Gauge-Higgs unification with broken flavour symmetry

Dissertation
zur Erlangung des Doktorgrades
des Fachbereichs Physik
der Universität Hamburg

vorgelegt von
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Hamburg 2007

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Datum der Disputation:	18. Mai 2007
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Abstract

We study a five-dimensional Gauge-Higgs unification model on the orbifold $S^1/\mathbb{Z}_2$ based on the extended standard model (SM) gauge group $SU(2)_L \times U(1)_Y \times SO(3)_F$. The group $SO(3)_F$ is treated as a chiral gauged flavour symmetry. Electroweak-, flavour- and Higgs interactions are unified in one single gauge group $SU(7)$. The unified gauge group $SU(7)$ is broken down to $SU(2)_L \times U(1)_Y \times SO(3)_F$ by orbifolding and imposing Dirichlet and Neumann boundary conditions. The compactification scale of the theory is $\mathcal{O}(1)$ TeV. Furthermore, the orbifold $S^1/\mathbb{Z}_2$ is put on a lattice. This setting gives a well-defined staring point for renormalisation group (RG) transformations. As a result of the RG-flow, the bulk is integrated out and the extra dimension will consist of only two points: the orbifold fixed points. The model obtained this way is called an effective bilayered transverse lattice model. Parallel transporters (PT) in the extra dimension become nonunitary as a result of the blockspin transformations. In addition, a Higgs potential $V(\Phi)$ emerges naturally. The PTs can be written as a product $e^{A_y} e^{\eta} e^{A_y}$ of unitary factors e^{A_y} and a selfadjoint factor e^{η}. The reduction $\mathbf{48} \to \mathbf{35} + \mathbf{6} + \bar{\mathbf{6}} + \mathbf{1}$ of the adjoint representation of $SU(7)$ with respect to $SU(6) \supset SU(2)_L \times U(1)_Y \times SO(3)_F$ leads to three $SU(2)_L$ Higgs doublets: one for the first, one for the second and one for the third generation. Their zero modes serve as a substitute for the SM Higgs. When the extended SM gauge group $SU(2)_L \times U(1)_Y \times SO(3)_F$ is spontaneously broken down to $U(1)_{em}$, an exponential gauge boson mass splitting occurs naturally. At a first step $SU(2)_L \times U(1)_Y \times SO(3)_F$ is broken to $SU(2)_L \times U(1)_Y$ by VEVs for the selfadjoint factor e^{η}. This breaking leads to masses of flavour changing $SO(3)_F$ gauge bosons much above the compactification scale. Such a behaviour has no counterpart within the customary approximation scheme of an ordinary orbifold theory. This way tree-level flavour-changing-neutral-currents are naturally suppressed. In a second step the electroweak gauge group $SU(2)_L \times U(1)_Y$ is broken to $U(1)_{em}$ by VEVs for the unitary factors e^{A_y} at the electroweak scale. This breaking is equivalent to a Wilson line breaking. Making some simplifying assumptions we also calculate fermion masses and CKM mixing angles. As for the gauge bosons an exponential fermion mass splitting occurs naturally. Fermion masses and mixing angles are determined by the VEVs for e^{η} and e^{A_y} of PTs for quarks and leptons. The model predicts a large Higgs sector consisting of altogether 30 Higgs particles. The model in its simplest form also predicts the (too small) weak mixing angle $\theta_W = 0.125$.

Zusammenfassung

Wir untersuchen ein fünfdimenisonales Eich-Higgs Vereinigungsmodell auf der Orbifold $S^1/\mathbb{Z}_2$ basierend auf der erweiterten Standardmodell (SM) Eichgruppe $SU(2)_L \times U(1)_Y \times SO(3)_F$. Die Gruppe $SO(3)_F$ wird behandelt als chirale geeichte Flavoursymmetrie. Elektroschwache-, Flavour- und Higgswechselwirkungen sind in einer einzigen Eichgruppe $SU(7)$ vereinigt. Die Vereinigungsgruppe $SU(7)$ wird durch Orbifolding und Dirichlet- und Neumannrandbedingungen auf $SU(2)_L \times U(1)_Y \times SO(3)_F$ gebrochen. Die Kompaktifizierungsskala der Theorie ist $\mathcal{O}(1)$ TeV. Weiterhin setzen wir die Orbifold $S^1/\mathbb{Z}_2$ auf ein Gitter. Dieser Rahmen gibt einen wohldefinierten Startpunkt für die Betrachtung von Renormierungsgruppentransformationen. Als Ergebnis des Renormierungsgruppenflusses wird der Bulk ausintegriert und die Extradimension besteht aus nur zwei Punkten: Die Fixpunkte der Orbifold. Wir nennen das auf diese Weise erhaltene Modell ein effektives, transverses Zweischichtmodell. Als ein Ergebnis Blockspintransformationen werden Paralleltransporter (PT) in der Extradimension nichtunitär. Zusätzlich entsteht ein Higgspotential auf natürliche Art und Weise. Die PT können geschrieben werden als ein Produkt $e^{A_y} e^{\eta} e^{A_y}$ von unitären Faktoren e^{A_y} und einem selbstadjungierten Faktor e^{η}. Die Reduktion $\mathbf{48} \rightarrow \mathbf{36} + \mathbf{6} + \bar{\mathbf{6}} + \mathbf{1}$ der adjungierten Darstellung von $SU(7)$ bezüglich $SU(6)$ führt auf drei $SU(2)_L$ Higgsdoublets: Eines für die erste, eines für die zweite und eines für die dritte Generation. Ihre Nullmoden dienen als Ersatz für das SM Higgs. Wenn die erweiterte SM Eichgruppe $SU(2)_L \times U(1)_Y \times SO(3)_F$ spontan zu $U(1)_{em}$ gebrochen wird, entsteht eine exponentielle Aufspaltung der Eichbosonenmassen auf auf natürliche Art und Weise. Dies führt auf Flavoureichbosonen mit Massen weit oberhalb der Kompaktifizierungsskala. Solch ein Verhalten hat keine Entsprechung innerhalb der herkömmlichen Näherungen einer Orbifoldtheorie. Flavourveränderne neutrale Ströme sind auf natürliche Art und Weise unterdrückt. Die elektroschwache Eichgruppe $SU(2)_L \times U(1)_Y$ wird durch Vakuumerwartungswerte für die unitären Faktoren e^{A_y} bei der elektroschwachen Brechungsskala auf $U(1)_{em}$ gebrochen. Ausserdem berechnen wir unter vereinfachenden Annahmen Fermionenmassen und die CKM Matrix. Wie für Eichbosonen, so ersteht auch für Fermionen eine exponentielle Massenaufspaltung. Fermionenmassen und Mischungswinkel sind festgelegt durch Vakuumerwartungswerte für e^{η} und e^{A_y} von PTn für Quarks und Leptonen. Das Modell sagt insgesamt 30 Higgsteilchen voraus. In seine einfachsten Version sagt das Modell den (zu kleinen) schwachen Mischungswinkel $\theta_W = 0.125$ voraus.

Contents

Chapter 1

Introduction

During the last ten years much attention has been paid to gauge theories in higher dimensions. One of the strongest motivations for extra dimensions is based on the very attractive idea that gauge and Higgs fields can be unified in higher dimensions [61, 34]. Gauge bosons and Higgs fields arise from the four-dimensional and extra components of higher-dimensional gauge fields, respectively. This scenario is called Gauge-Higgs unification [32, 2, 17, 11, 73, 54, 29]. The gauge group in this class of models must be larger than the Standard model (SM) gauge group in order to obtain Higgs fields which transform according to the fundamental representation of $SU(2)_L$. The larger amount of gauge symmetry can be reduced to the SM one by compactifying the extra dimensions on an orbifold. Orbifolding [36, 69] is a technique used to break a gauge group without the use of Higgs fields. It has many applications not only in Gauge-Higgs unification models but also in GUT breaking [35, 7].

The SM is extremely successful in reproducing all the available data up to currently accessible energies. However, it has serious unsolved problems. One of the biggest problems is the stability of the electroweak scale against quadratically divergent corrections to the Higgs mass. This problem, called hierarchy problem, suggests the presence of new physics at the TeV scale [1, 18, 3]. In Gauge-Higgs unification models tree level Higgs masses are forbidden by higher-dimensional gauge invariance. For infinite large extra dimensions the masslessness of Higgs fields should hold to any order of perturbation theory. However, for compact extra dimensions radiative corrections generate finite mass terms for the Higgs $\sim 1/R$, where $1/R$ is the compactification scale of the theory. In Gauge-Higgs unification models the compactification scale is usually set to $\mathcal{O}(1)$ TeV. This way Gauge-Higgs unification models on orbifolds give a solution for the hierarchy problem. Electroweak symmetry breaking occurs radiatively in this class of models and is equivalent to a Wilson line symmetry breaking [76, 48] or Hosotani breaking [42, 41, 43]. Matter fields can be introduced either as bulk fields [11] in representations of the unified gauge group or as boundary fields [17] localised at the orbifold fixed points where the unified gauge group is broken to its subgroup.

Another central problem of the SM is the arbitrariness of the Yukawa couplings and the related problem of the strength of the CKM (and PMNS) matrix elements. This is called the flavour problem: The question why there are three families of quarks and leptons in the SM and how they get their masses and mixing angles. In the literature, there are many postulated forms of Yukawa matrices [22, 21, 23]. In order to understand their origin one can try to apply a family symmetry G_f connecting different generations. Some candidates for a family symmetry group are continuous groups like $U(1)$ [49], $SU(2)$ [13, 55], $SU(3)$ [52] or $SO(3)$ [27, 78, 51] or discrete groups like S_3 [33], S_4 [59] or A_4 [58]. The groups $SU(2)$, $SO(3)$ and $SU(3)$ as well as S_4 and A_4 have the advantage over the groups $U(1)$ and S_3 that they have irreducible three dimensional representations into which the three families of the SM can fit. If one adds to the SM gauge group $G_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y$ a gauged flavour group G_f, e.g. $SU(3)$ or $SO(3)$, one is faced with the problem that the latter leads to flavour-changing-neutral-currents (FCNC). However, FCNC are highly suppressed in the SM due to the GIM-mechanism. There are experimental lower bounds on the masses of such flavour gauge bosons of $\mathcal{O}(10^3) - \mathcal{O}(10^5)$ TeV, which is much above the electroweak breaking scale (but far below the GUT scale). Hence a Gauge-Higgs unification model, which also includes a gauged family symmetry, with a compactification scale $\mathcal{O}(1)$ TeV will in general fail [62], because it leads to unsuppressed FCNC. The reason is that flavour gauge bosons in this scenario will get masses at most of the order of the compactification scale, i.e. $\mathcal{O}(1)$ TeV.

A possible solution to this problem is to built a Gauge-Higgs unification model with the help of nonunitary parallel transporters (PTs). Gauge theories with nonunitary PTs were first examined in [60, 57]. They are based on the idea to abandon unitarity of PTs. In this class of theories PTs are no longer elements of a (unitary [1]) gauge group G but are rather elements of a holonomy group H. The holonomy group H is typically noncompact and larger than the unitary gauge group one has started with. Nonunitary PTs occur naturally in effective theories as a result of the renormalisation group (RG) flow [57]. One starts in a fundamental theory with conventional (unitary) PTs. Blockspin transformations will in general lead to nonunitary PTs. The most exciting property of gauge theories with nonunitary PTs is that an exponential mass hierarchy appears naturally when the local gauge symmetry is spontaneously broken by a Higgs mechanism. In [56] it has been shown that an exponential flavour mass splitting for quarks can be obtained this way.

In this thesis we show that also exponential (flavour) gauge bosons masses can be obtained when the PTs in the extra dimension become nonunitary. This opens up the possibility of suppressing tree-level FCNC by large flavour gauge boson masses.

[1] i.e. a compact gauge group whose finite dimensional representations are unitary.

We will present a Gauge-Higgs unification model, which includes a gauged flavour symmetry, with nonunitary PTs in the extra dimension. It will be consistent with existing experimental constraints on FCNC. The compactification scale of the theory is $\mathcal{O}(1)$ TeV.

The thesis is organised as follows. In chapter 2 we review orbifolds [36, 69] in one extra dimension. For this analysis, we will refer to the space group $\mathbb{D}_\infty$ [64]. In comparison with the more ad hoc definitions in the literature, the definition of orbifolds in terms of space groups is attractive. The reason is that all properties of the orbifold, in particular the orbifold space-time and the various relations the projection matrices and twist matrices have to fulfil, can be derived directly from the defining space group. Furthermore, we review the issue of gauge symmetry breaking [37] through orbifolding and consider also familiar orbifold constructions in orbifold GUTs. We will work out the Fourier mode expansions and zero modes on the orbifold $S^1/\mathbb{Z}_2$, which will be useful for the topics discussed in chapter 3. In addition, we review continuous Wilson line breaking, also know as Hosotani breaking [43, 28].

In chapter 3 we describe how an effective transverse lattice model can be obtained from an ordinary $S^1/\mathbb{Z}_2$ orbifold model. We start with the five-dimensional space-time $M^4 \times S^1/\mathbb{Z}_2$ where M^4 is the four-dimensional Minkowski space-time and $S^1/\mathbb{Z}_2$ is the orbifold. Furthermore we put the orbifold $S^1/\mathbb{Z}_2$ on a lattice. Hence the four-dimensional Minkowski space-time will remain continuous and only the extra dimension is latticized. Such a scenario is known as a transverse lattice and it occurs naturally in deconstruction theories [39, 5, 6]. This setting gives a well-defined starting point for RG transformations. Starting with this latticized extra dimension one can calculate the RG-flow. The endpoint of the RG flow will be an extra dimension, which consists of only two points: the two orbifold fixed points. The bulk is completely integrated out. We call the model obtained this way an effective bilayered transverse lattice model (eBTLM). The PTs Φ in the extra dimension from one orbifold fixed point to the other will be nonunitary as a result of the blockspin transformations. They can be interpreted as Higgs fields. When Φ becomes nonunitary, a Higgs potential $V(\Phi)$ naturally emerges. We will discuss in detail the physical interpretation of an eBTLM. It will turn out that for trivial minimum of the Higgs potential and trivial orbifold projection an eBTLM equals an ordinary $S^1/\mathbb{Z}_2$ orbifold model with trivial orbifold projection, if one truncates the Fourier mode expansion for all fields in the $S^1/\mathbb{Z}_2$ orbifold model at the first Kaluza-Klein mode. In order to handle also non-trivial minima of the Higgs potential and non-trivial orbifold projections we formulate orbifold conditions for nonunitary PTs Φ and consider spontaneous symmetry breaking. As an application, we analyse in detail an eBTLM based on the (flavour) gauge group $SU(2)$. The most exciting result is that exponential gauge boson masses can occur for some of the first excited *and* the zero mode gauge bosons, when the gauge group $SU(2)$ is broken spontaneously. This behaviour has no counterpart within the customary approximation scheme of an ordinary orbifold theory.

In chapter 4 we present a realistic Gauge-Higgs unification model, which includes a chiral gauged $SO(3)_F$ flavour symmetry. This model is based on the gauge group $SU(7)$. The gauge group $SU(7)$ unifies electroweak-, flavour- and Higgs interactions. Colour will be ignored. As an intermediate step the model also unifies weak- and flavour interactions in the gauge group $SU(6)_L \subset SU(7)$. Zero modes of the extra-dimensional component of the five-dimensional gauge fields, transforming according to the fundamental representation of $SU(2)_L$ and carrying the hypercharge $\frac{1}{2}$, will serve as a substitute for the SM Higgs. The theory will include three $SU(2)_L$ Higgs doublets, one for the first, one for the second and one for the third generation. They generate the unitary part of the nonunitary bulk parallel transporter Φ. We break $SU(7)$ again down to $SU(6)_L \times U(1)_Y$ by orbifolding. The gauge symmetry breaking $SU(6)_L \times U(1)_Y \rightarrow SU(2)_L \times U(1)_Y \times SO(3)_F$ can be achieved by demanding additional Dirichlet- and Neumann boundary conditions for the $SU(6) \times U(1)_Y$ gauge fields. When spontaneous symmetry breaking occurs, the $SO(3)_F$ flavour symmetry is broken by vacuum expectation values (VEVs) for the selfadjoint part of Φ. This way the flavour gauge bosons can receive very large masses in comparison to the compactification scale $1/R = \mathcal{O}(1)$ TeV. Hence tree-level FCNC are naturally suppressed due to the large flavour gauge boson masses. The electroweak gauge symmetry $SU(2)_L \times U(1)_Y$ is broken to $U(1)_{em}$ by VEVs for the three $SU(2)_L$ Higgs doublets. We calculate all gauge boson masses in the model in terms of the minimum Φ_{min} of the Higgs potential $V(\Phi)$. The model will also make a prediction for the weak mixing angle θ_W.

In chapter 5 we will calculate the fermion masses and the CKM mixing matrix in the $SU(7)$ model under some simplifying assumptions. We assume that nonunitary parallel transporters for gauge fields, quarks and leptons are different.

In chapter 6 will draw our conclusions and discuss possible extensions of the $SU(7)$ model.

Chapter 2

Orbifolds in one extra dimension, Fourier mode expansion and the Hosotani mechanism

In this chapter we review orbifolds [36, 37, 69, 72] and gauge symmetry breaking through orbifolding in one extra dimension. In contrast to the literature, we will define orbifolds in terms of one-dimensional space groups [64]. The definition of orbifolds in terms of space groups is attractive since all properties of the orbifold can be derived directly from the defining space group. Furthermore we will work out the Fourier mode expansions and zero modes on the orbifold $S^1/\mathbb{Z}_2$ which will be useful for the topics discussed in chapter 3. In addition, we review continuous Wilson line breaking also know as Hosotani breaking [43, 28, 46, 45]. In the following section we sketch the basic ideas [36] of orbifolding.

2.1 The meaning of orbifolding

We consider a quantum field theory (QFT) with gauge group G in $D = d + 4$ dimensions, where d denote the number of extra dimensions. The QFT is defined on $M = M^4 \times C$, where M^4 is the four-dimensional Minkowski spacetime and C is a smooth manifold. Let

$$x^M = (x^\mu, y^m) \quad \mu = 0, \ldots, 3 \quad m = 1, \ldots, d \tag{2.1}$$

denote the coordinates of the D-dimensional space, where x_μ and y^m are the coordinates on M^4 and C, respectively.

We suppose that both the manifold C and the QFT possess a symmetry under a discrete group $\mathcal{K}$, i.e.

1. $\mathcal{K}$ acts on the manifold C as

$$\mathcal{K} : y \to \tau_k(y) \tag{2.2}$$

where $y = (y^m)$ and τ_k constitute a representation of $\mathcal{K}$ on C.

2. $\mathcal{K}$ acts on the field space as

$$\mathcal{K} : \Phi_{(i)} \to P_{k\,(ij)} \Phi_{(j)} \ , \tag{2.3}$$

where Φ is a vector containing all fields of the theory and P_k is a matrix representation of $\mathcal{K}$ on the field space.

With the symmetry group $\mathcal{K}$ at hand we can now construct the space $C/\mathcal{K}$ by identifying points y and $\tau_k(y)$ that belong to the same orbit

$$y \equiv \tau_k(y) \ . \tag{2.4}$$

According to the action of $\mathcal{K}$ on C there are two possibilities

1. $\mathcal{K}$ acts freely on C, i.e.

$$\tau_k(y) \neq y \quad \forall y \in C \ , \forall k \in \mathcal{K} \ , k \neq 1 \ . \tag{2.5}$$

This means that non-trivial elements of $\mathcal{K}$ move all points of C. The space $C/\mathcal{K}$ is then again a smooth manifold.

2. $\mathcal{K}$ acts non freely on C, i.e. the action of $\mathcal{K}$ on C has fixed points

$$\tau_k(y) = y \quad \text{for some } y \in C \quad k \neq 1 \ . \tag{2.6}$$

The resulting space $C/\mathcal{K}$ is not a smooth manifold but it has singularities at the fixed points. Such a space is known as an *orbifold*.

We set $C = \mathbb{R}^d$ and consider the quotient space $\mathbb{R}^d/\mathcal{K}$. Note that for $\mathcal{K}$ we cannot choose any arbitrary discrete group. Instead of that $\mathcal{K}$ is restricted to be a d-dimensional space group. A d-dimensional space group is defined as a discrete group of isometries of $\mathbb{R}^d$.

Definition 1 (Orbifold) *Let $\mathcal{K}$ be a space group in d-dimensions acting* **non freely** *on $\mathbb{R}^d$. We define an orbifold in d extra dimensions to be the quotient space*

$$\mathbb{R}^d/\mathcal{K} \ . \tag{2.7}$$

Remarks: i) Space groups are also known as crystallographic groups and their classification is known for dimensions $d \leq 6$.
ii) Since orbifolds are defined as quotient spaces $\mathbb{R}^d/\mathcal{K}$, their classification follows directly from the classification of the space groups $\mathcal{K}$.

Recall that $\mathcal{K}$ is assumed to be a symmetry of both $\mathbb{R}^d$ and the QFT. We declare that only field configurations invariant under the actions (2.2) and (2.3) are physical. This means that we demand

$$\Phi_{(i)}(x^\mu, \tau_k(y)) = P_{k\,(ij)}\Phi_{(j)}(x^\mu, y) \ . \tag{2.8}$$

In general the action of $\mathcal{K}$ on the fields can make use of all symmetries of the QFT. This means that P_k can involve gauge transformations, discrete parity transformations and in the supersymmetric case, R-symmetry transformations [37]. In this thesis we consider the case where P_k involves gauge transformations and restrict ourselves to orbifolds in one extra dimension, i.e. we take $d = 1$ in (2.7).

2.2 One-dimensional orbifolds

Let us first consider all possible space groups in one dimension and as a start do not care whether they act freely on $\mathbb{R}$ or not. The real line $\mathbb{R}$ has two possible isometries, the translation t and the π-rotation r. The one-dimensional space groups are therefore [64]

$$\begin{aligned} \mathbb{Z} &= \langle t \rangle \ , \\ \mathbb{D}_\infty &= \langle t, r \mid r^2 = 1, (tr)^2 = 1 \rangle \supseteq \mathbb{Z}, \mathbb{Z}_2 \ , \end{aligned} \tag{2.9}$$

where $\mathbb{Z}_2 = \langle r \mid r^2 = 1 \rangle$. The space groups (2.9) are defined in a purely algebraic way, i.e. initially we do not specify a particular representation of them. Instead of that we define a set of generators and list the relations among them. This way the space groups (2.9) are uniquely defined. Take for example the space group $\mathbb{D}_\infty$. It is generated by a translation t and a π-rotation r. The generators r and t fulfil the relations $r^2 = 1$ and $(tr)^2 = 1$. It is important and we will make use of this fact later that the choice of the generators in (2.9) is not unique [64]. For instance the space group $\mathbb{D}_\infty$ can be defined equally in terms of two π-rotations

$$\mathbb{D}_\infty = \langle r, r' \mid r^2 = r'^2 = 1 \rangle \ , \tag{2.10}$$

with $r' = tr$. Note that $rr' \neq r'r$.
Remark: $\mathbb{D}_\infty$ may have representations P, P' of r, r' on the field space, which are not faithful. For instance, one can have representations P, P' fulfilling $PP' = P'P$. In fact we will consider this possibility later in section 2.4.

For each $\mathcal{K}$ let $\mathcal{K}'$ be the largest subgroup of $\mathcal{K}$ that does not include translations. Thus we can rewrite (2.7) for $d = 1$ as

$$\mathbb{R}/\mathcal{K} = S^1/\mathcal{K}' \, , \tag{2.11}$$

where S^1 is the circle. The circle S^1 is the quotient space $\mathbb{R}/\mathbb{Z}$ and it is constructed by identifying the points

$$y \to y + 2\pi R \, , \tag{2.12}$$

on $\mathbb{R}$. Here y denotes the coordinate on $\mathbb{R}$ and R is the compactification radius, i.e. the radius of S^1. In (2.12) we have given a particular representation of t on $\mathbb{R}$. Since in one dimension there exist only two space groups, namely $\mathbb{Z}$ and $\mathbb{D}_\infty$, we arrive at the two one-dimensional compact spaces

$$S^1 = \mathbb{R}/\mathbb{Z} \quad , \quad S^1/\mathbb{Z}_2 = \mathbb{R}/\mathbb{D}_\infty \, . \tag{2.13}$$

We will see later that only $S^1/\mathbb{Z}_2$ has fixed points and is therefore the only one-dimensional orbifold.

2.2.1 Gauge symmetry breaking through orbifolding

We consider now the case where $\mathcal{K}$ acts on the space of gauge fields. If $\mathcal{K}$ is a symmetry of the gauge action P_k will act as a gauge transformation. To be more precise, consider a five-dimensional gauge field $A_M = A_M^A T^A$ where T^A are the generators of G, $M \in (\mu, y)$ and $A = 1, \ldots, \dim(G)$. Let the generators T^A be normalised such that $\mathrm{tr}\left(T^A T^B\right) = \frac{1}{2}\delta_{AB}$. The five-dimensional Yang-Mills action reads

$$S_{5D} = \int d^4x dy \; \mathrm{tr}\left(-\frac{1}{2} F_{MN} F^{MN}\right) \, , \tag{2.14}$$

where $F_{MN} = F_{MN}^A T^A$, $F_{MN}^A = \partial_M A_N^A - \partial_N A_M^A + g_5 f^{ABC} A_M^B A_N^C$, $M, N \in (\mu, y)$, $\left[T^A, T^B\right] = i f^{ABC} T^C$ and g_5 denote the five-dimensional gauge coupling constant. The T^A are considered here as a matrix representation of the generators of G. Under a gauge transformation $\Omega(x^\mu, y) \in G$ on the covering space $\mathbb{R}$ the five-dimensional gauge field $A_M(x^\mu, y)$ transforms as

$$A_M(x^\mu, y) \to A'_M(x^\mu, y) = \Omega(x^\mu, y) A_M(x^\mu, y) \Omega(x^\mu, y)^{-1} - \frac{i}{g} \Omega(x^\mu, y) \partial_M \Omega(x^\mu, y)^{-1} \, . \tag{2.15}$$

We represent r and t on $\mathbb{R}$ by

$$y \to -y \, , \tag{2.16}$$

$$y \to y + 2\pi R \, , \tag{2.17}$$

respectively, and on $\mathfrak{g} = \text{Lie}\, G$ by

$$A_M(x^\mu, y) \to P\ A_M(x^\mu, y)\ P^{-1}\ , \tag{2.18}$$
$$A_M(x^\mu, y) \to T\ A_M(x^\mu, y)\ T^{-1}\ , \tag{2.19}$$

respectively. Note that we have restricted here to the case where the action of t and r on $\mathfrak{g} = \text{Lie}\, G$ can be written as an inner automorphism. According to (2.8) we demand [1]

$$A_\mu(x^\mu, -y) = P\ A_\mu(x^\mu, y)\ P^{-1} \tag{2.20}$$
$$A_y(x^\mu, -y) = -P\ A_y(x^\mu, y)\ P^{-1} \tag{2.21}$$
$$A_M(x^\mu, y + 2\pi R) = T\ A_M(x^\mu, y)\ T^{-1}\ . \tag{2.22}$$

It follows that

$$F_{\mu\nu}(x^\mu, -y) = P\ F_{\mu\nu}(x^\mu, y)\ P^{-1} \tag{2.23}$$
$$F_{\mu y}(x^\mu, -y) = -P\ F_{\mu y}(x^\mu, y)\ P^{-1} \tag{2.24}$$
$$F_{MN}(x^\mu, y + 2\pi R) = T\ F_{MN}(x^\mu, y)\ T^{-1}\ . \tag{2.25}$$

Thus (2.14) is invariant under the action of $\mathbb{D}_\infty$. The conditions (2.20), (2.21) are known as boundary conditions and the condition (2.22) is known as periodicity condition.

Let us discuss the issue of gauge symmetry breaking due to the boundary condition (2.20) and the periodicity condition (2.22) for the four-dimensional components $A_\mu(x^\mu, y)$ of the five-dimensional gauge field $A_M(x^\mu, y)$. First (2.20) and (2.22) can alternatively be understood in terms of local gauge symmetry breaking at the various fixed points of the orbifold. This reinterpretation comes out if one takes into

[1] Note that the minus sign in (2.21) is needed in order to maintain the gauge covariance for $F_{\mu y}$, i.e.

$$\begin{aligned}
F_{\mu y}(x^\mu, -y) &= \partial_\mu A_y(x^\mu, -y) - \partial_y A_\mu(x^\mu, -y) - ig_5 \left[A_\mu(x^\mu, -y), A_y(x^\mu, -y)\right] \\
&= -\partial_\mu (P A_y(x^\mu, y) P^{-1}) - \partial_y (P A_\mu(x^\mu, y) P^{-1}) + ig_5 P \left[A_\mu(x^\mu, y), A_y(x^\mu, y)\right] P^{-1} \\
&= -P \left(\partial_\mu A_y(x^\mu, y) - \partial_y A_\mu(x^\mu, y) - ig_5 \left[A_\mu(x^\mu, y), A_y(x^\mu, y)\right]\right) P^{-1} \\
&= -P F_{\mu y}(x^\mu, y) P^{-1}\ .
\end{aligned}$$

If we instead of (2.21) demand

$$A_y(x^\mu, -y) = P\ A_y(x^\mu, y)\ P^{-1}\ ,$$

we get

$$\begin{aligned}
F_{\mu y}(x^\mu, -y) &= \partial_\mu A_y(x^\mu, -y) - \partial_y A_\mu(x^\mu, -y) - ig_5 \left[A_\mu(x^\mu, -y), A_y(x^\mu, -y)\right] \\
&= \partial_\mu (P A_y(x^\mu, y) P^{-1}) - \partial_y (P A_\mu(x^\mu, y) P^{-1}) + ig_5 P \left[A_\mu(x^\mu, y), A_y(x^\mu, y)\right] P^{-1} \\
&= P \left(\partial_\mu A_y(x^\mu, y) + \partial_y A_\mu(x^\mu, y) + ig_5 \left[A_\mu(x^\mu, y), A_y(x^\mu, y)\right]\right) P^{-1}\ .
\end{aligned}$$

account that a generic fixed point $y_i \in \mathbb{R}$ is left fixed by an element $k' \in \mathcal{K}'$ only modulo a suitable translation in the covering space $\mathbb{R}$, i.e.

$$y_i = k'(y_i) + n_i \cdot 2\pi R \, , \tag{2.26}$$

where $n_i \in \mathbb{N}$ depend on the particular fixed point y_i. Thus we conclude that the effective orbifold projection P_i, assigned to the fixed point y_i, is given by

$$P_i = T^{n_i} P \, . \tag{2.27}$$

The boundary condition for the four-dimensional gauge fields at a given fixed point y_i then reads

$$A_\mu(x^\mu, y_i - y) = P_i \; A_\mu(x^\mu, y_i + y) \; P_i^{-1} \, . \tag{2.28}$$

This formula shows explicitly that

- the gauge group G is broken *locally* at the orbifold fixed point y_i to the centraliser of P_i in G
$$H_i = \{g \in G \mid P_i g = g P_i\} \tag{2.29}$$
- away from the fixed points, i.e. in the bulk, the gauge group G remains unbroken.

The *globally* unbroken gauge group H, i.e. the gauge group of the low energy four-dimensional effective theory, is given by the intersection

$$H = \cap_i H_i \, . \tag{2.30}$$

It is remarkable that this reinterpretation follows directly from the fact that the definition of the space group generators t, r in (2.9) is not unique. In fact one can always redefine the generators t and r such that to every fixed point y_i of the orbifold one can assign one generator of the space group. In the next section we will discuss this topic for the orbifold $S^1/\mathbb{Z}_2$.

2.3 The orbifold $S^1/\mathbb{Z}_2$

The orbifold $S^1/\mathbb{Z}_2 = \mathbb{R}/\mathbb{D}_\infty$ is the quotient space of the real line modulo $\mathbb{D}_\infty$. Recall that $\mathbb{D}_\infty$ is defined as

$$\mathbb{D}_\infty = \langle t, r \mid r^2 = 1, (tr)^2 = 1 \rangle \, . \tag{2.31}$$

This space group has two generators, the translation t and the reflection r. We represent t on $\mathbb{R}$ by

$$y \to y + 2\pi R \, . \tag{2.32}$$

Thus we arrive as an intermediate step at the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Figure 2.1 shows the representation of t on $\mathbb{R}$ and the resulting space S^1. Note that t acts freely on $\mathbb{R}$

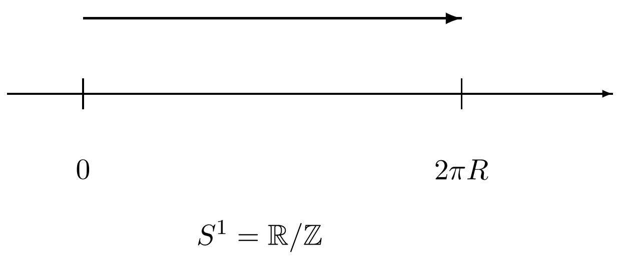


Figure 2.1: Representation of t on $\mathbb{R}$ (thick black arrow) and the resulting space S^1.

and thus S^1 possesses no fixed points. Consequently S^1 is not an orbifold. In order to arrive at the orbifold $S^1/\mathbb{Z}_2$ we represent r on $\mathbb{R}$ by

$$y \to -y \; , \tag{2.33}$$

i.e. we divide the circle S^1 by a $\mathbb{Z}_2$ transformation. Figure 2.2 shows the representation of t and r on $\mathbb{R}$ and the resulting space $S^1/\mathbb{Z}_2$. Due to the definition of $\mathbb{D}_\infty$

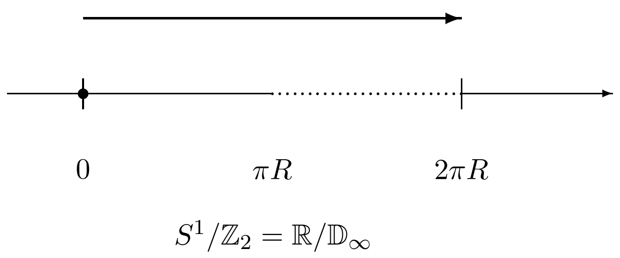


Figure 2.2: Representation of t (thick black arrow) and r (black dot at $y = 0$) on $\mathbb{R}$ and the resulting space $S^1/\mathbb{Z}_2$.

(2.31), the following relations hold

$$r^2 = (tr)^2 = 1 \quad , \quad t = (tr)r \quad , \quad trt = r \; . \tag{2.34}$$

The gauge fields have to fulfil the boundary conditions

$$\begin{aligned} A_\mu(x^\mu, -y) &= P\; A_\mu(x^\mu, y)\; P^{-1} \\ A_y(x^\mu, -y) &= -P\; A_y(x^\mu, y)\; P^{-1} \end{aligned} \tag{2.35}$$

and the periodicity condition

$$A_M(x^\mu, y + 2\pi R) = T\; A_M(x^\mu, y)\; T^{-1} \; . \tag{2.36}$$

The orbifold $S^1/\mathbb{Z}_2$ has two fixed points $y_1 = 0$ and $y_2 = \pi R$, where $y_1 = 0$ is invariant under the group element r

$$y_1 = 0 \xrightarrow{r} 0 = y_1 \tag{2.37}$$

and $y_2 = \pi R$ is invariant under the group element tr

$$y_2 = \pi R \xrightarrow{r} -\pi R \xrightarrow{t} \pi R = y_2 \; . \tag{2.38}$$

This means that in (2.26) we have $n_1 = 0$ and $n_2 = 1$. The corresponding effective projections are therefore

$$P_1 = P \quad , \quad P_2 = TP \; . \tag{2.39}$$

Consequently we can rewrite the orbifold boundary conditions (2.35) and the periodicity condition (2.36) as

$$\begin{aligned} A_\mu(x^\mu, -y) &= P_1 \; A_\mu(x^\mu, y) \; P_1^{-1} \\ A_y(x^\mu, -y) &= -P_1 \; A_y(x^\mu, y) \; P_1^{-1} \; , \end{aligned} \tag{2.40}$$

$$\begin{aligned} A_\mu(x^\mu, \pi R - y) &= P_2 \; A_\mu(x^\mu, \pi R + y) \; P_2^{-1} \\ A_y(x^\mu, \pi R - y) &= -P_2 \; A_y(x^\mu, \pi R + y) \; P_2^{-1} \; . \end{aligned}$$

Due to (2.34), the projection matrices P_1 and P_2 fulfil

$$P_1^2 = P_2^2 = 1 \quad , \quad T = P_2 P_1 \quad , \quad T P_1 T = P_1 \; . \tag{2.41}$$

The resulting physical space $S^1/\mathbb{Z}_2$ is the interval $[0, \pi R]$.

At $y = 0$, the gauge group G is broken to the centraliser of P_1 in G

$$H_1 = \{ g \in G \mid P_1 g = g P_1 \} \tag{2.42}$$

and at $y = \pi R$ is broken to the centraliser of P_2 in G

$$H_2 = \{ g \in G \mid P_2 g = g P_2 \} \; . \tag{2.43}$$

The low energy four-dimensional gauge group is given by the intersection

$$H = H_1 \cap H_2 = \{ g \in G \mid P_1 g = g P_1 \; \wedge P_2 g = g P_2 \} \; . \tag{2.44}$$

It is remarkable that in general

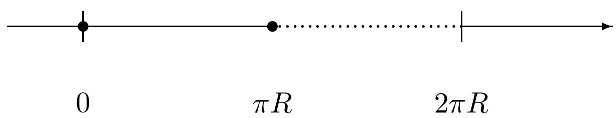
$$[P_1, P_2] \neq 0 \tag{2.45}$$

This allows to reduce the *rank* of $\mathfrak{g}$, i.e. *rank* $\mathfrak{h} <$ *rank* $\mathfrak{g}$, where $\mathfrak{h} = LieH$.

In the last section we have argued that this reinterpretation follows directly from the fact that the definition of the space group generators in $\mathbb{D}_\infty$ is not unique. In fact we can rewrite

$$\mathbb{D}_\infty = \langle r, r' \mid r^2 = r'^2 = 1 \rangle \ , \tag{2.46}$$

with $r' = tr$. Remember that $rr' \neq r'r$. In definition (2.46) the two space group generators r and r' are *directly assigned* to the fixed points $y_1 = 0$ and $y_2 = \pi R$, respectively. Figure 2.3 shows the representation of r and r' on $\mathbb{R}$, and the resulting space $S^1/\mathbb{Z}_2$. The orbifold defined by (2.31) and leading to the boundary condition



$$S^1/\mathbb{Z}_2 = \mathbb{R}/\mathbb{D}_\infty$$

Figure 2.3: Representation of r (black dot at $y = 0$) and r' (black dot at $y = \pi R$) on $\mathbb{R}$ and the resulting space $S^1/\mathbb{Z}_2$.

(2.35) and the periodicity condition (2.36) is known as $S^1/\mathbb{Z}_2$ with twisted boundary conditions.

2.3.1 The orbifold $S^1/\mathbb{Z}_2 \times \mathbb{Z}_2'$

In the literature, especially in orbifold GUTs, one is often faced with the orbifold $S^1/\mathbb{Z}_2 \times \mathbb{Z}_2'$ [35]. It is constructed as follows. The starting point is a circle S^1 of radius R'. We divide S^1 by two $\mathbb{Z}_2$ transformations

$$\mathbb{Z}_2 : \quad y \to -y \quad , \quad \mathbb{Z}_2' : \quad y' \to -y' \ , \tag{2.47}$$

where $y' = y - \pi R'/2$. In this case the resulting physical space is the interval $[0, \pi R'/2]$.

The gauge fields have to fulfil the boundary conditions

$$\begin{aligned} A_\mu(x^\mu, -y) &= P \ A_\mu(x^\mu, y) \ P^{-1} \\ A_y(x^\mu, -y) &= -P \ A_y(x^\mu, y) \ P^{-1} \ , \\ \\ A_\mu(x^\mu, -y') &= P' A_\mu(x^\mu, y') P'^{\,-1} \\ A_y(x^\mu, -y') &= -P' A_y(x^\mu, y') P'^{\,-1} \ . \end{aligned} \tag{2.48}$$

The projection matrices P and P' fulfil

$$P^2 = P'^{\,2} = 1 \; . \tag{2.49}$$

If one compares the resulting physical space $S^1/\mathbb{Z}_2 \times \mathbb{Z}'_2 = [0, \pi R'/2]$ generated by the two reflections (2.47) with the resulting physical space $S^1/\mathbb{Z}_2 = [0, \pi R]$ generated by the translation (2.32) and the reflection (2.33) and the boundary conditions (2.48) and (2.40), one observes that the orbifold $S^1/\mathbb{Z}_2 \times \mathbb{Z}'_2$ is equivalent to the orbifold $S^1/\mathbb{Z}_2$ with twisted boundary conditions if we take

$$R' = 2R \; . \tag{2.50}$$

Therefore we will not distinguish between the orbifold $S^1/\mathbb{Z}_2 \times \mathbb{Z}'_2$ and the orbifold $S^1/\mathbb{Z}_2$ with twisted boundary conditions.

In general, $[P, P'] \neq 0$. However in orbifold GUTs it is assumed [35] that projection matrices P and P' commute

$$[P, P'] = 0 \; . \tag{2.51}$$

This means that is this case the representation P, P' of $\mathbb{D}_\infty$ on the field space is not faithful. Due to (2.51) the *rank* of $\mathfrak{g}$ is not reduced.

2.4 Continuous versus discrete Wilson line breaking

In this section we give an interpretation of the twist matrix T in terms of Wilson lines [72, 30]. Remember that the five-dimensional gauge field $A_M(x)$ has to fulfil the periodicity condition

$$A_M(x^\mu, y + 2\pi R) = T \; A_M(x^\mu, y) \; T^{-1} \; . \tag{2.52}$$

The twist matrix T can always be interpreted as a Wilson line W

$$W = \exp\left(2\pi i g R \langle A_y \rangle\right) \; , \tag{2.53}$$

where $\langle A_y \rangle$ is a constant VEV for $A_y(x^\mu, y)$. However W and therefore $\langle A_y \rangle$ must be compatible with the boundary condition for $A_y(x^\mu, y)$ (2.35). This means that according to (2.34) the orbifold projection P and the Wilson line W has to fulfil the consistency condition

$$(WP)^2 = 1 \; . \tag{2.54}$$

In general W, and therefore $\langle A_y \rangle$, need not to commute with P. To be more precise, suppose that boundary conditions for A_y are given

$$A_y(x^\mu, -y) = -P \; A_y(x^\mu, y) \; P^{-1} \; . \tag{2.55}$$

Three possibilities [72] can occur

1. Let $\{T^a\}$ denote the set of generators of G, which fulfil simultaneously

$$[P, T^a] = 0 \quad , \quad [WP, T^a] = 0 \, . \tag{2.56}$$

Note, that these generators create the four-dimensional unbroken gauge group H due to (2.42) and (2.43). The relations (2.56) imply $[W, T^a] = 0$. Thus (2.53) can be written as

$$W = \exp\left(2\pi i g R \sum_a \langle A_y^a \rangle \, T^a\right) \tag{2.57}$$

i.e. $A_y = \sum_a A_y^a T^a$. The Wilson line (2.57) commutes with every $T^b \in \{T^a\}$

$$[\exp\left(2\pi i g R \sum_a \langle A_y^a \rangle \, T^a\right), T^b] = 0 \, . \tag{2.58}$$

Due to the minus sign in the boundary conditions (2.55) A_y is *odd* under P. Since $[P, T^a] = 0$, W also commutes with P

$$[P, W] = 0 \, . \tag{2.59}$$

Together with (2.54) this yields

$$W^2 = 1 \, . \tag{2.60}$$

Thus $\langle A_y^a \rangle$ in (2.57) can take only special values compatible with (2.60). Therefore the Wilson line constructed from $A_y = A_y^a T^a$ is called a *discrete Wilson line*. Note, that because of (2.59) a discrete Wilson line symmetry breaking preserve the *rank*, i.e. $rank\ \mathfrak{h} = rank\ \mathfrak{g}$ [2].

2. Let $\{T^{\hat{a}}\}$ denote the set of generators of G which fulfil simultaneously

$$\{P, T^{\hat{a}}\} = 0 \quad , \quad \{WP, T^{\hat{a}}\} = 0 \, . \tag{2.61}$$

This implies $[W, T^{\hat{a}}] = 0$. Thus (2.53) can be written as

$$W = \exp\left(2\pi i g R \sum_{\hat{a}} \langle A_y^{\hat{a}} \rangle T^{\hat{a}}\right) \tag{2.62}$$

i.e. $A_y = \sum_{\hat{a}} A_y^{\hat{a}} T^{\hat{a}}$. The Wilson line (2.62) commutes [3] with every $T^{\hat{b}} \in \{T^{\hat{a}}\}$

$$[\exp\left(2\pi i g R \sum_{\hat{a}} \langle A_y^{\hat{a}} \rangle T^{\hat{a}}\right), T^{\hat{b}}] = 0 \, . \tag{2.63}$$

[2]Recall that $\mathfrak{h} = LieH$ and $\mathfrak{g} = LieG$.

[3]Let W be given by (2.62). According to (2.61) we have for any $T^{\hat{b}} \in \{T^{\hat{a}}\}$: $WPT^{\hat{b}} = -WT^{\hat{b}}P \stackrel{!}{=} T^{\hat{b}}WP$. Thus $[W, T^{\hat{b}}] = 0$ for every $T^{\hat{b}}$.

Due to the minus sign in the boundary conditions (2.55) A_y is *even* under P. Since $\{P, T^{\hat{a}}\} = 0$, W does not commute with P

$$[P, W] \neq 0 \, . \tag{2.64}$$

In this case the VEV for A_y can be an arbitrary constant. Therefore, we call the Wilson line constructed from $A_y = \sum_{\hat{a}} A_y^{\hat{a}} T^{\hat{a}}$ a *continuous Wilson line.* Due to (2.64), a continuous Wilson line induces a spontaneous *rank* reducing gauge symmetry breaking, i.e. *rank* $\mathfrak{h} <$ *rank* $\mathfrak{g}$.

3. The remaining generators of G, which are even under one effective projection and odd under the other, can never give rise to a consistent Wilson line W.

Remarks: i) Following the line of thinking of section 2.3.1, the orbifold $S^1/\mathbb{Z}_2$ with continuous Wilson line breaking is equivalent to the orbifold $S^1/\mathbb{Z}_2 \times \mathbb{Z}_2'$ if we allow the orbifold projection P' (2.49) to depend on a continuous parameter. In this case rank reduction is also possible on the orbifold $S^1/\mathbb{Z}_2 \times \mathbb{Z}_2'$.

Example 1 *Let $G = SU(3)$ be the bulk gauge group and let $T^A = \lambda^A$ be the Gell-Mann matrices generating $SU(3)$. We break $G = SU(3)$ down to $H_1 = SU(2) \times U(1)$ at the orbifold fixed point $y_1 = 0$ by choosing*

$$P_1 = \exp(\pi i \lambda_3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{2.65}$$

where H_1 is generated by $\{T^a\}$, $a = 1, 2, 3, 8$ and the coset G/H_1 is generated by $\{T^{\hat{a}}\}$, $\hat{a} = 4, 5, 6, 7$. Note that $P_1 \in G$.

Let us first consider a discrete Wilson line, for example

$$W = \exp(\pi i(\lambda_3 + \sqrt{3}\lambda_8)/2) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \tag{2.66}$$

This Wilson line leads to the breaking $H_1 \to H = U(1) \times U(1)$ generated by T^3, T^8. Alternatively, we can directly assign the projection P_2 to the fixed point $y_2 = \pi R$

$$P_2 = W P_1 = \exp(\pi i(2\lambda_3 + \sqrt{3}\lambda_8)/2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \tag{2.67}$$

Thus G is broken at the orbifold fixed point $y_2 = \pi R$ down to $H_2 = SU(2) \times U(1)$ generated by $\{T^{a'}\}$, $a = 3, 6, 7, 8$. In fact, $H = H_1 \cap H_2$ is generated by T^3, T^8. For $T^a \in \{T^3, T^8\}$ the following relations hold

$$[P_1, T^a] = [W P_1, T^a] = [W, T^a] = 0 \, . \tag{2.68}$$

The projection matrices fulfil

$$P_1^2 = P_2^2 = 1\ . \tag{2.69}$$

In particular we have

$$W^2 = 1\ . \tag{2.70}$$

Let us construct the Wilson line W explicitly. Since $A_y = \sum_a A_y^a T^a = A_y^3 T^3 + A_y^8 T^8$ we have

$$W = \exp(2\pi i g R \langle A_y^3 \rangle T^3 + 2\pi i g R \langle A_y^8 \rangle T^8)\ . \tag{2.71}$$

We can built four different discrete Wilson lines

$$\langle A_y^3 \rangle = 0\ ,\ \langle A_y^8 \rangle = 0\ \rightarrow\ W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\ , \tag{2.72}$$

$$\langle A_y^3 \rangle = \frac{1}{2gR}\ ,\ \langle A_y^8 \rangle = 0\ \rightarrow\ W = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\ , \tag{2.73}$$

$$\langle A_y^3 \rangle = -\frac{1}{2gR}\ ,\ \langle A_y^8 \rangle = \frac{\sqrt{3}}{4gR}\ \rightarrow\ W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\ , \tag{2.74}$$

$$\langle A_y^3 \rangle = \frac{1}{4gR}\ ,\ \langle A_y^8 \rangle = \frac{\sqrt{3}}{4gR}\ \rightarrow\ W = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^4\ . \tag{2.75}$$

We see that the Wilson line (2.66) is obtained from the choice $\langle A_5^3 \rangle = \frac{1}{4gR}$ and $\langle A_5^8 \rangle = \frac{\sqrt{3}}{4gR}$ in (2.71). Note that the orbifold projection P_1 (2.65) commutes with

[4] We stress that is possible to rewrite (2.71) is terms of equivalent sets of generators of $SU(3)$ such that one VEV equals $\frac{1}{2gR}$ while the other equals 0, i.e.

- (2.74) can be rewritten as

$$W = \exp(2\pi i g R \langle A_y^{\eta} \rangle \eta + 2\pi i g R \left\langle A_y^{\eta'} \right\rangle \eta') = \exp(\pi i\ \eta)$$

  where $\eta = \frac{1}{2}\left(\sqrt{3}\lambda_8 - \lambda_3\right)$, $\eta' = \frac{1}{2}\left(\lambda_8 - \sqrt{3}\lambda_3\right)$ and $\langle A_y^{\eta} \rangle = \frac{1}{2gR}$ and $\left\langle A_y^{\eta'} \right\rangle = 0$.

- (2.75) can be rewritten as

$$W = \exp(2\pi i g R \langle A_y^{\rho} \rangle \rho + 2\pi i g R \left\langle A_y^{\rho'} \right\rangle \rho') = \exp(\pi i\ \rho)$$

  where $\rho = \frac{1}{2}\left(\lambda_3 + \sqrt{3}\lambda_8\right)$, $\rho' = \frac{1}{2}\left(\sqrt{3}\lambda_3 + \lambda_8\right)$ and $\langle A_y^{\rho} \rangle = \frac{1}{2gR}$ and $\left\langle A_y^{\rho'} \right\rangle = 0$.

