UNIVERSITÀ DEGLI STUDI DI MILANO

Facoltà di Scienze Matematiche Fisiche e Naturali

Dipartimento di Fisica

Corso di Dottorato di Ricerca in Fisica, Astrofisica e Fisica Applicata

Ciclo XIX

TESI DI DOTTORATO DI RICERCA:

Sudakov resummation in QCD

Settore scientifico: FIS/02

Tutor: Prof. Stefano FORTE
Referee: Prof. Giovanni RIDOLFI
Coordinatore: Prof. Gianpaolo BELLINI

Tesi di dottorato di:
Paolo BOLZONI
Matr. R05430

Anno Accademico 2005-2006

Sudakov resummation in QCD

Paolo Bolzoni

Abstract

In this PhD thesis, we analyze and generalize the renormalization group approach to the resummation of large logarithms in the perturbative expansion due to soft and collinear multiparton emissions. In particular, we present a generalization of this approach to prompt photon production. It is interesting to see that also with the more intricate two-scale kinematics that characterizes prompt photon production in the soft limit, it remains true that resummation simply follows from general kinematic properties of the phase space. Also, this approach does not require a separate treatment of individual colour structures when more than one colour structure contributes to fixed order results. However, the resummation formulae obtained here turn out to be less predictive than previous results: this depends on the fact that here neither specific factorization properties of the cross section in the soft limit is assumed, nor that soft emission satisfies eikonal-like relations. We also derive resumation formulae to all logarithmic accuracy and valid for all values of rapidity for the prompt photon production and the Drell-Yan rapidity distributions. We show that for the fixed-target experiment E866/NuSea, the NLL resummation corrections are comparable to NLO fixed-order corrections and are crucial to obtain agreement with the data. Finally we outline also possible future applications of the renormalization group approach.

Sudakov resummation in QCD

Paolo Bolzoni

Contents

Introduction

The current theory that describes the strong subnuclear interactions is the non-abelian gauge theory of the SU(3) local symmetry group. This theory, called Quantum Chromo-Dynamics (QCD), has been tested to great accuracy in many experiments. Its perturbative regime will be crucial at the imminent high-energy proton-proton collider LHC. The main target of LHC, is to find signals of the Standard Model Higgs boson and of new physics, i.e. supersymmetry, new interactions predicted by great unified theories, extra-dimensions, new gauge bosons and mini-black holes. In order to accomplish all this, an excellent understanding of QCD is necessary, both because LHC is a proton collider and because QCD backgrounds must be described accurately.

In perturbative QCD (pQCD), it is well known that, when one approaches to the boundary of the phase space, the cross section receives logarithmical-enhanced contributions of soft and collinear origin at all orders. These large tems at order $O(\alpha_s^n)$ of the strong coupling constant in pQCD have, in general, the form

$$\alpha_s^n \left[\frac{\log^m(1-z)}{1-z} \right]_+, \quad m \le 2n - 1,$$

where z is a parameter that becomes close to 1 near to the phase space boundary and "+" denotes the plus distribution. These terms become important in the limit $z \to 1$ spoiling the validity of the perturbative fixed-order QCD expansion. Hence, they should be resummed to all-orders of pQCD in order to get accurate predictions of the cross sections of QCD processes. An example of the importance that can have resummation is given by the Higgs boson production at LHC near the threshold of its production. In this case z is given by the ratio of the searched Higgs mass over the center-of-mass energy of the partons in the colliding hadrons and the large logarithms arise from soft-gluon emissions.

These large terms have been resummed a long time ago for the classes of inclusive hadronic processes of the type of deep-inelastic scattering (DIS) and Drell-Yan (DY) [1, 2, 3]. Threshold resummation of inclusive processes can affect significantly cross sections and the extraction of parton densities [4, 5]. For the case of small transverse momentum distributions in Drell-Yan processes, it has been shown that resummation is necessary to reproduce the correct behavior of the cross section [6].

These results has been obtained using the eikonal approaximation of Ref.[2] which generalizes to QCD the Sudakov exponentiation of soft photons emissions in electro-dynamics [7] or assuming suitable factorization properties of the QCD cross section [1]. More recently two other approaches to resummation have been proposed. The first is the renormalization group approach of Ref.[8] and the other is the effective

field theoretic (EFT) approach of Refs.[9, 10]. There the resummation of the large logarithms for full inclusive deep-inelastic and Drell-Yan processes is obtained.

