Quantum Aspects of the Free Electron Laser

Dissertation

zur Erlangung des Doktorgrades

des Departments Physik

der Universität Hamburg

vorgelegt von

Roberto Gaiba

aus Bologna

Hamburg

2007

Gutachter des Dissertation:	Dr. Helmut Mais
	Prof. Dr. Jochen Bartels
Gutachter der Disputation:	Dr. Helmut Mais
	Prof. Dr. Klaus Fredenhagen
Datum der Disputation:	27.02.2007
Vorsitzender des Prüfungsausschusses:	Dr. Hans Dierk Rüter
Vorsitzender des Promotionsausschusses:	Prof. Dr. Günter Huber
MIN-Dekan des Department Physik :	Prof. Dr. Arno Früwald

Abstract

We study the role of Quantum Mechanics in the physics of Free Electron Lasers. While the Free Electron Laser (FEL) is usually treated as a classical device, we review the advantages of a quantum formulation of the FEL. We then show the existence of a regime of operation of the FEL that can only be described using Quantum Mechanics: if the dimensionless quantum parameter $\bar{\rho}$ is smaller than 1, then in the 1-dimensional approximation the Hamiltonian that describes the FEL becomes equivalent to the Hamiltonian of a two-level system coupled to a radiation field. We give analytical and numerical solutions for the photon statistics of a Free Electron Laser operating in the quantum regime under various approximations. Since in the quantum regime the momentum of the electrons is discrete, we give a description of the electrons in phase space by introducing the Discrete Wigner Function.

We then drop the assumption of a mono-energetic electron beam and describe the general case of a initial electron energy spread $G(\gamma)$. Numerical analysis shows that the FEL quantum regime is observed only when the width of the initial momentum distribution is smaller than the momentum of the emitted photons. Both the analytical results in the linear approximation and the numerical simulations show that only the electrons close to a certain resonant energy start to emit photons. This generates the so-called Hole-burning effect in the electrons energy distribution, as it can be seen in the simulations we provide.

Finally, we present a brief discussion about a fundamental uncertainty relation that ties the electron energy spread and the electron bunching.

Zusammenfassung

In dieser Arbeit untersuchen wir Quantenaspekte des FEL. Normalerweise wird der FEL klassisch (im Rahmen der Vlasov-Maxwell Theorie) beschrieben. Es gibt aber ein Regime - definiert durch den Quantenparameter $\bar{\rho} \ll 1$ - wo Quanteneffekte wichtig werden. Im eindimensionalen Fall wird gezeigt, wie sich in diesem Fall der FEL durch ein Zwei-Niveau-System, das an das Strahlungfeld gekoppelt ist, beschreiben lässt. Numerische und analytische Lösungen für die Photon-Statistik werden präsentiert und diskutiert. Für die Beschreibung der Elektronen benutzen wir wegen der gequantelten Energie der Teilchen die diskrete Wigner-Funktion.
Im zweiten Teil der Arbeit wird die Annahme eines mono-energetischen Elektronenstrahls fallen gelassen, und wir beschreiben den allgemeineren Fall einer Energieverteilung $G(\gamma)$. Verschiedene Effekte u.a. "Hole-Burning" werden beschrieben und diskutiert. Die numerische Behandlung zeigt, dass das FEL Quantenregime nur beobachtet wird, wenn die Breite der anfänglichen Elektronen-Impulsverteilung kleiner ist als der Impuls der emittierten Photonen.
Abschliessend diskutieren wir fundamentale Quanten-Begenzungen für die Energiebreite und das "Bunching" der Elektronen.

Contents

Acknowledgements

I would like to thank Dr. Helmut Mais for his precious help in writing this thesis, his patience for dealing with my many trips to and from Italy during these years and for making it possible for me to work at DESY for my PhD.

I wish to thank Prof. Rodolfo Bonifacio for the possibility to work on the Quantum FEL together with him and his group and all I have learnt from this experience.

A special thank you to Dr. Nicola Piovella for the huge and continuous help and support he has given me, without which this thesis would have not been possible.

Thanks to Dr. Mary Cola and Luca Volpe for the work together, the many talks together and the support.

I also wish to thank Prof. Jochen Bartels for accepting to referee this thesis and the help and support given as the head of the Graduiertenkolleg.

This work was supported by the Graduiertenkolleg 'Zukünftige Entwicklungen in der Teilchenphysik'. My deep gratitude for having fed me for three long years!

A very heartfelt thank you also to Prof. Kobayashi Maru for the many insightful talks and discussions. Koi wa shian no hoka.

And finally some very important personal thanks to my parents whose merits are so many that I would have to write another thesis just for those, grazie di tutto!

-

Introduction

The Free Electron Laser (FEL) is "*a source of powerful and tunable coherent radiation, potentially able to cover those regions of the electromagnetic spectrum which are not accessible to conventional lasers*" [1]. This description explains the great interest in the scientific community that surrounds the Free Electron Laser, a coherent light source that is theoretically tunable in a continuous way to any possible wavelength. These special properties of the FEL come from its basic design: a relativistic electron beam is injected in the periodic magnetic field of an undulator magnet, that is a series of magnets of alternating polarities usually called wiggler. As a consequence of the Lorentz force, the electrons undulate transversally and thus start to radiate in the longitudinal direction, generally in a incoherent way due to their different position and velocity. The coupling between this spontaneous radiation and the undulating motion of the electrons generates a ponderomotive potential traveling along the wiggler. If its phase velocity is close enough to the velocity of the electron beam, then a resonant process takes place that leads to the formation of bunches in the electron beam. The electrons in the same bunch will then radiate coherently. Since the wavelength of the emitted radiation will depend on various variables such as the electron energy, the wiggler magnetic field and its period, this shows how it is possible to tune the FEL in a continuous way by changing these parameters.

Even if the first proposal of an FEL device [2] was written starting from from quantum mechanics, the Free Electron Laser is considered by most as a completely classical device and there are extensive works describing the FEL in the framework of classical physics [3, 4]. Still, there have been many approaches to a quantum mechanical description of the FEL [5, 6, 7, 8]. This is partly due to the fact that some of its properties can only be completely explained in a quantum framework, like full spontaneous emission. Another reason is that some topics of interest in Free Electron Laser physics are intrinsically quantum in nature, like photon statistics.

Following that line, my original aim was to study those properties of the FEL that require a quantum mechanical treatment and to show how quantum mechanics has a place in the analysis of the FEL. In addition to an outline of how quantum physics allows to obtain certain results that cannot be described with a purely classical analysis, the main and most important part of this thesis is dedicated to the description and analysis of a *Quantum Regime* of the FEL. In this regime, the FEL has a very different physical behaviour that can only be explained and described using quantum mechanics. This particular property of the FEL was found by R. Bonifacio and his research group of the University of Milano while I was beginning to work on my thesis. The collaboration with this group allowed me to further delve into this topic. Thus what distinguishes this work from most past quantum approaches is that not only quantum mechanics is used to describe the FEL, but completely new physics in the context of the Free Electron Laser are explored.

In the first chapter of this thesis I introduce the basic physics of the FEL, its key parameters and its main known properties. In particular the FEL universal dimensionless scaling, that will be adopted all through this thesis, is introduced: in this scaling all the FEL variables are dimensionless quantities, to ease notation and calculations. A brief summary of its principal regimes of operation is discussed there, together with a review of what makes the FEL interesting compared to atomic lasers.

In the second chapter I give a short review of some past descriptions of the FEL, that were done using quantum physics and that can be found in literature. These works show the importance of quantum physics in the analysis of the FEL. In particular I describe the Hamiltonian approach that is the starting point of my work in the following chapters: the FEL quantum Hamiltonian is derived under some assumptions from the classical Hamiltonian that describes the interaction between relativistic electrons and an electro-magnetic field, by going to the electron rest frame and quantizing the electromagnetic field.

The introductory part of the third chapter is dedicated to how the quantum regime of the FEL was found: from the FEL quantum Hamiltonian a model is derived where the electron degrees of freedom are described by a Wigner function, a quasi-probability distribution function usually used to describe semi-classical systems in phase space. Numerical analysis of the evolution equation of the Wigner function shows that the FEL electrons behave, within a certain range of its physical parameters, as a two-level system coupled to a radiation field. Specifically, it is possible to define a *Quantum FEL Parameter* $\bar{\rho}$ whose value specifies if the Free Electron Laser is in its classical regime ($\bar{\rho} \gg 1$) or in its quantum regime ($\bar{\rho} \ll 1$). Since this particular behaviour first appeared only through numerical simulations, I show in this chapter, as my own contribution, how the FEL evolution equations reduce to those of a two-level system when $\bar{\rho} \ll 1$.

In the second part of the third chapter, I review the dynamics and properties of a two-level system coupled to radiation. The solutions of the FEL equations in this regime are discussed, under different approximations. Then, I show a numerical simulation of the two-level system, obtained by diagonalizing the Hamiltonian after having rewritten it in matrix form: this method [9] of numerical analysis is a mathematical tool that, up to my knowledge, was never used before in this context.

The last part of this chapter introduces another mathematical tool, the *discrete* Wigner function [10]. In the quantum regime the momentum of the FEL electrons becomes a discrete variable, and thus the normal Wigner function, set in the classical phase space of continuous variables, is not well suited to describe the FEL; the less known discrete Wigner function has to be used for systems with discrete momentum.

In the fourth and last chapter of my thesis I discuss a more realistic model of the FEL, introducing the complication of a generic energy distribution for the electrons of the FEL. I describe how an energy distribution can be implemented in the models used in the previous chapters and how to solve the resulting systems of equations. In addition to that, it is shown how an important FEL parameter, that measures the bunching of the electrons in the beam, is connected to the energy spread of the electrons by an inequality relation that is derived by the Heisenberg uncertainty principle.

Chapter 1

Classical FEL Description

1.1 Introduction to FEL physics

The FEL is a device that uses a relativistic beam of electrons passing through a transverse periodic magnetic field to produce and amplify electromagnetic radiation [3, 4, 2]. The periodic magnetic field is provided by the *wiggler*, an insertion device usually realized as two arrays of permanent magnets with alternating polarities, or as two helical coils with current circulating in opposite directions. The wavelength of the emitted radiation depends on the period of the wiggler, the strength of its magnetic field and the electron energy. This means that the FEL can be continuously tuned in frequency, ranging from microwaves to X-rays: this tunability is one of the main advantages of FELs over atomic lasers, where the wavelength is defined by the difference between the energy levels of its active medium. Another advantage of the FEL over the atomic lasers is that all its main processes happen in vacuum: in this way there are no thermal dispersion effects caused by an active medium and very high power levels and very short wavelengths can be obtained.

The FEL radiation can be characterized in two ways: basic *spontaneous emission* from the direct interaction of the electron beam with the wiggler magnetic field, and *stimulated emission* which happens when a radiation field copropagates with the electron beam under certain conditions. We will introduce the basic physics of these two processes using a one-dimensional model, as both can be fully understood and explored even without taking into account three-dimensional effects, like the electron emittance, the focusing induced by the wiggler or the transverse structure of the electron beam.

1.1.1 FEL Spontaneous Emission

The FEL radiation is synchrotron radiation, i.e. the radiation emitted by an electric charge moving at relativistic speed when a transverse force is applied to it. This radiation is much more intense, bigger by a factor γ^2 (where γ is the electron energy in rest mass units), than the one emitted when the electrons are accelerated along the longitudinal axis.

In the FEL this transverse force is the Lorentz force

$$\vec{F} = \frac{e}{c}\,\vec{v} \times \vec{B}_w = e\,\vec{\beta} \times \vec{B}_w$$

generated by the wiggler magnetic field $\vec{B}_w$ on electrons, which are traveling at speed $\vec{v}$ along the wiggler. Since the magnets of the wiggler have a periodic alternating polarity, the electrons will wiggle, i.e. oscillate transversally (see picture) along the same longitudinal trajectory. We identify the z-axis with this axis of oscillation. Due to this configuration the emitted radiation will be confined in a narrow cone along the z-axis, of angle of order $\sim 1/\gamma\sqrt{N_w}$, where N_w is the number of wiggler periods (i.e. the number of magnets) [1].

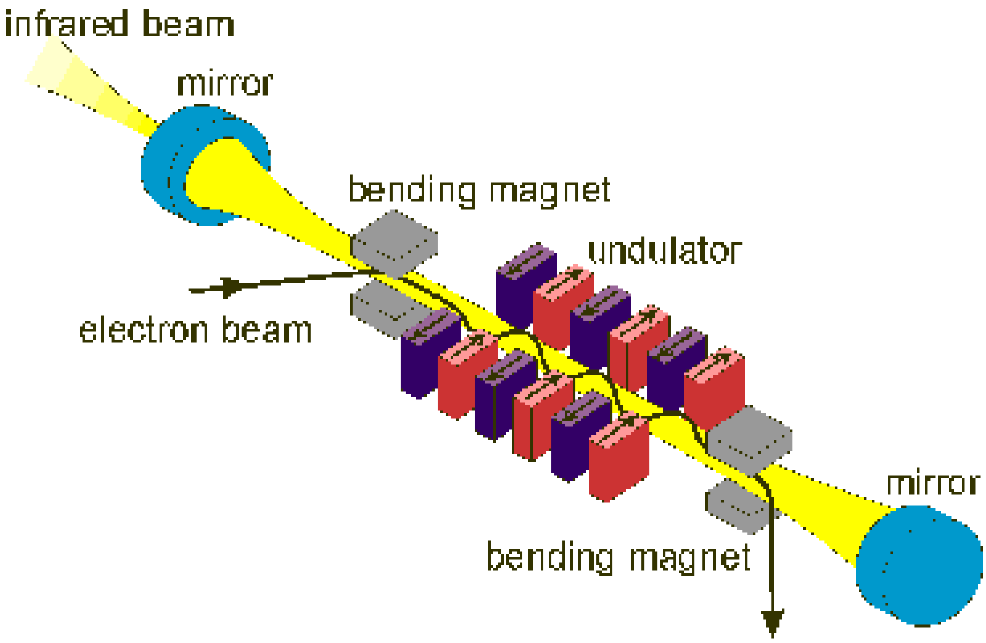

Figure 1.1: *FEL Scheme*

The intensity of the spontaneous emission is proportional to the electron current and its on-axis spectral distribution has the following shape[1] [1]:

$$\frac{\mathrm{d}^2 I}{\mathrm{d}\Omega\,\mathrm{d}\omega} \propto \mathrm{sinc}^2\left(\pi N_w \frac{\omega - \omega_s}{\omega_s}\right) \tag{1.1}$$

This shows that the spontaneous emission linewidth can be reduced simply by increasing the number of wiggler periods N_w, as

$$\frac{\Delta\omega}{\omega} \simeq \frac{1}{N_w} \tag{1.2}$$

It also means that there is a 'spontaneous' frequency $\omega_s = 2\pi c/\lambda_s$ where the spectrum is peaked. The wavelength λ_s must be equal to the slippage that develops between the radiation and the electrons while the electrons advance by one wiggler period λ_w. At this particular wavelength, the radiation emitted by the electrons at each wiggler period is in phase with that emitted at every other period, giving positive interference and thus the maximum peak of the radiation. This requires that the phase $\theta = (k_s + k_w)z - \omega_s t$ is constant along the wiggler, i.e.

$$0 = \frac{\mathrm{d}\theta}{\mathrm{d}z} = k_w - k_s\left(\frac{1}{\beta_\parallel} - 1\right) \tag{1.3}$$

which gives

$$\lambda_s = \frac{1 - \beta_\parallel}{\beta_\parallel}\lambda_w \simeq \frac{\lambda_w}{2\gamma_\parallel^2} \tag{1.4}$$

Another widely used formula for the FEL wavelength is found solving the first Lorentz-Newton equation, which describes the momentum of a relativistic charged particle moving inside an electromagnetic field characterized by the vector potential $\vec{\mathrm{A}}_w$:

$$mc\frac{\mathrm{d}(\gamma\vec{\beta})}{\mathrm{d}t} = e\left(\vec{E_w} + \vec{\beta} \times \vec{B_w}\right) \tag{1.5}$$

where

$$\vec{E}_w = -\frac{1}{c}\frac{\partial\vec{\mathrm{A}}_w}{\partial t} \tag{1.6}$$

$$\vec{B}_w = \vec{\nabla} \times \vec{\mathrm{A}}_w \tag{1.7}$$

From equation (1.5) we can get an expression for the electron transverse velocity (under the approximation that $\beta_\perp(0) = 0$),

$$\vec{\beta}_\perp = -\frac{\vec{a}_w}{\gamma} \tag{1.8}$$

[1] We use the compact notation $\mathrm{sinc}(x) \equiv \frac{\sin x}{x}$.

where the dimensionless wiggler field

$$\vec{a}_w \equiv \frac{e}{mc^2}\vec{\mathrm{A}}_w = \frac{a_w}{\sqrt{2}}\left(\hat{\mathrm{e}}\, e^{-ik_w z} + c.c.\right) \tag{1.9}$$

has been introduced. Here a_w is the important wiggler (or undulator) parameter

$$a_w = \frac{e\lambda_w B_w}{2\pi mc^2} \simeq 0.93 \cdot B_w[T] \cdot \lambda_w[cm] \tag{1.10}$$

From (1.4) and (1.8) it is possible to get

$$\lambda_s \simeq \lambda_w \frac{1 + a_w^2}{\gamma^2} \tag{1.11}$$

This result shows the high tunability of the FEL: the radiation peak wavelength can be changed by varying different parameters, either the electron energy γ, the wiggler magnetic field B_w or its period λ_w.

1.1.2 FEL Stimulated Emission

Stimulated emission takes place when a radiation field co-propagating with the electron beam is inserted in the wiggler. This field must have a wavelength close to the resonance, as from Eq. (1.11):

$$\lambda_s \simeq \lambda_w \frac{1 + a_w^2}{\gamma^2} \tag{1.12}$$

Equivalently, given the wavelength λ of the radiation field, the particular electron energy resonating with it can be defined as

$$\gamma_r = \sqrt{\frac{(1 + a_w^2)}{2}\frac{\lambda_w}{\lambda}} \tag{1.13}$$

This is important because, as already said about the peak wavelength of the stimulated emission, when the electron energy and the radiation wavelength follow the relation (1.13), the relative phase θ between the transverse oscillations of the electrons and the radiation remains constant. Depending on the value of this relative phase, one of two opposite processes can take place for each electron:

- the electron radiates, losing energy to the field and thus decelerating
- the electron is accelerated by the field, taking energy from it

This can be seen by solving the other Newton-Lorentz equation coupled to (1.5), i.e. the one describing the energy exchange between the electron and the field:

$$mc\frac{\mathrm{d}\gamma}{\mathrm{d}t} = e\vec{E}\cdot\vec{\beta}_\perp \tag{1.14}$$

This must be combined with the equation for the transverse velocity of the electrons (1.8).

The total radiation field must be taken into account instead of only the one from the wiggler:

$$\vec{\beta}_\perp = -\frac{(\vec{a}_w + \vec{a}_\lambda)}{\gamma} = -\frac{\vec{a}_{tot}}{\gamma} \tag{1.15}$$

where the dimensionless radiation field is given by

$$\vec{a}_\lambda \equiv \frac{e}{mc^2}\vec{\mathrm{A}}_\lambda = -\frac{i}{\sqrt{2}}\left(\hat{\mathrm{e}}\,a_\lambda\,e^{i\theta} - c.c.\right) \tag{1.16}$$

The resulting equation is

$$\frac{\mathrm{d}\gamma^2}{\mathrm{d}z} = -\frac{2\pi}{\lambda}a_w\left(a_\lambda e^{i\theta} + a_\lambda^* e^{-i\theta}\right) \tag{1.17}$$

Equation (1.17) shows how the value of θ rules if $\mathrm{d}\gamma^2/\mathrm{d}z$ is positive or negative, i.e. if the electrons accelerate or decelerate.

If the first of these two processes dominates, then the inserted radiation field is amplified by the FEL: this is what is called a *FEL amplifier*. The radiation field to be amplified can be the very same field created through spontaneous emission by the FEL itself, if the wiggler is long enough or if two mirrors at both ends of the wiggler reflect the radiation back and forth along it: this last case is what is called a *FEL oscillator*.

1.2 1D FEL Equations

In the previous section we have used the Newton-Lorentz equations for a charged particle moving at relativistic speed in an electro-magnetic field with the only aim to find some key resonance conditions and give a qualitative picture of the FEL radiating processes. Now we will describe a set of closed equations representing the evolution of the whole system.

The equations used until now only describe the electron dynamics; to close them we need another equation for the evolution of the electromagnetic field. The variables we use are the electron phase $\theta = (k_\lambda + k_w)z - \omega t$, the electron energy γ and the dimensionless radiation field amplitude a_λ (the wiggler field a_w is externally fixed and as such a parameter).

We take the one-dimensional approximation, which means we neglect any dependence on transverse spatial coordinates: our variables depend only on the direction of propagation z and the time t.

As sketched before, from Eqs.(1.5) and (1.14) we can derive the evolution of γ and θ under certain physical assumptions:

- the wiggler field is much greater than the radiation field, $a_w \gg |a_\lambda(z,t)|$
- the relativistic limit $1-\beta_\| \ll 1$, which also means that $\mathrm{d}z \simeq c\,\mathrm{d}t$
- during the interaction with the e.m. field, the electron energy stays very close to the resonant energy, i.e. what is known as the Compton limit $\dfrac{\gamma-\gamma_r}{\gamma_r} \ll 1$
- we assume a helical wiggler and circularly polarized radiation, i.e. in Eq.(1.16) the unit vector is $\hat{\mathrm{e}} = (\hat{x}+i\hat{y})/\sqrt{2}$

In this way we obtain

$$\frac{\mathrm{d}\gamma}{\mathrm{d}z} = -\frac{k_\lambda a_w}{2\gamma_r}\left(a_\lambda e^{i\theta} + c.c.\right) \tag{1.18}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = 2k_w\frac{\gamma-\gamma_r}{\gamma_r} \tag{1.19}$$

The evolution of a_λ is derived from the wave equation for an e.m. field driven by an electron current:

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{\mathrm{A}}_\lambda = \frac{e}{mc^2}\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{a}_\lambda = -\frac{4\pi}{c}\vec{J}_\perp \tag{1.20}$$

where $\vec{J}_\perp$ is the transverse component of the electron current (for a beam of N electrons):

$$\vec{J}_\perp = e\sum_{j=1}^{N} c\vec{\beta}_\perp\,\delta\Big(\vec{x}-\vec{x}_j(t)\Big) \tag{1.21}$$

Substituting (1.21) in (1.20) we get

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{a}_\lambda = -\frac{4\pi e^2}{mc^2}\sum_{j=1}^{N}\vec{\beta}_\perp\,\delta\Big(\vec{x}-\vec{x}_j(t)\Big) \tag{1.22}$$

We take for this field the Slowly Varying Envelope Approximation (SVEA) [1]

$$\begin{aligned} \left|\frac{\partial a_\lambda}{\partial z}\right| &\ll |k a_\lambda| \\ \left|\frac{\partial a_\lambda}{\partial t}\right| &\ll |\omega a_\lambda| \end{aligned}$$

This approximation allows to reduce the evolution equation to first order:

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] a_\lambda e^{i\theta} \simeq 2ik\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) a_\lambda e^{i\theta} \tag{1.23}$$

After substituting the transverse velocity $\vec{\beta}_\perp$ from (1.15), the final equation becomes

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) a_\lambda = \frac{k a_w}{2\gamma_r^2}\frac{\omega_p^2}{\omega_\lambda^2}\langle e^{-i\theta}\rangle \tag{1.24}$$

where we have introduced the plasma frequency

$$\omega_p \equiv \sqrt{\frac{4\pi e^2 n_e}{m_e}} \tag{1.25}$$

n_e is the electron density and the brackets $\langle ... \rangle$ represent the average over all the electrons:

$$\langle f(\theta,\gamma)\rangle = \frac{1}{N}\sum_{j=1}^{N} f(\theta_j,\gamma_j)$$

1.2.1 Universal Scaling

It is possible and convenient to introduce a dimensionless scaling, called the universal scaling, for all the quantities in the 1D FEL equations, so that no experimental parameters appear explicitely. This scaling will be used in this thesis unless explicitely stated otherwise, to make formulae less cumbersome and clearer to read.

$$\rho \equiv \frac{1}{\gamma_r}\left(\frac{a_w \omega_p}{4 k_w c}\right)^{2/3} \text{(the FEL parameter)} \tag{1.26}$$

$$p_j \equiv \frac{\gamma_j - \gamma_0}{\rho\gamma_r} \simeq \frac{\gamma_j - \gamma_r}{\rho\gamma_r} \tag{1.27}$$

$$A \equiv \frac{\omega}{\omega_p\sqrt{\rho\gamma_r}}\, a \tag{1.28}$$

$$l_g \equiv \frac{\lambda_w}{4\pi\rho} \text{ (the gain length)} \tag{1.29}$$

$$l_c \equiv \frac{\lambda_r}{4\pi\rho} \text{ (the cooperation length)} \tag{1.30}$$

$$\bar{z} \equiv \frac{z}{l_g} \tag{1.31}$$

$$z_1 \equiv \frac{z - v_{\|}t}{l_c} \tag{1.32}$$

$\bar{z}$ is simply the longitudinal position measured in gain lengths, while z_1 is a scaled "retarded time".

The parameter ρ is extremely important and is often called the fundamental FEL parameter, since the different FEL regimes are related to the value of ρ. We will see in later chapters how a slightly modified parameter, $\bar{\rho} \propto \rho$, is introduced when dealing with quantum effects in the FEL.

Using the universal scaling the FEL equations assume then the more compact form

$$\frac{\partial}{\partial \bar{z}} p_j = -\left(Ae^{i\theta_j} + c.c.\right) \tag{1.33}$$

$$\frac{\partial}{\partial \bar{z}} \theta_j = p_j \tag{1.34}$$

$$\left(\frac{\partial}{\partial \bar{z}} + \frac{\partial}{\partial z_1}\right) A = \left\langle e^{-i\theta} \right\rangle \tag{1.35}$$

where j goes from 1 to N, the number of electrons.

This dimensionless form of the FEL equations has the advantage of being solvable without having to specify the operating parameters - and once they are solved, the scaling can be reversed to find the real physical quantities needed for an experimental set-up.

This 1D model is sometimes called the Maxwell-pendulum model, as it can be easily rewritten in a pendulum-like form: writing explicitely the real and imaginary parts of the field $A = |A| \exp(i\phi)$ and combining Eqs.(1.33) with (1.34) we get

$$\frac{\partial^2 \theta_j}{\partial \bar{z}^2} = -2|A(\bar{z}, z_1)| \cos(\theta_j + \phi(\bar{z}, z_1)) \tag{1.36}$$

The amplitude A and phase ϕ are not constant, their evolution is determined by Eq.(1.35); due to this, equation (1.36) does not describe an ordinary pendulum.

1.3 Electron Bunching

We have initially described spontaneous and stimulated emission as phenomena in a single particle picture, treating all electrons independently. This applies for short FEL amplifiers with low electron current, where there is not enough time for the initial random distribution of electron momenta around γ_r to change appreciably. In this case, called the *Madey's small-signal regime*, there will be some low gain in radiation power if the average energy of the electron beam is slightly above resonance, $\langle\gamma\rangle_0 > \gamma_r$, as a little more than half electrons will decelerate and radiate while a little less than half electron will accelerate and absorb photons. In this regime the radiated power is proportional to the number of electrons N.

However, if the wiggler is long enough and/or the electron current high enough, then the electron beam will start to bunch: electrons faster than γ_r will decelerate, slower electrons will accelerate, so that the electron energy will be driven toward resonance. This energy modulation becomes space modulation, i.e. the electrons start to bunch in packets on the scale of the radiation wavelength (*microbunching*), around a phase that produces gain. As we have seen, a peaked phase distribution means that the electrons will emit coherent synchrotron radiation at the resonant wavelength.

In the next section we will see that under certain conditions the scheme mentioned above manifests itself in a collective instability in the system, where the electrons keep self-bunching more and more until saturation effects take place, with a consequent exponential growth of radiation; in this high-gain regime the radiated power scales as $N^{4/3}$.

The variable that represents how much the electrons become bunched is called the *bunching parameter* or just bunching, and is given by

$$b \equiv \langle e^{-i\theta} \rangle \tag{1.37}$$

b is the measure of the longitudinal modulation of the electron beam on the scale of the radiation wavelength. A bunching of zero represents a completely random distribution of phases, while an ideal bunching of $b = 1$ can only be possible with all electrons perfectly in phase. Eq.(1.35) shows how the bunching directly drives the evolution of the radiation field.

1.4 The steady-state regime

Equation (1.35), describing the evolution of the radiation field A, contains two derivatives, one in $\bar{z}$ and one in the retarded time z_1; this latter derivative represents the propagation effects that arise from the difference between the velocity of the electrons and of the radiation, i.e. the so-called *slippage*. If the slippage remains small during the interaction time inside the wiggler then these effects and thus the derivative over z_1 can be neglected.

It is possible to show that this is the case when the slippage length $N_w\lambda_s$ is much smaller than the electron bunch length l_b: reversing the universal scaling, the ratio between the coefficients of the time derivative and those of the space derivative is $N_w\lambda_s/l_b \equiv S$, defined as the slippage parameter. When $S \ll 1$ the time derivative in Eq.(1.35) can be dropped and only space-dependence remains in the system's equations: one can follow the *steady-state* evolution of the system as it moves along the z-axis of the wiggler, therefore the name of this regime.

From Eq.(1.27) it follows that at time zero the average dimensionless momentum p is given by

$$\langle p \rangle_0 = \frac{\langle \gamma \rangle_0 - \gamma_r}{\rho \gamma_r} \equiv \delta \tag{1.38}$$

We define this value as the *detuning parameter* δ. It is particularly useful to redefine our variables so that the initial condition is zero:

$$p'_j = p_j - \delta \tag{1.39}$$

$$\theta'_j = \theta_j - \delta \bar{z} \tag{1.40}$$

$$A' = A e^{i\delta \bar{z}} \tag{1.41}$$

In this way the detuning parameter disappears from the initial conditions and enters explicitely into the evolution equations, where we drop for brevity the primes:

$$\frac{\mathrm{d}}{\mathrm{d}\bar{z}} p_j = -\left(A e^{i\theta_j} + c.c.\right) \tag{1.42}$$

$$\frac{\mathrm{d}}{\mathrm{d}\bar{z}} \theta_j = p_j \tag{1.43}$$

$$\frac{\mathrm{d}}{\mathrm{d}\bar{z}} A = \left\langle e^{-i\theta} \right\rangle + i\delta A \tag{1.44}$$

1.4.1 Linear approximation: exponential gain

The system of equations (1.42)-(1.44) can now be solved in the linear approximation, taking a small initial condition A_0 for the field and assuming a totally unbunched ($\langle \exp[-i\theta] \rangle_0 = 0$) and perfectly cold, on resonance ($p_j(0) = 0$ for every j) electron beam. Differentiating in succession (1.42)-(1.44) and keeping only the linear terms we obtain

$$\frac{\mathrm{d}^3}{\mathrm{d}\bar{z}^3} A - i\delta \frac{\mathrm{d}^2}{\mathrm{d}\bar{z}^2} A - iA = 0$$

Its solution is

$$A(\bar{z}) = A_0 \sum_{j=1}^{3} c_j e^{i\lambda_j \bar{z}} \tag{1.45}$$

where λ_j are the three solutions to the cubic equation

$$\lambda^3 - \delta\lambda^2 + 1 = 0 \tag{1.46}$$

If Eq.(1.46) admits three real roots, then the system is stable, as $|A|$ will remain constant. But if it instead has only one real root and two complex-conjugate ones, then one of the latter two will cause an exponential growth of the field until non-linear effects cannot be ignored anymore.

If instead of choosing an ideal cold beam where all the electrons have the same energy $\langle\gamma\rangle_0$ we take an initial energy distribution $f(p_0)$, then (1.46) is replaced by the more general equation [1, 12]

$$\lambda - \delta + \int_{-\infty}^{+\infty} \frac{f(p_0)}{(\lambda + p_0)^2}\, \mathrm{d}p_0 = 0 \tag{1.47}$$

The cold case is re-derived taking the delta distribution $f(p_0) = \delta(p_0)$. Another interesting simple case is that of a rectangular distribution of half-width σ

$$f(p_0) = \begin{cases} \dfrac{1}{2\sigma} & \text{if} \quad -\sigma < p_0 < \sigma \\ \\ 0 & \qquad \text{otherwise} \end{cases}$$

which gives

$$(\lambda - \delta)\left[\lambda^2 - \left(\frac{\sigma}{\rho\gamma_r}\right)^2\right] + 1 = 0 \tag{1.48}$$

It is convenient in this case to introduce the *energy spread parameter*

$$\mu \equiv \frac{\sigma}{\rho\gamma_r}$$

The exponential behaviour of $A(\bar{z})$ is determined by the imaginary part of the roots of Eq.(1.48). Fig.1.2 shows $|\mathrm{Im}\lambda|$ as a function of the detuning parameter δ for different values of the spread μ.