Note that with the definitions for η, η', ρ and ρ' above the three equivalent sets of generators of $SU(3)$ read: $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_5, \lambda_7, \lambda_8\}$, $\{\lambda_1, \lambda_2, \rho, \lambda_4, \lambda_5, \lambda_5, \lambda_7, \rho'\}$ and $\{\lambda_1, \lambda_2, \lambda_4, \lambda_5, \eta, \lambda_6, \lambda_7, \eta'\}$

every discrete Wilson line W. This shows explicitly that discrete Wilson line breaking is rank preserving.

Next we choose a continuous Wilson line, e.g.

$$W(\alpha) = \exp(2\pi i\ \alpha\lambda_7) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\ 2\pi\alpha & \sin\ 2\pi\alpha \\ 0 & -\sin\ 2\pi\alpha & \cos\ 2\pi\alpha \end{pmatrix} . \tag{2.76}$$

Let the parameter α be limited to $0 < \alpha < 1$ but otherwise arbitrary. This Wilson line leads to the breaking $H_1 \to H = U(1)$. Alternatively, we again directly assign the projection P_2 to the fixed point $y_2 = \pi R$

$$P_2(\alpha) = W(\alpha)P_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos\ 2\pi\alpha & \sin\ 2\pi\alpha \\ 0 & \sin\ 2\pi\alpha & \cos\ 2\pi\alpha \end{pmatrix} . \tag{2.77}$$

P_2 now depends on α. For $T^{\hat{a}} \in \{T^{\hat{7}}\}$ the following relations hold

$$\{P_1, T^{\hat{a}}\} = \{WP_1, T^{\hat{a}}\} = [W, T^{\hat{a}}] = 0 . \tag{2.78}$$

Therefore $A_y = \sum_{\hat{a}} A_y^{\hat{a}} T^{\hat{a}} = A_y^{\hat{7}} T^{\hat{7}}$. The projection matrices P_1 and P_2 fulfil

$$P_1^2 = P_2^2 = 1 . \tag{2.79}$$

But now we obviously have

$$W^2 \neq 1 . \tag{2.80}$$

Let us again explicitly construct the Wilson line W. Since $A_y = A_y^{\hat{7}} T^{\hat{7}}$ we have

$$W = \exp(2\pi i g R < A_y^{\hat{7}} > T^{\hat{7}}) = \exp(2\pi i\ \alpha\lambda_7) , \tag{2.81}$$

where

$$< A_y^{\hat{7}} >= \frac{\alpha}{gR} . \tag{2.82}$$

For $0 < \alpha \ll 1$ the VEV for A_y (2.82) can be much smaller than the compactification scale $1/R$.

2.5 Fourier expansion and zero modes on $S^1/\mathbb{Z}_2$

In this section, we discuss the Fourier mode expansions of $A_\mu(x^\mu, y)$ and $A_y(x^\mu, y)$ on the orbifold $S^1/\mathbb{Z}_2$. Recall that the gauge fields have to fulfil the boundary conditions

$$A_\mu(x^\mu, -y) = P\ A_\mu(x^\mu, y)\ P^{-1} \tag{2.83}$$

$$A_y(x^\mu, -y) = -P\ A_y(x^\mu, y)\ P^{-1} . \tag{2.84}$$

and the periodicity condition

$$A_M(x^\mu, y + 2\pi R) = W\ A_M(x^\mu, y)\ W^{-1}\ , \tag{2.85}$$

where W is the corresponding Wilson line. In general, three cases can arise

1. $W = 1$. This means that we admit the trivial periodicity condition, i.e.

$$A_M(x^\mu, y + 2\pi R) = A_M(x^\mu, y)\ . \tag{2.86}$$

The boundary condition (2.83) breaks the bulk gauge group G down to its subgroup H_{y_1}

$$H_{y_1} = \{g \in G \mid Pg = gP\} \tag{2.87}$$

at the fixed point $y_1 = 0$. Let $\{T^a\}$ denote the set of generators creating H_{y_1} and let $\{T^{\hat{a}}\}$ denote the set of generators creating the coset G/H. In following we call T^a the unbroken generators and $T^{\hat{a}}$ the broken generators, respectively.

According to (2.83) and (2.84) unbroken gauge $A^a_\mu(x^\mu, y)$ and the scalar fields $A^{\hat{a}}_y(x^\mu, y)$ [5] are even functions, i.e.

$$\begin{aligned} A^a_\mu(x^\mu, -y) &= A^a_\mu(x^\mu, y) \\ A^{\hat{a}}_y(x^\mu, -y) &= A^{\hat{a}}_y(x^\mu, y)\ . \end{aligned} \tag{2.88}$$

Thus we can Fourier expand

$$\begin{aligned} A^a_\mu(x^\mu, y) &= \frac{1}{\sqrt{2\pi R}} A^{a\,(0)}_\mu(x^\mu) + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A^{a\,(n)}_\mu(x^\mu) \cos(\frac{ny}{R}) \\ A^{\hat{a}}_y(x^\mu, y) &= \frac{1}{\sqrt{2\pi R}} A^{\hat{a}(0)}_y(x^\mu) + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A^{\hat{a}(n)}_y(x^\mu) \cos(\frac{ny}{R})\ . \end{aligned} \tag{2.89}$$

Since $\cos(\frac{ny}{R})$ is $2\pi R$-periodic, $A^a_\mu(x^\mu, y)$ and $A^{\hat{a}}_y(x^\mu, y)$ fulfil also the periodicity condition (2.86). Note that for the scalar fields $A_y(x^\mu, y)$ the situation is opposite (a and $\hat{a}$ are interchanged) due to the relative minus sign in the boundary conditions (2.83) and (2.84), respectively.

The Fourier coefficients $A^{a\,(n)}_\mu(x^\mu)$ and $A^{\hat{a}(n)}_y(x^\mu)$ are given by

$$\begin{aligned} A^{a\,(n)}_\mu(x^\mu) &= \frac{1}{\sqrt{\pi R}} \int_0^{\pi R} A^a_\mu(x^\mu, y) \cos(\frac{ny}{R})\ dy \\ A^{\hat{a}(n)}_y(x^\mu) &= \frac{1}{\sqrt{\pi R}} \int_0^{\pi R} A^{\hat{a}}_y(x^\mu, y) \cos(\frac{ny}{R})\ dy\ . \end{aligned} \tag{2.90}$$

[5]Note that from a four-dimensional point of view the gauge fields $A_y(x^\mu, y)$ are seen as scalar fields. Therefore we will also call $A_y(x^\mu, y)$ scalar fields.

The zero modes read

$$A_\mu^{a\,(0)}(x^\mu) = \frac{1}{\sqrt{2\pi R}} \int_0^{\pi R} A_\mu^a(x^\mu, y)\; dy \qquad (2.91)$$
$$A_y^{\hat{a}\,(0)}(x^\mu) = \frac{1}{\sqrt{2\pi R}} \int_0^{\pi R} A_y^{\hat{a}}(x^\mu, y)\; dy \; .$$

On the other hand, according to (2.83) and (2.84), broken gauge $A_\mu^{\hat{a}}(x^\mu, y)$ and the scalar fields $A_y^a(x^\mu, y)$ are odd functions, i.e.

$$A_\mu^{\hat{a}}(x^\mu, -y) = -A_\mu^{\hat{a}}(x^\mu, y) \qquad (2.92)$$
$$A_y^a(x^\mu, -y) = -A_y^a(x^\mu, y) \; .$$

Thus we can Fourier expand

$$A_\mu^{\hat{a}}(x^\mu, y) = \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A_\mu^{\hat{a}\,(n)}(x^\mu) \sin(\frac{ny}{R}) \qquad (2.93)$$
$$A_y^a(x^\mu, y) = \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A_y^{a\,(n)}(x^\mu) \sin(\frac{ny}{R}) \; .$$

Again since $\sin(\frac{ny}{R})$ is $2\pi R$-periodic, $A_\mu^{\hat{a}}(x^\mu, y)$ and $A_y^a(x^\mu, y)$ fulfil also the periodicity condition (2.86).

The Fourier coefficients $A_\mu^{\hat{a}\,(n)}(x^\mu)$ and $A_y^{a\,(n)}(x^\mu)$ are given by

$$A_\mu^{\hat{a}\,(n)}(x^\mu) = \frac{1}{\sqrt{\pi R}} \int_0^{\pi R} A_\mu^{\hat{a}}(x^\mu, y) \sin(\frac{ny}{R})\; dy \qquad (2.94)$$
$$A_y^{a\,(n)}(x^\mu) = \frac{1}{\sqrt{\pi R}} \int_0^{\pi R} A_y^a(x^\mu, y) \sin(\frac{ny}{R})\; dy \; .$$

Note that in contrast to (2.89) in (2.93) no zero modes occurs. Since only (2.89) contains zero modes the gauge group of the low energy four-dimensional effective theory is given by (2.87).

2. W is a discrete Wilson line. This means that $W^2 = 1$. We know that a discrete Wilson line W commutes with the orbifold projection P

$$[P, W] = 0 \; . \qquad (2.95)$$

Therefore W and P have a common set of eigenfunctions. In order to find their eigenfunctions we first look at the periodicity condition

$$A_M(x^\mu, y + 2\pi R) = W\; A_M(x^\mu, y)\; W^{-1} \; . \qquad (2.96)$$

Due to $W^2 = 1$ the five-dimensional gauge field $A_M(x^\mu, y)$ splits into an even part

$$A^a_M(x^\mu, y + 2\pi R)T^a = + A^a_M(x^\mu, y)T^a \ , \tag{2.97}$$

where the $\{T^a\}$ satisfy $[W, T^a] = 0$, and an odd part

$$A^{\hat{a}}_M(x^\mu, y + 2\pi R)T^{\hat{a}} = - A^{\hat{a}}_M(x^\mu, y)T^{\hat{a}} \tag{2.98}$$

where the $\{T^{\hat{a}}\}$ satisfy $\{W, T^{\hat{a}}\} = 0$. Taking further into account that the orbifold projection P acts at $y_1 = 0$ according to

$$\begin{aligned} A_\mu(x^\mu, -y) &= P\ A_\mu(x^\mu, y)\ P^{-1} \\ A_y(x^\mu, -y) &= -P\ A_y(x^\mu, y)\ P^{-1} \ , \end{aligned} \tag{2.99}$$

we can Fourier expand [29]

$$\begin{aligned} A^{(+,+)}_\mu(x^\mu, y) &= \frac{1}{\sqrt{2\pi R}} A^{(+,+)(0)}_\mu(x^\mu) + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A^{(+,+)(n)}_\mu(x^\mu) \cos(\frac{ny}{R}) \ , \\ A^{(+,-)}_\mu(x^\mu, y) &= \frac{1}{\sqrt{\pi R}} \sum_{n=0}^{\infty} A^{(+,-)(n)}_\mu(x^\mu) \sin(\frac{ny}{R}) \ , \\ A^{(-,+)}_\mu(x^\mu, y) &= \frac{1}{\sqrt{\pi R}} \sum_{n=0}^{\infty} A^{(-,+)(n)}_\mu(x^\mu) \cos(\frac{(n+1/2)y}{R}) \ , \\ A^{(-,-)}_\mu(x^\mu, y) &= \frac{1}{\sqrt{\pi R}} \sum_{n=0}^{\infty} A^{(-,-)(n)}_\mu(x^\mu) \sin(\frac{(n+1/2)y}{R}) \ . \end{aligned} \tag{2.100}$$

The superscript $(\pm, \pm)$ denotes the eigenvalue of W and P, respectively. This means that $A^{(+,+)}_\mu(x^\mu, y) = A^{++}_\mu(x^\mu, y)T^{++}$ commutes with W and P

$$[W, T^{++}] = [P, T^{++}] = 0 \ , \tag{2.101}$$

$A^{(+,-)}_\mu(x^\mu, y) = A^{+-}_\mu(x^\mu, y)T^{+-}$ commutes with W and anticommutes with P

$$[W, T^{+-}] = \{P, T^{+-}\} = 0 \ . \tag{2.102}$$

$A^{(-,+)}_\mu(x^\mu, y) = A^{-+}_\mu(x^\mu, y)T^{-+}$ anticommutes with W and commutes with P

$$\{W, T^{-+}\} = [P, T^{-+}] = 0 \ , \tag{2.103}$$

and $A^{(-,-)}_\mu(x^\mu, y) = A^{--}_\mu(x^\mu, y)T^{--}$ anticommutes with W and P

$$\{W, T^{--}\} = \{P, T^{--}\} = 0 \ . \tag{2.104}$$

The expansion for A_y is done in the same way but, due to (2.99), with the opposite eigenvalue for W and P.

The Fourier coefficient for $A_\mu^{++(n)}(x^\mu)$ is given by

$$A_\mu^{++(n)}(x^\mu) = \frac{1}{\sqrt{\pi R}} \int_0^{\pi R} A_\mu^{++}(x^\mu, y) \cos(\frac{ny}{R})\, dy \; . \tag{2.105}$$

All other Fourier coefficients can be obtained in an analogous manner. The zero modes are contained in $A_\mu^{(+,+)(0)}(x^\mu, y)$ and read

$$A_\mu^{(+,+)(0)}(x^\mu) = A_\mu^{++(0)}(x^\mu) T^{++} \; . \tag{2.106}$$

The Fourier coefficient $A_\mu^{++(0)}(x^\mu)$ is given by

$$A_\mu^{++(0)}(x^\mu) = \frac{1}{\sqrt{2\pi R}} \int_0^{\pi R} A_\mu^{++}(x^\mu, y)\, dy \; . \tag{2.107}$$

The low energy four-dimensional unbroken gauge group H is created by the generators $\{T^{++}\}$. Note that the generators $\{T^{++}\}$ commute with W and P, i.e.

$$[W, T^{++}] = [P, T^{++}] = 0 \; . \tag{2.108}$$

If we switch to the reinterpretation of $S^1/\mathbb{Z}_2$ in terms of the two effective orbifold projections $P_1 = P$ and $P_2 = WP$ [6] the orbifold boundary conditions read (2.40)

$$\begin{aligned} A_\mu(x^\mu, -y) &= P_1\; A_\mu(x^\mu, y)\; P_1^{-1} \\ A_y(x^\mu, -y) &= -P_1\; A_y(x^\mu, y)\; P_1^{-1} \; , \end{aligned} \tag{2.109}$$

$$\begin{aligned} A_\mu(x^\mu, \pi R - y) &= P_2\; A_\mu(x^\mu, \pi R + y)\; P_2^{-1} \\ A_y(x^\mu, \pi R - y) &= -P_2\; A_y(x^\mu, \pi R + y)\; P_2^{-1} \; . \end{aligned}$$

These boundary conditions lead to the Fourier expansion

$$\begin{aligned} A_\mu^{(+,+)}(x^\mu, y) &= \frac{1}{\sqrt{2\pi R}} A_\mu^{(+,+)(0)}(x^\mu) + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A_\mu^{(+,+)(n)}(x^\mu) \cos(\frac{ny}{R}) \; , \\ A_\mu^{(+,-)}(x^\mu, y) &= \frac{1}{\sqrt{\pi R}} \sum_{n=0}^{\infty} A_\mu^{(+,-)(n)}(x^\mu) \cos(\frac{(n+1/2)y}{R}) \; , \\ A_\mu^{(-,+)}(x^\mu, y) &= \frac{1}{\sqrt{\pi R}} \sum_{n=0}^{\infty} A_\mu^{(-,+)(n)}(x^\mu) \sin(\frac{(n+1/2)y}{R}) \; , \\ A_\mu^{(-,-)}(x^\mu, y) &= \frac{1}{\sqrt{\pi R}} \sum_{n=0}^{\infty} A_\mu^{(-,-)(n)}(x^\mu) \sin(\frac{(n+1)y}{R}) \; . \end{aligned} \tag{2.110}$$

[6] Note that due to $[P, W] = 0$ we have $[P, WP] = 0$ and thus also $P_1 = P$ and $P_2 = WP$ have a common set of eigenfunctions

The subscript $(\pm,\pm)$ denote the eigenvalues of (P_1, P_2). The connection to (2.100) becomes apparent if we looks at the different eigenvalues

W	P	P_1	P_2
$+$	$+$	$+$	$+$
$+$	$-$	$-$	$-$
$-$	$+$	$+$	$-$
$-$	$-$	$-$	$+$

(2.111)

Recall again that $P_1 = P$ and $P_2 = WP$. The gauge group G is broken to its subgroup H_1 generated by $\{T^{++}, T^{+-}\}$ at $y_1 = 0$ and to its subgroup H_2 generated by $\{T^{++}, T^{-+}\}$ at $y_2 = \pi R$. The low energy four-dimensional unbroken gauge group H is thus generated by $\{T^{++}\}$ and we have

$$H = H_1 \cap H_2 \; . \tag{2.112}$$

2.5.1 Fourier expansion and zero modes on $S^1/\mathbb{Z}_2 \times \mathbb{Z}_2'$

Since the orbifold $S^1/\mathbb{Z}_2 \times \mathbb{Z}_2'$ is familiar in orbifold GUTs, we shortly discuss their Fourier mode expansion. We recall that the gauge fields have to fulfil the boundary conditions (2.48)

$$\begin{aligned} A_\mu(x^\mu, -y) &= P\; A_\mu(x^\mu, y)\; P^{-1} \\ A_y(x^\mu, -y) &= -P\; A_y(x^\mu, y)\; P^{-1} \; , \end{aligned} \tag{2.113}$$

$$\begin{aligned} A_\mu(x^\mu, -y') &= P'\; A_\mu(x^\mu, y')\; P'^{\,-1} \\ A_y(x^\mu, -y') &= -P'\; A_y(x^\mu, y')\; P'^{\,-1} \; . \end{aligned}$$

Remember that $y' = y - \pi R'/2$ and $\mathbb{Z}_2 : \; y \to -y$, $\mathbb{Z}_2' : \; y' \to -y'$. The orbifold $S^1/\mathbb{Z}_2 \times \mathbb{Z}_2'$ has the two fixed points $y = 0$ and $y = \pi R'/2$. Fourier expanding yields

$$\begin{aligned} A_\mu^{(+,+)}(x^\mu, y) &= \frac{1}{\sqrt{\pi R'}} A_\mu^{(+,+)(0)}(x^\mu) + \frac{1}{\sqrt{\pi R'/2}} \sum_{n=1}^{\infty} A_\mu^{(+,+)(n)}(x^\mu) \cos(\frac{2ny}{R'}) \; , \\ A_\mu^{(+,-)}(x^\mu, y) &= \frac{1}{\sqrt{\pi R'/2}} \sum_{n=0}^{\infty} A_\mu^{(+,-)(n)}(x^\mu) \cos(\frac{(2n+1)y}{R'}) \; , \\ A_\mu^{(-,+)}(x^\mu, y) &= \frac{1}{\sqrt{\pi R'/2}} \sum_{n=0}^{\infty} A_\mu^{(-,+)(n)}(x^\mu) \sin(\frac{(2n+1)y}{R'}) \; , \\ A_\mu^{(-,-)}(x^\mu, y) &= \frac{1}{\sqrt{\pi R'/2}} \sum_{n=0}^{\infty} A_\mu^{(-,-)(n)}(x^\mu) \sin(\frac{(2n+2)y}{R'}) \; . \end{aligned} \tag{2.114}$$

The expansion for A_y is again done in the same way but with the opposite eigenvalue for W and P. We already know that the orbifold $S^1/\mathbb{Z}_2 \times \mathbb{Z}_2'$ is equivalent to the orbifold $S^1/\mathbb{Z}_2$ with twisted boundary conditions. To show this equivalence also for the Fourier mode expansion we have to keep in mind that

$$R' = 2R \; . \tag{2.115}$$

Indeed, if we insert (2.115) in (2.114) we recover (2.110). The Fourier mode expansion (2.114) well known in orbifold GUTs [35, 31].

3. W is a continuous Wilson line. In this case the Wilson line W and the projection P do not commute

$$[P, W] \neq 0 \; . \tag{2.116}$$

Therefore P and W do not have a common set of eigenfunctions.

2.6 Continuous Wilson line breaking and the Hosotani mechanism on the orbifold $S^1/\mathbb{Z}_2$

In section 2.4 we have seen that a continuous Wilson line is given by

$$W = \exp\left(2\pi i g R \sum_{\hat{a}} \langle A_y^{\hat{a}} \rangle T^{\hat{a}}\right) \tag{2.117}$$

where $T^{\hat{a}} \in H_W$ and

$$H_W = \{T^{\hat{a}} \in G \mid \{T^{\hat{a}}, P_1\} = \{T^{\hat{a}}, P_2\} = 0\} \; . \tag{2.118}$$

We consider x- and y-independent modes of

$$A_y = \sum_{\hat{a}} A_y^{\hat{a}} T^{\hat{a}} \quad , \; T^{\hat{a}} \in H_W \; . \tag{2.119}$$

They correspond to Wilson line phases via [28]

$$\theta_{\hat{a}} := g\pi R A_y^{\hat{a}} \; . \tag{2.120}$$

Wilson line phases are part of the Hosotani mechanism [43]. The Hosotani mechanism is used in a series of papers [28, 44, 46, 45, 73, 54]. Here we describe the main ingredients [28, 44] of the Hosotani mechanism:

- Wilson line phases $\theta_{\hat{a}}$ along noncontractible loops become physical degrees of freedom which cannot be gauged away once boundary conditions are given. They yield vanishing field strengths such that they appear as degenerate vacua at the classical level.

- The degeneracy of the classical vacuum is in general lifted by quantum effects. Let $V_{eff} = V_{eff}(\theta_{\hat{a}})$ be the effective potential for the Wilson line phases $\theta_{\hat{a}}$. Then the true physical vacuum is given by those configurations of the Wilson line phases $\theta_{\hat{a}}$ which minimise V_{eff}.

- Suppose that the effective potential V_{eff} is minimised at nontrivial configurations of the Wilson line phases. Then the gauge symmetry is spontaneously broken by radiative corrections. This part of the mechanism is called *Wilson line symmetry breaking*. Gauge fields in four dimensions whose gauge symmetry is spontaneously broken get masses from nonvanishing VEVs for the Wilson line phases. In addition, some matter fields also acquire masses.

- All zero modes of the extra-dimensional component of the higher dimensional gauge field become massive. Their masses are given by the second derivatives of V_{eff} up to numerical constants.

- The physical symmetry of the theory is determined by orbifold boundary conditions and the VEVs of the Wilson line phases.

Chapter 3

Effective Theories and nonunitary parallel transporters

In this chapter we describe how an effective bilayered transverse lattice model can be obtained from an ordinary $S^1/\mathbb{Z}_2$ orbifold model via renormalisation group (RG) transformations. We start with a five-dimensional space-time $M^4 \times S^1/\mathbb{Z}_2$, which is the product of the four-dimensional Minkowski space-time M^4 and the orbifold $S^1/\mathbb{Z}_2$. Recall that the orbifold $S^1/\mathbb{Z}_2$ is obtained by dividing the circle S^1 with radius R by a $\mathbb{Z}_2$ transformation. The resulting space is the interval $[0, \pi R]$. Let G be the bulk gauge group. In order to obtain a well-defined starting point for RG transformations we put the orbifold $S^1/\mathbb{Z}_2$ on a lattice. Thus the four-dimensional Minkowski space-time M^4 remains continuous while the extra dimension is latticized. Such a scenario is known as a transverse lattice and it occurs naturally in deconstruction models [39, 14, 15]. Starting with this latticized extra dimension we calculate the RG-flow. The endpoint of the RG-flow will be an extra dimension which consists of only two points: the two orbifold fixed points $y = 0$ and $y = \pi R$. The bulk is completely integrated out. The effective theory obtained this way will be called an *effective bilayered transverse lattice model* (eBTLM). We call the four-dimensional boundary at the fixed point $y = 0$ the L-boundary and the four-dimensional boundary at the fixed point $y = \pi R$ the R-boundary. PTs Φ in the extra dimension from the L- to the R-boundary (and vice versa) become nonunitary as a result of the blockspin transformation. They take their values in a Lie group H which is typically noncompact and larger than the unitary gauge group G we have started with. We always consider the case where G is the maximal compact subgroup of H. It will turn out that these nonunitary PTs Φ can be interpreted as Higgs fields. In this chapter we will also formulate orbifold conditions for nonunitary PTs Φ. As an application, we analyse in detail an eBTLM based on the gauge group $SU(2)$.

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3.1 $S^1/\mathbb{Z}_2$ orbifold model on a lattice

We consider a one-dimensional lattice Γ with lattice spacing a. The points of Γ have the coordinate $y = a\, n_y$, where $n_y = -N_y + 1, \ldots, N_y$, $N_y \in \mathbb{N}_*$. If we identify the points $y = -aN_y$ and $y = aN_y$, Γ will possesses the translation invariance

$$t:\ n_y \to n_y + 2N_y \quad \Longleftrightarrow \quad y \to y + 2\pi R\ . \tag{3.1}$$

Thus the physical extension of Γ is $2\pi R = 2N_y a$. We define the reflection r on Γ by

$$r:\ n_y \to -n_y \quad \Longleftrightarrow \quad y \to -y\ . \tag{3.2}$$

Figure 3.1 shows the representation of the orbifold reflection r on Γ. The orbifold

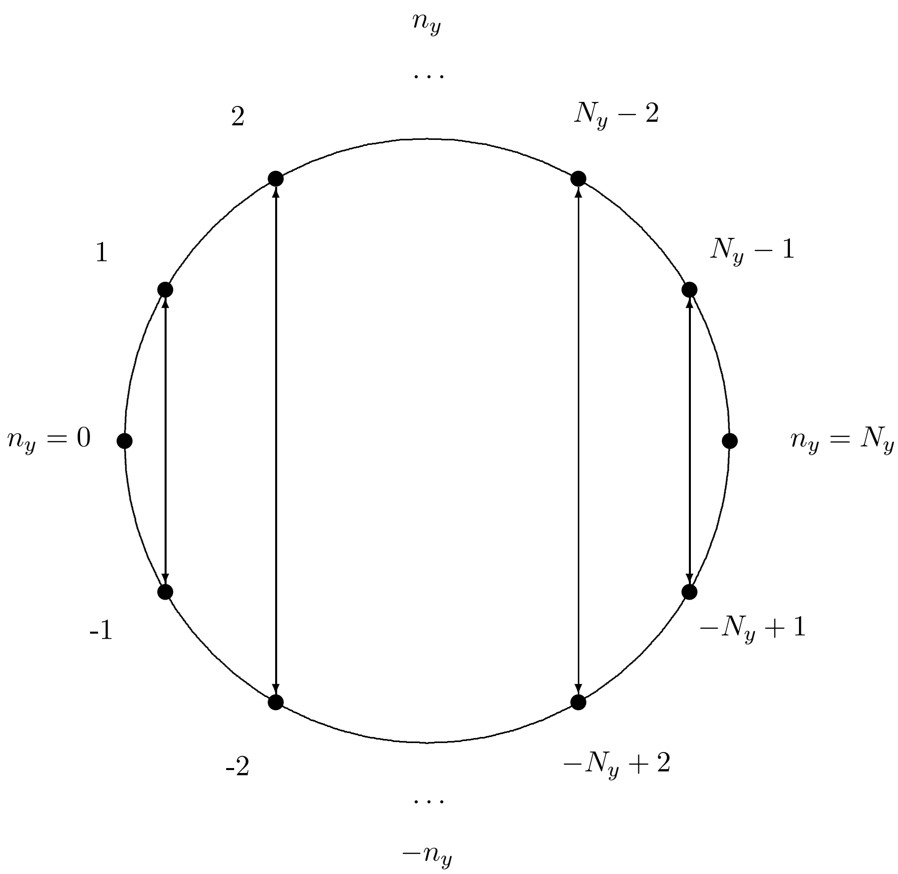

Figure 3.1: Representation of the $S^1/\mathbb{Z}_2$ orbifold reflection r on the lattice Γ.

has two fixed points $n_y = 0$, invariant under r, and $n_y = N_y$, invariant under tr. In terms of the coordinate y they read $y = 0$ and $y = \pi R$. After the identification (3.2) the resulting space is the latticized interval $[0, N_y]$.

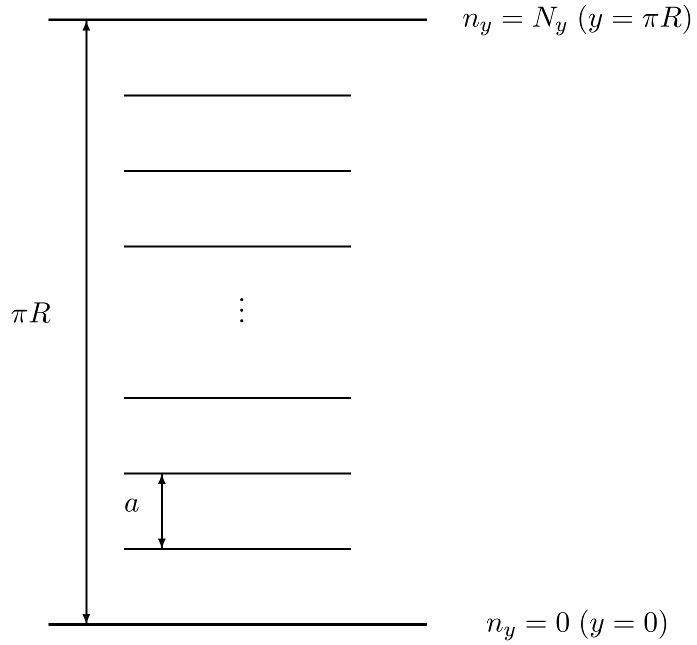

Figure 3.2: The latticized interval $[0, N_y]$. The two orbifold fixed points are $n_y = 0$ $(y = 0)$ and $n_y = N_y$ $(y = \pi R)$.

3.2 From an orbifold model on the latticized interval to an effective bilayered transverse lattice model via renormalisation group transformations

In this section we sketch the basic ideas which lead to an effective bilayered transverse lattice model. We start with the latticized interval $\Delta = [0, N_y]$, where $N_y \gg 1$, and take it as the fundamental lattice. The bilayered transverse lattice model is treated as an effective theory, which leads at a coarser scale to the same expectation values as the $S^1/\mathbb{Z}_2$ orbifold theory on Δ, however, with less degrees of freedom. The transition from a theory on the latticized interval Δ to a theory on a bilayered transverse lattice is given by RG transformations. Let ϕ be an unitary PT in the fundamental theory and let ϕ' a PT in the effective theory. The transition from ϕ to ϕ' is given by a blockspin operator $\mathcal{C}$

$$\phi' = \mathcal{C}\phi \, . \tag{3.3}$$

To be more precise, let ϕ' be the PT from a point $x \in \Delta'$ to a point $y \in \Delta'$ along a path $C : x \to y$, where Δ' is a coarser latticized interval than Δ. The blockspin is given by

$$\phi' = \sum_{C:x\to y} \rho(C)\phi(C) \ , \tag{3.4}$$

where the sum goes over all paths $C : x \to y$ in Δ and $\rho(C)$ is a weight factor. Thus ϕ' will be in the linear span of the unitary bulk gauge group G. The blockspin ϕ' polar decomposes into a unitary part U and a selfadjoint part S

$$\phi' = U\ S\ . \tag{3.5}$$

If it is possible to integrate out the selfadjoint part S we will recover a local effective theory with unitary PTs ϕ'. We assume that this procedure will fail after n steps in the sense that the theory would acquire bad locality properties [57] and the effective theory needs nonunitary PTs for its locality.

Let us consider the family of latticized intervals $\{\Delta^0, \Delta^1, \ldots, \Delta^m\}$ [1], where $\Delta^0 = \Delta$ is the fundamental latticized interval $[0, N_y]$ and Δ^m, $m > n$ is the bilayered transverse lattice. Obviously Δ^m is the coarsest latticized interval as it consists of only two points. Hence an eBTLM can always be interpreted as the endpoint of a RG-flow. The PTs Φ [2] in the extra dimension are nonunitary as a consequence of the blockspin transformation (3.4). They can be interpreted as Higgs fields. When Φ becomes nonunitary a Higgs potential $V(\Phi)$ naturally emerges [68]. The nonunitary PTs Φ take their values in a Lie group H which is typically noncompact and larger than the unitary bulk gauge group G. We call H the holonomy group. We always consider the case where G is the maximal compact subgroup of H. As already mentioned above, the extra dimension consists of only two points which are the orbifold fixed points $n_y = 0$ ($y = 0$) and $n_y = N_y$ ($y = \pi R$). We call the four-dimensional boundary at the fixed point $y = 0$ the L-boundary and the four-dimensional boundary at the fixed point $y = \pi R$ the R-boundary. G_R denotes the gauge group of the R-boundary, and G_L denotes the gauge group of the L-boundary. In principle, an orbifold breaking can lead to different gauge groups G_L and G_R at the boundaries R and L. In the following however we will restrict ourselves to the case where $G_L = G_R = G_0$. The gauge group G_0 is the subgroup of G left unbroken by the orbifold projection P i.e. the centraliser of P in G. We call G_0 the orbifold unbroken gauge group.

In the simplest approximation, the four-dimensional effective Lagrangian of an eBTLM reads

$$\mathcal{L}_{4D} = -\frac{1}{4}\sum_{i=L,R} F^a_{i\mu\nu}F^{i\mu\nu a} + \mathrm{tr}\left[(D_\mu\Phi)^\dagger\,(D_\mu\Phi)\right] + V(\Phi)\ , \tag{3.6}$$

[1] Note that in the context of RG transformations one usually considers a family of hypercubic lattices $\{\Lambda^0, \ldots, \Lambda^i, \Lambda^{i+1}, \ldots\}$.

[2] In the following we write Φ instead of ϕ' for nonunitary PTs

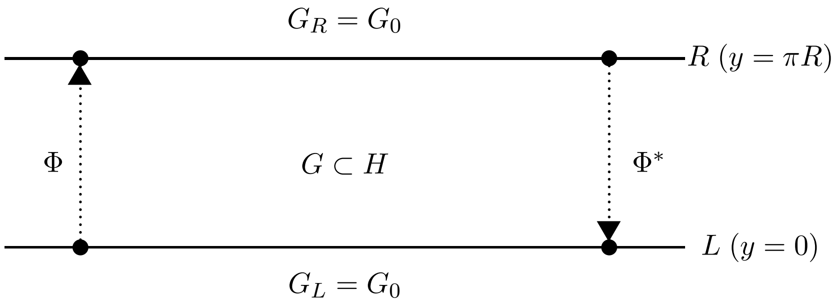

Figure 3.3: Effective bilayered transverse lattice model (eBTLM).

where the covariant derivative is given by

$$D_\mu \Phi = \partial_\mu \Phi + ig\left(A^R_\mu \Phi - \Phi A^L_\mu\right)\ . \tag{3.7}$$

We discuss the terms in (3.6):

- The term $F^a_{i\mu\nu} F^{i\mu\nu a}$ is a Yang-Mills term for the boundary gauge fields A^R_μ and A^L_μ. Let $\mathfrak{g}_0 = \mathrm{Lie}\, G_0$. Then $A^R_\mu, A^L_\mu \in \mathfrak{g}_0$.
- The term $\mathrm{tr}\left[(D_\mu\Phi)^\dagger (D_\mu\Phi)\right]$ is the kinetic term for Φ. It will lead to a mass term for the boundary gauge fields A^R_μ and A^L_μ.
- The term $V(\Phi)$ is the Higgs potential. If $V(\Phi)$ takes its minimum at non-trivial Φ_{min} the orbifold unbroken gauge group G_0 is spontaneously broken.

We will see that under certain circumstances the effective four-dimensional Lagrangian (3.6) equals the effective four-dimensional Lagrangian of a corresponding $S^1/\mathbb{Z}_2$ continuum [3] orbifold model.

3.3 Effective bilayered transverse lattice model, ordinary Higgs mechanism and renormalisability

In this section, we start to work out the correspondence between an eBTLM and a $S^1/\mathbb{Z}_2$ continuum orbifold model by investigating two simple examples. In both examples the orbifold projection P is chosen to be trivial. Thus the bulk gauge

[3]By continuum we mean that $S^1/\mathbb{Z}_2$ is treated as usual as the quotient space $\mathbb{R}/\mathbb{D}_\infty$ and not as the latticized interval $[0, N_y]$.

group G remains unbroken. In the first example we consider Abelian gauge theory, i.e. we set $G = U(1)$, while in the second example we consider non-Abelian gauge theory, e.g. we set $G = SU(N)$. It will turn out that the Lagrangian of an eBTLM equals the effective four-dimensional Lagrangian of an $S^1/\mathbb{Z}_2$ continuum orbifold model if we truncate the Kaluza-Klein (KK) expansion for all fields in the $S^1/\mathbb{Z}_2$ continuum orbifold model at the first excited KK mode. In addition, we need to make certain assumptions about the minimum of the Higgs potential $V(\Phi)$ in the eBTLM. We will also demonstrate the close analogy between an $S^1/\mathbb{Z}_2$ continuum orbifold model with truncated KK-mode expansion and the *ordinary* Higgs mechanism of four-dimensional gauge theories [19]. Since the ordinary Higgs mechanism of four-dimensional gauge theories preserves renormalisability, we conclude that also the truncated $S^1/\mathbb{Z}_2$ continuum orbifold model is renormalisable and therewith the corresponding eBTLM. As already mentioned in the introduction of this chapter, for *trivial orbifold projection* P and *trivial minimum of the Higgs potential* $V(\Phi)$ there is a close analogy between an eBTLM and deconstruction models [39, 14, 15].

3.3.1 Abelian gauge theory

In the first example we consider Abelian gauge theory, i.e. we start with the bulk gauge group $G = U(1)$. Figure 3.4 summarises the setup. The effective four-

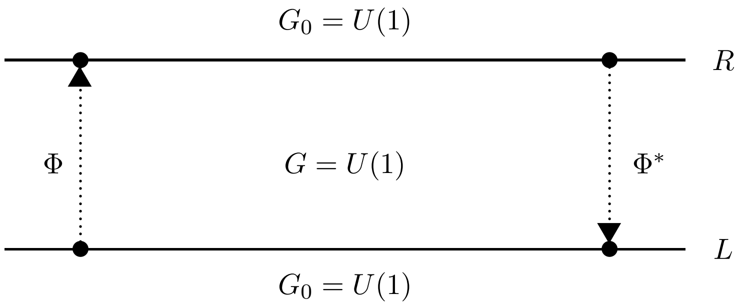

Figure 3.4: Effective bilayered transverse lattice model for the bulk gauge group $G = U(1)$ and trivial orbifold projection $P = 1$.

dimensional Lagrangian (3.6) reads

$$\mathcal{L}_{4D} = -\frac{1}{4} F^L_{\mu\nu} F^{L\mu\nu} - \frac{1}{4} F^R_{\mu\nu} F^{R\mu\nu} + (D_\mu \Phi)^\dagger (D_\mu \Phi) + V(\Phi) \qquad (3.8)$$

where

$$F^L_{\mu\nu} = \partial_\mu A^L_\nu - \partial_\nu A^L_\mu \quad , \quad F^R_{\mu\nu} = \partial_\mu A^R_\nu - \partial_\nu A^R_\mu \ , \tag{3.9}$$

$$D_\mu \Phi = \partial_\mu \Phi + ig \left(A^R_\mu \Phi - \Phi A^L_\mu \right) \ . \tag{3.10}$$

The fields A^L_μ and A^R_μ are $U(1)$ gauge fields on the L- and R-boundary respectively.

Suppose that the Higgs potential $V(\Phi)$ takes its minimum at Φ_{min} where

$$\Phi_{min} = \frac{1}{2}\, \rho_{min} \ , \quad \rho_{min} \in \mathbb{R}^+_* \tag{3.11}$$

and $\mathbb{R}^+_* = \mathbb{R}^+/\{0\}$. Inserting (3.11) in (3.10) yields for the kinetic term

$$(D_\mu \Phi_{min})^\dagger (D_\mu \Phi_{min}) = \frac{1}{4}\, g^2 \rho^2_{min} \left(A^R_\mu - A^L_\mu \right)^2 \ . \tag{3.12}$$

This is a mass term for the $U(1)$ gauge fields A^R_μ and A^L_μ. Defining $A := (A^R_\mu, A^L_\mu)$, we can rewrite

$$\mathcal{L}_{mass} = \frac{1}{4}\, g^2 \rho^2_{min} \left(A^R_\mu - A^L_\mu \right)^2 = A\ M\ A^t \ , \tag{3.13}$$

where M is the mass-squared matrix

$$M = \frac{1}{4}\, g^2 \rho^2_{min} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \ , \tag{3.14}$$

and A^t is the transpose of A. The gauge fields A^R_μ and A^L_μ can be expressed as real linear combinations of their mass eigenstates $A^{(0)}_\mu$ and $A^{(1)}_\mu$

$$A^R_\mu = \frac{1}{\sqrt{2}} \left(A^{(0)}_\mu + A^{(1)}_\mu \right) \ , \tag{3.15}$$
$$A^L_\mu = \frac{1}{\sqrt{2}} \left(A^{(0)}_\mu - A^{(1)}_\mu \right) \ .$$

In this new basis the mass-squared matrix M is diagonal. We obtain

$$\begin{aligned} \mathcal{L}_{mass} &= A\ M\ A^t = \frac{1}{4}\, g^2 \rho^2_{min} \left(A^R_\mu - A^L_\mu \right)^2 \\ &= \frac{1}{8}\, g^2 \rho^2_{min} \left(A^{(0)}_\mu + A^{(1)}_\mu - A^{(0)}_\mu - A^{(1)}_\mu \right)^2 \\ &= \frac{1}{2}\, g^2 \rho^2_{min} \left(A^{(1)}_\mu \right)^2 \end{aligned} \tag{3.16}$$

which leads to the mass

$$m = g \rho_{min} \tag{3.17}$$

for the gauge field $A_\mu^{(1)}$, while the gauge field $A_\mu^{(0)}$ remains massless. In the basis of mass eigenstates (3.15) the Lagrangian (3.8) reads

$$\mathcal{L}_{4D} = -\frac{1}{4}\left(\partial_\mu A_\nu^{(0)} - \partial_\nu A_\mu^{(0)}\right)^2 - \frac{1}{4}\left(\partial_\mu A_\nu^{(1)} - \partial_\nu A_\mu^{(1)}\right)^2 + \frac{1}{2}\, m^2 \left(A_\mu^{(1)}\right)^2 \ . \quad (3.18)$$

This Lagrangian describes two Abelian gauge fields $A_\mu^{(0)}$ and $A_\mu^{(1)}$, where $A_\mu^{(0)}$ is a massless field and the field $A_\mu^{(1)}$ is massive with mass $m = g\rho_{min}$.

We compare this result to an $S^1/\mathbb{Z}_2$ continuum orbifold model. Let $G = U(1)$ be the bulk gauge group. The five-dimensional Lagrangian [4] reads

$$\mathcal{L}_{5D} = -\frac{1}{4} F_{MN} F^{MN} \ , \quad (3.19)$$

where

$$F_{MN} = \partial_M A_N - \partial_N A_M \ . \quad (3.20)$$

The boundary conditions for the gauge fields read

$$A_\mu(x^\mu, -y) = P\ A_\mu(x^\mu, y)\ P^{-1} \quad (3.21)$$

$$A_y(x^\mu, -y) = -P\ A_y(x^\mu, y)\ P^{-1} \ . \quad (3.22)$$

We take the trivial orbifold projection

$$P = 1 \quad (3.23)$$

and Fourier expand

$$A_\mu(x^\mu, y) = \frac{1}{\sqrt{2\pi R}} A_\mu^{(0)}(x^\mu) + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A_\mu^{(n)}(x^\mu) \cos(\frac{ny}{R}) \quad (3.24)$$

$$A_y(x^\mu, y) = \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A_y^{(n)}(x^\mu) \sin(\frac{ny}{R}) \ . \quad (3.25)$$

Truncating this expansion at $n = 1$ yields

$$A_\mu(x^\mu, y) = \frac{1}{\sqrt{2\pi R}} A_\mu^{(0)}(x^\mu) + \frac{1}{\sqrt{\pi R}} A_\mu^{(1)}(x^\mu) \cos(\frac{y}{R}) \quad (3.26)$$
$$A_y(x^\mu, y) = \frac{1}{\sqrt{\pi R}} A_y^{(1)}(x^\mu) \sin(\frac{y}{R}) \ .$$

The field strength F_{MN} consists of two parts

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \qquad (3.27) \\ &= \frac{1}{\sqrt{2\pi R}} \left[\left(\partial_\mu A_\nu^{(0)} - \partial_\nu A_\mu^{(0)}\right) + \left(\partial_\mu A_\nu^{(1)} - \partial_\nu A_\mu^{(1)}\right) \cdot \sqrt{2} \cos(\frac{y}{R})\right] \ , \\ F_{\mu y} &= \partial_\mu A_y - \partial_y A_\mu = \frac{1}{\sqrt{\pi R}} \left[\partial_\mu A_y^{(1)} \cdot \sin(\frac{y}{R}) + A_\mu^{(1)} \frac{1}{R} \sin(\frac{y}{R})\right] \ , \end{aligned}$$

[4] Recall that $M, N \in (\mu, y)$, where $\mu = 0, 1, 2, 3$.

where we have inserted the truncated KK-mode expansion (3.26). We insert $F_{\mu\nu}$ and $F_{\mu y}$ into the five-dimensional Lagrangian (3.19) and integrate over the circle S^1. This yields

$$\begin{aligned}\mathcal{L}_{4D} &= \int_0^{2\pi R}\{-\frac{1}{4}F_{MN}F^{MN}\}\,dy = \int_0^{2\pi R}\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2}F_{\mu y}F^{\mu y}\}\,dy \\ &= -\frac{1}{4}\left(\partial_\mu A_\nu^{(0)}-\partial_\nu A_\mu^{(0)}\right)^2-\frac{1}{4}\left(\partial_\mu A_\nu^{(1)}-\partial_\nu A_\mu^{(1)}\right)^2+\frac{1}{2}\left(\partial_\mu A_y^{(1)}+\frac{1}{R}A_\mu^{(1)}\right)^2 .\end{aligned} \tag{3.28}$$

This Lagrangian describes two Abelian gauge fields $A_\mu^{(0)}$ and $A_\mu^{(1)}$, where the field $A_\mu^{(0)}$ is massless and the field $A_\mu^{(1)}$ is massive. We define

$$B_\mu^{(1)} = A_\mu^{(1)} + R\,\partial_\mu A_y^{(1)} , \tag{3.29}$$

and express the Lagrangian (3.28) in terms of $A_\mu^{(0)}$ and $B_\mu^{(1)}$

$$\mathcal{L}_{4D} = -\frac{1}{4}\left(\partial_\mu A_\nu^{(0)}-\partial_\nu A_\mu^{(0)}\right)^2-\frac{1}{4}\left(\partial_\mu B_\nu^{(1)}-\partial_\nu B_\mu^{(1)}\right)^2+\frac{1}{2}\frac{1}{R^2}\left(B_\mu^{(1)}\right)^2 . \tag{3.30}$$

We observe that the field $B_\mu^{(1)}$ has mass $1/R$.

We compare this result to the ordinary $U(1)$ Abelian Higgs model [65]. Let

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mid D_\mu\phi\mid^2 - V(\phi) , \tag{3.31}$$

with $D_\mu = \partial_\mu + ieA_\mu$, be the Lagrangian of a complex scalar field ϕ coupled both to itself and an electromagnetic field. The potential $V(\Phi)$ is chosen to be of the form

$$V(\phi) = -\mu^2\,(\phi^*\phi) + \lambda^2\,(\phi^*\phi)^2 \tag{3.32}$$

where $\mu^2 > 0$. With the minimum of $V(\phi)$ at

$$\langle\phi\rangle = \frac{1}{\sqrt{2}}\phi_0 \tag{3.33}$$

where $\phi_0 = \mu/\lambda$. We expand the complex field $\phi(x)$ around the minimum ϕ_0 as

$$\phi(x) = \frac{1}{\sqrt{2}}\left(\phi_0+\phi_1+i\phi_2\right) . \tag{3.34}$$

We insert this expansion into the kinetic term $\mid D_\mu\phi\mid^2$. Thus we obtain

$$\mid D_\mu\phi\mid^2 = \frac{1}{2}\left(\partial_\mu\phi_1\right)^2+\frac{1}{2}\left(\partial_\mu\phi_2\right)^2+e\phi_0\,A_\mu\partial^\mu\phi_2+\frac{1}{2}e^2\phi_0^2A_\mu A^\mu+\dots , \tag{3.35}$$

where we have omitted terms cubic and quartic in the fields A_μ, ϕ_1 and ϕ_2. We compare this result with

$$\frac{1}{2}\left(\partial_\mu A_y^{(1)} + \frac{1}{R}A_\mu^{(1)}\right)^2 \tag{3.36}$$

from (3.28). This apparently coincides with the Abelian Higgs model if we identify [19]

$$e\,\phi_0 \quad \Longleftrightarrow \quad \frac{1}{R}\,. \tag{3.37}$$

In addition, the comparison of (3.35) with (3.36) shows that the first excited KK-mode gauge field $A_y^{(1)}$ plays the role of the Goldstone boson ϕ_2 [19]. It is therefore natural to go to unitary gauge, i.e. we set

$$A_y^{(1)} = 0\,. \tag{3.38}$$

In the context of gauge theories in extra dimensions this gauge is known as axial gauge and we will from now on call (3.38) axial gauge. In axial gauge the Lagrangian (3.28) reads

$$\mathcal{L}_{4D} = -\frac{1}{4}\left(\partial_\mu A_\nu^{(0)} - \partial_\nu A_\mu^{(0)}\right)^2 - \frac{1}{4}\left(\partial_\mu A_\nu^{(1)} - \partial_\nu A_\mu^{(1)}\right)^2 + \frac{1}{2}\,\frac{1}{R^2}\left(A_\mu^{(1)}\right)^2\,, \tag{3.39}$$

since $B_\mu^{(1)} = A_\mu^{(1)}$ for $A_y^{(1)} = 0$. Due to the close analogy of the $S^1/\mathbb{Z}_2$ continuum orbifold model with truncated KK-mode expansion and the ordinary Higgs mechanism, we conclude that the truncated $S^1/\mathbb{Z}_2$ continuum orbifold model is renormalisable.

