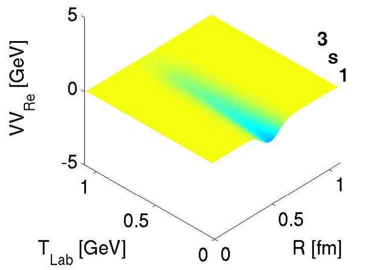
a. Dirac reference potential



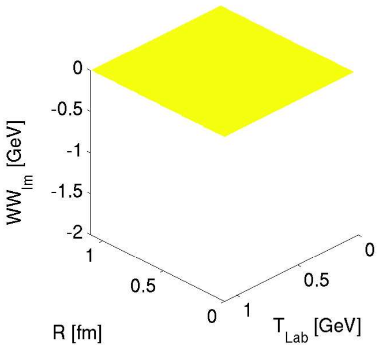
b. Saxon-Wood potential



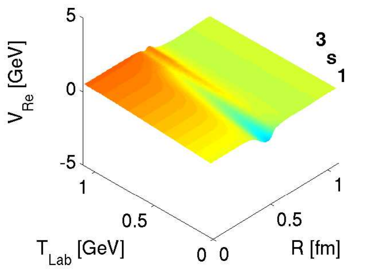
c. Dirac+Saxon.W (a.+b.)



d. OMP real potential

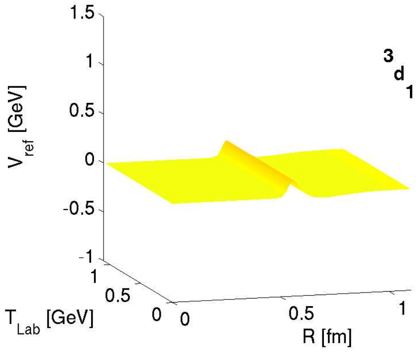


e. OMP imaginary potential

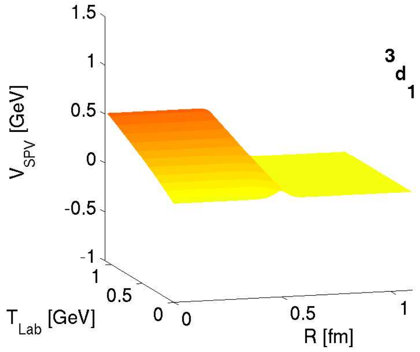


f. full real (a.+b.+d.)

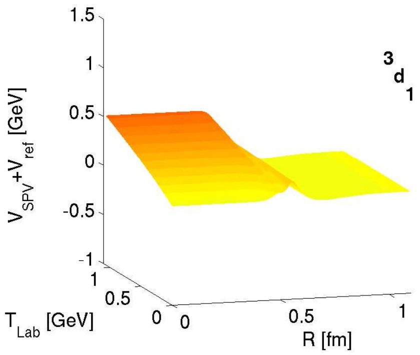
Figure 7.45: Three dimensional representation of the potential strengths for $^3S_1$, $T = 0$. (*Continuation of Fig. 7.33*).



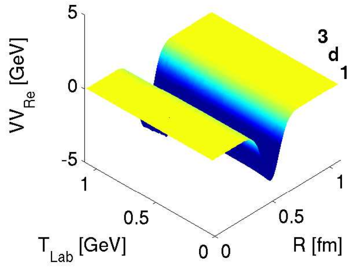
a. Dirac reference potential



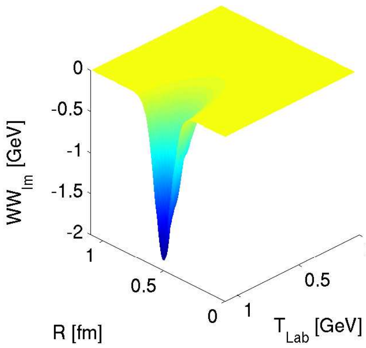
b. Saxon-Wood potential



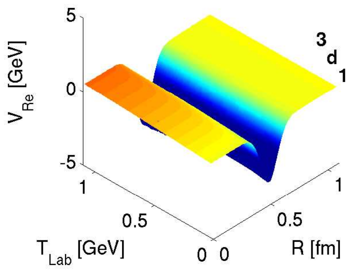
c. Dirac+Saxon.W (a.+b.)



d. OMP real potential

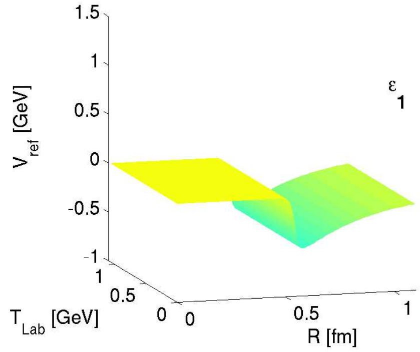


e. OMP imaginary potential

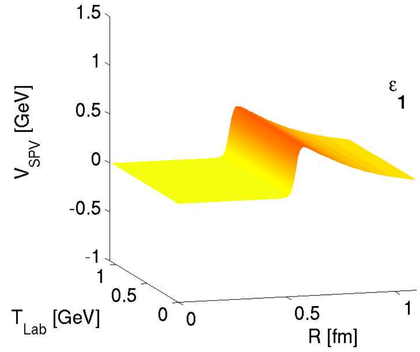


f. full real (a.+b.+d.)

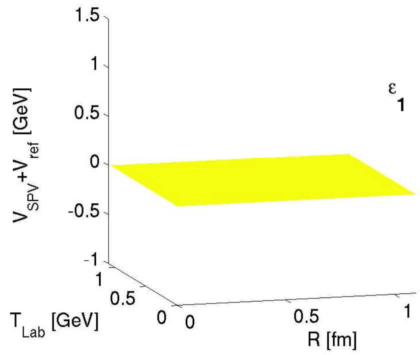
Figure 7.46: Three dimensional representation of the potential strengths for ${}^3D_1$, $T = 0$. (*Continuation of Fig. 7.33*).



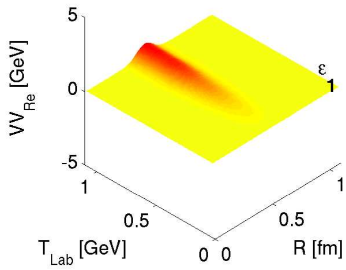
a. Dirac reference potential



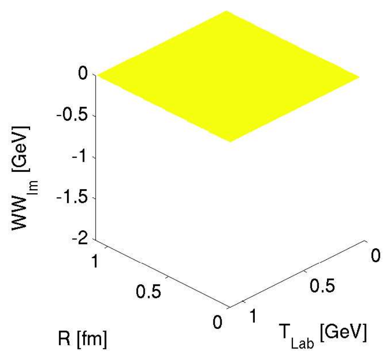
b. Saxon-Wood potential



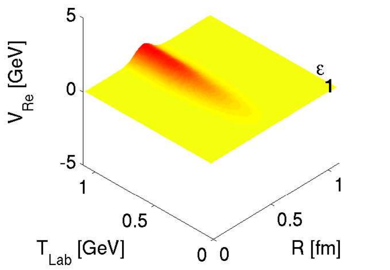
c. Dirac+Saxon.W (a.+b.)



d. OMP real potential

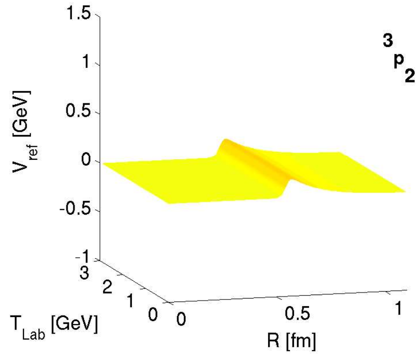


e. OMP imaginary potential

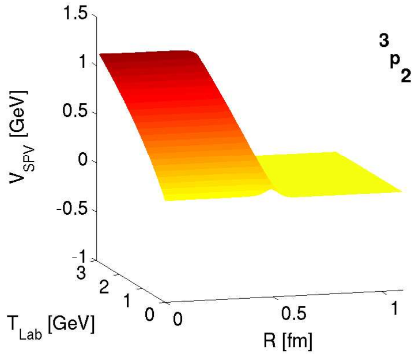


f. full real (a.+b.+d.)