In this thesis we will concentrate mostly on the study of the renormalization group approach of Ref.[8] and on its applications and generalizations. This approach has the advantage of being valid to all logarithmic orders, and self-contained, in that it does not require any factorization beyond the standard factorization of collinear singularities. It relies on an essentially kinematical analysis of the phase space for the given process in the soft limit, which is used to establish the result that the dependence on the resummation variable only appears through a given fixed dimensionful combination. This provides a second dimensionful variable, along with the hard scale of the process, which can be resummed using standard renormalization group techniques. Beyond the leading log level, the resummed result found within this approach turns out to be somewhat less predictive than the result obtained with the other methods. In the other approaches references resummed results at a certain logarithmic accuracy is fully determined by a ceratin fixed order computation, whereas a higher fixed order computation is needed to determine all coefficients in the resummed formula of Ref.[8]. The more predictive result is recovered within this approach if the dependence of the perturbative coefficients on the two dimensionful variables factorizes, i.e. if the two-scale factorization mentioned above holds.

We shall show the generalization of the renormalization group approach to the resummation of the inclusive transverse momentum spectrum of prompt photons produced in hadronic collisions in the region where the transverse momentum is close to its maximal value. Prompt photon production is a less inclusive process than Drell-Yan or deep-inelastic scattering, and it is especially interesting from the point of view of the renormalization group approach, because the large logs which must be resummed turn out to depend on two independent dimensionful variables, on top of the hard scale of the process: hence, prompt photon production is characterized by three scales. The possibility that the general factorization Ref.[11] might extend to prompt photon production was discussed in Ref.[12], based on previous generalizations [13] of factorization, and used to derive the corresponding resummed results. Resummation formulae for prompt photon production in the approach of Ref.[2] were also proposed in Ref.[14], and some arguments which might support such resummation were presented in Ref.[15]. Our treatment will provide a full proof of resummation to all logarithmic orders. Our resummation formula does not require the factorization proposed in Refs.[12, 14], and it is accordingly less predictive. Because of the presence of two scales, it is also weaker than the result of Ref.[8] for DIS and Drell-Yan production. Increasingly more predictive results are recovered if increasingly restrictive forms of factorization hold.

Moreover, we shall prove an all-logarithmic orders resummation formula for differential rapidity of Drell-Yan and prompt photon production processes. The differential rapidity Drell-Yan cross section is used for the extraction of the ratio $\bar{d}/\bar{u}$ of parton densities. The accurate knowledge of these functions is needed to study Higgs boson production and the asymmetry $W^{\pm}$. The resummation of Drell-Yan rapidity distributions was first considered in 1992 [16]. At that time, it was suggested a resummation formula for the case of zero rapidity. Very recently, thanks to the analysis of the full

NLO calculation of the Drell-Yan rapidity distribution, it has been shown [17], that the result given in [16] is valid at next-to-leading logarithmic accuracy (NLL) for all rapidities. In this thesis, we shall give a simple proof of an all-order resummation formula valid for all values of rapidity. To do this, we will use the technique of the double Fourier-Mellin moments developed in [18]. In particular, we will show that the resummation can be reduced to that of the rapidity-integrated process, which is given in terms of a dimensionless universal function for both DY and $W^{\pm}$ and Z^0 production. Then, we implement numerically the resummation formula and give predictions of the full rapidity-dependent NLL Drell-Yan cross section for the case of the fixed-target E866/NuSea experiment. We find that resummation at the NLL level is necessary and that its agreement with the experimental data is better than the NNLO calculation of Ref.[19]. In this case, we find also that the NLL resummation reduced the cross section instead of enhancing it for the parameter choices of this experiment. Threshold corrections to Higgs, Z^0 and $W^{\pm}$ production rapidity distributions at high energy hadron colliders have also been studied in [20, 21].