Looking at Fig.1.2, the following remarks can be done:

1. the absolute maximum gain takes place at resonance, $\delta = 0$

2. an energy spread ($\mu > 0$) implies a lower growth rate, as well as a detuning shift

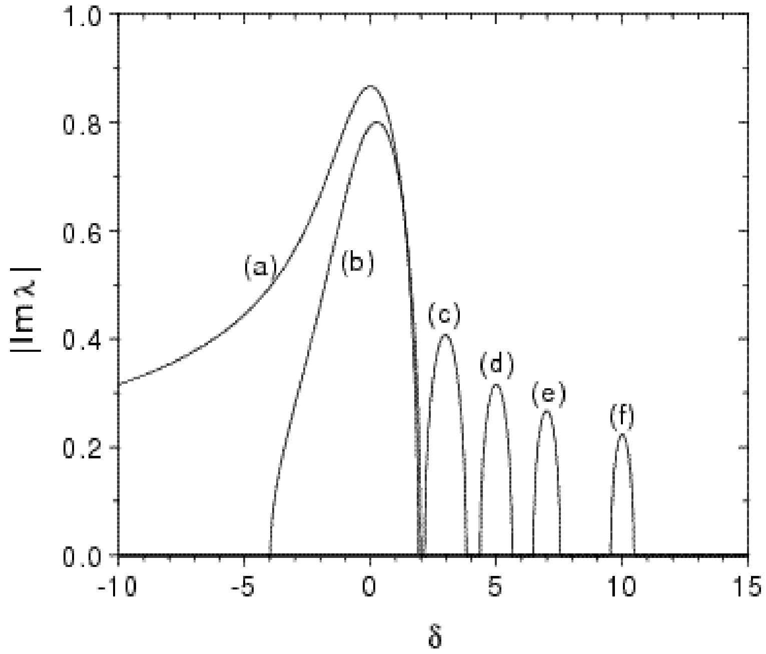

Figure 1.2: *FEL growth parameter as a function of the detuning δ for different values of the energy spread μ: (a) $\mu = 0$, (b) $\mu = 0.5$, (c) $\mu = 1$, (d) $\mu = 5$, (e) $\mu = 7$, (f) $\mu = 10$. Taken from [1].*

3. given a spread μ, the optimal gain occurs for the specific detuning $\delta = \mu$, while a detuning too far away from μ produces no exponential gain

This means that the exponential gain is obtained only with an electron beam that follows the contraints

$$\begin{aligned} \frac{\sigma}{\gamma_r} &\leq \rho \\ \frac{\langle\gamma\rangle_0 - \gamma_r}{\gamma_r} &\leq \rho \end{aligned}$$

This is one of the reasons why ρ is called the fundamental FEL parameter.

A simple example of exponential gain is the solution of the cold beam case at resonance; then the form of the scaled field intensity is

$$|A|^2(\bar{z}) = \frac{|A|_0^2}{9}\left[4\cosh^2\left(\frac{\sqrt{3}}{2}\bar{z}\right) + 4\cos\left(\frac{3}{2}\bar{z}\right)\cosh\left(\frac{\sqrt{3}}{2}\bar{z}\right) + 1\right] \tag{1.49}$$

1.4.2 Collective behaviour

The 1D FEL equations (1.42-1.44) can be solved numerically: the solution for the field intensity is shown in Fig.1.3, where again we have used the simple case of an unbunched, cold beam on resonance. This numerical analysis shows that the initial exponential growth reaches its peak due to saturation when $|A|^2 \sim O(1)$, as confirmed by several FEL experiments. Due to the universal scaling, this result is independent of the chosen physical parameters. Since the field intensity is proportional to $|E|^2/\rho n$, where n is the electron density, then from the definition of ρ we have

$$O(1) = |A|^2 \propto \frac{|E|^2}{n^{4/3}}$$

i.e. $|E|^2 \propto n^{4/3}$ instead of n: this implies the existence of a collective behaviour in the electron beam, as it does not grow linearly with the electron density as would happen in the case of N independent processes.

The independence of the saturated field amplitude from its initial value gives us another important information, related again to the importance of the ρ parameter. The efficiency η of the FEL in converting the electron kinetic energy into radiation can be defined as the ratio between the radiation power and the beam power, which gives

$$\eta \equiv \frac{P_{rad}}{P_{beam}} = \rho|A|^2 \tag{1.50}$$

Since at saturation the dimensionless field amplitude $|A|$ will be ~ 1, independently of other physical parameters (at least in the approximations assumed until now), then the measure of FEL efficiency will be given by ρ.

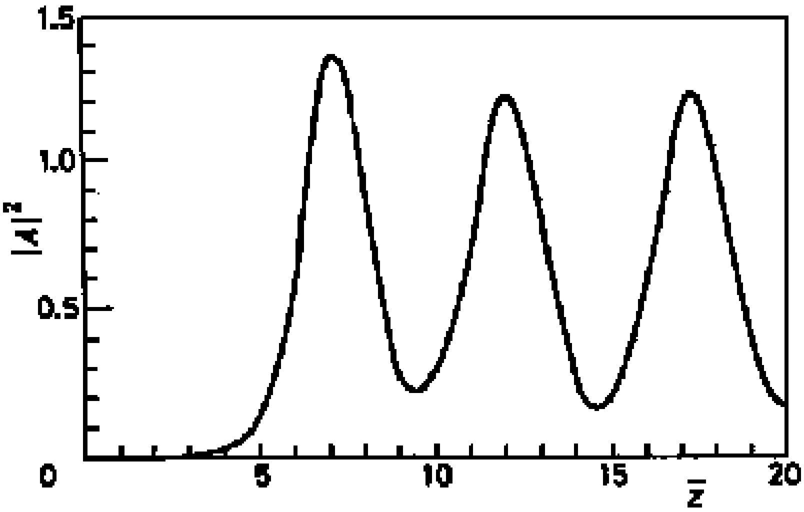

Figure 1.3: *Field intensity* $|A|^2(\bar{z})$*, numerical solution from Eqs.(1.42-1.44).* $N = 100$*,* $Re(A_0) = Im(A_0) = 0.01$ *Taken from [1].*

1.4.3 SASE: Self Amplified Spontaneous Emission

The assumptions leading to Eq.(1.49) implied an initial input signal A_0. In general, the obtained results seem to indicate that without any initial signal ($A_0 = 0$) there would be no emission. However, even in the classical picture, the free electron laser can radiate spontaneously in the high gain regime, starting from a small random bunching b_0 given by some noise in the electron phases: while the initial average bunching is still zero, in some regions there can be some oscillations from which the exponential growth can start.

Instead of directly looking for a differential equation for the field A, we repeat the same steps but using the bunching $b = \langle \exp(-i\theta) \rangle$ as our variable. In the same way as $|A|^2$, the squared bunching $|b|^2$ grows exponentially along the wiggler,

$$|b|^2 = \frac{|b_0|^2}{9} \exp\left(\sqrt{3}\bar{z}\right) \tag{1.51}$$

as expected given Eq.(1.44). This self-bunching process, that starts-up from noise until saturation and that generates the exponential emission of radiation, is at the core of SASE.

This result is obtained shifting the initial condition requirement from the field to the bunching of the beam: instead of having a finite initial condition for the field, a finite initial condition for the bunching is used. This is still a limit of the classical picture: to find a completely spontaneous emission without any seeding of any kind we will need to approach the FEL problem in a quantum framework.

1.5 The superradiant regime

The steady-state regime that we just described was based on the assumption of a negligible slippage: the wiggler wasn't long enough to make the difference in velocity between the electrons and the radiation relevant, so that all sections of the electron beam evolved identically. If we drop that assumption, and take into account the z_1 derivative in the system equations (1.33)-(1.35), then a different kind of high-gain regime is found, called *superradiant regime* [1, 13, 14], where the peak power scales as n^2, the square of the electron density.

This is possible because, contrary to the steady-state regime, now the radiation propagates with respect to the electrons, interacting with different sections of the beam: this means that near the trailing edge of the electron pulse there will a region of length $N_w\lambda_s$, called the *slippage region*, where the electrons will radiate without being affected by the radiation produced by the other electrons behind them. Thus the steady-state saturation will not take place and the superradiant spike will keep evolving until it will surpass all of the electron beam.

In general, if the electron pulse is long enough, the superradiant spikes will appear together with the steady-state radiation: there will be superradiance in the slippage region and the usual steady-state emission in the remaining part of the beam. It is still possible to observe pure superradiance by tuning the system out of resonance: in fact, while the steady-state regime requires a certain closeness to resonance to produce exponential gain, the superradiant behaviour is always on resonance with the electrons, being coherent spontaneous emission. So, when the system will be detuned in such a way to prevent steady-state radiation, the superradiant spike will travel over nearly unperturbed electrons, extracting energy from them with an even greater efficiency than in the steady-state resonance.

These properties of the superradiant regime can be shown by finding a solution of

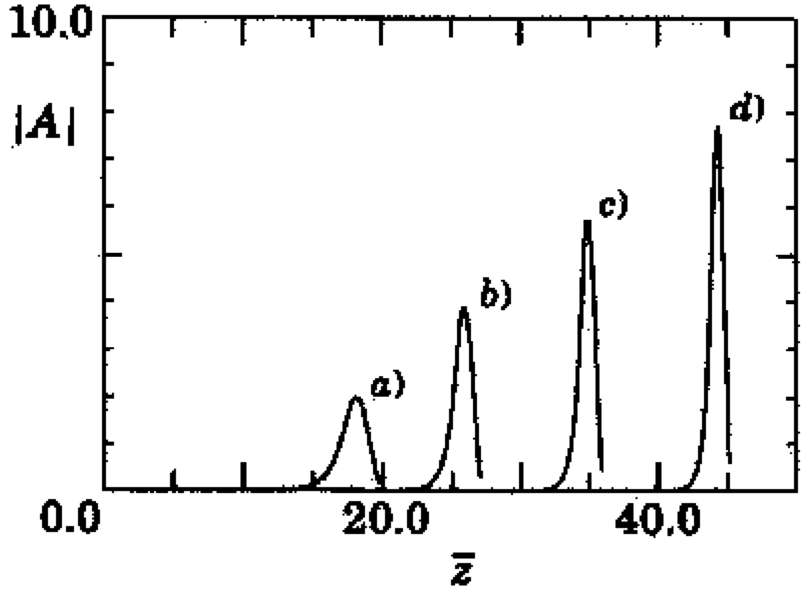

Figure 1.4: $|A|^2(\bar{z})$ *for* $A_0 = 10^{-4}$, $\delta = 4$ *and: (a)* $z_1 = 10$, *(b)* $z_1 = 20$, *(c)* $z_1 = 30$, *(d)* $z_1 = 40$ *Taken from [12].*

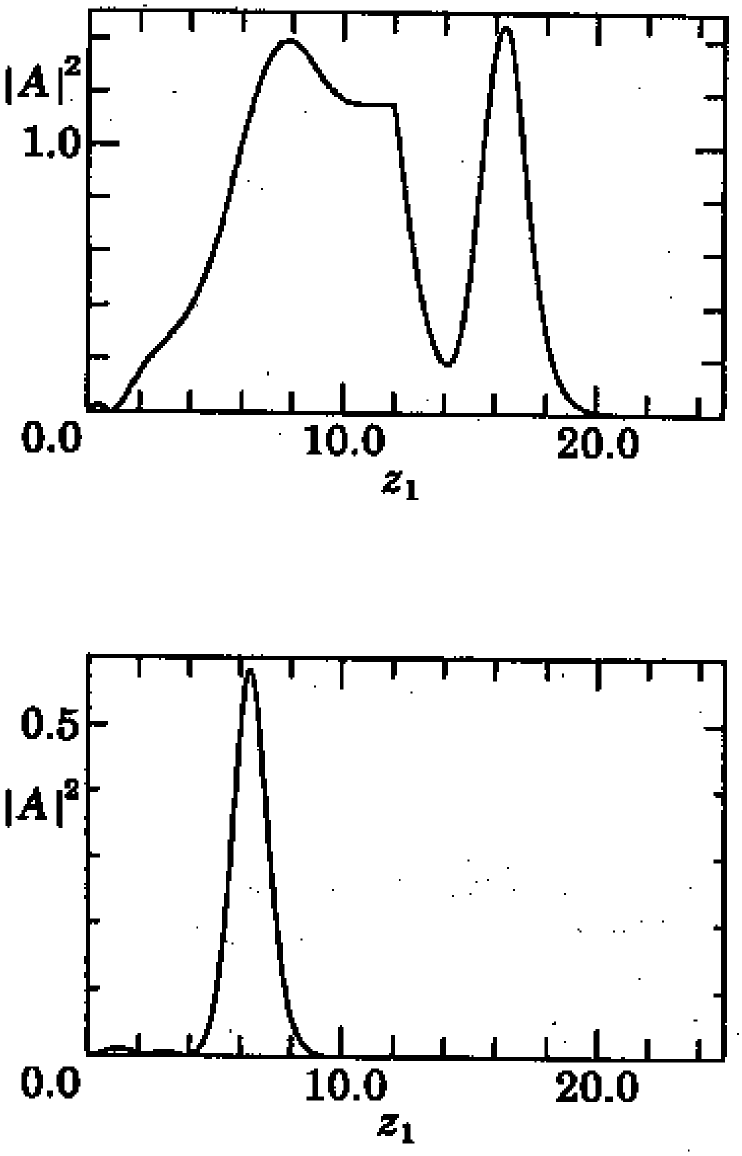

Figure 1.5: *SUPERRADIANCE: (Top) Emitted field at resonance* $\delta = 0$. *(Bottom) Emitted field out of resonance,* $\delta = 2$*; only the superradiant pulse is amplified. Taken from [12].*

the general 1D FEL equations (1.33)-(1.35). It is possible to simplify them through the following substitutions

$$\begin{aligned} A(\bar{z}, z_1) &= z_1 A_1(y) \\ \theta(\bar{z}, z_1) &= \theta(y) \\ y &\equiv \sqrt{z_1}(\bar{z} - z_1) \end{aligned}$$

This reduces the system to ordinary differential equations

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}y^2} = -\left(A_1 e^{i\theta} + c.c.\right) \tag{1.52}$$

$$\frac{y}{2}\frac{\mathrm{d}A_1}{\mathrm{d}y} = \langle e^{-i\theta}\rangle - A_1 \tag{1.53}$$

The numerical analysis of these equations (Fig.1.5) shows that

- the field is indeed superradiant, that is $|E|^2 \propto n^2$ instead of $n^{4/3}$
- the amplitude $|A|$ of the superradiant peak is proportional to z_1
- the width of the superradiant pulse is inversely proportional to $\sqrt{z_1}$

Chapter 2

First approaches to the Quantum FEL

The original proposal of an FEL [2] was formulated in a quantum framework; in this early picture, an FEL was treated as light amplification induced by stimulated Compton scattering. However, most of the theoretical work for the FEL was formulated classicaly and the results obtained through quantum theory were rederived using classical electrodynamics [16, 17]. Indeed, since most of the properties of existing FELs can be analyzed and found remaining in a strictly classical framework, it is widely accepted that the FEL is an essentially classical device [18].

Anyway, a quantum mechanical analysis of the FEL starting from first principles has been adopted more than once in literature. Such an approach is not only very interesting from a theoretical point of view, but there are also some very practical reasons to use quantum mechanics instead of classical mechanics when dealing with the FEL:

- As seen in the previous chapter, a classical treatment yields an e.m. potential $A(t)$ proportional to its initial value A_0; this means that spontaneous emission, i.e. start-up from vacuum, is not possible unless somehow a small fluctuation is introduced by hand as a non-zero initial potential. A quantum treatment instead includes real spontaneous emission from quantum noise.

- Some physical questions require a quantum mechanical approach: for example, photon statistics and the quantum coherence properties of the emitted radiation.

- While for all existing FEL devices classical physics is a perfectly safe approximation, quantum effects could and do become relevant when operating in different regions for the range of the experimental parameters. In particular for very short wavelengths, the electron recoil from the emission of a photon cannot be neglected anymore when compared to the emission linewidth, making a quantum mechanical treatment of the electron-photon interaction necessary.

In this chapter we will provide a summary of some of the past treatments of the FEL using quantum physics.

2.1 The Quantum FEL Hamiltonian

Most of the quantum approaches [5, 6, 7, 8] to the free electron laser are based on a Hamiltonian formalism describing the energy exchange process between the electron beam and the radiation field. The electrons are treated using operators corresponding to the phase space variables (z, p), in a frame such that the electron motion is non-relativistic [19], while the laser field is represented by the photon annihilation and creation operators. The quantum Hamiltonian derived in this section will be the starting point of our own work in the next chapter.

We start from the classical Hamiltonian describing N relativistic electrons interacting with an electro-magnetic field; we neglect space-charge effects, and assume that the electrons do not directly interact among each other. This gives the Hamiltonian

$$H = \sum_{j=1}^{N} \sqrt{m_e^2 c^4 + \left[c\vec{P}_j + e\vec{A}\right]^2} \tag{2.1}$$

Here c is the speed of light and m_e, $-e$ are the mass and charge of the electron. $\vec{P}$ is the canonical momentum and $\vec{A} = \vec{A}_w + \vec{A}_\lambda$ the total field vector potential given by the wiggler field $\vec{A}_w$ and the laser field $\vec{A}_\lambda$. The axis $\hat{z}$ is taken as the direction of propagation of the fields.

The electrons travel along this direction as well and their momentum remains perpendicular to the field vector potential at all times, i.e. $\vec{P} \cdot \vec{A} = 0$ [22], and thus the term in (2.1) containing P and A becomes

$$\left(\vec{P} + \vec{A}\right)^2 = c^2 \left(P_z^2 + P_\perp^2\right) + e^2 \left(|A_w|^2 + |A_\lambda|^2 + 2\vec{A}_w \cdot \vec{A}_\lambda\right)$$

We then make the following considerations:

- The electron longitudinal momentum is much greater than the transverse momentum, $P_z \gg P_\perp$, so that we can neglect $P_\perp^2$.
- The intensity of the wiggler field is much bigger than the laser field's, $|A_w|^2 \gg |A_\lambda|^2$ so that again we can neglect the latter term.
- The wiggler field $|A_w|^2$ is a constant so we can just include it in the $m_e^2c^4$ term, renormalizing the electron mass as

  $$m \equiv m_e\sqrt{1+\frac{e^2}{m^2c^4}|A_w|^2} = m_e\sqrt{1+a_w^2}$$

  where $a_w = e|A_w|/mc^2$ is the wiggler parameter as defined in Eq.(1.10).
- The wiggler field is written as in (1.9)

  $$\vec{A}_w = \frac{mc^2}{e}\frac{a_w}{\sqrt{2}}\left(\hat{\mathrm{e}}\, e^{-ik_w z} + c.c.\right) \tag{2.2}$$

  with a_w being the wiggler parameter, while the radiation field is decomposed over the quantization volume V [21, 8]

  $$\vec{A}_\lambda = -i\sqrt{\frac{2\pi\hbar c}{k_\lambda V}}\left(a_\lambda \hat{\mathrm{e}}\, e^{-ik_\lambda(z-ct)} - c.c.\right) \tag{2.3}$$

  a_λ is the dimensionless field amplitude. We will quantize it to obtain the photon creation and annihilation operators.

  (Here we have supposed that the radiation field is monoenergetic, i.e. all radiation has a single wavelength, but every step of this derivation could be repeated using a multi-mode field $\vec{A}_{rad} = \sum_\lambda \vec{A}_\lambda$)

Thus the Hamiltonian becomes

$$\begin{aligned} H &= \sum_{j=1}^{N}\sqrt{m^2c^4 + c^2P_j^2 + 2e^2\vec{A}_w\cdot\vec{A}_\lambda} = \\ &= mc^2\sum_{j=1}^{N}\sqrt{1+\frac{P_j^2}{m^2c^2} + i\frac{2\varepsilon}{mc^2}\left(a_\lambda e^{i\theta_j} - a_\lambda^* e^{-i\theta_j}\right)} \end{aligned}$$

where $P_j \equiv P_{z,j}$ is the longitudinal momentum, $\theta_j = (k_\lambda + k_w)z_j - k_\lambda ct$ is the FEL phase and

$$\varepsilon = \sqrt{\frac{e^2\hbar\lambda c}{2V}}\, a_w \ll mc^2 \tag{2.4}$$

We have included a_w in the coupling energy ε because we're not interested in quantizing the wiggler field. Since $|A_w|^2 \gg |A_\lambda|^2$ it is possible to treat it as a classical field [22].

Until now we were in the laboratory frame; we make then a Lorentz transformation, going to a frame whose velocity with respect to the laboratory frame is given by the initial average energy γ_0 of the electrons:

$$\beta' \equiv \sqrt{1 - \frac{1}{\gamma_0^2}} \tag{2.5}$$

This frame is called the Bambini-Renieri frame [19].

The electron beam is injected in the wiggler with an energy very close to the resonant one, so that in this frame it is nearly at rest. Thus it is possible to treat it as non-relativistic beam ($P'_z \ll mc$): using $\sqrt{1+\epsilon} \simeq 1 + \epsilon/2$ we get

$$H = \sum_{j=1}^{N} \left[\frac{P_j'^2}{2m} + i\varepsilon \left(a_\lambda e^{i\theta'_j} - c.c. \right) \right] \tag{2.6}$$

where P' and θ' are the variables in the Bambini-Renieri frame.

Before quantizing both the field amplitude and the electron variables, we renormalize the Hamiltonian to a dimensionless form, introducing the fundamental FEL *quantum* parameter $\bar{\rho}$ [5, 20]:

$$\bar{\rho} \equiv \frac{mc\gamma_r}{\hbar k_\lambda} \rho = q\,\rho \tag{2.7}$$

where we introduced the *quantum recoil parameter*

$$q = mc\gamma_r/\hbar k \tag{2.8}$$

i.e. the ratio between the electron and photon momenta (we drop the λ label from k from now on, $k \equiv k_\lambda$), and ρ is the FEL parameter defined in the first chapter

$$\rho = \frac{1}{\gamma_r} \left(\frac{a_w \omega_p}{4 k_w c} \right)^{2/3}$$

We express the momentum in units of photon momentum $\hbar k$

$$\bar{p}_j \equiv q\, \frac{\gamma_j - \gamma_r}{\gamma_r} = \frac{mc(\gamma_j - \gamma_r)}{\hbar k} \tag{2.9}$$

This dimensionless momentum $\bar{p}$ is different from the dimensionless momentum p defined in the universal scaling (1.27), since

$$\bar{p} = \bar{\rho} p \tag{2.10}$$

The final dimensionless Hamiltonian is

$$H = \sum_{j=1}^{N} \left[\frac{\bar{p}_j^2}{2\bar{\rho}} - i\sqrt{\frac{\bar{\rho}}{N}} \left(a_\lambda e^{i\theta_j} - a_\lambda^\dagger e^{-i\theta_j} \right) \right] \tag{2.11}$$

In Eq.(2.11) we have already transformed $\bar{p}$, θ and a_λ into operators, using the commutation relations

$$[\theta_j, \bar{p}_k] = i\,\delta_{jk} \tag{2.12}$$

$$[a_\lambda, a_\lambda^\dagger] = 1 \tag{2.13}$$

Now a_λ and $a_\lambda^\dagger$ are the annihilation and creation operators of the laser photons.

It is useful to redefine the three variables of the system such that the average momentum distribution is centered on γ_0:

$$\hat{p}_j = \bar{p}_j - \bar{\rho}\delta \tag{2.14}$$

$$\hat{\theta}_j = \theta_j - \delta z \tag{2.15}$$

$$a = a_\lambda e^{i\delta z} \tag{2.16}$$

where again $\delta = \dfrac{\gamma_0 - \gamma_r}{\rho\gamma_r}$. In fact it can be checked that

$$\langle \hat{p}_j \rangle = \frac{mc(\gamma_j - \gamma_r)}{\hbar k} - \bar{\rho}\frac{\gamma_0 - \gamma_r}{\rho\gamma_r} = \frac{mc(\gamma_j - \gamma_0)}{\hbar k}$$

We substitute the definitions (2.14)-(2.16) in (2.11) and use the integral of motion

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\{ \sum_j \bar{p}_j + a^\dagger a \right\} = 0 \tag{2.17}$$

to get the following dimensionless Hamiltonian

$$H = \sum_{j=1}^{N} \left[\frac{\hat{p}_j^2}{2\bar{\rho}} - i\sqrt{\frac{\bar{\rho}}{N}} \left(a e^{i\hat{\theta}_j} - a^\dagger e^{-i\hat{\theta}} \right) \right] - \delta a^\dagger a \tag{2.18}$$

From this Hamiltonian we derive the equations of motions for the electron and photon operators:

$$\frac{\mathrm{d}\hat{p}_j}{\mathrm{d}t} = -i[\hat{p}_j, H] = -\sqrt{\frac{\bar{\rho}}{N}} \left(a_\lambda e^{i\theta_j} + a_\lambda^\dagger e^{-i\theta_j} \right) \tag{2.19}$$

$$\frac{\mathrm{d}\theta_j}{\mathrm{d}t} = -i[\theta_j, H] = \frac{\hat{p}_j}{\bar{\rho}} \tag{2.20}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -i[a, H] = i\delta a + \sqrt{\frac{\bar{\rho}}{N}} \sum_j e^{-i\theta_j} \tag{2.21}$$

In the following sections of this chapter, we will give a short summary of some past quantum approaches to the FEL that, while not directly related to our own work found in the next chapters, can give a general overview of how quantum mechanics is connected to the FEL.

2.2 Collective Operators

A linear analysis of the Hamiltonian (2.18) can be done introducing two collective operators [21, 22, 23], the bunching operator

$$\mathrm{B} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-i\hat{\theta}_j} \tag{2.22}$$

and the symmetrized momentum bunching

$$\mathrm{P} = \frac{1}{2\sqrt{N}} \sum_{j=1}^{N} \left(\hat{p}_j e^{-i\hat{\theta}_j} + e^{-i\hat{\theta}_j} \hat{p}_j \right) \tag{2.23}$$

Notice that in contrast to other past works [21, 22], this definition of P is correctly symmetrized [23].

The equations of motion for these operators can be deduced from (2.19)-(2.21), so that one obtains

$$\frac{\mathrm{dB}}{\mathrm{d}t} = -\frac{i}{\bar{\rho}}\mathrm{P} \tag{2.24}$$

$$\frac{\mathrm{dP}}{\mathrm{d}t} = -\frac{i}{4\bar{\rho}}\mathrm{B} - \sqrt{\bar{\rho}}a \tag{2.25}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = i\delta a + \sqrt{\bar{\rho}}\mathrm{B} \tag{2.26}$$

In these equations the higher order terms have been neglected, specifically those proportional to

$$\frac{1}{\sqrt{N}} \sum_j p_j e^{-i\theta_j} p_j$$

and

$$\frac{a^\dagger}{\sqrt{N}} \sum_j e^{-2i\theta_j}$$

since in the linear approximation they do not give any contribution to the expectation values $\langle B \rangle$ and $\langle P \rangle$ [23].

The closed linear system given by (2.24)-(2.26) has solutions of the form

$$\mathrm{B}(t) \propto e^{i\lambda t}$$

where the λ are the solutions of the dispersion relation

$$(\lambda - \delta)\left(\lambda^2 - \frac{1}{4\bar{\rho}^2}\right) + 1 = 0 \tag{2.27}$$

This equation corresponds to the classical cubic equation (1.47) for the case of a square energy distribution of width $1/2\bar{\rho}$, i.e. in non-dimensionless units a spread in momentum of $\hbar k/2$. This can be interpreted as the intrinsic momentum spread due to quantum mechanics.

I will continue the analysis of the quantum Hamiltonian (2.18) in Chapter 3. I will now give a review of previous works about a few FEL topics that will be very important in this work:

- The Interaction Picture Hamiltonian and Perturbation Theory
- Photon Statistics
- The Bosonic Approximation for the electrons
- Two-Level Systems

2.3 Interaction Picture: the Time-evolution Operator

Photon statistics is a very important issue for free electron lasers and strongly connected to quantum mechanics; most of the work of this thesis indeed revolves around photon statistics. In the next chapter we will use the Interaction Picture to study photon statistics in the quantum regime of FEL; a previous similar study was done by Becker and Zubairy [8], using the time-evolution operator formalism. This operator can be used to find transition probabilities: if the system is in the state $|i\rangle$ at the time t_i, then the probability of finding it in the state $|f\rangle$ at time t_f is

$$P(i|f) = |\langle f|S(t_f, t_i)|i\rangle|^2 \tag{2.28}$$

The formal definition of the time-evolution operator is

$$S(t_f,t_i) = \mathcal{T}\exp\left\{-\frac{i}{\hbar}\int_{t_i}^{t_f} H_{IP}(t)\mathrm{d}t\right\} \tag{2.29}$$

where $\mathcal{T}$ is the Dyson time-ordering operator and $H_{IP}(t)$ is the Interaction Picture Hamiltonian, given by

$$H_{IP}(t) = e^{iH_0t/\hbar}H_I e^{-iH_0t/\hbar} \tag{2.30}$$

The full system Hamiltonian is given by the sum of the kinetic energy H_0 and the interaction potential H_I:

$$\begin{aligned} H &= H_0 + H_I &(2.31)\\ H_0 &= \frac{\hat{p}^2}{2m} + \hbar\omega(a_l^\dagger a_l + a_w^\dagger a_w) &(2.32)\\ H_I &= i\hbar g\left(a_l^\dagger a_w e^{-ik\hat{z}} - a_w^\dagger a_l e^{ik\hat{z}}\right) &(2.33)\end{aligned}$$

In this Hamiltonian the creation and annihilation operators of both the laser field a_l and the wiggler field a_w are used, and a specific frame has been chosen so that the laser and wiggler frequencies coincide, $\omega_l = \omega_w \equiv \omega$. *It is important to notice that the approximation of a one-electron Hamiltonian has been taken.*

The usual commutation rules apply

$$\begin{aligned} [a_l, a_l^\dagger] &= [a_w, a_w^\dagger] = 1\\ [\hat{p}, \hat{z}] &= -i\hbar \end{aligned}$$

Inserting (2.32) and (2.33) into (2.30) the interaction picture Hamiltonian obtained is

$$\begin{aligned} H_{IP}(t) &= i\hbar g\left(a_l^\dagger a_w e^{-ik\hat{z}} e^{-i(\hbar k^2+2k\hat{p})t/m} - h.c.\right)\\ &= i\hbar g\sqrt{N_w}\left(A^\dagger e^{-i(\hbar k^2+2k\hat{p})t/m} - A e^{i(\hbar k^2+2k\hat{p})t/m}\right) &(2.34)\end{aligned}$$

In (2.34) the semiclassical limit for the wiggler field has been taken, for the same reasons expressed when deriving (2.18) - mostly summarized by the fact that $|\langle a_w\rangle|^2 \gg |\langle a_l\rangle|^2$. The operator $A \equiv a_l e^{ik\hat{z}}$ has been introduced for simplicity; notice that still $[A, A^\dagger] = 1$ and $A^\dagger A = a_l^\dagger a_l$.

The time-evolution operator (2.29) cannot be exactly derived for the Hamiltonian (2.34): this can be solved by treating p as a c-number instead of an operator, but this extreme approximation would mean neglecting the electron recoil during the photon emission and it is well known that this would amount to getting no gain.