We compare the Lagrangian (3.28) of the truncated $S^1/\mathbb{Z}_2$ continuum orbifold model in axial gauge (3.39) with the Lagrangian of the corresponding eBTLM (3.18). If we require

$$g\rho_{min} = \frac{1}{R}\,, \tag{3.40}$$

both Lagrangian's equal and hence both theories describe the same physics. Thus we conclude that also the eBTLM (3.18) is *renormalisable.*

3.3.2 Non-Abelian gauge theory

In the second example we consider non-Abelian gauge theory, i.e. we start for example with the bulk gauge group $G = SU(N)$. Figure 3.5 summarises the setup. The effective four-dimensional Lagrangian reads

$$\mathcal{L}_{4D} = -\frac{1}{4}F^i_{L\mu\nu}F^{Li\mu\nu} - \frac{1}{4}F^i_{R\mu\nu}F^{Ri\mu\nu} + \mathrm{tr}\left[(D_\mu\Phi)^\dagger\,(D_\mu\Phi)\right] + V(\Phi)\,, \tag{3.41}$$

where

$$F^{Li}_{\mu\nu} = \partial_\mu A^{Li}_\nu - \partial_\nu A^{Li}_\mu + g f^{ijk} A^{Lj}_\nu A^{Lk}_\mu\,,\; F^R_{\mu\nu} = \partial_\mu A^{Ri}_\nu - \partial_\nu A^{Ri}_\mu + g f^{ijk} A^{Lj}_\nu A^{Lk}_\mu\,, \tag{3.42}$$

$$D_\mu\Phi = \partial_\mu\Phi + ig\left(A^R_\mu\Phi - \Phi A^L_\mu\right) = \partial_\mu\Phi + ig\left(A^{Ri}_\mu L_i\,\Phi - \Phi\,A^{Li}_\mu L_i\right)\,. \tag{3.43}$$

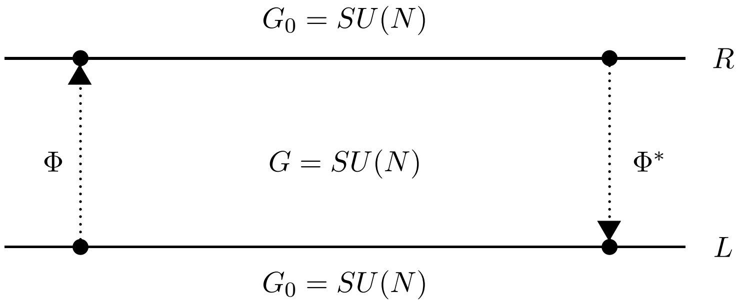

Figure 3.5: Effective bilayered transverse lattice model for the bulk gauge group $G = SU(N)$ and trivial orbifold projection and $P = \text{diag}(1, \ldots, 1)$.

The L_i denote the generators of G normalised as $\text{tr}(L_i L_j) = \frac{1}{2}\delta_{ij}$. Let $V(\Phi)$ take its minimum at Φ_{min} with

$$\Phi_{min} = \rho_{min} \, \frac{1}{\sqrt{2}} \mathbf{1}_N \ , \tag{3.44}$$

where $\mathbf{1}_N$ is the $N \times N$ unit matrix and $\rho_{min} \in \mathbb{R}^+_*$. Inserting (3.44) in (3.43) yields for the kinetic term

$$\text{tr}\left[(D_\mu \Phi_{min})^\dagger \, (D_\mu \Phi_{min}) \right] = \frac{1}{4} \, g^2 \rho_{min}^2 \left(A_\mu^{Ri} - A_\mu^{Li} \right)^2 \ . \tag{3.45}$$

Defining $A := (A_\mu^{Ri}, A_\mu^{Li})$, we can rewrite

$$\mathcal{L}_{mass} = \frac{1}{4} \, g^2 \rho_{min}^2 \left(A_\mu^{Ri} - A_\mu^{Li} \right)^2 = A \ M \ A^t \ , \tag{3.46}$$

where M is the mass-squared matrix

$$M = \frac{1}{4} \, g^2 \rho_{min}^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \ , \tag{3.47}$$

and A^t is the transpose of A. The gauge fields A_μ^{Ri} and A_μ^{Li} can be expressed as real linear combinations of their mass eigenstates $A_\mu^{i(0)}$ and $A_\mu^{i(1)}$

$$A_\mu^{Ri} = \frac{1}{\sqrt{2}} \left(A_\mu^{i(0)} + A_\mu^{i(1)} \right) \ , \tag{3.48}$$

$$A_\mu^{Li} = \frac{1}{\sqrt{2}} \left(A_\mu^{i(0)} - A_\mu^{i(1)} \right) \ .$$

In this new basis, the mass squared matrix M is diagonal. We obtain

$$\begin{aligned}\mathcal{L}_{mass} &= \frac{1}{4}\, g^2\rho_{min}^2 \left(A_\mu^{Ri} - A_\mu^{Li}\right)^2 \\ &= \frac{1}{8}\, g^2\rho_{min}^2 \left(A_\mu^{i(0)} + A_\mu^{i(1)} - A_\mu^{i(0)} - A_\mu^{i(1)}\right)^2 \\ &= \frac{1}{2}\, g^2\rho_{min}^2 \left(A_\mu^{i(1)}\right)^2 \;.\end{aligned} \tag{3.49}$$

This leads to the common mass term

$$m = g\rho_{min} \tag{3.50}$$

for all gauge fields $A_\mu^{i(1)}$. The gauge fields $A_\mu^{i(0)}$ remain massless.

We calculate the Lagrangian (3.41) in the basis of mass eigenstates (3.48). The Yang-Mills term reads

$$\begin{aligned}\mathcal{L}_{YM} &= -\frac{1}{4}\left(\partial_\mu A_\nu^{Li} - \partial_\nu A_\mu^{Li} + g f^{ijk} A_\mu^{Lj} A_\nu^{Lk}\right)^2 - \frac{1}{4}\left(\partial_\mu A_\nu^{Rl} - \partial_\nu A_\mu^{Rl} + g f^{lmn} A_\mu^{Rm} A_\nu^{Rn}\right)^2 \\ &= -\frac{1}{8}\Big(\partial_\mu A_\nu^{i(0)} - \partial_\nu A_\mu^{i(0)} - \left(\partial_\mu A_\nu^{i(1)} - \partial_\nu A_\mu^{i(1)}\right) \\ &+ \frac{g}{\sqrt{2}} f^{ijk}\left(A_\mu^{j(0)}A_\nu^{k(0)} - A_\mu^{j(0)}A_\nu^{k(1)} - A_\mu^{j(1)}A_\nu^{k(0)} + A_\mu^{j(1)}A_\nu^{k(1)}\right)\Big)^2 \\ &- \frac{1}{8}\Big(\partial_\mu A_\nu^{l(0)} - \partial_\nu A_\mu^{l(0)} + \left(\partial_\mu A_\nu^{l(1)} - \partial_\nu A_\mu^{l(1)}\right) \\ &+ \frac{g}{\sqrt{2}} f^{lmn}\left(A_\mu^{m(0)}A_\nu^{n(0)} + A_\mu^{m(0)}A_\nu^{n(1)} + A_\mu^{m(1)}A_\nu^{n(0)} + A_\mu^{m(1)}A_\nu^{n(1)}\right)\Big)^2 \;.\end{aligned} \tag{3.51}$$

We isolate the zero-mode. This yields

$$\mathcal{L}_0 = -\frac{1}{4}\left(\partial_\mu A_\nu^{i(0)} - \partial_\nu A_\mu^{i(0)} + \frac{g}{\sqrt{2}} f^{ijk} A_\mu^{j(0)} A_\nu^{k(0)}\right)^2 \;. \tag{3.52}$$

If we define the coupling constant

$$\tilde{g} = \frac{g}{\sqrt{2}} \;, \tag{3.53}$$

the zero mode has the canonical four-dimensional kinetic term with field strength

$$F_{\mu\nu}^{i(0)} = \partial_\mu A_\nu^{i(0)} - \partial_\nu A_\mu^{i(0)} + \tilde{g} f^{ijk} A_\mu^{j(0)} A_\nu^{k(0)} \;. \tag{3.54}$$

In contrast, for the first KK-mode we obtain

$$\mathcal{L}_1 = -\frac{1}{4}\left(\partial_\mu A_\nu^{i(1)} - \partial_\nu A_\mu^{i(1)}\right)^2 \;. \tag{3.55}$$

Note that a term $-\frac{1}{4}\left(\partial_\mu A_\nu^{i(1)} - \partial_\nu A_\mu^{i(1)} + g f^{ijk} A_\mu^{j(1)} A_\nu^{k(1)}\right)^2$ for the first excited KK-mode does not occur due to the relative minus sign for $\partial_\mu A_\nu^{i(1)} - \partial_\nu A_\mu^{i(1)}$ and $\partial_\mu A_\nu^{l(1)} - \partial_\nu A_\mu^{l(1)}$ in (3.51). However there will the term

$$-\frac{1}{4}\tilde{g}^2 f^{ijk} f^{imn} A_\mu^{j(1)} A_\nu^{k(1)} A_\mu^{m(1)} A_\nu^{n(1)} \ , \tag{3.56}$$

which describes the self-interaction of the first excited KK-modes. In addition, we obtain the following interaction terms among the zero mode and the first excited mode linear in $\tilde{g}$

$$\begin{aligned} \mathcal{L}_{\tilde{g}} &= -\frac{1}{4}\,\tilde{g}\,\left(\left(\partial_\mu A_\nu^{i(0)} - \partial_\nu A_\mu^{i(0)}\right) f^{ijk}\ A_\mu^{j(1)} A_\nu^{k(1)}\right. \\ &+ \left.\left(\partial_\mu A_\nu^{i(1)} - \partial_\nu A_\mu^{i(1)}\right) f^{ijk}\ A_\mu^{j(0)} A_\nu^{k(1)} + \left(\partial_\mu A_\nu^{i(1)} - \partial_\nu A_\mu^{i(1)}\right) f^{ijk}\ A_\mu^{j(1)} A_\nu^{k(0)}\right) \end{aligned} \tag{3.57}$$

and quadratic in $\tilde{g}$

$$\begin{aligned} \mathcal{L}_{\tilde{g}^2} &= -\frac{1}{4}\,\tilde{g}^2\ f^{ijk} f^{imn}\ \left(A_\mu^{j(0)} A_\nu^{k(1)} A_\mu^{m(0)} A_\nu^{n(1)} + A_\mu^{j(1)} A_\nu^{k(0)} A_\mu^{m(1)} A_\nu^{n(0)}\right) \\ &- \frac{1}{2}\,\tilde{g}^2\ f^{ijk} f^{imn}\ \left(A_\mu^{j(0)} A_\nu^{k(0)} A_\mu^{m(1)} A_\nu^{n(1)} + A_\mu^{j(0)} A_\nu^{k(1)} A_\mu^{m(1)} A_\nu^{n(0)}\right) \ . \end{aligned} \tag{3.58}$$

We compare this result to an $S^1/\mathbb{Z}_2$ continuum orbifold model. Let $G = SU(N)$ be the bulk gauge group. The five-dimensional Lagrangian reads

$$\mathcal{L}_{5D} = -\frac{1}{4} F^a_{MN} F^{aMN} \ , \tag{3.59}$$

where

$$F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + g_5 f^{abc} A^b_M A^c_N \ . \tag{3.60}$$

In this equation, g_5 is the five-dimensional gauge coupling constant. The boundary conditions for the gauge fields read

$$A_\mu(x^\mu, -y) = P\ A_\mu(x^\mu, y)\ P^{-1} \tag{3.61}$$

$$A_y(x^\mu, -y) = -P\ A_y(x^\mu, y)\ P^{-1} \ . \tag{3.62}$$

As in the Abelian case, we take the trivial orbifold projection

$$P = \mathrm{diag}(1, \ldots, 1) \ . \tag{3.63}$$

The Fourier expansion up to the first KK-mode reads

$$A^a_\mu(x^\mu, y) = \frac{1}{\sqrt{2\pi R}} A_\mu^{a(0)}(x^\mu) + \frac{1}{\sqrt{\pi R}} A_\mu^{a(1)}(x^\mu) \cos(\frac{y}{R}) \tag{3.64}$$

$$A^a_y(x^\mu, y) = \frac{1}{\sqrt{\pi R}} A_y^{a(1)}(x^\mu) \sin(\frac{y}{R}) \ . \tag{3.65}$$

The field strength F_{MN} consists of two parts

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_5 f^{abc} A^b_\mu A^c_\nu \qquad (3.66)\\ &= \frac{1}{\sqrt{2\pi R}}\left[\left(\partial_\mu A^{a(0)}_\nu - \partial_\nu A^{a(0)}_\mu\right) + \left(\partial_\mu A^{a(1)}_\nu - \partial_\nu A^{a(1)}_\mu\right)\cdot\sqrt{2}\cos(\frac{y}{R})\right]\\ &+ \frac{g_5}{2\pi R} f^{abc}\left[A^{b(0)}_\mu(x^\mu) + A^{b(1)}_\mu(x^\mu)\sqrt{2}\cos(\frac{y}{R})\right]\\ &\left[A^{c(0)}_\nu(x^\mu) + A^{c(1)}_\nu(x^\mu)\sqrt{2}\cos(\frac{y}{R})\right] ,\\ F^a_{\mu y} &= \partial_\mu A^a_y - \partial_y A^a_\mu + g_5 f^{abc} A^b_\mu A^c_y \qquad (3.67)\\ &= \frac{1}{\sqrt{\pi R}}\partial_\mu A^{a(1)}_y \cdot \sin(\frac{y}{R}) - \frac{1}{\sqrt{\pi R}} A^{a(1)}_\mu \frac{1}{R}\sin(\frac{y}{R})\\ &+ \frac{g_5}{2\pi R} f^{abc}\left[A^{b(0)}_\mu(x^\mu) + A^{b(1)}_\mu(x^\mu)\sqrt{2}\cos(\frac{y}{R})\right]\left[A^{a(1)}_y(x^\mu)\sqrt{2}\sin(\frac{y}{R})\right] . \end{aligned}$$

The second term in (3.67) will lead to mass terms for the gauge fields $A^{a(1)}_\mu$. As in the Abelian case, we compare this result to the ordinary non-Abelian Higgs model. The result is analogous to the Abelian case. In particular, the first excited KK-mode gauge fields $A^{a(1)}_y$ play again the role of Goldstone bosons. Therefore we go to axial gauge

$$A^{a(1)}_y = 0 . \qquad (3.68)$$

Due to the close analogy of the non-Abelian $S^1/\mathbb{Z}_2$ continuum orbifold model with truncated KK-mode expansion and the non-Abelian ordinary Higgs mechanism, we can conclude that also the non-Abelian truncated $S^1/\mathbb{Z}_2$ continuum orbifold model is renormalisable. In axial gauge (3.67) becomes

$$F^a_{\mu y} = -\frac{1}{\sqrt{\pi R}} A^{a(1)}_\mu \frac{1}{R}\sin(\frac{y}{R}) . \qquad (3.69)$$

We insert (3.66) and (3.69) into the five-dimensional Lagrangian (3.59) and integrate over the circle S^1. This yields

$$\begin{aligned} \mathcal{L}_{4D} &= \int_0^{2\pi R}\{-\frac{1}{4}F^a_{MN}F^{aMN}\} = \int_0^{2\pi R}\{-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2}F^a_{\mu y}F^{a\mu y}\}\, dy \qquad (3.70)\\ &= -\frac{1}{4}\left(\partial_\mu A^{a(0)}_\nu - \partial_\nu A^{a(0)}_\mu + \frac{g_5}{\sqrt{2\pi R}} f^{abc} A^{b(0)}_\mu A^{c(0)}_\nu\right)^2 - \frac{1}{4}\left(\partial_\mu A^{a(1)}_\nu - \partial_\nu A^{a(1)}_\mu\right)^2\\ &+ \frac{1}{2}\frac{1}{R^2}\left(A^{a(1)}_\mu\right)^2 + \mathcal{L}'_{g_5} + \mathcal{L}'_{g_5^2} , \end{aligned}$$

where $\mathcal{L}'_{g_5}$ and $\mathcal{L}'_{g_5^2}$ are interaction terms. Note that the zero mode has the canonical field strength

$$F^{a(0)}_{\mu\nu} = \partial_\mu A^{a(0)}_\nu - \partial_\nu A^{a(0)}_\mu + g_4 f^{abc} A^{b(0)}_\mu A^{c(0)}_\nu \qquad (3.71)$$

if we identify

$$g_4 = \frac{g_5}{\sqrt{2\pi R}} , \tag{3.72}$$

where g_4 is the four-dimensional effective gauge coupling constant. This relation is well-known from higher-dimensional gauge theories [74]. Note that while g_4 is dimensionless, g_5 has mass dimension $-1/2$. The comparison of (3.71) and (3.52) yields the relation

$$g_4 = \tilde{g} = \frac{g}{\sqrt{2}} \quad \Longrightarrow \quad g = \frac{g_5}{\sqrt{\pi R}} . \tag{3.73}$$

Finally we compare the Lagrangian (3.70) of the non-Abelian truncated $S^1/\mathbb{Z}_2$ continuum orbifold model in axial gauge (3.68) with the Lagrangian of the corresponding eBTLM. First, using

$$\int_0^{2\pi R} \cos(\frac{y}{R})\, dy = \int_0^{2\pi R} \sin(\frac{y}{R})\, dy = \int_0^{2\pi R} \cos^3(\frac{y}{R})\, dy = 0 , \tag{3.74}$$

$$\int_0^{2\pi R} \cos^2(\frac{y}{R})\, dy = \pi R \quad , \quad \int_0^{2\pi R} \cos^4(\frac{y}{R})\, dy = \frac{3}{4}\pi R \tag{3.75}$$

and inserting (3.72) an elementary but lengthy calculation shows that the interaction terms $\mathcal{L}'_{g_5}$ and $\mathcal{L}'_{g_5^2}$ in (3.70) equal (3.57) and (3.58) (including the term (3.56)), respectively. Second, if we require as in the Abelian case

$$g\rho_{min} = \frac{1}{R} , \tag{3.76}$$

both Lagrangian's equal and hence both theories describe the same physics. Therefore we conclude that also the eBTLM is *renormalisable*.

3.4 Orbifold conditions for nonunitary parallel transporters in the effective bilayered tranverse lattice model

In the last section we have restricted to the case where the orbifold projection P is trivial. In addition, we have made certain assumptions about the minimum Φ_{min} of the Higgs potential $V(\Phi)$, see (3.11) and (3.44). As a result, the gauge group G remained unbroken and the zero mode gauge fields remained massless.

In this section we determine orbifold conditions for nonunitary parallel transporters Φ. As a result, we can also handle non-trivial minima of the Higgs potential $V(\Phi)$ and non-trivial orbifold projections P. At first, we recall some standard facts about Lie algebras, which can be found in [53].

Theorem 1 (Cartan decomposition 1) *[53] Let H be a real semi-simple Lie group with Lie algebra $\mathfrak{h}$. Then $\mathfrak{h}$ has a Cartan involution θ. A Cartan involution θ of $\mathfrak{h}$ leads to an eigenspace decomposition*

$$\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{p} \tag{3.77}$$

of $\mathfrak{h}$ such that

$$[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}, \quad [\mathfrak{g}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{g} \tag{3.78}$$

and $\mathfrak{g}, \mathfrak{p}$ are $+1$ and -1 eigenspaces of θ, i.e.

- $\theta\, X = X \quad$ *for* $X \in \mathfrak{g}$
- $\theta\, X = -X \quad$ *for* $X \in \mathfrak{p} \quad .$

Let κ be the Killing form of $\mathfrak{h}$. Then $\mathfrak{g}$ and $\mathfrak{p}$ are orthogonal under κ and κ is negative definite on $\mathfrak{g}$ and positive definite on $\mathfrak{p}$.

Remark: If $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{h}$ then $\mathfrak{g} \otimes i\mathfrak{p}$ is a compact real form of its complexification $(\mathfrak{h})^{\mathbb{C}}$.

Theorem 2 (Cartan decomposition 2) *[53] Let H be a real semi-simple Lie group with Lie algebra $\mathfrak{h}$, let θ be a Cartan decomposition of its Lie algebra $\mathfrak{h}$ and let $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Suppose H has a finite centre, then G is the maximal compact subgroup of H and G has Lie algebra $\mathfrak{g}$. The elements of H can be written as*

$$h = g \exp X \ , \quad g \in G \ , X \in \mathfrak{p} \tag{3.79}$$

This decomposition is called the global Cartan decomposition.

Remark: The global Cartan decomposition generalises the polar decomposition of matrices.

Definition 2 *Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal Abelian Lie algebra in $\mathfrak{p}$ and let A be the corresponding subgroup of H. We call A a maximal noncompact Abelian subgroup of H.*

Remark: A is not unique and we will make use of this fact later.

Let us recall the definition of the Weyl group $W(G, A)$ of the pair (G, A). We use the notations above. Let W^* be the normaliser of $\mathfrak{a}$ in G, i.e.

$$W^* = \{g \in G \mid Ad(g)\mathfrak{a} \subset \mathfrak{a}\} \ , \tag{3.80}$$

where $Ad(g)\mathfrak{a} \subset \mathfrak{a}$ means that for all $x \in \mathfrak{a}$ we have $gxg^{-1} \in \mathfrak{a}$ [5]. Let W be the centraliser of a in G, i.e.

$$W = \{g \in G \mid Ad(g)x = x \text{ for all } x \in \mathfrak{a}\} \ . \tag{3.81}$$

Their quotient group is the Weyl group

$$W(G, A) = W^*/W \ . \tag{3.82}$$

Theorem 3 (Uniqueness of KAK-decomposition) *Let H be a reductive Lie group, G the maximal compact subgroup of H, A a maximal noncompact Abelian subgroup of H and $\mathfrak{a}$ the corresponding Abelian subspace in $\mathfrak{p}$. Then every element $h \in H$ admits a decomposition*

$$h = k_1 a k_2^{-1} \tag{3.83}$$

where $k_1, k_2 \in G$ and $a \in A$. In this decomposition

- *a is unique up to conjugation with elements of $W(G, A)$*
- *Given $a \in A$, let $W_a = \{g \in G \mid gag^{-1} = a\}$. Then k_1 and k_2 are unique up to right multiplication by an element of W_a, i.e.*

$$k_1 \to k_1' = k_1 k \quad , \quad k_2 \to k_2' = k_2 k \tag{3.84}$$

where $k \in W_a$.

Proof 1 *The existence of the decomposition can be found in [53]. It is based on the global Cartan decomposition $H = G \exp \mathfrak{p}$ and the equality $\mathfrak{p} = \cup_{g \in G} Ad(g)\mathfrak{a}$. Let us come to the uniqueness of (3.83). First, the proof for nonuniqueness of a can be found in [53]. Given now $a \in A$. Suppose*

$$k_1 a k_2^{-1} = k_1' a k_2'^{-1} \ . \tag{3.85}$$

If $\tilde{k}_1 = k_1'^{-1} k_1$ and $\tilde{k}_2 = k_2^{-1} k_2'$, then $\tilde{k}_1 a \tilde{k}_2 = a$ and therefore $(\tilde{k}_1 \tilde{k}_2)(\tilde{k}_2^{-1} a \tilde{k}_2) = a$. By the uniqueness of the global Cartan decomposition (3.79) it follows that $\tilde{k}_1 \tilde{k}_2 = 1$ and $\tilde{k}_2^{-1} a \tilde{k}_2 = a$. Thus $\tilde{k}_2 \in W_a$. Then

$$\tilde{k}_2 = k_2^{-1} k_2' \Longrightarrow k_2' = k_2 \tilde{k}_2 = k_2 k \ , \tag{3.86}$$

with $k = \tilde{k}_2 \in W_a$. In addition

$$\tilde{k}_1 = k_1'^{-1} k_1 \Longrightarrow k_1' = k_1 \tilde{k}_1^{-1} = k_1 \tilde{k}_2 = k_1 k \ , \tag{3.87}$$

because $\tilde{k}_1 \tilde{k}_2 = 1$.

[5] For matrixgroups the adjoint action Ad can be written as $Ad(g)x = gxg^{-1}$

Definition 3 *We call $a \in A$ generic if $W_a = W$.*

Corollary 1 *i) (generic case) Given $a \in A$ in (3.83). If a is generic, it follows that*

$$k_1' = k_1 w \ , \quad k_2' = k_2 w \tag{3.88}$$

in (3.85) where $w \in W$.
ii) (unitary case) Let $a = 1$ in (3.83). Then

$$k_1 k_2^{-1} = k_1' k_2'^{-1} \tag{3.89}$$

allows

$$k_1' = k_1 g \ , \quad k_2' = k_2 g \tag{3.90}$$

where $g \in G$ that is $W_a = G$ for $a = 1$.

Proof 2 *i) We know that $\tilde{k}_2^{-1} a \tilde{k}_2 = a$ where $\tilde{k}_2 = W_a$. Now let a be generic. Thus $\tilde{k}_2 \in W$.*
ii) is obvious since in this case $W_a = G$.

3.4.1 Orbifold conditions for nonunitary parallel transporters

We now determine orbifold conditions for nonunitary PTs. Let G be the unitary gauge group of the bulk and let $\mathfrak{g}$ be its Lie algebra. Any orbifold projection $P \in G$ can be written as an exponential of some Lie algebra element and is therefore contained in some $U(1)$ subgroup of G. If we start with this $U(1)$ subgroup, we can construct a maximal torus $T \subset G$. By $\mathfrak{t}$ we denote the Lie algebra of T. Let $\mathfrak{t} = \text{Lie } T$. Let $\{H_i\}_{i=1}^r$, with $r = rank\ \mathfrak{g}$, denote the generators of $\mathfrak{t}$. Since $P \in T$ by construction we can always write

$$P = \exp(-2\pi i\ \vec{V} \cdot \vec{H}) \ , \tag{3.91}$$

where $\vec{V}$ is a shift vector [6] and $\vec{H} = (H_1, H_2, \ldots, H_r)$. The shift vector $\vec{V}$ is an element of the weight space of $\mathfrak{g}$. We consider the case where $\mathfrak{g}$ can be obtained from a complex Lie algebra $\mathfrak{h}$, i.e.

$$\mathfrak{h} = \mathfrak{g} \oplus i\mathfrak{g} \ . \tag{3.92}$$

An important example is $\mathfrak{h} = \mathfrak{sl}(N, \mathbb{C})$. In this case $\mathfrak{g} = \mathfrak{su}(N)$.

Let $\mathfrak{a}$ be a maximal Abelian Lie algebra in $i\mathfrak{g}$ and let A be its corresponding subgroup in G according to Definition 2. The KAK-decomposition (3.83) holds for any choice of $\mathfrak{a}$. It is natural to make the special choice

$$\mathfrak{a} = i\mathfrak{t} \ . \tag{3.93}$$

[6] Every possible orbifold projection P can be specified by a corresponding shift vector $\vec{V}$. Shift vectors are listed in the literature for many gauge groups, see e.g. [9].

Let $\eta \in \mathfrak{a}$. Then we have by construction

$$P\eta P^{-1} = \eta \; . \tag{3.94}$$

Thus $P \in W_{e^\eta}$.

Example 2 *Let $G = SU(N)$ and $P^2 = 1$. Without loss of generality we can write P as*

$$P = diag(\underbrace{1,\ldots,1}_{N-m},\underbrace{-1,\ldots,-1}_{m}) \tag{3.95}$$

for $1 \leq m \leq N$ and m is restricted to be an even integer. Let $\mathfrak{h} = \mathfrak{sl}(N,\mathbb{C})$. The Cartan decomposition of $\mathfrak{sl}(N,\mathbb{C})$ reads $\mathfrak{sl}(N,\mathbb{C}) = \mathfrak{su}(N) + i\mathfrak{su}(N)$. We are free to choose

$$\mathfrak{a} = \{\eta = diag(a_1,\ldots,a_N)\} \subset i\mathfrak{su}(N) \; , \tag{3.96}$$

where $\sum a_i = 0$, $a_i \in \mathbb{R}$. It follows that

$$P\eta P^{-1} = \eta \; . \tag{3.97}$$

Thus $P \in W_{e^\eta}$.

Let $V_L(V_R)$ be the fibre over the $L(R)$-boundary. The parallel transporter Φ is a map $\Phi: \; V_R \to V_L$. In addition, the parallel transporter Φ^* in the backwards direction is a map $\Phi^*: \; V_L \to V_R$. If we identify V_L and V_R via a map $i: V_L \to V_R$ [60] ($i^{-1}: V_R \to V_L$), there remains the freedom that $\Phi \in H$ transforms under a unitary gauge transformation according to

$$\Phi \mapsto S(x)\Phi S(x)^{-1} \; , \tag{3.98}$$

where $S(x) \in G$. Hence we can require for $\Phi, \Phi^* \in H$ and $P \in T \subset G$ the orbifold condition

$$\Phi = P\,\Phi^* P^{-1} \; , \tag{3.99}$$

where $P^* = P^{-1}$. According to (3.83), we can write $\Phi \in H$ as

$$\Phi = U_L\, e^\eta\, U_R^* \; , \tag{3.100}$$

where $\eta \in \mathfrak{a} = i\mathfrak{t}$, $\mathfrak{t} = \text{Lie}\; T$ and $U_L, U_R \in G$. We insert (3.100) in (3.99). Consequently we get for the right-hand side of (3.99)

$$P\Phi^* P^{-1} = PU_R e^{\eta^*} U_L^* P^{-1} = PU_R e^{\eta} U_L^* P^{-1} \; . \tag{3.101}$$

The second equation holds since e^η is selfadjoint. According to Theorem 3 however, the decomposition (3.100) is not unique. The comparison of (3.100) with (3.101) tells us that there is a $K \in W_{e^\eta}$ such that

$$U_L = PU_R K \tag{3.102}$$

$$U_R^* = K^* U_L^* P^{-1} \; . \tag{3.103}$$

Let us consider (3.103). We obtain

$$U_R^* = K^* U_L^* P^{-1} \Longrightarrow U_R = P U_L K \Longrightarrow U_L = P^{-1} U_R K^{-1} \ . \tag{3.104}$$

We now restrict to involutive P. On the orbifold $S^1/\mathbb{Z}_2$ this restriction is empty because P fulfils already $P^2 = 1$. In addition, because in one extra dimension there exits only one orbifold, namely $S^1/\mathbb{Z}_2$, the following is true for all orbifold models in one extra dimension. Since $P^2 = 1$ it follows that $P = P^{-1}$. Then (3.104) and (3.102) are compatible if

$$K^2 = 1 \ . \tag{3.105}$$

This result shows that K can be interpreted as a projection. Recapitulating, we can write $\Phi \in H$ as

$$\Phi = U_L \, e^{\eta} \, K U_L^* P^{-1} \ , \tag{3.106}$$

where $K, P \in W_{e^\eta}$ and $K^2 = P^2 = 1$. This result motivates the following

Definition 4 (Sharpened orbifold condition for nonunitary Φ) *Let H be a reductive Lie group, G the maximal compact subgroup of H, A a maximal noncompact Abelian subgroup of H and $\mathfrak{a}$ the corresponding maximal Abelian Lie Algebra in $\mathfrak{p}$. $\Phi \in H$ can be decomposed according to Theorem 3 as*

$$\Phi = U_L \, e^{\eta} \, U_R^* \ , \tag{3.107}$$

where $\eta \in \mathfrak{a}$, $U_L, U_R \in G$ and $\mathfrak{a}$ is not unique. Given $P \in G$. Then we demand that

1. $$P \eta P^{-1} = \eta \tag{3.108}$$

2. *η, U_L, U_R satisfy the condition*

$$U_R = P U_L P^{-1} \tag{3.109}$$

for a suitable choice of $\mathfrak{a}$.

Remark: For a complex Lie group H one can choose a maximal Abelian Lie Algebra $\mathfrak{a}$ such that (3.108) is automatically fulfilled.

Corollary 2 *If Φ fulfils the sharpened orbifold condition and P is involutive, then Φ also satisfies $\Phi = P \Phi^* P^{-1}$.*

Proof 3 *If Φ fulfils the sharpened orbifold condition, we can decompose*

$$\Phi = U_L \, e^{\eta} \, U_R^* \ , \tag{3.110}$$

where $\eta \in \mathfrak{a}$, $U_L, U_R \in G$, $P \in G$ *with* $P\eta P^{-1} = \eta$ *and* $U_R = PU_LP^{-1}$. *We get*

$$P\Phi^* P^{-1} = P\, U_R\, e^{\eta}\, U_L^*\, P^{-1} \overset{!}{=} P\, U_R\, P^{-1}\, e^{\eta}\, P\, U_L^*\, P^{-1}\ , \tag{3.111}$$

where in the second step we have used that $P\eta P^{-1} = \eta$. *Since* $U_R = PU_LP^{-1}$ *and* $P^2 = 1$, *we further obtain*

$$P\, U_R\, P^{-1}\, e^{\eta}\, P\, U_L^*\, P^{-1} = U_L\, e^{\eta}\, U_R^* = \Phi\ . \tag{3.112}$$

Let $\Phi \in H$ fulfil the sharpened orbifold condition. Then Φ can be decomposed as

$$\Phi = U_L e^{\eta} U_R^* = U_L\, e^{\eta}\, PU_L^* P^{-1}\ , \tag{3.113}$$

where $P\eta P^{-1} = \eta$. P acts on G through an automorphism on its Lie algebra $\mathfrak{g}$. Let G_0 be the centraliser of P in G. Since P is an involutive automorphism $\mathfrak{g}$ splits as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1\ , \tag{3.114}$$

where $[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_0, [\mathfrak{g}_0, \mathfrak{g}_1] \subseteq \mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_0$ and $\mathfrak{g}_0 = \text{Lie}\ G_0$. G_0 is called the orbifold unbroken gauge group. $\mathfrak{g}_1$ is the orthogonal complement of $\mathfrak{g}_0$ and may be viewed as the tangent vector to the coset space G/G_0. Let $g_0 \in G_0$ and $A_y \in \mathfrak{g}_1$. Then g_0 and A_y fulfil

$$Pg_0P^{-1} = g_0\ , \tag{3.115}$$
$$PA_yP^{-1} = -A_y\ . \tag{3.116}$$

We can decompose U_L (at least in a small neighbourhood of the identity) as

$$U_L = g_0\, e^{A_y} \tag{3.117}$$

according to the action of P on G. We insert this decomposition into (3.113) and obtain

$$\begin{aligned}
\Phi &= g_0 e^{A_y}\, e^{\eta}\, P(g_0 e^{A_y})^* P^{-1} && (3.118)\\
&= g_0 e^{A_y}\, e^{\eta}\, Pe^{A_y^*} \underbrace{P^{-1}P}_{=1} g_0^* P^{-1} && (3.119)\\
&= g_0 e^{A_y}\, e^{\eta}\, Pe^{A_y^*} P^{-1}\, g_0^* && (3.120)\\
&= g_0 e^{A_y} g_0^{-1}\, g_0 e^{\eta} g_0^{-1}\, g_0 e^{A_y} g_0^{-1} && (3.121)\\
&= e^{A_y'}\, e^{\eta'}\, e^{A_y'}\ , && (3.122)
\end{aligned}$$

where

$$A_y' = g_0 A_y g_0^{-1} \in \mathfrak{g}_1 \tag{3.123}$$
$$\eta' = g_0 \eta g_0^{-1} \in \mathfrak{a}' = Ad(g_0)\mathfrak{a}\ . \tag{3.124}$$

We summarise this result in the following

Theorem 4 *Suppose that $\Phi \in H$ fulfils the sharpened orbifold condition. Then Φ can be decomposed as*

$$\Phi = U_L e^{\eta} U_R^* = U_L \, e^{\eta} \, P U_L^* P^{-1} \, , \tag{3.125}$$

where $\eta \in \mathfrak{a}$, $U_L, U_R \in G$, $\mathfrak{a}$ such that $P\eta P^{-1} = \eta$ and U_L, U_R such that $U_R = PU_L P^{-1}$. Let $P \in G$ be involutive and let $\mathfrak{g} = Lie\ G$ split as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{3.126}$$

according to the action of P on $\mathfrak{g}$. Then Φ can be written as

$$\Phi = e^{A_y} \, e^{\eta} \, e^{A_y} \, , \tag{3.127}$$

and

$$PA_y P^{-1} = -A_y \, , \tag{3.128}$$
$$P\eta P^{-1} = \eta \, , \tag{3.129}$$

where $A_y \in \mathfrak{g}_1$ and $\eta \in \mathfrak{a}$. Since Φ in H fulfils the sharpened orbifold condition P has the property

$$P \in W_{e^{\eta}} \, . \tag{3.130}$$

Remarks: i) For unitary Φ, that is $e^{\eta} = 1$, we have $W_{e^{\eta}} = G$ and therefore $P \in G$. Φ can be written as

$$\Phi = e^{2A_y} \, . \tag{3.131}$$

Thus if $e^{\eta} = 1$ we recover the conventional orbifold case.
ii) If $a = e^{\eta}$ is generic then Corollary 1 yield $W_{e^{\eta}} = W$. Thus $P \in W$.

Example 3 *We consider the Lie algebra $\mathfrak{h} = sl(3, \mathbb{C})$. It possesses the Cartan decomposition $sl(3, \mathbb{C}) = su(3) \oplus isu(3)$. Let $\mathfrak{a} = \{\eta = diag(a_1, a_2, a_3)\}$ be a maximal Abelian Lie algebra of $\mathfrak{p}$, where $\sum a_i = 0$, $a_i \in \mathbb{R}$. In addition, let $\lambda_i, i = 1, \ldots, 8$ denote the Gell-Mann matrices generating $SU(3)$. Let $\eta \in \mathfrak{a}$ be generic, i.e. $\eta = (a_1, a_2, a_3)$ where the a_i are all distinct. Then $W_{e^{\eta}} = W$ is the torus T consisting of all diagonal matrices in $SU(3)$. Since $P \in T$, the orbifold projection P has to be a diagonal matrix. For example we can choose $P \in T \subset G$ as*

$$P = \exp(i\pi\sqrt{3}\lambda_8) = diag(-1, -1, 1) \, . \tag{3.132}$$

Note that this choice for P breaks the unitary gauge group $G = SU(3)$ down to $G_0 = SU(2) \times U(1)$.

3.5 Spontaneous symmetry breaking

In this section, we discuss the topic of spontaneous symmetry breaking in detail. First we introduce some terminology.

Definition 5 *Let $G_{0\eta}$ be the centraliser of $\eta \in \mathfrak{a}$ in G_0, i.e.*

$$G_{0\eta} = \{g \in G_0 \mid Ad(g)\eta = \eta\} \ . \tag{3.133}$$

$G_{0\eta}$ is called the unbroken subgroup of G_0 with respect to η.

In the context of orbifold- and spontaneous symmetry breaking one is usually faced with the situation where first the bulk gauge group G is broken by orbifolding to G_0 at high energies and second G_0 is broken further spontaneously to $G_{0\eta}$. We schematically write

$$G \xrightarrow{P} G_0 \xrightarrow{\eta} G_{0\eta} \ . \tag{3.134}$$

Let us consider the case where $P = 1$. For $P = 1$ the unitary gauge group G remains unbroken and $\mathfrak{g}_0$ the Lie algebra of G_0 equals $\mathfrak{g}$. Suppose Φ fulfils the sharpened orbifold condition. Thus we can write Φ according to Theorem 4 as

$$\Phi = U_L e^{\eta} U_L^* \ , \tag{3.135}$$

where $\eta \in \mathfrak{a}$, $U_L, U_L^* \in G$ and $\mathfrak{a}$ such that $P\eta P = \eta$. This follows directly from the decomposition (3.125) with $P = 1$. Let us consider the Higgs potential

$$V(\Phi) = V(U_L e^{\eta} U_L^*) \ . \tag{3.136}$$

Since Φ transforms under a unitary gauge transformation as

$$\Phi \mapsto S(x)\Phi S(x)^{-1} \ , \tag{3.137}$$

where $S(x) \in G$, the unitary factors $U_L, U_L^* \in G$ in (3.136) can be transformed away. As a consequence the Higgs potential $V(\Phi)$ can be written as a function which depends only on η, and we have

Corollary 3 *Suppose that the Higgs potential $V(\Phi)$ is G-invariant, i.e.*

$$V(S(x)\Phi S(x)^{-1}) = V(\Phi) \tag{3.138}$$

for all $S(x) \in G, \Phi \in H$. Then there exists a function $\mathcal{V}$ on $\mathfrak{a}$ such that

$$V(\Phi) = V(g e^{\eta} g^*) = \mathcal{V}(\eta) \quad \textit{for all}\ g \in G, \eta \in \mathfrak{a} \tag{3.139}$$

and

$$\mathcal{V}(\eta) = \mathcal{V}(w(\eta)) \quad \textit{for all}\ w \in W(G, A) \ . \tag{3.140}$$

Thus $W(G,A)$ is a discrete group of symmetries of the Higgs potential $V(\Phi)$.

Definition 6 *Let $\mathcal{W}_\eta \subset W(g,A)$ be the subgroup of elements which leave η invariant, i.e.*

$$\mathcal{W}_\eta = \{\omega \in W(H,A) \mid \omega(\eta) = \eta\} \ . \tag{3.141}$$

We note that $\mathcal{W}_\eta = (G_{0\eta} \cap W^*)/W$.

Definition 7 *We call $\eta \in \mathfrak{a}$ generic if $G_{0\eta} = W$.*

Corollary 4 *If $\eta \in \mathfrak{a}$ is generic, then $\mathcal{W}_\eta$ is trivial.*

The proof is obvious.

Example 4 *We consider the Lie algebra $\mathfrak{h} = sl(3,\mathbb{C})$. It possesses the Cartan decomposition $sl(3,\mathbb{C}) = su(3) \oplus isu(3)$. Let $\mathfrak{a} = \{\eta = diag(a_1,a_2,a_3)\}$ be a maximal Abelian Lie algebra of $\mathfrak{p}$, where $\sum a_i = 0$, $a_i \in \mathbb{R}$. In addition, let $\lambda_i, i = 1,\ldots,8$ denote the Gell-Mann matrices generating $SU(3)$.*
First, let $\eta \in \mathfrak{a}$ be generic, i.e. $\eta = (a_1,a_2,a_3)$ where the a_i are all distinct. Then $G_{0\eta} = T = W$ is a Cartan subgroup of $SU(3)$. Since $G_{0\eta} = W$, it follows that $\mathcal{W}_\eta$ is trivial.
Second, let $\eta \in \mathfrak{a}$ be nongeneric, e.g. $\eta = (a_1,a_2,a_3)$ where $a_1 = a_2$. In this case the unbroken subgroup $G_{0\eta}$ of $G = SU(3)$ is generated by $\{\lambda_1,\lambda_2,\lambda_3,\lambda_8\}$ and consequently $G_{0\eta} = SU(2)\times U(1)$. The Weyl group $\mathcal{W}_\eta$ is $\mathcal{S}_2$, that is the permutation group of the two elements (a_1,a_2).

Next we consider the case where $P \neq 1$. Then $\mathfrak{g} = \text{Lie } G$ splits as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{3.142}$$

according to the action of P on $\mathfrak{g}$. Hence the orbifold unbroken gauge group G_0 has Lie algebra $\mathfrak{g}_0$. Suppose $\Phi \in H$ fulfils the sharpened orbifold condition. According to Theorem 4, we can write

$$\Phi = e^{A_y}\, e^{\eta}\, e^{A_y} \ , \tag{3.143}$$

and

$$PA_yP^{-1} = -A_y \ , \tag{3.144}$$

$$P\eta P^{-1} = \eta \ , \tag{3.145}$$

where $A_y \in \mathfrak{g}_1$ and $\eta \in \mathfrak{a}$. We consider the Higgs potential

$$V(\Phi) = V(e^{A_y}\, e^{\eta}\, e^{A_y}) \ . \tag{3.146}$$

Since G is broken to G_0, Φ transforms under a unitary gauge transformation as

$$\Phi \mapsto S_0(x)\Phi S_0(x)^{-1} \,, \tag{3.147}$$

where $S_0(x) \in G_0$. In contrast to the case where $P = 1$, the unitary factors e^{A_y} in (3.146) cannot be gauged away due to the lack of gauge invariance. Therefore the Higgs potential $V(\Phi)$ depends also A_y and we have

Theorem 5 *Suppose that $\Phi \in H$ can be written as*

$$\Phi = e^{A_y}\, e^{\eta}\, e^{A_y} \,, \tag{3.148}$$

and

$$PA_yP^{-1} = -A_y \,, \tag{3.149}$$
$$P\eta P^{-1} = \eta \,, \tag{3.150}$$

where $A_y \in \mathfrak{g}_1$, $\eta \in \mathfrak{a}$ for a suitable choice of $\mathfrak{a}$. The action of $P \in W_{e^\eta}$ leads to a split $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where the orbifold unbroken gauge group G_0 has Lie algebra $\mathfrak{g}_0$. Then

1. *the Higgs potential $V(\Phi)$ is G_0-invariant*

$$V(S_0(x)\Phi S_0(x)^{-1}) = V(\Phi) \tag{3.151}$$

   for all $S_0(x) \in G_0, \Phi \in H$.

2. *there exists a function $\mathcal{V}$ on $\mathfrak{a} \times \mathfrak{g}_1$ such that*

$$V(\Phi) = V(e^{A_y}\, e^{\eta}\, e^{A_y}) = \mathcal{V}(\eta, A_y) \,. \tag{3.152}$$

   Due to (3.151) we have

$$\mathcal{V}(\eta', A'_y) = \mathcal{V}(\eta, A_y) \,, \tag{3.153}$$

   when $\eta' \in \mathfrak{a}' = Ad(g_0)\mathfrak{a}$, $A'_y = g_0 A_y g_0^{-1} \in \mathfrak{g}_1$ for some $g_0 \in G_0$.

3. *in (3.152) A_y cannot be gauged away because $S_0(x)$ in (3.151) is restricted to $G_0 \subset G$.*

3.6 The customary approximation scheme of a truncated $S^1/\mathbb{Z}_2$ orbifold model

In section 3.3 we have obtained that for a non-Abelian gauge theory with gauge group $G = SU(N)$ and trivial orbifold projection P

1. an eBTLM with minimum of the Higgs potential $V(\Phi)$ of the form

$$\Phi_{min} = \rho_{min} \frac{1}{\sqrt{2}} \mathbf{1}_N , \tag{3.154}$$

   see (3.44), leads to the common mass term

$$m = g\rho_{min} \tag{3.155}$$

   for all first excited KK mode gauge fields.

2. such an eBTLM with Φ_{min} given by (3.154) equals a $S^1/\mathbb{Z}_2$ continuum orbifold model with trivial orbifold projection and a Fourier mode expansion truncated for all fields at the first Kaluza-Klein mode, if we require

$$g\rho_{min} = \frac{1}{R} , \tag{3.156}$$

   see (3.76).

Note that in this equation g is the dimensionless four-dimensional effective gauge coupling constant of the eBTLM and thus ρ_{min} has mass dimension 1.

We will now derive (3.154). In the following calculations we restrict ourselves for simplicity to the bulk gauge group $G = SU(2)$. However, the results of this section can be generalised to $G = SU(N)$ in a straightforward way. For $G = SU(2)$, we assume that the holonomy group is given by $H = \mathbb{R}^+_* \, SL(2,\mathbb{C})$, with $\mathbb{R}^+_* = \mathbb{R}^+/\{0\}$. Let $\Phi \in H$ fulfil the sharpened orbifold condition. Then $\Phi \in H$ can be written according to Theorem 4 as

$$\Phi = \rho \, U_L e^{\eta} U_R^* = \rho \, U_L \, e^{\eta} \, P U_L^* P^{-1} , \tag{3.157}$$

where $\eta \in \mathfrak{a}$, $U_L, U_R \in SU(2)$, $\rho \in \mathbb{R}^+_*$, $\mathfrak{a}$ such that $P\eta P^{-1} = \eta$ and U_L, U_R such that $U_R = PU_L P^{-1}$. We have to make a choice for $\mathfrak{a}$. Since H is complex we can always choose a maximal Abelian Lie algebra $\mathfrak{a} \subset i\mathfrak{su}(2)$, such that $P\eta P^{-1} = \eta$ is automatically fulfilled. Without loss of generality suppose P is diagonal. Then

$$\mathfrak{a} = \{\eta = diag(a_1, a_2)\} \quad , \quad a_1 = -a_2 \, , \quad a_i \in \mathbb{R} \tag{3.158}$$

is a maximal Abelian Lie algebra of $i\mathfrak{su}(2)$ and $P\eta P^{-1} = \eta$ for all $\eta \in \mathfrak{a}$. We first focus on the case where P is trivial, i.e. $P = \mathrm{diag}(1, 1)$. Hence (3.157) reads

$$\Phi = \rho \, U_L \, e^{\eta} \, U_L^* . \tag{3.159}$$

Let us consider the Higgs potential

$$V(\Phi) = V(\rho \, U_L \, e^{\eta} \, U_L^*) . \tag{3.160}$$

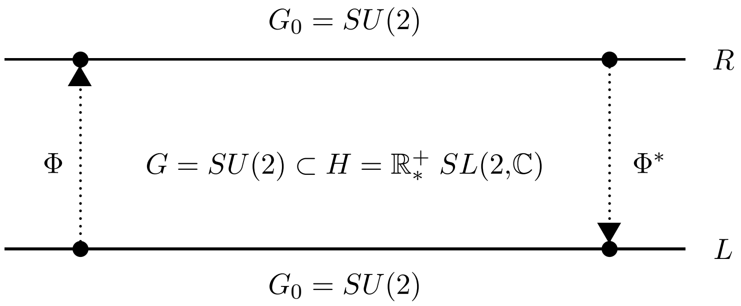

Figure 3.6: Effective bilayered transverse lattice model for bulk gauge group $G = SU(2)$ and trivial orbifold projection $P = \mathrm{diag}(1,1)$.

Φ transforms under a unitary gauge transformation according to $\Phi \to S(x)\Phi S(x)^{-1}$, where $S(x) \in SU(2)$. Consequently the unitary factor U_L in (3.160) can be transformed away. Thus, according to Corollary 3, the Higgs potential $V(\phi)$ depends only on η and ρ

$$V(\rho\, U_L\, e^{\eta}\, U_L^*) = V(\rho\, e^{\eta}) = \mathcal{V}(\rho, \eta)\ . \tag{3.161}$$

Let $V(\Phi)$ assume its minimum at Φ_{min}. According to (3.161) we can parametrise any Φ_{min} as

$$\Phi_{min} = \rho_{min} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{a_1} & 0 \\ 0 & e^{a_2} \end{pmatrix}\ ,\ a_1 = -a_2\ ,\ a_i \in \mathbb{R}\ . \tag{3.162}$$

Note that the a_i in (3.162) are dimensionless parameters. Since P is trivial, the bulk gauge group $G = SU(2)$ remains unbroken at both boundaries. Figure 3.6 summarises the setting. In order to arrive at (3.154) for $N = 2$, we set $a_1 = -a_2 = 0$ and (3.162) becomes

$$\Phi_{min} = \rho_{min} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ . \tag{3.163}$$

We know from section 3.3 that (3.163) leads to the mass term

$$\mathcal{L}_{mass} = tr \left[(D_\mu \Phi_{min})^\dagger\, (D_\mu \Phi_{min}) \right] = \frac{1}{2}\, g^2 \rho_{min}^2 \left(A_\mu^{i(1)} \right)^2\ , \tag{3.164}$$

with $i = 1, 2, 3$, i.e. only the first excited KK mode gauge fields $A_\mu^{i(1)}$ become massive with common mass $m = g\rho_{min}$. All zero mode gauge fields $A_\mu^{i(0)}$ remain massless. Therefore we make the following

Definition 8 *We call*

$$\Phi_{min} = \rho_{min} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.165}$$

the trivial minimum of the Higgs potential $V(\Phi)$ for $G = SU(2)$.