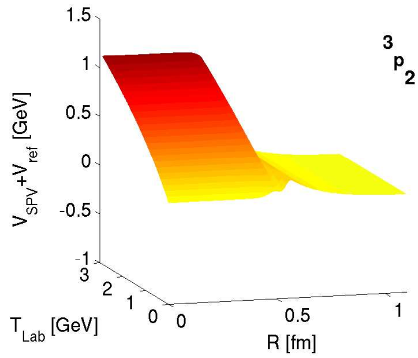
Figure 7.47: Three dimensional representation of the potential strengths for ϵ_1, $T = 0$. (*Continuation of Fig. 7.33*).



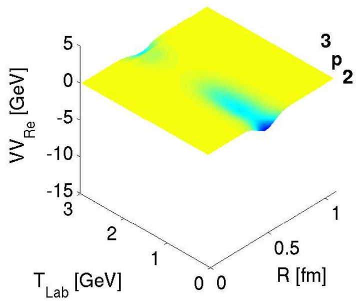
a. Dirac reference potential



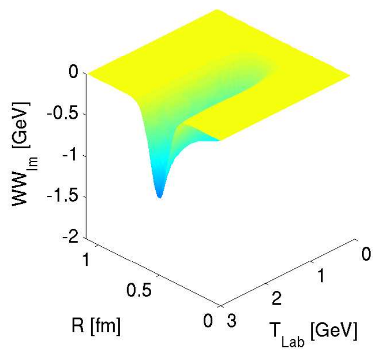
b. Saxon-Wood potential



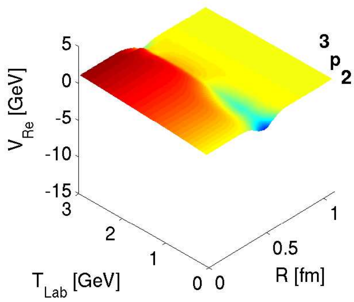
c. Dirac+Saxon.W (a.+b.)



d. OMP real potential

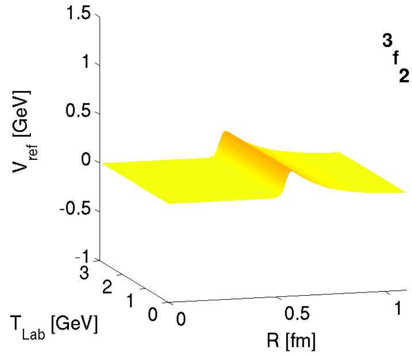


e. OMP imaginary potential

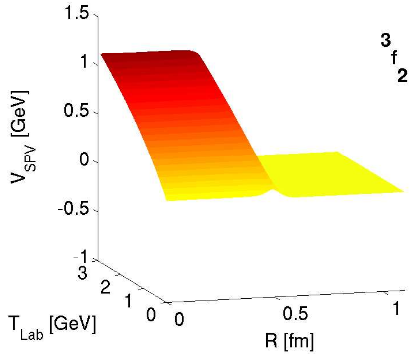


f. full real (a.+b.+d.)

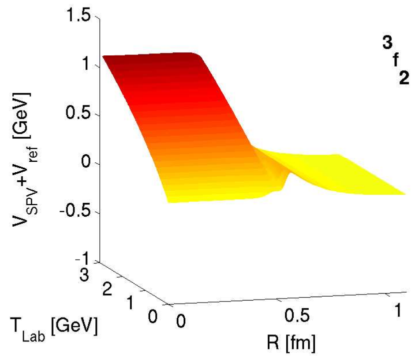
Figure 7.48: Three dimensional representation of the potential strengths for ${}^3P_2$, $T = 1$. (*Continuation of Fig. 7.33*).



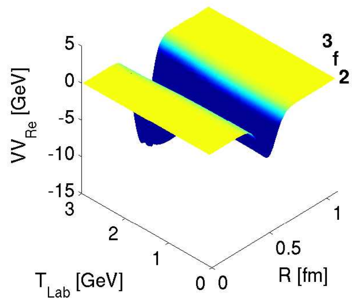
a. Dirac reference potential



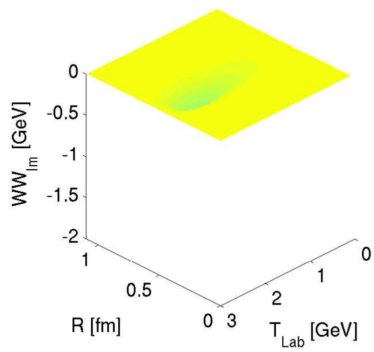
b. Saxon-Wood potential



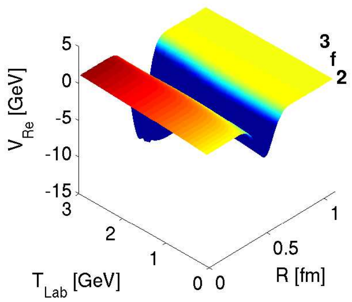
c. Dirac+Saxon.W (a.+b.)



d. OMP real potential

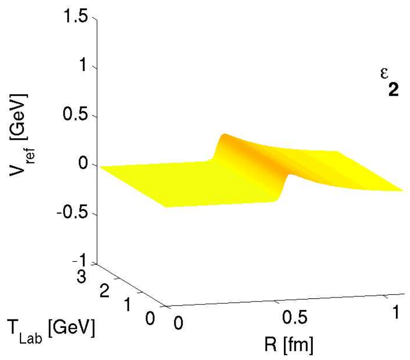


e. OMP imaginary potential

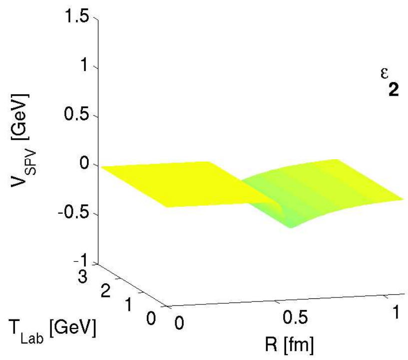


f. full real (a.+b.+d.)

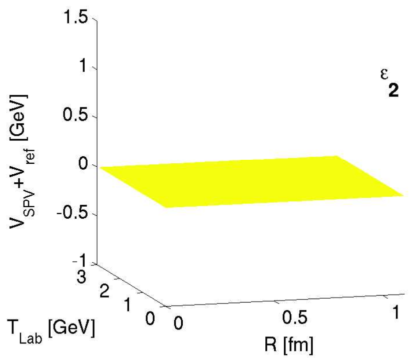
Figure 7.49: Three dimensional representation of the potential strengths for $^3F_2$, $T = 1$. (*Continuation of Fig. 7.33*).



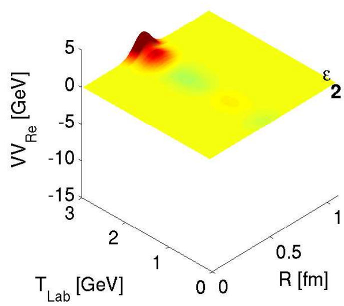
a. Dirac reference potential



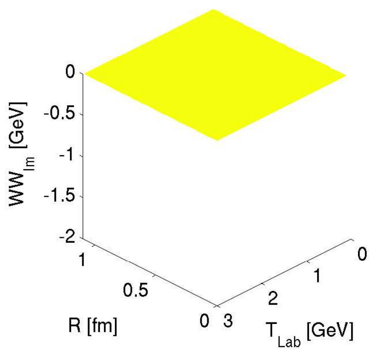
b. Saxon-Wood potential



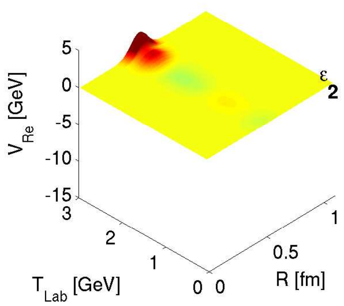
c. Dirac+Saxon.W (a.+b.)



d. OMP real potential



e. OMP imaginary potential



f. full real (a.+b.+d.)

Figure 7.50: Three dimensional representation of the potential strengths for ϵ_2, $T = 1$. (*Continuation of Fig. 7.33*).