Becker and Zubairy try to get around this problem by expanding the time-evolution operator around a c-number central value p_0, up to the first order in $(\hat{p} - p_0)$:

$$S(t_f, t_i) \simeq S_0(t_f, t_i) + S_1(t_f, t_i) \tag{2.35}$$

$$S_0(t_f, t_i) = \mathcal{T} \exp\left\{ -\frac{i}{\hbar} \int_{t_i}^{t_f} H_{IP}(t) \mathrm{d}t \right\}\Bigg|_{\hat{p}=p_0} \tag{2.36}$$

$$S_1(t_f, t_i) = -\frac{i}{\hbar} \int_{t_i}^{t_f} S_0(t_f, t)\mathcal{T}(\hat{p} - p_0) \left\{ \frac{\partial}{\partial p} H_{IP}(t)\Big|_{\hat{p}=p_0} \right\} S_0(t, t_i)\mathrm{d}t \tag{2.37}$$

S_0 is what would be obtained by treating the momentum as a c-number, while in S_1 the quantum electron recoil is taken into account up to first order. *The limit of this treatment is that this linear approximation can only hold for the small-signal regime.*

The time ordering of the zeroeth order term S_0 can be easily calculated, since in the case of a c-number momentum p_0 the different times commutator is a c-number too:

$$[H_{IP}(t_1), H_{IP}(t_2)] = 2ig^2 N_w \sin\left[\frac{k}{m} \left(p_0 + \frac{\hbar k}{2} \right) (t_1 - t_2) \right]$$

The final result for S_0 is then

$$S_0(t_f, t_i) = \exp\left[i\Theta(t_f, t_i)\right] \exp\left[J^*(t_f, t_i)A^\dagger\right] \exp\left[-J(t_f, t_i)A\right] \exp\left[-\frac{|J(t_f, t_i)|^2}{2} \right] \tag{2.38}$$

where

$$\Theta(t_f, t_i) = \frac{g^2 N_w}{\hbar^2} \left[\frac{(t_f - t_i)}{\beta} + \frac{\cos(\beta t_f)\sin(\beta t_i) - \sin(\beta t_f)\cos(\beta t_i)}{\beta^2} \right] \tag{2.39}$$

$$J(t_f, t_i) = -i\frac{g\sqrt{N_w}}{\beta} \left(e^{i\beta t_f} - e^{i\beta t_i} \right) \tag{2.40}$$

$$\beta = \frac{k}{m} \left(p_0 + \frac{\hbar k}{2} \right) \tag{2.41}$$

For simplicity the initial and final times are taken to be $t_f = T$ and $t_i = -T$, where $2T = L_w/c$ is the interaction time along the wiggler length L_w. This way the formulas simplify to

$$\Theta(T, -T) = \frac{g^2 N_w}{\hbar^2} \left[\frac{2T}{\beta} + \frac{\sin(2\beta T)}{\beta^2} \right] \tag{2.42}$$

$$J \equiv J(T, -T) = J^*(T, -T) = \frac{2g\sqrt{N_w}}{\beta} \sin(\beta T) \tag{2.43}$$

Using (2.38) in (2.37) one gets the first order term $S_1(T,-T)$:

$$\begin{aligned}S_1(T,-T) \simeq\ & S_0(T,-T)\Big\{-ig\sqrt{N_w}\frac{k}{m}\int_{-T}^{T} e^{-i\beta t}\Big[\\ & \{\hat{p}-p_0-\hbar k[|J(t,-T)|^2+J^*(t,-T)A^\dagger+J(t,-T)A]\}[A^\dagger+J(t,-T)] \\ & -[A+J(T,t)]\{\hat{p}-p_0-\hbar k[|J(T,t)|^2+J^*(T,t)A+J(T,t)A^\dagger]\}\Big]\mathrm{d}t\Big\}\end{aligned} \tag{2.44}$$

The resulting time-evolution operator depends on the choice of the central momentum p_0, but it is possible to show instead that, up to first order in the recoil, this choice is arbitrary. This can be seen by taking the partial derivative of $S(T,-T)$ over p_0, which after some algebra gives

$$\frac{\partial S(T,-T)}{\partial p_0} = O\left(\frac{k}{m}\right)^2$$

Now that $S(T,-T)$ is known, it is possible to use it together with (2.28) to get transition probabilities, in particular to study photon statistics. The momentum eigenstates $|p,n\rangle$ are chosen as the physical states, where n represents the number of laser photons. It can be seen then that

$$\hat{p}|p,n\rangle = p|p,n\rangle \tag{2.45}$$

$$A|p,n\rangle = \sqrt{n}|p+\hbar k, n-1\rangle \tag{2.46}$$

$$A^\dagger|p,n\rangle = \sqrt{n+1}|p-\hbar k, n+1\rangle \tag{2.47}$$

since $e^{ik\hat{z}}$ acts a translation operator for the electron momentum. As an example, an initial state with zero photons is taken, and momentum $p = p_0 + \hbar k/2$. Then the photon distribution function is given by

$$\begin{aligned}P(n) &= |\langle p-n\hbar k, n|S(T,-T)|p,0\rangle|^2 \\ &= \frac{J^{2n}}{n!}e^{-J^2}\left\{1-\frac{\hbar k^2}{mJ}\frac{\partial J}{\partial\beta}[n^2-(2n+1)J^2+J^4]\right\}\end{aligned} \tag{2.48}$$

The average photon number and the photons distribution spread can be now calculated:

$$\langle n\rangle = J^2 - \frac{\hbar k^2}{m}J\frac{\partial J}{\partial\beta} + \delta \tag{2.49}$$

$$\Delta n^2 = J^2 - \frac{\hbar k^2}{m}(J+2J^3)\frac{\partial J}{\partial\beta} + \delta \tag{2.50}$$

$$\begin{aligned}\delta \equiv\ & ig\sqrt{N_w}\frac{\hbar k^2}{m}J\int_{-T}^{T} e^{-i\beta t} \\ & \times\left\{2|J(T,t)|^2+2|J(t,-T)|^2-J^2(T,t)-J^2(t,-T)\right\}\mathrm{d}t\end{aligned}$$

There was no need to include fluctuations in the inital conditions to get spontaneous emission, while it would have been necessary if using classical physics.

Putting together the above equations, most terms cancel out and the following equation is obtained:

$$\Delta n^2 - \langle n \rangle = -\frac{2\hbar k^2}{m} J^3 \frac{\partial J}{\partial \beta} \tag{2.51}$$

Eqs.(2.51) and (2.43) show another quantum effect that can't be found using classical tools: the spontaneous emission radiation field is bunched for $\beta > 0$, i.e. when the electrons are above resonance; and antibunched when below resonance, $\beta < 0$.

Even in the approximation of a single electron and up to first order in the quantum recoil, a quantum mechanical treatment of the FEL allows to get information otherwise barred to a classical analysis.

2.4 Dirac Equation

While most of the quantum treatments of the FEL usually started from the Hamiltonian (2.18) in the Bambini-Renieri frame (2.5) to avoid the complications of relativistic quantum mechanics, an approach was made by deriving the FEL interaction Hamiltonian by general quantum electrodynamics [24].

The final result of this analysis shows how, under certain approximations, Quantum Electro Dynamics leads to the usual Hamiltonian (2.18). Here we briefly summarize its derivation. The starting point is the QED Hamiltonian representing the interaction between a radiation field and charged particles:

$$H_I = e \int j^\nu A_\nu^{[l]} \, \mathrm{d}^3 x \tag{2.52}$$

Here j^ν is the four-vector current density operator, $A_\nu^{[l]}$ is the four-vector potential of the emitted radiation and e the charge of the electron. Upper and lower indexes are considered to be summed over when they're repeated. The current density is given by

$$j^\nu = \bar{\Psi} \gamma^\nu \Psi \tag{2.53}$$

where the γ^ν are the Dirac matrices and Ψ the field operators defined by the solutions of the Dirac equation:

$$\Psi = \sum_p \psi_p(x^\nu) b_p \tag{2.54}$$

The b_p are the fermion creation operators, the label p representing momentum. The ψ_p are the solutions of the Dirac equation [24]. The total four-vector radiation potential is the sum of the laser field $A_\nu^{[l]}$ and the wiggler field $A_\nu^{[w]}$. The latter is given by $A^{[w]\nu} = (0, \vec{A}_w)$, where $\vec{A}_w$ is the wiggler vector potential. For a circularly polarized static wiggler along the $\hat{z}$ axis, this is

$$\vec{A}_w = i\left(\frac{\hat{x}+i\hat{y}}{2}\right) A_w e^{-i\phi} + c.c. \tag{2.55}$$

$$\phi = k_\nu^{[w]} x^\nu = k_w z \tag{2.56}$$

$k_\nu^{[w]}$ being the wave (four)vector of the wiggler field; $k_\nu = (0,0,0,k_w)$ for a static wiggler. In this case, with the approximation that for relativistic electrons

$$p^2 \gg e^2 A_w^2$$

the Dirac equation can be solved and one obtains

$$\psi_p \propto \left(1 + \frac{e\left(k_\nu \gamma^\nu\right)\left(A_\nu^{[w]}\gamma^\nu\right)}{2k_\nu^{[w]} p^\nu}\right) \exp\left\{-\frac{i}{\hbar} p_\nu x^\nu - \frac{i}{2\hbar}\int_0^{k_w z} \frac{e^2 A_w^2}{p_\nu k^\nu}\,\mathrm{d}\phi\right\} \tag{2.57}$$

From ψ_p it is possible to get j^ν and thus $j^\nu A_\nu^{[l]}$. Given the approximation of a perfectly longitudinal momentum

$$p^\nu = \left(\frac{E_p}{c}, 0, 0, p_z\right)$$

$$E_p = \gamma m c^2$$

it simplifies to

$$j^\nu A_\nu^{[l]} = \frac{eA^{[w]\nu} A_\nu^{[l]}}{m\gamma V} \sum_{p,p'} \exp -i\left\{\frac{1}{\hbar}(p_\nu - p'_\nu)x^\nu + \int_0^{kz} \frac{e^2 A_w^2}{2\hbar}\left(\frac{1}{k^\nu p_\nu} - \frac{1}{k^\nu p'_\nu}\right)\mathrm{d}\phi\right\} b_{p'}^\dagger b_p \tag{2.58}$$

where m is the mass of the electrons, $\gamma = \sqrt{1 + (p_z/mc)^2}$ the relativistic factor and V the quantization volume.

The laser field is written as the general superposition of all the plane-wave modes a_k travelling along the z-axis:

$$A^{[l]\nu} = (0, \vec{A}_l) \tag{2.59}$$

$$\vec{A}_l = i\left(\frac{\hat{x}+i\hat{y}}{2}\right)\sqrt{\frac{\hbar}{\epsilon_0 c \tilde{k}_l V}} \sum_{k_l} e^{ik_l(z-ct)} a_{k_l} + h.c. \tag{2.60}$$

where one has assumed that all the modes frequencies k_l of the laser field are all close to the central value $\tilde{k}_l$, so that $|k - \tilde{k}| \ll \tilde{k}$ for all k_l. The operators a_k and

$a_k^\dagger$ are then the annihilation and creation operators for the laser field photons of frequency ck_l. For the case of a single mode FEL, this reduces to

$$\vec{A}_l = i\left(\frac{\hat{x}+i\hat{y}}{2}\right)\sqrt{\frac{\hbar}{\epsilon_0 c k_l V}}\, e^{ik_l(z-ct)}a_l + h.c. \tag{2.61}$$

Substituting (2.58) and (2.61) into (2.52) and then integrating over d^3x gives a more explicit form of the interaction Hamiltonian. The integral over $\mathrm{d}\phi$ gives an integral over z between $z=0$ and $z=L$, where L is the length of the wiggler:

$$\begin{aligned} H_I &= \frac{e^2 A_w}{2m\gamma L}\sqrt{\frac{\hbar}{\epsilon_0 c k_l V}}\sum_{p,p'} e^{-i\{(E_p-E_{p'})/\hbar+ck\}t} b_{p'}^\dagger b_p a_l \\ & \int_0^L \exp\left\{i\left[\left(1+\frac{a_w^2}{2\gamma^2}\right)\frac{p-p'}{\hbar}+k_w+k_l\right]z\right\}\mathrm{d}z + h.c. \end{aligned} \tag{2.62}$$

The integral over z of the imaginary exponential can be approximated by a δ distribution, which gives the condition

$$p' = p + \frac{\hbar(k_w+k_l)}{1+a_w^2/2\gamma^2} \tag{2.63}$$

One can define the detuning parameter[1]

$$\mu \equiv \frac{(\gamma-\gamma_0)}{\gamma_0^3}\, k_l \tag{2.64}$$

Under the assumption $\mu\gamma_0^2/k_l \ll 1$, one can linearly expand $(E_p - E_{p'})/\hbar$, so that H_I simplifies to

$$H_I = \frac{e^2 A_w}{2m\gamma_0}\sqrt{\frac{\hbar}{\epsilon_0 c k_l V}}\sum_{\mu}\exp\left\{i\left[\mu+\frac{q}{1+a_w^2}\right]ct\right\} b_{\mu+2q}^\dagger b_\mu a_l \tag{2.65}$$

where $q = \hbar k_w k_l / mc\gamma_0$ is the quantum recoil parameter.

Until now the electrons have been treated as fermions. This means they must obey the exclusion principle, i.e. no two electrons can have the same quantum numbers; this includes longitudinal and transverse momentum. Still, it is physically acceptable to take all the following assumptions:

- There is no electron with both the same longitudinal and transverse momentum as another electron.

[1] Notice that this isn't the same detuning parameter δ that we have defined in (1.38) and which will be used in the rest of this thesis.

- The transverse momenta of all the electrons are negligible, so that the approximation taken in (2.58) is respected.
- The longitudinal momentum spread is negligible, so that all the electrons can be considered to have the same longitudinal momentum.

This allows to treat the electrons as bosons, since the labels that *distinguish* them (to satisfy the exclusion principle) are neglected. The interaction Hamiltonian (2.65) can then be rewritten by replacing the fermionic operator $b^{\dagger}_{\mu+2q}b_{\mu}$ with a bosonic momentum-shift operator:

$$H_I = \hbar g \sum_{j=1}^{N} e^{i\theta_j} e^{i(\mu_j+q)ct} a_l + h.c. \tag{2.66}$$

Here

- $g = \dfrac{e^2 A_w}{2\hbar m\gamma_0}\sqrt{\dfrac{\hbar}{\epsilon_0 c k_l V}}$ is the coupling constant.
- θ_j is an operator defined such that $e^{i\theta_j}$ raises the momentum μ of the quantity $2q$, i.e. $[\mu_j, e^{i\theta_j}] = 2qe^{i\theta_j}$.

This interaction Hamiltonian is equivalent to (2.34), generalized to the case of N electrons, if θ is identified with $k\hat{z}$, $\hat{\mu}c$ with $k\hat{p}/m$ and q with $\hbar k^2/2mc$.

2.4.1 Second order perturbation theory

The interaction Hamiltonian (2.66) was used by Gea-Banacloche [24] in the attempt to solve the evolution equation of the density matrix σ:

$$\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = -\frac{i}{\hbar}[H_I(t), \sigma(t)] \tag{2.67}$$

If the coupling constant g is small, this equation can be iterated to obtain a perturbation series. Up to first order it becomes

$$\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = -\frac{i}{\hbar}[H_I(t), \sigma(0)] - \frac{1}{\hbar^2}\int_0^t [H_I(t), [H_I(t'), \sigma(t')]]\mathrm{d}t' \tag{2.68}$$

In the 0th order term the initial density matrix is factorized as

$$\sigma(0) \simeq \sigma_{el}(0) \otimes \sigma_{ph}(0) \tag{2.69}$$

representing the lack of correlations between the radiation field and the injected electrons at time zero. A strong approximation is taken instead in the 1st order term, assuming that

$$\sigma(t') \simeq \sigma_{el}(0) \otimes \sigma_{ph}(t') \tag{2.70}$$

This amounts to neglecting any higher-order changes in the electron system; in other words the electron recoil is taken into account only up to first order. Substituting (2.70) into (2.68) the zeroeth order disappears due to the diagonality of $\sigma(0)$ and the equation for the photon density matrix becomes

$$\frac{\mathrm{d}\sigma_{ph}(t)}{\mathrm{d}t} = -\frac{1}{\hbar^2}\int_0^t Tr_{el}[H_I(t), [H_I(t'), \sigma_{el}(0) \otimes \sigma_{ph}(t')]]\mathrm{d}t' \tag{2.71}$$

Using the interaction Hamiltonian (2.66) the final equation is

$$\begin{aligned}\frac{\mathrm{d}\sigma_{ph}(t)}{\mathrm{d}t} &= -g^2 \int_0^t \left\{ \left\langle \sum_{j=1}^{N} e^{i(\mu-q)c(t-t')} \right\rangle \left[aa^\dagger \sigma_{ph}(t') - a^\dagger \sigma_{ph}(t')a\right] \right. \\ &\left. + \left\langle \sum_{j=1}^{N} e^{-i(\mu+q)c(t-t')} \right\rangle \left[a^\dagger a \sigma_{ph}(t') - a\sigma_{ph}(t')a^\dagger\right] \right\} \mathrm{d}t' + h.c. \end{aligned} \tag{2.72}$$

The brackets $\langle ... \rangle$ represents the average over the electron degrees of freedom. Assuming that the electrons' momenta follow a distribution $G(\mu)$, those averages become

$$\left\langle \sum_{j=1}^{N} e^{\mp i(\mu\pm q)c(t-t')} \right\rangle = N \int G(\mu) e^{\mp i(\mu\pm q)c(t-t')}\mathrm{d}\mu \tag{2.73}$$

Once the evolution equation for the density matrix is obtained, it is then possible to get the expectation value of any operator $\hat{O}$ acting on the photon states simply using the well known identity

$$\langle \hat{O} \rangle = \mathrm{Tr}\{\sigma_{ph}\hat{O}\} \tag{2.74}$$

For example, in the case of start-up from vacuum ($\sigma_{ph} = |0\rangle\langle 0|$), the average number of photons emitted after a single pass in the wiggler is

$$\langle a^\dagger a \rangle = 4g^2T^2N \int \left(\frac{\sin[(\mu-q)cT]}{(\mu-q)cT} \right)^2 G(\mu)\mathrm{d}\mu \tag{2.75}$$

where $T = L_w/2c$ is half the interaction time. The number of photons grows as the square of the interaction time: the lack of an exponential growth is the main limit of using perturbation theory, since it will only give powers of T up to the order to which the series was calculated to.

2.5 Two-level Klein-Gordon equation

We'll see in the next chapter how a quantum regime for FEL is found under certain conditions for the physical parameters; in such regime the free electron laser behaves as a discrete two-level system (the electrons) coupled to radiation (the laser field). Smetanin et al. [25, 26] have studied a model of a two-level FEL, starting from a description of the free electron laser through the approximated Klein-Gordon equation:

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial z^2} + \frac{2e^2}{\hbar^2}\vec{A}_w \cdot \vec{A}_l + \frac{m^2c^4}{\hbar^2}\right)\Psi(z,t) = 0 \tag{2.76}$$

where $\Psi(z,t)$ is the electron wave-function and $\vec{A}_w$, $\vec{A}_l$ are respectively the wiggler and laser (classical) radiation fields.

The electron momentum is discretized, to represent excitations of the discrete energy levels of an anharmonic oscillator [15]

$$p_n = p_0 + n\hbar(\omega_w + \omega_l)/c$$

where p_0 is the initial momentum, ω_w and ω_l are the frequencies of the wiggler field and the laser field. The electron wave function can then be expanded as a superpositions of wave functions each corresponding to one of those different energy levels:

$$\Psi(z,t) = \sum_n \sqrt{\frac{mc^2}{2VE_n}}\, c_n \exp\left(\frac{i}{\hbar}[p_n z - E_n t]\right) \tag{2.77}$$

where $E_n = \sqrt{p_n^2c^2 + m^2c^4}$ is the energy of the momentum level n. Inserting (2.77) in the Klein-Gordon equation (2.76) gives a system of equations describing an anharmonic oscillator with infinite levels:

$$i\frac{\mathrm{d}c_n}{\mathrm{d}t} = g_n c_{n-1} e^{-i\Delta_n t} + g_{n+1} c_{n+1} e^{i\Delta_{n+1} t} \tag{2.78}$$

where

$$\begin{aligned} g_n &\equiv \frac{e^2 A_w A_l \sqrt{E_n E_{n-1}}}{\hbar} \\ \Delta_n &\equiv \omega_l - \omega_w + \frac{E_{n-1} - E_n}{\hbar} \end{aligned}$$

Under certain conditions, the imaginary exponent Δ_n becomes extremely big when $n \neq 0$, so that all $n \neq 0$ terms give no contribution to the solution of (2.78). This reduces the infinite levels system described by (2.78) to a two-level system.

In detail, using the definitions of E_n and p_n, one can obtain

$$\Delta_n \simeq -\frac{4\pi N_w}{T}\left\{\frac{E_0 - E_l + n\hbar(\omega_l - \omega_w)}{E_l}\right\} \tag{2.79}$$

where N_w is the number of wiggler periods and T is the half interaction time $L_w/2c$. E_l is the emission line energy of the laser radiation, given by

$$E_l = \frac{\hbar(\omega_l - \omega_w)}{2} + mc^2\sqrt{\frac{\omega_l + \omega_w}{2\omega_w}} \tag{2.80}$$

If the electrons are injected in the FEL with an energy E_0 very close to E_l, then Δ_n simplifies as

$$\Delta_n \simeq -\frac{4\pi N_w}{T}\frac{n\hbar(\omega_l - \omega_w)}{E_0} \tag{2.81}$$

The imaginary exponentials in (2.78) are then rapidly oscillating when

$$\Delta_n T \gg 2\pi \tag{2.82}$$

so that the condition on the physical parameters becomes

$$2N_w\frac{\hbar(\omega_l - \omega_w)}{E_0} \gg 1 \tag{2.83}$$

When this happens, only the level $n = 0$ is stable and the equations (2.78) reduce to the two-level system

$$\frac{\mathrm{d}c_0}{\mathrm{d}t} = -ig_0 c_{-1} e^{-i\Delta_0 t} \tag{2.84}$$

$$\frac{\mathrm{d}c_{-1}}{\mathrm{d}t} = -ig_0 c_0 e^{i\Delta_0 t} \tag{2.85}$$

Notice that $\Delta_0 \neq 0$ since $E_0 - E_l$ can be neglected and Eq.(2.81) used only when $n \neq 0$.

To close the system the equations (2.84)-(2.85) must be coupled with an equation for the radiation fields. This is given by the Maxwell equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\vec{A}_w + \vec{A}_l\right) = -\frac{4\pi}{c}\vec{j} \tag{2.86}$$

where $\vec{j}$ is the electron current

$$\vec{j} = \frac{2e^2}{mc}(\vec{A}_w + \vec{A}_l)\langle\Psi^*\Psi\rangle_{el} \tag{2.87}$$

The wiggler field is assumed to be an constant and the approximation of a slowly-varying laser field is taken. Thus the equations (2.86) and (2.86) reduce to

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)A_l=i\frac{2\pi e^2A_w}{\omega_l\sqrt{E_0E_{-1}}}\int_0^\infty G(E_0)c_0c_{-1}\mathrm{d}E_0 \tag{2.88}$$

where $G(E_0)$ is the initial energy distribution of the electron beam.

An approximate solution of this problem is given [26] by neglecting Δ_0, so that the equations (2.84)-(2.85) have no explicit time dependence. This gives

$$c_0(z) = c_0(0)\cos[\chi(z)] \tag{2.89}$$

$$c_{-1}(z) = c_{-1}(0)\sin[\chi(z)] \tag{2.90}$$

where $\chi(z)$ is the solution of the partial differential equation

$$\frac{\partial^2\chi}{\partial z^2}=\left(\frac{2\pi n_e e^4E_w^2}{\hbar\omega_l^2\omega_w m^2c^2}\right)\sin\chi \tag{2.91}$$

where n_e is the electron density and $E_w=-\nabla A_w$ is the wiggler field amplitude. The solution of the equation for $\chi(z)$ can be written in terms of elliptic functions, but that is left out since it is not interesting for our current purposes.

In the next chapter the physics of two-level system and its implications for the FEL will be discussed at length.

Chapter 3

The Quantum Regime of FEL

The main core of this work revolves around the *quantum regime* of the free electron laser. I will show how the quantum FEL parameter $\bar{\rho}$, i.e. the product of the quantum recoil parameter q (2.8) and the classical FEL parameter ρ (1.26), rules the dynamics of the free electron laser: we will see that when $\bar{\rho} \ll 1$ a new completely quantum mechanical behaviour takes place, where the dynamics of the system is the same as that of a two-level system.

3.1 Occupation Number Representation

Our starting point is the Hamiltonian (2.18),

$$H = \sum_{j=1}^{N} \left[\frac{\hat{p}_j^2}{2\bar{\rho}} - i\sqrt{\frac{\bar{\rho}}{N}} \left(a e^{i\hat{\theta}_j} - a^\dagger e^{-i\hat{\theta}} \right) \right] - \delta a^\dagger a \tag{3.1}$$

I consider a base of eigenstates $|p\rangle$ of the momentum operator $\hat{p}$ so that $\hat{p}|p\rangle = p|p\rangle$ (where $\hat{p}$ is an operator and p a c-number). Since

$$[\hat{p}, e^{\pm i\hat{\theta}}] = \pm e^{\pm i\hat{\theta}} \tag{3.2}$$

then it follows that

$$\begin{aligned}
\hat{p}\left(e^{\pm i\hat{\theta}}|p\rangle\right) &= \left(e^{\pm i\hat{\theta}}\hat{p} + [\hat{p}, e^{\pm i\hat{\theta}}]\right)|p\rangle \\
&= (p \pm 1)\left(e^{\pm i\hat{\theta}}|p\rangle\right) \\
&\Downarrow \\
e^{\pm i\hat{\theta}}|p\rangle &= |p \pm 1\rangle
\end{aligned}$$

i.e. $\exp(\pm i\hat{\theta})$ acts as a raising/lowering momentum operator. The variation of ± 1 for the dimensionless momentum p corresponds to a variation of $\hbar k$ in dimensional units, where k is the wavenumber of the radiation field.

Analyzing the interaction part of the Hamiltonian (3.1), two opposite physical processes can be identified:

- $a^\dagger e^{-i\hat{\theta}_j} \rightarrow$ the j^{th} electron loses $\hbar k$ of momentum, and one photon is created
- $a e^{+i\hat{\theta}_j} \rightarrow$ absorption of one photon by the j^{th} electron and consequent momentum gain of $\hbar k$

This means that the electron momenta can only vary by steps of $\hbar k$: barring an initial energy spread (which I will take into account in Chapter 4), I assume that all the electron momenta are multiples of $\hbar k$, labeling the momentum eigenstates by a discrete number, $|n\rangle \equiv |p\rangle$. I can then define the occupation number states $|N_n\rangle$, where N_n is the number of electrons with dimensionless momentum n, and $\sum_n N_n = N$.

I neglect the fermionic nature of the electrons, so that I can define the bosonic creation and annihilation operators $\hat{c}_n^\dagger, \hat{c}_n$ for these states. Treating the electrons as bosons is justified since:

1. I assume that the effects caused by electron spin are negligible in our case.
2. The electrons are extremely far apart from each other in the beam, given its low density, so that charge effects from direct electron-electron interaction do not take place

I use the bosonic commutation rules for the electron operators $\hat{c}_n^\dagger, \hat{c}_n$:

$$[\hat{c}_n, \hat{c}_m^\dagger] = \delta_{nm} \tag{3.3}$$

Their action on the occupation number states is

$$\begin{aligned} \hat{c}_n|N_n\rangle &= \sqrt{N_n}|N_n - 1\rangle \\ \hat{c}_n^\dagger|N_n\rangle &= \sqrt{N_n + 1}|N_n + 1\rangle \\ \hat{c}_n^\dagger\hat{c}_n|N_n\rangle &= N_n|N_n\rangle \end{aligned}$$

I wish to rewrite the Hamiltonian (3.1) in the occupation numbers representation, using the operators $\hat{c}, \hat{c}^\dagger$. To do that, I introduce the scalar field operator $\hat{\Psi}(\theta)$:

$$\hat{\Psi}(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \hat{c}_n \langle\theta|n\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \hat{c}_n e^{in\theta} \tag{3.4}$$

where $|\theta\rangle$ and $|n\rangle$ are respectively the eigenstates of the operators $\hat{\theta}$ and $\hat{p}$. Using (3.3) I get the field commutation rules:

$$[\hat{\Psi}(\theta), \hat{\Psi}^\dagger(\theta')] = \frac{1}{2\pi}\sum_{n,m} e^{i(n\theta - m\theta')}[\hat{c}_n, \hat{c}_m^\dagger] = \frac{1}{2\pi}\sum_n e^{in(\theta-\theta')} = \delta(\theta - \theta') \tag{3.5}$$

I rewrite the Hamiltonian using the formula

$$\hat{H} = \int_0^{2\pi} \hat{\Psi}^\dagger(\theta) H\left(\theta, -i\frac{\partial}{\partial\theta}, a, a^\dagger\right)\hat{\Psi}(\theta)\mathrm{d}\theta \tag{3.6}$$

The main blocks of (3.1) are transformed in the following way:

$$\begin{aligned}
\sum_{j=1}^N \hat{p}_j^2 \quad \longrightarrow \quad & \int_0^{2\pi} \frac{1}{\sqrt{2\pi}}\sum_n \hat{c}_n^\dagger e^{-in\theta}\left(-i\frac{\partial}{\partial\theta}\right)^2 \frac{1}{\sqrt{2\pi}}\sum_m \hat{c}_m e^{im\theta}\mathrm{d}\theta \\
&= \frac{1}{2\pi}\sum_{n,m}\hat{c}_n^\dagger\hat{c}_m \int_0^{2\pi} m^2 e^{i(n-m)\theta}\mathrm{d}\theta \\
&= \sum_{n=-\infty}^{+\infty} n^2 \hat{c}_n^\dagger \hat{c}_n
\end{aligned}$$

and similarly

$$\begin{aligned}
\sum_{j=1}^N e^{+i\theta_j} \quad &\longrightarrow \quad \sum_{n=-\infty}^{+\infty} \hat{c}_{n-1}\hat{c}_n^\dagger \\
\sum_{j=1}^N e^{-i\theta_j} \quad &\longrightarrow \quad \sum_{n=-\infty}^{+\infty} \hat{c}_n\hat{c}_{n-1}^\dagger
\end{aligned}$$

The right-hand side operators represent the same physical processes as the left-hand side ones. The first one gives the total electron kinetic energy, the second and the third terms describe the increase and the reduction of the electron momentum by $\hbar k$.

The Hamiltonian becomes

$$\hat{H} = \sum_{n=-\infty}^{+\infty}\left\{\frac{1}{2\bar{\rho}}n^2\hat{c}_n^\dagger\hat{c}_n + i\sqrt{\frac{\bar{\rho}}{N}}\left(a^\dagger\hat{c}_n\hat{c}_{n-1}^\dagger - a\hat{c}_n^\dagger\hat{c}_{n-1}\right)\right\} - \delta a^\dagger a \tag{3.7}$$

The Heisenberg equations for $\hat{c}$ and a are

$$\frac{\mathrm{d}\hat{c}_n}{\mathrm{d}t} = -i[\hat{c}_n, H] = -i\frac{n^2}{2\bar{\rho}}\hat{c}_n + \sqrt{\frac{\bar{\rho}}{N}}\left(a^\dagger\hat{c}_{n+1} - a\hat{c}_{n-1}\right) \tag{3.8}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -i[a, H] = i\delta a + \sqrt{\frac{\bar{\rho}}{N}}\sum_n \hat{c}_n^\dagger\hat{c}_{n+1} = i\delta a + gB \tag{3.9}$$

where $g \equiv \sqrt{\dfrac{\bar{\rho}}{N}}$ is the coupling constant and

$$B \equiv \sum_{n=-\infty}^{+\infty} \hat{c}_n^\dagger \hat{c}_{n+1} = \int_0^{2\pi} \hat{\Psi}^\dagger(\theta) e^{-i\theta} \hat{\Psi}(\theta) \mathrm{d}\theta \tag{3.10}$$

is called the bunching operator in analogy to (1.37). The “time” t I have used is actually

$$t \equiv \bar{z} = \frac{4\pi\rho}{\lambda_w} z = \frac{z}{l_g}$$

as defined in Eq.(1.31), i.e. the longitudinal position measured in gain lengths. This definition of a dimensionless time variable will be used for the rest of the chapter.701

3.2 Evolution in Phase Space

I wish to describe our system in the phase space, through the use of the Wigner function. I start from the definition:

$$W(x,p,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi p} \psi^* \left(x - \frac{\hbar}{2}\xi, t \right) \psi \left(x + \frac{\hbar}{2}\xi, t \right) \mathrm{d}\xi \tag{3.11}$$

where $\psi(x,t)$ is the Schroedinger wave function. W has the dimensions of the inverse of a momentum, since ξp must be a dimensionless quantity.