From section 3.3.2 we also know that a truncated $S^1/\mathbb{Z}_2$ orbifold model with bulk gauge group $G = SU(2)$ and trivial orbifold projection $P = \text{diag}(1,1)$ leads to the mass term (3.70)

$$\mathcal{L}_{mass} = \frac{1}{2} \frac{1}{R^2} \left(A_\mu^{i(1)} \right)^2 , \tag{3.166}$$

with $i = 1, 2, 3$. If we insert $g\rho_{min} = 1/R$ in (3.164), (3.166) and (3.164) coincide. In fact we know from the discussion in section 3.3.2 that also the effective four-dimensional Lagrangian of the truncated $S^1/\mathbb{Z}_2$ continuum orbifold model equals the effective four-dimensional Lagrangian of the corresponding eBTLM. Therefore we conclude

Proposition 1 *An $S^1/\mathbb{Z}_2$ continuum orbifold model with bulk gauge group $G = SU(2)$, trivial orbifold projection $P = diag(1,1)$ and a Fourier mode expansion for all gauge fields truncated at the first excited Kaluza-Klein mode in axial gauge equals an effective bilayered transverse lattice model with bulk gauge group $G = SU(2)$, trivial orbifold projection $P = diag(1,1)$ and trivial minimum of the Higgs potential $V(\Phi)$.*

Remark: i) In general the minimum Φ_{min} of the Higgs potential is given by

$$\Phi_{min} = \rho_{min} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{a_1} & 0 \\ 0 & e^{a_2} \end{pmatrix} , \tag{3.167}$$

where $a_1 = -a_2 \neq 0$. Therefore an eBTLM with nonunitary parallel transporter Φ is richer in its physical content than a truncated $S^1/\mathbb{Z}_2$ continuum orbifold model.

3.7 Beyond the customary approximation scheme of a truncated $S^1/\mathbb{Z}_2$ orbifold model: Exponential gauge boson masses

In this section we consider the case where the minimum Φ_{min} of the Higgs potential $V(\Phi)$ is non-trivial, i.e.

$$\Phi_{min} = \rho_{min} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{a_1} & 0 \\ 0 & e^{a_2} \end{pmatrix} , a_1 = -a_2 \neq 0 . \tag{3.168}$$

We calculate the mass term for the $SU(2)$ gauge bosons by computing the kinetic term

$$\mathcal{L}_{mass} = tr \left[(D_\mu \Phi_{min})^\dagger (D_\mu \Phi_{min}) \right] . \tag{3.169}$$

We start with the covariant derivative

$$D_\mu\Phi = \partial_\mu\Phi + ig\left(A_\mu^R\Phi - \Phi A_\mu^L\right) \ , \tag{3.170}$$

where

$$A_\mu^R = A_\mu^{Ri} t_i \quad , \quad A_\mu^L = A_\mu^{Li} t_i \ , \tag{3.171}$$

$t_i = \frac{1}{2}\tau_i$ and τ_i denote the Pauli matrices. Note that $\mathrm{tr}\,(t_i t_j) = \frac{1}{2}\delta_{ij}$. We transform

$$\begin{aligned} A_\mu^{Ri} &= \frac{1}{\sqrt{2}}\left(A_\mu^{i(0)} + A_\mu^{i(1)}\right) \\ A_\mu^{Li} &= \frac{1}{\sqrt{2}}\left(A_\mu^{i(0)} - A_\mu^{i(1)}\right) \ . \end{aligned} \tag{3.172}$$

Recall that $A_\mu^{i(0)}$ and $A_\mu^{i(1)}$ denote mass eigenstates. The covariant derivative (3.170) reads in terms of these mass eigenstates

$$D_\mu\Phi = \partial_\mu\Phi + i\frac{g}{\sqrt{2}}A_\mu^{i(0)}\,[t_i,\Phi] + i\frac{g}{\sqrt{2}}A^{i(1)}\{t_i,\Phi\} \ , \tag{3.173}$$

where $[,]$ and $\{,\}$ denote the commutator and anticommutator, respectively. In order to calculate $[t_i,\Phi_{min}]$ and $\{t_i,\Phi_{min}\}$, respectively, it is convenient to add the generator $t_0 = \frac{1}{2}\mathbf{1}_2$ to the generators of $SU(2)$. The set $\{t_i\}, i = 0,\ldots,3$ is a basis of the Lie algebra $u(2)$ of $U(2)$. We can expand every diagonal 2×2 matrix ϕ is terms of t_3 and t_0 as

$$\phi = \phi_0 t_0 + \phi_3 t_3 = \frac{1}{2}\begin{pmatrix} \phi_0+\phi_3 & 0 \\ 0 & \phi_0-\phi_3 \end{pmatrix} \ . \tag{3.174}$$

Using this expansion we rewrite Φ_{min} as

$$\Phi_{min} = \rho_{min}\frac{1}{\sqrt{2}}\begin{pmatrix} e^{a_1} & 0 \\ 0 & e^{a_2} \end{pmatrix} =: \frac{1}{2}\begin{pmatrix} \phi_0+\phi_3 & 0 \\ 0 & \phi_0-\phi_3 \end{pmatrix} \ , \tag{3.175}$$

where

$$\phi_0 = \rho_{min}\frac{1}{\sqrt{2}}\left(e^{a_1} + e^{a_2}\right) \tag{3.176}$$

$$\phi_3 = \rho_{min}\frac{1}{\sqrt{2}}\left(e^{a_1} - e^{a_2}\right) \ . \tag{3.177}$$

For the commutators and anticommutators we obtain

$$[t_i,\Phi_{min}] = [t_i,\phi_0 t_0 + \phi_3 t_3] = \phi_0 \underbrace{[t_i,t_0]}_{=0} + \phi_3 \underbrace{[t_i,t_3]}_{=i\epsilon_{i3k}t_k} \ , \tag{3.178}$$

$$\{t_i,\Phi_{min}\} = \{t_i,\phi_0 t_0 + \phi_3 t_3\} = \phi_0 \underbrace{\{t_i,t_0\}}_{=t_i} + \phi_3 \underbrace{\{t_i,t_3\}}_{\delta_{i3}t_0} \ . \tag{3.179}$$

Inserting $[t_i, \Phi_{min}]$ and $\{t_i, \Phi_{min}\}$ in (3.173) yields

$$D_\mu \Phi_{min} = -\frac{g}{\sqrt{2}}\, A_\mu^{i(0)}\; \phi_3 \epsilon_{i3k} t_k + i \frac{g}{\sqrt{2}}\, A_\mu^{i(1)}\; (\phi_0 t_i + \phi_3 t_0 \delta_{i3}) \; . \tag{3.180}$$

Taking the adjoint $(D_\mu \Phi_{min})^\dagger = -\frac{g}{\sqrt{2}}\, A_\mu^{i(0)}\; \phi_3 \epsilon_{i3k} t_k - i \frac{g}{\sqrt{2}}\, A_\mu^{i(1)}\, (\phi_0 t_i + \phi_3 t_0 \delta_{i3})$ and multiplying $(D_\mu \Phi_{min})^\dagger$ by $D_\mu \Phi_{min}$ we obtain

$$\begin{aligned} & (D_\mu \Phi_{min})^\dagger\, (D_\mu \Phi_{min}) \\ = \;& \frac{1}{2} g^2\; A_\mu^{i(0)} A_\mu^{\tilde{i}(0)}\; \phi_3^2 \epsilon_{i3k} \epsilon_{\tilde{i}3\tilde{k}} t_k t_{\tilde{k}} && (3.181) \\ + \;& \frac{1}{2} g^2\; A_\mu^{i(1)} A_\mu^{\tilde{i}(1)}\; \left(\phi_0^2 t_i t_{\tilde{i}} + \frac{1}{2} \phi_0 \phi_3 \left(t_i \delta_{i3} \delta_{\tilde{i}3} \right) + \phi_3^2 t_0^2 \delta_{i3} \delta_{\tilde{i}3} \right) && (3.182) \\ + \;& i\, \frac{1}{2} g^2\; A_\mu^{i(0)} A_\mu^{\tilde{i}(1)}\; \phi_3 \phi_0\; \epsilon_{i3k} \left[t_k, t_{\tilde{i}} \right] && (3.183) \end{aligned}$$

with $i, \tilde{i} = 1, 2, 3$. As tr $t_i = 0$ for $i = 1, 2, 3$, the mixed term (3.183) vanish after taking the trace.

Let us focus on the mass term (3.181) for the zero mode $A_\mu^{i(0)}$

$$\frac{1}{2} g^2\; A_\mu^{i(0)} A_\mu^{\tilde{i}(0)}\; \phi_3^2 \epsilon_{i3k} \epsilon_{\tilde{i}3\tilde{k}} t_k t_{\tilde{k}} \; . \tag{3.184}$$

Since $\mathrm{tr}\,(t_i t_j) = \frac{1}{2} \delta_{ij}$ we get after taking the trace

$$\frac{1}{4} g^2\; A_\mu^{i(0)} A_\mu^{\tilde{i}(0)}\; \phi_3^2\; \left(\epsilon_{i3k} \epsilon_{\tilde{i}3k} \right) \; . \tag{3.185}$$

For $i = 3$ this term vanishes and thus the corresponding gauge field $A_\mu^{3(0)}$ remains massless. With $\epsilon_{132}^2 = \epsilon_{231}^2 = 1$ we obtain for $i = 1, 2$

$$\frac{1}{4} g^2\; \left(A_\mu^{i(0)} \right)^2\; \phi_3^2 \; . \tag{3.186}$$

Next we consider the mass term (3.182) for the first excited mode $A_\mu^{i(1)}$

$$\frac{1}{2} g^2\; A_\mu^{i(1)} A_\mu^{\tilde{i}(1)}\; \left(\phi_0^2 t_i t_{\tilde{i}} + \frac{1}{2} \phi_0 \phi_3 \left(t_i \delta_{i3} \delta_{\tilde{i}3} \right) + \phi_3^2 t_0^2 \delta_{i3} \delta_{\tilde{i}3} \right) \; . \tag{3.187}$$

The second term vanishes after taking the trace. With tr $t_0^2 = \frac{1}{2}$ and $\mathrm{tr}\,(t_i t_j) = \frac{1}{2} \delta_{ij}$ for $i, j \in \{1, 2, 3\}$ we obtain

$$\frac{1}{4} g^2\; \left(A_\mu^{j(1)} \right)^2\; \phi_0^2 + \frac{1}{4} g^2\; \left(A_\mu^{3(1)} \right)^2\; \left(\phi_0^2 + \phi_3^2 \right) \; , \tag{3.188}$$

where $j = 1, 2$. Recapitulating we have obtained

$$\begin{aligned} tr\left[(D_\mu\Phi_{min})^\dagger(D_\mu\Phi_{min})\right] &= \frac{1}{4}g^2\left(A_\mu^{i(0)}\right)^2\phi_3^2 + \frac{1}{4}g^2\left(A_\mu^{i(1)}\right)^2\phi_0^2 \\ &+ \frac{1}{4}g^2\left(A_\mu^{3(1)}\right)^2\left(\phi_0^2+\phi_3^2\right) \end{aligned}$$

with $i = 1, 2$. Inserting ϕ_0 and ϕ_3 (3.176) into (3.189) we get the final result

$$\begin{aligned} tr\left[(D_\mu\Phi_{min})^\dagger(D_\mu\Phi_{min})\right] &= \frac{1}{8}g^2\rho_{min}^2\left(e^{a1}-e^{a2}\right)^2\left(A_\mu^{i(0)}\right)^2 \\ &+ \frac{1}{8}g^2\rho_{min}^2\left(e^{a1}+e^{a2}\right)^2\left(A_\mu^{i(1)}\right)^2 \\ &+ \frac{1}{4}g^2\rho_{min}^2\left(e^{2a1}+e^{2a2}\right)\left(A_\mu^{3(1)}\right)^2 \end{aligned} \quad (3.189)$$

with $i = 1, 2$. Table 1 summarises the result

Table 1

Field	*Mass squared*
$A_\mu^{i(0)}$ $i = 1, 2$	$\frac{1}{8}g^2\rho_{min}^2\left(e^{a_1}-e^{a_2}\right)^2$
$A_\mu^{3(0)}$	0
$A_\mu^{i(1)}$ $i = 1, 2$	$\frac{1}{8}g^2\rho_{min}^2\left(e^{a_1}+e^{a_2}\right)^2$
$A_\mu^{3(1)}$	$\frac{1}{4}g^2\rho_{min}^2\left(e^{2a_1}+e^{2a_2}\right)$

(3.190)

Discussion: i) We observe that only the zero mode gauge field $A_\mu^{3(0)}$ remains massless. This follows from the fact that t_3 commutes with Φ_{min}

$$[t_3, \Phi_{min}] = 0\ . \quad (3.191)$$

Thus the $U(1)$ subgroup of $SU(2)$ generated by t_3 remains always unbroken. Note that also $[P, t_3] = 0$. We have the spontaneous symmetry breaking scheme

$$SU(2) \xrightarrow{\langle\eta\rangle} U(1) \quad (3.192)$$

for $a_1 = -a_2 \neq 0$.
ii) For $a_1 = -a_2 = 0$ in (3.190) we recover (3.164).
iii) For (3.190) there are two cases of special interest:

1. Limit of small a_1, i.e. $0 < a_1 \ll 1$:. In this case it is possible to find a corresponding orbifold model as an approximation. We will discuss this case in detail in the next section.

2. Limit of large a_1, i.e. $a_1 \gg 1$: In this case gauge boson masses can be very large in comparison to the compactification scale $g\rho_{min} = 1/R$. This behaviour has no counterpart within the customary approximation scheme of an orbifold model. We will discuss this case in detail in section 3.9.

3.8 Linear approximation and corresponding truncated $S^1/\mathbb{Z}_2$ orbifold model with an additional scalar field in the adjoint representation of $SU(2)$

Let $0 < a_1 \ll 1$ in (3.190). Thus we can approximate

$$e^{a_i} \approx 1 + a_i \tag{3.193}$$

for $i = 1, 2$. In this approximation we obtain

$$\begin{aligned} e^{a_1} - e^{a_2} &= a_1 - a_2 = 2a_1 \ , \\ e^{a_1} + e^{a_2} &= a_1 + a_2 + 2 = 2 \ , \end{aligned} \tag{3.194}$$

where we have used that $a_1 = -a_2$. Inserting (3.194) in (3.190) we obtain

Table 2

Field	*Mass squared*
$A_\mu^{i(0)} \quad i = 1, 2$	$\frac{1}{2} g^2 \rho_{min}^2 \ a_1^2$
$A_\mu^{3(0)}$	0
$A_\mu^{i(1)} \quad i = 1, 2$	$\frac{1}{2} g^2 \rho_{min}^2$
$A_\mu^{3(1)}$	$\frac{1}{2} g^2 \rho_{min}^2$

(3.195)

Discussion: i) The zero mode gauge fields $A_\mu^{i(0)}$, for $i = 1, 2$, get small masses in comparison to the compactification scale $g\rho_{min} = 1/R$.
ii) The first excited KK-mode gauge fields $A_\mu^{i(1)}$ get the common mass term $g\rho_{min} = 1/R$. This result is just what one would expect from the customary approximation scheme of a truncated $S^1/\mathbb{Z}_2$ continuum orbifold model.

(3.195) suggests that there exits a corresponding $S^1/\mathbb{Z}_2$ orbifold model which at least approximately describes an eBTLM in the limit of small a_2. In fact, let us consider a $S^1/\mathbb{Z}_2$ continuum orbifold model with bulk gauge group $G = SU(2)$. In addition, we introduce a bulk scalar field $\phi(x^\mu, y)$ transforming according to the adjoint representation of $SU(2)$. The five-dimensional Lagrangian reads

$$\mathcal{L}_{5D} = -\frac{1}{4} F^a_{MN} F^{aMN} + \mid D_M \phi^a \mid^2 \ , \tag{3.196}$$

where

$$F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + g_5 f^{abc} A^b_M A^c_N \tag{3.197}$$

and

$$D_M \phi^a = \partial_M \phi^a + g_5 f^{abc} A^b_M \phi^c \ . \tag{3.198}$$

The boundary conditions read

$$A_\mu(x^\mu, -y) = P\ A_\mu(x^\mu, y)\ P^{-1} \tag{3.199}$$
$$A_y(x^\mu, -y) = -P\ A_y(x^\mu, y)\ P^{-1} \tag{3.200}$$
$$\phi(x^\mu, -y) = P\ \phi(x^\mu, y)\ P^{-1}\ . \tag{3.201}$$

We choose the trivial orbifold projection

$$P = \mathrm{diag}(1, 1)\ . \tag{3.202}$$

Thus $G = SU(2)$ remains unbroken. The Fourier mode expansion up to the first KK-mode reads

$$A_\mu^a(x^\mu, y) = \frac{1}{\sqrt{2\pi R}} A_\mu^{a(0)}(x^\mu) + \frac{1}{\sqrt{\pi R}} A_\mu^{a(1)}(x^\mu)\cos(\frac{y}{R}) \tag{3.203}$$
$$A_y^a(x^\mu, y) = \frac{1}{\sqrt{\pi R}} A_y^{a(1)}(x^\mu)\sin(\frac{y}{R}) \tag{3.204}$$
$$\phi^a(x^\mu, y) = \frac{1}{\sqrt{2\pi R}}\phi^{a(0)}(x^\mu) + \frac{1}{\sqrt{\pi R}}\phi^{a(1)}(x^\mu)\cos(\frac{y}{R})\ . \tag{3.205}$$

We insert the KK-mode expansion for ϕ (3.205) in the covariant derivative for ϕ (3.198). This yields

$$\begin{aligned} D_M\phi^a &= \frac{1}{\sqrt{2\pi R}}\partial_\mu\phi^{a(0)} + \frac{1}{\sqrt{\pi R}}\partial_\mu\phi^{a(1)}\cdot\cos(\frac{y}{R}) + \frac{g_5}{2\pi R} f^{abc}\left[A_\mu^{b(0)}(x^\mu)\right. \\ &+ \left. A_\mu^{b(1)}(x^\mu)\sqrt{2}\cos(\frac{y}{R})\right]\left[\phi^{c(0)}(x^\mu) + \phi^{c(1)}(x^\mu)\sqrt{2}\cos(\frac{y}{R})\right] \\ &+ \frac{1}{\sqrt{\pi R}}\phi^{a(1)}\frac{1}{R}\cdot\sin(\frac{y}{R}) + \frac{g_5}{2\pi R} f^{abc}\left[A_y^{b(1)}(x^\mu)\sqrt{2}\cos(\frac{y}{R})\right] \\ &\quad \left[\phi^{c(0)}(x^\mu) + \phi^{c(1)}(x^\mu)\sqrt{2}\cos(\frac{y}{R})\right] \end{aligned} \tag{3.206}$$

We assume that ϕ gets a VEV in its diagonal direction

$$\phi\ \longrightarrow\ \langle\ \phi^{3(0)}\ \rangle\ . \tag{3.207}$$

Inserting this VEV in (3.206) and imposing axial gauge we obtain

$$D_M\phi^a = \frac{g_5}{2\pi R} f^{ab3}\left[A_\mu^{b(0)}(x^\mu)\langle\ \phi^{3(0)}\ \rangle + A_\mu^{b(1)}(x^\mu)\langle\ \phi^{3(0)}\ \rangle\sqrt{2}\cos(\frac{y}{R})\right]\ . \tag{3.208}$$

We already see that this term will vanish for $b = 3$. The field $A_\mu^{3(0)}$ will therefore remain massless. We insert (3.208) into the five-dimensional Lagrangian (3.196) and integrate over the circle S^1. This yields

$$\begin{aligned} \mathcal{L}_{mass}^{\phi} &= \int_0^{2\pi R} \mid D_M\phi^a \mid^2 \\ &= g_4^2\left(A_\mu^{b(0)}(x^\mu)\right)^2\langle\ \phi^{3(0)}\ \rangle^2 + 2g_4^2\left(A_\mu^{b(1)}(x^\mu)\right)^2\langle\ \phi^{3(0)}\ \rangle^2 \end{aligned} \tag{3.209}$$

where $b = 1, 2$ and we have inserted (3.72)

$$g_4 = \frac{g_5}{\sqrt{2\pi R}} \ . \tag{3.210}$$

The Yang-Mills term in (3.196) yield a mass term for the gauge fields $A_\mu^{a(1)}$ as usual (3.70)

$$\mathcal{L}_{mass}^{ym} = \int_0^{2\pi R} \{-\frac{1}{2} F_{\mu y}^a F^{a\mu y}\} \, dy = \frac{1}{2} \, \frac{1}{R^2} \left(A_\mu^{a(1)} \right)^2 \tag{3.211}$$

with $a = 1, 2, 3$. Recapitulating we have obtained the following mass terms for the truncated $S^1/\mathbb{Z}_2$ orbifold model

$$\begin{aligned} \mathcal{L}_{mass}^{orbifold} &= g_4^2 \left(A_\mu^{b(0)} \right)^2 \langle \, \phi^{3(0)} \, \rangle^2 + 2 g_4^2 \left(A_\mu^{b(1)} \right)^2 \langle \, \phi^{3(0)} \, \rangle^2 \\ &+ \frac{1}{2} \, \frac{1}{R^2} \left(A_\mu^{b(1)} \right)^2 + \frac{1}{2} \, \frac{1}{R^2} \left(A_\mu^{3(1)} \right)^2 \end{aligned} \tag{3.212}$$

where $b = 1, 2$.

From (3.195) we read off the mass terms in the corresponding eBTLM

$$\mathcal{L}_{mass}^{eBTLM} = \frac{1}{2} \, g^2 \rho_{min}^2 a_1^2 \, \left(A_\mu^{j(0)} \right)^2 + \frac{1}{2} \, g^2 \rho_{min}^2 \, \left(A_\mu^{b(1)} \right)^2 + \frac{1}{2} \, g^2 \rho_{min}^2 \, \left(A_\mu^{3(1)} \right)^2 \ . \tag{3.213}$$

Inserting the identification $g\rho_{min} = 1/R$ we obtain

$$\mathcal{L}_{mass}^{eBTLM} = \frac{1}{2} \, \frac{a_1^2}{R^2} \, \left(A_\mu^{j(0)} \right)^2 + \frac{1}{2} \, \frac{1}{R^2} \, \left(A_\mu^{b(1)} \right)^2 + \frac{1}{2} \, \frac{1}{R^2} \, \left(A_\mu^{3(1)} \right)^2 \ . \tag{3.214}$$

The comparison of (3.214) with (3.212) yields

- The mass term for the gauge field $A_\mu^{3(1)}$ coincides in both models .
- Since the zero KK modes of all fields are expected to be much lighter than their first KK excitation, we assume

  $$\langle \, \phi^{3(0)} \, \rangle \ll \frac{1}{g_4 R} \ . \tag{3.215}$$

  Thus the masses for the gauge fields $A_\mu^{1,2(1)}$ are approximately equal in both models.
- Setting

  $$\langle \, \phi^{3(0)} \, \rangle = \frac{a_1}{g_4 R} \ , \tag{3.216}$$

  both models yield the same mass terms for the gauge fields $A_\mu^{b(0)}$ with $b = 1, 2$. Note that $0 < a_1 \ll 1$, which is compatible with the assumption (3.215). Since a_1 and g_4 are dimensionless and $1/R$ has mass dimension 1, the VEV $\langle \, \phi^{3(0)} \, \rangle$ has mass dimension 1.

Proposition 2 *An $S^1/\mathbb{Z}_2$ continuum orbifold model with bulk gauge group $G = SU(2)$, an additional scalar field ϕ transforming according to the adjoint representation of $G = SU(2)$, trivial orbifold projection $P = diag(1,1)$ and a Fourier mode expansion for all fields truncated at the first excited Kaluza-Klein mode in axial gauge gives an approximation to an effective bilayered transverse lattice model with bulk gauge group $G = SU(2)$, trivial orbifold projection $P = diag(1,1)$ and minimum of the Higgs potential at*

$$\Phi_{min} = \rho_{min} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{a_1} & 0 \\ 0 & e^{a_2} \end{pmatrix} , \; a_2 = -a_1 , \tag{3.217}$$

in the limit of small a_1 ($0 < a_1 \ll 1$), if the scalar field ϕ gets the VEV

$$\phi \;\longrightarrow\; \langle\, \phi^{3(0)} \,\rangle = \frac{a_1}{g_4 R} . \tag{3.218}$$

3.9 Large gauge boson masses from spontaneous symmetry breaking

Let $a_2 \gg 1$ in (3.190). Since $e^{a_2} = e^{-a_1}$ and $a_2 \gg 1$, it follows $e^{a_1} \approx 0$ and we obtain the following mass squared terms

Table 3

Field	*Mass squared*
$A_\mu^{i(0)} \quad i = 1,2$	$\frac{1}{8}\, g^2 \rho_{min}^2\, e^{2a_1}$
$A_\mu^{3(0)}$	0
$A_\mu^{i(1)} \quad i = 1,2$	$\frac{1}{8}\, g^2 \rho_{min}^2\, e^{2a_1}$
$A_\mu^{3(1)}$	$\frac{1}{4}\, g^2 \rho_{min}^2\, e^{2a_1}$

(3.219)

We recognise that for $a_1 \gg 1$ the gauge field masses show an exponential dependence on a_1 and can therefore be very large. *It is remarkable that already the zero mode gauge fields $A_\mu^{1,2(0)}$ can have masses much above the compactification scale $1/R$.* This behaviour has no counterpart within the customary approximation scheme of an ordinary orbifold model.

Proposition 3 *An effective bilayered transverse lattice model with bulk gauge group $G = SU(2)$, trivial orbifold projection $P = diag(1,1)$ and minimum of the Higgs potential at*

$$\Phi_{min} = \rho_{min} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{a_1} & 0 \\ 0 & e^{a_2} \end{pmatrix} , \; a_2 = -a_1 , \tag{3.220}$$

in the limit of large a_1 ($a_1 \gg 1$) allows masses for some zero mode and first excited KK-mode gauge fields, which are much larger than the compactification scale $g\rho_{min} = 1/R$.

3.10 Effective bilayered transverse lattice model and continuous Wilson line breaking

In this section we consider the case where the gauge group $G = SU(2)$ is broken via orbifolding to its subgroup $G_0 = U(1)$. We embed the orbifold projection P in G by setting

$$P = \exp(2\pi i\, t_3) = diag(1, -1)\ . \tag{3.221}$$

This choice for P breaks $G = SU(2)$ down to $G_0 = U(1)$ where G_0 is generated by t_3. As in section 3.7 we assume that the holonomy group H is given by $\mathbb{R}_*^+\ SL(2,\mathbb{C})$, with $\mathbb{R}_*^+ = \mathbb{R}^+/\{0\}$. The action of P on G leads to the split

$$\mathfrak{su}(2) = \mathfrak{u}(1) \oplus \mathfrak{su}(2)/\mathfrak{u}(1)\ , \tag{3.222}$$

where $\mathfrak{u}(1) = \text{Lie}\ G_0$. Figure 3.7 summarises the setting.

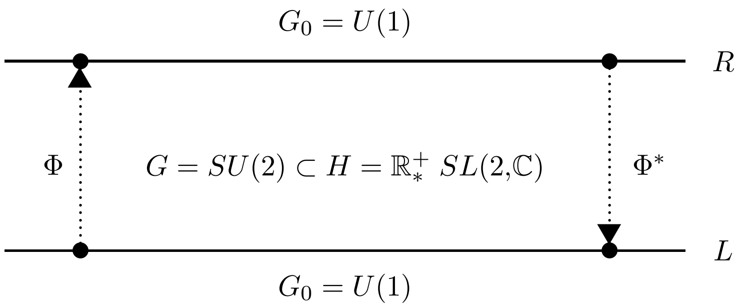

Figure 3.7: Effective bilayered transverse lattice model for bulk gauge group $G = SU(2)$ and non-trivial orbifold projection $P = \text{diag}(1, -1)$. The bulk gauge group $G = SU(2)$ is broken to its subgroup $G_0 = U(1)$ via orbifolding.

Let $\Phi \in H$ fulfil the sharpened orbifold condition. Then Φ can be written according to Theorem 4 as

$$\Phi = \rho\ e^{A_y}\ e^{\eta}\ e^{A_y}\ , \tag{3.223}$$

where

$$P A_y P^{-1} = -A_y\ , \tag{3.224}$$

$$P \eta P^{-1} = \eta\ , \tag{3.225}$$

$\eta \in \mathfrak{a}$, for an appropriate choice of $\mathfrak{a}$, and $A_y \in \mathfrak{su}(2)/\mathfrak{u}(1)$. As in section 3.7 we choose

$$\mathfrak{a} = \{\eta = diag(a_1, a_2)\}\quad ,\quad a_1 = -a_2\ ,\quad a_i \in \mathbb{R}\ . \tag{3.226}$$

Thus we have $P\eta P^{-1} = \eta$ for all $\eta \in \mathfrak{a}$.

Let us consider the Higgs potential

$$V(\Phi) = V(\rho\, e^{A_y}\, e^{\eta}\, e^{A_y}) \; . \tag{3.227}$$

According to Theorem 5, $V(\Phi)$ is invariant under unitary gauge transformations

$$V(S_0(x)\Phi S_0(x)^{-1}) = V(\Phi) \tag{3.228}$$

where $S_0(x) \in G_0$. Thus e^{A_y} in (3.227) cannot be gauged away. Consequently the Higgs potential $V(\Phi)$ depends on ρ, η and A_y

$$V(\Phi) = V(\rho\, e^{A_y}\, e^{\eta}\, e^{A_y}) = \mathcal{V}(\rho, \eta, A_y) \; . \tag{3.229}$$

The unitary factor e^{A_y} in (3.223) can be written as

$$\exp\left(2\pi i\; gR\; \mathcal{A}_y^{(0)}\right) \; , \tag{3.230}$$

where $\mathcal{A}_y^{(0)} \in \mathfrak{su}(2)/\mathfrak{u}(1)$ is the zero of the extra-dimensional component of the five-dimensional gauge field. We can expand

$$\mathcal{A}_y^{(0)} = \mathcal{A}_y^{1(0)} t_1 + \mathcal{A}_y^{2(0)} t_2 \; . \tag{3.231}$$

Let us consider the case where $\mathcal{A}_y$ assume a VEV. Without loss of generality we lay this VEV in the t_1-direction, i.e.

$$\mathcal{A}_y \to \langle \mathcal{A}_y^{1(0)} \rangle t_1 \; . \tag{3.232}$$

Inserting (3.232) in (3.230) we get

$$W = \exp(2\pi i\; gR\; \langle \mathcal{A}_y^{1(0)} \rangle t_1) \; . \tag{3.233}$$

This is a Wilson line, compare with (2.53), and since $[P, t_1] \neq 0$ it does not commute with P. Thus the VEV for $\langle \mathcal{A}_y^{1(0)} \rangle$ can be an arbitrary constant and thus W is a continuous Wilson line, compare with (2.62).

However $\mathcal{A}_y^{(0)}$ is not in its canonical four-dimensional form. Therefore we make the following

Definition 9 *The unitary factor in the decomposition (3.223) is given by*

$$\exp(A_y) = \exp(i\; g_4 R\; \mathcal{A}_y^{(0)}) \; . \tag{3.234}$$

Remarks: i) In (3.234) $\mathcal{A}_y^{(0)}$ is the zero of the extra-dimensional component of the five-dimensional gauge field in its canonical four-dimensional form and can be interpreted as a usual four-dimensional Higgs field, and g_4 is the four-dimensional effective gauge coupling constant. This definition is convenient because the kinetic term $\mathcal{L}_{mass} = tr\left[(D_\mu \Phi_{min})^\dagger (D_\mu \Phi_{min})\right]$ should involve only rescaled four-dimensional terms, compare with section 3.3.2.
ii) In contrast to (3.230) we have rescaled $\mathcal{A}_y^{(0)}$ [7] by a factor 2π.
iii) e^{A_y} and g_4 are dimensionless, R has mass dimension -1 and $\mathcal{A}_y^{(0)}$ has mass dimension 1.

Within Definition 9 the Wilson line (3.233) becomes

$$W = \exp(i\; g_4 R\; \langle \mathcal{A}_y^{1(0)} \rangle t_1)\;. \tag{3.235}$$

It is convenient to rewrite the VEV $\langle \mathcal{A}_y^{1(0)} \rangle$ as

$$\langle \mathcal{A}_y^{1(0)} \rangle = \frac{\alpha_1}{g_4 R}\;, \tag{3.236}$$

where $0 < \alpha_1 < 1$ is a dimensionless parameter. Inserting (3.236) in (3.235) the Wilson line becomes

$$W = \exp(i\; g_4 R\; \langle \mathcal{A}_y^{1(0)} \rangle t_1) = \exp(i\; \alpha_1 t_1)\;. \tag{3.237}$$

A VEV for $\mathcal{A}_y^{(0)}$ is usually much smaller than the compactification scale $1/R$. Thus $0 < \alpha_1 \ll 1$ in (3.236) and we can approximate

$$W = \exp(i\; \alpha_1 t_1) \approx 1 + i\; \alpha_1 t_1 = \begin{pmatrix} 1 & i\frac{\alpha_1}{2} \\ i\frac{\alpha_1}{2} & 1 \end{pmatrix}\;. \tag{3.238}$$

According to (3.227) we can parametrise the minimum Φ_{min} of the Higgs potential as

$$\begin{aligned} \Phi_{min} &= \rho_{min} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i\frac{\alpha_1}{2} \\ i\frac{\alpha_1}{2} & 1 \end{pmatrix} \begin{pmatrix} e^{a_1} & 0 \\ 0 & e^{a_2} \end{pmatrix} \begin{pmatrix} 1 & i\frac{\alpha_1}{2} \\ i\frac{\alpha_1}{2} & 1 \end{pmatrix} \\ &= \rho_{min} \frac{1}{2} \begin{pmatrix} e^{a_1} & (e^{a_1} + e^{a_2})\, i\frac{\alpha_1}{2} \\ -(e^{a_1} + e^{a_2})\, i\frac{\alpha_1}{2} & e^{a_2} \end{pmatrix} + \mathcal{O}\left(\alpha_1^2\right)\;. \end{aligned} \tag{3.239}$$

In the following since $0 < \alpha_1 \ll 1$ we neglect terms of $\mathcal{O}\left(\alpha_1^2\right)$. We calculate the mass terms for the gauge fields $A_\mu^{3(0)}$ and $A_\mu^{3(1)}$. The covariant derivative reads (3.173)

$$D_\mu \Phi = \partial_\mu \Phi + i\frac{g}{\sqrt{2}} A_\mu^{3(0)} \left[t_3, \Phi\right] + i\frac{g}{\sqrt{2}} A^{3(1)} \{t_3, \Phi\} \tag{3.240}$$

[7] In an orbifold theory $\mathcal{A}_y$ and its zero mode $\mathcal{A}_y^{(0)}$ is related by $\mathcal{A}_y = \frac{1}{\sqrt{2\pi R}} \mathcal{A}_y^{(0)}$ (3.264).

where $[,]$ and $\{,\}$ denote the commutator and anticommutator, respectively. We restrict ourselves to the case where $a_1 = -a_2 = 0$. Then (3.239) becomes

$$\Phi_{min} = \rho_{min} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i\alpha_1 \\ i\alpha_1 & 1 \end{pmatrix} . \tag{3.241}$$

We can expand Φ_{min} in terms of t_0 and t_1 as

$$\begin{aligned} \Phi_{min} &= \phi_0 t_0 + \phi_1 t_1 && (3.242)\\ &= \frac{1}{2} \begin{pmatrix} \phi_0 & \phi_1 \\ \phi_1 & \phi_0 \end{pmatrix} := \rho_{min} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i\alpha_1 \\ i\alpha_1 & 1 \end{pmatrix} . && (3.243) \end{aligned}$$

where

$$\begin{gathered} \phi_0 = \sqrt{2}\rho_{min} , \qquad (3.244)\\ \phi_1 = i\phi_1' , \quad \phi_1' = \sqrt{2}\rho_{min}\, \alpha_1 . \end{gathered}$$

For the commutators $[t_3, \Phi_{min}]$ and anticommutators $\{t_3, \Phi_{min}\}$ we obtain

$$[t_3, \Phi_{min}] = \phi_0 \underbrace{[t_3, t_0]}_{=0} + \phi_1 \underbrace{[t_3, t_1]}_{=it_2} \tag{3.245}$$

$$\{t_3, \Phi_{min}\} = \phi_0 \underbrace{\{t_3, t_0\}}_{=t_3} + \phi_1 \underbrace{\{t_3, t_1\}}_{=0} . \tag{3.246}$$

Inserting $[t_3, \Phi_{min}]$ and $\{t_3, \Phi_{min}\}$ in (3.240) we find

$$D_\mu \Phi_{min} = -i \frac{g}{\sqrt{2}}\, \phi_1' t_2\, A_\mu^{3(0)} + i \frac{g}{\sqrt{2}}\, \phi_0 t_3\, A_\mu^{3(1)} . \tag{3.247}$$

Taking the adjoint $(D_\mu \Phi_{min})^\dagger = i\frac{g}{\sqrt{2}}\ \phi_1' t_2\ A_\mu^{3(0)} - i\frac{g}{\sqrt{2}}\ \phi_0 t_3\ A_\mu^{3(1)}$, multiplying $(D_\mu \Phi_{min})^\dagger$ with $D_\mu \Phi_{min}$ we obtain

$$\begin{aligned} & (D_\mu \Phi_{min})^\dagger (D_\mu \Phi_{min}) && (3.248)\\ = \;& \frac{1}{2}\, g^2\, \phi_1'^2 t_1^2 \left(A_\mu^{3(0)}\right)^2 + \frac{1}{2}\, g^2\, \phi_0^2 t_3^2 \left(A_\mu^{3(1)}\right)^2 - \frac{1}{2}\, g^2\, \phi_0 \phi_1'\ [t_2, t_3]\ A_\mu^{3(0)} A_\mu^{3(1)} \end{aligned}$$

First we observe that the mixed term vanishes after taking the trace. The mass term for the zero mode $A_\mu^{3(0)}$ becomes after taking the trace

$$\frac{1}{4} g^2\, \phi_1'^2 \left(A_\mu^{3(0)}\right)^2 , \tag{3.249}$$

and the mass term for the first excited mode $A_\mu^{3(1)}$ in (3.248) becomes after taking the trace

$$\frac{1}{4}\, g^2 \phi_0^2 \left(A_\mu^{3(1)}\right)^2 . \tag{3.250}$$

Recapitulating we have obtained

$$\mathrm{tr}\left[(D_\mu\Phi_{min})^\dagger\,(D_\mu\Phi_{min})\right] = \frac{1}{4}\,g^2\phi_1'^2\,\left(A_\mu^{3(0)}\right)^2 + \frac{1}{4}\,g^2\phi_0^2\,\left(A_\mu^{3(1)}\right)^2\,. \quad (3.251)$$

Inserting ϕ_0 and ϕ_1 (3.244) we finally arrive at

$$\mathrm{tr}\left[(D_\mu\Phi_{min})^\dagger\,(D_\mu\Phi_{min})\right] = \frac{1}{2}\,g^2\rho_{min}^2\alpha_1^2\,\left(A_\mu^{3(0)}\right)^2 + \frac{1}{2}\,g^2\rho_{min}^2\,\left(A_\mu^{3(1)}\right)^2\,. \quad (3.252)$$

Table 4 summarises the result

Table 4

Field	*Mass squared*
$A_\mu^{3(0)}$	$g^2\rho_{min}^2\,\alpha_1^2$
$A_\mu^{3(1)}$	$g^2\rho_{min}^2$

(3.253)

Discussion: i) The fields $A_\mu^{1,2(0)}$ and $A_\mu^{1,2(1)}$ are integrated out due to the choice of the orbifold projection (3.221).
ii) The mass for the zero mode gauge boson $A_\mu^{3(0)}$ is

$$m = \frac{\alpha_1}{R}\,, \quad (3.254)$$

where we have inserted $g\rho_{min} = 1/R$ in (3.252). For $0 < \alpha_1 \ll 1$ the mass of $A_\mu^{3(0)}$ is much lower than the compactification scale $1/R$.
iii) For $\langle\mathcal{A}_y^{1(0)}\rangle \neq 0$, the orbifold unbroken gauge group $G_0 = U(1)$ is completely broken and we have the breaking scheme

$$SU(2) \xrightarrow{P} U(1) \xrightarrow{\langle\mathcal{A}_y^{1(0)}\rangle} \emptyset\,. \quad (3.255)$$

Thus the *rank* of the gauge group $G = SU(2)$ is reduced. This follows from the fact that

$$W = \exp(i\;g_4R\;\langle\mathcal{A}_y^{1(0)}\rangle t_1) \quad (3.256)$$

is a continuous Wilson line.
iv) The first excited KK-mode gauge bosons $A_\mu^{3(1)}$ acquire the mass $g\rho_{min} = 1/R$. This result is just what one would expect from the customary approximation scheme of a truncated $S^1/\mathbb{Z}_2$ continuum orbifold model.

We compare this result to a $S^1/\mathbb{Z}_2$ continuum orbifold model with bulk gauge group $G = SU(2)$ and non-trivial orbifold projection P. The five-dimensional Lagrangian reads

$$\mathcal{L}_{5D} = -\frac{1}{4}F^a_{MN}F^{aMN}\,, \quad (3.257)$$

where

$$F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + g_5 f^{abc}A^b_M A^c_N\,. \quad (3.258)$$

The boundary conditions read

$$A_\mu(x^\mu,-y) = P\ A_\mu(x^\mu,y)\ P^{-1} \tag{3.259}$$
$$A_y(x^\mu,-y) = -P\ A_y(x^\mu,y)\ P^{-1}\ . \tag{3.260}$$

We break $G = SU(2)$ down to $U(1)$ by choosing

$$P = \mathrm{diag}(1,-1)\ , \tag{3.261}$$

compare with (3.221). The Fourier mode expansion up to the first Kaluza-Klein mode reads

$$A^3_\mu(x^\mu,y) = \frac{1}{\sqrt{2\pi R}}A^{3(0)}_\mu(x^\mu) + \frac{1}{\sqrt{\pi R}}A^{3(1)}_\mu(x^\mu)\cos(\frac{y}{R})\ , \tag{3.262}$$
$$A^{1,2}_\mu(x^\mu,y) = \frac{1}{\sqrt{\pi R}}A^{1,2(1)}_\mu(x^\mu)\sin(\frac{y}{R})\ , \tag{3.263}$$
$$A^{1,2}_y(x^\mu,y) = \frac{1}{\sqrt{2\pi R}}A^{1,2(0)}_y(x^\mu) + \frac{1}{\sqrt{\pi R}}A^{1,2(1)}_y(x^\mu)\cos(\frac{y}{R})\ , \tag{3.264}$$
$$A^3_y(x^\mu,y) = \frac{1}{\sqrt{\pi R}}A^{3(1)}_y(x^\mu)\sin(\frac{y}{R})\ . \tag{3.265}$$

We calculate $F^a_{\mu y}$ in axial gauge, i.e. we set $A^{a(1)}_y = 0$ for $a = 1,2,3$. The result is

$$\begin{aligned} F^a_{\mu y} &= \partial_\mu A^a_y - \partial_y A^a_\mu + g_5 f^{abc} A^b_\mu A^c_y \\ &= \frac{1}{\sqrt{2\pi R}}\partial_\mu A^{1,2(0)}_y(x^\mu) + \frac{1}{\sqrt{\pi R}}A^{3(1)}_\mu \frac{1}{R}\sin(\frac{y}{R}) - \frac{1}{\sqrt{\pi R}}A^{1,2(1)}_\mu \frac{1}{R}\cos(\frac{y}{R}) \\ &+ \frac{g_5}{2\pi R} f_{a3c}\left[A^{3(0)}_\mu(x^\mu) + A^{3(1)}_\mu(x^\mu)\sqrt{2}\cos(\frac{y}{R})\right]\left[A^{c(0)}_y(x^\mu)\right] \\ &+ \frac{g_5}{\sqrt{2\pi R}}\left(\left[A^{1(1)}_\mu(x^\mu)\sin(\frac{y}{R})\right]\left[A^{2(0)}_y(x^\mu)\right] - \left[A^{2(1)}_\mu(x^\mu)\sin(\frac{y}{R})\right]\left[A^{1(0)}_y(x^\mu)\right]\right) \end{aligned} \tag{3.266}$$

with $c = 1,2$. We assume that A_y gets a VEV in its t_1 direction

$$A_y \ \longrightarrow\ \langle\ A^{1(0)}_y\ \rangle\ . \tag{3.267}$$

Inserting this VEV in (3.266), we obtain [8]

$$\begin{aligned} F^a_{\mu y} &= \frac{1}{\sqrt{\pi R}}A^{3(1)}_\mu \frac{1}{R}\sin(\frac{y}{R}) - \frac{1}{\sqrt{\pi R}}A^{1,2(1)}_\mu \frac{1}{R}\cos(\frac{y}{R}) \\ &+ \frac{g_5}{2\pi R}\left[A^{3(0)}_\mu(x^\mu)\langle\ A^{1(0)}_y\ \rangle + A^{3(1)}_\mu(x^\mu)\sqrt{2}\cos(\frac{y}{R})\langle\ A^{1(0)}_y\ \rangle\right] \\ &- \frac{g_5}{\sqrt{2\pi R}}\left[A^{1(1)}_\mu(x^\mu)\sin(\frac{y}{R})\right]\langle\ A^{1(0)}_y\ \rangle\ . \end{aligned} \tag{3.268}$$

[8] Note that $[t_i,t_j] = i f_{ijk} t_k$ with $f_{ijk} = \epsilon_{ijk}$

Inserting $F^a_{\mu y}$ in the five-dimensional Lagrangian (3.257) and integrating over the circle S^1 we find

$$\begin{aligned}
\mathcal{L}^{orbifold}_{mass} &= \int_0^{2\pi R} \{-\frac{1}{2} F^a_{\mu y} F^{a\mu y}\}\, dy \qquad (3.269)\\
&= \frac{1}{2}\, g_4^2 \left(A^{3(0)}_\mu(x^\mu)\right)^2 \langle\, A^{1(0)}_y \,\rangle^2 + \frac{1}{2}\, g_4^2 \left(A^{3(1)}_\mu(x^\mu)\right)^2 \langle\, A^{1(0)}_y \,\rangle^2\\
&+ \frac{1}{2}\, g_4^2 \left(A^{1(1)}_\mu(x^\mu)\right)^2 \langle\, A^{1(0)}_y \,\rangle^2 + \frac{1}{2}\, \frac{1}{R^2} \left(A^{a(1)}_\mu\right)^2
\end{aligned}$$

for $a = 1, 2, 3$. We compare this result to (3.252)

$$\begin{aligned}
\mathcal{L}^{eBTLM}_{mass} &= \frac{1}{2}\, g^2 \rho^2_{min} \alpha_1^2 \left(A^{3(0)}_\mu\right)^2 + \frac{1}{2}\, g^2 \rho^2_{min} \left(A^{3(1)}_\mu\right)^2\\
&= \frac{1}{2}\, \frac{\alpha_1^2}{R^2} \left(A^{3(0)}_\mu\right)^2 + \frac{1}{2}\, \frac{1}{R^2} \left(A^{3(1)}_\mu\right)^2 \qquad (3.270)
\end{aligned}$$

where we have inserted $g\rho_{min} = 1/R$ in the second step. The comparison of (3.269) with (3.270) yield

- The zero KK modes of all fields are expected to be much lighter than their first KK excitation. Therefore we can assume

$$\langle\, A^{1(0)}_y \,\rangle \ll \frac{1}{g_4 R}\ . \qquad (3.271)$$

  Then the mass of the gauge field $A^{3(1)}_\mu$ is approximately equal in both models.

- For

$$\langle\, A^{1(0)}_y \,\rangle = \frac{\alpha_1}{g_4 R}\ , \qquad (3.272)$$

  both models yield the same mass term for $A^{3(0)}_\mu$, i.e.

$$m = \frac{\alpha_1}{R}\ . \qquad (3.273)$$

  This is a consequence of the fact that we have rescaled the extra-dimensional vector potential in the eBTLM such that the VEV for $A^{2(0)}_y$ is given by (3.272). Note that $0 < \alpha_2 \ll 1$ which is compatible with the assumption (3.271).

Proposition 4 *An $S^1/\mathbb{Z}_2$ continuum orbifold model with bulk gauge group $G = SU(2)$, non-trivial orbifold projection $P = diag(1, -1)$ and a Fourier mode expansion for all fields truncated at the first excited Kaluza-Klein mode in axial gauge gives an approximation to an effective bilayered transverse lattice model with bulk gauge group $G = SU(2)$, non-trivial orbifold projection $P = diag(1, -1)$ and the minimum of the Higgs potential at*

$$\Phi_{min} = \rho_{min} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i\alpha_1 \\ i\alpha_1 & 1 \end{pmatrix} \qquad (3.274)$$

with $0 < \alpha_1 \ll 1$.

Chapter 4

$SU(7)$ unified model

4.1 Introduction: Why $SU(7)$?

In this chapter we will present a realistic five-dimensional Gauge-Higgs unification model based on the unified gauge group $SU(7)$. The gauge group $SU(7)$ unifies electroweak-, flavour- and Higgs interactions in one single gauge group. Colour will be ignored. In the following we will outline the basic considerations that will lead to the unified gauge group $SU(7)$.

Let us start with the electroweak gauge group of the SM

$$SU(2)_L \times U(1)_Y \; . \tag{4.1}$$

In order to proceed we need to add a suitable flavour gauge group. In chapter 1 we have given an overview of flavour groups discussed in the literature. In particular, there are the four continuous flavour groups: $SU(2)_F$, $SU(3)_F$, $SO(3)_F$ and $U(1)_F$. Note that a possible flavour gauge group should remain unbroken by the orbifold projection P. The considerations of the last chapter suggest that the flavour gauge group in our model should be $SO(3)_F$. There are three reasons that motivates this choice:

1. We want to explain naturally why there are three generations in the SM. The flavour gauge groups $SU(2)_F$, $SU(3)_F$ and $SO(3)_F$ possess all an irreducible three dimensional representation in which the three generations of the SM can fit. For this reason we exclude $U(1)_F$.

2. Masses for all flavour gauge fields must be very large $\mathcal{O}(10^3) - \mathcal{O}(10^5)$ TeV in comparison to the electroweak breaking scale $\mathcal{O}(246)$ GeV in order to suppress tree-level FCNC. Thus for a compactification scale $1/R$ of the theory of $\mathcal{O}(1)$ TeV such flavour gauge fields must receive masses from VEVs for the selfadjoint part of Φ. In section 3.9, see Proposition 3, we have obtained that in the limit of large a_1 gauge field masses for some *zero* and first excited KK mode gauge

fields are much larger than the compactification scale $1/R$. This behaviour is what we need here. Note that a Wilson line breaking of the flavour gauge group would lead to flavour gauge field masses below the compactification scale $1/R$.

The three-dimensional representation of $SU(2)_F$ is not faithful (faithful representations of $SU(2)_F$ have even dimension). However the three-dimensional representation of $SU(2)_F$ is a faithful representation of $SO(3)_F$. The three generations of the SM can fit is this three-dimensional representation of $SO(3)_F$. If we embed $SO(3)_F$ into $SU(7)$ in an appropriate way it is possible for a suitable minimum of the Higgs potential that all $SO(3)_F$ gauge fields can receive masses from VEVs for the selfadjoint part of Φ much above the compactification scale $1/R$. The three generations of the SM can also fit is the three-dimensional representation of $SU(3)_F$. In this chapter we discuss also the embedding of $SU(3)_F$ into $SU(7)$. Note that we have the embedding scheme: $SO(3)_F \subset SU(3)_F \subset SU(7)$. However for the embedding of $SU(3)_F$ into $SU(7)$ there remains at least an $U(1)_F \times U(1)_F$ left unbroken by VEVs for the selfadjoint part of Φ. Note again that we do not want to break the flavour gauge group by orbifolding. Thus in this setting we exclude $SU(3)_F$.