Table 7.7: Values of the OMP3 potentials strengths for the single channels $^1S_0$, $^1D_2$, $^1G_4$ ($T = 1$) GWU/VPI SAID. The values calculated in the Eq. (6.71).

	$^1S_0$		$^1D_2$		$^1G_4$	
T_{Lab}(MeV)	V_0	W_0	V_0	W_0	V_0	W_0
200.0	-259.140	-0.290	-317.055	28.082	-4.369	-0.729
300.0	127.180	-0.463	-404.961	12.056	-46.016	0.399
400.0	476.603	-1.902	-431.259	-69.757	-86.119	0.655
500.0	789.076	-6.584	-412.549	-182.512	-122.769	-3.230
600.0	1065.215	-15.737	-363.114	-289.911	-154.600	-13.008
700.0	1306.225	-29.543	-295.055	-370.895	-180.736	-28.556
800.0	1513.827	-47.304	-218.423	-421.749	-200.741	-48.412
900.0	1690.179	-67.803	-141.352	-451.167	-214.566	-70.589
1000.0	1837.801	-89.677	-70.194	-473.266	-222.503	-93.313
1100.0	1959.499	-111.712	-9.650	-501.437	-225.129	-115.507
1200.0	2058.286	-133.023	37.092	-544.453	-223.258	-136.966
1300.0	2137.312	-153.105	68.231	-605.102	-217.891	-158.261
1400.0	2199.779	-171.792	83.212	-680.954	-210.162	-180.455
1500.0	2248.873	-189.152	82.598	-766.464	-201.288	-204.740
1600.0	2287.681	-205.371	67.932	-855.505	-192.523	-232.087
1700.0	2319.118	-220.635	41.608	-943.524	-185.100	-262.998
1800.0	2345.853	-235.070	6.735	-1028.744	-180.184	-297.391
1900.0	2370.228	-248.715	-32.994	-1112.155	-178.822	-334.631
2000.0	2394.182	-261.550	-73.441	-1196.403	-181.888	-373.680
2100.0	2419.180	-273.550	-110.149	-1283.983	-190.040	-413.279
2200.0	2446.131	-284.740	-138.480	-1375.396	-203.658	-452.127
2300.0	2475.314	-295.228	-153.749	-1467.978	-222.805	-488.935
2400.0	2506.301	-305.187	-151.352	-1555.986	-247.166	-522.361
2500.0	2537.883	-314.777	-126.906	-1632.124	-276.005	-550.819
2600.0	2567.989	-324.046	-76.379	-1689.926	-308.110	-572.307
2700.0	2593.616	-332.846	3.775	-1725.326	-341.741	-584.537
2800.0	2610.746	-340.930	116.487	-1734.120	-374.585	-585.842
2900.0	2614.275	-348.411	263.933	-1699.975	-403.697	-577.623
3000.0	2597.935	-356.920	447.405	-1564.955	-425.459	-569.398

Table 7.8: Values of the OMP3 potentials strengths for the triplet channels $^3P_0$, $^3P_1$, $^3F_3$ $(T = 1)$ GWU/VPI SAID. The values calculated in the Eq. (6.71).

	$^3P_0$		$^3P_1$		$^3F_3$	
T_{Lab}(MeV)	V_0	W_0	V_0	W_0	V_0	W_0
200.0	-81.883	-0.961	407.409	1.413	-120.689	12.218
300.0	162.446	3.971	679.522	1.759	-127.031	39.568
400.0	415.644	0.445	911.738	-0.523	-90.805	7.054
500.0	666.280	-20.046	1108.429	-3.735	-27.079	-84.406
600.0	905.344	-58.284	1273.705	-7.740	51.693	-209.847
700.0	1126.028	-108.415	1411.413	-14.284	135.477	-338.178
800.0	1323.515	-161.227	1525.135	-25.980	216.471	-443.642
900.0	1494.759	-207.597	1618.192	-44.990	288.919	-511.510
1000.0	1638.274	-241.052	1693.639	-72.032	348.919	-539.186
1100.0	1753.921	-258.996	1754.268	-105.970	394.237	-534.068
1200.0	1842.687	-262.651	1802.609	-144.030	424.114	-509.588
1300.0	1906.475	-256.019	1840.926	-182.502	439.084	-480.721
1400.0	1947.889	-244.334	1871.219	-217.700	440.778	-460.007
1500.0	1970.017	-232.467	1895.226	-246.938	431.739	-454.854
1600.0	1976.218	-223.703	1914.419	-269.262	415.235	-466.513
1700.0	1969.908	-219.144	1930.006	-285.773	395.065	-490.740
1800.0	1954.343	-217.837	1942.931	-299.418	375.373	-519.852
1900.0	1932.405	-217.542	1953.874	-314.259	360.462	-545.576
2000.0	1906.388	-215.871	1963.249	-334.311	354.600	-561.895
2100.0	1877.782	-211.451	1971.206	-362.165	361.834	-567.071
2200.0	1847.061	-204.655	1977.631	-397.665	385.802	-564.087
2300.0	1813.465	-197.575	1982.144	-437.005	429.542	-559.087
2400.0	1774.785	-193.031	1984.100	-472.611	495.304	-557.897
2500.0	1727.152	-192.779	1982.591	-494.152	584.362	-561.546
2600.0	1664.819	-195.601	1976.441	-490.959	696.826	-562.785
2700.0	1579.949	-196.696	1964.210	-455.963	831.449	-547.059
2800.0	1462.395	-190.733	1944.192	-391.034	985.444	-503.195
2900.0	1299.492	-182.210	1914.418	-313.315	1154.289	-451.311
3000.0	1075.837	-208.261	1872.649	-261.670	1331.546	-498.079

Table 7.9: Values of the OMP3 potentials strengths for the coupled channels $^3P_2$, $^3F_2$, $^3PF_2$ ($T = 1$) GWU/VPI SAID. The values calculated in the Eq. (6.71).

	$^3P_2$		$^3F_2$		$^3PF_2$	
T_{Lab}(MeV)	V_0	W_0	V_0	W_0	V_0	W_0
200.0	-154.526	-0.686	-782.756	0.104	-13.008	0.00
300.0	-123.450	1.010	-733.547	-0.095	-9.359	0.00
400.0	-102.523	0.998	-691.454	-0.097	-5.170	0.00
500.0	-88.884	-1.947	-655.228	-0.095	-0.991	0.00
600.0	-80.170	-7.612	-623.965	-0.362	2.744	0.00
700.0	-74.473	-15.277	-597.061	-1.007	5.719	0.00
800.0	-70.304	-24.138	-574.170	-1.964	7.714	0.00
900.0	-66.553	-33.449	-555.160	-3.078	8.613	0.00
1000.0	-62.445	-42.539	-540.072	-4.208	8.382	0.00
1100.0	-57.509	-50.803	-529.075	-5.290	7.071	0.00
1200.0	-51.528	-57.707	-522.425	-6.361	4.805	0.00
1300.0	-44.510	-62.822	-520.422	-7.531	1.770	0.00
1400.0	-36.639	-65.875	-523.367	-8.930	-1.785	0.00
1500.0	-28.243	-66.798	-531.518	-10.641	-5.567	0.00
1600.0	-19.751	-65.758	-545.051	-12.652	-9.235	0.00
1700.0	-11.652	-63.147	-564.011	-14.829	-12.416	0.00
1800.0	-4.460	-59.533	-588.275	-16.927	-14.710	0.00
1900.0	1.330	-55.566	-617.508	-18.630	-15.695	0.00
2000.0	5.280	-51.860	-651.117	-19.627	-14.940	0.00
2100.0	7.047	-48.880	-688.211	-19.688	-12.009	0.00
2200.0	6.425	-46.865	-727.558	-18.729	-6.471	0.00
2300.0	3.382	-45.804	-767.544	-16.853	2.096	0.00
2400.0	-1.899	-45.524	-806.125	-14.322	14.095	0.00
2500.0	-8.981	-45.854	-840.788	-11.485	29.900	0.00
2600.0	-17.132	-46.881	-868.510	-8.658	49.852	0.00
2700.0	-25.287	-49.203	-885.710	-5.988	74.250	0.00
2800.0	-32.010	-54.079	-888.210	-3.406	103.343	0.00
2900.0	-35.449	-63.252	-871.193	-0.774	137.323	0.00
3000.0	-33.303	-78.161	-829.155	1.563	176.316	0.00

Table 7.10: Values of the OMP3 potentials strengths for the single channels $^1P_1$, $^1F_3$, $^1H_5$ $(T = 0)$ GWU/VPI SAID. The values calculated in the Eq. (6.71).