In our dimensionless phase space variables, (3.11) becomes

$$W(\theta,\bar{p},t) = \frac{1}{2\pi} \frac{1}{\hbar k} \int_{-\infty}^{\infty} e^{-i\eta\bar{p}} \psi^* \left(\theta - \frac{\eta}{2}, t \right) \psi \left(\theta + \frac{\eta}{2}, t \right) \mathrm{d}\eta \tag{3.12}$$

where the integration variable η is now a dimensionless quantity, and $\bar{p}$ again is the momentum expressed in units of $\hbar k$, as defined in (2.9). Notice that W has still the dimensions of an inversed momentum.

An evolution equation for the Wigner function was found in [20], by using the theory of Preparata [5] as the starting point.

3.2.1 Preparata Equations

Preparata proposed [5] that the behaviour of the electrons in the FEL can be described by a single wave function ψ, in the case of a large electron number, $N \to \infty$. The wave function ψ is derived from the field operator $\hat{\Psi}$ defined in (3.4): by building a quantum field theory for $\hat{\Psi}$, it turns out that the quantum fluctuations around the saddle point path are proportional to $1/\sqrt{N}$ and as such disappear in the limit $N \to \infty$. The dynamics of the electrons are then governed by the Schroedinger equation

$$\frac{\partial\psi}{\partial t} = \frac{i}{2\bar{\rho}}\frac{\partial^2\psi}{\partial\theta^2} - \bar{\rho}\left(Ae^{i\theta} - A^*e^{-i\theta}\right)\psi \tag{3.13}$$

where

$$\psi = \frac{1}{\sqrt{N}}\,\hat{\Psi} \tag{3.14}$$

so that ψ is normalized to unity:

$$\int_0^{2\pi}|\psi(\theta)|^2\mathrm{d}\theta = 1 \tag{3.15}$$

Eq.(3.13) is closed by coupling it to the equation for the scaled radiation field A

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \int_0^{2\pi}|\psi(\theta)|^2e^{-i\theta}\mathrm{d}\theta + i\delta A \tag{3.16}$$

Here A has too been scaled,

$$A = \frac{1}{\sqrt{\bar{\rho}N}}\,a \tag{3.17}$$

The steps used by Preparata to derive (3.13) are briefly summarized in the Appendix.

3.2.2 Evolution of the Wigner Function

Using the Preparata equation (3.13), one can find a closed differential equation for the Wigner function [20]. Its derivation can be found in Appendix B.

$$\frac{\partial W(\theta,\bar{p},t)}{\partial t} = -\frac{\bar{p}}{\bar{\rho}}\frac{\partial W(\theta,\bar{p},t)}{\partial\theta} + \bar{\rho}\left(Ae^{i\theta} - A^*e^{-i\theta}\right)\left[W\left(\theta,\bar{p}+\frac{1}{2},t\right) - W\left(\theta,\bar{p}-\frac{1}{2},t\right)\right] \tag{3.18}$$

The term $\bar{p} \pm \frac{1}{2}$ represents the electron momentum increased or reduced by $\hbar k$.

Using (2.10), $\bar{p} = \bar{\rho}p$, one gets the final form of the evolution equation for the Wigner function:

$$\frac{\partial W(\theta,p,t)}{\partial t} = -p\frac{\partial W(\theta,p,t)}{\partial\theta} + \bar{\rho}\left(Ae^{i\theta} + A^*e^{-i\theta}\right)\left[W\left(\theta,p+\frac{1}{2\bar{\rho}},t\right) - W\left(\theta,p-\frac{1}{2\bar{\rho}},t\right)\right] \tag{3.19}$$

To close the system, (3.19) has to be coupled to an equation for the evolution of the field A. I use the Wigner function property

$$\int_{-\infty}^{\infty}W(\theta,p,t)\mathrm{d}p = |\psi(\theta,t)|^2 \tag{3.20}$$

and substitute it in the other Preparata equation, (3.16), obtaining

$$\frac{\mathrm{d}A}{\mathrm{d}t} = i\delta A + \int_{-\infty}^{\infty} \mathrm{d}p \int_{-\infty}^{+\infty} \mathrm{d}\theta \, W(\theta, p, t) e^{-i\theta} \tag{3.21}$$

The phase space of this system is periodic over the variable θ, so that problems arise [30] when using the Wigner function (3.12), since the integral runs from $-\infty$ to $+\infty$ instead of fixed periodic boundaries. This problem cannot be solved by simply changing the boundaries of the integral, since then the Wigner function would not have all of its required properties as a quasi-probability density distribution. A way to solve this problem will be presented at the end of this chapter, using the *discrete* Wigner function introduced by Bizarro [10].

3.2.3 Classical Limit: Vlasov Equation

The evolution equation for the Wigner function is found directly from the Preparata equation for the ψ. It is possible to show that Eq.(3.19) has a classical limit that appears when $\bar{\rho}$ tends to infinity, that is indeed the expected physical result [20]. One first observes that

$$\lim_{\bar{\rho}\to\infty} \bar{\rho} \left[W\left(\theta, p + \frac{1}{2\bar{\rho}}, t\right) - W\left(\theta, p - \frac{1}{2\bar{\rho}}, t\right) \right] = \frac{\partial W(\theta, p, t)}{\partial p} \tag{3.22}$$

from the very definition of a partial derivative. Also, from (1.42) and (1.43) one has

$$\dot{\theta} = p \tag{3.23}$$

$$\dot{p} = -\left(A e^{i\theta} + A^* e^{-i\theta}\right) \tag{3.24}$$

Substituting these equations into Eq.(3.19) one gets

$$\frac{\partial W(\theta, p, t)}{\partial t} + \dot{\theta} \frac{\partial W(\theta, p, t)}{\partial \theta} + \dot{p} \frac{\partial W(\theta, p, t)}{\partial p} = 0 \tag{3.25}$$

i.e. the *Vlasov Equation* for a classical system.

While we have taken the limit for $\bar{\rho} \to \infty$, the real requirement is only that $1/\bar{\rho}$ is much smaller than the scale of variation of p. Going back to physical non-dimensionless quantities by reverting the universal scaling, this is equivalent to saying that the momentum $\hbar k$ has to be negligible compared to the momentum of the electrons!

When this is not the case, then it will not be possible to approximate that finite difference as a derivative and the system will not be described by the classical equations of the FEL.

3.3 The Density Matrix and the Quantum Regime

In analogy to (3.4), ψ can be expanded in a Fourier series:

$$\psi(\theta,t)=\frac{1}{\sqrt{2\pi}}\sum_n c_n(t)e^{in\theta} \tag{3.26}$$

This time the c_n are not operators but c-numbers, namely the probability amplitude of finding an electron in the state $|n\rangle$, i.e. with dimensionless momentum n. Inserting this expansion in the Preparata equation (3.13) one gets the evolution for the c_n,

$$\frac{\mathrm{d}c_n}{\mathrm{d}t}=-\frac{i}{2\bar{\rho}}n^2c_n-\bar{\rho}(Ac_{n-1}-A^*c_{n+1}) \tag{3.27}$$

With the definition of the density matrix in the momentum representation,

$$\sigma_{m,n}=c_m^*c_n \tag{3.28}$$

one obtains

$$\begin{aligned}\frac{\mathrm{d}\sigma_{m,n}}{\mathrm{d}t} &= \frac{i}{2\bar{\rho}}(m^2-n^2)\sigma_{m,n}\\ &\quad +\bar{\rho}\left[A(\sigma_{m+1,n}-\sigma_{m,n-1})+A^*(\sigma_{m,n+1}-\sigma_{m-1,n})\right]\end{aligned} \tag{3.29}$$

The equation for the scaled radiation field can be rewritten as well using the density matrix, inserting (3.26) in (3.16):

$$\frac{\mathrm{d}A}{\mathrm{d}t}=\sum_n\sigma_{n-1,n}+i\delta A \tag{3.30}$$

The system (3.29)-(3.30) can be solved numerically. These simulations show that the FEL behaves as a two-level system in momentum space, when $\bar{\rho}\ll 1$. This can be seen in Fig.3.1: only the states of momentum $n=0$ and $n=-1$ are occupied.

3.4 Interaction Picture Approach

The numerical results shown in the previous section are based on the assumption that the Preparata approach is valid, namely that a single scalar field can be used to describe the N electrons interacting with a common radiation field.

I use here a different approach to show how the FEL reduces to a two-level system, starting from the Hamiltonian (3.7). The only requirement of this approach is to take the bosonic approximation for the electrons.

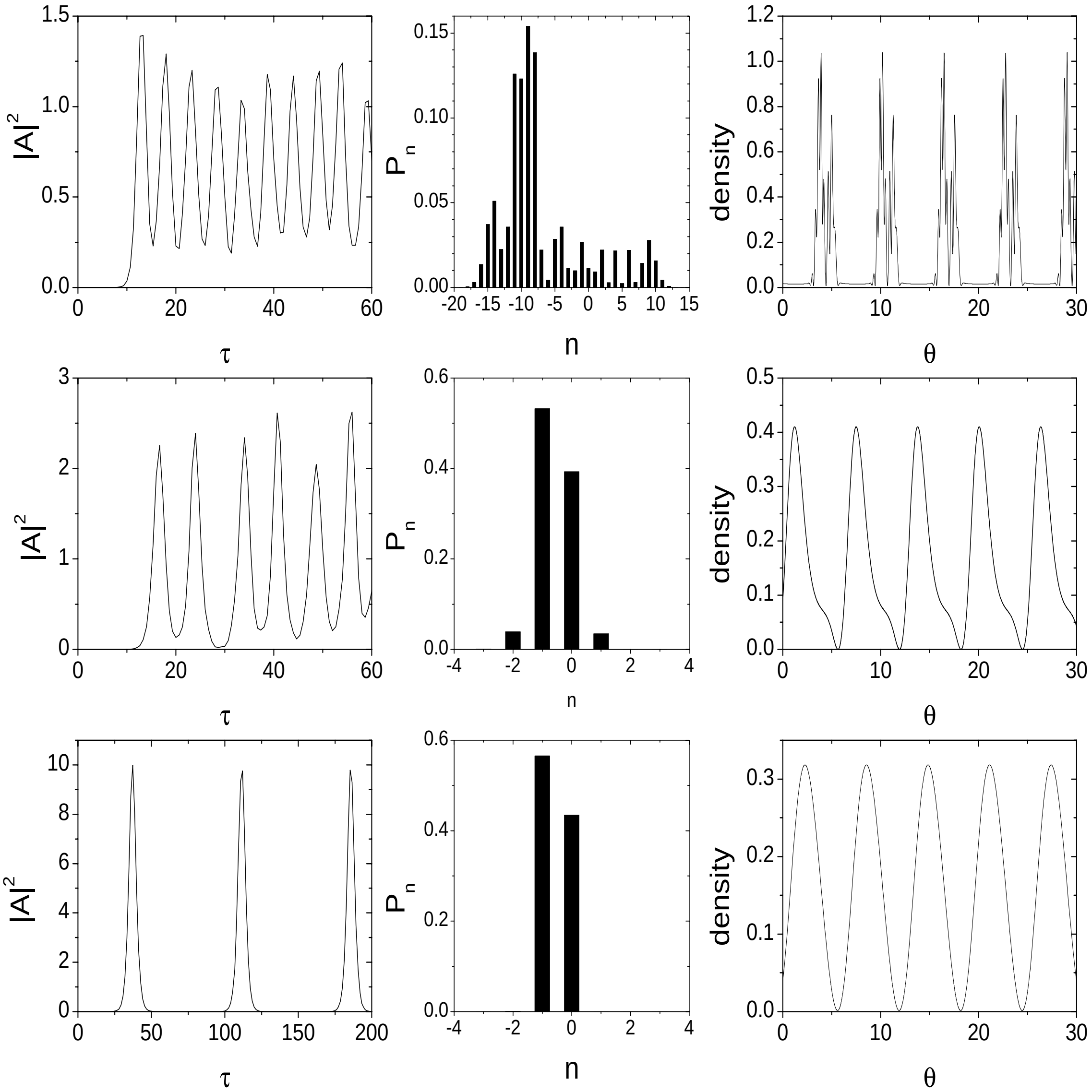

Figure 3.1: *Numerical solutions of the system (3.29)-(3.30) for* $\delta = 1$, $A(0) = 10^{-4}$, $c_n(0) = \delta_{n0}$ *and: (top row)* $\bar{\rho} = 10$, *(middle row)* $\bar{\rho} = 1$, *(bottom row)* $\bar{\rho} = 0.2$. *Numerical simulation taken from [20]. The left column shows the emitted radiation* $|A|^2(t)$. *The middle column shows the occupation probabilities* $P_n = |c_n|^2$ *for the electron momentum levels. The right column shows* $|\psi(\theta)|^2$ *when* $|A|^2(t)$ *is at his first peak.*

My starting point is the *Interaction Picture* formalism, which can be used when the Hamiltonian is the sum of two parts:

$$\hat{H} = \hat{H}_0 + \hat{H}_I \tag{3.31}$$

where $\hat{H}_I$ is the interaction potential. In our case one has

$$\hat{H}_0 = \frac{1}{2\bar{\rho}} \sum_{n=-\infty}^{+\infty} n^2 \hat{c}_n^\dagger \hat{c}_n - \delta a^\dagger a \tag{3.32}$$

and

$$\hat{H}_I = i\sqrt{\frac{\bar{\rho}}{N}} \sum_{n=-\infty}^{+\infty} \left(a^\dagger \hat{c}_n \hat{c}_{n-1}^\dagger - a \hat{c}_n^\dagger \hat{c}_{n-1} \right) \tag{3.33}$$

It is then possible to define the interaction picture Hamiltonian

$$\hat{H}_{IP}(t) = e^{i\hat{H}_0 t}\, \hat{H}_I\, e^{-i\hat{H}_0 t} \tag{3.34}$$

This is particularly useful because it allows to write perturbation series involving only the interaction potential, which is usually proportional to a small coupling constant. For an operator $\hat{M}$ (which is not explicitely time-dependent) the perturbation expansion is

$$\begin{aligned} \hat{M}_{HP}(t) &= \hat{M}_{IP}(t) + i\int_0^t \mathrm{d}t_1 [H_{IP}(t_1), \hat{M}_{IP}(t)] \\ &\quad + i^2 \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 [H_{IP}(t_1), [H_{IP}(t_2), \hat{M}_{IP}(t)]] + ... \end{aligned} \tag{3.35}$$

where

$$\hat{M}_{IP}(t) = e^{i\hat{H}_0 t}\, \hat{M}_{SP}\, e^{-i\hat{H}_0 t} \tag{3.36}$$

is the operator in the Interaction Picture ($\hat{M}_{SP}$ is the operator in the time-independent form of the Schroedinger picture). The time evolution of $\hat{M}_{IP}$ is usually easy to find as it depends only on the non-interacting Hamiltonian $\hat{H}_0$:

$$i\frac{\mathrm{d}}{\mathrm{d}t}\hat{M}_{IP} = [\hat{M}_{IP}, \hat{H}_0]$$

A similar perturbation series can be written also for the transition amplitudes from an initial state $|i\rangle$ to a final state $|f\rangle$ at time t:

$$\phi(f, t|i, 0) = \langle f| \sum_{m=0}^{\infty} \left(\frac{1}{i}\right)^m \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{m-1}} \mathrm{d}t_m H_{IP}(t_1) H_{IP}(t_2) ... H_{IP}(t_m)|i\rangle \tag{3.37}$$

or for the density matrix[1]:

$$\sigma_{IP}(t) = \sigma_0 + \frac{1}{i}\int_0^t \mathrm{d}t_1[H_{IP}(t_1), \sigma_0] + \frac{1}{i^2}\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2[H_{IP}(t_1), [H_{IP}(t_2), \sigma_0]] + ... \tag{3.38}$$

Using (3.32) and (3.33) in (3.34) (see Appendix C for the details) one gets the interaction picture Hamiltonian for this system:

$$\hat{H}_{IP}(t) = i\sqrt{\frac{\bar{\rho}}{N}} \sum_{n=-\infty}^{\infty} \left\{ a^\dagger \hat{c}_n \hat{c}_{n-1}^\dagger \exp\left[i\left(\frac{2n-1}{2\bar{\rho}} + \delta\right) t\right] - h.c. \right\} \tag{3.39}$$

The time-dependence of $\hat{H}_{IP}(t)$ comes from an imaginary exponential and its Hermitian conjugate, whose argument is proportional to

$$\frac{2n-1}{2\bar{\rho}} + \delta = \frac{1}{2\bar{\rho}}\left(\frac{mc(\gamma_n - \gamma_r)}{\hbar k} - 1\right)$$

where $mc\gamma_n$ is the momentum in physical units corresponding to the discrete momentum n.

Each order of the perturbation series (3.35)-(3.38) is then related to the integral

$$\int_0^\tau \exp\left\{\pm i\left(\frac{2n-1}{2\bar{\rho}} + \delta\right) t\right\} \mathrm{d}t = \int_0^\tau \exp\left\{\pm \frac{i}{2\bar{\rho}}\left(\frac{mc(\gamma_n - \gamma_r)}{\hbar k} - 1\right) t\right\} \mathrm{d}t \tag{3.40}$$

When $\bar{\rho}$ becomes small, the imaginary exponential in (3.40) rapidly oscillates, so that its contribution averages to zero. The only case when this does not happen is when the numerator of the argument of the exponential is of the same order of $\bar{\rho}$, or smaller. When $\bar{\rho}$ is smaller than unity, there is only one value of n that makes the exponential stationary. This value, that I name $\tilde{n}$, is the one that makes the argument of the exponential equal to zero:

$$\frac{2\tilde{n}-1}{2\bar{\rho}} + \delta = 0$$

$$\Downarrow$$

$$\tilde{n} = -\bar{\rho}\,\delta + \frac{1}{2}$$

In terms of the energy γ this corresponds to

$$mc\gamma_{\tilde{n}} = mc\gamma_r + \frac{\hbar k}{2} \tag{3.41}$$

[1]At time $t = 0$ the three different pictures coincide, so that $\sigma_0 \equiv \sigma_{HP} = \sigma_{SP}(0) = \sigma_{IP}(0)$.

Thus of the sum over n in (3.39) only the $\tilde{n}$-th term gives any contribution to the perturbation series, and the system behaves as if its interaction picture Hamiltonian were

$$\hat{H}_{IP} = i\sqrt{\frac{\bar{\rho}}{N}}\left(a^\dagger\, \hat{c}_{\tilde{n}}\hat{c}^\dagger_{\tilde{n}-1} - a\, \hat{c}^\dagger_{\tilde{n}}\hat{c}_{\tilde{n}-1}\right) \tag{3.42}$$

When the system is tuned as in the simulations seen in Fig.3.1, the detuning is

$$\delta = \frac{1}{2\bar{\rho}} \quad \text{i.e.} \quad mc\gamma_0 = mc\gamma_r + \frac{\hbar k}{2} \tag{3.43}$$

then the two momentum levels allowed for the electrons are

$$\begin{aligned} \tilde{n} &\rightarrow mc\gamma_0 \\ \tilde{n}-1 &\rightarrow mc\gamma_0 - \hbar k \end{aligned}$$

Thus, at resonance, an electron will only be able to emit a single photon, shifting from the initial state to a lower energy state. Once in the lower energy state, it will not be able to emit any more photons, but only to absorb one to go back to its initial energy state.

3.5 Two-level systems

The Hamiltonian (3.42) I have just found is indeed the well-known interaction Hamiltonian of a two-level system coupled to radiation [28, 29]. For such a system, both $\hat{H}_0$ and $\hat{H}_I$ are constants of motion, as $[\hat{H}_0, \hat{H}_I] = 0$. This implies that the interaction picture Hamiltonian is time independent. It can be rewritten in a more compact form:

$$H_I = \hat{H}_{IP} = ig\left(a^\dagger\, R^- - a\, R^+\right) \tag{3.44}$$

where I have defined the operators

$$R^+ \equiv \hat{c}^\dagger_{\tilde{n}}\hat{c}_{\tilde{n}-1} \tag{3.45}$$

$$R^- \equiv \hat{c}_{\tilde{n}}\hat{c}^\dagger_{\tilde{n}-1} \tag{3.46}$$

and the coupling constant $g \equiv \sqrt{\frac{\bar{\rho}}{N}}$.

The $R^\pm$ operators follow the angular momentum algebra

$$[R^+, R^-] = D \tag{3.47}$$

$$[D, R^\pm] = \pm 2R^\pm \tag{3.48}$$

Here D is the population difference operator

$$D \equiv \hat{c}^{\dagger}_{\tilde{n}}\hat{c}_{\tilde{n}} - \hat{c}^{\dagger}_{\tilde{n}-1}\hat{c}_{\tilde{n}-1} \tag{3.49}$$

According to (3.44), of all the electrons in the FEL, only those with momentum $\tilde{n}$ or $\tilde{n}-1$ interact with the photons of the laser field: thus I can consider all electrons to be either in the energy level corresponding to $\tilde{n}$ or $\tilde{n}-1$. An electron can go from the lower level to the higher level, absorbing a photon, or from the higher level to the lower level, emitting a photon.

This gives two constants of motion:

$$\hat{N} \equiv \hat{c}^{\dagger}_{\tilde{n}}\hat{c}_{\tilde{n}} + \hat{c}^{\dagger}_{\tilde{n}-1}\hat{c}_{\tilde{n}-1} \tag{3.50}$$

i.e. the total number of electrons, and

$$\hat{M} \equiv \frac{1}{2}D + a^{\dagger}a \tag{3.51}$$

This equation implies that everytime a photon is created or destroyed, an electron has changed its level and thus the population difference has changed by 2.

A pure state of this system is labeled by three variables: n_2 and n_1 as the number of electrons respectively in the higher and lower state, and n as the number of photons. In this way

$$\hat{N}|n_2, n_1, n\rangle = N|n_2, n_1, n\rangle \quad \text{where} \quad N = n_1 + n_2 \tag{3.52}$$

$$\hat{M}|n_2, n_1, n\rangle = M|n_2, n_1, n\rangle \quad \text{where} \quad M = n + \frac{1}{2}(n_2 - n_1) \tag{3.53}$$

Since N and M are constants, I can write n_2 and n_1 as functions of the number of photons n:

$$n_1 = \frac{N}{2} - (M - n) \tag{3.54}$$

$$n_2 = \frac{N}{2} + (M - n) \tag{3.55}$$

Thus one can label a state with only one variable: $|n\rangle \equiv |n, N, M\rangle$. The operators in the interaction Hamiltonian act on these states by raising or lowering its label:

$$aR^{+}|n\rangle = \alpha_n|n-1\rangle \tag{3.56}$$

$$a^{\dagger}R^{-}|n\rangle = \alpha_{n+1}|n+1\rangle \tag{3.57}$$

where

$$\alpha_n = \sqrt{n\, n_1(n_2+1)} = \sqrt{n\left(\frac{N}{2} + M - n + 1\right)\left(\frac{N}{2} - M + n\right)} \tag{3.58}$$

3.5.1 Perturbation Series

I use the two-level system Hamiltonian H_I in the perturbation series (3.38). H_{IP} is now time-independent, so the series simplifies greatly:

$$\begin{aligned}\sigma(t) &= \sigma_0 + \frac{1}{i}[H_{IP},\sigma_0]\int_0^t \mathrm{d}t_1 + \frac{1}{i^2}[H_{IP},[H_{IP},\sigma_0]]\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2 + \ldots = \\ &= \sigma_0 - it[H_{IP},\sigma_0] + \frac{(-it)^2}{2!}[H_{IP},[H_{IP},\sigma_0]] + \ldots = \\ &= \sum_{m=0}^{\infty}\frac{(-it)^m}{m!}\sum_{k=0}^{m}(-1)^k\begin{pmatrix} m \\ k\end{pmatrix} H_{IP}^k\,\sigma_0\,H_{IP}^{m-k} = \\ &= \sum_{m=0}^{\infty}\left(t\sqrt{\frac{\bar\rho}{N}}\right)^m\sum_{k=0}^{m}\frac{(-1)^k}{k!(m-k)!}\left(a^\dagger R^- - a\,R^+\right)^k\ \sigma_0\ \left(a^\dagger R^- - a\,R^+\right)^{m-k}\end{aligned} \tag{3.59}$$

From the density matrix it is possible to extract the expectation value of any operator by taking the trace of their product over the system degrees of freedom:

$$\langle X\rangle = \mathrm{Tr}\{\sigma(t)X\} \tag{3.60}$$

In the classical approach briefly described in the first chapter, I was looking for the time evolution of the scaled field intensity $|A|^2$. Its equivalent in this fully quantum analysis is given by the average number of emitted laser photons. Using (3.59) and (3.60) one gets

$$\begin{aligned}\langle a^\dagger a\rangle_t &= \mathrm{Tr}\{\sigma(t)a^\dagger a\} = \\ &= \sum_{m=0}^{\infty}\left(t\sqrt{\frac{\bar\rho}{N}}\right)^m\sum_{k=0}^{m}\langle n_0|\frac{\left(a\,R^+ - a^\dagger R^-\right)^k a^\dagger a\left(a^\dagger R^- - a\,R^+\right)^{m-k}}{k!(m-k)!}|n_0\rangle\end{aligned} \tag{3.61}$$

The states $|n\rangle$ are orthonormal, i.e. $\langle n|m\rangle = \delta_{nm}$. This together with (3.56) and (3.57) means that

$$\langle n|\left(a^\dagger R^- - a\,R^+\right)^k|n\rangle = 0$$

when k is odd. Thus, only the even terms of the series (3.61) do not vanish:

$$\langle a^\dagger a\rangle_t = \sum_{m=0}^{\infty}\left(t\sqrt{\frac{\bar\rho}{N}}\right)^{2m}\sum_{k=0}^{2m}\langle n_0|\frac{\left(a\,R^+ - a^\dagger R^-\right)^k a^\dagger a\left(a^\dagger R^- - a\,R^+\right)^{2m-k}}{k!(2m-k)!}|n_0\rangle \tag{3.62}$$

This infinite series cannot be exactly summed, but it is possible to cut it at some order for times t small enough. At second order, this gives

$$\langle a^\dagger a\rangle_t \simeq n_0 + \left(\alpha_{n_0+1}^2 - \alpha_{n_0}^2\right)\frac{\bar\rho}{N}t^2 \tag{3.63}$$

With an initial state where all electrons are in the upper level and there are no laser photons (spontaneous emission), $M = N/2$ and $n_0 = 0$, the emitted photons are

$$\langle a^\dagger a \rangle_t \simeq \bar{\rho} t^2 \tag{3.64}$$

This result shows how spontaneous emission is naturally found in a quantum picture, even when there is no initial field.

It is possible to get a rough estimate of the time region of validity for the second order approximation, i.e. for which times it is correct to cut the perturbation series. This is done by taking the greater term in the series (3.62), which is

$$\left(\frac{\bar{\rho}\, t^2}{N}\right)^m \frac{\langle n_0| \left(aR^+\right)^m a^\dagger a \left(a^\dagger R^-\right)^m |n_0\rangle}{m!^2} = \left(\frac{\bar{\rho}\, t^2}{N}\right)^m \frac{(n_0+m)}{m!^2} \left(\prod_{j=1}^{m} \alpha_{n_0+j}\right)^2 \tag{3.65}$$

For the case of $M = N/2$ and $n_0 = 0$, the coefficient α_n can be approximated as

$$\alpha_n \simeq n\sqrt{N}$$

where I also have taken the assumption that $N \gg n$. The highest term in the sum over k in the series (3.62) is then

$$\left(\frac{\bar{\rho}\, t^2}{N}\right)^m \frac{m N^m m!^2}{m!^2} = m \left(\bar{\rho}\, t^2\right)^m \tag{3.66}$$

To be able to cut the series at $m = 1$ I need to show that the order $m = 2$ is much smaller than the order $m = 1$. Approximately, this is true when $t \ll \bar{\rho}^{-1/2}$. Thus the smaller the parameter $\bar{\rho}$ is, the longer it will be valid the second order approximation. This is in agreement with the fact that the coupling parameter in the interaction Hamiltonian is proportional to $\sqrt{\bar{\rho}}$.

However, cutting the perturbation series at some order cannot give the typical exponential growth behaviour that is typical of the SASE; to investigate it in the quantum regime I need to use the two-level Hamiltonian in a different way.

3.5.2 Transition Amplitudes

Since the Hamiltonian in the interaction picture is time-independent, the perturbation series (3.37), that allows to calculate transition amplitudes, assumes a simple and compact form:

$$\begin{aligned}
\phi(f,t|i,0) &= \langle f|\sum_{m=0}^{\infty}\left(\frac{1}{i}\right)^m\int_0^t dt_1\int_0^{t_1}dt_2...\int_0^{t_{m-1}}dt_m H_{IP}(t_1)H_{IP}(t_2)...H_{IP}(t_m)|i\rangle \\
&= \langle f|\sum_{m=0}^{\infty}(-iH_I)^m\int_0^t dt_1\int_0^{t_1}dt_2...\int_0^{t_{m-1}}dt_m|i\rangle \\
&= \langle f|\sum_{m=0}^{\infty}\frac{(-iH_It)^m}{m!}|i\rangle \\
&= \langle f|e^{-iH_It}|i\rangle
\end{aligned} \tag{3.67}$$

Again, I am mainly interested in photon statistics, specifically the transition amplitude from an initial state with n_0 photons and a final one with n photons:

$$\phi_n(t) \equiv \langle n|e^{-iH_It}|n_0\rangle$$

Instead of directly trying to get an explicit expression for $\phi_n(t)$, I look for a finite difference - differential equation, that could be solved numerically or even analitically under some approximations. I differentiate the transition amplitude with respect to time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_n(t) = \langle n|(-iH_I)e^{-iH_It}|n_0\rangle$$

and with (3.56) and (3.57), I obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\phi_n(t) &= \langle n|g(a^\dagger R^- - aR^+)e^{-iH_It}|n_0\rangle \\
&= g\left\{\langle n+1|\alpha_{n+1} - \langle n-1|\alpha_n\right\}e^{-iH_It}|n_0\rangle \\
&= g\alpha_{n+1}\phi_{n+1}(t) - g\alpha_n\phi_{n-1}(t)
\end{aligned} \tag{3.68}$$

where α_n is given by (3.58). The ϕ_n will have to be normalized according to

$$\sum_n |\phi_n(t)|^2 = 1$$

I make the following physical assumptions:

1. At time zero all the electrons are in the excited energy state $\tilde{n}$:

$$n_2 = N \qquad n_1 = 0$$

2. I consider only spontaneous radiation, that is the initial number of photons n_0 is equal to zero.

3. The number of electrons is very large, $N \gg 1$.

Then $M = N/2$ and equation (3.68) simplifies to

$$\dot{\phi}_n(\tau) = \sqrt{N-n}\Big[(n+1)\phi_{n+1}(\tau) - n\phi_{n-1}(\tau)\Big] \tag{3.69}$$

where I have also introduced the scaled time τ:

$$\tau \equiv gt = \sqrt{\frac{\bar{\rho}}{N}}\frac{z}{l_g} \tag{3.70}$$

Once the probability amplitudes $\phi_n(\tau)$ are found, the average number of photons emitted at time τ is given by

$$\langle n\rangle_\tau = \sum_{n=0}^{N} n|\phi_n(\tau)|^2$$

The sum over n runs from zero to N because those are the only allowed physical states: once all electrons have shifted to the lower state and N photons have been emitted, no more energy can be transferred from the electron beam to the laser field.

In general, if I had taken a different distribution of electrons in the two levels and a different initial number of photons, the two extremes of the sum would have been different, namely

$$\begin{aligned} n_{\min} &= \begin{cases} n_0 - \dfrac{N}{2} + M & \text{if} \quad n_0 > \dfrac{N}{2} - M \\ 0 & \text{otherwise} \end{cases} \\ n_{\max} &= n_0 + \frac{N}{2} + M \end{aligned}$$

This is consistent with $n_{\min} = 0$ and $n_{\max} = N$ for our chosen case of $n_0 = 0$ and $M = N/2$.

I will now apply some mathematical tools, described in [28, 29], to give two approximated solutions to Eq.(3.69), one for short times and a more general one.