3. The three-dimensional representation of $SO(3)_F$ is anomaly-free while the three-dimensional representation of $SU(3)_F$ is not anomaly-free. This is an additional reason why we exclude $SU(3)_F$. We will discuss the issue of anomaly cancellation in the $SU(7)$ model in detail in section 4.5.

If we add the flavour gauge group $SO(3)_F$ to the electroweak gauge group of the SM we arrive at

$$SU(2)_L \times U(1)_Y \times SO(3)_F \ . \tag{4.2}$$

Since VEVs for the selfadjoint part of Φ break only the flavour gauge group, i.e.

$$SU(2)_L \times U(1)_Y \times SO(3)_F \stackrel{\langle\eta\rangle}{\to} SU(2)_L \times U(1)_Y \ , \tag{4.3}$$

there remains an unbroken electroweak gauge group. The question is now:

- How can we include electroweak symmetry breaking in the model?

Note that we do not want to introduce extra Higgs fields in the model (besides the nonunitary parallel transporters Φ). The answer to this question is the following: First we embed the flavour gauge group $SO(3)_F$ into $SU(3)_F$. Consequently we arrive at $SU(2)_L \times U(1)_Y \times SU(3)_F$. The purpose is to unify weak- and flavour interactions in one single gauge group $SU(6)_L$ [67]. The embedding of $SU(2)_L \times SU(3)_F$ in $SU(6)_L$ is a special maximal one. The flavour gauge group $SU(3)_F$ itself appears only at an intermediate step towards the unified gauge group $SU(6)_L$ and not as an unbroken symmetry for the reasons mentioned above. Second, we embed $SU(6)_L \times U(1)_Y$ into $SU(7)$ in an appropriate way

$$SU(6)_L \times U(1)_Y \subset SU(7) \; , \tag{4.4}$$

and thus arrive at the unified gauge group $SU(7)$.

Starting with the unified bulk gauge group $G = SU(7)$ on the five-dimensional space-time $M^4 \times S^1/\mathbb{Z}_2$, we put $S^1/\mathbb{Z}_2$ on a lattice, calculate the RG-flow and consequently arrive at an eBTLM with unitary bulk gauge group $G = SU(7)$ and holonomy group $H = \mathbb{R}_*^+ \; SL(7,\mathbb{C})$ [1], with $\mathbb{R}_*^+ = \mathbb{R}^+/\{0\}$. This procedure was explained in detail in the last chapter. The model contains nonunitary PTs $\Phi \in H$ in the extra dimension.

The main idea is now to choose a non-trivial orbifold projection P. Via a non-trivial orbifold projection P the unified gauge group $SU(7)$ is broken down to $SU(6)_L \times U(1)_Y$ at the orbifold fixed points, i.e.

$$SU(7) \xrightarrow{P} SU(6)_L \times U(1)_Y \; . \tag{4.5}$$

At first view, this seems to be curious: First embedding $SU(6)_L \times U(1)_Y$ into $SU(7)$ and one step later breaking $SU(7)$ again down to $SU(6)_L \times U(1)_Y$. However, there is a bonus. If Φ fulfils the sharpened orbifold condition we can write $\Phi \in H$ as

$$\Phi = \rho \; e^{A_y} e^{\eta} e^{A_y} \tag{4.6}$$

where $\eta \in \mathfrak{a}$ for an appropriate choice of $\mathfrak{a}$, $A_y \in \mathfrak{su}(6) \oplus \mathfrak{u}(1)$ and $\rho \in \mathbb{R}_*^+$. The Higgs potential $V(\Phi)$ will therefore also depend on A_y:

$$V(\Phi) = V(\rho \; e^{A_y} e^{\eta} e^{A_y}) = \mathcal{V}(\rho, \eta, A_y) \; . \tag{4.7}$$

Note that A_y in $V(\Phi)$ cannot be gauged away because $G = SU(7)$ is broken to $G_0 = SU(6)_L \times U(1)_Y$. The gauge group $G_0 = SU(6)_L \times U(1)_Y$ is broken further to $SU(2)_L \times U(1)_Y \times SO(3)_F$ by imposing Dirichlet and Neumann boundary conditions.

We will show that zero modes $A_y^{(0)}$ of A_y have the required properties to serve as a substitute for the SM Higgs. In particular they are $SU(2)_L$ doublets and carry hypercharge 1/2. In contrast to the SM, the model includes three Higgs doublets, one for the first, one for the second and one for the third generation.

If the zero modes $A_y^{(0)}$ acquire VEVs in their $SU(2)_L$ down component, the electroweak gauge group $SU(2)_L \times U(1)_Y$ is broken down to $U(1)_{em}$:

$$SU(2)_L \times U(1)_Y \xrightarrow{\langle A_y^{(0)} \rangle} U(1)_{em} \tag{4.8}$$

[1] Note that the linear span of $SU(7)$ reads $\mathbb{R}_*^+ \; SL(7,\mathbb{C})$ and thus H is unique.

This breaking is equivalent to Wilson line breaking or Hosotani breaking. Recapitulating we have the following spontaneous symmetry breaking pattern:

$$SU(2)_L \times U(1)_Y \times SO(3)_F \xrightarrow{\langle\eta\rangle} SU(2)_L \times U(1)_Y \xrightarrow{\langle A_y^{(0)}\rangle} U(1)_{em} \; , \tag{4.9}$$

where the breaking of $SO(3)_F$ takes place at energies much above the compactification scale $1/R = \mathcal{O}(1)$ TeV. This way tree-level FCNC are naturally suppressed. The electroweak gauge bosons $W^\pm, Z$ receive masses only from VEVs for $A_y^{(0)}$. Their masses will therefore be $\mathcal{O}(246)$ GeV.

4.2 Family unification in $SU(6)_L \times U(1)_Y$

As already mentioned in the introduction $SU(6)_L$ unifies [67] the weak gauge group $SU(2)_L$ of the SM with the flavour gauge group $SU(3)_F$. Note that $SU(2)_L \times SU(3)_F$ is a special maximal subgroup of $SU(6)_L$. Since $SU(6)_L \times U(1)_Y$ is broken to $SU(2)_L \times U(1)_Y \times SO(3)_F$ by orbifold and additional boundary conditions its full meaning is of no importance for us. However, because $SU(2)_L$ and $SO(3)_F \subset SU(3)_F$ are subgroups of $SU(6)_L$ we will discuss a model [25, 16] based on the gauge group $SU(6)_L \times U(1)_Y$ shortly on its own. The gauge group $SU(6)_L$ has 35 generators L_i which in the $SU(2)_L \times SU(3)_F$ basis can be written as

- $$L_i = \frac{1}{2\sqrt{3}}\, \sigma_i \otimes \mathbf{1}_3 \; , \tag{4.10}$$

  where $\sigma_i, i = 1, 2, 3$ are the Pauli matrices and $\frac{1}{2}\sigma_i \otimes \mathbf{1}_3$ are the generators of $SU(2)_L$. The symbol $\mathbf{1}_3$ stand for the 3×3 unit matrix.

- $$L_{i'} = \frac{1}{2\sqrt{2}}\, \mathbf{1}_2 \otimes \lambda_j \; , \tag{4.11}$$

  where $i' = 4, \ldots, 11$ [2], $j = 1, \ldots, 8$, λ_j are the Gell-Mann matrices and $\frac{1}{2}\mathbf{1}_2 \otimes \lambda_j$ are the generators of $SU(3)_F$. The symbol $\mathbf{1}_2$ stand for the 2×2 unit matrix.

- $$L_{i''} = \frac{1}{2\sqrt{2}}\, \sigma_i \otimes \lambda_j \; , \tag{4.12}$$

  where $i'' = 12, \ldots, 35$ [3], $i = 1, 2, 3$ and $j = 1, \ldots, 8$.

[2] $L_4 = \frac{1}{2\sqrt{2}}\, \mathbf{1}_2 \otimes \lambda_1, \ldots, L_{11} = \frac{1}{2\sqrt{2}}\, \mathbf{1}_2 \otimes \lambda_8$

[3] $L_{12} = \frac{1}{2\sqrt{2}}\, \sigma_1 \otimes \lambda_1, \ldots, L_{19} = \frac{1}{2\sqrt{2}}\, \sigma_1 \otimes \lambda_8, L_{20} = \frac{1}{2\sqrt{2}}\, \sigma_2 \otimes \lambda_1, \ldots, L_{27} = \frac{1}{2\sqrt{2}}\, \sigma_2 \otimes \lambda_8, L_{28} = \frac{1}{2\sqrt{2}}\, \sigma_3 \otimes \lambda_1, \ldots, L_{35} = \frac{1}{2\sqrt{2}}\, \sigma_3 \otimes \lambda_8$

Note that all generators of $SU(6)_L$ are equally normalised as

$$\mathrm{tr}\,(L_i L_j) = \frac{1}{2}\delta_{ij}\ . \tag{4.13}$$

The gauge group $SU(6)_L \times U(1)_Y$ gives rise to 36 gauge bosons: 35 are linked to the generators of $SU(6)_L$ and one is linked to the generator of $U(1)_Y$. Besides the standard model gauge bosons there are 32 extra gauge bosons which can be divided into four groups

- 12 charged gauge bosons associated to the generators $\frac{1}{2\sqrt{2}}\sigma_i \otimes \lambda_j$ where $i = 1, 2$ and $j = 1, 2, 4, 5, 6, 7$. These gauge bosons perform transitions among families. They couple to *family changing charged currents* (FCCC).
  Example: In the $SU(6)_L \times U(1)_Y$ one introduces left-handed quarks in the fundamental representation **6** of $SU(6)_L$, i.e. $q_L = (u, c, t, d, s, b)_L$. We pick as an example the generator $L_{15} = \frac{1}{2\sqrt{2}}\sigma_1 \otimes \lambda_4$. Ignoring the normalisation of L_{15}, the corresponding family changing charged current reads

$$\bar{q}_L\ \gamma_\mu\ (\sigma_1 \otimes \lambda_4)\ q_L = \bar{u}\gamma_\mu b + \bar{t}\gamma_\mu d + \bar{d}\gamma_\mu t + \bar{b}\gamma_\mu u\ . \tag{4.14}$$

  This means that the corresponding gauge bosons perform the transitions $u \leftrightarrow b$ and $t \leftrightarrow d$.

- 4 charged gauge bosons associated to the generators $\frac{1}{2\sqrt{2}}\sigma_i \otimes \lambda_j$ where $i = 1, 2$ and $j = 3, 8$. These gauge bosons make no transitions among families but their couplings are *family dependent*. They couple to *non-universal family diagonal charged currents* (NUFDCC).
  Example: Using the notations above, we pick as an example $L_{14} = \frac{1}{2\sqrt{2}}\sigma_1 \otimes \lambda_3$. The corresponding gauge boson perform the transitions $u \leftrightarrow d$ and $s \leftrightarrow c$.

- 12 neutral gauge bosons associated to the generators $\frac{1}{2\sqrt{2}}\sigma_3 \otimes \lambda_j$ and $\mathbf{1}_2 \otimes \lambda_j$ where $j = 1, 2, 4, 5, 6, 7$. These gauge bosons perform transitions among families and couple to *flavour changing neutral currents* (FCNC).
  Example: Using the notations above, we pick as an example $L_9 = \frac{1}{2\sqrt{2}}\mathbf{1}_2 \otimes \lambda_6$. The corresponding gauge bosons perform the transitions $u \leftrightarrow t$ and $d \leftrightarrow c$.

- 4 neutral gauge bosons associated to the generators $\frac{1}{2\sqrt{2}}\sigma_3 \otimes \lambda_j$ and $\frac{1}{2\sqrt{2}}\mathbf{1}_2 \otimes \lambda_j$ where $j = 3, 8$. These gauge bosons make no transition among families but their couplings are *family dependent*. They couple to *non-universal family diagonal neutral currents* (NUFDNC). Example: Using the notations above, we pick as an example $L_{30} = \frac{1}{2\sqrt{2}}\sigma_3 \otimes \lambda_3$. The corresponding gauge bosons perform the transitions $u \leftrightarrow u$, $d \leftrightarrow d$, $c \leftrightarrow c$ and $s \leftrightarrow s$.

After the additional symmetry breaking by imposing Dirichlet and Neumann boundary conditions

$$SU(6)_L \times U(1)_Y \to SU(2)_L \times U(1)_Y \times SO(3)_F \ , \tag{4.15}$$

besides the SM gauge group, only the flavour gauge group $SO(3)_F$ survives. It is generated by

$$\frac{1}{2}\,\mathbf{1}_2 \,\otimes\, \lambda_j \ , \tag{4.16}$$

where $j = 2, 5, 7$. Thus the corresponding gauge bosons lead to FCNC. We note that since the bulk is completely integrated out the $SU(7)$ model leads only to FCNC. FCCC, NUFDCC and NUFDNC are absent.

4.3 Embedding of $SU(6) \times U(1)_Y$ in $SU(7)$

In this section we define the generators of $SU(7)$. The unified gauge group $SU(7)$ is broken again down to $SU(6) \times U(1)_Y$ via orbifolding. This orbifold breaking can be achieved by choosing e.g. the orbifold projection $P = \mathrm{diag}(-1, -1, -1, -1, -1, -1, 1)$. If we embed the generators of $SU(6) \times U(1)_Y$ in $SU(7)$ as upper 6×6 matrices and $U(1)_Y$ in $SU(7)$ as a diagonal 7×7 matrix an orbifold breaking by $P = \mathrm{diag}(-1, -1, -1, -1, -1, -1, 1)$ leave the $SU(6) \times U(1)_Y$ subgroup of $SU(7)$ unbroken. To be more precise, take for example the $SU(6)_L$ generator

$$L_1 = \frac{1}{2\sqrt{3}}\,(\sigma_1 \times \mathbf{1}_3) = \frac{1}{2\sqrt{3}} \begin{pmatrix} \mathbf{0}_3 & \mathbf{1}_3 \\ \mathbf{1}_3 & \mathbf{0}_0 \end{pmatrix} \ , \tag{4.17}$$

where $\mathbf{1}_3(\mathbf{0}_3)$ stands for the 3×3 unit(zero) matrix. L_1 is embedded in $SU(7)$ as

$$\tilde{L}_1 = \frac{1}{2\sqrt{3}} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \ , \tag{4.18}$$

and P acts on $\tilde{L}_1$ as

$$\tilde{L}_1 \to P\tilde{L}_1P^{-1} = \tilde{L}_1 \ . \tag{4.19}$$

The same relation holds for all other generators of the $SU(6)_L \times U(1)_Y$ subgroup of $SU(7)$. In order to simplify notations we drop the tilde and write the generators of the $SU(6)_L$ subgroup of $SU(7)$ just as 6×6 matrices. However by this notation we always mean that they are embedded in $SU(7)$ as described above. In the following, we choose for all generators of $SU(7)$ the normalisation

$$Tr\,(L_i L_j) = \frac{1}{2}\delta_{ij}\ . \tag{4.20}$$

Let again $\sigma_i, i = 1,\dots,3$ denote the Pauli matrices, $\lambda_j, j = 1,\dots,8$ the Gell-Mann matrices, $\mathbf{1}_2$ the 2×2 unit matrix and $\mathbf{1}_3$ the 3×3 unit matrix, respectively.

The gauge group $SU(7)$ has 48 generators

- 36 generators belonging to the $SU(6)_L$ subgroup of $SU(7)$:

$$L_i = \frac{1}{2\sqrt{3}}\,\sigma_i \otimes \mathbf{1}_3 \quad , \quad L_{i'} = \frac{1}{2\sqrt{2}}\,\mathbf{1}_2 \otimes \lambda_j \quad , \quad L_{i''} = \frac{1}{2\sqrt{2}}\,\sigma_i \otimes \lambda_j\ , \tag{4.21}$$

  compare (4.10), (4.11) and (4.12), where $i = 1,2,3$, $i' = 4,\dots,11$, $i'' = 12,\dots,35$ and $j = 1,\dots,8$ [4]. Note that $L_1,\dots,L_{35}$ are embedded in $SU(7)$ as upper 6×6 matrices as described above.

- 1 generator belonging to the $U(1)_Y$ subgroup of $SU(7)$:

$$L_{36} = \frac{1}{2\sqrt{21}} diag(1,1,1,1,1,1,-6)\ . \tag{4.22}$$

  Note that L_{36} commutes with $L_1,\dots,L_{35}$.

- 12 generators belonging to the coset $SU(7)/SU(6)_L \times U(1)_Y$:

$$L_{37} = \frac{1}{2}\begin{pmatrix} 0&0&0&0&0&0&\mathbf{1}\\ 0&0&0&0&0&0&\mathbf{0}\\ 0&0&0&0&0&0&\mathbf{0}\\ 0&0&0&0&0&0&\mathbf{0}\\ 0&0&0&0&0&0&\mathbf{0}\\ 0&0&0&0&0&0&\mathbf{0}\\ \mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&0 \end{pmatrix}, \dots,\ L_{48} = \frac{1}{2}\begin{pmatrix} 0&0&0&0&0&0&\mathbf{0}\\ 0&0&0&0&0&0&\mathbf{0}\\ 0&0&0&0&0&0&\mathbf{0}\\ 0&0&0&0&0&0&\mathbf{0}\\ 0&0&0&0&0&0&\mathbf{0}\\ 0&0&0&0&0&0&-\mathbf{i}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{i}&0 \end{pmatrix}$$

Next we identify the generators of weak gauge group $SU(2)_L$, the hypercharge $U(1)_Y$ and flavour gauge group $SO(3)_F$ as follows

- $SU(2)_L$: The weak generators of the SM are identified with

$$T_i = \sqrt{3}L_i = \frac{1}{2}\,\sigma_i\ \otimes\ \mathbf{1}_3\ , \tag{4.23}$$

  where $i = 1,2,3$.

[4] $L_4 = \frac{1}{2\sqrt{2}}\,\mathbf{1}_2 \otimes \lambda_1,\dots,L_{11} = \frac{1}{2\sqrt{2}}\,\mathbf{1}_2 \otimes \lambda_8, L_{12} = \frac{1}{2\sqrt{2}}\,\sigma_1 \otimes \lambda_1,\dots,L_{19} = \frac{1}{2\sqrt{2}}\,\sigma_1 \otimes \lambda_8, L_{20} = \frac{1}{2\sqrt{2}}\,\sigma_2\ \otimes\ \lambda_1,\dots,L_{27} = \frac{1}{2\sqrt{2}}\,\sigma_2\ \otimes\ \lambda_8, L_{28} = \frac{1}{2\sqrt{2}}\,\sigma_3\ \otimes\ \lambda_1,\dots,L_{35} = \frac{1}{2\sqrt{2}}\,\sigma_3\ \otimes\ \lambda_8$

- $U(1)_Y$: The hypercharge generator of the SM is identified with

$$Y = \sqrt{21}L_{36} = \frac{1}{2}\text{diag}(1,1,1,1,1,1,-6) \ . \tag{4.24}$$

- $SO(3)_F$: The generators of the flavour gauge group $SO(3)_F$ will be identified with

$$H_j = \sqrt{2}L_{i'} = \frac{1}{2}\,\mathbf{1}_2 \ \otimes\ \lambda_j \ , \tag{4.25}$$

  where $i' = 5, 8, 10,\ j = 2, 5, 7$.

Since the hypercharge operator is normalised as

$$Y = \sqrt{21}L_{36} = \frac{1}{2}\text{diag}(1,1,1,1,1,1,-6) \ , \tag{4.26}$$

we can define the electric charge operator as usual

$$Q = T_3 + Y \ . \tag{4.27}$$

4.4 Matter fields in the $SU(7)$ model

In this section we come to the fermionic content of the $SU(7)$ model. After symmetry breaking by orbifolding and imposing Dirichlet and Neumann boundary conditions the orbifold fixed points possess the gauge symmetry

$$SU(2)_L \times U(1)_Y \times SO(3)_F \ . \tag{4.28}$$

If we put matter fields at the orbifold fixed points, i.e. as brane fields, they have to transform according to the unbroken gauge group (4.28) only and not according to the unified gauge group $SU(7)$. Therefore we can introduce the SM matter at the orbifold fixed points without any difficulty. In the following, by $(\mathbf{X}, Y, \mathbf{Z})$ we denote the irreducible representations of $SU(2)_L \times U(1)_Y \times SO(3)_F$ where Y denotes the hypercharge.

- Left-handed quarks localised on the L-boundary

$$q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L = \begin{pmatrix} u \\ c \\ t \\ d \\ s \\ b \end{pmatrix}_L \ : \ (\mathbf{2}, \frac{1}{6}, \mathbf{3}) \ . \tag{4.29}$$

- Left-handed leptons are localised on the L-boundary

$$l_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \\ e \\ \mu \\ \tau \end{pmatrix}_L \ : \ (\mathbf{2}, -\frac{1}{2}, \mathbf{3}) \ . \tag{4.30}$$

- Right-handed quarks localised on the R-boundary

$$u_R = (u_R, c_R, t_R) \ : \ (\mathbf{1}, \frac{2}{3}, \mathbf{1}) \ , \tag{4.31}$$

$$d_R = (d_R, s_R, b_R) \ : \ (\mathbf{1}, -\frac{1}{3}, \mathbf{1}) \ . \tag{4.32}$$

We put u_R and d_R together in a vector of two components

$$q_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix} = \begin{pmatrix} u_R \\ c_R \\ t_R \\ d_R \\ s_R \\ b_R \end{pmatrix} \ . \tag{4.33}$$

- Right-handed leptons localised on the R-boundary

$$\nu_R = (\nu_{R1}, \nu_{R2}, \nu_{R3}) \ : \ (\mathbf{1}, 0, \mathbf{1}) \ , \tag{4.34}$$

$$e_R = (e_R, \mu_R, \tau_R) \ : \ (\mathbf{1}, -1, \mathbf{1}) \ . \tag{4.35}$$

We put ν_R and e_R together in a vector of two components

$$l_R = \begin{pmatrix} \nu_R \\ e_R \end{pmatrix} = \begin{pmatrix} \nu_{R1} \\ \nu_{R2} \\ \nu_{R3} \\ e_R \\ \mu_R \\ \tau_R \end{pmatrix} \ . \tag{4.36}$$

Figure 4.1 summarises the assignment of matter fields.
Remarks: i) Note that we have introduced also right-handed neutrinos (4.34) in the model. The reason is that we want to be able to give neutrinos a mass. This topic

will be discussed in detail in the next subsection.
ii) Since left-handed matter transforms according to the **3** representation of $SO(3)_F$ while right-handed matter transforms according to the **1** representation of $SO(3)_F$, the $SU(7)$ model is a model with a chiral gauged flavour symmetry. The reason why we need right-handed matter to transform according to the **1** representation of $SO(3)_F$ will also be discussed in the next subsection.

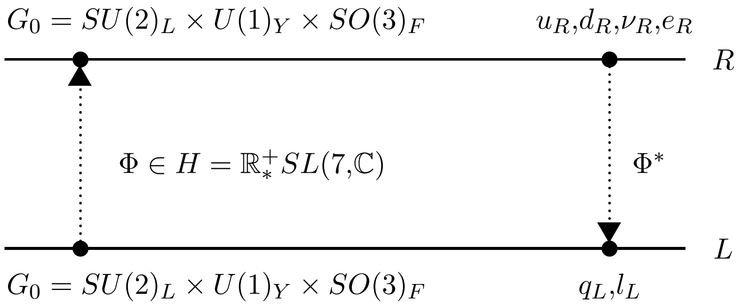

Figure 4.1: Assignment of SM matter fields in the eBTLM with bulk gauge group $SU(7)$. The unified gauge group $SU(7)$ is broken to $SU(2)_L \times U(1)_Y \times SO(3)_F$ by orbifolding and imposing Dirichlet and Neumann boundary conditions. The holonomy group reads $H = \mathbb{R}_*^+ SL(7,\mathbb{C})$. Since left- and right-handed matter transforms different under $SO(3)_F$ the $SU(7)$ model is a model with a chiral $SO(3)_F$ gauged flavour symmetry.

4.4.1 Neutrino masses and the see-saw mechanism

In 1998 the Super-Kamiokande experiment [24] showed that muon neutrinos undergo flavour oscillations. This implies that also neutrinos like charged fermions are massive. In order to give neutrinos a mass we have introduced right-handed neutrinos ν_R (4.34) in the model. However, if one introduces an ordinary Dirac mass term for neutrinos

$$\mathcal{L}_{mass}^{(\nu)} = m_\nu \bar{\nu}_L \nu_R \ , \tag{4.37}$$

one needs tiny Yukawa couplings which is a quite unnatural assumption. A possible solution for this problem is the so called see-saw mechanism [26]. The see-saw mechanism involves the introduction of an additional Majorana mass term for ν_R. In the SM a Majorana mass term for ν_R is possible since right-handed neutrinos carry no colour, weak isospin or hypercharge and thus are gauge-singlets. Note that a Majorana mass term breaks lepton number symmetry.

In the $SU(7)$ model there is an additional flavour gauge group $SO(3)_F$. However, since we have introduced ν_R in the

$$(\mathbf{1}, 0, \mathbf{1}) \tag{4.38}$$

of $SU(2)_L \times U(1)_Y \times SO(3)_F$, right-handed neutrinos are in particular $SO(3)_F$ singlets. Therefore a Majorana mass term for ν_R is allowed and we can write

$$\mathcal{L}^{(\nu)}_{mass} = m_{dirac}\bar{\nu}_L\nu_R + \frac{1}{2}M_{major}\bar{\nu^c_R}\nu_R + h.c. \;, \tag{4.39}$$

where ν^c_R is the CP conjugate of ν_R (representing left-handed antineutrinos), m_{dirac} is the 3×3 Dirac mass matrix and the M_{major} is the heavy 3×3 Majorana mass matrix. It is important that M_{major} is not generated by a nonunitary parallel transporter. Therefore Majorana masses can be several orders of magnitude larger than ordinary quark and lepton masses. An attractive assumption is that M_{major} may be generated somewhere at the GUT scale [10, 63]. In contrast, Dirac masses for all leptons (see Proposal 1 on page 111 for details) are given through Yukawa interactions by the nonunitary parallel transporter Φ^{lepton}. In this setup the see-saw mechanism works as usual. We note that if we had introduced right-handed SM matter q_R and l_R in the **3** of $SO(3)_F$ a Majorana mass term for ν_R would not be possible and the see-saw mechanism would not work.

4.5 Anomalies in the $SU(7)$ model

In this section, we discuss the topic of anomalies and anomaly cancellation in the $SU(7)$ model. Since we have introduced (chiral) SM matter at *different* orbifold fixed points the cancellation of anomalies in the $SU(7)$ model is in contrast to the SM not automatic. Before we come in detail to the $SU(7)$ model we first discuss the issue of anomaly cancellation in orbifold models more generally. For simplicity we focus on five-dimensional orbifolds. In orbifold theories two types of anomalies can arise:

- four-dimensional anomalies intrinsic to the orbifold fixed points.
- five-dimensional anomalies intrinsic to the bulk.

For the low energy consistency of the theory, it is necessary that both the anomaly at the orbifold fixed points *and* the anomaly in the bulk cancels. Let us assume that the four-dimensional anomaly in the effective low-energy theory cancels. We then may ask: Is the cancellation of the four-dimensional anomaly sufficient to cancel also the five-dimensional anomaly? As it has be worked out by Arkani-Hamed and others [4, 71, 66] this is indeed the case. More precisely, for a collection of five-dimensional fermions all one has to care about is that their *zero modes* form an anomaly-free representation of the low-energy four-dimensional gauge group. This means that the five-dimensional anomaly is independent of the details of the physics in the bulk.

4.5.1 Anomaly cancellation mechanisms in the $SU(7)$ model

We now come in detail to anomaly cancellation mechanisms in the $SU(7)$ model. In section 4.4, we have introduced chiral SM matter at the orbifold fixed points. For the following discussion let us initially ignore the flavour gauge group $SO(3)_F$. We consider two different scenarios.

In the first scenario, we put *both* left- and right-handed SM matter on the same orbifold fixed point. Without loss of generality let this orbifold fixed point be the L-boundary. Figure 4.2 summarises the setting. We observe that all anomalies

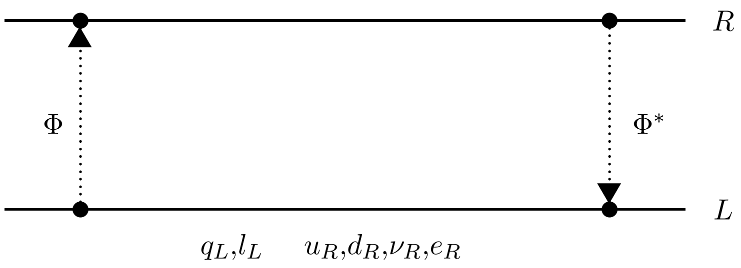

Figure 4.2: Anomaly cancellation scenario: Left- and right-handed SM matter on the same orbifold fixed point.

cancel *locally*, in particular at the L-boundary, thanks to the usual cancellation of anomalies in the SM. Note that this scenario is *not* assumed in the $SU(7)$ model.

In the second scenario, we put left-handed SM matter on the L-boundary and right-handed SM matter on the R-boundary. This scenario is known as chiral delocalisation [38]. This is exactly what we have adopted in the $SU(7)$ model. This means that since the anomaly of three $U(1)$ gauge bosons is nonzero [47]

$$\mathrm{Tr}\left[Y_L^3\right] = -\frac{2}{9}\ ,\ \mathrm{Tr}\left[Y_R^3\right] = -\frac{2}{9}\ , \tag{4.40}$$

this scenario leads to localised SM anomalies at the different orbifold fixed points. Figure 4.3 summarises the setting. At first sight, this scenario leads to an inconsistent theory. However, one can introduce a bulk Chern-Simons term with a jumping coefficient [8, 66, 71, 4, 38] in order to *locally* cancel the SM anomalies arising from the three $U(1)$ gauge bosons. Note that this anomaly cancellation mechanism works only if the *integrated* anomaly, i.e. the sum over all local contributions to the anomaly, vanishes. Due to (4.40), this is indeed the case. The work of Arkani-Hamed and others [4] describes a mechanism how the Chern-Simons term can be

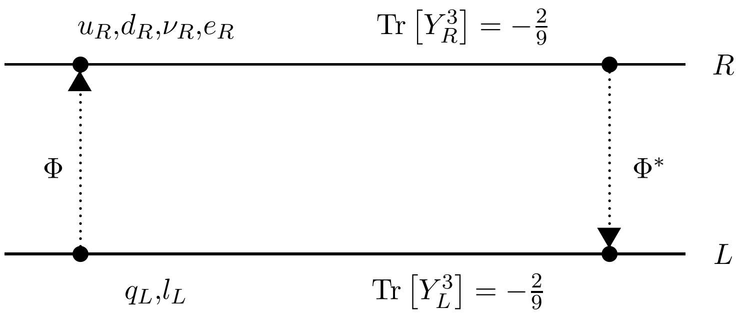

Figure 4.3: Anomaly cancellation scenario: Left-handed SM matter on the L-boundary, right-handed SM matter on the R-boundary. The emerging anomalies are inscribed.

generated by integrating out massive bulk fermions which transforms according to an anomaly-free representation of the low energy four-dimensional gauge group.

4.5.2 Contributions to the anomaly from the flavour gauge group $SO(3)_F$

In this subsection, we include the flavour gauge group $SO(3)_F$ in our considerations. In section 4.4, we have introduced left-handed SM matter in the **3** of $SO(3)_F$ and right-handed matter SM in the **1** of $SO(3)_F$. The **3** of $SO(3)_F$ is formed by the generators

$$H_j = \frac{1}{2}\,\mathbf{1}_2 \,\otimes\, \lambda_j \;, \tag{4.41}$$

where $j = 2, 5, 7$. Since they fulfil

$$\mathrm{Tr}\left[\{H_i, H_j\}H_k\right] = 0 \;, \tag{4.42}$$

where $i, j, k \in \{2, 5, 7\}$, the **3** of $SO(3)_F$ is an anomaly-free representation of $SO(3)_F$. Thus the $SU(7)$ model with left- and right-handed matter introduced as in section 4.4 is *free of anomalies.*

Remarks: i) Concerning anomalies it doesn't matter that we have introduced left-handed SM matter and right-handed SM matter in different representation of $SO(3)_F$.
ii) If we had used the flavour gauge group $SU(3)_F$ instead of $SO(3)_F$ the model would contain a non-vanishing anomaly due to

$$Tr\left[\{\lambda_i, \lambda_j\}\lambda_k\right] = 4 i d_{ijk} \;, \tag{4.43}$$

where d_{ijk} are the completely symmetric coefficients of $\mathfrak{su}_3$.

4.6 Orbifold breaking in the $SU(7)$ model and the electroweak Higgs

In this section we describe how a symmetry breaking by a non-trivial orbifold projection P will not only break the unitary gauge group $SU(7)$ down to $SU(6)_L \times U(1)_Y$ but also leads to non-trivial unitary factors e^{A_y} in the decomposition $\Phi = \rho\, e^{A_y}\, e^{\eta}\, e^{A_y}$. In order to break $SU(7)$ down to $SU(6)_L \times U(1)_Y$ we choose

$$P = \mathrm{diag}(-1,-1,-1,-,1-,1,-1,1) \ . \tag{4.44}$$

The branching rule for the adjoint representation **48** of $SU(7)$ with respect to $SU(6)_L$ reads

$$\mathbf{48} \to \mathbf{35} + \mathbf{6} + \bar{\mathbf{6}} + \mathbf{1} \ , \tag{4.45}$$

where **35** is the adjoint representation of $SU(6)_L$, **1** is the trivial representation of $SU(6)_L$ and **6** ($\bar{\mathbf{6}}$) is the fundamental (complex conjugate fundamental) representation of $SU(6)_L$. According to the action of P on $\mathfrak{g} = \mathfrak{su}(7)$ we have

$$\mathfrak{g}_0 = \mathbf{35} + \mathbf{1} \ , \quad \mathfrak{g}_1 = \mathbf{6} + \bar{\mathbf{6}} \ . \tag{4.46}$$

For $G = SU(7)$ the corresponding holonomy group reads $H = \mathbb{R}_*^+ SL(7,\mathbb{C})$ [5]. The Cartan decomposition for $\mathfrak{sl}(7,\mathbb{C}) = \mathrm{Lie}\ SL(7,\mathbb{C})$ reads

$$\mathfrak{sl}(7,\mathbb{C}) = \mathfrak{su}(7) + i\mathfrak{su}(7) = \mathfrak{g} + i\mathfrak{g} \ . \tag{4.47}$$

We choose an $\mathfrak{a} \subset i\mathfrak{su}(7)$ such that $P\eta P = \eta$ is automatically fulfilled. In fact for

$$\mathfrak{a} = \{\eta = \mathrm{diag}(a_1,a_2,a_3,a_4,a_5,a_6,a_7)\} \quad , \quad \sum_i a_i = 0 \ , \quad a_i \in \mathbb{R} \ , \tag{4.48}$$

$P\eta P = \eta$ holds for any $\eta \in \mathfrak{a}$. Let $\Phi \in \mathbb{R}_*^+ SL(7,\mathbb{C})$ fulfil the sharpened orbifold condition. Then Φ can be written as

$$\Phi = \rho\, e^{A_y}\, e^{\eta}\, e^{A_y} \ , \tag{4.49}$$

where

$$P A_\mu^{L(R)} P^{-1} = A_\mu^{L(R)} \ , \tag{4.50}$$

$$P A_y P^{-1} = -A_y \ , \tag{4.51}$$

$$\tag{4.52}$$

for $A_\mu^{L(R)} \in \mathbf{35} + \mathbf{1}$, $A_y \in \mathbf{6} + \bar{\mathbf{6}}$ and $\rho \in \mathbb{R}_*^+$. This means that the generators $L_1, \ldots, L_{36}$ of the $SU(6)_L \times U(1)_Y$ subgroup of $SU(7)$ remain unbroken and the gauge fields $A_\mu^{L,R}$ can be written as

$$A_\mu^{L(R)} = \sum_{a=1}^{36} A_\mu^{L(R)a} L_a \ . \tag{4.53}$$

[5]Note that the linear span of $SU(7)$ reads $\mathbb{R}_*^+\ SL(7,\mathbb{C})$ and thus H is unique, compare with (3.4)

4.6.1 The electroweak Higgs

Let us consider the unitary factor e^{A_y} in (4.49). According to Definition 9 (see page 71), e^{A_y} can be written as

$$\exp(A_y) = \exp(i\, g_4 R\, \mathcal{A}_y^{(0)}) \; , \tag{4.54}$$

where $\mathcal{A}_y^{(0)}$ is the zero mode of the extra-dimensional component of the five-dimensional gauge field in its canonical four-dimensional form. With $A_y^{(0)} \in \mathbf{6}+\bar{\mathbf{6}}$ we can expand $\mathcal{A}_y^{(0)}$ as

$$\mathcal{A}_y^{(0)} = \sum_{\hat{a}=37}^{48} \mathcal{A}_y^{\hat{a}(0)} L_{\hat{a}} \; , \tag{4.55}$$

where $L_{37}, \ldots, L_{48}$ are the generators of the coset space $SU(7)/SU(6)_L \times U(1)_Y$. We determine the hypercharge of $\mathcal{A}_y^{(0)}$. Looking at the generators $L_{37}, \ldots, L_{48}$, e.g.

$$L_{37} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \; , \tag{4.56}$$

we see that $\mathcal{A}_y^{(0)}$ carries the hypercharge

$$Y = \frac{1}{2} \; . \tag{4.57}$$

Next we consider the fundamental representation $\mathbf{6}$ of $SU(6)$. The fundamental representation $\mathbf{6}$ will remain irreducible when restricted to $SU(2)_L \times SU(3)_F$ ($SO(3)_F$)

$$\mathbf{6} = (\mathbf{2}, \mathbf{3}) \; . \tag{4.58}$$

This means that $\mathcal{A}_y^{(0)}$ is a doublet with respect to $SU(2)_L$ and a triplet with respect to $SU(3)_F$ ($SO(3)_F$), respectively. We explicitly write

$$\begin{aligned}
\mathcal{A}_y^{(0)} &= \sum_{\hat{a}=37}^{48} \mathcal{A}_y^{\hat{a}(0)} L_{\hat{a}} \qquad (4.59)\\
&= \frac{1}{2}\begin{pmatrix}
\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathcal{A}_y^{1+(0)} \\ \mathcal{A}_y^{2+(0)} \\ \mathcal{A}_y^{3+(0)} \end{pmatrix} \\
\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathcal{A}_y^{10(0)} \\ \mathcal{A}_y^{20(0)} \\ \mathcal{A}_y^{30(0)} \end{pmatrix} \\
\begin{pmatrix} \mathcal{A}_y^{1+(0)*} & \mathcal{A}_y^{2+(0)*} & \mathcal{A}_y^{3+(0)*} \end{pmatrix} & \begin{pmatrix} \mathcal{A}_y^{10(0)*} & \mathcal{A}_y^{20(0)*} & \mathcal{A}_y^{30(0)*} \end{pmatrix} &
\end{pmatrix},
\end{aligned}$$

where

$$\mathcal{A}_y^{1+(0)} = \mathcal{A}_y^{37(0)} - i\mathcal{A}_y^{38(0)}\ ,\ \mathcal{A}_y^{2+(0)} = \mathcal{A}_y^{39(0)} - i\mathcal{A}_y^{40(0)}\ ,\ \mathcal{A}_y^{3+(0)} = \mathcal{A}_y^{41(0)} - i\mathcal{A}_y^{42(0)}$$
$$\mathcal{A}_y^{10(0)} = \mathcal{A}_y^{43(0)} - i\mathcal{A}_y^{44(0)}\ ,\ \mathcal{A}_y^{20(0)} = \mathcal{A}_y^{45(0)} - i\mathcal{A}_y^{46(0)}\ ,\ \mathcal{A}_y^{30(0)} = \mathcal{A}_y^{47(0)} - i\mathcal{A}_y^{48(0)}\ .$$

Thus $\mathcal{A}_y^{1+(0)}$, $\mathcal{A}_y^{2+(0)}$, $\mathcal{A}_y^{3+(0)}$ form the up-component of the doublet while $\mathcal{A}_y^{10(0)}$, $\mathcal{A}_y^{20(0)}$, $\mathcal{A}_y^{30(0)}$ form the down-component of the doublet. We see from (4.59) that the $SU(7)$ model contains three $SU(2)_L$ doublets, one for each flavour. This comes out if we we consider the mass terms for quarks and leptons, respectively. In the next chapter we will discuss the topic of fermion masses in detail. However, anticipating a little, the mass term for quarks is given by

$$\bar{q}_L \Phi q_R = \overline{(u\ c\ t\ d\ s\ b)}_L \left(\rho\ e^{A_y}\ e^{\eta}\ e^{A_y}\right) \begin{pmatrix} u_R \\ c_R \\ t_R \\ d_R \\ s_R \\ b_R \end{pmatrix} . \qquad (4.60)$$

We see that

$$H_1 = \begin{pmatrix} \mathcal{A}_y^{1+(0)} \\ \mathcal{A}_y^{10(0)} \end{pmatrix} = \begin{pmatrix} \mathcal{A}_y^{37(0)} - i\mathcal{A}_y^{38(0)} \\ \mathcal{A}_y^{43(0)} - i\mathcal{A}_y^{44(0)} \end{pmatrix} \qquad (4.61)$$

is a $SU(2)_L$ Higgs doublet coupled to u, d, ν_e, e,

$$H_2 = \begin{pmatrix} \mathcal{A}_y^{2+(0)} \\ \mathcal{A}_y^{20(0)} \end{pmatrix} = \begin{pmatrix} \mathcal{A}_y^{39(0)} - i\mathcal{A}_y^{40(0)} \\ \mathcal{A}_y^{45(0)} - i\mathcal{A}_y^{46(0)} \end{pmatrix} \qquad (4.62)$$

is a $SU(2)_L$ Higgs doublet coupled to c, s, ν_μ, μ and

$$H_3 = \begin{pmatrix} \mathcal{A}_y^{3+(0)} \\ \mathcal{A}_y^{30(0)} \end{pmatrix} = \begin{pmatrix} \mathcal{A}_y^{41(0)} - i\mathcal{A}_y^{42(0)} \\ \mathcal{A}_y^{47(0)} - i\mathcal{A}_y^{48(0)} \end{pmatrix} \tag{4.63}$$

is a $SU(2)_L$ Higgs doublet coupled to t, b, ν_τ, τ. Thus the model includes three $SU(2)_L$ doublets $\{H_i\}$, $i = 1, 2, 3$, one for each flavour. Finally we determine the electric charge of $\mathcal{A}_y^{(0)}$ and $\{H_i\}$, respectively. For the up-component $\mathcal{A}_y^{1+(0)}, \mathcal{A}_y^{2+(0)}, \mathcal{A}_y^{3+(0)}$ of the $\{H_i\}$ we obtain

$$Q = T_3 + Y = \frac{1}{2} + \frac{1}{2} = 1 \ , \tag{4.64}$$

and for the down-component $\mathcal{A}_y^{10(0)}, \mathcal{A}_y^{20(0)}, \mathcal{A}_y^{30(0)}$ of the $\{H_i\}$ we obtain

$$Q = T_3 + Y = \frac{1}{2} - \frac{1}{2} = 0 \ . \tag{4.65}$$

We summarise: The zero modes of extra-dimensional component of the five-dimensional gauge field $\mathcal{A}_y^{(0)}$ have the following properties

- They appear from the four-dimensional point of view as scalar fields.
- They are doublets with respect to the weak SM gauge group $SU(2)_L$.
- They carry hypercharge $\frac{1}{2}$.
- Their $SU(2)_L$ down-component is electrically neutral and their $SU(2)_L$ up-component has electric charge $+1$.
- They include three $SU(2)_L$ doublets $\{H_i\}$, $i = 1, 2, 3$, one for first, one for the second and one for the third generation.

Conclusion: *The zero modes of the extra-dimensional component of the five-dimensional gauge field* $\mathcal{A}_y^{(0)} = \sum_{\hat{a}=37}^{48} \mathcal{A}_y^{\hat{a}(0)} L_{\hat{a}}$ *are a substitute for the SM Higgs.* $\mathcal{A}_y^{(0)}$ *includes three Higgs doublets* $\{H_i\}$, $i = 1, 2, 3$, *one for each flavour. They generate the unitary factors* e^{A_y} *via* $A_y = i\, g_4 R\, \mathcal{A}_y^{(0)}$ *in the decomposition* $\Phi = \rho\, e^{A_y} e^{\eta} e^{A_y}$.

Making use of the residual $SU(6)_L \times U(1)_Y$ global symmetry, it is possible to transform away the up-components of the $SU(2)_L$ doublets

$$\mathcal{A}_y^{(0)} = \sum_{\hat{a}=37}^{48} \mathcal{A}_y^{\hat{a}(0)} L_{\hat{a}} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{A}_y^{1+(0)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{A}_y^{2+(0)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{A}_y^{3+(0)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{A}_y^{10(0)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{A}_y^{20(0)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{A}_y^{30(0)} \\ \mathcal{A}_y^{1+(0)*} & \mathcal{A}_y^{2+(0)*} & \mathcal{A}_y^{3+(0)*} & \mathcal{A}_y^{10(0)*} & \mathcal{A}_y^{20(0)*} & \mathcal{A}_y^{30(0)*} & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\mathcal{A}_y^{(0)} = \sum_{\hat{a}=43}^{48} \mathcal{A}_y^{\hat{a}(0)} L_{\hat{a}} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{A}_y^{10(0)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{A}_y^{20(0)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{A}_y^{30(0)} \\ 0 & 0 & 0 & \mathcal{A}_y^{10(0)*} & \mathcal{A}_y^{20(0)*} & \mathcal{A}_y^{30(0)*} & 0 \end{pmatrix} . \quad (4.66)$$

This transformation results in a vanishing mass term for the photon and is in analogy to the SM.

4.7 Additional gauge symmetry breaking by Dirichlet and Neumann boundary conditions

In this section we describe how the symmetry breaking

$$SU(6)_L \times U(1)_Y \to SU(2)_L \times U(1)_Y \times SO(3)_F \quad (4.67)$$

can be achieved by imposing Dirichlet and Neumann boundary conditions for the gauge fields (4.53) which are unaffected by the orbifold projection P. The reason why we need this additional symmetry breaking is that the orbifold $S^1/\mathbb{Z}_2$ possesses only one orbifold projection P. Note that the underlying orbifold in the $SU(7)$ model is consider as the orbifold $S^1/\mathbb{Z}_2$ with twisted boundary conditions, see discussion in section 2.3, and thus we have one orbifold projection P given by (4.44) and one continuous Wilson line W given by (4.98).

In the next subsection we first describe the issue of gauge symmetry breaking through Dirichlet and Neumann boundary conditions more generally. We define our theory in five dimensions between two parallel branes. One brane is located at $y = 0$ and the other brane is located at $y = \pi R$. The two branes are considered as four-dimensional boundaries. This way we can compare gauge symmetry breaking through Dirichlet and Neumann boundary conditions to gauge symmetry breaking on the orbifold $S^1/\mathbb{Z}_2$. As in the orbifold case y denote the coordinate of the extra dimension.

4.7.1 Gauge symmetry breaking by Dirichlet and Neumann boundary conditions and its relation to gauge symmetry through orbifolding

Let G be the gauge group we want to break with Lie algebra $\mathfrak{g}$. In addition, let G_0 be the subgroup of G we want to obtain as the unbroken gauge group with Lie algebra $\mathfrak{g}_0$. By $\{T^A\}$ we denote the set of generators creating G and by $\{T^a\}$ the set of generators creating G_0. We consider the split

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 \ , \tag{4.68}$$

where $\mathfrak{g}_1$ generate the coset space G/G_0. By $\{T^{\hat{a}}\}$ we denote the set of generators of the coset space G/G_0. In order to achieve the symmetry breaking $G \to G_0$ we demand

$$A_\mu^{\hat{a}} = 0 \quad , \quad \partial_y A_\mu^a = 0 \tag{4.69}$$

at both boundaries $y = 0$ and $y = \pi R$.

- $A_\mu^{\hat{a}} = 0$ are Dirichlet boundary conditions for the broken gauge fields $A_\mu^{\hat{a}}$.
- $\partial_y A_\mu^a = 0$ are Neumann boundary conditions for the unbroken gauge fields A_μ^a.

We compare this gauge symmetry breaking through boundary conditions to the gauge symmetry breaking on the orbifold $S^1/\mathbb{Z}_2$. Recall that on the orbifold $S^1/\mathbb{Z}_2$ gauge and scalar fields have to fulfil the boundary conditions (2.83), (2.84)

$$A_\mu(x^\mu, -y) = P\ A_\mu(x^\mu, y)\ P^{-1} \tag{4.70}$$

$$A_y(x^\mu, -y) = -P\ A_y(x^\mu, y)\ P^{-1} \tag{4.71}$$

and the periodicity condition (2.85)

$$A_M(x^\mu, y + 2\pi R) = W\ A_M(x^\mu, y)\ W^{-1} \ , \tag{4.72}$$

In following discussion we admit the trivial periodicity condition, i.e. we set $W = 1$ in (4.72). The boundary condition (4.70) breaks the bulks gauge group G down to G_0'

$$G_0' = \{g \in G \mid Pg = gP\} \tag{4.73}$$

at $y = 0$. Let $\{T^{a'}\}$ denote the set of generators creating G'_0 and let $\{T^{\hat{a}'}\}$ denote the set of generators creating the coset space G/G'_0. According to (4.70) and (4.71) unbroken gauge $A_\mu^{a'}(x^\mu, y)$ and the scalar fields $A_y^{\hat{a}'}(x^\mu, y)$ are even functions, i.e.

$$
\begin{aligned}
A_\mu^{a'}(x^\mu, -y) &= A_\mu^{a'}(x^\mu, y) \\
A_y^{\hat{a}'}(x^\mu, -y) &= A_y^{\hat{a}'}(x^\mu, y) \; .
\end{aligned}
\tag{4.74}
$$

Thus we can Fourier expand

$$
\begin{aligned}
A_\mu^{a'}(x^\mu, y) &= \frac{1}{\sqrt{2\pi R}} A_\mu^{a'(0)}(x^\mu) + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A_\mu^{a'(n)}(x^\mu) \cos(\frac{ny}{R}) \\
A_y^{\hat{a}'}(x^\mu, y) &= \frac{1}{\sqrt{2\pi R}} A_y^{\hat{a}'(0)}(x^\mu) + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A_y^{\hat{a}'(n)}(x^\mu) \cos(\frac{ny}{R}) \; .
\end{aligned}
\tag{4.75}
$$

On the other hand, according to (4.70) and (4.71) broken gauge $A_\mu^{\hat{a}'}(x^\mu, y)$ and the scalar fields $A_y^{a'}(x^\mu, y)$ are odd functions, i.e.

$$
\begin{aligned}
A_\mu^{\hat{a}'}(x^\mu, -y) &= -A_\mu^{\hat{a}'}(x^\mu, y) \\
A_y^{a'}(x^\mu, -y) &= -A_y^{a'}(x^\mu, y) \; .
\end{aligned}
\tag{4.76}
$$

Thus we can Fourier expand

$$
\begin{aligned}
A_\mu^{\hat{a}'}(x^\mu, y) &= \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A_\mu^{\hat{a}'(n)}(x^\mu) \sin(\frac{ny}{R}) \\
A_y^{a'}(x^\mu, y) &= \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A_y^{a'(n)}(x^\mu) \sin(\frac{ny}{R}) \; .
\end{aligned}
\tag{4.77}
$$

The expansions (4.77) lead to Dirichlet boundary conditions for broken gauge and scalar fields, i.e.

$$
\begin{aligned}
A_\mu^{\hat{a}'}(x^\mu, 0) &= A_\mu^{\hat{a}'}(x^\mu, \pi R) = 0 \\
A_y^{a'}(x^\mu, 0) &= A_y^{a'}(x^\mu, \pi R) = 0 \; .
\end{aligned}
\tag{4.78}
$$

The expansions (4.75) lead to Neumann boundary conditions for unbroken gauge fields and scalar fields, i.e.

$$
\begin{aligned}
\partial_y A_\mu^{a'}(x^\mu, 0) &= \partial_y A_\mu^{a'}(x^\mu, \pi R) = 0 \\
\partial_y A_y^{\hat{a}'}(x^\mu, 0) &= \partial_y A_y^{\hat{a}'}(x^\mu, \pi R) = 0 \; .
\end{aligned}
\tag{4.79}
$$

The advantage of gauge symmetry breaking by boundary conditions is that unlike in the orbifold case, one can obtain *any* subgroup G_0 of G. In contrast, in the orbifold case only very special subgroups of G compatible with the action of P on the Lie algebra of $\mathfrak{g}$ can be obtained.