	$^1P_1$		$^1F_3$		$^1H_4$	
T_{Lab}(MeV)	V_0	W_0	V_0	W_0	V_0	W_0
200.0	460.453	0.156	-94.307	0.00	-284.79450	0.00
250.0	562.608	0.186	-69.417	0.00	-276.60664	0.00
300.0	649.552	-0.011	-45.964	0.00	-271.17457	0.00
350.0	726.158	-0.316	-23.057	0.00	-266.22250	0.00
400.0	796.110	-0.739	-0.064	0.00	-260.22159	0.00
450.0	862.039	-1.683	23.426	0.00	-252.28500	0.00
500.0	925.658	-4.103	47.635	0.00	-242.06290	0.00
550.0	987.892	-9.492	72.620	0.00	-229.63751	0.00
600.0	1049.021	-19.733	98.314	0.00	-215.41817	0.00
650.0	1108.809	-36.844	124.550	0.00	-200.03633	0.00
700.0	1166.638	-62.696	151.098	0.00	-184.24061	0.00
750.0	1221.650	-98.762	177.693	0.00	-168.79184	0.00
800.0	1272.873	-145.988	204.066	0.00	-154.35808	0.00
850.0	1319.363	-204.811	229.980	0.00	-141.40968	0.00
900.0	1360.334	-275.351	255.257	0.00	-130.11427	0.00
950.0	1395.296	-357.748	279.810	0.00	-120.23185	0.00
1000.0	1424.187	-452.544	303.677	0.00	-111.00980	0.00
1050.0	1447.512	-560.973	327.052	0.00	-101.07790	0.00
1100.0	1466.474	-684.896	350.312	0.00	-88.34341	0.00
1150.0	1483.109	-826.073	374.056	0.00	-69.88606	0.00
1200.0	1500.425	-984.323	399.130	0.00	-41.85311	0.00

Table 7.11: Values of the OMP3 potentials strengths for the triplet channels $^3D_2$, $^3G_4$, $^3I_6$, $(T = 0)$ GWU/VPI SAID. The values calculated in the Eq. (6.71).

	$^3D_2$		$^3G_4$		$^3I_6$	
T_{Lab}(MeV)	V_0	W_0	V_0	W_0	V_0	W_0
200.0	-766.39	0.00	-125.440	0.00	-211.320	0.00
250.0	-843.96	0.00	-185.387	0.00	-231.363	0.00
300.0	-878.06	0.00	-238.122	0.00	-249.697	0.00
350.0	-879.69	0.00	-283.196	0.00	-266.503	0.00
400.0	-857.93	0.00	-320.605	0.00	-281.899	0.00
450.0	-820.14	0.00	-350.717	0.00	-295.940	0.00
500.0	-772.13	0.00	-374.205	0.00	-308.641	0.00
550.0	-718.34	0.00	-391.972	0.00	-319.978	0.00
600.0	-662.00	0.00	-405.083	0.00	-329.904	0.00
650.0	-605.33	0.00	-414.693	0.00	-338.360	0.00
700.0	-549.65	0.00	-421.977	0.00	-345.283	0.00
750.0	-495.64	0.00	-428.055	0.00	-350.622	0.00
800.0	-443.43	0.00	-433.929	0.00	-354.342	0.00
850.0	-392.82	0.00	-440.404	0.00	-356.441	0.00
900.0	-343.46	0.00	-448.022	0.00	-356.958	0.00
950.0	-294.99	0.00	-456.990	0.00	-355.986	0.00
1000.0	-247.21	0.00	-467.107	0.00	-353.680	0.00
1050.0	-200.31	0.00	-477.697	0.00	-350.271	0.00
1100.0	-154.96	0.00	-487.535	0.00	-346.073	0.00
1150.0	-112.56	0.00	-494.777	0.00	-341.500	0.00
1200.0	-75.35	0.00	-496.889	0.00	-337.071	0.00

Table 7.12: Values of the OMP3 potentials strengths for the coupled channels $^3S_1$, $^3D_1$, $^3SD_1$, ($T = 0$) GWU/VPI SAID. The values calculated in the Eq. (6.71).

	$^3S_1$		$^3D_1$		$^3SD_1$	
T_{Lab}(MeV)	V_0	W_0	V_0	W_0	V_0	W_0
200.0	-105.367	0.00	-556.062	-0.309	11.140	0.00
250.0	-93.988	0.00	-537.185	-0.385	14.701	0.00
300.0	-84.402	0.00	-520.524	0.098	18.824	0.00
350.0	-76.296	0.00	-505.798	0.475	23.584	0.00
400.0	-69.366	0.00	-493.000	0.283	28.969	0.00
450.0	-63.321	0.00	-482.318	-0.353	34.895	0.00
500.0	-57.890	0.00	-474.060	-0.966	41.218	0.00
550.0	-52.831	0.00	-468.570	-1.269	47.754	0.00
600.0	-47.935	0.00	-466.154	-1.573	54.289	0.00
650.0	-43.038	0.00	-467.000	-2.805	60.596	0.00
700.0	-38.022	0.00	-471.101	-6.128	66.451	0.00
750.0	-32.826	0.00	-478.175	-12.416	71.645	0.00
800.0	-27.452	0.00	-487.586	-22.018	76.000	0.00
850.0	-21.971	0.00	-498.270	-35.180	79.384	0.00
900.0	-16.533	0.00	-508.650	-53.112	81.727	0.00
950.0	-11.370	0.00	-516.565	-78.864	83.034	0.00
1000.0	-6.807	0.00	-519.184	-115.822	83.399	0.00
1050.0	-3.264	0.00	-512.935	-159.546	83.023	0.00
1100.0	-1.270	0.00	-493.422	-175.928	82.226	0.00
1150.0	-1.464	0.00	-455.345	-54.868	81.464	0.00
1200.0	-4.603	0.00	-392.430	475.971	81.341	0.00

Conclusions

Medium energy NN scattering, as is accepted for low energy nuclear physics in general, is determined from proton, nucleon and meson degrees of freedom in the long range soft interaction sector, the quark gluon degrees of freedom govern the short distance hard processes. The identification and parameterization, of the combined long and short range NN domains, is the topic of this thesis. The formalism for two coupled Dirac equations, within constraint instant form dynamics, is used to study the NN interaction. Explicitly energy dependent coupled channel potentials, for use in partial wave Schrödinger like equations, with nonlinear and complicated derivative terms, result. The interactions are inspired by meson exchange of π, η, ρ, ω and σ mesons for which we adjust coupling constants. This yields, in the first instant, high quality fits to the Arndt phase shifts 0 to 300 MeV. Second, the potentials show a universal, independent from angular momentum, core potential which is generated with the relativistic meson exchange dynamics. Extrapolations towards higher energies, up to T_{Lab} equal 3 GeV, allow to separate a QCD dominated short range zone as well as inelastic nucleon excitation mechanism contributing to meson production. A local optical model, in addition to the meson exchange Dirac potential, produces agreement between theoretical and phase shifts data. The optical model potentials reflect a short lived complex multi hadronic intermediate structure formation, $r \sim 0.5$ fm, of which the optical model parameters give a consistent picture. For future work, the here presented phenomenological access encourages a more microscopic and detailed use of QCD.

Appendix A

Tools, Definitions and Conventions

A.1 Majorana Representation

The four γ matrices have real elements and satisfy

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}, \tag{A.1}$$

with $g_{00} = -1$.