3.5.3 Short time solution

A solution of (3.69) can be found as long as the number of photons n can be neglected compared to the total electron number N. This is valid long before the first saturation peak of the FEL, where usually n is of the order of $\sim N$. Eq.(3.69) simplifies to

$$\dot{\phi}_n(\tau) = \sqrt{N}\Big[(n+1)\phi_{n+1}(\tau) - n\phi_{n-1}(\tau)\Big] \tag{3.71}$$

and $\sqrt{N}$ can be absorbed redefining time. Then the solution is (see [28, 29])

$$\phi_n(\tau) = (-i)^n \tanh^n(\sqrt{N}\,\tau)\,\mathrm{sech}(\sqrt{N}\,\tau) \tag{3.72}$$

and the average photon number is

$$\langle n\rangle_\tau = \sinh^2(\sqrt{N}\,\tau) \tag{3.73}$$

Since this solution is only valid for $n \ll N$, I can use (3.73) to specify this time:

$$\sinh^2(\sqrt{N}\,\tau) \ll N \quad \Rightarrow \quad \tau \ll \tau_s = \frac{\ln 2N}{2\sqrt{N}}$$

In terms of physical units, this is equal to

$$z \ll z_s \simeq \sqrt{\frac{N}{\bar\rho}}\frac{\ln 2N}{2\sqrt{N}}\,l_g = \frac{\ln 2N}{2\sqrt{\bar\rho}}\,l_g \tag{3.74}$$

For an electron beam of $N \sim 10^6$ electrons and a quantum parameter $\bar\rho \sim 0.1$ this gives an estimate of

$$z_s \simeq 23\,l_g \tag{3.75}$$

3.5.4 Long time solution: classical radiation field

A solution valid at all times for (3.69) is possible, provided that I make further approximations. At the end, we will see how these approximations effectively correspond to treating the radiation field as classical. Through some substitutions the differential equation can be rewritten, into a simpler form; the details of the passages required to do this are in Appendix D. After these manipulations, one gets the equation

$$\dot\Lambda(\vartheta_n,\tau) = G(\vartheta_n)\Big\{\Lambda(\vartheta_{n-1},\tau) - \Lambda(\vartheta_{n+1},\tau)\Big\} \tag{3.76}$$

where

$$\Lambda_n(t) \equiv (-i)^n\sqrt{G(n)}\,\phi_n(t) \tag{3.77}$$

The functions $\Lambda(\vartheta_n,\tau)$ and the physical amplitudes $\phi_n(\tau)$ are directly related, so that finding one univocally gives the other. $G(\vartheta_n)$ is a fixed function, defined in (D.3). The new variable ϑ_n is

$$\vartheta_n \equiv \arcsin\sqrt{\frac{n}{N}} \tag{3.78}$$

The key passage now is to transform the discrete variable ϑ_n into the continuous variable ϑ. This is acceptable because the variation of ϑ_n is very small:

$$\Delta\vartheta_n = \vartheta_{n+1} - \vartheta_n \simeq \frac{1}{\sqrt{(n+1)(N-n+1)}} \tag{3.79}$$

in fact

$$\Delta\vartheta_n \leq \frac{1}{\sqrt{N}} \ll 1$$

Thus I assume that the "angle" ϑ_n has continuous variation, and Eq.(3.76) becomes

$$\frac{\partial\Lambda(\vartheta,\tau)}{\partial\tau} = -G(\vartheta)\Delta(\vartheta)\frac{\partial\Lambda(\vartheta,\tau)}{\partial\vartheta} \tag{3.80}$$

where

$$\Delta(\vartheta) = \frac{1}{\sqrt{(1+N\sin^2\vartheta)(1+N\cos^2\vartheta)}} \tag{3.81}$$

$$G(\vartheta) = (1+N\cos^2\vartheta)\sqrt{1+N\sin^2\vartheta} \tag{3.82}$$

This equation can be solved by first changing to the variable

$$u(\vartheta) = \int_0^{\vartheta} \frac{\mathrm{d}\vartheta'}{G(\vartheta')\Delta(\vartheta')}$$

so that it takes the simple form

$$\frac{\partial\Lambda}{\partial\tau} = -\frac{\partial\Lambda}{\partial u}$$

Its solution is

$$\Lambda(u,\tau) = \Lambda(u-\tau,0) \equiv \Lambda_0(u-\tau)$$

The function $|\Lambda(u,\tau)|^2$ can be intepreted as the probability distribution in the space u, as it can be seen by

$$\int |\Lambda(u,\tau)|^2 \mathrm{d}u = \sum_n |\phi_n(\tau)|^2 = 1$$

where I have used the definition (3.77) of $\Lambda(u,\tau)$.

It is then possible to use $|\Lambda_0(u-\tau)|^2$ to calculate all the moments of the photon distribution :

$$\langle n^k\rangle_t = \sum_n n^k|\phi_n(t)|^2 = \int n^k(u)|\Lambda_0(u-gt)|^2\mathrm{d}u \tag{3.83}$$

The quantity $n(u)$ is found by reversing the definitions of ϑ_n and $u(\vartheta)$:

$$n(u) = N\,\mathrm{cn}^2\left(\sqrt{\bar{\rho}}\,(u-T), 1-\frac{1}{N}\right) \tag{3.84}$$

where $\mathrm{cn}(x,k)$ is one the Jacobi elliptic functions, and

$$T \simeq \frac{\ln(4\sqrt{N})}{\sqrt{N}} \tag{3.85}$$

The initial condition for the probability amplitudes can be approximated by a pure state with zero photons, i.e. $\phi_n(0) = \delta_{n0}$. Going to the space of ϑ and then u, this initial condition remains sharply peaked around a central value, so that

$$\Lambda_0(u-\tau) \simeq \delta(u-\tau)$$

Inserting this into (3.83) and using (3.84), I finally obtain the mean photon number:

$$\langle n\rangle_t = \int n(u)\delta(u-\tau)\mathrm{d}u = N\,\mathrm{cn}^2\left(\sqrt{\bar{\rho}}\,(t-T), 1-\frac{1}{N}\right) \tag{3.86}$$

When the parameter k of the function $\mathrm{cn}(x,k)$ is very close to 1, as in our case, its shape becomes that of an infinite series of identical sharp spikes. Each peak is separated from its neighbour by $2T$. Each spike in this train of pulses has the form of an hyperbolic secant:

$$\begin{aligned} \langle n\rangle_j &\simeq N\,\mathrm{sech}^2\Big\{\sqrt{\bar{\rho}}\,[t-(2j+1)T]\Big\} \\ j &= 0,1,\ldots \end{aligned} \tag{3.87}$$

Comparing this result to the numerical simulations of Eqs.(3.29) and (3.30) one can see that there is a perfect match.

It is important to clarify the meaning of the approximations I used to obtain this result. The equations (3.29)-(3.30) used a classical radiation field, represented by the complex number variable A. Equation (3.86) instead comes from a model where the annihilation and creation operators $a, a^\dagger$ are used, i.e. a quantum discrete field. Why then did I find the same result for $|A|^2$ and $\langle a^\dagger a\rangle$?

The answer lies in the approximation of a continuous ϑ instead of a discrete ϑ_n, essentially neglecting the discreteness of the photon number. It is then no surprise that this corresponds to treating the radiation field as a classical one.

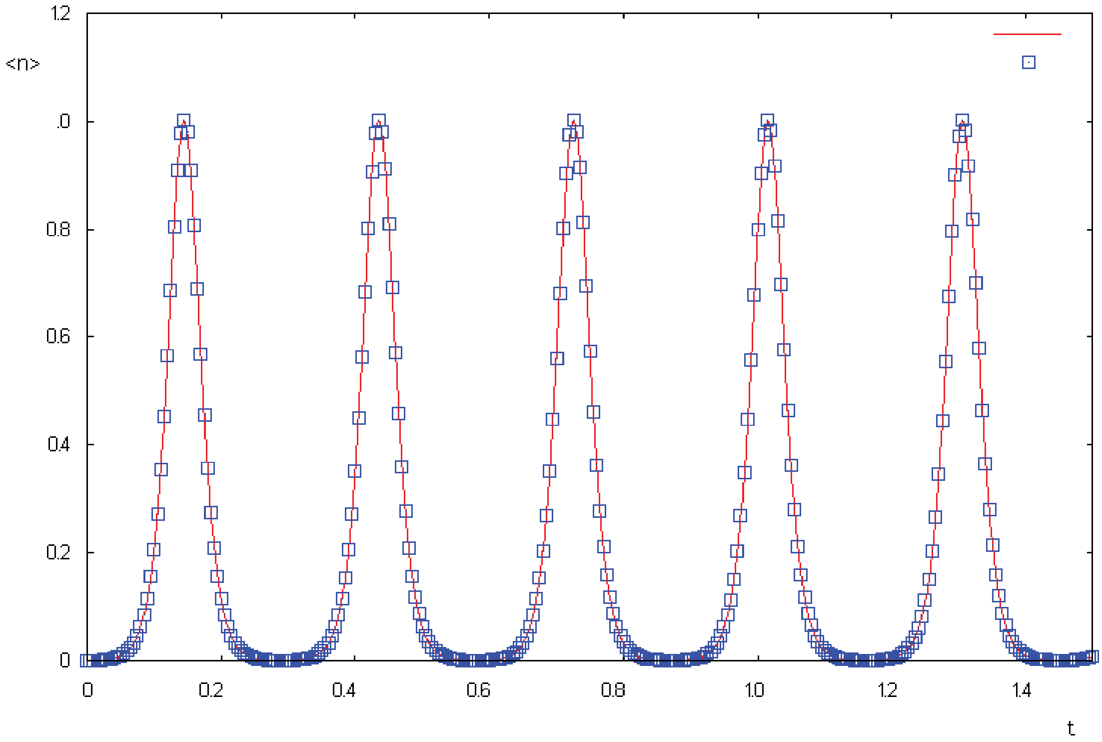

Figure 3.2: *The solid line is the analytical solution (3.86). The square dots are the numerical solutions of the system (3.29)-(3.30).* $\bar{\rho} = 0.1$

3.5.5 Diagonalization of the Interaction Hamiltonian

I wish to avoid the approximation of a classical radiation field, as this could hide some quantum effects rising from the discreteness of the photon number.

An interesting approach to the problem of finding the transition amplitudes for a two-level system coupled to radiation is due to D.F. Walls and R. Barakat [9]. Briefly, since in such a system H_0 and H_I commute, it is possible to build a diagonal representation of H, as H_0 and H_I have the same eigenstates. Once this diagonal representation is found, it is straightforward to solve the associated Schroedinger equation and get all the transition probabilities.

I assume again the case of a fully excited electron beam ($n_2 = N$, $n_1 = 0$) and no initial radiation field ($n_0 = 0$). The physical states (that are eigenstates of the Hamiltonian) are

$$|n\rangle \equiv |n; n_2 = N - n; n_1 = n\rangle$$

The Hamiltonian is given by taking (3.32) and (3.33) for the electron levels 0 and -1:

$$H_0 = \frac{1}{2\bar{\rho}}\left(\hat{c}_{-1}^{\dagger}\hat{c}_{-1} - a^{\dagger}a\right) \tag{3.88}$$

$$H_I = i\sqrt{\frac{\bar{\rho}}{N}}\left(a^{\dagger}\,\hat{c}_0\hat{c}_{-1}^{\dagger} - a\,\hat{c}_0^{\dagger}\hat{c}_{-1}\right) \tag{3.89}$$

Notice that the system is at resonance, $\delta = \dfrac{1}{2\bar{\rho}}$.

Since $H_0|n\rangle = 0$, I can take $H \equiv H_I$. Grouping the states $|n\rangle$ in one single vector

$$\overrightarrow{\Psi} = \begin{pmatrix} |0\rangle \\ |1\rangle \\ \vdots \\ |N\rangle \end{pmatrix} \tag{3.90}$$

the Hamiltonian can take the matrix form $H\overrightarrow{\Psi} = \mathbb{H}\overrightarrow{\Psi}$:

$$\mathbb{H} = i\sqrt{\frac{\bar{\rho}}{N}} \begin{bmatrix} 0 & \alpha_1 & 0 & 0 & 0 & . & & & & \\ -\alpha_1 & 0 & \alpha_2 & 0 & 0 & . & & & & \\ 0 & -\alpha_2 & 0 & \alpha_3 & 0 & . & & & & \\ 0 & 0 & -\alpha_3 & 0 & . & & & & & \\ . & . & . & . & & & & & & \\ & & & & & & & & . & . \\ & & & & & & & . & \alpha_{N-1} & 0 \\ & & & & & & . & -\alpha_{N-1} & 0 & \alpha_N \\ & & & & & & . & 0 & -\alpha_N & 0 \end{bmatrix} \tag{3.91}$$

This matrix was derived from (3.56)-(3.57):

$$H_I|n\rangle = i\sqrt{\frac{\bar{\rho}}{N}}(\alpha_{n+1}|n+1\rangle - \alpha_n|n-1\rangle)$$

where α_n in this particular case is

$$\alpha_n = n\sqrt{N-n+1}$$

$\mathbb{H}$ can be diagonalized, i.e. there is an orthonormal matrix $\mathbb{U}$ such that

$$\mathbb{U}\mathbb{H}\mathbb{U}^\top = \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_N) \tag{3.92}$$

Since the matrix $\mathbb{H}$ is imaginary and anti-symmetrical, its eigenvalues λ_j are all real. Notice that the λ_j already "include" the factor $i\sqrt{\bar{\rho}/N}$. The eigenstates of $\mathbb{H}$ are then given by

$$\overrightarrow{\Phi} = \mathbb{U}\overrightarrow{\Psi} \tag{3.93}$$

where the components of the vector $\overrightarrow{\Phi}$ are the single eigenstates for each eigenvalue λ_j:

$$\overrightarrow{\Phi} = \begin{pmatrix} |\lambda_0\rangle \\ |\lambda_1\rangle \\ \vdots \\ |\lambda_N\rangle \end{pmatrix} \tag{3.94}$$

so that

$$H|\lambda_j\rangle = \lambda_j|\lambda_j\rangle \tag{3.95}$$

These states are orthonormal:

$$\langle\lambda_j|\lambda_l\rangle = \delta_{jl} \tag{3.96}$$

I can then expand any physical state $|n\rangle$ as a linear combination of these eigenstates, using (3.93):

$$|n\rangle = \sum_{j=0}^{N} U_{jn} |\lambda_j\rangle \tag{3.97}$$

where U_{jn} are the elements of the matrix $\mathbb{U}$.

I am ready now to evaluate the transition probabilities of our system: the probability of having n photons at time t is

$$|\phi_n(t)|^2 = \langle n|e^{-iHt}|0\rangle = \left| \sum_{j=0}^{N} U_{j0} U_{jn} e^{-i\lambda_j t} \right|^2 \tag{3.98}$$

where I had to expand both the initial and final state over the $|\lambda_j\rangle$.

While (3.98) is an analytically exact result, the only way to obtain the transformation matrix $\mathbb{U}$ is through numerical computation. The big advantage of this procedure is that once the components U_{jn} and consequently the eigenvalues λ_j are determined, the evolution of the system at all times is exactly known, with no need of further calculations.

3.5.6 Numerical computation: quantum effects

Computing numerically the matrix elements U_{jn} shows an interesting effect: using equation (3.98) to get the transition probabilities and then evaluating the average photon number, a quite different result from (3.86) is found. As it can seen in Fig.3.4, the numerically found $\langle n\rangle$ starts by closely following the hyperbolic spike, but at some point before saturation it starts to change and from the third spike onwards the two results completely diverge.

This is due to the fact that this time the discreteness of the photon number was not neglected, and thus quantum effects arise. Zooming out the numerical calculation up to longer times, it is possible to see that the photon number undergoes fixed periodical revivals, so that the seemingly quite chaotic behaviour on the short time scale is replaced by a pattern on a longer time scale.

3.6 The discrete Wigner function

I will now go back to a description of the FEL in phase space, in order to obtain further physical information about its quantum regime.

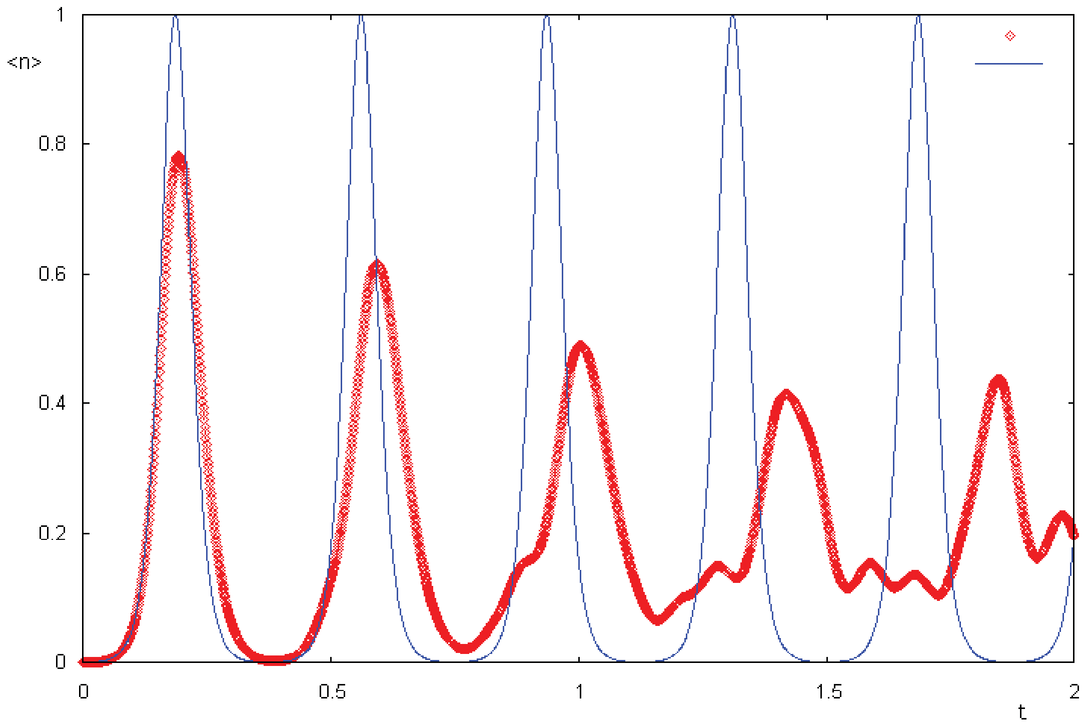

Figure 3.3: *The solid line is the normalized photon number (3.86) found neglecting its discreteness (semiclassical approximation). The Xs are the numerical data given by (3.98). The first spike is quite similar, but soon they completely differ.*

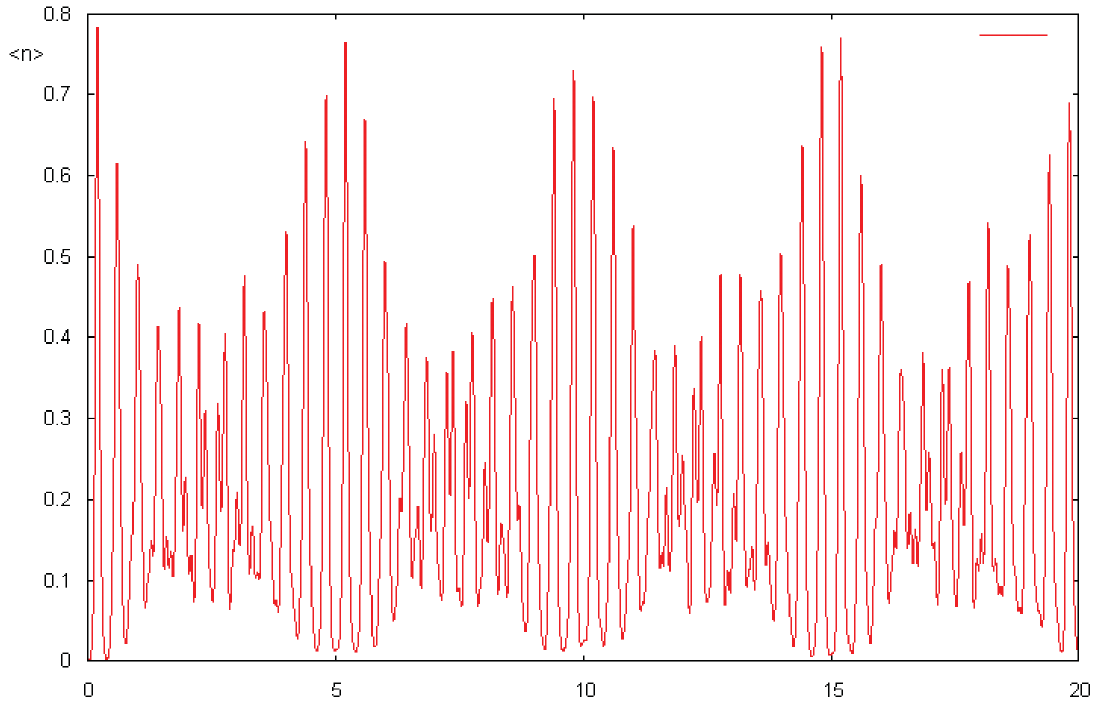

Figure 3.4: *Numerical data from (3.98), the photon number on a long time scale. A pattern of quantum revivals appear.*

The definition (3.11) of the Wigner function, introduced to describe the evolution of the quantum FEL, is normally used when dealing with non-periodic variables (q, p). It is known [30] that problems arise with the Wigner function when applying it to periodical variables, as it is in our case[2]. This can already be seen in (3.11), where the integral over η runs from $-\infty$ to $+\infty$, but η is then summed or subtracted from the angle θ, which ranges in the interval $[0, 2\pi)$ due to the periodicity of the wiggler.

To solve this problem, it is possible to define a *discrete* Wigner function, following the work of Bizarro [10]:

$$W_k(\theta) = \frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} e^{-2ik\theta'}\psi^*(\theta-\theta')\psi(\theta+\theta')\mathrm{d}\theta' \tag{3.99}$$

The momentum is now discrete instead of continuous, as represented by the label k. It can be verified that this definition keeps the required properties of the Wigner function as a quasi-probability distribution, such as:

$$\int_{-\pi}^{+\pi} W_k(\theta,t)\mathrm{d}\theta = |c_k(t)|^2 \tag{3.100}$$

$$\sum_{k=-\infty}^{+\infty} W_k(\theta,t) = |\psi(\theta,t)|^2 \tag{3.101}$$

i.e. summing over one of the two variables give the probability distribution of the other. This implies the normalization of the discrete Wigner function

$$\sum_{k=-\infty}^{+\infty}\int_{-\pi}^{+\pi} W_k(\theta,t)\mathrm{d}\theta = 1 \tag{3.102}$$

It is possible and useful to rewrite $W_k(\theta, t)$ as a function of the transition amplitudes $c_k(t)$ using the Fourier expansion (3.26) of the Schroedinger wavefunction:

$$\psi(\theta,t) = \frac{1}{\sqrt{2\pi}}\sum_n c_n(t)e^{in\theta}$$

Inserting this in the definition of $W_k(\theta, t)$ one obtains[3]

$$W_k(\theta,t) = \frac{1}{2\pi}\sum_{n,m} c_n^*(t)c_m(t)e^{-i(n-m)\theta}\mathrm{sinc}\left[\left(k-\frac{n+m}{2}\right)\pi\right]$$

[2]The periodicity of θ comes from the fact that the wiggler potential is itself periodical.

[3]$\mathrm{sinc}(x) \equiv \frac{\sin x}{x}$.

$$\begin{aligned}
&= \sum_{\mu=0}^{1} \sum_{n=-\infty}^{+\infty} \operatorname{sinc}\left[\left(k-n-\frac{\mu}{2}\right)\pi\right] w_{n+\mu/2}(\theta,t) \\
&= w_k(\theta,t) + \sum_{n=-\infty}^{+\infty} \operatorname{sinc}\left[\left(k-n-\frac{1}{2}\right)\pi\right] w_{n+1/2}(\theta,t) \qquad (3.103)
\end{aligned}$$

where

$$w_{n+\mu/2}(\theta,t) \equiv \frac{1}{2\pi}\sum_{j=-\infty}^{+\infty} c^*_{n+j+\mu}(t)c_{n-j}(t)e^{i(2j+\mu)\theta} \quad ; \quad \mu=0,1 \qquad (3.104)$$

I will call the functions $w_n(\theta,t)$ (i.e. for $\mu=0$) integer and the functions $w_{n+1/2}(\theta,t)$ (i.e. for $\mu=1$) half-integer. I will show now the relevant properties of these functions and why they are important for the analysis of the two-level FEL.

3.6.1 Some properties

These integer and half-integer functions $w_{n+\mu/2}$ are orthogonal to each other:

$$\int_{-\pi}^{+\pi} w_m(\theta,t)w_{n+\frac{1}{2}}(\theta,t)\mathrm{d}\theta = 0 \qquad \forall\, n,m$$

and contain all the information needed to determine $W_k(\theta,t)$. The classical probabilities for the momentum and the phase can be derived directly from the $w_{k+\mu/2}(\theta,t)$ too:

$$\begin{aligned}
|c_m(t)|^2 &= \int_{-\pi}^{+\pi}\{w_m(\theta,t)+w_{m+\frac{1}{2}}(\theta,t)\}\mathrm{d}\theta \qquad (3.105)\\
|\psi(\theta,t)|^2 &= \sum_{m=-\infty}^{+\infty}\{w_m(\theta,t)+w_{m+\frac{1}{2}}(\theta,t)\} \qquad (3.106)
\end{aligned}$$

The expectation value of any quantum operator $\hat{X}$ can be expressed through the $w_{m+\mu/2}(\theta,t)$:

$$\langle\hat{X}\rangle = \sum_{\mu=0}^{1}\sum_{m=-\infty}^{+\infty}\int_{-\pi}^{+\pi} X_{m+\mu/2}(\theta)\, w_{m+\mu/2}(\theta)\mathrm{d}\theta \qquad (3.107)$$

where the function $X_{m+\mu/2}(\theta)$ is given by

$$\begin{aligned}
X_{m+\mu/2}(\theta) &\equiv \int_{-\pi}^{+\pi} e^{-i(2m+\mu)\theta'}\langle\theta+\theta'|\hat{X}|\theta-\theta'\rangle\mathrm{d}\theta' \\
&\equiv \sum_{m'=-\infty}^{+\infty} e^{-i(2m+\mu)\theta'}\langle m-m'|\hat{X}|m+m'+\mu\rangle
\end{aligned}$$

3.6.2 Evolution equations

The importance of the integer and half-integer Wigner functions $w_{k+\mu/2}(\theta, t)$ is that it is possible to find a partial differential equation describing their evolution, while the same cannot be accomplished for the normal Wigner function $W_k(\theta, t)$.

I use the equations (3.27) for the amplitudes $c_n(t)$ together with the definition (3.104) to obtain

$$\frac{\partial w_{m+\mu/2}}{\partial t} = -\left(\frac{m+\mu/2}{\bar{\rho}}\right)\frac{\partial w_{m+\mu/2}}{\partial \theta} + \bar{\rho}\left(Ae^{i\theta} + A^* e^{-i\theta}\right)\left\{w_{m+\frac{\mu+1}{2}} - w_{m+\frac{\mu-1}{2}}\right\} \tag{3.108}$$

This shows how each function $w_{k+\mu/2}$ is connected to its two neighbouring functions $w_{m+\frac{\mu+1}{2}}$ and $w_{m+\frac{\mu-1}{2}}$. It is important to notice that in this section I have reverted to using the classical field A to represent the laser field, instead of the annihilation and creation operators a and $a^\dagger$. This has been done for simplicity and to allow the numerical simulation of the Wigner functions evolution.

The bunching of the electron beam can be written in this new formalism as well, using Eq.(3.107):

$$\langle e^{-i\theta}\rangle = \sum_{m=-\infty}^{+\infty}\int_{-\pi}^{+\pi} e^{-i\theta} w_{m+1/2}(\theta, t)\mathrm{d}\theta \tag{3.109}$$

so that Eq.(3.108) can be closed by coupling it to the radiation field by substituting (3.109) in (1.44):

$$\frac{\mathrm{d}A}{\mathrm{d}t} = i\delta A + \sum_{m=-\infty}^{+\infty}\int_{-\pi}^{+\pi} e^{-i\theta} w_{m+1/2}(\theta, t)\mathrm{d}\theta \tag{3.110}$$

3.6.3 Two-level approximation

To apply this new formalism to the two-level system regime I consider that only c_0 and c_{-1} have non-zero values, that is

$$\psi(\theta, t) = \frac{1}{\sqrt{2\pi}}\left\{c_0(t) + c_{-1}(t)e^{-i\theta}\right\}$$

Then from the definition of $w_{k+\mu/2}$ I have to take only the terms with $\mu = 0$ and $k = 0, -1$, or $\mu = 1$ and $k = -1$

$$\begin{aligned} w_0 &= \frac{1}{2\pi}c_0^* c_0 \\ w_{-1} &= \frac{1}{2\pi}c_{-1}^* c_{-1} \\ w_{-1/2}(\theta) &= \frac{1}{2\pi}\left\{c_{-1}^* c_0 e^{-i\theta} + c_0^* c_{-1} e^{i\theta}\right\} \end{aligned}$$

and the discrete Wigner function becomes

$$W_m(\theta) = w_m + \frac{(-1)^m}{(m+\frac{1}{2})\pi} w_{-1/2}(\theta) \tag{3.111}$$

The Wigner function appears now divided in a classical and a quantum part: the w_m are the classical probabilities of finding an electron in the momentum state m, and the $w_{m+1/2}(\theta)$ represent the quantum interference between the two states. For $m \neq 0, -1$ the Wigner function is made up only of interference terms proportional to $w_{-1/2}$.

I use as new variables the normalized population difference D and the bunching B defined in (3.10):

$$D = n_0 - n_{-1} = 2\pi\left(w_0 - w_{-1}\right) \tag{3.112}$$

$$B = \langle e^{-i\theta}\rangle = \int_{-\pi}^{+\pi} e^{-i\theta} w_{-1/2}(\theta)\mathrm{d}\theta = c_0 c_{-1}^* \tag{3.113}$$

where $n_0 \equiv c_0^* c_0$ and $n_{-1} \equiv c_{-1}^* c_{-1}$ are the occupation probabilities of the two momentum states.

From the equations (3.27) for the $c_n(t)$ and (3.110) for the field A one gets a set of Maxwell-Bloch type equations:

$$\begin{aligned} \dot{D} &= -2\bar{\rho}(AB^* + A^*B) \\ \dot{B} &= \bar{\rho}AD + \frac{i}{2\bar{\rho}}B \\ \dot{A} &= B + i\delta A \end{aligned}$$

I can rescale these quantities absorbing the $\bar{\rho}$ and δ parameters:

$$A' = \sqrt{\bar{\rho}}Ae^{-i\delta t} \tag{3.114}$$

$$B' = Be^{-i\delta t} \tag{3.115}$$

$$t' = \sqrt{\bar{\rho}}t \tag{3.116}$$

$$\theta' = \theta - \delta t \tag{3.117}$$

The equations for the rescaled variables are

$$\dot{B}' = -i\left(\delta - \frac{1}{2\bar{\rho}}\right)B' + DA' \tag{3.118}$$

$$\dot{D} = -2\left(A'B'^* + A'^*B'\right) \tag{3.119}$$

$$\dot{A}' = B' \tag{3.120}$$

At resonance $\delta = 1/2\bar{\rho}$ the three variables can be taken consistently as real since all of their derivates will be real as well. Then the previous equations become

$$\dot{B}' = A'D \tag{3.121}$$
$$\dot{D} = -4A'B' \tag{3.122}$$
$$\dot{A}' = B' \tag{3.123}$$

This system of equations has two integrals of motion:

$$\Lambda \equiv A'^2 + \frac{D}{2} \tag{3.124}$$
$$\Upsilon \equiv \frac{\sqrt{D^2 + 4B'^2}}{2} \tag{3.125}$$

It can be solved (see D.12 to D.23) under the approximation of $\Lambda^2 \simeq \Upsilon^2$. The solution for the squared field potential is

$$|A|^2(t) = \frac{2\Lambda}{\bar{\rho}} \operatorname{sech}^2 \left\{ \sqrt{2\bar{\rho}\Lambda}(t - t_{\max}) \right\} \tag{3.126}$$

The time $t_{\max}$ is the point where $|A|^2$ reaches its maximum. The values of $t_{\max}$ and Λ depend on the choice of the initial conditions $A'(0)$ and $D(0)$. The population difference D and the bunching B can then be found using the integrals of motion (3.124) and (3.125). As previously seen in (3.87), the field intensity grows in time as a squared hyperbolic secant, whose argument is proportional to $\sqrt{\bar{\rho}}$. Thus the smaller $\bar{\rho}$ is, the slower will be the exponential growth of the emitted radiation.