4.7.2 Gauge symmetry breaking by Dirichlet and Neumann boundary conditions in the $SU(7)$ model

Let us return to the $SU(7)$ model. We consider the breaking

$$SU(6)_L \times U(1)_Y \to SU(2)_L \times U(1)_Y \times SO(3)_F \; . \tag{4.80}$$

This breaking can be achieved by demanding

$$A_\mu^{\hat{a}} = 0 \quad , \quad \partial_y A_\mu^a = 0 \; , \tag{4.81}$$

for

$$\begin{aligned} A_\mu^a &\in \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(3) \\ A_\mu^{\hat{a}} &\in \mathfrak{su}(6) \oplus \mathfrak{u}(1) / \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(3) \end{aligned} \tag{4.82}$$

at both boundaries L and R.

4.8 Gauge coupling unification and the weak mixing angle

The gauge group $SU(7)$ unifies inter alia the weak gauge group $SU(2)_L$ and the hypercharge gauge group $U(1)_Y$ of the SM. This means that in the unified theory there exists only one five-dimensional gauge coupling constant which we denote by

$$g_5^{SU(7)} \; . \tag{4.83}$$

Therefore it is possible to calculate the effective four-dimensional coupling constants for $SU(2)_L$ and $U(1)_Y$, respectively, and thus the weak mixing angle θ_W of the SM. Recall that in the SM the covariant derivative reads [47]

$$D_\mu = \partial_\mu + ig \, \mathbf{W}_\mu \cdot \mathbf{t} + ig' \, W_0^\mu t_0 \; , \tag{4.84}$$

where g and g' are the four-dimensional coupling constants of $SU(2)_L$ and $U(1)_Y$, respectively. The weak mixing angle θ_W is given by

$$\sin^2 \theta_W = \frac{g'^2}{g^2 + g'^2} \; . \tag{4.85}$$

In order to compute θ_W in the $SU(7)$ model we have to determine the effective four-dimensional coupling constants

$$g \equiv g_4^{SU(2)_L} \quad , \quad g' \equiv g_4^{U(1)_Y} \; . \tag{4.86}$$

To calculate (4.86) we have to take into account the normalisation of the generators T_3 and Y, see (4.23) and (4.24). Thus due to (4.23) the five-dimensional gauge coupling constant $g_5^{SU(2)_L}$ of the $SU(2)_L$ subgroup of $SU(7)$ in related to the five-dimensional gauge coupling constant $g_5^{SU(7)}$ by

$$g_5^{SU(2)_L} = \frac{g_5^{SU(7)}}{\sqrt{3}} \tag{4.87}$$

and due to (4.24) the five-dimensional gauge coupling constant $g_5^{U(1)_Y}$ of the $U(1)_Y$ subgroup of $SU(7)$ is related to the five-dimensional gauge coupling constant $g_5^{SU(7)}$ by

$$g_5^{U(1)_Y} = \frac{g_5^{SU(7)}}{\sqrt{21}} \ . \tag{4.88}$$

In addition, due to (3.72) an effective four-dimensional gauge coupling constant g_4 is related to a five-dimensional gauge coupling constant g_5 via

$$g_4 = \frac{g_5}{\sqrt{2\pi R}} \ , \tag{4.89}$$

where R is the compactification radius. Thus we obtain the following four-dimensional effective SM coupling constants

$$g = g_4^{SU(2)_L} = \frac{g_5^{SU(7)}}{\sqrt{6\pi R}} \quad , \quad g' = g_4^{U(1)_Y} = \frac{g_5^{SU(7)}}{\sqrt{42\pi R}} \ , \tag{4.90}$$

where we have inserted (4.87) and (4.88). Inserting further (4.90) in (4.85) we obtain for the weak mixing angle in the $SU(7)$ model:

$$\sin^2\theta_W^{SU(7)} = 0.125 \ . \tag{4.91}$$

We compare this result with the experimental value [20]

$$\sin^2\theta_W^{\mathrm{exp}} \approx 0.23 \ . \tag{4.92}$$

We see that the obtained value for θ_W is too small by approximately a factor of two. This problem can however be solved by starting with a slightly different unified gauge group. This issue will be discussed in the outlook.

4.9 The minimum of the Higgs potential $V(\Phi)$

We consider the Higgs potential

$$V(\Phi) = V(\rho\, e^{A_y}\, e^{\eta}\, e^{A_y}) \ . \tag{4.93}$$

According to Theorem 5 $V(\Phi)$ is invariant under unitary gauge transformations

$$V(S_0(x)\Phi S_0(x)^{-1}) = V(\Phi) \tag{4.94}$$

where $S_0(x) \in SU(2)_L \times SO(3)_F \times U(1)_Y$. Thus e^{A_y} in (4.93) cannot be gauged away and the Higgs potential $V(\Phi)$ depends on ρ, η and A_y. The unitary factor e^{A_y} in (4.93) is given by

$$e^{A_y} = e^{i\, g_4 R\, \mathcal{A}_y^{(0)}} \tag{4.95}$$

where

$$\mathcal{A}_y^{(0)} = \sum_{\hat{a}=43}^{48} \mathcal{A}_y^{\hat{a}(0)} L_{\hat{a}} \ , \tag{4.96}$$

compare with (4.66). Note that $\mathcal{A}_y^{(0)}$ denote the neutral components of the three electroweak Higgs doublets (4.61), (4.62) and (4.63). We consider now the case where the $\mathcal{A}_y^{(0)}$ assume a VEV. Without loss of generality we lay it in the $L_{\hat{4}\hat{3}}$- $L_{\hat{4}\hat{5}}$- and $L_{\hat{4}\hat{7}}$-direction, i.e.

$$\mathcal{A}_y \to \langle \mathcal{A}_y^{(0)} \rangle = \sum_{\hat{a}=\hat{4}\hat{3},\hat{4}\hat{5},\hat{4}\hat{7}} \langle \mathcal{A}_y^{\hat{a}(0)} \rangle L_{\hat{a}} \ . \tag{4.97}$$

Inserting (4.97) in (4.95) we get

$$W = e^{i\, g_4 R\, \sum_{\hat{a}} \langle \mathcal{A}_y^{\hat{a}(0)} \rangle L_{\hat{a}}} \ . \tag{4.98}$$

Since $[P, L_{\hat{a}}] \neq 0$ for $\hat{a} = \hat{4}\hat{3}, \hat{4}\hat{5}, \hat{4}\hat{7}$ all VEVs $\langle \mathcal{A}_y^{\hat{a}(0)} \rangle$ can be arbitrary constants and thus W is a continuous Wilson line. We parametrise the VEVs $\langle \mathcal{A}_y^{\hat{a}(0)} \rangle$ as

$$\langle \mathcal{A}_y^{\hat{4}\hat{3}(0)} \rangle = \frac{\alpha_{\hat{4}\hat{3}}}{g_4 R} \quad , \quad \langle \mathcal{A}_y^{\hat{4}\hat{5}(0)} \rangle = \frac{\alpha_{\hat{4}\hat{5}}}{g_4 R} \quad , \quad \langle \mathcal{A}_y^{\hat{4}\hat{7}(0)} \rangle = \frac{\alpha_{\hat{4}\hat{7}}}{g_4 R} \tag{4.99}$$

where the $\alpha_{\hat{a}}$ are dimensionless parameters. Inserting (4.99) in (4.98) we get

$$W = e^{i\, \sum_{\hat{a}} \alpha_{\hat{a}} L_{\hat{a}}} \ . \tag{4.100}$$

The VEVs $\langle \mathcal{A}_y^{\hat{a}(0)} \rangle$ are much smaller than the compactification scale $1/R$. Thus $0 < \alpha_{\hat{a}} \ll 1$ in (4.100) and we can approximate

$$W = e^{i\, \sum_{\hat{a}} \alpha_{\hat{a}} L_{\hat{a}}} \approx \mathbf{1} + \sum_{\hat{a}} i \alpha_{\hat{a}} L_{\hat{a}} \tag{4.101}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & i\frac{\alpha_{\hat{4}\hat{3}}}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & i\frac{\alpha_{\hat{4}\hat{5}}}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & i\frac{\alpha_{\hat{4}\hat{7}}}{2} \\ 0 & 0 & 0 & i\frac{\alpha_{\hat{4}\hat{3}}}{2} & i\frac{\alpha_{\hat{4}\hat{5}}}{2} & i\frac{\alpha_{\hat{4}\hat{7}}}{2} & 1 \end{pmatrix} . \tag{4.102}$$

According to (4.93) we can parameterise the minimum Φ_{min} of the Higgs potential as

$$\Phi_{min} = \rho_{min}\frac{1}{\sqrt{2}} \begin{pmatrix} e^{a_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{a_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{a_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{a_4} & 0 & 0 & i\alpha'_{\hat{4}3} \\ 0 & 0 & 0 & 0 & e^{a_5} & 0 & i\alpha'_{\hat{4}5} \\ 0 & 0 & 0 & 0 & 0 & e^{a_6} & i\alpha'_{\hat{4}7} \\ 0 & 0 & 0 & i\alpha'_{\hat{4}3} & i\alpha'_{\hat{4}5} & i\alpha'_{\hat{4}7} & e^{a_7} \end{pmatrix} + \mathcal{O}(\alpha_a^2) \tag{4.103}$$

where $\sum_{i=1}^{7} a_i = 0$, $a_i \in \mathbb{R}$ and

$$\alpha'_{\hat{4}3} = \frac{\alpha_{\hat{4}3}}{2}\left(e^{a_4} + e^{a_7}\right) \quad , \quad \alpha'_{\hat{4}5} = \frac{\alpha_{\hat{4}5}}{2}\left(e^{a_5} + e^{a_7}\right) \quad , \quad \alpha'_{\hat{4}7} = \frac{\alpha_{\hat{4}7}}{2}\left(e^{a_6} + e^{a_7}\right) . \tag{4.104}$$

In the following we neglect terms of $\mathcal{O}(\alpha_a^2)$. *In order to have a spontaneous symmetry breaking we assume that $V(\Phi)$ is minimised at non-trivial Φ_{min}, i.e. we assume*

$$a_i \neq 0 \quad , \quad \alpha'_{\hat{a}} \neq 0 \tag{4.105}$$

for $i = 1, \ldots, 7$ and $\hat{a} = \hat{4}3, \hat{4}5, \hat{4}7$ in (4.103). For later use we make the following

Definition 10 *We call a minimum of the Higgs potential Φ_{min} quasi $\mathcal{S}_2$ symmetric if*

$$a_i = a_j \tag{4.106}$$

for the pairs $(i, j) = (1, 4), (2, 5), (3, 6)$.

Note that fluctuations of a_i and α_a around the minimum Φ_{min} (4.103) give rise to altogether 10 Higgs particles. This topic will be discussed in the outlook.

4.10 Calculation of gauge field masses in the $SU(7)$ model

In this section we calculate the masses of all gauge fields in the $SU(7)$ model. Recall that $SU(2)_L \times U(1)_Y \times SO(3)_F$ is left unbroken by the orbifolding and imposing Dirichlet and Neumann boundary conditions. Consequently we have to compute

$$D_\mu \Phi_{min} = i\frac{g}{\sqrt{2}}\, A_\mu^{i(0)}\, [L_i, \Phi_{min}] + i\frac{g}{\sqrt{2}}\, A_\mu^{i(1)} \{L_i, \Phi_{min}\} , \tag{4.107}$$

for the generators $\{L_i\}$ with $i = 1, 2, 3, 5, 8, 10, 36$. Note that $T_j = \sqrt{3}L_j$ (4.23), $Y = \sqrt{21}L_{36}$ (4.24) and $H_k = \sqrt{2}L_l$ (4.25) with $j = 1, 2, 3$, $k = 2, 5, 7$ and $l = 5, 8, 10$ denote the generators of the weak gauge group, the hypercharge and the $SO(3)_F$ flavour gauge group, respectively. In the following calculations we can us either the set $\{L_i\}$ or the set $\{T_j, Y, H_k\}$ in (4.107). However only the $\{L_i\}$ are normalised as $\mathrm{tr}\,(L_i L_j) = \frac{1}{2}\delta_{ij}$ and imply conventional normalisation of the four-dimensional kinetic terms for $A_\mu^{i(0)}$ and $A_\mu^{i(1)}$ as we have shown explicitly in section 3.3.2. Therefore we have to use the set $\{L_i\}$ instead of $\{T_j, Y, H_k\}$. Φ_{min} in (4.107) is given by equation (4.103). *We assume now that Φ_{min} is quasi $\mathcal{S}_2$ symmetric.* The reason will be explained in the next subsection. We can expand a quasi $\mathcal{S}_2$ symmetric Φ_{min} as

$$\Phi_{min} = \Phi_{min}^{diag} + \Phi_{min}^{offdiag} = \sum_{j=1,2,3,4} \phi_j\, \tilde{L}_j + \sum_{k=\hat{43},\hat{45},\hat{47}} \phi_k\, L_k\ . \tag{4.108}$$

where

$$\tilde{L}_1 = \frac{1}{2}\,\mathrm{diag}(1,0,0,1,0,0) = \frac{1}{2}\,\mathrm{diag}(l_1, l_1) \quad , \quad l_1 = \mathrm{diag}(1,0,0) \tag{4.109}$$

$$\tilde{L}_2 = \frac{1}{2}\,\mathrm{diag}(0,1,0,0,1,0) = \frac{1}{2}\,\mathrm{diag}(l_2, l_2) \quad , \quad l_2 = \mathrm{diag}(0,1,0) \tag{4.110}$$

$$\tilde{L}_3 = \frac{1}{2}\,\mathrm{diag}(0,0,1,0,0,1) = \frac{1}{2}\,\mathrm{diag}(l_3, l_3) \quad , \quad l_3 = \mathrm{diag}(0,0,1) \tag{4.111}$$

$$\tilde{L}_4 = \frac{1}{2}\,\mathrm{diag}(0,0,0,0,0,0,1) \tag{4.112}$$

Remarks: i) By writing a matrix just as a 6×6 matrix we mean that this matrix is embedded in a 7×7 matrix as an upper 6×6 matrix. This convention is the same convention as we have made for the generators $L_1, \ldots, L_{35}$ of $SU(6)_L \subset SU(7)$ and we will use this convention in the following calculations.
ii) $\sum_{j=1,2,3,4} \phi_j\, \tilde{L}_j$ form the diagonal part Φ_{min}^{diag} of Φ_{min} while $\sum_{k=\hat{43},45,\hat{47}} \phi_k\, L_k$ form the off-diagonal part $\Phi_{min}^{offdiag}$ of Φ_{min}.
iii) The $\tilde{L}_i$, $i = 1,2,3,4$, are normalised such that $\mathrm{tr}\,(\tilde{L}_i\tilde{L}_j) = \frac{1}{2}\delta_{ij}$.
The ϕ_j and ϕ_k in (4.108) are given by

$$\phi_1 = \sqrt{2}\rho_{min}e^{a_1}\ ,\ \phi_2 = \sqrt{2}\rho_{min}e^{a_2}\ \ \phi_3 = \sqrt{2}\rho_{min}e^{a_3}\ ,\ \phi_4 = \sqrt{2}\rho_{min}e^{a_7} \tag{4.113}$$

$$\phi_k = i\phi_k'\ , \quad \phi_k' = \sqrt{2}\rho_{min}\alpha_k' \tag{4.114}$$

for $k = \hat{43}, \hat{45}, \hat{47}$. We then have to calculate the commutators $[L_i, \Phi_{min}]$ and anticommutators $\{L_i, \Phi_{min}\}$ for $\{L_i\}$, $i = 1,2,3,5,8,10,35$, insert the results of this computation in (4.107) and evaluate the covariant derivative $D_\mu\Phi_{min}$. After taking the adjoint $(D_\mu\Phi_{min})^\dagger$, multiplying the adjoint with $(D_\mu\Phi_{min})$ and taking the trace we obtain

$$\mathcal{L}_{\mathrm{mass}} = tr\left[(D_\mu\Phi_{min})^\dagger\,(D_\mu\Phi_{min})\right] \tag{4.115}$$

We remind the reader that the basis of mass eigenstates $A_\mu^{i(0)}$ and $A_\mu^{i(1)}$ is already diagonal and no mixed terms between the zero mode and the first excited mode in (4.115) mode occur.

4.10.1 Mass term for the SM weak gauge fields from the off-diagonal part of Φ_{min}

In this subsection we calculate mass squared for the the zero mode gauge fields $A_\mu^{i(0)}$ associated to the generators L_i $(i = 1, 2)$ and weak generators T_i $(i = 1, 2)$, respectively. First since $\left[L_i, \tilde{L}_j\right] = 0$ for $j = 1, 2, 3, 4$ we see that the diagonal part of Φ_{min} gives no contribution to the mass of zero mode gauge fields $A_\mu^{i(0)}$. *This is a consequence of the $\mathcal{S}_2$ quasi symmetry of the minimum Φ_{min} of the Higgs potential.* Therefore the fields $A_\mu^{i(0)}$ get their mass only from $\Phi_{min}^{offdiag}$. The covariant derivative reads

$$\begin{aligned}
D_\mu \Phi_{min}^{offdiag} &= i\frac{g}{\sqrt{2}}\, A_\mu^{i(0)} \left[L_i, \Phi_{min}^{offdiag}\right] \\
&= i\frac{g}{\sqrt{2}}\, A_\mu^{1(0)} \left(\phi_{\hat{43}} \underbrace{\left[L_1, L_{\hat{43}}\right]}_{=\frac{1}{2\sqrt{3}} i L_{38}} + \phi_{\hat{45}} \underbrace{\left[L_1, L_{\hat{45}}\right]}_{=\frac{1}{2\sqrt{3}} i L_{40}} + \phi_{\hat{47}} \underbrace{\left[L_1, L_{\hat{47}}\right]}_{=\frac{1}{2\sqrt{3}} i L_{42}} \right) \\
&+ i\frac{g}{\sqrt{2}}\, A_\mu^{2(0)} \left(\phi_{\hat{43}} \underbrace{\left[L_2, L_{\hat{43}}\right]}_{=-\frac{1}{2\sqrt{3}} i L_{37}} + \phi_{\hat{45}} \underbrace{\left[L_2, L_{\hat{45}}\right]}_{=-\frac{1}{2\sqrt{3}} i L_{39}} + \phi_{\hat{47}} \underbrace{\left[L_2, L_{\hat{47}}\right]}_{=-\frac{1}{2\sqrt{3}} i L_{41}} \right) \\
&= -i\frac{g}{2\sqrt{6}}\, A_\mu^{1(0)} \left(\phi'_{\hat{43}} L_{38} + \phi'_{\hat{45}} L_{40} + \phi'_{\hat{47}} L_{42}\right) \\
&+ i\frac{g}{2\sqrt{6}}\, A_\mu^{2(0)} \left(\phi'_{\hat{43}} L_{37} + \phi'_{\hat{45}} L_{39} + \phi'_{\hat{47}} L_{41}\right)\ .
\end{aligned}$$

Taking the adjoint $(D_\mu \Phi_{min}^{offdiag})^\dagger$, multiplying $(D_\mu \Phi_{min}^{offdiag})^\dagger$ and $(D_\mu \Phi_{min}^{offdiag})$ and taking the trace we arrive at

$$\mathrm{tr}\left[\left(D_\mu \Phi_{min}^{offdiag}\right)^\dagger \left(D_\mu \Phi_{min}^{offdiag}\right)\right] = \frac{1}{24}\, g^2 \rho_{min}^2 \sum_{a=\hat{43},\hat{45},\hat{47}} \alpha_a'^2 \, \left(A_\mu^{i(0)}\right)^2 \tag{4.116}$$

with $i = 1, 2$. Thus the mass squared of the gauge fields $A_\mu^{i(0)}$ reads

$$m^2 = \frac{1}{12}\, g^2 \rho_{min}^2 \sum_{a=\hat{43},\hat{45},\hat{47}} \alpha_a'^2 = \frac{1}{12}\, \frac{1}{R^2} \sum_{a=\hat{43},\hat{45},\hat{47}} \alpha_a'^2\ , \tag{4.117}$$

where we have inserted $g\rho_{min} = 1/R$ in the second step. For $0 < \alpha'_a \ll 1$ we deduce that $1/R = \mathcal{O}(1)$ TeV so that $m = m_W = 80.4$ GeV [20]. Therefore

The zero mode gauge fields $A_\mu^{1,2(0)}$ are identified with the SM weak gauge fields $W_\mu^{1,2}$.

4.10.2 Mass term for the SM Z gauge field from off-diagonal part of Φ_{min}

We consider the linear combination of the gauge fields $A_\mu^{3(0)}$ and $A_\mu^{36(0)}$ associated to the generators L_3, L_{36}

$$\begin{aligned} A_\mu^{(0)} &= A_\mu^{36(0)} \cos\theta_W^{SU(7)} + W_\mu^{3(0)} \sin\theta_W^{SU(7)} \\ Z_\mu^{(0)} &= -A_\mu^{36(0)} \sin\theta_W^{SU(7)} + W_\mu^{3(0)} \cos\theta_W^{SU(7)} \end{aligned} \tag{4.118}$$

where $\theta_W^{SU(7)}$ is the weak mixing angle in the $SU(7)$ model given by equation (4.91). The gauge field $Z_\mu^{(0)}$ get its mass, like the weak gauge fields $W_\mu^{1,2}$, only from $\Phi_{min}^{offdiag}$ and the gauge field $A_\mu^{(0)}$ remains massless. The relation between the mass m_Z of the gauge field $Z_\mu^{(0)}$ the mass squared m_W (4.117) of the weak gauge fields $W_\mu^{1,2}$ is given by

$$m_Z = \frac{m_W}{\cos\theta_W^{SU(7)}} \tag{4.119}$$

Therefore
The zero mode gauge field $Z_\mu^{(0)}$ are identified with the SM Z gauge field Z_μ and the zero mode gauge field $A_\mu^{(0)}$ are identified with the SM photon field A_μ.

The relation (4.119) is familiar in the SM. However according to (4.119) the mass of the Z gauge field turns out to be too low which is a consequence of then wrong weak mixing angle $\theta_W^{SU(7)}$.

4.10.3 Mass term for the first excited KK mode of the SM weak gauge fields from diagonal part of Φ_{min}

In this subsection we calculate the mass squared for the first excited KK mode of SM weak gauge fields $W_\mu^{1,2}$. Since we have identified $W_\mu^{1,2} = A_\mu^{1,2(0)}$ the first excited KK mode gauge fields of the weak gauge fields $W_\mu^{1,2}$ are $A_\mu^{1,2(1)}$. Therefore in the following we write $W_\mu^{1,2(1)} = A_\mu^{1,2(1)}$. To calculate the mass squared of $W_\mu^{1,2(1)}$, we consider the scenario where $a_i \gg 1$ for some a_i in Φ_{min}^{diag}. Hence the off-diagonal part $\Phi_{min}^{offdiag}$ will give only a very small contribution to the mass of $W_\mu^{1,2(1)}$. For this reason we neglect it in the following calculations. Consequently we have to compute

$$D_\mu \Phi_{min}^{diag} = i\frac{g}{\sqrt{2}}\, W_\mu^{i(1)} \{L_i, \Phi_{min}^{diag}\} \tag{4.120}$$

with $i = 1, 2$. We calculate the anticommutators

$$\begin{aligned}
\{L_1, \Phi_{min}^{diag}\} &= \phi_1\{L_1, \tilde{L}_1\} + \phi_2\{L_1, \tilde{L}_2\} + \phi_3\{L_1, \tilde{L}_3\} \\
&= \phi_1 \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & l_1 \\ l_1 & 0 \end{pmatrix} + \phi_2 \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & l_2 \\ l_2 & 0 \end{pmatrix} + \phi_3 \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & l_3 \\ l_3 & 0 \end{pmatrix} \\
\{L_2, \Phi_{min}^{diag}\} &= \phi_1\{L_1, \tilde{L}_1\} + \phi_2\{L_1, \tilde{L}_2\} + \phi_3\{L_1, \tilde{L}_3\} \\
&= \phi_1 \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & -il1 \\ il_1 & 0 \end{pmatrix} + \phi_2 \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & -il_2 \\ il_2 & 0 \end{pmatrix} + \phi_3 \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & -il_3 \\ il_3 & 0 \end{pmatrix} ,
\end{aligned} \tag{4.121}$$

where 0 denotes the 3×3 zero matrix. Recall that $l_1 = \text{diag}(1, 0, 0)$ and $l_2 = \text{diag}(0, 1, 0)$, see (4.109). Inserting (4.121) in (4.120), taking the adjoint $(D_\mu \Phi_{min}^{diag})^\dagger$, multiplying the adjoint with $D_\mu \Phi_{min}^{diag}$ and finally taking the trace we get

$$\text{tr}\left[(D_\mu \Phi_{min}^{diag})^\dagger \left(D_\mu \Phi_{min}^{diag}\right)\right] = \frac{1}{12}\, g^2 \, \left(\phi_1^2 + \phi_2^2 + \phi_3^2\right) \, \left(W_\mu^{1,2(1)}\right)^2 \tag{4.122}$$

Note that $tr\ (\tilde{L}_i \tilde{L}_j) = \frac{1}{2}\delta_{ij}$. Inserting the expression (4.113) for ϕ_i we finally get

$$\text{tr}\left[(D_\mu \Phi_{min}^{diag})^\dagger \left(D_\mu \Phi_{min}^{diag}\right)\right] = \frac{1}{6}\, g^2 \rho_{min}^2 \, \left(e^{2a_1} + e^{2a_2} + e^{3a_2}\right) \left(W_\mu^{1,2(1)}\right)^2 \tag{4.123}$$

Thus the mass squared for the first excited KK mode of the SM weak gauge fields $W_\mu^{1,2}$ reads

$$m^2_{W^{1,2(1)}} = \frac{1}{3}\frac{1}{R^2}\left(e^{2a_1} + e^{2a_2} + e^{3a_2}\right) \tag{4.124}$$

where we have inserted $g\rho_{min} = 1/R$.

If $a_i \gg 1$ for at least one $i = 1, 2, 3$ the first excited KK mode of the SM weak gauge fields $W_\mu^{1,2}$ receives very large masses from the Higgs mechanism in comparison to the compactification scale $1/R$.

4.10.4 Mass term for the first excited KK mode of the SM Z gauge field and SM photon field from diagonal part of Φ_{min}

The first excited KK mode of the SM Z gauge field $Z_\mu^{(1)}$ and the first excited KK mode of the photon field $A_\mu^{(1)}$ is a linear combination of the fields $A_\mu^{3(1)}$ and $A_\mu^{36(1)}$ associated to the generators L_3 and L_{36}

$$\begin{aligned}
A_\mu^{(1)} &= A_\mu^{36(1)} \cos\theta_W^{SU(7)} + A_\mu^{3(1)} \sin\theta_W^{SU(7)} \\
Z_\mu^{(1)} &= -A_\mu^{36(1)} \sin\theta_W^{SU(7)} + A_\mu^{3(1)} \cos\theta_W^{SU(7)}
\end{aligned} \tag{4.125}$$

As in the last section we consider the scenario, where $a_i \gg 1$ for some a_i in Φ_{min}^{diag}. Hence the off-diagonal part $\Phi_{min}^{offdiag}$ will again give only a very small contribution to the mass of $Z^{(1)}_{mu}$ and $A^{(1)}_\mu$. For this reason we neglect it in the following calculations. Consequently we have to compute

$$D_\mu \Phi^{diag}_{min} = i\frac{g}{\sqrt{2}}\, W^{3(1)}_\mu \{L_3, \Phi^{diag}_{min}\} + i\frac{g}{\sqrt{2}}\, B^{(1)}_\mu \{L_{36}, \Phi^{diag}_{min}\} \tag{4.126}$$

We calculate the anticommutators

$$\begin{aligned}
\{L_3, \Phi^{diag}_{min}\} &= \phi_1\{L_3, \tilde{L}_1\} + \phi_2\{L_3, \tilde{L}_2\} + \phi_3\{L_3, \tilde{L}_3\} \qquad (4.127)\\
&= \phi_1 \frac{1}{2\sqrt{3}} \begin{pmatrix} l_1 & 0 \\ 0 & -l_1 \end{pmatrix} + \phi_2 \frac{1}{2\sqrt{3}} \begin{pmatrix} l_2 & 0 \\ 0 & -l_2 \end{pmatrix} + \phi_3 \frac{1}{2\sqrt{3}} \begin{pmatrix} l_3 & 0 \\ 0 & -l_3 \end{pmatrix}\\
\{L_{36}, \Phi^{diag}_{min}\} &= \phi_1\{L_{36}, \tilde{L}_1\} + \phi_2\{L_{36}, \tilde{L}_2\} + \phi_3\{L_{36}, \tilde{L}_3\} + \phi_4\{L_{36}, \tilde{L}_4\}\\
&= \phi_1 \frac{1}{2\sqrt{21}} \begin{pmatrix} l_1 & 0 \\ 0 & l_1 \end{pmatrix} + \phi_2 \frac{1}{2\sqrt{21}} \begin{pmatrix} l_2 & 0 \\ 0 & l_2 \end{pmatrix} + \phi_3 \frac{1}{2\sqrt{21}} \begin{pmatrix} l_3 & 0 \\ 0 & l_3 \end{pmatrix}\\
&- \phi_4 \frac{1}{2\sqrt{21}} \,\mathrm{diag}(0,0,0,0,0,0,6)
\end{aligned}$$

Inserting this equations in (4.126), taking the adjoint $(D_\mu \Phi^{diag}_{min})^\dagger$, multiplying the adjoint with $(D_\mu \Phi^{diag}_{min})$ and finally taking the trace we get

$$\begin{aligned}
\mathrm{tr}\left[(D_\mu \Phi^{diag}_{min})^\dagger \left(D_\mu \Phi^{diag}_{min}\right)\right] &= \frac{1}{12}\, g^2 \left(\phi_1^2 + \phi_2^2 + \phi_3^2\right) \left(A^{3(1)}_\mu\right)^2\\
&+ \frac{1}{84}\, g^2 \left(\phi_1^2 + \phi_2^2 + \phi_3^2 + 18\phi_4^2\right) \left(A^{36(1)}_\mu\right)^2
\end{aligned}$$

Inserting the expression (4.113) for ϕ_i we get

$$\begin{aligned}
\mathrm{tr}\left[(D_\mu \Phi^{diag}_{min})^\dagger \left(D_\mu \Phi^{diag}_{min}\right)\right] &= \frac{1}{6}\, g^2 \rho^2_{min} \left(e^{2a_1} + e^{2a_2} + e^{2a_3}\right) \left(A^{3(1)}_\mu\right)^2\\
&+ \frac{1}{42}\, g^2 \rho^2_{min} \left(e^{2a_1} + e^{2a_2} + e^{2a_3} + 18 e^{2a_7}\right) \left(A^{36(1)}_\mu\right)^2
\end{aligned}$$

If we assume that $a_7 \ll 1$ we can neglect e^{2a_7} for at least one of the a_1, a_2 or a_3 large and thus after transforming $A^{36(1)}_\mu = A^{(1)}_\mu \cos\theta^{SU(7)}_W - Z^{(1)}_\mu \sin\theta^{SU(7)}_W$, $A^{3(1)}_\mu = A^{(1)}_\mu \sin\theta^{SU(7)}_W + Z^{(1)}_\mu \cos\theta^{SU(7)}_W$ which follows from (4.125) we get the final result

$$\begin{aligned}
\mathrm{tr}\left[(D_\mu \Phi^{diag}_{min})^\dagger \left(D_\mu \Phi^{diag}_{min}\right)\right] &= \frac{1}{6}\, g^2 \rho^2_{min} \left(e^{2a_1} + e^{2a_2} + e^{2a_3}\right) \left(Z^{(1)}_\mu\right)^2\\
&+ \frac{1}{42}\, g^2 \rho^2_{min} \left(e^{2a_1} + e^{2a_2} + e^{2a_3}\right) \left(A^{(1)}_\mu\right)^2
\end{aligned} \tag{4.128}$$

Thus the mass squared for the first excited KK mode of the SM Z gauge field and the SM photon field reads

$$m^2_{Z^{(1)}} = \frac{1}{3}\frac{1}{R^2}\left(e^{2a_1} + e^{2a_2} + e^{3a_2}\right) \quad , \quad m^2_{\gamma^{(1)}} = \frac{1}{21}\frac{1}{R^2}\left(e^{2a_1} + e^{2a_2} + e^{2a_2}\right) \tag{4.129}$$

where we have inserted $g\rho_{min} = 1/R$.

If $a_i \gg 1$ for at least one $i = 1,2,3$ the first excited KK mode of the SM Z gauge field and the first excited KK mode of the SM photon field receive very large masses from the Higgs mechanism in comparison to the compactification scale $1/R$.

4.10.5 Mass term for $SO(3)_F$ flavour gauge fields from diagonal part of Φ_{min}

In this subsection we calculate the mass squared for the zero and the first excited KK mode of the flavour gauge fields $A_\mu^{j(0,1)}$ associated to the generators L_j, where $j = 5,8,10$. Since the corresponding flavour gauge bosons couple to FCNC its mass terms must be very large $\mathcal{O}(10^3) - \mathcal{O}(10^5)$ TeV. Therefore, in order to be consistent with experiment, we consider again the case where $a_i \gg 1$ for some a_i in Φ_{min}^{diag}. Hence the contribution to the mass squared for the flavour gauge fields $A_\mu^{j(0,1)}$ from the off-diagonal part of Φ_{min} can be neglected. Consequently we get have to compute

$$D_\mu \Phi_{min}^{diag} = i\frac{g}{\sqrt{2}}\, A_\mu^{j(0)} \left[A_\mu^j, \Phi_{min}^{diag}\right] + i\frac{g}{\sqrt{2}}\, A_\mu^{j(1)} \{A_\mu^j, \Phi_{min}^{diag}\} \tag{4.130}$$

where $j = 5,8,10$. We calculate the commutators

$$\begin{aligned}
\left[L_5, \Phi_{min}^{diag}\right] &= \phi_1 \left[L_5, \tilde{L}_1\right] + \phi_2 \left[L_5, \tilde{L}_2\right] + \phi_3 \left[L_5, \tilde{L}_3\right] \\
&= \frac{i}{4\sqrt{2}}\phi_1 \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} - \frac{i}{4\sqrt{2}}\phi_2 \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} \\
\left[L_8, \Phi_{min}^{diag}\right] &= \phi_1 \left[L_8, \tilde{L}_1\right] + \phi_2 \left[L_8, \tilde{L}_2\right] + \phi_3 \left[L_8, \tilde{L}_3\right] \\
&= \frac{i}{4\sqrt{2}}\phi_1 \begin{pmatrix} \lambda_4 & 0 \\ 0 & \lambda_4 \end{pmatrix} - \frac{i}{4\sqrt{2}}\phi_3 \begin{pmatrix} \lambda_4 & 0 \\ 0 & \lambda_4 \end{pmatrix} \\
\left[L_{10}, \Phi_{min}^{diag}\right] &= \phi_1 \left[L_{10}, \tilde{L}_1\right] + \phi_2 \left[L_{10}, \tilde{L}_2\right] + \phi_3 \left[L_{10}, \tilde{L}_3\right] \\
&= \frac{i}{4\sqrt{2}}\phi_2 \begin{pmatrix} \lambda_6 & 0 \\ 0 & \lambda_6 \end{pmatrix} - \frac{i}{4\sqrt{2}}\phi_3 \begin{pmatrix} \lambda_6 & 0 \\ 0 & \lambda_6 \end{pmatrix}
\end{aligned} \tag{4.131}$$

and anticommutators

$$\begin{aligned}\{L_5, \Phi^{diag}_{min}\} &= \phi_1\{L_5, \tilde{L}_1\} + \phi_2\{L_5, \tilde{L}_2\} + \phi_3\{L_5, \tilde{L}_3\} \\ &= \frac{1}{4\sqrt{2}}\phi_1 \begin{pmatrix}\lambda_2 & 0 \\ 0 & \lambda_2\end{pmatrix} + \frac{1}{4\sqrt{2}}\phi_2 \begin{pmatrix}\lambda_2 & 0 \\ 0 & \lambda_2\end{pmatrix} \\ \{L_8, \Phi^{diag}_{min}\} &= \phi_1\{L_8, \tilde{L}_1\} + \phi_2\{L_8, \tilde{L}_2\} + \phi_3\{L_8, \tilde{L}_3\} \\ &= \frac{1}{4\sqrt{2}}\phi_1 \begin{pmatrix}\lambda_5 & 0 \\ 0 & \lambda_5\end{pmatrix} + \frac{1}{4\sqrt{2}}\phi_3 \begin{pmatrix}\lambda_5 & 0 \\ 0 & \lambda_5\end{pmatrix} \\ \{L_{10}, \Phi^{diag}_{min}\} &= \phi_1\{L_{10}, \tilde{L}_1\} + \phi_2\{L_{10}, \tilde{L}_2\} + \phi_3\{L_{10}, \tilde{L}_3\} \\ &= \frac{1}{4\sqrt{2}}\phi_2 \begin{pmatrix}\lambda_7 & 0 \\ 0 & \lambda_7\end{pmatrix} + \frac{1}{4\sqrt{2}}\phi_3 \begin{pmatrix}\lambda_7 & 0 \\ 0 & \lambda_7\end{pmatrix}\end{aligned} \quad (4.132)$$

Let us first calculate the mass squared for the zero mode flavour gauge fields $A^{j(0)}_\mu$. Inserting (4.131) in (4.130) we get

$$\begin{aligned}D_\mu \Phi^{diag}_{min} &= -\frac{1}{8}\, g \left(A^{5(0)}_\mu \left(\phi_1 \begin{pmatrix}\lambda_1 & 0 \\ 0 & \lambda_1\end{pmatrix} - \phi_2 \begin{pmatrix}\lambda_1 & 0 \\ 0 & \lambda_1\end{pmatrix}\right)\right. \\ &+ A^{8(0)}_\mu \left(\phi_1 \begin{pmatrix}\lambda_4 & 0 \\ 0 & \lambda_4\end{pmatrix} - \phi_3 \begin{pmatrix}\lambda_4 & 0 \\ 0 & \lambda_4\end{pmatrix}\right) \\ &+ \left. A^{10(0)}_\mu \left(\phi_1 \begin{pmatrix}\lambda_6 & 0 \\ 0 & \lambda_6\end{pmatrix} - \phi_3 \begin{pmatrix}\lambda_6 & 0 \\ 0 & \lambda_6\end{pmatrix}\right)\right)\end{aligned}$$

Taking the adjoint $(D_\mu \Phi^{diag}_{min})^\dagger$ multiplying with $(D_\mu \Phi^{diag}_{min})$ and taking the trace we obtain

$$\begin{aligned}&\mathrm{tr}\left[\left(D_\mu \Phi^{diag}_{min}\right)^\dagger \left(D_\mu \Phi^{diag}_{min}\right)\right] \\ = &\frac{1}{16}\, g^2 \left(\left(A^{2(0)}_\mu\right)^2 (\phi_1 - \phi_2)^2 + \left(A^{5(0)}_\mu\right)^2 (\phi_1 - \phi_3)^2 + \left(A^{7(0)}_\mu\right)^2 (\phi_2 - \phi_3)^2 \right)\end{aligned}$$

Inserting the expressions for ϕ_1, ϕ_2 and ϕ_3 (4.113) we finally get

$$\begin{aligned}\mathrm{tr}\left[\left(D_\mu \Phi^{diag}_{min}\right)^\dagger \left(D_\mu \Phi^{diag}_{min}\right)\right]_{\text{zero mode}} &= \frac{1}{8}\, g^2 \rho^2_{min} \left(e^{a1} - e^{a2}\right)^2 \left(A^{5(0)}_\mu\right)^2 \\ &+ \frac{1}{8}\, g^2 \rho^2_{min} \left(e^{a1} - e^{a3}\right)^2 \left(A^{8(0)}_\mu\right)^2 \\ &+ \frac{1}{8}\, g^2 \rho^2_{min} \left(e^{a3} - e^{a2}\right)^2 \left(A^{10(0)}_\mu\right)^2\end{aligned}$$

To summarise we have obtained

Field	Mass squared
$A^{5(0)}_\mu$	$\frac{1}{4}\,\frac{1}{R^2}\,\left(e^{a1} - e^{a2}\right)^2$
$A^{8(0)}_\mu$	$\frac{1}{4}\,\frac{1}{R^2}\,\left(e^{a1} - e^{a3}\right)^2$
$A^{10(0)}_\mu$	$\frac{1}{4}\,\frac{1}{R^2}\,\left(e^{a2} - e^{a3}\right)^2$

(4.133)

where we have inserted $g\rho_{min} = 1/R$.

For an appropriate choice of the a_i*, so that in particular* $a_1 \neq a_2$*,* $a_1 \neq a_3$*,* $a_2 \neq a_3$ *and two* $a_i \gg 1$*, the zero mode flavour gauge fields will receive very large masses from the Higgs mechanism in comparison to the compactification scale* $g\rho_{min} = 1/R$*.*

Next we calculate the masses for the first excited KK mode flavour gauge fields $A_\mu^{j(1)}$. Inserting (4.132) in (4.130) we get

$$\begin{aligned} D_\mu \Phi_{min}^{diag} &= i\,\frac{1}{8}\,g\,\left(A_\mu^{5(1)}\left(\phi_1\begin{pmatrix}\lambda_2 & 0\\ 0 & \lambda_2\end{pmatrix} + \phi_2\begin{pmatrix}\lambda_2 & 0\\ 0 & \lambda_2\end{pmatrix}\right)\right.\\ &\quad + A_\mu^{8(1)}\left(\phi_1\begin{pmatrix}\lambda_5 & 0\\ 0 & \lambda_5\end{pmatrix} + \phi_3\begin{pmatrix}\lambda_5 & 0\\ 0 & \lambda_5\end{pmatrix}\right)\\ &\quad \left. + A_\mu^{10(1)}\left(\phi_1\begin{pmatrix}\lambda_7 & 0\\ 0 & \lambda_7\end{pmatrix} + \phi_3\begin{pmatrix}\lambda_7 & 0\\ 0 & \lambda_7\end{pmatrix}\right)\right) \end{aligned}$$

Taking the adjoint $(D_\mu \Phi_{min}^{diag})^\dagger$ multiplying with $(D_\mu \Phi_{min}^{diag})$ and taking the trace we get

$$\begin{aligned} &\mathrm{tr}\left[\left(D_\mu \Phi_{min}^{diag}\right)^\dagger \left(D_\mu \Phi_{min}^{diag}\right)\right]\\ &= \frac{1}{16}\,g\,\left(\left(A_\mu^{5(1)}\right)^2\,(\phi_1+\phi_2)^2 + \left(A_\mu^{8(1)}\right)^2\,(\phi_1+\phi_3)^2 + \left(A_\mu^{10(1)}\right)^2\,(\phi_2+\phi_3)^2\right) \end{aligned}$$

Inserting the expressions for ϕ_1, ϕ_2 and ϕ_3 (4.113) we finally get

$$\begin{aligned} \mathrm{tr}\left[\left(D_\mu \Phi_{min}^{diag}\right)^\dagger \left(D_\mu \Phi_{min}^{diag}\right)\right]_{\text{first excited mode}} &= \frac{1}{8}\,g^2\rho_{min}^2\,\left(e^{a1}+e^{a2}\right)^2\left(A_\mu^{5(1)}\right)^2\\ &+ \frac{1}{8}\,g^2\rho_{min}^2\,\left(e^{a1}+e^{a3}\right)^2\left(A_\mu^{8(1)}\right)^2\\ &+ \frac{1}{8}\,g^2\rho_{min}^2\,\left(e^{a3}+e^{a2}\right)^2\left(A_\mu^{10(1)}\right)^2 \end{aligned}$$

To summarise we have obtained

Field	Mass squared
$A_\mu^{5(1)}$	$\frac{1}{4}\,\frac{1}{R^2}\,\left(e^{a1}+e^{a2}\right)^2$
$A_\mu^{8(1)}$	$\frac{1}{4}\,\frac{1}{R^2}\,\left(e^{a1}+e^{a3}\right)^2$
$A_\mu^{10(1)}$	$\frac{1}{4}\,\frac{1}{R^2}\,\left(e^{a2}+e^{a3}\right)^2$

(4.134)

where we have inserted $g\rho_{min} = 1/R$.

4.11 Suppression of tree-level FCNC

Looking at the different results (4.117), (4.119), (4.124), (4.129), (4.134) and (4.133), we observe that the strongest constraint in the model is the suppression of tree-level FCNC. This suppression leads to the following conditions

1.
$$a_1 \neq a_2 \ , \ a_1 \neq a_3 \ , \ a_2 \neq a_3 \tag{4.135}$$

2.
$$a_i \gg 1 \tag{4.136}$$

   for at least for two a_i

for the diagonal part of Φ_{min}, see (4.133). These conditions are also sufficient to give large masses to the first excited KK mode of SM gauge bosons, see (4.124) and (4.129), and to the first excited KK mode of flavour gauge boson, see (4.134).

Chapter 5

Fermion masses and CKM mixing matrix in the $SU(7)$ model

In this chapter we describe how fermion masses are generated by the Higgs mechanism in the context of nonunitary parallel transporters Φ. In general, fermion masses are given by Yukawa interactions. For example a term

$$\bar{q}_L \Phi q_R \, , \tag{5.1}$$

where q_L and q_R are given by (4.29) and (4.33), respectively, will lead to a mass term for quarks. However, since quarks and leptons have different masses their mass terms cannot be generated by the same nonunitary parallel transporter Φ. In addition we will see that if the Higgs potential $V(\Phi)$ is minimised at a quasi $\mathcal{S}_2$ symmetric Φ_{min} it is not possible to obtain the correct quark and lepton masses nor the correct CKM mixing matrix. Therefore we make the following

Proposition 1 *We introduce a nonunitary parallel transporter Φ^{quark} which leads to a mass term for the quarks via*

$$\bar{q}_L \; \Phi^{quark} \; q_R \tag{5.2}$$

and a nonunitary parallel transporter Φ^{lepton} which leads to a mass terms for the leptons via

$$\bar{l}_L \; \Phi^{lepton} \; l_R \tag{5.3}$$

and make a clear distinction between Φ^{quark} and Φ^{lepton}. In particular $\Phi^{quark} \neq \Phi^{lepton} \neq \Phi^{gauge}$ where $\Phi^{gauge} = \Phi$ is the nonunitary parallel transporter which leads to masses for the gauge bosons.

It is important that we have three different parallel transporters in the theory. This way we can get different quark and lepton masses as it is observed in nature. We may speculate that at the GUT scale $\Phi^{quark} = \Phi^{lepton}$.

Since we have three different parallel transporters in the theory, we also have three Higgs potentials $V(\Phi^{gauge}) = V(\Phi)$, $V(\Phi^{quark})$ and $V(\Phi^{lepton})$. The minimum of $V(\Phi^{quark})$ and $V(\Phi^{lepton})$ is denoted by Φ^{quark}_{min} and Φ^{lepton}_{min}, respectively. According to (4.103) they can be parameterised as

$$\Phi^{quark}_{min} = \rho^{quark}_{min} \begin{pmatrix} e^{a_1^{quark}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{a_2^{quark}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{a_3^{quark}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{a_4^{quark}} & 0 & 0 & i\alpha^{quark\,\prime}_{\hat{43}} \\ 0 & 0 & 0 & 0 & e^{a_5^{quark}} & 0 & i\alpha^{quark\,\prime}_{\hat{45}} \\ 0 & 0 & 0 & 0 & 0 & e^{a_6^{quark}} & i\alpha^{quark\,\prime}_{\hat{47}} \\ 0 & 0 & 0 & i\alpha^{quark\,\prime}_{\hat{43}} & i\alpha^{quark\,\prime}_{\hat{45}} & i\alpha^{quark\,\prime}_{\hat{47}} & e^{a_7^{quark}} \end{pmatrix} \tag{5.4}$$

and

$$\Phi^{lepton}_{min} = \rho^{lepton}_{min} \begin{pmatrix} e^{a_1^{lepton}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{a_2^{lepton}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{a_3^{lepton}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{a_4^{lepton}} & 0 & 0 & i\alpha^{lepton\,\prime}_{\hat{43}} \\ 0 & 0 & 0 & 0 & e^{a_5^{lepton}} & 0 & i\alpha^{lepton\,\prime}_{\hat{45}} \\ 0 & 0 & 0 & 0 & 0 & e^{a_6^{lepton}} & i\alpha^{lepton\,\prime}_{\hat{47}} \\ 0 & 0 & 0 & i\alpha^{lepton\,\prime}_{\hat{43}} & i\alpha^{lepton\,\prime}_{\hat{45}} & i\alpha^{lepton\,\prime}_{\hat{47}} & e^{a_7^{lepton}} \end{pmatrix} . \tag{5.5}$$

The factor $1/\sqrt{2}$ is absorbed in ρ^{quark}_{min} and ρ^{lepton}_{min}, respectively. We assume that $V(\Phi^{quark})$ and $V(\Phi^{lepton})$ is minimised at non-trivial Φ^{quark}_{min} and Φ^{lepton}_{min}, respectively, i.e. we assume

$$a_i^{quark} \neq 0 \quad , \quad \alpha^{quark\prime}_{\hat{a}} \neq 0 \quad , \quad a_i^{lepton} \neq 0 \quad , \quad \alpha^{lepton\prime}_{\hat{a}} \neq 0 \tag{5.6}$$

for $i = 1, \ldots, 7$ and $\hat{a} = \hat{43}, \hat{45}, \hat{47}$ in (5.4) and (5.5). Note that in general $a_i^{quark} \neq a_i^{lepton} \neq a_i^{gauge} = a_i$ and $\alpha^{quark\prime}_{\hat{a}} \neq \alpha^{lepton\prime}_{\hat{a}} \neq \alpha^{gauge\prime}_{\hat{a}} = \alpha'_{\hat{a}}$. *However, in contrast to Φ^{gauge}_{min}, we do not assume that Φ^{quark}_{min} and Φ^{lepton}_{min} are quasi $\mathcal{S}_2$ symmetric.* The minimum Φ^{quark}_{min} and Φ^{lepton}_{min} of the Higgs potential $V(\Phi^{quark})$ and $V(\Phi^{lepton})$, respectively, will fix the quark and lepton masses. In addition, Φ^{quark}_{min} also fixes the CKM mixing matrix. In the next section we will review the CKM matrix in the SM and in particular some familiar parametrisations of the CKM matrix. This review will also clarify the notation and conventions we use in this chapter.