The sixteen basic matrices $\gamma_i \quad (i = 1, 2, ...16)$ are $1, \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3, \gamma_\mu, \gamma_5\gamma_\mu, \gamma_\mu\gamma_\nu$ $(\mu < \nu)$. They are all real and $\gamma_5^2 = -1, \gamma_5^T = -\gamma_5$, where T denotes the ordinary transposed of a matrix. [566]

For any four by four matrix Γ define the adjoint

$$\tilde{\Gamma} = -\gamma_0 \Gamma^T \gamma_0. \tag{A.2}$$

then, for any two hermitian anticommuting spinors α_1 and α_2,

$$\bar{\alpha}_1 \gamma \alpha_2 = \bar{\alpha}_2 \tilde{\Gamma} \alpha_1, \quad \bar{\alpha} = \alpha \gamma^0. \tag{A.3}$$

Now $\tilde{\gamma}_i = \gamma_i$ for $1, \gamma_5, \gamma_5\gamma_\mu$, while $\tilde{\gamma}_i = -\gamma_i$ for $\gamma_\mu, \gamma_\mu\gamma\nu(\mu < \nu)$. The 16 matrices γ_i have squares equal to ± 1. For any γ_i define γ^i so that $\gamma_i\gamma^i = +1$. The following rearrangement formula is then valid

$$(\bar{\alpha}_1 \psi)\alpha_2 = -\frac{1}{4}\sum_i (\bar{\alpha}_1 \gamma_i \alpha_2)\gamma^i \psi, \tag{A.4}$$

where α_1, α_2 and ψ are any three spinors. The minus sign comes from the anticommutation property of the spinors.

A.2 The Fierz Transformation

Fierz transformation [236] is a name given to the expression of a certain product of nondiagonal matrix elements of Dirac Γ-matrices as an expansion into products of diagonal matrix elements, such as

$$(\bar{a}\,\Gamma_i\,b)(\bar{b}\,\Gamma_j\,a) = \sum_{k,l=1}^{16} c_{kl}(\bar{a}\,\Gamma_k\,a)(\bar{b}\,\Gamma_l\,b). \tag{A.5}$$

Here Γ_i stands for one of the 16 Dirac matrices $\{\mathbf{1}, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu}\}$ constituting a linearly independent basis in the space of complex 4×4 matrices. The matrix elements denote products in Dirac-index space only, i. e.,

$$(\bar{a}\,\Gamma_i\,b) = \bar{\psi}_a(\mathbf{r},t)\,\Gamma_i\,\psi_b(\mathbf{r},t). \tag{A.6}$$

The Fierz transformation is useful for expressing exchange matrix elements in terms of densities, currents, and other diagonal ones, which greatly eases their use in, for example, relativistic mean-field theories.

In the context of nonlinear self-coupling of meson fields, higher-order versions of the Fierz transformation have become of interest. If we express the order as the number of Γ-matrices involved, the above Eq. (A.5) is of second order, and third

$$(\bar{a}\,\Gamma_i\,b)(\bar{b}\,\Gamma_j\,c)(\bar{c}\,\Gamma_k\,a) = \sum_{l,m,n=1}^{16} c_{lmn}(\bar{a}\,\Gamma_l\,a)(\bar{b}\,\Gamma_m\,b)(\bar{c}\,\Gamma_n\,c), \tag{A.7}$$

and fourth order:

$$(\bar{a}\,\Gamma_i\,b)(\bar{b}\,\Gamma_j\,c)(\bar{c}\,\Gamma_k\,d)(\bar{d}\,\Gamma_l\,a) = \sum_{m,n,p,q=1}^{16} c_{mnpq}(\bar{a}\,\Gamma_m\,a)(\bar{b}\,\Gamma_n\,b)(\bar{c}\,\Gamma_p\,c)(\bar{d}\,\Gamma_q\,d). \tag{A.8}$$

In many applications, the wave function indices will all be summed over, so that it is sufficient to deal with an expression symmetrized over the indices. Thus, on the left-hand side of Eq. (A.5) in the second-order case we can write

$$\sum_{ab}(\bar{a}\,\Gamma_i\,b)(\bar{b}\,\Gamma_j\,a) = \frac{1}{2}\sum_{ab}\left[(\bar{a}\,\Gamma_i\,b)(\bar{b}\,\Gamma_j\,a) + (\bar{b}\,\Gamma_i\,a)(\bar{a}\,\Gamma_j\,b)\right], \tag{A.9}$$

and the right-hand side of Eq. (A.5) will be symmetrized exactly in the same way.

Using the notation $\sum_{\{abc\ldots\}}$ to refer to the sum over all permutations of the symbols $a, b, c\ldots$, we can reformulate the Fierz transformation problem for the symmetrized matrix element as

$$\sum_{\{ab\}}(\bar{a}\,\Gamma_i\,b)(\bar{b}\,\Gamma_j\,a) = \sum_{\{ab\}}\sum_{kl} c_{kl}(\bar{a}\,\Gamma_k\,a)(\bar{b}\,\Gamma_l\,b), \tag{A.10}$$

where the factor $\frac{1}{2}$ has been dropped on both sides.

It is important to realize that because of the product structure, the right-hand side is symmetric under an exchange of the Γ matrices as well. It is thus useful to introduce a notation for symmetrized terms,

$$\Gamma_k \otimes \Gamma_l = \sum_{\{ab\}} (\bar{a}\,\Gamma_k\,a)(\bar{b}\,\Gamma_l\,b) = (\bar{a}\,\Gamma_k\,a)(\bar{b}\,\Gamma_l\,b) + (\bar{a}\,\Gamma_l\,a)(\bar{b}\,\Gamma_k\,b), \tag{A.11}$$

which can easily be generalized to higher order, for example, in third order the symmetrized problem becomes

$$\sum_{\{abc\}} (\bar{a}\,i\,b)(\bar{b}\,j\,c)(\bar{c}\,k\,a) = \sum_{l\le m\le n} c_{lmn}\Gamma_l \otimes \Gamma_m \otimes \Gamma_n, \tag{A.12}$$

with

$$\Gamma_l \otimes \Gamma_m \otimes \Gamma_n = \sum_{\{abc\}} (\bar{a}\,\Gamma_l\,a)(\bar{b}\,\Gamma_m\,b)(\bar{c}\,\Gamma_n\,c), \tag{A.13}$$

where symmetrization could equivalently be carried out in the indices l, m, n instead of a, b, c. Terms of fourth and higher orders are defined analogously.

The second-order Fierz transformation as defined in Eq. (A.5) can be viewed as a system of equations obtained by comparing coefficients in the $4^4 = 16^2$ dimensional space spanned by the spinors ψ_a, ψ_b, $\bar{\psi}_a$, and $\bar{\psi}_b$. The coefficients are given by the components of the Γ-matrices and thus can be expressed as complex integers. The unknowns c_{kl} are 16×16 in number, so that we have exactly the right number of equations, and since the Γ-matrices form a basis for the 4×4 complex matrices, the decomposition (A.5) is always possible.

The solution of this system of linear equations can be carried out using the standard Gauss elimination algorithm, provided that the coefficients are not treated in floating point arithmetic, but as exact complex fractions.

For third-order Fierz transformations the dimension of the system of equations is 16^3 and it is 16^4 for fourth order, that is, the complexity in going from second order to fourth order is in the ratio 1:16:256. For the latter case practical solution would require substantial computing resources, but fortunately in cases of practical interest the number of terms in the expansion can be reduced substantially by symmetry and invariance requirements. In the symmetrized case of the preceeding section, for example, the dimension in fourth order is reduced by the number of permutations 4!.