Finally, we shall discuss the application of the renormalization group approach to the resummation another class of large logarithms that arise in Drell-Yan processes for small tranverse momentum distribution of the produces lepton pairs. These logarithmic-enhanced terms at order $O(\alpha_s^n)$ have, in general, the form

$$\alpha_s^n \left[\frac{\log^m(q_\perp^2)}{q_\perp^2} \right]_+ , \quad m \leq 2n - 1,$$

where $q_\perp$ is the transverse momentum of the produced Drell-Yan pair. Also in this case the resummed results using the renormalization group approach are less predictive than results obtained with the approach of Ref.[1], as it is shown in Ref.[6]. Furthermore the conditions that reduce our results to those of Ref.[6] in terms of factorization properties is still an interesting open question.

This thesis is organized as follows. In Chapter 1, we review the basics concepts of pQCD. In particular the construction of the QCD Lagrangian, the asymptotic freedom of strong interactions, the structure of the cross section when initial state hadrons are present and the evolution equation of the parton densities. In Chapter 2, we discuss the importance of resummation at hadron colliders. Then, we describe how in the various approaches the large logarithms are exponentiated and resummed. They are the renormalization group approach, the eikonal approximation approach, the approach of non standard factorization properties and the effective field theoretic approach. In Chapter 3, we show in the detail the renormalization group approach to the resummation of all inclusive deep-inelastic and Drell-Yan processes and its generalization to the prompt photon process at large transverse photon momentum is shown in Chapter 4. In Chapter 5, we prove the all-logarithmic orders resummation formula for the Drell-Yan and prompt photon production processes. We show also the impact of the NLL resummation for the DY process at the E866/NuSea experiment and discuss the numerical results. Then in Chapter 6, we turn to discuss the generalization of the renormalization group approach to resummation in the case of the small transverse momentum differential cross section for the Drell-Yan process. Finally, in the last Chapter, we summarize and determine the predictive power of the

resummation formulae obtained with the different approaches, namely, the fixed-order computation needed to determine completely a resummation formula for an arbitrary logarithmic accuracy.

Chapter 1

General aspects of perturbative QCD

1.1   Quarks, Gluons and QCD

The quarks (the constituents of hadrons) are Dirac fermions. In the Standard Model (SM), as far as the electroweak interactions are concerned, the properties of quarks and leptons are similar. Indeed, as for the leptons, the six quark flavors are grouped into three $SU_L(2)$ left-handed doublets

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \quad \begin{pmatrix} c \\ s \end{pmatrix}_L, \quad \begin{pmatrix} t \\ b \end{pmatrix}_L \tag{1}$$

and six $SU(2)_L$ singlets, which are the right-handed parts of each flavor. Both, quarks and leptons, interact in a similar way with the electroweak gauge bosons (γ, $W^\pm$ and Z^0) of the group $SU_L(2) \times U_Y(1)$. The main difference is that each quark flavor eigenstate is a unitary mixing of the quark mass eigenstate, while, according to the SM, this is not the case for charged leptons and massless neutrinos. However, in this decade it has been proven that neutrinos have mass and that there is also mixing in the neutrino sector thanks to observations of their oscillations.

The peculiarity of quarks is that they have a specific property, the color charge, which is absent for leptons. Indeed, a quark of a given flavor has three different color states with equal masses and electroweak charges. The interaction of the quarks is mediated by the eight gauge bosons (gluons g) of the color group $SU_C(3)$. So, the quarks belong to the fundamental representation of $SU_C(3)$ and the gluons to the adjoint one. The gauge theory of this non-abelian group is called Quantum Chromodynamics (QCD) and is the current theory of strong interactions.

Specifically, a gauged $SU_C(3)$ transformation of a quark ($q_a(x)$ with $a = 1, 2, 3$) is given by

$$q_a(x) \quad \to \quad q'_a(x) = U_{ab}(x) q_b(x) \tag{2}$$

$$\bar{q}_a(x) \quad \to \quad \bar{q}'_a(x) = \bar{q}_b U^\dagger_{ba}(x), \tag{3}$$

where the 3×3 matrix $U_{ik}(x)$ is the fundamental representation of the $SU_C(3)$ group