I can now go back to the Wigner function (3.111): I rewrite it using the variables (3.113)-(3.112):

$$w_0(t) = \frac{1 + D(t)}{4\pi} \tag{3.127}$$
$$w_{-1}(t) = \frac{1 - D(t)}{4\pi} \tag{3.128}$$
$$w_{-1/2}(\theta, t) = \frac{1}{\pi} B(t) \cos\theta \tag{3.129}$$

and

$$W_0(\theta, t) = \frac{1}{2\pi} \left\{ \frac{1 + D(t)}{2} + \frac{4}{\pi} B(t) \cos\theta \right\} \tag{3.130}$$
$$W_{-1}(\theta, t) = \frac{1}{2\pi} \left\{ \frac{1 - D(t)}{2} + \frac{4}{\pi} B(t) \cos\theta \right\} \tag{3.131}$$

I have explicited the time-dependence to show more clearly how the evolution of the Wigner function is directly given by the solution of the system (3.121)-(3.123).

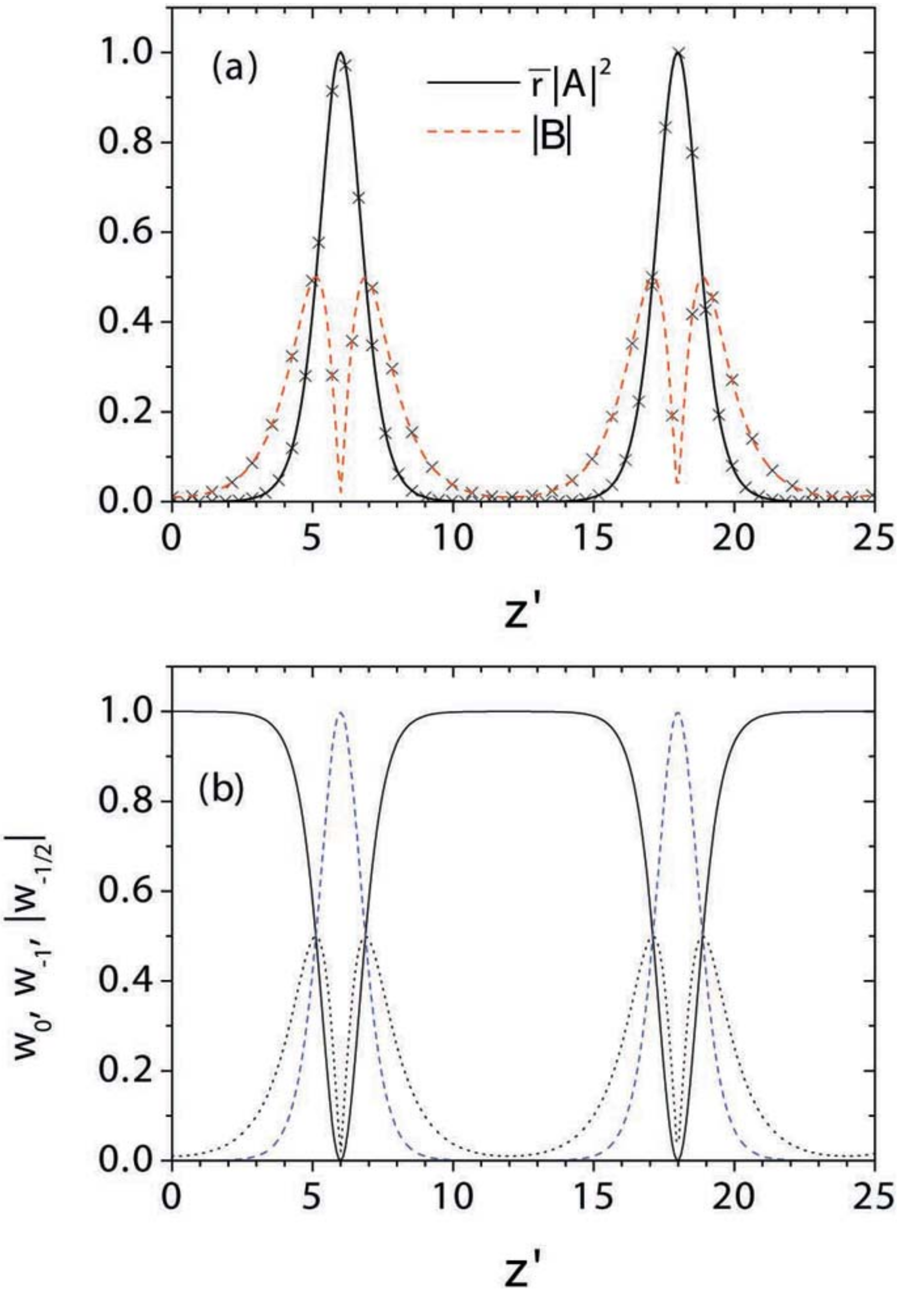

Figure 3.5: *Quantum Regime $\bar{\rho} = 0.1$, $\delta = 1/2\bar{\rho} = 5$. (a) Scaled field intensity $\bar{\rho}|A|^2$ and electron bunching $|B|$ vs $z' = \sqrt{\bar{\rho}}t$. . (b) $w_0(z')$ (continuous line), $w_{-1}(z')$ (dashed line), $w_{-1/2}(z')$ (dotted line).*

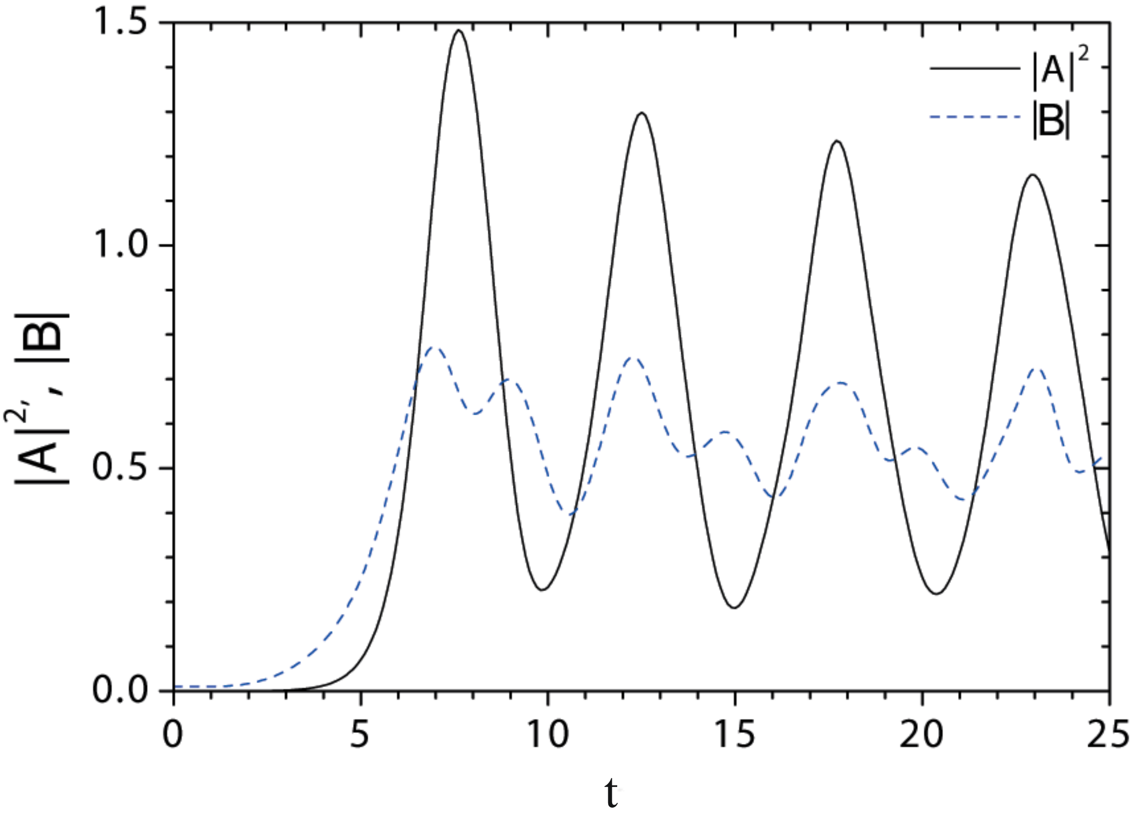

Figure 3.6: *Classical Regime* $\bar{\rho} = 5$, $\delta = 0$.

3.6.4 Negative Wigner Function

For a classical system, the Wigner function is positive everywhere in phase space and assumes the role of the probability density distribution, while for quantum system this does not hold true and it can become negative in certain zones of the phase space. It is then interesting to find out when $W_{0,-1}$ becomes negative, i.e. when the system displays a strictly quantum behaviour, due to the interference between the two levels. I use the integral of motion (3.125), assuming an initially unbunched beam ($B = 0$) and all electrons in the higher momentum level ($D = 1$), so that $D^2 + 4B^2 = 1$ and thus

$$|B| = \sqrt{\frac{1 - D^2}{4}} = \sqrt{n_0 n_{-1}}$$

Then I get

$$\begin{aligned} W_0(\theta, t) < 0 \quad \text{when} \quad \frac{n_{-1}}{n_0} > \alpha_\theta \equiv \frac{\pi^2}{16\cos^2\theta} \\ W_{-1}(\theta, t) < 0 \quad \text{when} \quad \frac{n_{-1}}{n_0} < \frac{1}{\alpha_\theta} \end{aligned}$$

or, using $n_0 + n_{-1} = 1$ to get the most compact form,

$$W_m(\theta, t) < 0 \quad \text{when} \quad n_m < \frac{1}{1 + \alpha_\theta} \tag{3.132}$$

where $n_m = c_m^* c_m$ is the number of electrons in the level m.

It is now possible to get the values of n_0 and n_{-1} that make *both* $W_0(\theta)$ and $W_{-1}(\theta)$ negative; since $n_0 + n_{-1} = 1$, one gets:

$$\left.\begin{array}{rcl} n_m & < & \dfrac{1}{1 + \alpha_\theta} \\ \\ 1 - n_m & < & \dfrac{1}{1 + \alpha_\theta} \end{array}\right\} \Rightarrow \frac{\alpha_\theta}{1 + \alpha_\theta} < n_m < \frac{1}{1 + \alpha_\theta} \tag{3.133}$$

which implies also that

$$\alpha_\theta < 1 \implies |\cos\theta| > \frac{\pi}{4} \tag{3.134}$$

Eq.(3.133) implies that there will be quantum interference between the two different momentum states only when the electrons are nearly evenly splitted between the two levels. This is in accordance with the requirement that the oscillating part of the discrete Wigner function is not negligible compared to its classical part: as

$w_{-1/2}(\theta)$ is proportional to the bunching $B = \pm\sqrt{n_0 n_{-1}}$, its maximum is given by $n_0 = n_{-1} = 1/2$.

Notice that even if W_0 and W_{-1} are negative, the total sum $\sum_k W_k$ still gives a positive value, that is $|\psi(\theta)|^2$, thanks to all the other infinite interference terms:

$$\begin{aligned}\sum_k W_k(\theta) &= w_0 + w_{-1} + \frac{w_{-1/2}(\theta)}{\pi} \sum_k \frac{(-1)^k}{\left(k+\frac{1}{2}\right)} \\ &= \frac{1}{2\pi}\left[1 + c_{-1}^* c_0 e^{-i\theta} + c_{-1} c_0^* e^{i\theta}\right] = |\psi(\theta)|^2 \end{aligned} \qquad (3.135)$$

Chapter 4

The Energy Spread

Until now we have considered a monoenergetic electron beam, where all electrons entered the FEL undulator with the same momentum: a single fixed parameter, δ, described the detuning between the resonant energy γ_r and the initial energy γ_0 of the electrons.

I wish to generalize the models used until now to include the more realistic situation of an initial broad distribution for the electron energy, where each electron will be allowed to have a different momentum upon entering the wiggler.

4.1 A simple inhomogeneous FEL model

A first approach starts from the Preparata model introduced in the last chapter, using a complex wavefunction $\psi(\theta, \bar{z}, z_1)$ to describe the electrons and a dimensionless classical radiation field $A(\bar{z}, z_1)$ for the photons. The evolution of the system is given by equations (3.13) and (3.16):

$$\frac{\partial \psi}{\partial \bar{z}} = \frac{i}{2\bar{\rho}} \frac{\partial^2 \psi}{\partial \theta^2} - \bar{\rho} \left(A e^{i\theta} - A^* e^{-i\theta} \right) \psi \tag{4.1}$$

$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} = \int_0^{2\pi} |\psi(\theta)|^2 e^{-i\theta} \mathrm{d}\theta + i\delta A \tag{4.2}$$

Notice that propagation has been included in the equation for the radiation field through the partial derivative in z_1, and that I have gone back to the notation of the universal scaling introduced in the first chapter, that is $\bar{z} = z/l_g$ is the longitudinal position measured in gain length and $z_1 = (z - v_\parallel t)/l_c$ is the dimensionless retarded time.

The dependence on the detuning δ is shifted from the equation for the field to the one for the wavefunction, through the substitution

$$\bar{A} = Ae^{-i\delta\bar{z}}$$

so that the equation for the field becomes

$$\frac{\partial \bar{A}}{\partial \bar{z}} + \frac{\partial \bar{A}}{\partial z_1} = \int_0^{2\pi} |\psi(\theta)|^2 e^{-i\theta} \mathrm{d}\theta \tag{4.3}$$

Before extending the model to take into account an energy distribution for the electrons, it is convenient to expand $\psi(\theta, \bar{z}, z_1)$ in a Fourier series:

$$\psi(\theta, \bar{z}, z_1) = \frac{1}{\sqrt{2\pi}} \sum_n c_n(\delta, \bar{z}, z_1) e^{in(\theta+\delta\bar{z})} \tag{4.4}$$

The coefficients $c_n(\delta, \bar{z}, z_1)$ give the complex probability amplitude to find an electron in the state of momentum n and detuning δ. That is, I am still using the discrete momenta n, but with a continuous distribution of different detunings to represent the different electron momenta. The evolution equation for the c_n is found by substituting (4.4) in the Schroedinger-like equation of the ψ:

$$\frac{\partial c_n(\delta)}{\partial \bar{z}} = -in\left(\frac{n}{2\bar{\rho}} + \delta\right) c_n(\delta) - \bar{\rho}[\bar{A}c_{n-1}(\delta) - \bar{A}^* c_{n+1}(\delta)] \tag{4.5}$$

This equation is coupled to the generalized version of the evolution equation for the radiation field:

$$\frac{\partial \bar{A}}{\partial \bar{z}} + \frac{\partial \bar{A}}{\partial z_1} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} G(\delta) c_{n-1}^*(\delta) c_n(\delta) \mathrm{d}\delta \tag{4.6}$$

The difference between (4.6) and the previous mono-energetic case (see (3.16) for example) is that this time the source of the radiation field, i.e. the right-hand side of (4.6), has been weighted over the distribution $G(\delta)$. This must be a normalized distribution centered around the resonant detuning $\delta_r = 1/2\bar{\rho}$, so that

$$\begin{aligned} \int_{-\infty}^{+\infty} G(\delta)\, \mathrm{d}\delta &= 1 \\ \int_{-\infty}^{+\infty} G(\delta)\, \delta\, \mathrm{d}\delta &= \frac{1}{2\bar{\rho}} \end{aligned}$$

In this model the electrons can only gain or lose momentum one quantum step $\hbar k$ at a time so that the detuning δ is a constant parameter for each electron, and thus the distribution $G(\delta)$ does not change in time.

4.1.1 Linear Analysis

The system of equations (4.5) and (4.6) can be solved under the linear approximation, taking no initial radiation field and all the electrons in the same momentum state as the initial conditions:

$$\begin{aligned} \bar{A}(\bar{z}=0, z_1=0) &= 0 \\ c_n(\bar{z}=0) &= \delta_{n0} \end{aligned}$$

It is important to remark that having all electrons in the same momentum state does not mean that all electrons have the same momentum: the detuning δ has to be taken into account to obtain the physical momentum of the electrons, which is given by

$$p = (n + 2\bar{\rho}\,\delta)\hbar k$$

In the linear approximation most of the electrons are considered to be in the initial momentum state $n = 0$, so that the Eqs.(4.5)-(4.6) are linearized around the state $c_0 = 1$. Then one obtains

$$\frac{\partial c_{+1}(\delta)}{\partial \bar{z}} = -i\left(\frac{1}{2\bar{\rho}} + \delta\right) c_{+1}(\delta) - \bar{\rho}\bar{A} \tag{4.7}$$

$$\frac{\partial c_{-1}(\delta)}{\partial \bar{z}} = -i\left(\frac{1}{2\bar{\rho}} - \delta\right) c_{-1}(\delta) + \bar{\rho}\bar{A} \tag{4.8}$$

$$\frac{\partial \bar{A}}{\partial \bar{z}} + \frac{\partial \bar{A}}{\partial z_1} = \int_{-\infty}^{+\infty} G(\delta)\left[c_{+1}(\delta) + c^*_{-1}(\delta)\right] \mathrm{d}\delta \tag{4.9}$$

To solve this system one performs the Laplace transform over $\bar{z}$ and the Fourier transform over z_1 of the radiation field

$$\begin{aligned} \tilde{A}(\lambda,\kappa) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}z_1 \int_0^{+\infty} \mathrm{d}\bar{z}\bar{A}(z_1,\bar{z})\, e^{-i(\lambda\bar{z}+\kappa z_1)} \\ &= -i\frac{\tilde{A}_0(\kappa)}{D(\lambda,\kappa)} \end{aligned} \tag{4.10}$$

where $\tilde{A}_0(\kappa)$ is the Fourier transform of $\bar{A}_0(z_1) \equiv \bar{A}(z_1, 0)$ and

$$D(\lambda,\kappa) = \lambda - \kappa + \int_{-\infty}^{+\infty} \frac{G(\delta)}{(\lambda+\delta)(\lambda+\delta+\frac{1}{\bar{\rho}})}\,\mathrm{d}\delta \tag{4.11}$$

I proceed by inverting the Laplace transform using residues, to get the Fourier transform of the radiation field:

$$\tilde{A}(\bar{z},\kappa) = \tilde{A}_0(\kappa)\sum_j \mathrm{Res}\left(\frac{e^{i\lambda_j\bar{z}}}{D(\lambda_j,\kappa)}\right) \tag{4.12}$$

where λ_j are the roots of the dispersion relation $D(\lambda_j, \kappa) = 0$ and κ represents the spectrum of the radiation field. The dispersion relation can be rewritten in the following way

$$\lambda - \kappa + \bar{\rho} \int_{-\infty}^{+\infty} \left[G\left(\delta + \frac{1}{2\bar{\rho}}\right) - G\left(\delta - \frac{1}{2\bar{\rho}}\right) \right] \frac{\mathrm{d}\delta}{\lambda + \delta + \frac{1}{2\bar{\rho}}} = 0 \tag{4.13}$$

I change variables, defining first the generalized central detuning

$$\Delta \equiv \kappa + \frac{1}{2\bar{\rho}} \tag{4.14}$$

and then shifting λ too by $1/2\bar{\rho}$:

$$\bar{\lambda} \equiv \lambda + \frac{1}{2\bar{\rho}} \tag{4.15}$$

Thus (4.13) becomes

$$\bar{\lambda} - \Delta + \bar{\rho} \int_{-\infty}^{+\infty} \left[G\left(\delta + \frac{1}{2\bar{\rho}}\right) - G\left(\delta - \frac{1}{2\bar{\rho}}\right) \right] \frac{\mathrm{d}\delta}{\bar{\lambda} + \delta} = 0 \tag{4.16}$$

In this way one can see how, in the limit $\bar{\rho} \to \infty$, the classical dispersion relation (1.47) is recovered, as the finite difference inside the integral becomes a derivative that can be eliminated integrating by parts:

$$\bar{\lambda} - \Delta + \int_{-\infty}^{+\infty} \frac{\mathrm{d}G(\delta)}{\mathrm{d}\delta} \frac{\mathrm{d}\delta}{(\bar{\lambda} + \delta)} = \bar{\lambda} - \Delta + \int_{-\infty}^{+\infty} \frac{G(\delta)}{(\bar{\lambda} + \delta)^2} \mathrm{d}\delta = 0 \tag{4.17}$$

Again, those roots whose imaginary part is negative will provide the exponential gain of the radiation field. An analytical solution of (4.13) is in general not possible, except for some specific cases or under some approximations. An interesting case is that of a broad distribution $G(\delta)$.

4.1.2 Broad distribution

It is expected that if $G(\delta)$ is a broad distribution then the imaginary part of λ is small: increasing the spread, the gain should diminish. Thus the integral can be simplified using the limit

$$\lim_{\epsilon \to 0} \left(\frac{1}{x + i\epsilon} \right) = P\left(\frac{1}{x}\right) - i\pi\delta(x)$$

Using it in (4.13), one obtains

$$\mathrm{Re}\lambda \simeq \Delta - \frac{1}{2\bar{\rho}} \tag{4.18}$$

$$\mathrm{Im}\lambda \simeq -\pi\bar{\rho}\left[G\left(\Delta + \frac{1}{2\bar{\rho}}\right) - G\left(\Delta - \frac{1}{2\bar{\rho}}\right)\right] \tag{4.19}$$

When $1/\bar{\rho}$ is much smaller than the width of $G(\delta)$ then classical limit is recovered,

$$\mathrm{Im}\lambda \simeq -\pi\frac{\mathrm{d}G(\Delta)}{\mathrm{d}\Delta} \tag{4.20}$$

Since $G(\Delta)$ can be assumed to be symmetrical around its center $1/2\bar{\rho}$ (for physical reasons, i.e. slower and faster electrons are equally distributed), its derivative will be instead antisymmetrical and thus the gain too.

In general, Eq.(4.19) shows that the more negligible the step $1/2\bar{\rho}$ is when compared to the width σ of the distribution $G(\delta)$, the smaller the difference in the parenthesis will be; thus the gain will be proportionally smaller. In physical terms, this means that if the initial energy spread is much greater than the photon momentum $\hbar k$, then the transition between the two momentum states 0 and -1 cannot be resolved by the system. This sets a limit on the electron energy spread to observe the quantum regime of the FEL, namely

$$\sigma \ll \frac{1}{2\bar{\rho}} \tag{4.21}$$

4.1.3 Lorentzian distribution

Eq.(4.13) can be solved exactly when $G(\delta)$ is a Lorentzian distribution:

$$G(\delta) = \frac{\sigma}{\pi\left[\sigma^2 + \left(\delta - \frac{1}{2\bar{\rho}}\right)^2\right]} \tag{4.22}$$

This way the integral in (4.13) has an analytical solution, giving the dispersion relation

$$(\lambda - \kappa)(\lambda + i\sigma)\left(\lambda + \frac{1}{\bar{\rho}} + i\sigma\right) + 1 = 0 \tag{4.23}$$

In the quantum regime, $\bar{\rho} \ll 1/\sigma$, the three roots of this dispersion relation can be evaluated approximately as

$$\lambda_1 \simeq -\frac{\kappa}{2}\left(1 + \frac{\sigma}{\Gamma}\right) - i\left(\frac{\Gamma - \sigma}{2}\right) \tag{4.24}$$

$$\lambda_2 \simeq -\frac{\kappa}{2}\left(1 - \frac{\sigma}{\Gamma}\right) + i\left(\frac{\Gamma - \sigma}{2}\right) \tag{4.25}$$

$$\lambda_3 \simeq i\sigma \tag{4.26}$$

where

$$\Gamma \equiv \left\{ \frac{\sqrt{(4\bar{\rho} + \sigma^2 - \kappa^2)^2 + 4\kappa^2\sigma^2} + 4\bar{\rho} + \sigma^2 - \kappa^2}{2} \right\}^{1/2} \tag{4.27}$$

Since for any $\bar{\rho} > 0$ one has $\Gamma \neq \sigma$, there will always be an exponential solution. Its gain will be given by

$$\mathrm{Im}\lambda = -\frac{|\Gamma - \sigma|}{2} \tag{4.28}$$

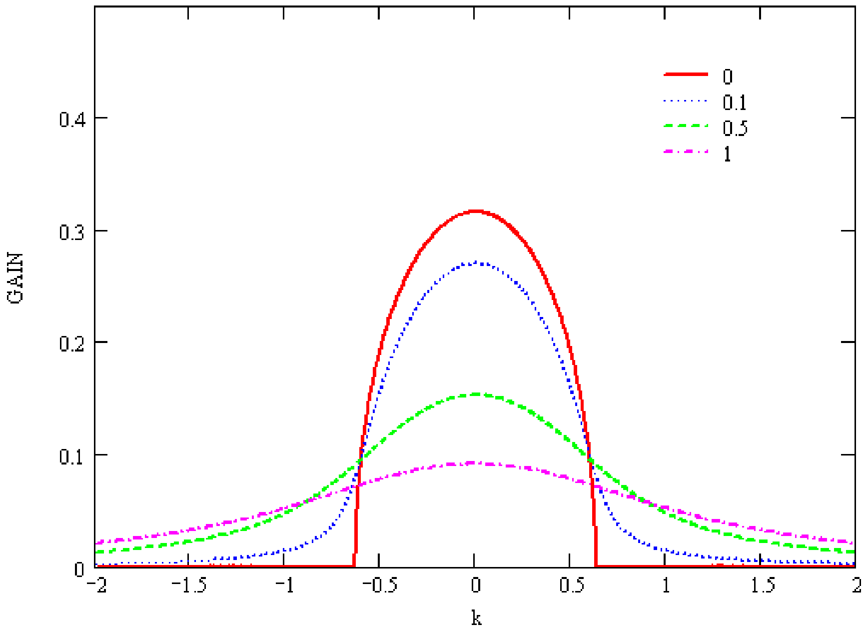

Figure 4.1: $|Im(\lambda)|$ *from (4.28) as a function of* κ. $\bar{\rho} = 0.1$ *and four different values of* σ*:* 0, 0.1, 0.5 *and* 1.

Assuming that the radiation field $A(\bar{z}, z_1)$ does not depend on z_1, that is neglecting the propagation with respect to the electron beam, then the spectrum in κ is single valued, $\kappa = 0$. This can be seen from the definition of κ, with the Fourier transform (4.10) of the radiation field. Then, the form of Γ simplifies:

$$\Gamma = \sqrt{\sigma^2 + 4\bar{\rho}} \tag{4.29}$$

If one also assumes $\bar{\rho} \ll \sigma^2/4$ in addition to the assumption of $\bar{\rho} \ll 1/2\sigma$ (4.21), i.e.

$$2\sqrt{\bar{\rho}} \ll \sigma \ll \frac{1}{2\bar{\rho}} \tag{4.30}$$

then

$$\Gamma \simeq \sigma + \frac{2\bar{\rho}}{\sigma}$$

These two inequalities do not contradict each other if $\bar{\rho}$ is of the order of or smaller than 0.1. The roots of the dispersion equation become

$$\begin{aligned} \lambda_1 &\simeq -i\frac{\bar{\rho}}{\sigma} \\ \lambda_2 &\simeq +i\frac{\bar{\rho}}{\sigma} \\ \lambda_3 &= +i\sigma \end{aligned}$$

and the dominating solution for the radiation field is

$$A(\bar{z}) \approx A(0)\frac{e^{i\lambda_1\bar{z}}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} = -\frac{\sigma^2 A(0)}{2\bar{\rho}\,(\bar{\rho}-\sigma^2)}\exp\left(\frac{\bar{\rho}}{\sigma}\,\bar{z}\right) \tag{4.31}$$

If instead the distribution $G(\delta)$ is so narrow that $\sigma \ll 2\sqrt{\bar{\rho}}$, then $\Gamma \simeq 2\sqrt{\bar{\rho}}$ and the roots are

$$\begin{aligned} \lambda_1 &\simeq -i\sqrt{\bar{\rho}} \\ \lambda_2 &\simeq +i\sqrt{\bar{\rho}} \\ \lambda_3 &= +i\sigma \end{aligned}$$

This gives the dominating solution

$$A(\bar{z}) \approx -\frac{A(0)}{2\bar{\rho}}\exp\left(\frac{\sqrt{\bar{\rho}}}{2}\,\bar{z}\right) \tag{4.32}$$

4.2 Wigner Function approach

The Wigner function formalism is particularly useful to describe the energy spread in the electron beam, since it does not require any particular retooling: the Wigner function already describes a distribution in phase and momentum, so that only different initial conditions have to be considered. I will show that it is possible to rederive in this way the results of the previous section, where the distribution $G(\delta)$ was introduced in the field equation as a generalization of the monoenergetic case. The starting point is given by the evolution equations (3.19)-(3.21):

$$\begin{aligned} \frac{\partial W(\theta,p,\bar{z})}{\partial \bar{z}} &= -p\frac{\partial W(\theta,p,\bar{z})}{\partial \theta} \qquad (4.33) \\ &\quad +\bar{\rho}\left(Ae^{i\theta}+A^*e^{-i\theta}\right)\left[W\left(\theta,p+\frac{1}{2\bar{\rho}},\bar{z}\right)-W\left(\theta,p-\frac{1}{2\bar{\rho}},\bar{z}\right)\right] \\ \frac{\mathrm{d}A}{\mathrm{d}\bar{z}} &= i\delta A + \int_{-\infty}^{\infty}\mathrm{d}p\int_0^{2\pi}\mathrm{d}\theta\; W(\theta,p,\bar{z})e^{-i\theta} \qquad (4.34) \end{aligned}$$

where I have already neglected propagation effects.

Since $W(\theta, p, \bar{z})$ is periodic in θ, we expand it in a Fourier series:

$$W(\theta, p, \bar{z}) = \frac{1}{\sqrt{2\pi}} \sum_n W_n(p, \bar{z}) e^{in\theta} \tag{4.35}$$

$W_{-n} = W_n^*$, since the Wigner function is real. One obtains the evolution equations for the coefficients W_n by substitution:

$$\begin{aligned}
\frac{\partial W_n(p, \bar{z})}{\partial \bar{z}} &= -inp\, W_n(p, \bar{z}) \\
&\quad + \bar{\rho} A \left[W_{n-1}\left(p + \frac{1}{2\bar{\rho}}, \bar{z}\right) - W_{n-1}\left(p - \frac{1}{2\bar{\rho}}, \bar{z}\right) \right] \\
&\quad + \bar{\rho} A^* \left[W_{n+1}\left(p + \frac{1}{2\bar{\rho}}, \bar{z}\right) - W_{n+1}\left(p - \frac{1}{2\bar{\rho}}, \bar{z}\right) \right] \qquad (4.36) \\
\frac{\mathrm{d}A}{\mathrm{d}\bar{z}} &= \int_{-\infty}^{+\infty} W_1(p, \bar{z}) \mathrm{d}p + i\delta A \qquad (4.37)
\end{aligned}$$

It has to be stressed that the W_n are not related to the discrete Wigner function introduced in the last chapter, and that the index n represents the spatial harmonics of the expansion of $W(\theta, p, \bar{z})$, and it is *not* some sort of discrete physical variable.

I assume an initially unbunched beam, i.e. independent on the phase θ; thus the only non-zero harmonic can be W_0:

$$W(\theta, p, 0) = W_0(p, 0) = G(p) \tag{4.38}$$

This initial condition is the equivalent of the detuning distribution $G(\delta)$, from the main definition of the Wigner function.

I show in Figs.4.2-4.4 some numerical solutions of the equations (4.36)-(4.37), with a Gaussian distribution and no radiation field as the initial conditions.

I have to point out that, as we wrote in the previous chapter, using the continuous Wigner function in a *periodic* phase space is not formally correct: this is a *phenomenological* approach, justified by the fact that in the linear regime it gives the same results as the previous approach.