The expansion into products of the diagonal matrix elements of the Γ-matrices is always possible, but usually is not the most useful expression of the Fierz transformation. To see this, let us look at an important special case: that of identity matrices on the left-hand side. The decomposition problem thus is

$$(\bar{a}\,b)(\bar{b}\,a) = \sum_{jk} c_{jk}(\bar{a}\,\Gamma_j\,a)(\bar{b}\,\Gamma_k\,b), \tag{A.14}$$

or, in symmetrized form,

$$\sum_{\{ab\}} (\bar{a}\, b)(\bar{b}\, a) = \sum_{jk} c_{jk} \Gamma_j \otimes \Gamma_k. \tag{A.15}$$

Since the left-hand side is a Dirac scalar, this means that the right-hand side also can contain only scalar combinations of Γ-matrices. The only scalar combinations built out of products of two Γ-matrices are $\mathbf{1} \otimes \mathbf{1}$, $\gamma_5 \otimes \gamma_5$, $\gamma_\mu \otimes \gamma^\mu$, $\gamma_5\gamma_\mu \otimes \gamma_5\gamma^\mu$, and $\sigma_{\mu\nu} \otimes \sigma^{\mu\nu}$, assuming the familiar index summation convention. The Fierz transformation problem in this case thus can be restated as (note that here because of the complete symmetry of all terms, the symmetrization can be omitted):

$$\begin{aligned}(\bar{a}\, b)(\bar{b}\, a) \;=\;& c_1(\bar{a}\, a)(\bar{b}\, b) + c_2(\bar{a}\, \gamma_5\, a)(\bar{b}\, \gamma_5\, b) + c_3(\bar{a}\, \gamma_\mu\, a)(\bar{b}\, \gamma^\mu\, b) \\ &+ c_4(\bar{a}\, \gamma_5\gamma_\mu\, a)(\bar{b}\, \gamma_5\gamma^\mu\, b) + c_5(\bar{a}\, \sigma_{\mu\nu}\, a)(\bar{b}\, \sigma^{\mu\nu}\, b).\end{aligned} \tag{A.16}$$

Note that symmetrization works slightly differently in this case: the scalar products sometimes make certain index combinations appear repeatedly in the expansion, but it is still sufficient to include only one ordering of the Γ matrices in the symmetrized terms.

Eq. (A.16) corresponds to 256 equations for the 5 unknown coefficients. Clearly most equations will be redundant; eliminating them from the statement of the problem, however, turns out to complicate the solution, but the high degree of redundancy provides a welcome check for completeness and consistency of the assumed decomposition.

The decomposition of the symmetrized term in second order is

$$\begin{aligned}(\bar{a}\, b)(\bar{b}\, a) \;=\;& \tfrac{1}{4}(\bar{a}\, a)(\bar{b}\, b) + \tfrac{1}{4}(\bar{a}\, \gamma_5\, a)(\bar{b}\, \gamma_5\, b) + \tfrac{1}{4}(\bar{a}\, \gamma_\mu\, a)(\bar{b}\, \gamma^\mu\, b) \\ &- \tfrac{1}{4}(\bar{a}\, \gamma_5\gamma_\mu\, a)(\bar{b}\, \gamma_5\gamma^\mu\, b) + \tfrac{1}{8}(\bar{a}\, \sigma_{\mu\nu}\, a)(\bar{b}\, \sigma^{\mu\nu}\, b).\end{aligned} \tag{A.17}$$

In third order the symmetrization is no longer trivial as it was in the second-order case. It may be surprising that terms with $\gamma_5\sigma_{\mu\nu}$ must be included; these can be equivalently formulated using the identity

$$\gamma_5\sigma_{\mu\nu} = \tfrac{\mathrm{i}}{2}\epsilon_{\kappa\lambda\mu\nu}\sigma^{\kappa\lambda}, \tag{A.18}$$

but retaining the matrix γ_5 makes the space-reversal properties of the terms more readily apparent. Note that either way our basis still consists of only 16 linearly independent matrices.

Appendix B

Technical Details

B.1 Clebsch-Gordan Coefficient

A mathematical symbol used to integrate products of three Spherical Harmonics. Clebsch-Gordan coefficients commonly arise in applications involving the addition of angular momentum in quantum mechanics [78]. If products of more than three Spherical Harmonics are desired, then a generalization known as Wigner 6j-Symbol or Wigner 9j-Symbol is used. The Clebsch-Gordan coefficients are written

$$C^{j}_{m_1 m_2} = (j_1 j_2 m_1 m_2 | j_1 j_2 j m),$$

and are defined by

$$\Psi_{JM} = \sum_{M=M_1+M_2} C^{J}_{M_1 M_2} \Psi_{M_1 M_2},$$

where $J \equiv J_1 + J_2$. The Clebsch-Gordan coefficients are sometimes expressed using the related Racah V-Coefficient

$$V(j_1 j_2 j; m_1 m_2 m), \tag{B.1}$$

or Wigner 3j-Symbol

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}.$$

Connections among the three are

$$(j_1 j_2 m_1 m_2 | j_1 j_2 m) = (-1)^{-j_1+j_2-m} \sqrt{2j+1} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix},$$

$$(j_1 j_2 m_1 m_2 | j_1 j_2 j m) = (-1)^{j+m} \sqrt{2j+1} V(j_1 j_2 j; m_1 m_2 \ -m) \tag{B.2}$$

$$V(j_1j_2j;m_1m_2m) = (-1)^{-j_1+j_2+j}\begin{pmatrix} j_1 & j_2 & j_1 \\ m_2 & m_1 & m_2 \end{pmatrix}. \tag{B.3}$$

They have the symmetry

$$(j_1j_2m_1m_2|j_1j_2jm) = (-1)^{j_1+j_2-j}(j_2j_1m_2m_1|j_2j_1jm), \tag{B.4}$$

and obey the orthogonality relationships

$$\sum_{j,m}(j_1j_2m_1m_2|j_1j_2jm)(j_1j_2jm|j_1j_2m_1'm_2') = \delta_{m_1m_1'}\delta_{m_2m_2'},$$

$$\sum_{m_1,m_2}(j_1j_2m_1m_2|j_1j_2jm)(j_1j_2j'm'|j_1j_2m_1m_2) = \delta_{jj'}\delta_{mm'}.$$

Racah W-Coefficient are written

$$(J_1J_2[J']J_3|J_1,J_2J_3[J'']) = \sqrt{(2J'+1)(2J''+1)}\,W(J_1J_2JJ_3;J'J''),$$

and

$$(J_1J_2[J']J_3|J_1J_3[J'']J_2) = \sqrt{(2J'+1)(2J''+1)}\,W(J_1'J_3J_2J'';JJ_1).$$

Wigner 3j-Symbol

The Wigner 3j-symbols have the symmetries

$$\begin{aligned}
\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} &= \begin{pmatrix} j_2 & j & j_1 \\ m_2 & m & m_1 \end{pmatrix} \\
&= \begin{pmatrix} j & j_1 & j_2 \\ m & m_1 & m_2 \end{pmatrix} = (-1)^{j_1+j_2+j}\begin{pmatrix} j_2 & j_1 & j \\ m_2 & m_1 & m \end{pmatrix} \\
&= (-1)^{j_1+j_2+j}\begin{pmatrix} j_1 & j & j_2 \\ m_1 & m & m_2 \end{pmatrix} \\
&= (-1)^{j_1+j_2+j}\begin{pmatrix} j & j_2 & j_1 \\ m & m_2 & m_1 \end{pmatrix} \\
&= (-1)^{j_1+j_2+j}\begin{pmatrix} j_1 & j_2 & j \\ -m_1 & -m_2 & -m \end{pmatrix}.
\end{aligned} \tag{B.5}$$

The symbols obey the orthogonality relations

$$\sum_{j,m}(2j+1)\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{pmatrix} = \delta_{m_1 m_1'}\delta_{m_2 m_2'}, \tag{B.6}$$

$$\sum_{m_1,m_2}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{pmatrix} = \delta_{jj'}\delta_{mm'}, \tag{B.7}$$

where δ_{ij} is the Kronecker Delta.

General formulas are very complicated, but some specific cases are

$$\begin{pmatrix} j_1 & j_2 & j_1+j_2 \\ m_1 & m_2 & -m_1-m_2 \end{pmatrix} = (-1)^{j_1-j_2+m_1+m_2}$$

$$\times\left[\frac{(2j_1)!(2j_2)!}{(2j_1+2j_2+1)!(j_1+m_1)!}\frac{(j_1+j_2+m_1+m_2)!(j_1+j_2-m_1-m_2)!}{(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}\right]^{1/2} \tag{B.8}$$

$$\begin{pmatrix} j_1 & j_2 & j \\ j_1 & -j_1- & m \end{pmatrix} = (-1)^{-j_1+j_2+m}$$

$$\times\left[\frac{(2j_1)!(-j_1+j_2+j)!}{(j_1+j_2+j+1)!(j_1-j_2+j)!}\frac{(j_1+j_2+m)!(j-m)!}{(j_1+j_2-j)!(-j_1+j_2-m)!(j+m)!}\right]^{1/2} \tag{B.9}$$

$$\begin{pmatrix} j_1 & j_2 & j \\ 0 & 0 & 0 \end{pmatrix} = \begin{cases} (-1)^g\sqrt{\frac{(2g-2j_1)(2g-2j_2)!(2g-2j)!}{(2g+1)!}}\frac{g!}{(g-j_1)!(g-j_2)!(g-j)!} \\ \quad \text{if } J=2g \\ 0 \\ \quad \text{if } J=2g+1, \end{cases} \tag{B.10}$$

for $J \equiv j_1+j_2+j$.