5.1 Quark masses and the CKM mixing matrix in the SM

In the SM the Yukawa interactions for quarks is given by [40]

$$\mathcal{L}_Y = -Y^d_{ij}\bar{Q}_{Li}\phi\, d_{Rj} - Y^u_{ij}\bar{Q}_{Li}\epsilon\phi^* u_{Rj} + h.c.\,, \tag{5.7}$$

where $Y^{u,d}$ are complex 3×3 matrices, ϕ is the SM Higgs field, i and j are generation labels, ϵ is the 2×2 antisymmetric tensor, Q_L are the left-handed quark doublets and d_R, u_R are the right-handed down- and up-type quark singlets, respectively. When ϕ acquires a VEV $\langle\phi\rangle = \binom{0}{v/\sqrt{2}}$ the Yukawa interactions yields Dirac mass terms for the quarks

$$M^u = \frac{vY^u}{\sqrt{2}} \quad , \quad M^d = \frac{vY^d}{\sqrt{2}}\,, \tag{5.8}$$

where M^u is the 3×3 up-type quark mass matrix and M^d is the 3×3 down-type quark mass matrix. Since M^u and M^d are given in the basis of flavour eigenstates we must diagonalise M^u and M^d by biunitary transformations

$$U^u_L M^u U^{u\dagger}_R = M^u_{diag} \tag{5.9}$$

$$U^d_L M^d U^{d\dagger}_R = M^d_{diag} \tag{5.10}$$

in order to move to the basis of mass eigenstates. The CKM matrix V_{CKM} is then given by

$$V_{CKM} = U^u_L U^{d\dagger}_L = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{5.11}$$

and transforms electroweak eigenstates (d', s', b') into mass eigenstates (d, s, b). Since $U^u_L, U^{d\dagger}_L$ are unitary 3×3 matrices V_{CKM} is a unitary 3×3 matrix too. This feature ensures the absence of tree-level FCNC in the SM.

5.1.1 Standard parametrisation and unitarity of the CKM matrix

Any unitary 3×3 matrix can be parametrised by three angles and six phases. Using the freedom to redefine the up- and down-type quarks fields one can remove five unphysical phases. The CKM matrix can therefore be written as the product of three Euler matrices

$$V_{CKM} = R_{23}U_{13}R_{12} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}. \tag{5.12}$$

where $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$ and δ_{13} is the CP violating phase. This parametrisation is known as the standard parametrisation [12]. The Euler matrices are given by

$$R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \quad , \quad U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \quad , \tag{5.13}$$

$$R_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where U_{13} is a complex Euler matrix and both R_{23} and R_{12} are real Euler matrices. The advantage of this parametrisation is that the mixing between two generations i, j vanishes if the corresponding mixing angle θ_{ij} it set to zero. Note that the standard parametrisation satisfies exactly the unitarity relations.

In the $SU(7)$ model we have tree-level FCNC mediated by the $SO(3)_F$ flavour gauge bosons. This tree-level FCNC violate the unitarity of the CKM matrix. However this violation is extremely small due to the large flavour gauge bosons masses. The current experimental constraints on the unitarity of the CKM matrix are [40]

$$\mid V_{ud} \mid^2 + \mid V_{us} \mid^2 + \mid V_{ub} \mid^2 -1 = -0.0008 \pm 0.0011 \quad \text{first row} \tag{5.14}$$

$$\mid V_{cd} \mid^2 + \mid V_{cs} \mid^2 + \mid V_{cb} \mid^2 -1 = -0.03 \pm 0.18 \quad \text{second row} \tag{5.15}$$

$$\mid V_{ud} \mid^2 + \mid V_{cd} \mid^2 + \mid V_{td} \mid^2 -1 = -0.001 \pm 0.005 \quad \text{first column} \tag{5.16}$$

The sum in the second column $\mid V_{us} \mid^2 + \mid V_{cs} \mid^2 + \mid V_{ts} \mid^2$ is practically identical to that in the second row, as the errors in both cases are dominated by $\mid V_{cs} \mid$. These experimental constraints show that the CKM matrix is almost unitary. For simplicity we treat V_{CKM} as a unitary 3×3 matrix in the following calculations.

5.1.2 Wolfenstein parametrisation of the CKM matrix

For later calculations we introduce another familiar parametrisation of the CKM matrix known as Wolfenstein parametrisation [77]. Following the observation that the mixing angles s_{12}, s_{13}, s_{23} fulfil the hierarchy $s_{13} \ll s_{23} \ll s_{12} \ll 1$ Wolfenstein proposed an expansion of the CKM matrix in terms of the four parameters λ, A, ρ and η defined by

$$s_{12} := \lambda \ , \quad s_{23} := A\lambda^2 \ , \quad s_{13} := A\lambda^3 \left(\rho - i\eta\right) \tag{5.17}$$

If we insert these definitions into (5.12) we obtain a parametrisation of the CKM matrix as a function of λ, A, ρ and η. If we expand all elements of the CKM matrix in powers of the small parameter λ we get

$$V_{CKM} = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}\left(\lambda^4\right) \tag{5.18}$$

Neglecting the terms of $\mathcal{O}\left(\lambda^4\right)$, this parametrisation is known as Wolfenstein parametrisation. In the following calculations we use the numerical values [40]

$$\begin{aligned} \lambda = s_{12} = 0.22 \ , \quad A\lambda^2 = s_{23} = 0.042 \\ A\lambda^3(\rho - i\eta) = s_{13}\, e^{-i\delta_{13}} \ , \quad s_{13} = 0.0039 \ , \quad \delta_{13} = \frac{2\pi}{3} = 60^\circ \end{aligned} \tag{5.19}$$

5.2 CKM mixing matrix and quark masses from Φ^{quark}_{min}

As already mentioned in the introduction the quark masses and the CKM matrix are determined by

$$\bar{q}_L \Phi^{quark}_{min} q_R \ , \tag{5.20}$$

where

$$\Phi^{quark}_{min} = \rho^{quark}_{min} \begin{pmatrix} e^{a^{quark}_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{a^{quark}_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{a^{quark}_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{a^{quark}_4} & 0 & 0 & i\alpha^{quark\,\prime}_{\hat{4}3} \\ 0 & 0 & 0 & 0 & e^{a^{quark}_5} & 0 & i\alpha^{quark\,\prime}_{\hat{4}5} \\ 0 & 0 & 0 & 0 & 0 & e^{a^{quark}_6} & i\alpha^{quark\,\prime}_{\hat{4}7} \\ 0 & 0 & 0 & i\alpha^{quark\,\prime}_{\hat{4}3} & i\alpha^{quark\,\prime}_{\hat{4}5} & i\alpha^{quark\,\prime}_{\hat{4}7} & e^{a^{quark}_7} \end{pmatrix} , \tag{5.21}$$

see (5.4). Φ^{quark}_{min} fixes not only the quark masses but also the CKM mixing matrix. This we will now be explained.

First, comparing with (5.8), the upper 3×3 submatrix of Φ^{quark}_{min} gives the up-type quark mass matrix

$$M^u = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} \tag{5.22}$$

where

$$m_u = \rho^{quark}_{min} e^{a^{quark}_1} \ , \quad m_c = \rho^{quark}_{min} e^{a^{quark}_2} \ , \quad m_t = \rho^{quark}_{min} e^{a^{quark}_3} \ . \tag{5.23}$$

Conclusion: *The up-type quark masses m_u, m_c, m_t are given by the diagonal part of Φ^{quark}_{min} only.*

This result is in contrast to the SM where both up- and down-type quark masses are given by the (same) SM Higgs doublet (5.7).

Second, the lower 4×4 submatrix of Φ^{quark}_{min}

$$M = \begin{pmatrix} \tilde{m}_d & 0 & 0 & i\,k_{\hat{43}}\,\langle \mathcal{A}_y^{\hat{43}(0)} \rangle^{quark} \\ 0 & \tilde{m}_s & 0 & i\,k_{\hat{45}}\,\langle \mathcal{A}_y^{\hat{45}(0)} \rangle^{quark} \\ 0 & 0 & \tilde{m}_b & i\,k_{\hat{47}}\,\langle \mathcal{A}_y^{\hat{47}(0)} \rangle^{quark} \\ i\,k_{\hat{43}}\,\langle \mathcal{A}_y^{\hat{43}(0)} \rangle^{quark} & i\,k_{\hat{45}}\,\langle \mathcal{A}_y^{\hat{45}(0)} \rangle^{quark} & i\,k_{\hat{47}}\,\langle \mathcal{A}_y^{\hat{47}(0)} \rangle^{quark} & m_x \end{pmatrix} \tag{5.24}$$

where

$$\tilde{m}_d = \rho^{quark}_{min} e^{a_4^{quark}} \quad , \quad \tilde{m}_s = \rho^{quark}_{min} e^{a_5^{quark}} \quad , \quad \tilde{m}_b = \rho^{quark}_{min} e^{a_6^{quark}} \quad , \quad m_x = \rho^{quark}_{min} e^{a_7^{quark}} \tag{5.25}$$

and

$$i\,k_{\hat{a}}\,\langle \mathcal{A}_y^{\hat{a}(0)} \rangle^{quark} = i\,\rho^{quark}_{min} \alpha^{quark\,\prime}_{\hat{a}} \tag{5.26}$$

for $\hat{a} = \hat{43}, \hat{45}, \hat{47}$ will lead to the down-type quark mass matrix M^d. In (5.24) we have introduced the VEVs

$$\langle \mathcal{A}_y^{\hat{43}(0)} \rangle^{quark} = \frac{\alpha^{quark}_{\hat{43}}}{g_4 R} \quad , \quad \langle \mathcal{A}_y^{\hat{45}(0)} \rangle^{quark} = \frac{\alpha^{quark}_{\hat{45}}}{g_4 R} \quad , \quad \langle \mathcal{A}_y^{\hat{47}(0)} \rangle^{quark} = \frac{\alpha^{quark}_{\hat{47}}}{g_4 R} \ . \tag{5.27}$$

in analogy with (4.99). The $k_{\hat{a}}$ in (5.24) are given by [1]

$$k_{\hat{43}} = \frac{\rho^{quark}_{min}}{\sqrt{2}\,\rho_{min}} \left(e^{a_4} + e^{a_7}\right) \ , \ k_{\hat{45}} = \frac{\rho^{quark}_{min}}{\sqrt{2}\,\rho_{min}} \left(e^{a_5} + e^{a_7}\right) \ , \ k_{\hat{47}} = \frac{\rho^{quark}_{min}}{\sqrt{2}\,\rho_{min}} \left(e^{a_6} + e^{a_7}\right) \tag{5.28}$$

In order to obtain M^d we have to bring M on block diagonal form by a unitary transformation

$$M \to U\ M\ U^\dagger = \begin{pmatrix} M^d & 0 \\ 0 & \tilde{m}_x \end{pmatrix} , \tag{5.29}$$

where M^d is the desired 3×3 down-type quark mass matrix. In a second step we diagonalise M^d by a second unitary transformation

$$U^d\ M^d\ U^{d\dagger} = M^d_{diag} \ . \tag{5.30}$$

Since M^u is already diagonal the corresponding transformation for the up-type mass matrix M^u is trivial

$$U^u\ M^u\ U^{u\dagger} = M^u_{diag} = M^u \ . \quad \longrightarrow \quad U^u = 1 \tag{5.31}$$

Thus the CKM matrix is given by

$$V_{CKM} = U^u U^{d\dagger} = U^{d\dagger} \ . \tag{5.32}$$

[1] This follows with $g\rho_{min} = 1/R$, $g = g_5/\sqrt{\pi R}$, $g_4 = g_5/\sqrt{2\pi R}$, $\alpha^{quark\prime}_{\hat{43}} = \alpha^{quark}_{\hat{43}} \left(e^{a_4} + e^{a_7}\right)$ $\alpha^{quark\prime}_{\hat{45}} = \alpha^{quark}_{\hat{45}} \left(e^{a_5} + e^{a_7}\right)$ and $\alpha^{quark\prime}_{\hat{47}} = \alpha^{quark}_{\hat{47}} \left(e^{a_6} + e^{a_7}\right)$

In the following calculations we use the numerical values [20]

$$\begin{aligned} m_u &= 1.5 \text{ to } 4.5 \text{ MeV} \quad , \quad m_d = 5 \text{ to } 8.5 \text{ MeV} \\ m_c &= 1.0 \text{ to } 1.4 \text{ GeV} \quad , \quad m_s = 80 \text{ to } 155 \text{ MeV} \\ m_t &= 174.3 \pm 5.1 \text{ GeV} \quad , \quad m_b = 4.0 \text{ to } 4.5 \text{ GeV} \end{aligned} \tag{5.33}$$

In particular for later calculations we make the (arbitrary) choice

$$m_d = 7.4 \text{ MeV} \quad , \quad m_s = 114.1 \text{ MeV} \quad , \quad m_b = 4250 \text{ MeV} . \tag{5.34}$$

5.3 CKM mixing matrix from M_d

According to (5.12) V_{CKM} can expressed as

$$V_{CKM} = R_{23} U_{13} R_{12} \tag{5.35}$$

where R_{23}, U_{13} and R_{12} are given by (5.13). We introduce the phase matrix

$$P_{13} = \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix} \tag{5.36}$$

where the phases ϕ_1 and ϕ_3 have to fulfil the constraint

$$\delta_{13} = \phi_1 - \phi_3 . \tag{5.37}$$

Recall that δ_{13} is the CP violating phase and occurs in

$$U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} . \tag{5.38}$$

Without loss of generality let ϕ_1 in (5.36) be arbitrary. The phase ϕ_3 in (5.36) is then fixed by the constraint (5.37). With the help of (5.36) we can rewrite

$$U_{13} = P_{13}^* R_{13} P_{13} \tag{5.39}$$

where R_{13} is the corresponding real Euler matrix given by

$$R_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} . \tag{5.40}$$

The CKM matrix V_{CKM} can then be written as the product

$$V_{CKM} = R_{23} P_{13}^* R_{13} P_{13} R_{12} . \tag{5.41}$$

Lemma 1 *Let M^d be given by*

$$M^d = \begin{pmatrix} \tilde{m}'_d & m_{12} & m_{13} \\ m^*_{12} & \tilde{m}'_s & m_{23} \\ m^*_{13} & m_{23} & m_b \end{pmatrix} \tag{5.42}$$

where

$$\begin{aligned} \tilde{m}'_d &= 13.4\ MeV \quad m_{12} = \hat{m}_{12} + \hat{m}_{13} s_{23} e^{i\,\delta_{13}} \approx 24.46\ MeV \\ \tilde{m}'_s &= 119.2\ MeV \quad m_{13} = \hat{m}_{12} s_{23} - \hat{m}_{13} e^{i\,\delta_{13}} \approx 16.65\ e^{i\,\frac{2\pi}{3}}\, MeV \\ m_b &= 4250\ MeV \quad m_{23} = 173.8\ MeV\,, \end{aligned} \tag{5.43}$$

and

$$\hat{m}_{12} = 24.8\ MeV \quad , \quad \hat{m}_{13} = 16.1\ MeV \quad , \quad \delta_{13} = \frac{2\pi}{3}\ . \tag{5.44}$$

The unitary transformation

$$V^\dagger_{CKM}\ M^d\ V_{CKM} = M^d_{diag}\ . \tag{5.45}$$

leads to the CKM matrix

$$V_{CKM} = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} \tag{5.46}$$

where $\lambda = s_{12} = 0.22$, $A\lambda^2 = s_{23} = 0.042$, $A\lambda^3(\rho - i\eta) = s_{13}\ e^{-i\delta_{13}}$, $s_{13} = 0.0039$ and $\delta_{13} = \frac{2\pi}{3}$.

The proof can be found in Appendix A.

5.4 Obtaining M^d from M

In this section we finally describe how the down-type quark mass matrix M^d is obtained from (5.24)

$$M = \begin{pmatrix} \tilde{m}_d & 0 & 0 & i\,k_{\hat{4}\hat{3}}\,\langle \mathcal{A}_y^{\hat{4}\hat{3}(0)}\rangle^{quark} \\ 0 & \tilde{m}_s & 0 & i\,k_{\hat{4}\hat{5}}\,\langle \mathcal{A}_y^{\hat{4}\hat{5}(0)}\rangle^{quark} \\ 0 & 0 & \tilde{m}_b & i\,k_{\hat{4}\hat{7}}\,\langle \mathcal{A}_y^{\hat{4}\hat{7}(0)}\rangle^{quark} \\ i\,k_{\hat{4}\hat{3}}\,\langle \mathcal{A}_y^{\hat{4}\hat{3}(0)}\rangle^{quark} & i\,k_{\hat{4}\hat{5}}\,\langle \mathcal{A}_y^{\hat{4}\hat{5}(0)}\rangle^{quark} & i\,k_{\hat{4}\hat{7}}\,\langle \mathcal{A}_y^{\hat{4}\hat{7}(0)}\rangle^{quark} & m_x \end{pmatrix}$$

It is convenient to introduce the following abbreviation

$$\langle \mathcal{A}_y^{\hat{a}} \rangle^q_k := k_{\hat{a}}\ \langle \mathcal{A}_y^{\hat{a}(0)}\rangle^{quark} \tag{5.47}$$

for $\hat{a} = \hat{43}, \hat{45}, \hat{47}$. With this abbreviation M reads

$$M = \begin{pmatrix} \tilde{m}_d & 0 & 0 & \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \, e^{i\,\frac{\pi}{2}} \\ 0 & \tilde{m}_s & 0 & \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \, e^{i\,\frac{\pi}{2}} \\ 0 & 0 & \tilde{m}_b & \langle \mathcal{A}_y^{\hat{47}} \rangle_k^q \, e^{i\,\frac{\pi}{2}} \\ \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \, e^{i\,\frac{\pi}{2}} & \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \, e^{i\,\frac{\pi}{2}} & \langle \mathcal{A}_y^{\hat{47}} \rangle_k^q \, e^{i\,\frac{\pi}{2}} & m_x \end{pmatrix} \tag{5.48}$$

where we have written all phases explicitly. As already explained in section 5.2 in order to obtain the 3×3 down-type quark mass matrix M^d we must transform M to block diagonal form

$$M \to U\; M\; U^\dagger = \begin{pmatrix} M^d & 0 \\ 0 & \tilde{m}_x \end{pmatrix} . \tag{5.49}$$

We write U as a product of a phase matrix and three Euler matrices

$$U^\dagger = P_{34} R_{34} R_{24} R_{14} \tag{5.50}$$

where

$$P_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\,\phi_3} & 0 \\ 0 & 0 & 0 & e^{i\,\phi_4} \end{pmatrix} , \quad R_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_{34} & s_{34} \\ 0 & 0 & -s_{34} & c_{34} \end{pmatrix} \tag{5.51}$$

$$R_{24} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{24} & 0 & s_{24} \\ 0 & 0 & 1 & 0 \\ 0 & -s_{24} & 0 & c_{24} \end{pmatrix} , \quad R_{14} = \begin{pmatrix} c_{14} & 0 & 0 & s_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_{14} & 0 & 0 & c_{14} \end{pmatrix}$$

Inserting this expansion in (5.49) we get

$$R_{14}^t R_{24}^t R_{34}^t P_{34}^* \; M \; P_{34} R_{34} R_{24} R_{14} = \begin{pmatrix} M^d & 0 \\ 0 & \tilde{m}_x \end{pmatrix} . \tag{5.52}$$

The matrix (5.48) has seven undetermined parameters: The diagonal elements $\tilde{m}_d, \tilde{m}_s, \tilde{m}_b, m_x$ and the off-diagonal elements $\langle \mathcal{A}_y^{\hat{43}} \rangle_k^q$, $\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q$, $\langle \mathcal{A}_y^{\hat{47}} \rangle_k^q$. The question is now:

- How do these parameters determine the CKM mixing angles s_{12}, s_{23}, s_{13} and the CP violating phase δ_{13} ?
- How do these parameters determine the down-type quark masses?
- What is the interpretation of the three off-diagonal elements $\langle \mathcal{A}_y^{\hat{43}} \rangle_k^q$, $\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q$, $\langle \mathcal{A}_y^{\hat{47}} \rangle_k^q$?

- Can we get the SM case?

The situation is complicated because we do not know anything about the explicit numerical values of these seven parameters.

In the next subsection we will consider the special but analytically exact solvable case, where we assume that

1. m_x is the dominant element in M, i.e. $m_x \gg \tilde{m}_d, \tilde{m}_s, \tilde{m}_b$ and $m_x \gg \langle \mathcal{A}_y^{\hat{a}} \rangle_k^q$ for $\hat{a} = \hat{43}, \hat{45}, \hat{47}$. We choose in the following calculations the (arbitrary) value $m_x = 10000\ m_b$. This assumption follows from the fact that m_x *may* be interpreted as the mass of a seventh quark. However we remind the reader that in the $SU(7)$ model m_x is *only* a parameter. It will turn out that the results of the following calculations are independent of the explicit value for m_x as long as m_x is the dominant element is M.

2. the d-quark masses are approximately given by the diagonal part of (5.48)

$$m_d \approx \tilde{m}_d \ , \quad m_s \approx \tilde{m}_s \ , \quad m_b \approx \tilde{m}_b \tag{5.53}$$

It will turn out during the following calculations that this assumption is nearly fulfilled for m_b and poorly fulfilled for m_s and m_d.

Since m_x is the dominant element in M it will turn out that all mixing angles in (5.51) are very small. Therefore we call this case *small mixing angle approximation.*

5.4.1 M^d from M in small mixing angle approximation

In (5.48) we choose the following numerical values

$$\begin{aligned} \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q &= 63910 \text{ MeV} \\ \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q &= 104240 \text{ MeV} \\ \langle \mathcal{A}_y^{\hat{47}} \rangle_k^q &= 70950 \text{ MeV} \end{aligned} \tag{5.54}$$

The reason for this choice will be become clear later. In addition, as already mentioned above we choose

$$m_x = 10000\ m_b = 4.25 \times 10^7 \text{ MeV} \tag{5.55}$$

The diagonal elements $\tilde{m}_d$, $\tilde{m}_s$ and $\tilde{m}_b$ will be fixed during the following computation. According to (5.52) the calculation is divided into four steps.

First step: We multiply M by the phase matrix

$$P_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\,\phi_3} & 0 \\ 0 & 0 & 0 & e^{i\,\phi_4} \end{pmatrix} \tag{5.56}$$

The purpose of this rephasing is to get a real angle θ_{34} in the next step. The complete transformation reads

$$\begin{aligned} M_1 &= P_{34}^{L*} M P_{34}^R \\ &= \begin{pmatrix} \tilde{m}_d & 0 & 0 & \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \, e^{i\,(\frac{\pi}{2}+\phi_4^R)} \\ 0 & \tilde{m}_s & 0 & \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \, e^{i\,(\frac{\pi}{2}+\phi_4^R)} \\ 0 & 0 & \tilde{m}_b \, e^{i\,(\phi_3^R-\phi_3^L)} & \langle \mathcal{A}_y^{\hat{47}} \rangle_k^q \, e^{i\,(\frac{\pi}{2}+\phi_4^R-\phi_3^L)} \\ \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \, e^{i\,(\frac{\pi}{2}-\phi_4^L)} & \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \, e^{i\,(\frac{\pi}{2}-\phi_4^L)} & \langle \mathcal{A}_y^{\hat{47}} \rangle_k^q \, e^{i\,(\frac{\pi}{2}+\phi_3^R-\phi_4^L)} & m_x \, e^{i\,(\phi_4^R-\phi_4^L)} \end{pmatrix} \end{aligned} \tag{5.57}$$

Concerning this transformation we give a comment. In (5.49) we have argued that M can be brought to block diagonal form via a unitary transformation. However within the approximations we will make during the following calculations we have to consider a multiplication of M from right with $P_{34}^R = diag(1,1,e^{i\,\phi_3^R},e^{i\,\phi_4^R})$ and from left with $P_{34}^{L*} = diag(1,1,e^{-i\,\phi_3^L},e^{-i\,\phi_4^L})$ and as a start have to make a distinction between ϕ_3^L and ϕ_3^R and between ϕ_4^L and ϕ_4^R, respectively. It will turns out during the following calculation that this distinction is necessary in order to get real mixing angles θ_{34} and θ_{24}. In addition we will compute explicit values for all phases. The resulting transformation (5.49) will turn out to be in fact biunitary within the approximations we will make during the following calculations. We introduce the following abbreviations

$$\begin{aligned} \delta_3 = \phi_3^R - \phi_3^L \quad &, \quad \delta_4 = \phi_4^R - \phi_4^L \\ \delta = \phi_4^R - \phi_3^L + \frac{\pi}{2} \quad &, \quad \tilde{\delta} = \phi_3^R - \phi_4^L + \frac{\pi}{2} \end{aligned} \tag{5.58}$$

Second step: We perform the rotation R_{34} on M_1. The purpose is to put zeroes in the $34, 43$ elements. The zeroes in the $34, 43$ elements are implemented by diagonalising the lower 34 block of M_1. This block is obtained by striking out the row and the column in which the unit elements of R_{34} appear. Thus we get the reduced rotation

$$R_{34}^t \begin{pmatrix} \tilde{m}_b \, e^{i\,\delta_3} & \langle \mathcal{A}_y^{\hat{47}} \rangle_k^q \, e^{i\,\delta} \\ \langle \mathcal{A}_y^{\hat{47}} \rangle_k^q \, e^{i\,\tilde{\delta}} & m_x \, e^{i\,\delta_4} \end{pmatrix} R_{34} := \begin{pmatrix} m_b \, e^{i\,\delta_b} & 0 \\ 0 & \tilde{m}_x \, e^{i\,\tilde{\delta}_x} \end{pmatrix} \tag{5.59}$$

where we have introduced two new phases δ_b and δ_x. From this matrix equation we obtain the mixing angle θ_{34}

$$\tan 2\theta_{34} = \frac{2\left(m_x \, e^{i\,\delta_4} \, \langle \mathcal{A}_y^{\hat{47}} \rangle_k^q \, e^{i\,\delta} + \tilde{m}_b \, e^{i\,\delta_3} \, \langle \mathcal{A}_y^{\hat{47}} \rangle_k^q \, e^{i\,\tilde{\delta}}\right)}{m_x^2 \, e^{i\,2\delta_4} - \tilde{m}_b^2 \, e^{i\,2\delta_3} + \left(\langle \mathcal{A}_y^{\hat{47}} \rangle_k^q\right)^2 \left(e^{i\,2\delta} - e^{i\,2\tilde{\delta}}\right)} \tag{5.60}$$

In order to simplify this equation let us fix

$$\delta = \tilde{\delta} \;\longrightarrow\; \phi_4^R - \phi_3^L = \phi_3^R - \phi_4^L \; . \tag{5.61}$$

Using this fixation θ_{34} can be written as

$$\tan 2\theta_{34} = \frac{2\,\langle \mathcal{A}_y^{\hat{47}} \rangle_k^q\; e^{i\,\delta}}{m_x\; e^{i\,\delta_4} - \tilde{m}_b\; e^{i\,\delta_3}}\ . \tag{5.62}$$

The requirement that the angle θ_{34} is real means that the numerator and the denominator must have equal phases. This leads to the condition

$$m_x \sin(\delta - \delta_4) = \tilde{m}_b \sin(\delta - \delta_3)\ . \tag{5.63}$$

Now we take into account that $m_x = 10000\ m_b \gg \tilde{m}_b$. This leads to the phase condition $\delta \approx \delta_4$ and we obtain

$$\delta = \phi_4^R - \phi_3^L + \frac{\pi}{2} \approx \phi_4^R - \phi_4^L \quad \longrightarrow \quad \phi_3^L \approx \phi_4^L + \frac{\pi}{2}\ . \tag{5.64}$$

Thus the mixing angle turns out to be

$$\theta_{34} \approx \frac{\langle \mathcal{A}_y^{\hat{47}} \rangle_k^q}{m_x} = 0.0017 \ll 1 \tag{5.65}$$

where we have inserted (5.54) and (5.55). For the diagonal elements one gets in small mixing angle approximation

$$m_b\; e^{i\,\delta_b} \approx \tilde{m}_b\; e^{i\,\delta_3} - \frac{\left(\langle \mathcal{A}_y^{\hat{47}} \rangle_k^q\right)^2}{m_x}\; e^{i\,\delta_4} \tag{5.66}$$

$$\tilde{m}_x\; e^{i\,\tilde{\delta}_x} \approx m_x\; e^{i\,\delta_4} + \frac{2\left(\langle \mathcal{A}_y^{\hat{47}} \rangle_k^q\right)^2}{m_x}\; e^{i\,\delta_4}\ . \tag{5.67}$$

We calculate from (5.54) and (5.55)

$$\frac{\left(\langle \mathcal{A}_y^{\hat{47}} \rangle_k^q\right)^2}{m_x} \approx 118\ . \tag{5.68}$$

Thus $m_x = 10000\ m_b$ is practically unchanged

$$\tilde{m}_x = m_x \tag{5.69}$$

and hence $\tilde{\delta}_x$ is given by

$$\tilde{\delta}_x = \delta_4\ . \tag{5.70}$$

Let us analyse equation (5.66). Since $\frac{\left(\langle \mathcal{A}_y^{\hat{47}} \rangle_k^q\right)^2}{m_x} \approx 118$ the second term can only give a small contribution to m_b. Therefore we approximately get

$$\delta_b \approx \delta_3\ . \tag{5.71}$$

The remaining elements are given by the reduced rotation in small mixing angle approximation

$$\begin{pmatrix} \tilde{m}_{13} & \tilde{m}_{14} \\ \tilde{m}_{23} & \tilde{m}_{24} \end{pmatrix} = \begin{pmatrix} 0 & \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \ e^{i\,(\frac{\pi}{2}+\phi_4^R)} \\ 0 & \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \ e^{i\,(\frac{\pi}{2}+\phi_4^R)} \end{pmatrix} \begin{pmatrix} 1 & \theta_{34} \\ -\theta_{34} & 1 \end{pmatrix} . \tag{5.72}$$

This leads to

$$\tilde{m}_{13} = \theta_{34} \ \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \ e^{i\,(\frac{3\pi}{2}+\phi_4^R)} \quad , \quad \tilde{m}_{14} = \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \ e^{i\,(\frac{\pi}{2}+\phi_4^R)} \tag{5.73}$$

$$\tilde{m}_{23} = \theta_{34} \ \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \ e^{i\,(\frac{3\pi}{2}+\phi_4^R)} \quad , \quad \tilde{m}_{24} = \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \ e^{i\,(\frac{\pi}{2}+\phi_4^R)} . \tag{5.74}$$

In addition

$$\begin{pmatrix} \tilde{m}_{31} & \tilde{m}_{32} \\ \tilde{m}_{41} & \tilde{m}_{42} \end{pmatrix} = \begin{pmatrix} 1 & -\theta_{34} \\ \theta_{34} & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \ e^{i\,(\frac{\pi}{2}-\phi_4^L)} & \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \ e^{i\,(\frac{\pi}{2}-\phi_4^L)} \end{pmatrix} \tag{5.75}$$

leads to

$$\tilde{m}_{31} = \theta_{34} \ \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \ e^{i\,(\frac{3\pi}{2}-\phi_4^L)} \quad , \quad \tilde{m}_{32} = \theta_{34} \ \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \ e^{i\,(\frac{3\pi}{2}-\phi_4^L)} \tag{5.76}$$

$$\tilde{m}_{41} = \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \ e^{i\,(\frac{\pi}{2}-\phi_4^L)} \quad , \quad \tilde{m}_{42} = \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \ e^{i\,(\frac{\pi}{2}-\phi_4^L)} . \tag{5.77}$$

The full transformation reads

$$\begin{aligned} M_2 &= R_{34}^t M_1 R_{34} \qquad (5.78) \\ &= \begin{pmatrix} \tilde{m}_d & 0 & |\,\tilde{m}_{13}\,| \ e^{i\,(\frac{3\pi}{2}+\phi_4^R)} & \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \ e^{i\,(\frac{\pi}{2}+\phi_4^R)} \\ 0 & \tilde{m}_s & |\,\tilde{m}_{23}\,| \ e^{i\,(\frac{3\pi}{2}+\phi_4^R)} & \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \ e^{i\,(\frac{\pi}{2}+\phi_4^R)} \\ |\,\tilde{m}_{13}\,| \ e^{i\,(\frac{3\pi}{2}-\phi_4^L)} & |\,\tilde{m}_{23}\,| \ e^{i\,(\frac{3\pi}{2}+\phi_4^R)} & m_b \ e^{i\,\delta_3} & 0 \\ \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \ e^{i\,(\frac{\pi}{2}-\phi_4^L)} & \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \ e^{i\,(\frac{\pi}{2}-\phi_4^L)} & 0 & m_x \ e^{i\,\delta_4} \end{pmatrix} \end{aligned}$$

Third step : We perform the real rotation R_{24} on M_2. The purpose is to put zeroes in the $24, 42$ elements. The zeroes in the $24, 42$ elements are implemented by diagonalising the lower middle block of M_2. This block is obtained by striking out the row and the column in which the unit elements of R_{24} appear. Thus we get the reduced rotation

$$R_{24}^t \begin{pmatrix} \tilde{m}_s & \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \ e^{i\,(\frac{\pi}{2}+\phi_4^R)} \\ \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \ e^{i\,(\frac{\pi}{2}-\phi_4^L)} & m_x \ e^{i\,\delta_4} \end{pmatrix} R_{24} := \begin{pmatrix} m_s \ e^{i\,\delta_s} & 0 \\ 0 & \tilde{m}_x \ e^{i\,\tilde{\delta}_x'} \end{pmatrix} . \tag{5.79}$$

where we have introduced two new phases δ_s and $\tilde{\delta}_x'$. From this matrix equation we obtain the mixing angle θ_{24}

$$\tan 2\theta_{24} = \frac{2\left(m_x \ e^{i\,\delta_4} \ \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \ e^{i\,(\frac{\pi}{2}+\phi_4^R)} + \tilde{m}_s \ \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \ e^{i\,(\frac{\pi}{2}-\phi_4^L)} \right)}{m_x^2 \ e^{i\,2\delta_4} - \tilde{m}_s^2 + \left(\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \right)^2 \ \left(e^{i\,2(\frac{\pi}{2}+\phi_4^R)} - e^{i\,2(\frac{\pi}{2}-\phi_4^L)} \right)} \tag{5.80}$$

Let us fix

$$\phi_4^R = -\phi_4^L \; . \tag{5.81}$$

Using this fixation θ_{24} can be written as

$$\tan 2\theta_{24} = \frac{2\,\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \; e^{i\,\delta'}}{m_x\, e^{i\,\delta_4} - \tilde{m}_s} \tag{5.82}$$

where $\delta' := \frac{\pi}{2} + \phi_4^R$. The requirement that θ_{24} is real means that the numerator and the denominator of this equation must have equal phases. This leads to the condition

$$m_x \sin(\delta' - \delta_4) = \tilde{m}_s \sin(\delta') \; . \tag{5.83}$$

Now we take into account that $m_x = 10000 m_b \gg \tilde{m}_s$. This leads to the phase condition $\delta' \approx \delta_4$ and the mixing angle θ_{24} turns out to be

$$\theta_{24} \approx \frac{\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q}{m_x} = 0.0025 \ll 1 \tag{5.84}$$

where we have inserted (5.54) and (5.55). The phase condition $\delta' \approx \delta_4$ now fixes the absolute value of ϕ_4^R and ϕ_4^L

$$\delta' = \frac{\pi}{2} + \phi_4^R \approx \delta_4 = \phi_4^R - \phi_4^L \stackrel{!}{=} 2\phi_4^R \quad \longrightarrow \quad \phi_4^R = \frac{\pi}{2} \tag{5.85}$$

where we have make use of (5.81) in the second step. Thus we obtain

$$\delta_4 = 2\phi_4^R = \pi \; , \tag{5.86}$$

and

$$\delta_3 = \phi_3^R - \phi_3^L = \phi_4^R + \phi_4^L - 2\phi_3^L \approx -2(\phi_4^L + \frac{\pi}{2}) = 0 \tag{5.87}$$

where we have used (5.61) in the first and (5.64) in the second step. We see that these values are in accordance with (5.63) and (5.83) for large m_x. The phases in P_{34}^R respectively P_{34}^L read

$$\phi_4^R = -\phi_4^L = \frac{\pi}{2} \quad , \quad \phi_3^R = -\phi_3^L = 0 \; . \tag{5.88}$$

With $\delta_3 = 0$ and $\delta_3 = \pi R$ equation (5.66) becomes

$$m_b\, e^{i\,\delta_b} \approx \tilde{m}_b\, e^{i\,\delta_3} - \frac{\left(\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \right)^2}{m_x}\, e^{i\,\delta_4} = \tilde{m}_b + \frac{\left(\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \right)^2}{m_x} \tag{5.89}$$

For $m_b = 4250$ MeV we obtain

$$\tilde{m}_b = 4132 \text{ MeV} \tag{5.90}$$

since $\frac{\left(\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q\right)^2}{m_x} = 105$. In addition we get

$$\delta_b = \delta_3 = 0 \ . \tag{5.91}$$

Next we determine the diagonal elements. Since $\theta_{24} \ll 1$ we get in the small mixing angle approximation

$$m_s e^{i\,\delta_s} \approx \tilde{m}_s - \frac{2\left(\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q\right)^2}{m_x}\, e^{i\,(\frac{\pi}{2}+\phi_4^R)} + \frac{\left(\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q\right)^2}{m_x}\, e^{i\,\delta_4} = \tilde{m}_s + \frac{\left(\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q\right)^2}{m_x} \tag{5.92}$$

and m_x is again practically unchanged. This leads to

$$\tilde{\delta}'_x = \delta_4 = \pi \ . \tag{5.93}$$

We calculate from (5.54) and (5.55)

$$\frac{\left(\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q\right)^2}{m_x} = 255 \text{ MeV} \ . \tag{5.94}$$

However, since $\tilde{m}_s$ cannot be negative we conclude that

$$m_s = m_s^\star > 255 \text{ MeV} \tag{5.95}$$

and the phase δ_s turns out to be

$$\delta_s = 0 \ . \tag{5.96}$$

The off-diagonal elements are given by the reduced rotation

$$\begin{pmatrix} \tilde{m}'_{12} & \tilde{m}'_{14} \\ \tilde{m}'_{32} & \tilde{m}'_{34} \end{pmatrix} = \begin{pmatrix} 0 & \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q e^{i\,\pi} \\ \theta_{34}\, \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q & 0 \end{pmatrix} \begin{pmatrix} 1 & \theta_{24} \\ -\theta_{24} & 1 \end{pmatrix} \ . \tag{5.97}$$

This leads to

$$\tilde{m}'_{12} = \theta_{24} \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \quad , \quad \tilde{m}'_{14} = \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q\, e^{i\,\pi} \tag{5.98}$$

$$\tilde{m}'_{32} = \theta_{34}\, \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \quad , \quad \tilde{m}'_{34} \approx 0 \ . \tag{5.99}$$

In addition

$$\begin{pmatrix} \tilde{m}'_{21} & \tilde{m}'_{23} \\ \tilde{m}'_{41} & \tilde{m}'_{43} \end{pmatrix} = \begin{pmatrix} 1 & -\theta_{24} \\ \theta_{24} & 1 \end{pmatrix} \begin{pmatrix} 0 & \theta_{34}\, \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \\ \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q\, e^{i\,\pi} & 0 \end{pmatrix} \tag{5.100}$$

leads to

$$\tilde{m}'_{21} = \theta_{24}\, \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \quad , \quad \tilde{m}'_{41} = \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q\, e^{i\,\pi} \tag{5.101}$$

$$\tilde{m}'_{23} = \theta_{34}\, \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \quad , \quad \tilde{m}'_{43} \approx 0 \ . \tag{5.102}$$

The whole transformation reads

$$M_3 = R_{24}^t M_2 R_{24} = \begin{pmatrix} \tilde{m}_d & \tilde{m}'_{12} & \tilde{m}'_{13} & \langle \mathcal{A}_y^{\hat{4}3} \rangle_k^q \, e^{i\,\pi} \\ \tilde{m}'_{12} & m_s^\star & \tilde{m}'_{23} & 0 \\ \tilde{m}'_{13} & \tilde{m}'_{23} & m_b & 0 \\ \langle \mathcal{A}_y^{\hat{4}3} \rangle_k^q \, e^{i\,\pi} & 0 & 0 & m_x \, e^{i\,\pi} \end{pmatrix} \tag{5.103}$$

where $\tilde{m}'_{13} = \theta_{34} \, \langle \mathcal{A}_y^{\hat{4}3} \rangle_k^q$. *At this state of the calculation it is remarkable that all off-diagonal elements $\tilde{m}_{12}$, $\tilde{m}_{13}$ and $\tilde{m}_{23}$ in the upper left 3×3 matrix are real and positive.*

Fourth step: We perform the real rotation R_{14} on M_3. The purpose is to put zeroes in the $14, 41$ elements The zeroes in the $14, 41$ elements are implemented by diagonalising the outer block of M_2. This block is obtained by striking out the row and the column in which the unit elements of R_{14} appear. Thus we get the reduced rotation

$$R_{14}^t \begin{pmatrix} \tilde{m}_u & \langle \mathcal{A}_y^{\hat{4}3} \rangle_k^q \, e^{i\,\pi} \\ \langle \mathcal{A}_y^{\hat{4}3} \rangle_k^q \, e^{i\,\pi} & m_x \, e^{i\,\pi} \end{pmatrix} R_{14} := \begin{pmatrix} m_d \, e^{i\,\delta_d} & 0 \\ 0 & m_x \, e^{i\,\pi} \end{pmatrix} \tag{5.104}$$

where we have introduced new phase δ_d. Again m_x will be practically unchanged and therefore we have already written $m_x \, e^{i\,\pi}$ in (5.104). From this matrix equation we obtain the mixing angle θ_{14}

$$\theta_{14} \approx \frac{\langle \mathcal{A}_y^{\hat{4}3} \rangle_k^q \, e^{i\,\pi}}{m_x \, e^{i\,\pi} - \tilde{m}_d} \approx \frac{\langle \mathcal{A}_y^{\hat{4}3} \rangle_k^q}{|\, m_x \,|} \tag{5.105}$$

Thus the mixing angle θ_{14} turns out to be

$$\theta_{14} \approx \frac{\langle \mathcal{A}_y^{\hat{4}3} \rangle_k^q}{m_x} = 0.0002 \ll 1 \tag{5.106}$$

where we have inserted (5.54) and (5.55). Next we determine the diagonal elements. Since $\theta_{14} \ll 1$ we obtain in the small mixing angle approximation

$$m_d \, e^{i\,\delta_d} \approx \tilde{m}_d - \frac{\left(\langle \mathcal{A}_y^{\hat{4}3} \rangle_k^q \right)^2}{m_x} \, e^{i\,\pi} = \tilde{m}_d + \frac{\left(\langle \mathcal{A}_y^{\hat{4}3} \rangle_k^q \right)^2}{m_x} \tag{5.107}$$

We calculate from (5.54) and (5.55)

$$\frac{\left(\langle \mathcal{A}_y^{\hat{4}3} \rangle_k^q \right)^2}{m_x} = 2.3 \text{ MeV} \tag{5.108}$$

This yields

$$\tilde{m}_d = 11.0 \text{ MeV} \quad , \quad \delta_d = 0 \; . \tag{5.109}$$

Finally, when calculating the off-diagonal elements, we observe that since $\theta_{14} \ll 1$ they are approximately unchanged. The whole transformation reads

$$M_4 = R_{14}^t M_3 R_{14} = \begin{pmatrix} m_d & \tilde{m}'_{12} & \tilde{m}'_{13} & 0 \\ \tilde{m}'_{12} & m_s^\star & \tilde{m}'_{23} & 0 \\ \tilde{m}'_{13} & \tilde{m}'_{23} & m_b & 0 \\ 0 & 0 & 0 & m_x\, e^{i\,\pi} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} M^d & 0 \\ 0 & \tilde{m}_x \end{pmatrix} \; . \tag{5.110}$$

Thus M^d reads

$$M^d = \begin{pmatrix} m_d & m_{12} & m_{13} \\ m_{12} & m_s^\star & m_{23} \\ m_{13} & m_{23} & m_b \end{pmatrix} \tag{5.111}$$

where

$$\begin{aligned} m_d = 13.4 \text{ MeV} \quad &, \quad m_{12} = \theta_{24} \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q = \frac{\langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q}{m_x} = 24.46 \text{ MeV} \\ m_s^\star > 255 \text{ MeV} \quad &, \quad m_{13} = \theta_{34} \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q = \frac{\langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \langle \mathcal{A}_y^{\hat{47}} \rangle_k^q}{m_x} = 16.65 \text{ MeV} \\ m_b = 4250 \text{ MeV} \quad &, \quad m_{23} = \theta_{34} \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q = \frac{\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \langle \mathcal{A}_y^{\hat{47}} \rangle_k^q}{m_x} = 173.8 \text{ MeV} \end{aligned} \tag{5.112}$$

Note that in the second step we have inserted

$$\theta_{24} = \frac{\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q}{m_x} \quad , \quad \theta_{34} = \frac{\langle \mathcal{A}_y^{\hat{47}} \rangle_k^q}{m_x} \; . \tag{5.113}$$

5.5 The meaning of the triplet VEV of $SO(3)_F$

We compare the result (5.112) with (5.43). We observe that

- The off-diagonal elements m_{12}, m_{13} and m_{23} are equal.
- The diagonal elements m_d and m_b are equal.
- The diagonal element $m_s^\star > 255$ MeV turns out to be at least two times bigger than $\tilde{m}'_s = 119.2$ MeV.
- There is no CP violation.

We stress that these results are only valid if m_x is the dominant element in M. We will discuss the case where m_x is lowered to $\mathcal{O}(\tilde{m}_s)$ at the end of this section. In this case it is possible to obtain the correct value for $\tilde{m}_s$ and a CP violation.

We calculate the CKM mixing matrix and the down-type quark masses from (5.112):

1. For the CKM mixing angles we obtain

$$s_{12} = 0.104 \quad , \quad s_{13} = 0.0039 \quad , \quad s_{23} = 0.043 \; . \tag{5.114}$$

   We compare these results to the SM case

$$s_{12}^{SM} = 0.22 \quad , \quad s_{13}^{SM} = 0.0039 \quad , \quad s_{13}^{SM} = 0.042 \; . \tag{5.115}$$

   We observe that s_{13} and s_{23} are in accordance with the SM case. However the Cabibbo angle s_{12} turns out to be small by approximately a factor of two. This is a consequence of the too large value for $m_s^\star$.

2. For the down-type quark masses we obtain

$$m_d = 10.7 \text{ MeV} \quad , \quad m_s = 251 \text{ MeV} \quad , \quad m_b = 4250 \text{ MeV} \; . \tag{5.116}$$

   We compare these results to the SM case

$$m_d = 5 \text{ to } 8.5 \text{ MeV} \quad , \quad m_s = 80 \text{ to } 155 \text{ MeV} \quad , \quad m_b = 4000 \text{ to } 4500 \text{ MeV} \; . \tag{5.117}$$

   We observe that only m_b is in accordance with the SM case.

We see that the too large value $m_s^\star$ in (5.112) not only influences s_{12} and m_s but also m_d. As already mentioned above in small mixing angle approximation the CP violating phase δ_{13} is zero.

The advantage of the small mixing angle approximation case is that ratios for the off-diagonal elements m_{12}, m_{13} and m_{23} of M^d are given by

$$\frac{m_{12}}{m_{13}} = \frac{\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q}{\langle \mathcal{A}_y^{\hat{47}} \rangle_k^q} \; , \quad \frac{m_{12}}{m_{23}} = \frac{\langle \mathcal{A}_y^{\hat{43}} \rangle_k^q}{\langle \mathcal{A}_y^{\hat{47}} \rangle_k^q} \; , \quad \frac{m_{13}}{m_{23}} = \frac{\langle \mathcal{A}_y^{\hat{43}} \rangle_k^q}{\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q} \; . \tag{5.118}$$

and involve only the triplet VEV of $SO(3)_F$.

Conclusion: *In small mixing angle approximation, i.e. $m_x \gg m_b, m_s, m_d$ and $m_x \gg \langle \mathcal{A}_y^{\hat{a}} \rangle_k^q$ for $\hat{a} = \hat{43}, \hat{45}, \hat{47}$, ratios for the off-diagonal elements m_{12}, m_{13}, m_{23} of M^d are given by rations of the triplet VEV of $SO(3)_F$ according to (5.118).*

Looking at the calculation for the CKM mixing angles s_{12}, s_{13}, s_{23} in Appendix A we obtain with (5.118) the ratios

$$\frac{s_{13}}{s_{23}} = \frac{\langle \mathcal{A}_y^{\hat{43}} \rangle_k^q}{\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q} \, , \quad \frac{s_{12}}{s_{13}} = \frac{\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \, m_b}{\langle \mathcal{A}_y^{\hat{47}} \rangle_k^q \, (m_s - m_d)} \, , \quad \frac{s_{12}}{s_{23}} = \frac{\langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \, m_b}{\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \, (m_s - m_d)} \tag{5.119}$$

Conclusion: *In small mixing angle approximation, i.e.* $m_x \gg m_b, m_s, m_d$ *and* $m_x \gg \langle \mathcal{A}_y^{\hat{a}} \rangle_k^q$ *for* $\hat{a} = \hat{43}, \hat{45}, \hat{47}$*, the CKM mixing angles* s_{12}, s_{13}, s_{23} *are determined by rations of the triplet VEV of* $SO(3)_F$ *plus a correction coming from the down-type masses according to (5.119).*

We note that these results are independent of the explicit value for m_x as long as $m_x \gg m_b, m_s, m_d$ and $m_x \gg \langle \mathcal{A}_y^{\hat{a}} \rangle_k^q$ for $\hat{a} = \hat{43}, \hat{45}, \hat{47}$. This can easily be seen as follows. Suppose we replace

$$m_x \to m_x \cdot k^2 \tag{5.120}$$

where $k \in \mathbb{R}$. The elements m_{12}, m_{13} and m_{23} are invariant under this replacement if we demand

$$\langle \mathcal{A}_y^{\hat{a}} \rangle_k^q \to \langle \mathcal{A}_y^{\hat{a}} \rangle_k^q \cdot k \tag{5.121}$$

$\hat{a} = \hat{43}, \hat{45}, \hat{47}$. However also

$$\frac{\left(\langle \mathcal{A}_y^{\hat{a}} \rangle_k^q \right)^2}{m_x} \tag{5.122}$$

is unchanged by this replacement.