1

that acts on an internal space defined at each space-time coordinate x. It satisfies

$$UU^\dagger = U^\dagger U = 1, \qquad \det(U) = 1. \tag{4}$$

In this section the sum over all the repeated indices is implicit and the sum over spinor indices is omitted for brevity. The usual exponential representation of the gauge transformation matrix in terms of the basis of matrices (the generators of $SU_C(3)$) of the corresponding algebra $su_C(3)$ is:

$$U(x) = e^{-\frac{i}{2}\vec{\alpha}(x)\cdot\vec{\lambda}} = e^{-\vec{\alpha}(x)\cdot\vec{t}}, \tag{5}$$

where $\vec{\alpha}(x) = (\alpha_1(x), \ldots, \alpha_8(x))$ are the eight arbitrary parameters of the gauge transformation, $\vec{\lambda}$ are the eight elements of the basis of the algebra $su_C(2)$ (or equivalently the eight generators of the group) and $\vec{t}$ are the eight color operators (in analogy to the spin operators of the group $SU(2)$). It is clear that, in order to respect Eq.(4), the generators of the group must be hermitian and traceless. The normalization of the color operators depends on the representation r

$$\mathrm{tr}(t_r^A t_r^B) = T_r \delta^{AB}. \tag{6}$$

The form chosen by Gell-Mann for the $su_C(3)$ basis in the fundamental representation:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{7}$$

With these definitions, the color matrices satisfy the following relations

$$[t^A, t^B] = if^{ABC} t^C \tag{8}$$

$$\mathrm{tr}(t^A t^B) = T_F \delta^{AB}, \quad T_F = \frac{1}{2} \tag{9}$$

where T_F is the normalization of the color matrices in the fundamental representation and f^{ABC} are called the structure constants of the algebra $su_C(3)$ which are totally antisymmetric in $\{A, B, C\}$. The independent non-vanishing structure constants are given by:

$$f^{123} = 1, \quad f^{147} = \frac{1}{2}, \quad f^{156} = -\frac{1}{2}, \quad f^{246} = \frac{1}{2}, \quad f^{257} = \frac{1}{2} \tag{10}$$

$$f^{345} = \frac{1}{2}, \quad f^{367} = -\frac{1}{2}, \quad f^{458} = \frac{\sqrt{3}}{2}, \quad f^{678} = \frac{\sqrt{3}}{2}. \tag{11}$$

Furthermore, the structure constants provide the adjoint representation of the $su_C(3)$ algebra (the one which has the same dimension of the algebra). Indeed, if we define the adjoint representation as $T_{BC}^A = -if^{ABC}$, we can verify explicitly that this is the adjoint representatio because

$$[T^A, T^B] = if^{ABC}T^C \tag{12}$$
$$\text{tr}(T^A T^B) = T_A \delta^{AB}, \quad T_A = 3 \tag{13}$$

The Casimir operator C_r (the one which commutes with all elements of the algebra), for a certain representation r, is constructed as

$$t_r^A t_r^A = C_r 1_{d_r \times d_r}. \tag{14}$$

Now, since the contraction of this last equation is equal to the contraction of Eq.(6), we get

$$d_r C_r = 8T_r. \tag{15}$$

In particular, we find the Casimir operators are given by

$$C_F = \frac{4}{3}, \qquad C_A = 3 \tag{16}$$

for the fundamental and the adjoint representation respectively.

We shall now show that the symmetry with respect to the gauge transformations Eqs.(2,5) (together with the Lorentz invariance), can be used as guiding principle to construct the QCD Lagrangian. We start from the usual Dirac free Lagrangian for each quark mass and color eigenstate

$$\mathcal{L}_D(x) = \bar{\psi}_{fa}(x) \left(i\slashed{\partial} - m_f\right) \psi_{fa}(x), \tag{17}$$

where f is the flavor index and a is the color quark index. This term is not gauge invariant. In fact under the gauge transformations Eqs.(2,5), the Dirac free lagrangian Eq.(17) transform as

$$\mathcal{L}_D(x) \to \mathcal{L}_D(x) + \bar{\psi}_{fb} \left[iU_{ba}^\dagger(x)\partial_\mu U_{ac}(x)\right] \gamma^\mu \psi_{fc}(x). \tag{18}$$

To restore gauge invariance, we introduce a gauge field matrix $A_{\mu ab}(x)$ made up of eight gauge fields $A_\mu^A(x)$ in this way:

$$A_{\mu ab}(x) = t_{ab}^A A_\mu^A. \tag{19}$$

We assign to this matrix field the following interaction Lagrangian

$$\mathcal{L}_I(x) = g_s \bar{\psi}_{fa} A_{\mu ab} \gamma^\mu \psi_{fb}. \tag{20}$$