For Spherical Harmonics $Y_{lm}(\theta,\phi)$,

$$Y_{l_1m_1}(\theta,\phi)Y_{l_2m_2}(\theta,\phi)$$

$$= \sum_{l,m}\sqrt{\frac{(2l_1+1)(2l_2+1)(2l+1)}{4\pi}}\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} Y^*_{lm}(\theta,\psi)\begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix}.$$

For values of l_3 obeying the Triangle Condition $\Delta(l_1l_2l_3)$,

$$\int Y_{l_1m_1}(\theta,\phi)Y_{l_2m_2}(\theta,\phi)Y_{l_3m_3}(\theta,\phi)\sin\theta\,d\theta\,d\phi$$

$$= \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

and

$$\frac{1}{2}\int P_{l_1}(\cos\theta)P_{l_2}(\cos\theta)P_{l_3}(\cos\theta)\sin\theta\, d\theta = \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}^2. \quad \text{(B.11)}$$

Wigner 6j-Symbol
A generalization of Clebsch-Gordan Coefficients and Wigner 3j-Symbol which arises in the coupling of three angular momenta. Let tensor operators $T^{(k)}$ and $U^{(k)}$ act, respectively, on subsystems 1 and 2 of a system, with subsystem 1 characterized by angular momentum j_1 and subsystem 2 by the angular momentum j_2. Then the matrix elements of the scalar product of these two tensor operators in the coupled basis $J = j_1 + j_2$ are given by

$$(\tau_1' j_1' \tau_2' j_2' J' M' | T^{(k)} \cdot U^{(k)} | \tau_1 j_1 \tau_2 j_2 J M)$$

$$= \delta_{JJ'}\delta_{MM'}(-1)^{j_1+j_2'+J} \begin{Bmatrix} J & j_2' & j_1' \\ k & j_1 & j_2 \end{Bmatrix} (\tau_1' j_1' || T^{(k)} || \tau_1 j_1)(\tau_2' j_2' || U^{(k)} || \tau_2 j_2), \quad \text{(B.12)}$$

where $\begin{Bmatrix} J & j_2' & j_1' \\ k & j_1 & j_2 \end{Bmatrix}$ is the Wigner 6j-symbol and τ_1 and τ_2 represent additional pertinent quantum numbers characterizing subsystems 1 and 2. The analytic forms of the 6j-symbol are written for simple cases

$$\begin{Bmatrix} a & b & c \\ 0 & c & b \end{Bmatrix} = \frac{(-1)^s}{\sqrt{(2b+1)(2c+1)}},$$

$$\begin{Bmatrix} a & b & c \\ 1 & c & b \end{Bmatrix} = \frac{2(-1)^{s+1}X}{\sqrt{2b(2b+1)(2b+42)2c(2c+1)(2c+2)}},$$

$$\begin{Bmatrix} a & b & c \\ 2 & c & b \end{Bmatrix} = \frac{2(-1)^s[3X(X-1) - 4b(b+1)c(c+1)]}{\sqrt{(2b-1)2b(2b+1)(2b+2)(2b+3)(2c-1)2c(2c+1)(2c+2)(2c+3)}},$$

where

$$s := a + b + c, \quad \text{(B.13)}$$
$$X := b(b+1) + c(c+1) - a(a+1). \quad \text{(B.14)}$$

Wigner 9j-Symbol
A generalization of Clebsch-Gordan Coefficients and Wigner 3j-Symbol and Wigner 6j-Symbol which arises in the coupling of four angular momenta and can be written in terms of the Wigner 3j-Symbol and Wigner 6j-Symbol. Let tensor operators $T^{(k_1)}$ and $U^{(k_2)}$ act, respectively, on subsystems 1 and 2. Then the reduced matrix element of the product$T^{(k_1)} \times U^{(k_2)}$ of these two irreducible operators in the coupled representation is given in terms of the reduced matrix elements of the individual operators in the uncoupled representation by

$$\begin{aligned}
&(\tau'\tau_1'j_1'\tau_2'j_2'J'||[T^{(k_1)} \times U^{(k_2)}]^{(k)}||\tau\tau_1 j_1\tau_2 j_2 J) \\
&= \sqrt{(2J+1)(2J'+1)(2k+1)} \sum_{\tau''} \begin{Bmatrix} j_1' & j_1 & k_1 \\ j_2' & j_2 & k_2 \\ J' & J & k \end{Bmatrix} \\
&\times (\tau'\tau_1'j_1'||T^{(k_1)}||\tau''\tau_1 j_1)(\tau''\tau_2'j_2'||U^{(k_2)}||\tau\tau_2 j_2), \qquad \text{(B.15)}
\end{aligned}$$

where $\begin{Bmatrix} j_1' & j_1 & k_1 \\ j_2' & j_2 & k_2 \\ J' & J & k \end{Bmatrix}$ is a Wigner 9j-symbol. The explicit formulas are

$$\begin{Bmatrix} a & b & C \\ d & e & F \\ G & H & J \end{Bmatrix} = \sum_x (-1)^{2x}(2x+1)$$

$$\times \begin{Bmatrix} a & b & C \\ F & J & x \end{Bmatrix} \begin{Bmatrix} d & e & F \\ b & x & H \end{Bmatrix} \begin{Bmatrix} G & H & J \\ x & a & d \end{Bmatrix},$$

$$\begin{Bmatrix} a & b & J \\ c & d & J \\ K & K & 0 \end{Bmatrix} = \frac{(-1)^{b+c+J+K}}{\sqrt{(2J+1)(2K+1)}} \begin{Bmatrix} a & b & J \\ d & c & K \end{Bmatrix},$$

$$\begin{Bmatrix} S & S & 1 \\ L & L & 2 \\ J & J & 1 \end{Bmatrix} = \frac{\begin{Bmatrix} S & L & J \\ L & S & 1 \end{Bmatrix} \begin{Bmatrix} J & L & S \\ L & J & 1 \end{Bmatrix}}{5 \begin{Bmatrix} 2 & L & L \\ L & 1 & 1 \end{Bmatrix}} + \frac{(-1)^{S+L+J+1}}{15(2L+1)} \frac{\begin{Bmatrix} S & J & L \\ J & S & 1 \end{Bmatrix}}{\begin{Bmatrix} 2 & L & L \\ L & 1 & 1 \end{Bmatrix}}.$$

The matrix elements of the tensor product of two operators are given by

$$[T^{(ki_1)} \times T^{(k_2)}]^{(k_3)}_{k_3} = \sum_{k_1 k_2} (R_1 k_1 R_2 k_2 | R_1 R_2 R_3 k_3) T^{(k_1)}_{k_1} T^{(k_2)}_{k_2}, \tag{B.16}$$

and for the scalar product are

$$(T^{(k)} \cdot U^{(k)}) = (-1)^k \sqrt{2k+1} [T^{(k)} \times U^{(k)}]^{(0)}_0. \tag{B.17}$$

Wigner-Eckart theorem:

$$< JM|T^{(\kappa)}_k|J'M' >= (-1)^{J-M} \begin{pmatrix} J & \kappa & J' \\ -M & k & M' \end{pmatrix} < J||T^{(\kappa)}||J' > . \tag{B.18}$$

This relation is known as the Wigner-Eckart theorem. The coefficient $< J||T^{(\kappa)}||J' >$ is called the reduced matrix element of $T^{(\kappa)}$ and is independent of the magnetic quantum numbers M, M' and k.