4.2.1 Linear Analysis

In the linear approximation I consider W_0 as nearly constant

$$W_0(p, \bar{z}) \simeq G(p) \tag{4.39}$$

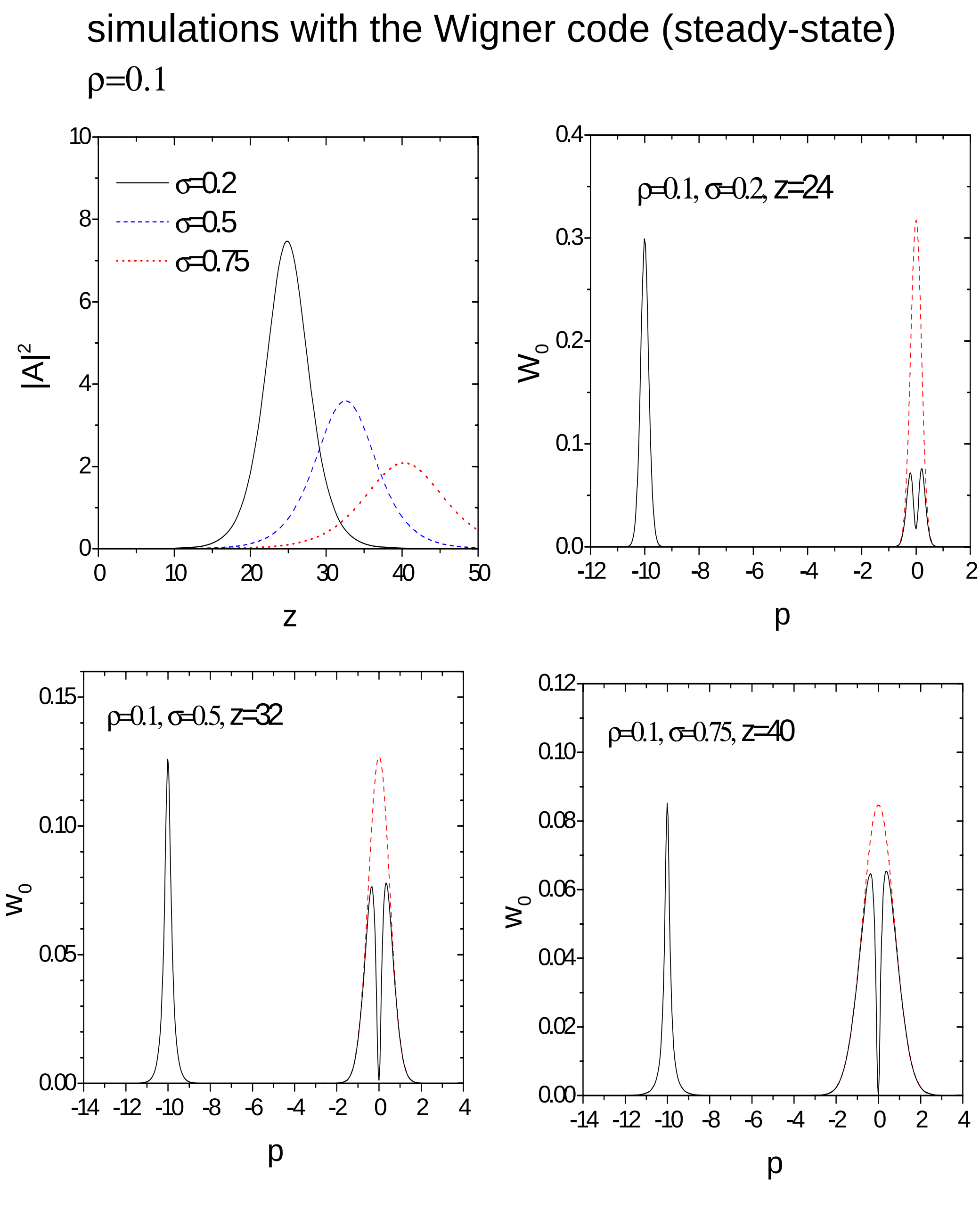

Figure 4.2: *Numerical simulations of (4.36)-(4.37) for* $\bar{\rho} = 0.1$. *It is possible to see very clearly the detached momentum levels and the "hole-burning" effect.*

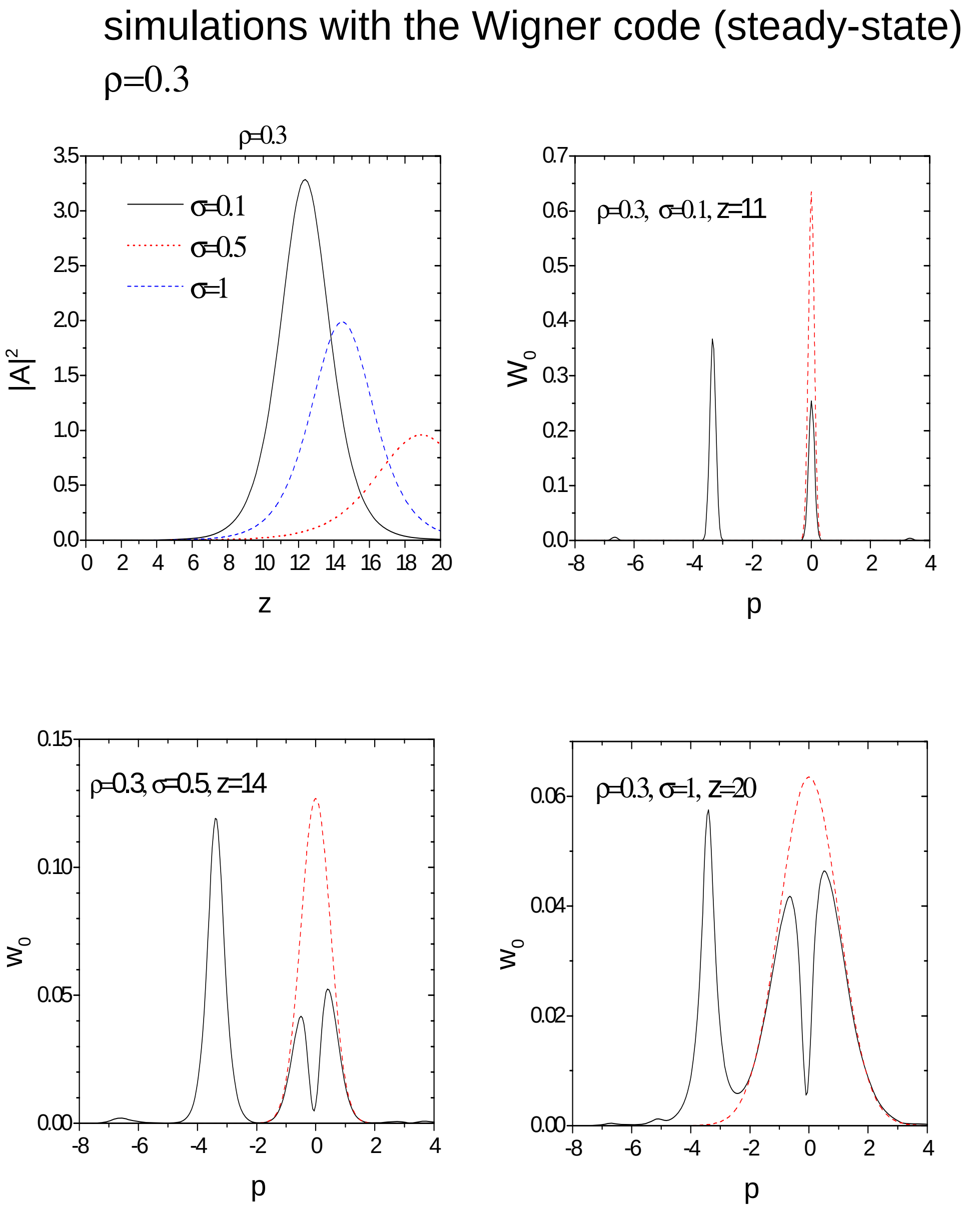

Figure 4.3: *Numerical simulations of (4.36)-(4.37) for* $\bar{\rho} = 0.3$*. It is still possible to see the "hole-burning" effect, but the momentum levels start to slightly overlap.*

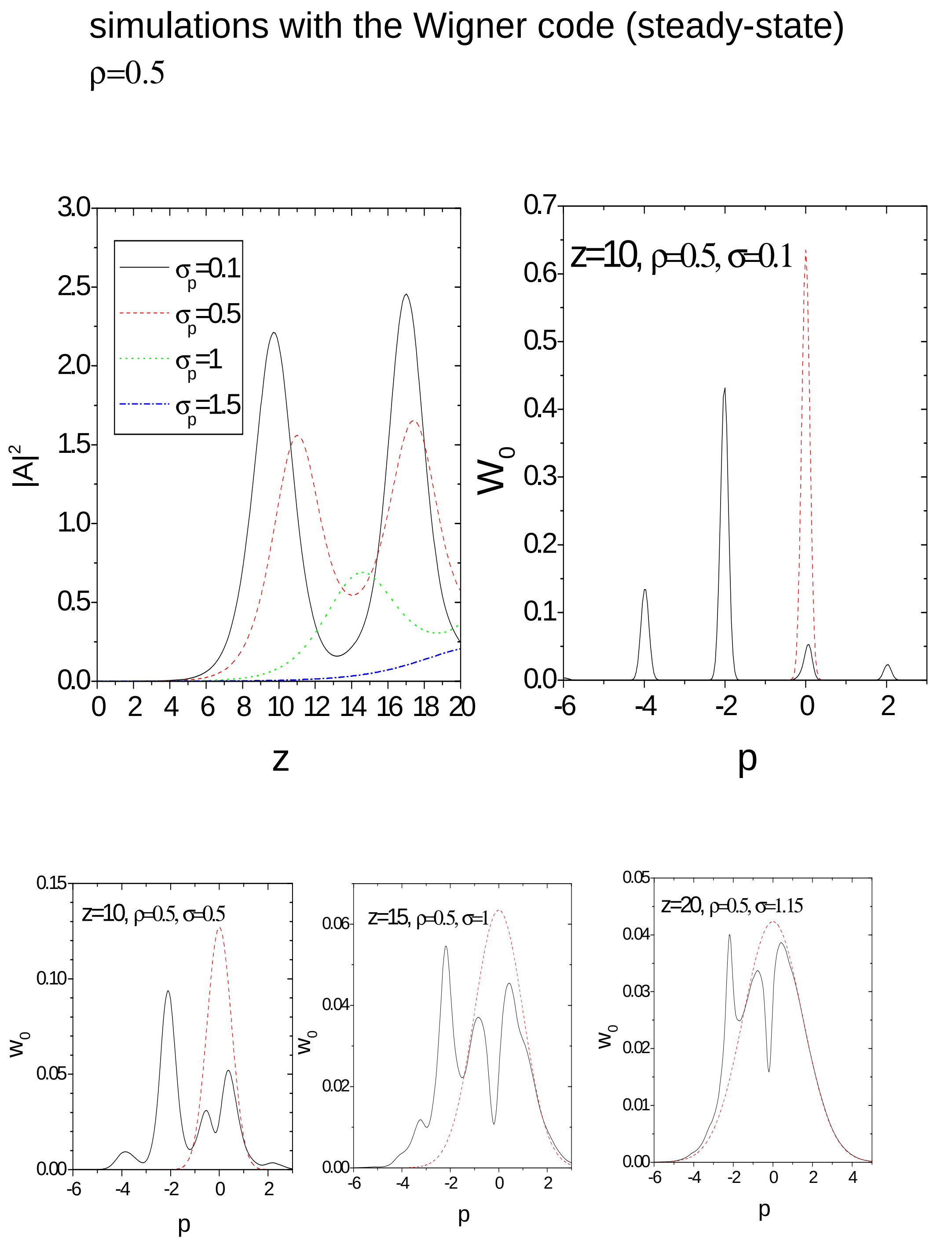

Figure 4.4: *Numerical simulations of (4.36)-(4.37) for $\bar{\rho} = 0.5$. It is not possible to distinguish the discrete momentum levels anymore.*

and neglect the higher harmonics when compared to it. This way only the equation for the first harmonic W_1 remains:

$$\frac{\partial W_1(p,\bar{z})}{\partial \bar{z}} = -ip\,W_1(p,\bar{z}) + \bar{\rho}A(\bar{z})\left[G\left(p+\frac{1}{2\bar{\rho}}\right) - G\left(p-\frac{1}{2\bar{\rho}}\right)\right] \tag{4.40}$$

Since the bunching and thus radiation field only depend on W_1, the Eqs.(4.40) and (4.37) form a closed set. To solve it, I use again the Laplace transform:

$$\begin{aligned}\tilde{A}(\zeta) &= \int_0^\infty e^{-\zeta\bar{z}}A(\bar{z})\mathrm{d}\bar{z}\\ \tilde{W}_1(p,\zeta) &= \int_0^\infty e^{-\zeta\bar{z}}W_1(p,\bar{z})\mathrm{d}\bar{z}\end{aligned}$$

Equations (4.37)-(4.40) become

$$\begin{aligned}(\zeta - i\delta)\tilde{A}(\zeta) &= A(0) + \int_{-\infty}^{+\infty}\tilde{W}_1(p,\zeta)\mathrm{d}p\\ (\zeta + ip)\tilde{W}_1(p,\zeta) &= \bar{\rho}\left[G\left(p+\frac{1}{2\bar{\rho}}\right) - G\left(p-\frac{1}{2\bar{\rho}}\right)\right]\tilde{A}(\zeta)\end{aligned}$$

that once combined yield

$$\tilde{A}(\zeta) = \tilde{A}(0)\left\{\zeta - i\delta - \bar{\rho}\int_{-\infty}^{+\infty}\frac{G\left(p+\frac{1}{2\bar{\rho}}\right) - G\left(p-\frac{1}{2\bar{\rho}}\right)}{\zeta + ip}\mathrm{d}p\right\}^{-1} \tag{4.41}$$

$$\tilde{W}_1(p,\zeta) = \frac{\bar{\rho}}{\zeta + ip}\left[G\left(p+\frac{1}{2\bar{\rho}}\right) - G\left(p-\frac{1}{2\bar{\rho}}\right)\right]\tilde{A}(\zeta) \tag{4.42}$$

From (4.41) it is possible to invert the Laplace transform, obtaining

$$A(\bar{z}) = \sum_j \frac{e^{i\bar{\lambda}_j\bar{z}}}{\prod_{l\neq j}(\bar{\lambda}_j - \bar{\lambda}_l)}A(0) \tag{4.43}$$

where the $\bar{\lambda}_j$ are the roots of the usual dispersion relation

$$\bar{\lambda} - \delta + \bar{\rho}\int_{-\infty}^{+\infty}\left[G\left(p+\frac{1}{2\bar{\rho}}\right) - G\left(p-\frac{1}{2\bar{\rho}}\right)\right]\frac{\mathrm{d}p}{\bar{\lambda} + p} = 0 \tag{4.44}$$

Thus both the previous approach, where I used a weighting over a distribution of detunings $G(\delta)$ in the Preparata equations, and this one, where the distribution $G(p)$ naturally appears as the initial condition of the Wigner function, lead to the same result, at least in the linear regime.

It is important to notice that in the starting equations (4.33)-(4.34) the detuning δ was left in the equation for the radiation field. While this eased the following calculations, it is just a mathematical tool. To get the physical radiation field we have to shift back the detuning into the electron variables, in this case the Wigner function. This translates into a shifting of the roots $\bar{\lambda}_j$:

$$\lambda_j = \bar{\lambda}_j - \delta$$

Notice that this way the notation is consistent with the one used in the first section of this chapter.

4.2.2 Hole-burning effect

Since the solution for the radiation field is the same as before, the assumption of a Lorentzian distribution leads to the same results:

$$\begin{aligned}
\lambda_1 &\simeq -\frac{1}{2}\left(\delta - \frac{1}{2\bar{\rho}}\right)\left(1 + \frac{\sigma}{\Gamma}\right) - i\left(\frac{\Gamma - \sigma}{2}\right) \\
\lambda_2 &\simeq -\frac{1}{2}\left(\delta - \frac{1}{2\bar{\rho}}\right)\left(1 - \frac{\sigma}{\Gamma}\right) + i\left(\frac{\Gamma - \sigma}{2}\right) \\
\lambda_3 &\simeq i\sigma
\end{aligned}$$

In addition to the evolution of the radiation field, one can now extract more information than in the previous case, by studying the evolution of the electron distribution through the Wigner function. From (4.35) one has

$$\int_0^{2\pi} W(\theta, p, \bar{z})\mathrm{d}\theta = W_0(p, \bar{z}) \equiv P(p) \tag{4.45}$$

so that W_0 represents the probability distribution in momentum space for the electrons. Instead of taking $W_0(p, \bar{z}) = G(p)$ I now wish to elaborate it to a better approximation. By inverting (4.42) one gets

$$\begin{aligned}
W_1(p, \bar{z}) &= -i\frac{\bar{\rho}}{\lambda_1 + p}\left[G\left(p + \frac{1}{2\bar{\rho}}\right) - G\left(p - \frac{1}{2\bar{\rho}}\right)\right]A(\bar{z}) \\
&\approx -i\frac{\bar{\rho}}{\lambda_1 + p}\left[G\left(p + \frac{1}{2\bar{\rho}}\right) - G\left(p - \frac{1}{2\bar{\rho}}\right)\right]\frac{A(0)e^{i\lambda_1\bar{z}}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}
\end{aligned} \tag{4.46}$$

where I have substituted only the exponential part of $A(\bar{z})$.

For simplicity I set the system at resonance $\delta = 1/2\bar{\rho}$, W_1 and the field A can be written as

$$
\begin{aligned}
A(\bar{z}) &= -\frac{2A(0)}{\Gamma(\Gamma+\sigma)} \exp\left(\frac{\Gamma-\sigma}{2}\,\bar{z}\right) \\
W_1(p,\bar{z}) &= \left[G\left(p+\frac{1}{2\bar{\rho}}\right) - G\left(p-\frac{1}{2\bar{\rho}}\right)\right] F\left(p+\frac{1}{2\bar{\rho}}\right) A(\bar{z})
\end{aligned}
$$

where

$$
F(p) = \sqrt{\frac{\Gamma-\sigma-2ip}{\Gamma-\sigma+2ip}} \tag{4.47}
$$

$$
G(p) = \frac{1}{\pi}\frac{\sigma}{\sigma^2+p^2} \tag{4.48}
$$

and $\Gamma = \sqrt{\sigma^2 + 4\bar{\rho}}$.

I take a Lorentzian distribution narrow enough to resolve the quantum step $\hbar k$ between adjacent two momentum levels, i.e. $\sigma \ll 1/2\bar{\rho}$. Thus in the linear regime the absorption line $G(p - 1/2\bar{\rho})$ will be very small compared to the emission line $G(p + 1/2\bar{\rho})$ and we can approximate W_1 as

$$
W_1(p,\bar{z}) \approx G\left(p+\frac{1}{2\bar{\rho}}\right) F\left(p+\frac{1}{2\bar{\rho}}\right) A(\bar{z}) \tag{4.49}
$$

Now one can use (4.49) in (4.36) for the case $n = 0$ to get an evolution equation for W_0:

$$
\frac{\partial W_0}{\partial \bar{z}} \simeq 2\bar{\rho}\,\mathrm{Re}\left\{|A|^2(\bar{z})\left[G\left(p+\frac{1}{\bar{\rho}}\right) F\left(p+\frac{1}{\bar{\rho}}\right) - G(p)F(p)\right]\right\} \tag{4.50}
$$

This can be integrated, obtaining

$$
W_0(p,\bar{z}) \approx G(p) - 2\bar{\rho}^2|A|^2(\bar{z})\left\{G(p)K(p) - G\left(p+\frac{1}{\bar{\rho}}\right) K\left(p+\frac{1}{\bar{\rho}}\right)\right\} \tag{4.51}
$$

where

$$
K(p) = \frac{1}{\sqrt{4p^2 + (\Gamma-\sigma)^2}}
$$

This shows the so-called *hole burning* effect, in which the electrons nearer to resonance, i.e. the center of the initial momentum distribution, shift to the displaced distribution around $p = -1/\bar{\rho}$.

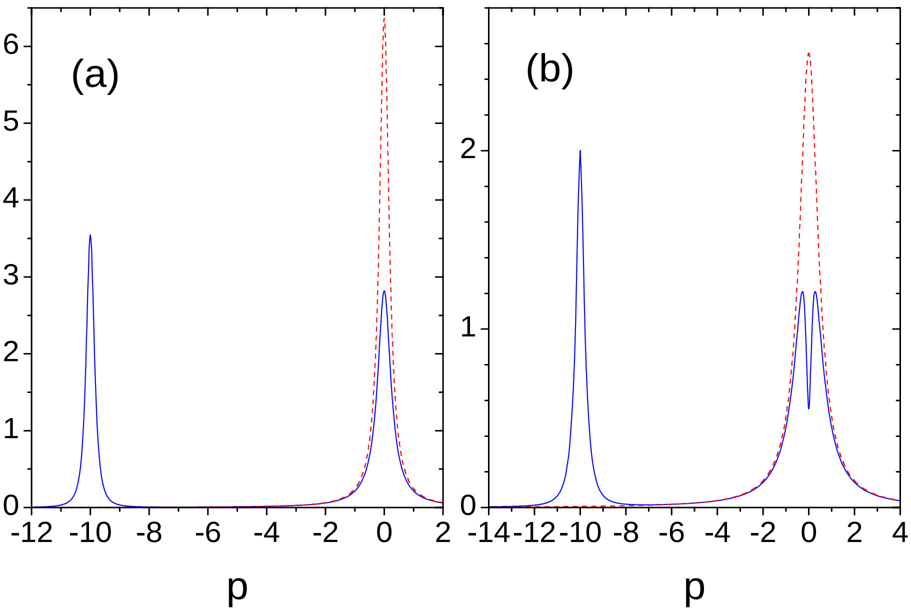

Figure 4.5: *Plot of (4.51): W_0 as a function of p at different times: first the lower momentum level appears (left), and then the forming of the hole (right).*

4.3 Bunching and energy spread uncertainty relations

Starting from the basic commutation relation $[\theta, p] = i$ between the phase and the momentum it is possible to show that some very general limitations link the bunching and the energy spread together. We have seen previously how the momentum p has discrete eigenvalues $n = 0, \pm 1, \ldots$ in units of $\hbar k$ and thus normalized eigenfunctions $(2\pi)^{-1/2}\exp(in\theta)$. It is known [30] that, assuming these discrete eigenstates, it is *not* possible to imply a uncertainty relation $\Delta\theta\Delta p \geq 1/2$ for the phase and momentum from their commutation relation. However, it is instead formally correct to use the periodic operators $\sin\theta$ and $\cos\theta$ to deduce uncertainty inequalities. The commutators with momentum for those functions are

$$[\sin\theta, p] = +i\cos\theta \tag{4.52}$$

$$[\cos\theta, p] = -i\sin\theta \tag{4.53}$$

From those follow the relations

$$\Delta p \cdot \Delta\sin\theta \geq \frac{|\langle\cos\theta\rangle|}{2} \tag{4.54}$$

$$\Delta p \cdot \Delta \cos\theta \quad \geq \quad \frac{|\langle \sin\theta \rangle|}{2} \tag{4.55}$$

which can be combined to get

$$(\Delta p)^2 \left[(\Delta \cos\theta)^2 + (\Delta \sin\theta)^2 \right] \geq \frac{\langle \cos\theta \rangle^2 + \langle \sin\theta \rangle^2}{4} \tag{4.56}$$

One can now use the definition of the bunching

$$B \equiv \left\langle e^{-i\theta} \right\rangle = \langle \cos\theta \rangle - i \langle \sin\theta \rangle \tag{4.57}$$

inserting it in (4.56) to obtain a lower limit on the energy spread:

$$\Delta p \geq \frac{|B|}{2\sqrt{1 - |B|^2}} \tag{4.58}$$

It is important to notice that this relation is independent from FEL dynamics, since it was derived from very general quantum mechanical first principles.

Eq.(4.58) can be reversed to get

$$|B| \leq \frac{\Delta p}{\sqrt{(\Delta p)^2 + 1/4}} \tag{4.59}$$

This relation sets an upper limit to how much bunched the electron beam can become inside a free electron laser, depending on the energy spread. In particular, the bunching can be close to unity $B \simeq 1$ only when $\Delta p \gg 1/2$, that is in physical units when the momentum spread is much bigger than the discrete linewidth $\hbar k/2$ given by the photon momentum.

It is interesting to note that the usual Heisenberg uncertainty principle is recovered when $\Delta\theta \ll 1$, since then $|B|^2 \simeq 1 - (\Delta\theta)^2$ and Eq.(4.58) gives

$$\Delta\theta \Delta p \geq 1/2$$

4.3.1 Minimum uncertainty states

It has been previously shown that it is impossible to have a normalized state that can minimize the general relation (4.56). It is however possible to define states [31] that minimize one, and only one, of the two inequalities (4.52) and (4.53).

We choose to minimize (4.52). The minimum uncertainty states are then the solution to the equation

$$(p + i\gamma \sin\theta) \, |\psi_\lambda(\gamma)\rangle = \lambda |\psi_\lambda(\gamma)\rangle \tag{4.60}$$

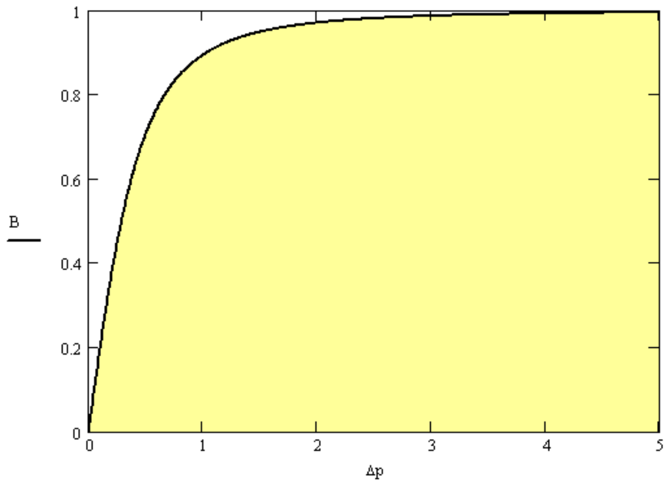

Figure 4.6: *The relation (4.59) between the bunching B and the energy spread Δp: the only allowed values are those below the solid curve.*

where γ and λ are real parameters that obey the relation

$$\lambda = \langle p \rangle + i\gamma \langle \sin\theta \rangle$$

The solution to Eq.(4.60) is given by

$$|\psi_\lambda(\gamma)\rangle = \frac{1}{\sqrt{I_0(2\gamma)}} \sum_n I_{n-\lambda}(\gamma)|n\rangle \tag{4.61}$$

where the $I_n(x)$ are the modified Bessel functions of order n and the states $|n\rangle$ are the discrete eigenstates of the dimensionless momentum p.

Using the properties of the modified Bessel functions one can show that

$$\langle p \rangle = \langle \psi_\lambda(\gamma)|p|\psi_\lambda(\gamma)\rangle = \lambda \in \mathbb{Z} \tag{4.62}$$

i.e. λ is the *discrete* expectation value of the momentum operator. From (4.62) one gets

$$\langle \sin\theta \rangle = 0 \tag{4.63}$$

and thus the bunching

$$B \equiv \langle e^{-i\theta} \rangle = \langle \cos\theta \rangle \tag{4.64}$$

While this makes λ and γ independent, one can find

$$\Delta p = \sqrt{\frac{\gamma}{2} \frac{I_1(2\gamma)}{I_0(2\gamma)}} \tag{4.65}$$

so that γ is just a function of the energy spread:

- for $\gamma = 0$ the Jackiw states reduce to the momentum eigenstates $|n\rangle$ where $\Delta p = 0$.
- for $\gamma \gg 1$ one obtains Gaussian wave packets with $\Delta\theta = 1/\sqrt{2\gamma}$ and $\Delta p = \sqrt{\gamma/2}$.

It can be double-checked (see the Appendix for the calculations) that the state (4.61) really minimize the uncertainty relation (4.52):

$$\Delta p \cdot \Delta \sin\theta = \frac{\langle\cos\theta\rangle}{2} \tag{4.66}$$

Conclusions

The aim of my work was to show which place quantum mechanics has in free electron laser physics and understand when a quantum treatment of the FEL is preferable instead of a classical one. In the first part of this thesis I gave a brief introduction to classical FELs physics and then I proceeded to present a summary of the past studies of the FEL using quantum physics. I used this summary to show how certain topics in FEL physics require a quantum treatment, in particular photon statistics and the problem of spontaneous emission (start-up from vacuum).

The main part of the thesis has been dedicated to the Quantum Regime of the FEL that was recently discovered and that is currently the object of studies and possible future experiments [20]. Numerical analysis has previously shown how the electrons of the FEL, in a certain range of the physical parameters (specifically when the Quantum FEL Parameter $\bar{\rho}$ is much smaller than unity, i.e. when the single photon momentum cannot be neglected compared to the electron momentum), behave as a two-level system coupled to a radiation field. The electron momentum has then to be treated as a discrete variable, varying only by fixed steps given by the photon momentum; in particular only two momentum levels are occupied by the electrons. I used then some tools from perturbation theory to show how the Hamiltonian of a two-level system coupled to radiation could be derived from the classical FEL Hamiltonian.

The two-level system Hamiltonian was the starting point of my further analysis: first, I used a semiclassical approximation [28] to obtain an analytical solution for the expectation number of the laser photons, that had the form of a train of identical exponential spikes, each separated from the next spike by the same constant spacing. This was shown to be in complete agreement with the numerical results from which the two-level system was initially observed. This semiclassical approximation implied neglecting the discreteness of the photon number, as had been as well the case in the original numerical simulations. I thus dropped such approximation, to obtain a description of the system where both the electrons and the laser field were treated quantum-mechanically. I used numerical analysis to investigate the system behaviour, applying some mathematical tools (concerning matrix diagonalization techniques [9]) that had never been used before in this context. This showed an effect that had been masked by the semiclassical approximation: instead of a train of identical spikes, this time the photon number featured a cyclical series of periodic revivals. This phenomenon was not further investigated, but it could be the object of a future study.

The results I had obtained up to this point mainly regarded the laser photons. Thus I then focused on the other part of the FEL system, the electrons. I used the Wigner Function formalism as my tool of choice to describe the electrons. To avoid losing the discreteness of the electron momentum, which is a fundamental characteristic of the quantum regime of the FEL, I introduced the discrete Wigner function [10] and its special properties, and then derived its evolution equation, coupled to the laser field, thus giving a description of the dynamics of the electron beam in phase space. This also allowed me to obtain a closed set of Maxwell-Bloch equations for the FEL, describing the evolution of the laser field, the electron bunching parameter and the population difference between the electron levels. I gave an analytical solution of this system of equations. Using the discrete Wigner function formalism I was also able to give an estimate of which distribution of electrons among the momentum levels would make the quantum effects more relevant.

In the last chapter of my thesis I dropped the approximation of a monoenergetic electron beam and considered the effect of an initial electron energy distribution in the quantum FEL. I approached the problem in two different ways, first using the probability amplitudes coefficients to describe the electrons and weighting them over a fixed energy distribution, and then using the Wigner function for a more general approach, where the energy distribution simply entered as the Wigner function initial condition. I showed how in the linear approximation both ways give the same results and in particular I studied the cases of a generic broad distribution and of a Lorentzian distribution, for which I gave analytical solutions. I also produced numerical simulations of the coupled Wigner function and laser field evolution equations for different parameters and initial conditions.

Finally, I showed how the electron bunching and the electron energy spread are connected by an uncertainty relation, such that there is an intrinsic quantum limit on how much the electron beam can become bunched, depending on the electron energy spread.

Bibliography

[1] R. Bonifacio, F. Casagrande, G. Cerchioni, L. De Salvo Souza, P. Pierini, N. Piovella, La Rivista del Nuovo Cimento, Vol.18 Issue 9, *Physics of the High-Gain FEL and Superradiance*(1990)

[2] J.M. Madey, J. Appl. Phys. 42, 1906 (1971)

[3] E.L. Saldin, E.A. Schneidmiller, M.V. Yurkov, *The Physics of Free Electron Lasers*, (Springer, Berlin Heidelberg 2000)

[4] C.A. Brau, *Free-Electron Lasers*, (Academic Press, New York 1990)

[5] G. Preparata, Phys. Rev. A 38, 233 (1988)

[6] Bonifacio R. and Casagrande F., Optics Comm. 50 (1984) 251.

[7] Bonifacio R., Pellegrini C. and Narducci L., Opt. Commun. 50 (1984) 373.