Let us discuss the alternative case where $m_x \approx \mathcal{O}(\tilde{m}_s)$. In this case it is possible to get a realistic value for m_s and the CP violating phase δ_{13}. However it is not possible to get an exact analytically solution. First from appendix B we see that for general m_x the absolute values for the off-diagonal elements of M^d are given by (B.31) and (B.24)

$$| \, m_{12} \, | = c_{14} s_{24} c_{34} \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \, , \quad | \, m_{13} \, | = c_{14} s_{34} \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \, , \quad | \, m_{23} \, | = c_{24} s_{34} \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q$$

and involve products of sine and cosine of the mixing angles θ_{14}, θ_{24}, and θ_{34}. Thus the rations (5.118) get modified and depend now also on these sine and cosine. Looking at the equation (B.20) that determines m_s

$$m_s \, e^{\delta_s} = \tilde{m}_s c_{24}^2 \, e^{\tilde{\delta}_s} - 2 \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q s_{24} c_{24} \, e^{\delta_{\hat{45}}} + \tilde{m}_x s_{24}^2 \, e^{\delta_{\tilde{m}_x}} \tag{5.123}$$

where δ_s, $\tilde{\delta}_s$, $\delta_{\hat{45}}$ and $\delta_{\tilde{m}_x}$ denote phase factors we see that it is possible to obtain a realistic for m_s in (5.112) for appropriate phase factors and a suitable mixing angle θ_{24}. In addition, we will get also non-vanishing phase factors in M^d that can lead to a CP violating phase δ_{13} in V_{CKM}. However since it is not possible to get an exact analytically solution we suggest to make a numerical analysis. This analysis has to clarify if it is possible to recover the SM case or not.

Chapter 6

Summary and outlook

In this thesis we have investigated a Gauge-Higgs-unification model in five dimensions with broken chiral $SO(3)_F$ flavour symmetry. The model is based on the gauge group $SU(7)$ which unifies electroweak-, flavour and Higgs interactions. The Higgs fields are identified with the zero mode of some extra components of the higher-dimensional gauge field. Hence the Higgs fields in this model are prevented from obtaining quadratically divergent corrections to their mass by the higher dimensional gauge symmetry. Therefore the model provides a solution to the hierarchy problem and will be valid at energy scales much above the electroweak breaking scale. The SM Higgs is replaced by three $SU(2)_L$ Higgs doublets H_1, H_2 and H_3 which couple to the first, second and third generation, respectively. The electroweak gauge group $SU(2)_L \times U(1)_Y$ is broken by these $\{H_i\}$ to $U(1)_{em}$. The model includes an anomaly-free chiral $SO(3)_F$ flavour symmetry which explains in a natural way why there are exactly three generations as it is observed in nature. All fermion masses and mixing angles are computable in principle and thus the $SU(7)$ model provides a solution to the flavour problem. The $SO(3)_F$ flavour symmetry is broken by additional Higgs fields coming from the selfadjoint part of Φ at energies much above the compactification scale. This way tree-level FCNC are naturally suppressed due to the large $SO(3)_F$ flavour gauge boson masses.

The $SU(7)$ model is an effective bilayered transverse lattice model with nonunitary parallel transporters in the extra dimension. In chapter 3 we have shown explicitly how an effective bilayered transverse lattice model can be obtained by starting with an ordinary $S^1/\mathbb{Z}_2$ model. In a first step we have put $S^1/\mathbb{Z}_2$ on a lattice. In a second step one has to calculate the renormalisation group flow. The endpoint of the RG-flow is an extra dimension which consist of only two points: the orbifold fixed points. The bulk is completely integrated out. As a result of the blockspin transformations the parallel transporters in the extra dimension become nonunitary. In addition, a Higgs potential emerges naturally. We have shown that for trivial orbifold projection P and trivial minimum of the Higgs potential the eBTLM equals a

$S^1/\mathbb{Z}_2$ continuum orbifold model with trivial orbifold projection P and KK-mode expansion truncated for all fields at the first KK mode. We have seen that the truncated $S^1/\mathbb{Z}_2$ model and consequently also the eBTLM is renormalisable.

We have analysed orbifold conditions for nonunitary parallel transporters. As an important result we have seen that for complex holonomy groups H it is always possible to choose a maximal noncompact Abelian subgroup A with Lie algebra $\mathfrak{a}$ such that $P\eta P^{-1} = \eta$ for $\eta \in \mathfrak{a}$. Using this, the nonunitary parallel transporter Φ can be written as

$$\Phi = e^{A_y}\, e^{\eta}\, e^{A_y} \ ,$$

with A_y such that $PA_yP^{-1} = -A_y$. It is essential that P is involutive. On $S^1/\mathbb{Z}_2$ this is always the case. We have seen that when spontaneous symmetry breaking occurs and the orbifold projection P is non-trivial the Higgs potential $V(\Phi)$ does not only depend on the selfadjoint factor e^{η} but also on the unitary factor e^{A_y}, i.e.

$$V(\Phi) = V(e^{A_y}\, e^{\eta}\, e^{A_y}) = \mathcal{V}(\eta, A_y) \ .$$

In chapter 3 we have furthermore analysed in detail an eBTLM based on the gauge group $SU(2)$. In particular we have studied two important cases, which provide a basis for the $SU(7)$ model:

1. For trivial orbifold projection, i.e $P = \mathrm{diag}(1,1)$, and non-trivial minimum Φ_{min} of the Higgs potential $V(\Phi)$ at

$$\Phi_{min} = \rho_{min} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{a_1} & 0 \\ 0 & e^{a_2} \end{pmatrix} \ , \tag{6.1}$$

   see (3.162), we have obtained (see Proposition 3) that in the limit of large a_2 (i.e. $a_2 \gg 1$) the eBTLM allows gauge boson masses for some zero mode and first excited KK mode gauge fields, which are much larger than the compactification scale (see 3.219). This behaviour has no counterpart in an ordinary $S^1/\mathbb{Z}_2$ model.

2. For non-trivial orbifold projection (3.221), i.e. $P = \mathrm{diag}(1,-1)$, we have seen that the Higgs potential $V(\Phi)$ depends also on the unitary factors e^{A_y}, and if the latter assume a VEV the gauge group $G_0 = U(1)$, which is left unbroken by the orbifold projection P, is spontaneously broken. This spontaneous symmetry breaking is however completely different from the spontaneous symmetry breaking by VEVs for the selfadjoint factor e^{η}. Indeed we have seen that a spontaneous symmetry breaking by a VEV for the unitary factors e^{A_y} equals a continuous Wilson line breaking or Hosotani breaking. This symmetry breaking allows the reduction of the *rank* of the underlying gauge group $G_0 = U(1)$, i.e. G_0 is completely broken.

Based on these two results, in chapter 4 we have formulated a realistic Gauge-Higgs unification model: The $SU(7)$ model. The group $SU(7)$ unifies electroweak-, flavour- and Higgs interactions. Colour was ignored. As an intermediate step the model also unifies weak- and flavour interactions in the gauge group $SU(6)_L \subset SU(7)$. We have shown that zero modes of the extra-dimensional component of the five-dimensional gauge fields transform according to the fundamental representation of $SU(2)_L$ and carry the hypercharge $\frac{1}{2}$. They serve as a substitute for the SM Higgs. The theory includes three $SU(2)_L$ Higgs doublets H_1 (4.61), H_2 (4.62) and H_3 (4.63). H_1 couples to the first, H_2 couples to the second and H_3 couples to the third generation.

We have calculated all zero- and first KK mode gauge boson masses associated to the gauge group $SU(2)_L \times U(1)_Y \times SO(3)_F$ in terms of the minimum Φ_{min} (4.103) of the Higgs potential. We have identified the SM gauge bosons, i.e. the W bosons, the Z boson and the photon, as zero mode gauge bosons of the electroweak gauge group $SU(2)_L \times U(1)_Y \subset SU(7)$. We have seen that the W and the Z boson get masses only from the off-diagonal part of (4.103). Thus their masses are $\mathcal{O}(246)$ GeV. The photon remains massless. All other gauge bosons receive masses mainly from the diagonal part of (4.103). The diagonal part of (4.103) reads

$$\Phi^{diag}_{min} = \rho_{min} \frac{1}{\sqrt{2}} \mathrm{diag}(e^{a_1}, e^{a_2}, e^{a_3}, e^{a_4}, e^{a_5}, e^{a_6}, e^{a_7}) \ . \tag{6.2}$$

We stress that in order to obtain a realistic model, it is important that:

- The minimum (4.103) of the Higgs potential is quasi $\mathcal{S}_2$ symmetric, i.e. we have $a_4 = a_1$, $a_5 = a_2$ and $a_6 = a_3$ in (4.103) and (6.2), respectively. Otherwise the W bosons will get masses from the diagonal part of Φ_{min} and thus will be very heavy.

- For a quasi $\mathcal{S}_2$ symmetric (4.103) and (6.2), respectively, we need $a_1 \neq a_2 \neq a_3$ and $a_i \gg 1$ for at least two a_i, $i = 1, 2, 3$. If these conditions are fulfilled, all zero mode- and first excited KK mode flavour gauge boson receive masses from spontaneous symmetry breaking much above the compactification scale and thus tree-level FCNC are naturally suppressed. If $a_i \gg 1$ already for one a_i, $i = 1, 2, 3$, all first excited KK modes of the SM gauge bosons, i.e. $W^{(1)}$, $Z^{(1)}$ and $\gamma^{(1)}$, get masses from spontaneous symmetry breaking much above the compactification scale and thus will be very heavy.

Furthermore we have calculated the weak mixing angle in the $SU(7)$ model which unfortunately turns out to be too small. Following Antoniadis, Benakli and Quiros [2] this problem could be solved as follows: We start with the larger gauge group $SU(7) \times U(1)''$. The larger gauge group $SU(7) \times U(1)''$ is broken by orbifolding and the additional boundary conditions to

$$SU(7) \times U(1)'' \stackrel{\text{P+b.c.}}{\longrightarrow} SU(2)_L \times U(1) \times SO(3)_F \times U(1)'' \ ,$$

i.e. the extra $U(1)''$ is unaffected by the orbifold projection and the additional boundary conditions. The hypercharge $U(1)_Y$ is identified as the sum of the $U(1)$ and $U(1)''$ charges. We then have a gauge field B_μ associated to the hypercharge and a gauge field A^X_μ associated to its orthonormal combination [2]. The additional $U(1)''$ comes equipped with an additional coupling constant g''_{5D}. Since g''_{5D} is undetermined we can set it to any desired value. This way we can restore the weak mixing angle of the SM [2]. Note that the additional $U(1)''$ is anomalous. However, these anomalies can be cancelled by a generalised Green-Schwarz mechanism [2].

In chapter 5 we have analysed how fermion masses and CKM mixing angles are generated by the Higgs mechanism in the context of nonunitary parallel transporters. We have seen that in order to explain why quarks and leptons have different masses, we need to introduce two additional nonunitary parallel transporters Φ^{Quark} and Φ^{Lepton}, and we have to make a clear distinction between Φ^{Quark}, Φ^{Lepton} and $\Phi^{Gauge} = \Phi$. Hence we have also three different Higgs potentials $V(\Phi^{Quark})$, $V(\Phi^{Lepton})$ and $V(\Phi^{Gauge})$ in the model. When spontaneous symmetry breaking occurs quark and lepton masses are given by the Yukawa interactions

$$\bar{q}_L \Phi^{Quark}_{min} q_R \quad , \quad \bar{q}_L \Phi^{Lepton}_{min} q_R \, .$$

The model has a large Higgs sector. The reason is that we have three different nonunitary parallel transporters in the model and therefore also three different Higgs potentials. The minima Φ^{Gauge}_{min}, Φ^{Quark}_{min} and Φ^{Lepton}_{min} are parametrised by altogether thirty parameters. For instance ten parameters $a^{quark}_1, \ldots, a^{quark}_7$ and $\alpha^{quark}_{\hat{4}\hat{3}}, \alpha^{quark}_{\hat{4}\hat{5}}, \alpha^{quark}_{\hat{4}\hat{7}}$ parametrise Φ^{Quark}_{min}. Fluctuations of $a^{quark}_1, \ldots, a^{quark}_7$ and $\alpha^{quark}_{\hat{4}\hat{3}}, \alpha^{quark}_{\hat{4}\hat{5}}, \alpha^{quark}_{\hat{4}\hat{7}}$ around the minimum Φ^{Quark}_{min} of $V(\Phi^{Quark})$ give rise to 10 Higgs particles:

- 3 Higgs particles which are associated to fluctuations of $\alpha^{quark}_{\hat{4}\hat{3}}, \alpha^{quark}_{\hat{4}\hat{5}}, \alpha^{quark}_{\hat{4}\hat{7}}$. Their mass squared is given by

$$m^2_{A^{\hat{a}(0)}_y} \sim \left(\frac{g_4}{R}\right)^2 \frac{\partial^2 V(\Phi^{quark})}{\partial \alpha_a}$$

  at the minimum Φ^{Quark}_{min} of $V(\Phi^{Quark})$.

- 7 Higgs particles which are associated to fluctuations of $a^{quark}_1, \ldots, a^{quark}_7$. Their mass squared is given by

$$m^2_{a_i} \sim \left(\frac{g_4}{R}\right)^2 \frac{\partial^2 V(\Phi^{quark})}{\partial a^{quark}_i}$$

  at the minimum Φ^{Quark}_{min} of $V(\Phi^{Quark})$.

For Φ_{min}^{Lepton} and Φ_{min}^{Gauge} the situation is analogous. Thus we conclude that the $SU(7)$ model predicts 30 Higgs particles which may be found at the LHC/ILC.

In chapter 5 we have also seen that the up-type quark masses m_u, m_c, m_t are given by the diagonal part of Φ_{min}^{Quark} only. This is an important result and stands in contrast to the SM case where up- and down-type masses are given by the same Higgs doublet. In particular, this means that we can produce Higgs particles associated to the fluctuations of $a_1^{quark}, \ldots, a_7^{quark}$ and originating from the selfadjoint part of Φ^{quark} exclusively by e.g. $t\bar{t}$ scattering.

In chapter 5 we have also investigated how the CKM mixing angles and the down-type quark masses are determined by the parameters $a_1^{quark}, \ldots, a_7^{quark}$ and $\alpha_{\hat{43}}^{quark}, \alpha_{\hat{45}}^{quark}, \alpha_{\hat{47}}^{quark}$. It turned out that for

$$\tilde{m}_d, \tilde{m}_s, \tilde{m}_b, \langle \mathcal{A}_y^{\hat{43}} \rangle_k^q, \langle \mathcal{A}_y^{\hat{45}} \rangle_k^q, \langle \mathcal{A}_y^{\hat{47}} \rangle_k^q \ll m_x \; , \tag{6.3}$$

where $\tilde{m}_d$, $\tilde{m}_s$, $\tilde{m}_b$ and m_x are given by (5.25) and $\langle \mathcal{A}_y^{\hat{a}} \rangle_k^q$ for $\hat{a} = \hat{43}, \hat{45}, \hat{47}$ are given by (5.47), it is possible to get an exact analytical solution for all down-type quark masses and CKM mixing angles. We have called this scenario small mixing angle approximation. As a result we have obtained the correct hierarchy for the CKM mixing angles, i.e.

$$s_{13} \ll s_{23} \ll s_{12} \; .$$

However the Cabibbo angles turn out to be too small by a factor of two. In addition, also the s-quark mass turned out to be too large by approximately a factor of two. We we have found that the off-diagonal entries in the down-type quark mass matrix (5.111)

$$M^d = \begin{pmatrix} \tilde{m}_d & m_{12} & m_{13} \\ m_{12} & \tilde{m}_s & m_{23} \\ m_{13} & m_{23} & \tilde{m}_b \end{pmatrix} \tag{6.4}$$

i.e. m_{12}, m_{13} and m_{23}, are given by the ratios

$$\frac{m_{12}}{m_{13}} = \frac{\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q}{\langle \mathcal{A}_y^{\hat{47}} \rangle_k^q} \; , \quad \frac{m_{12}}{m_{23}} = \frac{\langle \mathcal{A}_y^{\hat{43}} \rangle_k^q}{\langle \mathcal{A}_y^{\hat{47}} \rangle_k^q} \; , \quad \frac{m_{13}}{m_{23}} = \frac{\langle \mathcal{A}_y^{\hat{43}} \rangle_k^q}{\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q} \; .$$

These ratios give an interpretation for the triplet VEV of $SO(3)_F$. The triplet VEV of $SO(3)_F$ plus a correction coming from the down-type quark masses determine the CKM mixing angles via

$$\frac{s_{13}}{s_{23}} = \frac{\langle \mathcal{A}_y^{\hat{43}} \rangle_k^q}{\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q} \; , \quad \frac{s_{12}}{s_{13}} = \frac{\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \, m_b}{\langle \mathcal{A}_y^{\hat{47}} \rangle_k^q \, (m_s - m_d)} \; , \quad \frac{s_{12}}{s_{23}} = \frac{\langle \mathcal{A}_y^{\hat{43}} \rangle_k^q \, m_b}{\langle \mathcal{A}_y^{\hat{45}} \rangle_k^q \, (m_s - m_d)}$$

in small mixing angle approximation (6.3). In addition, in this case we have no CP violation.

We have argued that it is possible to cure the problem that $\tilde{m}_s$ is too large while s_{12} is too small by lowering m_x to $\mathcal{O}(\tilde{m}_s)$. In addition, in this case we get non-vanishing phase factors in M^d that can lead to CP violation. However it is not clear if we can recover the SM case or not.

The latter problem is a motivation for further investigations and extensions. First, since the case $m_x \sim \mathcal{O}(\tilde{m}_s)$ is analytically not exactly solvable, we suggest to make a detailed numerical analysis. This analysis has to clarify whether it is possible to recover the SM case or not. Suppose that this is not possible. Then of course the $SU(7)$ model in its simplest version is ruled out. However there are several possible extensions to the $SU(7)$ model. One possible extension is to consider the $SU(7)$ model in two extra dimensions. In this case, the underlying orbifold can be e.g. $T^2/\mathbb{Z}_2$. It is known [2, 70] that on the orbifold $T^2/\mathbb{Z}_2$ one has two independent Higgs doublets. In the case of the $SU(7)$ model this means that (5.24) can be non-symmetric, i.e.

$$M = \begin{pmatrix} \tilde{m}_d & 0 & 0 & i\,k_{\hat{43}}\,\langle\mathcal{A}_y^{\hat{43}(0)}\rangle^{quark\ 1} \\ 0 & \tilde{m}_s & 0 & i\,k_{\hat{45}}\,\langle\mathcal{A}_y^{\hat{45}(0)}\rangle^{quark\ 1} \\ 0 & 0 & \tilde{m}_b & i\,k_{\hat{47}}\,\langle\mathcal{A}_y^{\hat{47}(0)}\rangle^{quark\ 1} \\ i\,k_{\hat{43}}\,\langle\mathcal{A}_y^{\hat{43}(0)}\rangle^{quark\ 2} & i\,k_{\hat{45}}\,\langle\mathcal{A}_y^{\hat{45}(0)}\rangle^{quark\ 2} & i\,k_{\hat{47}}\,\langle\mathcal{A}_y^{\hat{47}(0)}\rangle^{quark\ 2} & m_x \end{pmatrix} \tag{6.5}$$

where $\langle\mathcal{A}_y^{\hat{43}(0)}\rangle^{quark\ 1}$, $\langle\mathcal{A}_y^{\hat{45}(0)}\rangle^{quark\ 1}$ and $\langle\mathcal{A}_y^{\hat{47}(0)}\rangle^{quark\ 1}$ denote VEVs for the neutral components of the first three $SU(2)_L$ Higgs doublets while $\langle\mathcal{A}_y^{\hat{43}(0)}\rangle^{quark\ 2}$, $\langle\mathcal{A}_y^{\hat{45}(0)}\rangle^{quark\ 2}$ and $\langle\mathcal{A}_y^{\hat{47}(0)}\rangle^{quark\ 2}$ denote VEVs for the neutral components of the second three $SU(2)_L$ Higgs doublets. In contrast to the $SU(7)$ model in five dimensions, $\langle\mathcal{A}_y^{\hat{a}(0)}\rangle^{quark\ 1}$ and $\langle\mathcal{A}_y^{\hat{a}(0)}\rangle^{quark\ 2}$ for $\hat{a} = \hat{43}, \hat{45}, \hat{47}$ are independent and can in particular get different VEVs, i.e. we could have

$$\langle\mathcal{A}_y^{\hat{a}(0)}\rangle^{quark\ 1} \neq \langle\mathcal{A}_y^{\hat{a}(0)}\rangle^{quark\ 2}$$

for $\hat{a} = \hat{43}, \hat{45}, \hat{47}$. Thus M will in general be non-symmetric. The down-type quark mass matrix M^d (6.4) in this case is obtained by first bringing M to block diagonal form by a biunitary transformation

$$M \to U^L\ M\ U^{R\dagger} = \begin{pmatrix} M^d & 0 \\ 0 & \tilde{m}_x \end{pmatrix} ,$$

where U^L and U^R are now different matrices that can be written as

$$U^{L\dagger} = P^L_{34} R^L_{34} R^L_{24} R^L_{14} \ , \quad U^{R\dagger} = P^R_{34} R^R_{34} R^R_{24} R^R_{14} \ ,$$

compare with (5.50). In contrast to the symmetric case we now have to determine four phases (two in the symmetric case) and six mixing angles (three in the symmetric case). For instance the mixing angles θ^L_{34} respectively θ^R_{34} are, ignoring phases, given by [1]

$$\tan 2\theta_{34}^L = \frac{2\left(m_x\langle\mathcal{A}_y^{\hat{47}}\rangle_k^{q\,1} + \tilde{m}_b\langle\mathcal{A}_y^{\hat{47}}\rangle_k^{q\,2}\right)}{m_x^2 - \tilde{m}_b^2 + \left(\langle\mathcal{A}_y^{\hat{47}}\rangle_k^{q\,2}\right)^2 - \left(\langle\mathcal{A}_y^{\hat{47}}\rangle_k^{q\,1}\right)^2}$$

$$\tan 2\theta_{34}^R = \frac{2\left(m_x\langle\mathcal{A}_y^{\hat{47}}\rangle_k^{q\,2} + \tilde{m}_b\langle\mathcal{A}_y^{\hat{47}}\rangle_k^{q\,1}\right)}{m_x^2 - \tilde{m}_b^2 + \left(\langle\mathcal{A}_y^{\hat{47}}\rangle_k^{q\,1}\right)^2 - \left(\langle\mathcal{A}_y^{\hat{47}}\rangle_k^{q\,2}\right)^2} .$$

Thus θ_{34}^L and θ_{34}^R are different if $\langle\mathcal{A}_y^{\hat{47}(0)}\rangle^{quark\ 1}$ and $\langle\mathcal{A}_y^{\hat{47}(0)}\rangle^{quark\ 2}$ get different VEVs. In a second step we diagonalise M^d by a second biunitary transformation

$$U_L^d\ M^d\ U_R^{d\dagger} = M_{diag}^d ,$$

and the CKM matrix is given by

$$V_{CKM} = U_R^{d\dagger} .$$

The matrix U_L^d involves three additional mixing angles θ_{12}^L, θ_{13}^L and θ_{23}^L and in principle two additional phases. However, only the CP violating phase and three mixing angles of the CKM matrix $V_{CKM} = U_R^{d\dagger}$ are of importance. We immediately see that in the non-symmetric case we have vastly more freedom to recover the correct CP violation of the SM and the correct down-type quark masses and CKM mixing angles. However such a model may reintroduce unsuppressed tree-level FCNC and thus a carful analysis is needed.

Second, we would like to calculate the RG-flow for the $SU(7)$ model. The result of this RG-flow will determine definite numerical values for all thirty parameters $a_i, \ldots, a_7$, $a_i^{quark}, \ldots, a_7^{quark}$, $a_i^{lepton}, \ldots, a_7^{lepton}$, $\alpha_{\hat{43}}, \alpha_{\hat{45}}, \alpha_{\hat{47}}$, $\alpha_{\hat{43}}^{quark}, \alpha_{\hat{45}}^{quark}, \alpha_{\hat{47}}^{quark}$ and $\alpha_{\hat{43}}^{lepton}, \alpha_{\hat{45}}^{lepton}, \alpha_{\hat{47}}^{lepton}$ parametrising Φ_{min}^{gauge}, Φ_{min}^{quark} and Φ_{min}^{lepton}, respectively. This means that we can calculate definite numerical values for

- all gauge boson masses: $W_\mu^{1,2}$, Z_μ, $W_\mu^{1,2(1)}$, $Z_\mu^{(1)}$, $\gamma^{(1)}$, $H_\mu^{j(0)}$ and $H_\mu^{j(1)}$
- the fermion masses: $m_u, m_c, m_t, m_d, m_s, m_b$ and e, μ, τ
- the mixing angles: s_{12}, s_{13} and s_{23} .

This is a remarkable feature of our model. Note that the neutrino masses are not determined by the results of the RG-flow.

In addition, the RG-flow will also determine the shape of the Higgs potentials $V(\Phi^{qauge})$, $V(\Phi^{quark})$ and $V(\Phi^{lepton})$. Therefore it would be possible to calculate

[1] Note that $\langle\mathcal{A}_y^{\hat{a}}\rangle_k^{q\,1} := k_{\hat{a}}\ \langle\mathcal{A}_y^{\hat{a}(0)}\rangle^{quark\ 1}$ and $\langle\mathcal{A}_y^{\hat{a}}\rangle_k^{q\,2} := k_{\hat{a}}\ \langle\mathcal{A}_y^{\hat{a}(0)}\rangle^{quark\ 2}$ for $\hat{a} = \hat{43}, \hat{45}, \hat{47}$.

the masses of all thirty Higgs particles in the $SU(7)$ model. This is also a remarkable feature of our model. However there is one question which has to been answered: What is the origin of the quasi $\mathcal{S}_2$ symmetry? Is it an accidental symmetry or is there any principle behind this?

Third, as already indicated above, we want to be able to extend our model to higher-dimensional orbifolds, e.g. two dimensional orbifolds like $T^2/\mathbb{Z}_2$. In principle the same steps which have lead to an eBTLM can be repeated for the orbifold $T^2/\mathbb{Z}_2$. As in the five-dimensional case, we can put $T^2/\mathbb{Z}_2$ on a lattice which must be two-dimensional and then calculate the renormalisation group flow. Also we should be able to determine orbifold conditions for nonunitary parallel transporters. Note that on $T^2/\mathbb{Z}_2$ the orbifold projection P is still involutive. However for two-dimensional orbifolds this not true in general. In particular, on the orbifold $T^2/\mathbb{Z}_3$, $T^2/\mathbb{Z}_4$ and $T^2/\mathbb{Z}_6$ the orbifold projection P has to fulfil $P^3 = 1$, $P^4 = 1$ and $P^6 = 1$, respectively. Therefore we have to reinvestigate orbifold conditions for nonunitary parallel transporters in the case of noninvolutive P.

Fourth we remind the reader that in the $SU(7)$ model we have completely ignored colour. If we include strong interactions we could ask whether it is possible to find a GUT extention for the $SU(7)$ model. A possible GUT group which is compatible with the assignments of fermions to the different fixed points of the orbifold is the Pati-Salam group

$$G_{PS} = SU(2)_L \times SU(2)_R \times SU(4)_c \; . \tag{6.6}$$

We can extend G_{PS} to

$$G_{extendedPS} = SU(6)_L \times SU(6)_R \times SU(4)_c \; . \tag{6.7}$$

The extension from $SU(2)_L$ to $SU(6)_L$ is exactly the same as in the $SU(7)$ model. In addition, the analogue extension can be made for $SU(2)_R$. It would be interesting to investigate how a Gauge-Higgs unification model can be built from $G_{extendedPS}$. In addition we have to determine the symmetry breaking pattern. We expect that such a GUT breaking is possible only on higher-dimensional orbifolds.

Appendix A

CKM mixing matrix from M^d

Lemma 2 *Let M^d be given by*

$$M^d = \begin{pmatrix} \tilde{m}'_d & m_{12} & m_{13} \\ m^*_{12} & \tilde{m}'_s & m_{23} \\ m^*_{13} & m_{23} & m_b \end{pmatrix} \tag{A.1}$$

where

$$\begin{aligned} \tilde{m}'_d &= 13.4\ MeV \quad m_{12} = \hat{m}_{12} + \hat{m}_{13} s_{23} e^{i\,\delta_{13}} \approx 24.46\ MeV \\ \tilde{m}'_s &= 119.2\ MeV \quad m_{13} = \hat{m}_{12} s_{23} - \hat{m}_{13} e^{i\,\delta_{13}} \approx 16.65\ e^{i\,\frac{2\pi}{3}}\,MeV \\ m_b &= 4250\ MeV \quad m_{23} = 173.8\ MeV\,, \end{aligned} \tag{A.2}$$

and

$$\hat{m}_{12} = 24.8\ MeV \quad , \quad \hat{m}_{13} = 16.1\ MeV \quad , \quad \delta_{13} = \frac{2\pi}{3}\ . \tag{A.3}$$

The unitary transformation

$$V^\dagger_{CKM}\, M^d\, V_{CKM} = M^d_{diag}\ . \tag{A.4}$$

leads to the CKM matrix

$$V_{CKM} = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} \tag{A.5}$$

where $\lambda = s_{12} = 0.22$, $A\lambda^2 = s_{23} = 0.042$, $A\lambda^3(\rho - i\eta) = s_{13}\ e^{-i\delta_{13}}$, $s_{13} = 0.0039$ and $\delta_{13} = \frac{2\pi}{3}$.

Proof 4 *We write V_{CKM} as a product of three Euler matrices and a phase matrix*

$$V_{CKM} = R_{23} P^*_{13} R_{13} P_{13} R_{12} \tag{A.6}$$

where

$$P_{13} = \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix} , \tag{A.7}$$

ϕ_1 *arbitrary and* ϕ_3 *such that* $\phi_1 - \phi_3 = \delta_{13} = \frac{2\pi}{3}$. *Inserting this expansion in (A.4) we obtain*

$$R_{12}^t P_{13}^* R_{13}^t P_{13} R_{23}^t \; M_d \; R_{23} P_{13}^* R_{13} P_{13} R_{12} = M_{diag}^d \tag{A.8}$$

The calculation will therefore be divided into five steps.

First step : *We perform the real rotation* R_{23} *on* M^d. *The purpose is to put zeroes in the* $23, 32$ *elements*

$$M_1^d = R_{23}^t M^d R_{23} = \begin{pmatrix} \tilde{m}_d' & \tilde{m}_{12} & \tilde{m}_{13} \\ \tilde{m}_{12} & \tilde{m}_s & 0 \\ \tilde{m}_{13}^* & 0 & \tilde{m}_b \end{pmatrix} \tag{A.9}$$

The zeroes in the $23, 32$ *elements are implemented by diagonalising the lower 23 block of* M^d. *This block is obtained by striking out the row and the column in which the unit element of* R_{23} *appears. Thus we get the reduced rotation*

$$R_{23}^t \begin{pmatrix} \tilde{m}_s' & m_{23} \\ m_{23} & m_b \end{pmatrix} R_{23} := \begin{pmatrix} \tilde{m}_s & 0 \\ 0 & \tilde{m}_b \end{pmatrix} . \tag{A.10}$$

This matrix equation fixes the mixing angle θ_{23}, *i.e.*

$$\tan 2\theta_{23} = \frac{2 m_{23}}{m_b - \tilde{m}_s'} . \tag{A.11}$$

Inserting (A.2) we obtain

$$s_{23} = 0.042 = A\lambda^2 \tag{A.12}$$

For the matrix elements $\tilde{m}_s$ *and* $\tilde{m}_b$ *we get from (A.10)*

$$\begin{aligned} \tilde{m}_s &= c_{23}^2 \tilde{m}_s' - 2 s_{23} c_{23} m_{23} + s_{23}^2 m_b \\ \tilde{m}_b &= s_{23}^2 \tilde{m}_s' + 2 s_{23} c_{23} m_{23} + c_{23}^2 m_b \; . \end{aligned} \tag{A.13}$$

Inserting (A.2) and (A.12) we obtain

$$\tilde{m}_s = 111.9 \; MeV \quad , \quad \tilde{m}_b \approx m_b \; . \tag{A.14}$$

The remaining elements of M_1^d *are given by*

$$(\tilde{m}_{12}, \tilde{m}_{13}) = (m_{12}, m_{13}) \, R_{23} \; , \tag{A.15}$$

and consequently, since

$$R_{23} = \begin{pmatrix} c_{23} & s_{23} \\ -s_{23} & c_{23} \end{pmatrix} , \tag{A.16}$$

we obtain

$$\begin{aligned} \tilde{m}_{12} &= m_{12} - s_{23}m_{13} = (=:)\hat{m}_{12} \\ \tilde{m}_{13} &= s_{23}m_{12} + m_{13} = (=:) \mid \hat{m}_{13} \mid e^{i\,\delta_{13}} \end{aligned} \tag{A.17}$$

where we have ignored terms of the order $\mathcal{O}(s_{23}^2)$. *Indeed, as indicated in the brackets, this equations define* m_{12} *and* m_{13} *in (A.2). In addition we get*

$$\begin{aligned} \tilde{m}_{21} &= m_{12}^* - s_{23}m_{13}^* = \hat{m}_{12} = \tilde{m}_{12} \\ \tilde{m}_{31} &= s_{23}m_{12}^* + m_{13}^* = \mid \hat{m}_{13} \mid e^{-i\,\delta_{13}} = \tilde{m}_{13}^* \,. \end{aligned} \tag{A.18}$$

In the following let us write all phases explicitly. Thus M_1^d *reads*

$$M_1^d = \begin{pmatrix} \tilde{m}_d' & \tilde{m}_{12} & \mid \tilde{m}_{13} \mid e^{-i\,\delta_{13}} \\ \tilde{m}_{12} & \tilde{m}_s & 0 \\ \mid \tilde{m}_{13} \mid e^{i\,\delta_{13}} & 0 & \tilde{m}_b \end{pmatrix} \tag{A.19}$$

where $\delta_{13} = \phi_1 - \phi_3$.

Second step: *We multiply* M_1^d *by the phase matrix* P_{13}. *The purpose of this rephasing is to put real values in the* $13, 31$ *elements in order to get a real angle* θ_{13} *in step three. Indeed*

$$M_2^d = P_{13}M_1^d P_{13}^* = \begin{pmatrix} \tilde{m}_d' & \mid \tilde{m}_{12} \mid e^{i\,\phi_1} & \tilde{m}_{13} \\ \mid \tilde{m}_{12} \mid e^{-i\,\phi_1} & \tilde{m}_s & 0 \\ \tilde{m}_{13} & 0 & \tilde{m}_b \end{pmatrix} \tag{A.20}$$

transforms away δ_{13} *in (A.19).*

Third step: *We perform the real rotation* R_{13} *on* M_2^d. *The purpose is to put zeroes in the* $13, 31$ *elements*

$$M_3^d = R_{13}^t M_2^d R_{13} = \begin{pmatrix} \tilde{m}_d & \mid \tilde{m}_{12}' \mid e^{i\,\phi_1} & 0 \\ \mid \tilde{m}_{12}' \mid e^{-i\,\phi_1} & \tilde{m}_s & 0 \\ 0 & 0 & \tilde{m}_b \end{pmatrix} . \tag{A.21}$$

The zeroes in the $13, 31$ *elements are implemented by diagonalising the outer 13 block of* M_2^d. *This block is obtained by striking out the row and the column in which the unit element of* R_{13} *appears. Thus we get the reduced rotation reads*

$$R_{13}^t \begin{pmatrix} \tilde{m}_d' & \tilde{m}_{13} \\ \tilde{m}_{13} & m_b \end{pmatrix} R_{13} := \begin{pmatrix} \tilde{m}_d & 0 \\ 0 & \tilde{m}_b \end{pmatrix} . \tag{A.22}$$

Note that all quantities in this matrix equation are real due to the phase transformation (A.20)in step two. This leads to the real mixing angle

$$s_{13} \approx \frac{\tilde{m}_{13}}{m_b - \tilde{m}'_d} \; . \tag{A.23}$$

Inserting (A.2) and (A.3) we obtain

$$s_{13} = 0.0039 = A\lambda^3\rho \; . \tag{A.24}$$

For the matrix elements $\tilde{m}_d$ and $\tilde{m}_b$ we obtain from (A.22)

$$\begin{aligned} \tilde{m}_d &= \tilde{m}'_d - 2s_{13}m_{13} \\ \tilde{m}_b &= m_b + 2s_{13}m_{13} \; . \end{aligned} \tag{A.25}$$

Inserting (A.2), (A.3) and (A.24) we obtain

$$\tilde{m}_d = 13.3 \; MeV \quad , \quad \tilde{m}_b \approx m_b \; . \tag{A.26}$$

The remaining elements of M_2^d read

$$\tilde{m}'_{12} = \tilde{m}_{12} \quad , \quad \tilde{m}'_{23} = s_{13}m_{12} = 0.097 \approx 0 \; . \tag{A.27}$$

Fourth step: *We multiply M_3^d by the phase matrix P_{13}. The purpose of this rephasing is to put real values in the $12, 21$ elements of (A.21) in order to get a real angle θ_{12} in step five. Indeed*

$$M_4^d = P_{13}^* M_3^d P_{13} = \begin{pmatrix} \tilde{m}_d & \tilde{m}'_{12} & 0 \\ \tilde{m}'_{12} & \tilde{m}_s & 0 \\ 0 & 0 & \tilde{m}_b \end{pmatrix} \tag{A.28}$$

transforms away ϕ_1 in the $12, 21$ elements of (A.21). Thus we are left with a complete real matrix M_4^d.

Fifth step: *We perform the real rotation R_{12} on M_4^d. The purpose is to put zeroes in the $12, 21$ elements*

$$M_{diag}^d = R_{12}^t M_4^d R_{12} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \tag{A.29}$$

The zeroes in the $12, 21$ elements are implemented by diagonalising the upper 12 block of M_4^d. This block is obtained by striking out the row and the column in which the unit element of R_{12} appears. Thus we get the reduced rotation

$$R_{12}^t \begin{pmatrix} \tilde{m}_d & \tilde{m}'_{12} \\ \tilde{m}'_{12} & \tilde{m}_s \end{pmatrix} R_{12} := \begin{pmatrix} m_d & 0 \\ 0 & m_s \end{pmatrix} \; . \tag{A.30}$$

This matrix equation fixes the mixing angle s_{12}, i.e.

$$\tan 2\theta_{12} = \frac{2\tilde{m}'_{12}}{\tilde{m}_s - \tilde{m}_d} \ . \tag{A.31}$$

Inserting (A.14), (A.26) and (A.3) we obtain

$$s_{12} = 0.22 = \lambda \tag{A.32}$$

Using the approximation $c_{12} = 0.973 \approx 1 - \frac{\lambda^2}{2}$ the reduced rotation R_{12} reads

$$R_{12} \approx \begin{pmatrix} 1 - \frac{\lambda^2}{2} & s_{12} \\ -s_{12} & 1 - \frac{\lambda^2}{2} \end{pmatrix} \ . \tag{A.33}$$

For the matrix elements $\tilde{m}_d$ and m_b one obtain

$$\begin{aligned} m_d &= \tilde{m}_d(1 - \frac{\lambda^2}{2})^2 - 2\lambda(1 - \frac{\lambda^2}{2})\tilde{m}'_{12} + \tilde{m}_s\lambda^2 \\ m_s &= \tilde{m}_d\lambda^2 + 2\lambda(1 - \frac{\lambda^2}{2})\tilde{m}'_{12} + \tilde{m}_s(1 - \frac{\lambda^2}{2})^2 \ . \end{aligned} \tag{A.34}$$

Inserting (A.14), (A.26),(A.3) and (A.32) we obtain

$$m_d = 7.4\ MeV \quad , \quad m_s = 114.1\ MeV \tag{A.35}$$

Note that $m_b = 4250$ is approximately unchanged by all transformations.

Appendix B

General procedure for obtaining M^d and V_{CKM} from M

We start with the 4×4 matrix

$$M = \begin{pmatrix} \tilde{m}_d & 0 & 0 & m_{14} \\ 0 & \tilde{m}_s & 0 & m_{24} \\ 0 & 0 & \tilde{m}_b & m_{34} \\ m_{14} & m_{24} & m_{34} & m_x \end{pmatrix} \qquad \text{(B.1)}$$

where m_{14}, m_{24} and m_{34} denote the off-diagonal elements of M. *In the following we ignore phases and treat M as real.* We must transform M on block diagonal form

$$M \to U\ M\ U^\dagger = \begin{pmatrix} M^d & 0 \\ 0 & \tilde{m}_x \end{pmatrix} , \qquad \text{(B.2)}$$

where M^d is the 3×3 down-quark mass matrix. We write U as a product of the real three Euler matrices

$$U^\dagger = R_{34} R_{24} R_{14} \qquad \text{(B.3)}$$

where

$$R_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_{34} & s_{34} \\ 0 & 0 & -s_{34} & c_{34} \end{pmatrix} \tag{B.4}$$

$$R_{24} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{24} & 0 & s_{24} \\ 0 & 0 & 1 & 0 \\ 0 & -s_{24} & 0 & c_{24} \end{pmatrix} \tag{B.5}$$

$$R_{14} = \begin{pmatrix} c_{14} & 0 & 0 & s_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_{14} & 0 & 0 & c_{14} \end{pmatrix} \tag{B.6}$$

$$\tag{B.7}$$

Inserting this expansion (B.2) reads

$$R^t_{14}R^t_{24}R^t_{34}MR_{34}R_{24}R_{14} = \begin{pmatrix} M_d & 0 \\ 0 & \tilde{m}_x \end{pmatrix} \tag{B.8}$$

The calculatation will be divided into three steps.

First step : We perform the real rotation R_{34} on M. The purpose is to put zeroes in the $34, 43$ elements

$$M_1 = R^t_{34}MR_{34} = \begin{pmatrix} \tilde{m}_d & 0 & \tilde{m}_{13} & \tilde{m}_{14} \\ 0 & \tilde{m}_s & \tilde{m}_{23} & \tilde{m}_{24} \\ \tilde{m}_{13} & \tilde{m}_{23} & m_b & 0 \\ \tilde{m}_{14} & \tilde{m}_{24} & 0 & \tilde{m}_x \end{pmatrix} \tag{B.9}$$

The zeroes in the $34, 43$ elements are implemented by diagonalising the lower 34 block of M. This block is obtained by striking out the row and the column in which the unit elements of R_{34} appear. Thus we get the reduced rotation

$$R^t_{34} \begin{pmatrix} \tilde{m}_b & m_{34} \\ m_{43} & m_x \end{pmatrix} R_{34} := \begin{pmatrix} m_b & 0 \\ 0 & \tilde{m}_x \end{pmatrix} . \tag{B.10}$$

From this matrix we get the mixing angle θ_{34}

$$\tan 2\theta_{34} = \frac{2m_{34}}{m_x - \tilde{m}_b} \tag{B.11}$$

and the diagonal elements

$$m_b = \tilde{m}_b c^2_{34} - 2m_{34}s_{34}c_{34} + m_x s^2_{34} \tag{B.12}$$

$$\tilde{m}_x = \tilde{m}_b s^2_{34} + 2m_{34}s_{34}c_{34} + m_x c^2_{34} . \tag{B.13}$$

The remaing elements are given by the reduced rotation

$$\begin{pmatrix} \tilde{m}_{13} & \tilde{m}_{14} \\ \tilde{m}_{23} & \tilde{m}_{24} \end{pmatrix} = \begin{pmatrix} 0 & m_{14} \\ 0 & m_{24} \end{pmatrix} \begin{pmatrix} c_{34} & s_{34} \\ -s_{34} & c_{34} \end{pmatrix} . \tag{B.14}$$

This leads to

$$\tilde{m}_{13} = -s_{34} m_{14} \quad , \quad \tilde{m}_{14} = c_{34} m_{14} \tag{B.15}$$
$$\tilde{m}_{23} = -s_{34} m_{24} \quad , \quad \tilde{m}_{24} = c_{34} m_{24} \tag{B.16}$$

Second step : We perform the real rotation R_{24} on M_1. The purpose is to put zeroes in the $24, 42$ elements

$$M_2 = R_{24}^t M_1 R_{24} = \begin{pmatrix} \tilde{m}_d & \tilde{m}'_{12} & \tilde{m}_{13} & \tilde{m}'_{14} \\ \tilde{m}'_{12} & m_s & \tilde{m}'_{23} & 0 \\ \tilde{m}_{13} & \tilde{m}'_{23} & m_b & \approx 0 \\ \tilde{m}'_{14} & 0 & \approx 0 & \tilde{m}'_x \end{pmatrix} \tag{B.17}$$

The zeroes in the $24, 42$ elements are implemented by diagonalising the lower middle block of M_1. This block is obtained by striking out the row and the column in which the unit elements of R_{24} appear. Thus we get the reduced rotation

$$R_{24}^t \begin{pmatrix} \tilde{m}_s & \tilde{m}_{24} \\ \tilde{m}_{24} & \tilde{m}_x \end{pmatrix} R_{24} := \begin{pmatrix} m_s & 0 \\ 0 & \tilde{m}'_x \end{pmatrix} . \tag{B.18}$$

From this matrix we get the mixing angle θ_{24}

$$\tan 2\theta_{24} = \frac{2\tilde{m}_{24}}{\tilde{m}_x - \tilde{m}_s} = \frac{2c_{34} m_{24}}{\tilde{m}_x - \tilde{m}_s} \tag{B.19}$$

and the diagonal elements

$$m_s = \tilde{m}_s c_{24}^2 - 2\tilde{m}_{24} s_{24} c_{24} + \tilde{m}_x s_{24}^2 \tag{B.20}$$
$$\tilde{m}'_x = \tilde{m}_s s_{24}^2 + 2\tilde{m}_{24} s_{24} c_{24} + \tilde{m}_x c_{24}^2 . \tag{B.21}$$

The remaing elements are given by the reduced rotation

$$\begin{pmatrix} \tilde{m}'_{12} & \tilde{m}'_{14} \\ \tilde{m}'_{23} & \tilde{m}'_{34} \end{pmatrix} = \begin{pmatrix} 0 & \tilde{m}_{14} \\ \tilde{m}_{23} & 0 \end{pmatrix} \begin{pmatrix} c_{24} & s_{24} \\ -s_{24} & c_{24} \end{pmatrix} . \tag{B.22}$$

This leads to

$$\tilde{m}'_{12} = -s_{24}\tilde{m}_{14} = -s_{24} c_{34} m_{14} \quad , \quad \tilde{m}'_{14} = c_{24}\tilde{m}_{14} = c_{24} c_{34} m_{14} \tag{B.23}$$
$$\tilde{m}'_{23} = c_{24}\tilde{m}_{23} = -c_{24} s_{34} m_{24} \quad , \quad \tilde{m}'_{34} = s_{24}\tilde{m}_{32} = -s_{24} s_{34} m_{24} \approx 0 \tag{B.24}$$

Third step : We perform the real rotation R_{14} on M_2. The purpose is to put zeroes in the $14, 41$ elements

$$M_3 = R_{14}^t M_2 R_{14} = \begin{pmatrix} m_d & \tilde{m}_{12}'' & \tilde{m}_{13}' & 0 \\ \tilde{m}_{12}'' & m_s & \tilde{m}_{23}' & \approx 0 \\ \tilde{m}_{13}' & \tilde{m}_{23}' & m_b & \approx 0 \\ 0 & \approx 0 & \approx 0 & \tilde{m}_x' \end{pmatrix} \tag{B.25}$$

The zeroes in the $14, 41$ elements are implemented by diagonalising the outer block of M_2. This block is obtained by striking out the row and the column in which the unit elements of R_{14} appear. Thus we get the reduced rotation

$$R_{14}^t \begin{pmatrix} \tilde{m}_d & \tilde{m}_{14}' \\ \tilde{m}_{14}' & \tilde{m}_x' \end{pmatrix} R_{14} := \begin{pmatrix} m_d & 0 \\ 0 & \tilde{m}_x'' \end{pmatrix} . \tag{B.26}$$

From this matrix we get the mixing angle θ_{14}

$$\tan 2\theta_{14} = \frac{2\tilde{m}_{14}'}{\tilde{m}_x' - \tilde{m}_d} = \frac{2c_{24}c_{34}m_{14}}{\tilde{m}_x' - \tilde{m}_d} \tag{B.27}$$

and the diagonal elements

$$m_d = \tilde{m}_d c_{14}^2 - 2\tilde{m}_{14}' s_{14} c_{14} + \tilde{m}_x s_{14}^2 \tag{B.28}$$

$$\tilde{m}_x'' = \tilde{m}_d s_{14}^2 + 2\tilde{m}_{24} s_{14} c_{14} + \tilde{m}_x c_{14}^2 . \tag{B.29}$$

The remaing elements are given by the reduced rotation

$$\begin{pmatrix} \tilde{m}_{12}'' & \tilde{m}_{24}'' \\ \tilde{m}_{13}' & \tilde{m}_{34}'' \end{pmatrix} = \begin{pmatrix} \tilde{m}_{12}' & 0 \\ \tilde{m}_{13}' & 0 \end{pmatrix} \begin{pmatrix} c_{14} & s_{14} \\ -s_{14} & c_{14} \end{pmatrix} . \tag{B.30}$$

This leads to

$$\begin{aligned} \tilde{m}_{12}'' = c_{14}\tilde{m}_{12}' = -c_{14}s_{24}c_{34}m_{14} \quad &, \quad \tilde{m}_{24}'' = s_{14}\tilde{m}_{12}' = s_{14}s_{24}c_{34}m_{14} \approx 0 \\ \tilde{m}_{13}' = c_{14}\tilde{m}_{13} = -c_{14}s_{34}m_{14} \quad &, \quad \tilde{m}_{34}'' = s_{14}\tilde{m}_{13} = s_{14}s_{34}m_{14} \approx 0 \end{aligned} \tag{B.31}$$

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Acknowledgements

First of all, I would like to thank my supervisor Prof. Gerhard Mack for his guidance and support.

Im a also very grateful to the members of my group Thorsten Prüstel, Falk Neugebohrn and Michael Röhrs for their friendship and many fruitful discussions about physics.

Diverse conversations broadened and deepened my physical knowledge. Hence I am greatly indebted to a number of colleagues, in particular Thorsten Prüstel, Falk Neugebohrn, Michael Röhrs, Florian Schwennsen, Martin Hentschinski, Frank Fugel and Thorben Kneesch.

It is also a pleasure to thank Sven Grosskreutz for his friendship and help.

Moreover, I am grateful to the members of the II. Institut für Theoretische Physik and the DESY theory group for creating a very pleasant and stimulating working atmosphere.

This work was supported by the Graduiertenkolleg "Zukünfitige Entwicklungen in der Teilchenphysik".

Finally, I would like to thank my parents for their love, support and trust.