B.2 Operators for Radial Form

In order to obtain the final eigenvalue radial equations of the wave equations Eqs. (5.257), it is useful for numerical calculation of the coupled and the uncoupled equations in the variable $\mathbf{r}$, we have used the following operators

1. $\mathbf{p^2}$
2. $(\sigma_1 \cdot \sigma_2)$ spin-spin term
3. $L \cdot (\sigma_1 + \sigma_2)$ spin-orbit angular momentum term
4. $L \cdot (\sigma_1 - \sigma_2)$ spin-orbit angular momentum difference term
5. $(\sigma_1 \cdot \hat{\mathbf{r}})(\sigma_2 \cdot \hat{\mathbf{r}})$ tensor term
6. $\hat{\mathbf{r}} \cdot \mathbf{p}$ Darwin term
7. $L \cdot (\sigma_1 \times \sigma_2)$ additional spin dependent terms
8. $(\sigma_1 \cdot \hat{\mathbf{r}})(\sigma_2 \cdot \mathbf{p}) + (\sigma_2 \cdot \hat{\mathbf{r}})(\sigma_1 \cdot \mathbf{p})$ spin independent terms.

We are listed the following expressions for numerical calculation:

$$O_r =< L||\hat{\mathbf{r}}_k||\mathcal{L} >= (-1)^L \sqrt{\hat{L}\hat{\mathcal{L}}\hat{k}} \begin{pmatrix} L & k & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix}, \tag{B.19}$$

$$O_L =< L'||\vec{L}||L >= \sqrt{L(L+1)\hat{L}} \delta_{L'L}, \tag{B.20}$$

$$O_\sigma =< s'||\sigma||s >= \sqrt{6}, \tag{B.21}$$

$$O_\nabla =< \mathcal{L}0||\nabla_0||L0 >=$$

$$\begin{cases} if \quad \mathcal{L} = L+1 \rightarrow & \frac{L+1}{\hat{L}(2L+3)}\left(\frac{\partial}{\partial r} - \frac{L}{r}\right) \\ if \quad \mathcal{L} = L-1 \rightarrow & \frac{L}{\hat{L}(2L-1)}\left(\frac{\partial}{\partial r} + \frac{L+1}{r}\right). \end{cases} \tag{B.22}$$

Now, one can use the Eq.(B.15), to drive a simple form of the operators

$$O_{\sigma_1\times\sigma_2} =< S'||[\sigma_1 \times \sigma_2]_k||S >= 6\sqrt{\hat{k}\hat{S}\hat{S}'} \begin{Bmatrix} 1/2 & 1/2 & S' \\ 1/2 & 1/2 & S \\ 1 & 1 & k \end{Bmatrix}, \tag{B.23}$$

$$O_{\hat{r}\times\hat{r}} =< L'||[\hat{\mathbf{r}} \times \hat{\mathbf{r}}]_k||L >=$$

$$(-1)^{L'+k+L}\sqrt{\hat{k}}\sum_{\mathcal{L}} < L'||\hat{\mathbf{r}}||\mathcal{L} >< \mathcal{L}||\hat{\mathbf{r}}||L > \begin{Bmatrix} 1 & 1 & k \\ L & L' & \mathcal{L} \end{Bmatrix}. \tag{B.24}$$

Finally, we becomes

1.

$$O_\Delta =< L'S'J|| - \hbar^2c^2\Delta||LSJ >= -\hbar^2c^2\left(\frac{d^2}{dr^2} - \frac{L(L+1)}{r^2}\right)\delta_{LL'}\delta_{SS'}, \tag{B.25}$$

2.

$$a) \quad O_{\sigma_1\sigma_2 a} =< L'S'J||(\sigma_1\cdot\sigma_2)||LSJ >= 2S(S+1) - 3, \tag{B.26}$$

$$b) \quad O_{\sigma_1\sigma_2 b} =< L'S'J||(\sigma_1\cdot\sigma_2)||LSJ >=$$

$$6(-1)^{L'+S+S'+J}\sqrt{\hat{S}'\hat{J}} \begin{Bmatrix} S' & J & L \\ J & S & 0 \end{Bmatrix} \begin{Bmatrix} 1/2 & 1/2 & S \\ 1/2 & 1/2 & 0 \end{Bmatrix}, \tag{B.27}$$

3.

$$O_{LS} =< L'S'J||(\vec{L}\cdot\vec{S})||LSJ >= (J(J+1) - L(L+1) - S(S+1)), \tag{B.28}$$

4.

$$O_{L\sigma_1\pm\sigma_2} =< L'S'J||\vec{L}\cdot(\sigma_1 \pm \sigma_2)||LSJ >=$$

$$\sqrt{6}(-1)^{S'+J+L+S}\sqrt{\hat{S}\hat{S}'} \begin{Bmatrix} L' & S' & J \\ S & L & 1 \end{Bmatrix} O_L \begin{Bmatrix} 1/2 & S' & 1/2 \\ S & 1/2 & 1 \end{Bmatrix} (1 \pm (-1)^{S'-S}) =$$

$$\sqrt{6}(-1)^{S'+J+L+S}\sqrt{L(L+1)\hat{L}\hat{S}\hat{S}'} \begin{Bmatrix} L' & S' & J \\ S & L & 1 \end{Bmatrix} \begin{Bmatrix} 1/2 & S' & 1/2 \\ S & 1/2 & 1 \end{Bmatrix} (1\pm(-1)^{S'-S}), \tag{B.29}$$

5.

$$O_{\sigma_1 r \sigma_2 r} =< L'S'J||(\sigma_1 \cdot \hat{\mathbf{r}})(\sigma_2 \cdot \hat{\mathbf{r}})||LSJ >= \left\{ \begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right\} O_{\sigma_1 \sigma_2 b} +$$

$$15 \left\{ \begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \end{array} \right\} \sqrt{2J+1} \left\{ \begin{array}{ccc} L' & S' & J \\ L & S & J \\ 2 & 2 & 0 \end{array} \right\} O_{\sigma_1 \times \sigma_2} O_{\hat{r} \times \hat{r}}, \tag{B.30}$$

6.

$$O_{\hat{\mathbf{r}} \cdot \vec{p}} =< L'S'J||(\hat{\mathbf{r}} \cdot \mathbf{p})||LSJ >$$

$$= (-1)^{L'+S'+J+1+2L} \hat{J} \sum_{\mathcal{L}} < L||\hat{\mathbf{r}}||\mathcal{L} >< \mathcal{L}0||\nabla_0||L0 > \left\{ \begin{array}{ccc} 1 & 1 & 0 \\ L & L & \mathcal{L} \end{array} \right\}, \tag{B.31}$$

7.

$$O_{L \cdot (\sigma_1 \times \sigma_2)} =< L'S'J||L \cdot (\sigma_1 \times \sigma_2)||LSJ >=$$

$$(-1)^{S'+J+L} \sqrt{\hat{J}} \left\{ \begin{array}{ccc} L' & S' & J \\ S & L & 1 \end{array} \right\} O_L O_{\sigma_1 \times \sigma_2}, \tag{B.32}$$

8.

$$O_{\sigma_1 r \sigma_2 p} =< L'S'J||(\sigma_1 \cdot \hat{\mathbf{r}})(\sigma_2 \cdot \mathbf{p}) + (\sigma_2 \cdot \hat{\mathbf{r}})(\sigma_1 \cdot \mathbf{p})||LSJ >=$$

$$\sum_k 3\hat{k}\hat{J} \left\{ \begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 0 \\ k & k & 0 \end{array} \right\} \left\{ \begin{array}{ccc} L' & S' & J \\ L & S & J \\ k & k & 0 \end{array} \right\} < L'||[\hat{\mathbf{r}} \times \mathbf{p}]_k||L >< S'||[\sigma_1 \times \sigma_2]_k||S > (1+(-1)^k)$$

$$= \sum_k 3\hat{k}\hat{J} \left\{ \begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 0 \\ k & k & 0 \end{array} \right\} \left\{ \begin{array}{ccc} L' & S' & J \\ L & S & J \\ k & k & 0 \end{array} \right\} O_{\hat{\mathbf{r}} \cdot \vec{p}} O_{\sigma_1 \times \sigma_2} (1 + (-1)^k). \tag{B.33}$$

Using above expressions, we can get their coupled and uncoupled radial eigenvalue equations from Eqs.(5.257).

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