[8] W. Becker, M.S. Zubairy, Phys. Rev. A 25, 2200 (1982)

[9] D.F. Walls, R. Barakat, Phys. Rev. A 1, 446 (1970)

[10] J.P. Bizarro, Phys. Rev. A 49, 3255 (1984)

[11] L.R. Elias, W.M. Fairbank, J.M. Madey, H.A. Schwettman, T.I. Smith, Phys. Rev. Lett. 36, 717 (1976)

[12] R. Bonifacio, R. Corsini, L. De Salvo, P. Pierini, N. Piovella, La Rivista del Nuovo Cimento, Vol.15 Issue 11 *New Effects in the Physics of the High-Gain FEL and Superradiance: a Proposed Experiment and Possible Applications*(1992)

[13] R. Bonifacio, B.W.J. McNeil, P. Pierini, Phys. Rev. A 40, 4467 (1989)

[14] R. Bonifacio, F. Casagrande, J. Opt. Soc. Am. B 2, 250 (1985)

[15] M.V. Fedorov, Prog. Quant. Electron. 7, 73 (1981)

[16] F.A. Hopf, P. Meystre, M.O. Scully, W.H. Louisell, Opt. Comm. 18, 413 (1976)

[17] *Novel Sources of Coherent Radiation*, (Addison-Wesley, Reading, Mass., 1978)

[18] T.C. Marshall, *Free-Electron Lasers*, (Mac-millan, 1985)

[19] A. Bambini, A. Renieri, Lettere al Nuovo Cimento 21, 399 (1978)

[20] Europhys. Lett., 69 (1), pp. 55 (2005)

[21] R. Bonifacio, F. Casagrande, Nucl. Instr. and Meth. in Phys. Res. A237, 168 (1985)

[22] C.B. Schroeder, C. Pellegrini, P. Chen, Phys. Rev. E 64, 056502 (2001)

[23] Phys.Rev.ST - Accelerators and Beams 9, 090701 (2006)

[24] J. Gea-Banacloche, Phys. Rev. A 31, 1607 (1985)

[25] I.V. Smetanin, Laser Phys. 7, 318 (1997)

[26] E.M. Belenov, S.V. Grigor'ev, A.V. Nazarkin, I.V. Smetanin, JEPT 78, 431 (1994)

[27] W. Becker, J. Gea-Banacloche, M.O. Scully, Phys. Rev. A 33, 2174 (1986)

[28] R. Bonifacio, G. Preparata, Phys. Rev. A 2, 336 (1970)

[29] M.E. Smithers, E.Y.C. Lu, Phys. Rev. A 9, 790 (1974)

[30] P. Carruthers and M.M. Nieto, Rev. of Mod. Phys. 40 (1968) 411

[31] R. Jackiw, Journ. of Math. Phys. 9, 339 (1968)

[32] J. Gea-Banacloche, M.O. Scully, Nucl. Instr. and Meth. in Phys. Res. A237, 100 (1985)

[33] Bonifacio R., De Salvo Souza L., Nucl. Instrum. and Meth. in Phys. Res. A 341 (1994) 360; Bonifacio R., De Salvo Souza L., Narducci L. and D'Angelo E.J., Phys. Rev.A 50 (1994) 1716.

[34] Bonifacio R., Optics Comm. 146 (1998) 236.

[35] Moore M.G., Zobay O. and Meystre P., Phys. Rev.A 60 (1999) 1491.

[36] Ling H.Y., Pu H., Baksmaty L. and Bigelow N.P., Phys. Rev.A 63 (2001) 053810.

[37] Piovella N., Cola M., Bonifacio R., Phys. Rev.A 67 (2003) 013817.

[38] Becker W. and McIver J.K., Phys. Rev.A 28 (1983) 1838.

[39] Bonifacio R., Cataliotti F.S., Cola M., Falani L., Fort C., Piovella N., Inguscio M., Optics Comm. 233 (2004) 155.

Glossary

- Wiggler (or Undulator) Parameter

$$a_w = \frac{e\lambda_w B_w}{2\pi mc^2} \simeq 0.93 \cdot B_w[T] \cdot \lambda_w[cm]$$

- FEL Parameter

$$\rho = \frac{1}{\gamma_r}\left(\frac{a_w\omega_p}{4k_w c}\right)^{2/3}$$

- Gain Length

$$l_g = \frac{\lambda_w}{4\pi\rho}$$

- Cooperation Length

$$l_c = \frac{\lambda_r}{4\pi\rho}$$

- Bunching

$$B = \langle e^{-i\theta}\rangle_{electrons}$$

- Quantum FEL Parameter

$$\bar{\rho} = \frac{mc\gamma_r}{\hbar k_\lambda}\,\rho$$

Appendix A

Derivation of the Preparata Equations

The starting point is (3.7), the FEL Hamiltonian written in the formalism of occupation numbers in Fock space:

$$\hat{H} = \sum_{n=-\infty}^{+\infty} \left\{ \frac{1}{2\bar{\rho}} n^2 \hat{c}_n^\dagger \hat{c}_n + i\sqrt{\frac{\bar{\rho}}{N}} \left(a^\dagger \hat{c}_n \hat{c}_{n-1}^\dagger - a \hat{c}_n^\dagger \hat{c}_{n-1} \right) \right\} - \delta a^\dagger a \tag{A.1}$$

From this picture I go back to the representation in terms of the scalar field operator (3.4):

$$\hat{\Psi}(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \hat{c}_n \langle \theta | n \rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \hat{c}_n e^{in\theta} \tag{A.2}$$

so that the Hamiltonian can be rewritten as

$$H = \int_0^{2\pi} \left[-\frac{1}{2\bar{\rho}} \hat{\Psi}^\dagger \frac{\partial^2}{\partial \theta^2} \hat{\Psi} + i\sqrt{\frac{\bar{\rho}}{N}} \left(a^\dagger e^{-i\theta} - a e^{i\theta} \right) \hat{\Psi}^\dagger \hat{\Psi} \right] \mathrm{d}\theta - \delta a^\dagger a \tag{A.3}$$

To this Hamiltonian it is possible to associate the following Lagrangian density:

$$\mathcal{L}(\theta) = \hat{\Psi}^\dagger(\theta) i \frac{\partial}{\partial t} \hat{\Psi}(\theta) + \frac{1}{2\bar{\rho}} \hat{\Psi}^\dagger \frac{\partial^2}{\partial \theta^2} \hat{\Psi}(\theta) - i\sqrt{\frac{\bar{\rho}}{N}} \left(a^\dagger e^{-i\theta} - a e^{+i\theta} \right) \hat{\Psi}^\dagger(\theta) \hat{\Psi}(\theta) + \frac{1}{2\pi} a^\dagger i \frac{\partial}{\partial t} a \tag{A.4}$$

The path-integral representation of the generating functional of this Lagrangian density is

$$\begin{aligned} Z[\theta, \theta^*; \alpha, \alpha^*] \quad = \quad & \int [\mathrm{d}a \, \mathrm{d}a^* \, \mathrm{d}\Psi \, \mathrm{d}\Psi^*] \exp \left\{ i \int_{-\infty}^{+\infty} dt \int_0^{2\pi} d\theta \right. \\ & \left. (L(t, \theta) + (\theta \Psi^* + \theta^* \Psi) + (\alpha a^* + \alpha^* a)) \right\} \end{aligned} \tag{A.5}$$

By rescaling a and Ψ as follows

$$A \equiv \frac{1}{\sqrt{\bar{\rho} N}}\, a \tag{A.6}$$

$$\psi \equiv \frac{1}{\sqrt{N}}\, \Psi \tag{A.7}$$

the generating functional becomes

$$\begin{aligned} Z[\theta, \theta^*; \alpha, \alpha^*] \;&=\; \int [\mathrm{d}A\, \mathrm{d}A^*\, \mathrm{d}\psi\, \mathrm{d}\psi^*] \exp\left\{ iN \int_{-\infty}^{+\infty} dt \int_0^{2\pi} d\theta \right. \\ &\left. \left(\bar{L}(t,\theta) + \frac{1}{\sqrt{N}}(\theta\psi^* + \theta^*\psi) + \sqrt{\frac{\bar{\rho}}{2N}}(\alpha A^* + \alpha^* A) \right) \right\} \end{aligned} \tag{A.8}$$

where

$$\bar{L}(t,\theta) = i\psi^* \frac{\partial}{\partial t}\psi + \frac{1}{\bar{\rho}}\psi^* \frac{\partial^2}{\partial \theta^2}\psi - \delta A^* A - \frac{i\bar{\rho}}{2}(A^* e^{-i\theta} - A e^{i\theta})\psi^*\psi + \frac{i\bar{\rho}}{4\pi} A^* \frac{\partial}{\partial t} A \tag{A.9}$$

From (A.8) it is possible to see how N takes the role usually of $1/\hbar$: then in the limit $N \gg 1$ the path integrals will be dominated by the classical trajectories found by the principle of stationary phase:

$$\delta \int \bar{L} = 0 \tag{A.10}$$

The equations of motion that minimize the action are

$$\frac{\partial \psi}{\partial t} = \frac{i}{2\bar{\rho}} \frac{\partial^2 \psi}{\partial \theta^2} - \bar{\rho}\left(A e^{i\theta} - A^* e^{-i\theta}\right)\psi \tag{A.11}$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \int_0^{2\pi} |\psi(\theta)|^2 e^{-i\theta} \mathrm{d}\theta + i\delta A \tag{A.12}$$

Appendix B

Derivation of the Wigner function evolution equation

I start from the form of the Wigner function given in (3.12):

$$W(\theta, \bar{p}, t) = \frac{1}{2\pi} \frac{1}{2\hbar k} \int_{-\infty}^{\infty} e^{-i\eta\bar{p}} \psi^* \left(\theta - \frac{\eta}{2}, t\right) \psi \left(\theta + \frac{\eta}{2}, t\right) \mathrm{d}\eta \tag{B.1}$$

I set

$$\begin{aligned} y &\equiv \theta + \frac{\eta}{2} \\ z &\equiv \theta - \frac{\eta}{2} \end{aligned}$$

and differentiate in respect to time

$$\frac{\partial W}{\partial t} = \frac{1}{2\pi} \frac{1}{2\hbar k} \int_{-\infty}^{\infty} e^{-i\eta\bar{p}} \left\{ \frac{\partial \psi^*(z)}{\partial t} \psi(y) + \psi^*(z) \frac{\partial \psi(y)}{\partial t} \right\} \tag{B.2}$$

I substitute then the Preparata equation (3.13) for the time derivative of the wave function:

$$\begin{aligned} \frac{\partial \psi(y)}{\partial t} &= \frac{i}{\bar{\rho}} \frac{\partial^2 \psi(y)}{\partial y^2} - \frac{\bar{\rho}}{2} \left(A e^{iy} - A^* e^{-iy} \right) \psi(y) \\ \frac{\partial \psi^*(z)}{\partial t} &= -\frac{i}{\bar{\rho}} \frac{\partial^2 \psi^*(z)}{\partial z^2} + \frac{\bar{\rho}}{2} \left(A e^{iz} - A^* e^{-iz} \right) \psi^*(z) \end{aligned}$$

obtaining

$$\frac{\partial W}{\partial t} = \frac{1}{2\pi} \frac{1}{2\hbar k} \left(\frac{i}{\bar{\rho}} \mathcal{A} + \frac{\bar{\rho}}{2} \mathcal{B} \right) \tag{B.3}$$

where

$$\mathcal{A} = \int_{-\infty}^{\infty} e^{-i\eta\bar{p}} \left\{ \frac{\partial^2\psi(y)}{\partial y^2}\psi^*(z) - \frac{\partial^2\psi^*(z)}{\partial z^2}\psi(y) \right\} \mathrm{d}\eta \tag{B.4}$$

$$\mathcal{B} = \int_{-\infty}^{\infty} e^{-i\eta\bar{p}} \left\{ A\left(e^{iz} - e^{iy}\right) - A^*\left(e^{-iz} - e^{-iy}\right) \right\} \psi^*(z)\psi(y)\mathrm{d}\eta \tag{B.5}$$

I have splitted the integral into two parts to calculate them one at a time. I start with $\mathcal{A}$; from the definition of y and z it follows that

$$\frac{\partial\eta}{\partial y} = -\frac{\partial\eta}{\partial z} = 2$$

so that

$$\begin{aligned} \frac{\partial^n\psi(y)}{\partial y^n} &= 2^n\frac{\partial^n\psi(y)}{\partial\eta^n} \\ \frac{\partial^n\psi^*(z)}{\partial z^n} &= (-2)^n\frac{\partial^n\psi^*(z)}{\partial\eta^n} \end{aligned}$$

and $\mathcal{A}$ becomes

$$\begin{aligned} \mathcal{A} &= 4\int_{-\infty}^{\infty} e^{-i\eta\bar{p}} \left\{ \frac{\partial^2\psi(y)}{\partial\eta^2}\psi^*(z) - \psi(y)\frac{\partial^2\psi^*(z)}{\partial\eta^2} \right\} \mathrm{d}\eta \\ &= 4\int_{-\infty}^{\infty} e^{-i\eta\bar{p}} \left\{ -\frac{\partial\psi(y)}{\partial\eta}\left[-i\bar{p}\psi^*(z) + \frac{\partial\psi^*(z)}{\partial\eta}\right] + \frac{\partial\psi^*(z)}{\partial\eta}\left[-i\bar{p}\psi(y) + \frac{\partial\psi(y)}{\partial\eta}\right] \right\} \mathrm{d}\eta \\ &= 4i\bar{p}\int_{-\infty}^{\infty} e^{-i\eta\bar{p}} \left\{ \frac{\partial\psi(y)}{\partial\eta}\psi^*(z) - \psi(y)\frac{\partial\psi^*(z)}{\partial\eta} \right\} \mathrm{d}\eta \\ &= 2i\bar{p}\int_{-\infty}^{\infty} e^{-i\eta\bar{p}} \left\{ \frac{\partial\psi(y)}{\partial y}\psi^*(z) + \psi(y)\frac{\partial\psi^*(z)}{\partial z} \right\} \mathrm{d}\eta \end{aligned}$$

where I have repeatedly used partial integration. I now substitute the partial derivatives in $\mathcal{A}$ using

$$\frac{\partial z}{\partial\theta} = \frac{\partial y}{\partial\theta} = 1$$

to get our final result for the first part of our equation:

$$\begin{aligned} \mathcal{A} &= 2i\bar{p}\int_{-\infty}^{\infty} e^{-i\eta\bar{p}} \left\{ \frac{\partial\psi(y)}{\partial\theta}\psi^*(z) + \psi(y)\frac{\partial\psi^*(z)}{\partial\theta} \right\} \mathrm{d}\eta \\ &= 2i\bar{p}\frac{\partial}{\partial\theta}\int_{-\infty}^{\infty} e^{-i\eta\bar{p}}\psi^*(z,t)\psi(y,t)\mathrm{d}\eta \\ &= 2i\bar{p}\,2\pi 2\hbar k\frac{\partial W(\theta,\bar{p},t)}{\partial\theta} \end{aligned} \tag{B.6}$$

I then pass on calculating $\mathcal{B}$:

$$
\begin{aligned}
\mathcal{B} &= \int_{-\infty}^{\infty} e^{-i\eta\bar{p}} \left\{ A\left(e^{iz} - e^{iy}\right) - A^*\left(e^{-iz} - e^{-iy}\right)\right\} \psi^*(z)\psi(y)\mathrm{d}\eta \\
&= \left(Ae^{i\theta} + A^* e^{-i\theta}\right)\left\{\int_{-\infty}^{\infty} e^{-i\eta\left(\bar{p}+\frac{1}{2}\right)}\psi^*(z)\psi(y)\mathrm{d}\eta - \int_{-\infty}^{\infty} e^{-i\eta\left(\bar{p}-\frac{1}{2}\right)}\psi^*(z)\psi(y)\mathrm{d}\eta\right\} \\
&= 2\pi 2\hbar k\left(Ae^{i\theta} + A^* e^{-i\theta}\right)\left\{W\left(\theta, \bar{p} + \frac{1}{2}, t\right) - W\left(\theta, \bar{p} - \frac{1}{2}, t\right)\right\} \qquad \text{(B.7)}
\end{aligned}
$$

Inserting (B.6) and (B.7) into (B.3) I get the final equation for the wigner function:

$$
\frac{\partial W(\theta, \bar{p}, t)}{\partial t} = -\frac{\bar{p}}{\bar{\rho}}\frac{\partial W(\theta, \bar{p}, t)}{\partial \theta} + \bar{\rho}\left(Ae^{i\theta} - A^* e^{-i\theta}\right)\left[W\left(\theta, \bar{p} + \frac{1}{2}, t\right) - W\left(\theta, \bar{p} - \frac{1}{2}, t\right)\right] \tag{B.8}
$$

Appendix C

Derivation of the Hamiltonian in the Interaction Picture

Here I show how the interaction picture Hamiltonian of the system (3.7) is

$$\hat{H}_{IP}(t) = i\sqrt{\frac{\bar{\rho}}{N}} \sum_{n=-\infty}^{\infty} \left\{ a^\dagger \hat{c}_n \hat{c}_{n-1}^\dagger \exp\left[i \left(\frac{2n-1}{2\bar{\rho}} + \delta \right) t \right] - h.c. \right\} \tag{C.1}$$

Starting from its definition,

$$\hat{H}_{IP}(t) = e^{i\hat{H}_0 t}\, \hat{H}_I\, e^{-i\hat{H}_0 t} \tag{C.2}$$

and substituting in it $\hat{H}_0$ and $\hat{H}_I$, the expression I get is

$$\exp\left\{ i \left[\sum_n \frac{n^2}{2\bar{\rho}} \hat{c}_n^\dagger \hat{c}_n - \delta a^\dagger a \right] t \right\} \left(a^\dagger \hat{c}_k \hat{c}_{k-1}^\dagger - a \hat{c}_k^\dagger \hat{c}_{k-1} \right) \exp\left\{ -i \left[\sum_m \frac{m^2}{2\bar{\rho}} \hat{c}_m^\dagger \hat{c}_m - \delta a^\dagger a \right] t \right\} \tag{C.3}$$

apart from the coupling constants $i\sqrt{\bar{\rho}/N}$ and a sum over k.

The various $\hat{c}$ operators all commute with a and $a^\dagger$, so that the exponentials can be splitted into a photon and an electron parts, each acting only on the corresponding operators of $\hat{H}_I$.

Using the following well-known theorem from operator algebra

$$e^{xA} B e^{-xA} = B + x[A, B] + \frac{x^2}{2!}[A, [A, B]] + \ldots \tag{C.4}$$

and the commutation rules

$$\begin{aligned} [a^\dagger a, a] &= -a \\ [a^\dagger a, a^\dagger] &= +a^\dagger \end{aligned}$$

I get

$$e^{-i\delta(a^\dagger a)t} a e^{i\delta(a^\dagger a)t} = a - i\delta t[a^\dagger a, a] + \frac{(-i\delta t)^2}{2!}[a^\dagger a, [a^\dagger a, a]] + ... = a \sum_n \frac{(i\delta t)^n}{n!} = a e^{i\delta t}$$

and

$$e^{-i\delta(a^\dagger a)t} a^\dagger e^{i\delta(a^\dagger a)t} = a^\dagger e^{-i\delta t}$$

Thus (C.3) becomes

$$\exp\left\{ i\frac{t}{2\bar{\rho}} \sum_n n^2 \hat{c}_n^\dagger \hat{c}_n \right\} \left(\hat{c}_k \hat{c}_{k-1}^\dagger a^\dagger e^{-i\delta t} - \hat{c}_k^\dagger \hat{c}_{k-1} a e^{i\delta t} \right) \exp\left\{ -i\frac{t}{2\bar{\rho}} \sum_m m^2 \hat{c}_m^\dagger \hat{c}_m \right\} \tag{C.5}$$

As $[\hat{c}_n^\dagger \hat{c}_n, \hat{c}_m^\dagger \hat{c}_m] = 0$, I can also split the sums in the remaining exponential:

$$\exp\left\{ i\frac{t}{2\bar{\rho}} \sum_n n^2 \hat{c}_n^\dagger \hat{c}_n \right\} = \prod_n \exp\left\{ i\frac{t}{2\bar{\rho}} n^2 \hat{c}_n^\dagger \hat{c}_n \right\} \tag{C.6}$$

The exponential $\exp\left\{ i\frac{t}{2\bar{\rho}} n^2 \hat{c}_n^\dagger \hat{c}_n \right\}$ commutes with $\hat{c}_k, \hat{c}_k^\dagger$ when $n \neq k$, so that the only terms that do not cancel each other are

$$\left(e^{itk^2 \hat{c}_k^\dagger \hat{c}_k / 2\bar{\rho}}\ \hat{c}_k\ e^{-itk^2 \hat{c}_k^\dagger \hat{c}_k / 2\bar{\rho}} \right) \left(e^{it(k-1)^2 \hat{c}_{k-1}^\dagger \hat{c}_{k-1} / 2\bar{\rho}}\ \hat{c}_{k-1}^\dagger\ e^{-it(k-1)^2 \hat{c}_{k-1}^\dagger \hat{c}_{k-1} / 2\bar{\rho}} \right)$$

$$= \left(e^{-itk^2/2\bar{\rho}}\ \hat{c}_k \right) \left(e^{it(k-1)^2/2\bar{\rho}}\ \hat{c}_{k-1}^\dagger \right)$$

$$= e^{-i(2k-1)t/2\bar{\rho}}\ \hat{c}_k \hat{c}_{k-1}^\dagger$$

and

$$\left(e^{itk^2 \hat{c}_k^\dagger \hat{c}_k / 2\bar{\rho}}\ \hat{c}_k^\dagger\ e^{-itk^2 \hat{c}_k^\dagger \hat{c}_k / 2\bar{\rho}} \right) \left(e^{it(k-1)^2 \hat{c}_{k-1}^\dagger \hat{c}_{k-1} / 2\bar{\rho}}\ \hat{c}_{k-1}\ e^{-it(k-1)^2 \hat{c}_{k-1}^\dagger \hat{c}_{k-1} / 2\bar{\rho}} \right)$$

$$= \left(e^{itk^2/2\bar{\rho}}\ \hat{c}_k^\dagger \right) \left(e^{-it(k-1)^2/2\bar{\rho}}\ \hat{c}_{k-1} \right)$$

$$= e^{i(2k-1)t/2\bar{\rho}}\ \hat{c}_k^\dagger \hat{c}_{k-1}$$

where I used again the theorem (C.4) and the commutation relations

$$\begin{aligned} [\hat{c}_k^\dagger \hat{c}_k, \hat{c}_k] &= -\hat{c}_k \\ [\hat{c}_k^\dagger \hat{c}_k, \hat{c}_k^\dagger] &= +\hat{c}_k^\dagger \end{aligned}$$

Now I have all the pieces of the interaction picture Hamiltonian: putting them together I get

$$\hat{H}_{IP}(t) = i\sqrt{\frac{\bar{\rho}}{N}} \sum_{k=-\infty}^{\infty} \left\{ a^{\dagger} \hat{c}_k \hat{c}_{k-1}^{\dagger} \exp\left[i\left(\frac{2k-1}{2\bar{\rho}} + \delta \right) t \right] - h.c. \right\} \tag{C.7}$$

Appendix D

Solution of the two-level equations

D.1 Transition Amplitudes

Here I show the necessary steps to transform Eq.(3.68) to Eq.(3.76). I start from

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_n(t) = g\Big(\alpha_{n+1}\phi_{n+1}(t) - \alpha_n\phi_{n-1}(t)\Big) \tag{D.1}$$

where

$$\alpha_n = \sqrt{n\,n_1(n_2+1)} = \sqrt{n\left(\frac{N}{2}+M-n+1\right)\left(\frac{N}{2}-M+n\right)}$$

I define the following two functions,

$$F(n) \equiv \frac{2\sqrt{2}}{n!}\,\Gamma\left(1+\frac{n}{2}\right) \tag{D.2}$$

and

$$G(n) \equiv F(n)\cdot F\left(\frac{N}{2}+M-n\right)\cdot F\left(\frac{N}{2}-M+n\right) \tag{D.3}$$

Then I perform a change of variables

$$\Lambda_n(t) \equiv (-i)^n\sqrt{G(n)}\,\phi_n(t) \tag{D.4}$$

This way the differential equation becomes

$$\dot{\Lambda}_n(t) = G(n)\Big\{\Lambda_{n-1}(t) - \Lambda_{n+1}(t)\Big\} \tag{D.5}$$

The final step concerns the definition of the “angle” ϑ_n:

$$\vartheta_n \equiv \arcsin\sqrt{\frac{2n}{N+2M}} \tag{D.6}$$

and substituting it inside the equation in the place of the discrete variable n:

$$\dot{\Lambda}(\vartheta_n, t) = G(\vartheta_n)\Big\{\Lambda(\vartheta_{n-1}, t) - \Lambda(\vartheta_{n+1}, t)\Big\} \tag{D.7}$$

This is done to ease the passage to the continuous limit, since the variation $\Delta\vartheta_n = \vartheta_{n+1} - \vartheta_n$ is very small for every n.

As a final note, the various functions and variables defined here assume of course a simpler form in the physical case of main interest for us, that is $M = N/2$:

$$\dot{\phi}_n(\tau) = \sqrt{N-n}\Big[(n+1)\phi_{n+1}(\tau) - n\phi_{n-1}(\tau)\Big] \tag{D.8}$$

$$G(n) = F^2(n)\, F(N-n) \tag{D.9}$$

$$\vartheta_n = \arcsin\sqrt{\frac{n}{N}} \tag{D.10}$$

$$\Delta\vartheta_n = \vartheta_{n+1} - \vartheta_n \simeq \frac{1}{\sqrt{(n+1)(N-n+1)}} \leq \frac{1}{\sqrt{N}} \ll 1 \tag{D.11}$$

D.2 Solution of the A-B-D System

I want to show the solution of the system

$$\dot{B} = AD \tag{D.12}$$

$$\dot{D} = -4AB \tag{D.13}$$

$$\dot{A} = B \tag{D.14}$$

This system has two integrals of motion:

$$\Lambda \equiv A^2 + \frac{D}{2} \tag{D.15}$$

$$\Upsilon \equiv \frac{\sqrt{D^2 + 4B^2}}{2} \tag{D.16}$$

These constants can be used to rewrite the bunching as

$$B = \sqrt{A^2(2\Lambda - A^2) + \Upsilon^2 - \Lambda^2} \tag{D.17}$$

Thus I obtain a single differential equation:

$$\dot{A} = \sqrt{-\left(A^2 - \Lambda + \Upsilon\right)\left(A^2 - \Lambda - \Upsilon\right)} \tag{D.18}$$

I take the case $\Upsilon^2 = \Lambda^2$, which reduces the equation to

$$\dot{A} = A\sqrt{2\Lambda - A^2} \tag{D.19}$$

This is solved by first taking the substitution $A = \sqrt{2\Lambda}\cos(x)$, which gives

$$\dot{x} = -\sqrt{2\Lambda}\cos(x) \tag{D.20}$$

and then $y = \sqrt{2\Lambda}\sin(x)$:

$$\dot{y} = y^2 - 2\Lambda \tag{D.21}$$

whose solution is

$$y = \sqrt{2\Lambda}\tanh[\sqrt{2\Lambda}(t - t_0)] \tag{D.22}$$

Since $y^2 + A^2 = 2\Lambda$, I obtain the final result for the field:

$$A^2 = 2\Lambda\left\{1 - \tanh^2[\sqrt{2\Lambda}(t - t_0)]\right\} = 2\Lambda\,\mathrm{sech}^2[\sqrt{2\Lambda}(t - t_0)] \tag{D.23}$$

Appendix E

Jackiw States

E.1 Bessel Functions

In this appendix I will need some properties of the modified Bessel functions. In the following equalities x is the argument of the Bessel functions unless otherwise specified, and the sums run from $-\infty$ to $+\infty$:

$$
\begin{aligned}
\sum t^n I_n &= \exp\left\{\frac{x}{2}\left(t+\frac{1}{t}\right)\right\} && \text{(E.1)}\\
I_{n-1} - I_{n+1} &= \frac{2n}{x} I_n && \text{(E.2)}\\
I_{n-1} + I_{n+1} &= 2\frac{d}{dx} I_n && \text{(E.3)}\\
I_{-n} &= I_n && \text{(E.4)}\\
\sum_k I_k(x) I_{m-k}(y) &= I_m(x+y) && \text{(E.5)}
\end{aligned}
$$

In particular from (E.2) follows

$$I_2 = I_0 - \frac{2}{x} I_1 \tag{E.6}$$

It is then possible to derive the following equalities:

$$
\begin{aligned}
\sum I_{n-\lambda} &= e^x && \text{(E.7)}\\
\sum I_{n-\lambda}^2 &= I_0(2x) && \text{(E.8)}\\
\sum n\, I_{n-\lambda} &= \sum n\, I_n + \lambda \sum I_n = \lambda e^x && \text{(E.9)}\\
\sum n\, I_{n-\lambda}^2 &= \lambda\, I_0(2x) && \text{(E.10)}\\
\sum n^2 I_{n-\lambda} &= \left(x+\lambda^2\right) e^x && \text{(E.11)}
\end{aligned}
$$

$$
\begin{aligned}
\sum n^2 I_{n-\lambda}^2 &= \sum \left(n^2 + 2n\lambda + \lambda^2\right) I_n^2 = \sum (n\, I_n)^2 + \lambda^2 I_0(2x) \\
&= \sum \left(\frac{x}{2}(I_{n-1} - I_{n+1})\right)^2 + \lambda^2 I_0(2x) \\
&= \frac{x^2}{4} \sum \left(I_{n-1}^2 + I_{n+1}^2 - 2I_{n-1}I_{n+1}\right) + \lambda^2 I_0(2x) \\
&= \frac{x^2}{2}[1 - I_2(2x)] + \lambda^2 I_0(2x) \\
&= I_0(2x) \cdot \left\{ \frac{x}{2} \frac{I_1(2x)}{I_0(2x)} + \lambda^2 \right\} \qquad \text{(E.12)} \\
\sum n\, I_{n-a} I_{n-b} &= \left(\frac{a+b}{2}\right) I_{a-b}(2x) \qquad \text{(E.13)}
\end{aligned}
$$

where I have made some intermediate steps more explicit.

E.2 Minimum Uncertainty

The definition (4.61) of the Jackiw states is

$$
\begin{aligned}
|\psi_\lambda(\gamma)\rangle &= \frac{1}{\sqrt{I_0(2\gamma)}} \sum_n I_{n-\lambda}(\gamma)|n\rangle \qquad \text{(E.14)} \\
p|n\rangle &= n|n\rangle \qquad \text{(E.15)}
\end{aligned}
$$

For simplicity I drop the label γ

$$
|\psi_\lambda\rangle \equiv |\psi_\lambda(\gamma)\rangle
$$

Notice that these states are not orthogonal:

$$
\langle\psi_\lambda|\psi_\mu\rangle = \frac{1}{I_0(2\gamma)} \sum_n I_{n-\lambda}(\gamma) I_{n-\mu}(\gamma) = \frac{I_{\lambda-\mu}(2\gamma)}{I_0(2\gamma)} \qquad \text{(E.16)}
$$

To show that the minimum uncertainty equality

$$
\Delta p \cdot \Delta \sin\theta = \frac{\langle\cos\theta\rangle}{2} \qquad \text{(E.17)}
$$

holds for these states, I need to calculate $\langle\sin^2\theta\rangle$, $\langle p^2\rangle$ and $\langle\cos\theta\rangle$, since I already know that $\langle p\rangle = \lambda$ and $\langle\sin\theta\rangle = 0$.

First of all I notice that

$$
e^{\pm i\theta}|\psi_\lambda\rangle = \frac{1}{\sqrt{I_0(2\gamma)}} \sum_n I_{n-\lambda}(\gamma)|n \pm 1\rangle = |\psi_{\lambda\pm 1}\rangle \qquad \text{(E.18)}
$$

Thus

$$\langle \cos\theta \rangle = B = \langle \psi_\lambda | e^{-i\theta} | \psi_\lambda \rangle = \frac{I_1(2\gamma)}{I_0(2\gamma)}$$

where I have used (E.16) and (E.18) together. In the same way by using the identity $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$ I get

$$\langle \sin^2\theta \rangle = -\frac{\langle \psi_\lambda | \psi_{\lambda+2} \rangle - 2\langle \psi_\lambda | \psi_\lambda \rangle + \langle \psi_\lambda | \psi_{\lambda-2} \rangle}{4} = \frac{1}{2}\left[1 - \frac{I_2(2\gamma)}{I_0(2\gamma)}\right] = \frac{\langle \cos\theta \rangle}{2\gamma} \quad \text{(E.19)}$$

where I also used (E.6) in the last passage. In same way I find

$$\langle p^2 \rangle = \frac{1}{I_0(2\gamma)} \sum_n n^2 I_{n-\lambda}^2(\gamma) = \frac{\gamma \langle \cos\theta \rangle}{2} + \lambda^2 \quad \text{(E.20)}$$

These are all the expectation values that I need:

$$\begin{aligned} (\Delta \sin\theta)^2 &= \langle \sin^2\theta \rangle - \langle \sin\theta \rangle^2 = \frac{\langle \cos\theta \rangle}{2\gamma} && \text{(E.21)} \\ (\Delta p)^2 &= \langle p^2 \rangle - \langle p \rangle^2 = \frac{\gamma \langle \cos\theta \rangle}{2} && \text{(E.22)} \end{aligned}$$

and thus

$$\Delta p \cdot \Delta \sin\theta = \frac{\langle \cos\theta \rangle}{2} \quad \text{(E.23)}$$