Here g_s is the gauge dimensionless coupling analogous to the electric charge in QED. In order to cancel the symmetry breaking term of Eq.(18), the gauge transformation of the field matrix $A_{\mu ab}(x)$ has to be

$$A_{\mu ab}(x) \to U_{ac}(x) A_{\mu cd}(x) U_{db}^\dagger(x) - \frac{i}{g_s} \partial_\mu U_{ac}(x) U_{cb}^\dagger(x). \tag{21}$$

Hence, with the introduction of the field matrix $A_{\mu\,ab}$, the sum $\mathcal{L}_D + \mathcal{L}_I$ is now gauge invariant. To complete the Lagrangian one has to add the pure gauge invariant term, which is

$$\mathcal{L}_G(x) = -\frac{1}{2}\,\text{tr}\left(G_{\mu\nu}(x)G^{\mu\nu}(x)\right), \tag{22}$$

where the gluon field-strength tensor is given by

$$G_{\mu\nu\,ab}(x) = \partial_\mu A_{\nu\,ab} - \partial_\nu A_{\mu\,ab} - ig_s\left[A_\mu, A_\nu\right]_{ab}. \tag{23}$$

Note that the third term of the gluon field-strength gives rise to the self interactions of gluons and that its origin stands in the fact that the gauge group $SU_C(3)$ is non-abelian. The final form of the QCD classical Lagrangian is obtained by adding the three pieces introduced above and using Eq.(9):

$$\mathcal{L}^{\text{cl}}_{QCD}(x) = -\frac{1}{4}G^A_{\mu\nu}(x)G^{A\,\mu\nu}(x) + \bar{\psi}_{fa}(x)(iD_{\mu\,ab}\gamma^\mu - m_f\delta_{ab})\psi_{fb}(x), \tag{24}$$

where

$$G^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + g_s f^{ABC}A^B_\mu A^C_\nu \tag{25}$$

and

$$D_{\mu ab} = \delta_{ab}\partial_\mu - ig_s A_{\mu\,ab} \tag{26}$$

is the covariant derivative in the sense that $D_{\mu\,ab}\psi_{fb}(x)$ transforms as $\psi_{fa}(x)$ under the gauge group.

The quantization of the classical gauge invariant theory of QCD Eq.(24), can be done with the Fadeev-Popov procedure. This procedure takes care of the fact that the equation of motion of the gluon field A^A_μ can not be inverted and this prohibits to find the propagator. However, this is a consequence of gauge invariance which implies that the physical massless gluon has only two polarizations/spin states whereas the field A^A_μ has four components. To make things work, an additional constraint on the gluon field is introduced, the so called gauge fixing condition which uses gauge invariance to define properly the gluon propagator. In QCD, however, this constraint is not linear and one should add specially designed fictitious particles (the so called Fadeev-Popov ghosts) which are Lorentz scalar anti-commuting fields and appear only in the loops. After this procedure (usually performed in the functional formalism), we have that the QCD Lagrangian from which we can calculate directly the Feynman rules is given by (see e.g. [22] page 514):

$$\mathcal{L}^{FP}_{QCD}(x) = \mathcal{L}^{cl}_{QCD}(x) - \frac{1}{2\lambda}(\partial^\mu\partial^\nu A^A_\mu(x)A^A_\nu(x)) - \bar{c}^A\partial^\mu D^{AC}_\mu c^C, \tag{27}$$

where λ is the gauge fixing parameter, c^A is the complex colored scalar ghost field and D^{AB}_μ is the covariant derivative in the adjoint representation:

$$D^{AB}_\mu = \delta^{AB}\partial_\mu - g_s f^{ABC}A^C_\mu. \tag{28}$$

The quark, gluon and ghost propagators and vertices for QCD are collected in Figure 1.1.
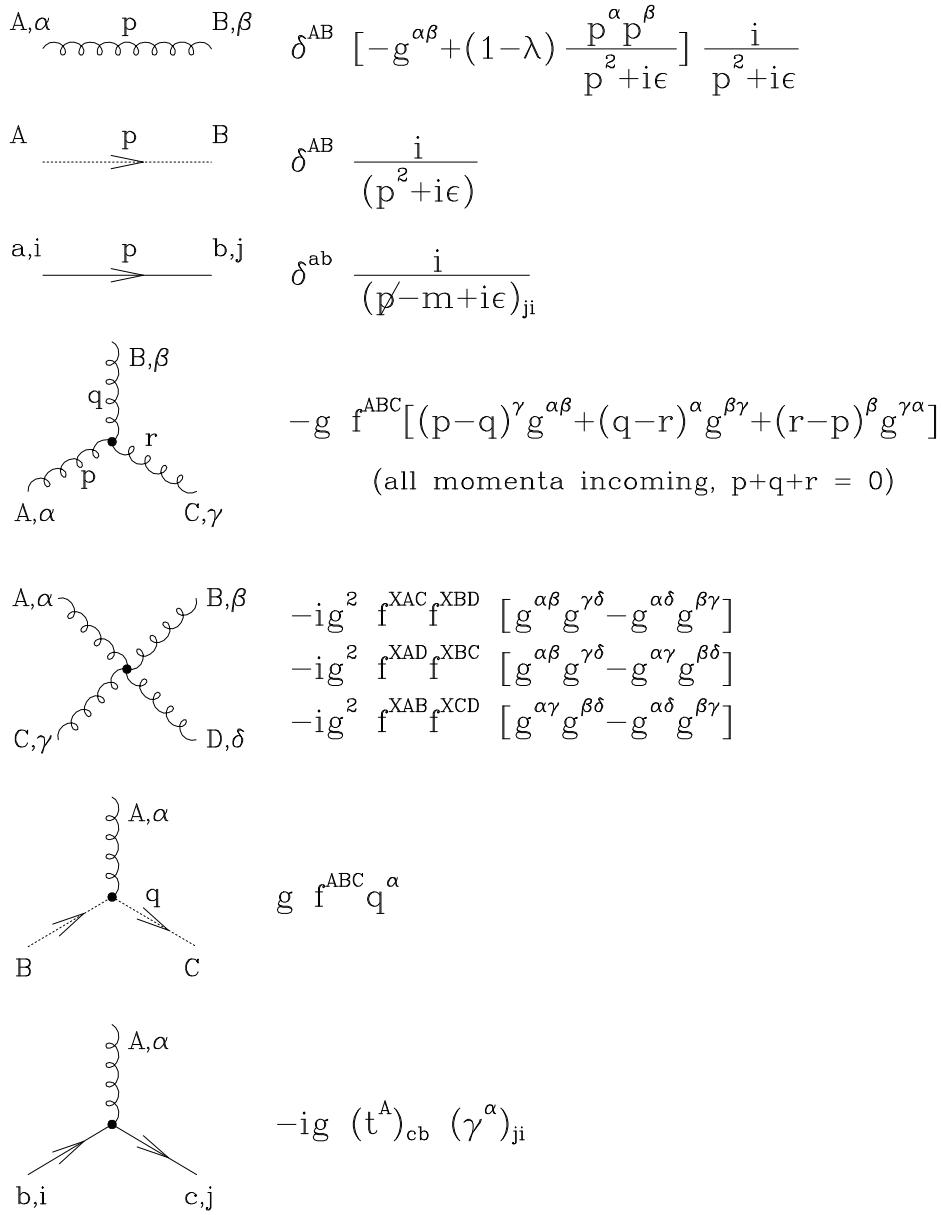
Figure 1.1: Feynman rules for QCD in a covariant gauge for gluons (curly lines), quarks (solid lines) and ghosts (dotted lines). Here A, B, C, D are the color indexes in the adjoint representation, a, b, c in the fundamental one, $\alpha, \beta, \gamma, \delta$ are the gluon polarization indexes and i, j are the spinorial indexes.

1.2   Asymptotic freedom and perturbative QCD

Feynman diagrams in QCD are obtained employing the vertices and propagators as building blocks. However, the use of diagrams makes sense only if the perturbative expansion in g_s is meaningful. To respect this condition, the coupling

$$\alpha_s = \frac{g_s}{4\pi}, \tag{29}$$

the QCD analog of the electromagnetic coupling $\alpha = e^2/4\pi$, has to be sufficiently small. We shall now show that the method of perturbation theory in QCD are useful at high energy. Indeed, the coupling constant is large at low energy and becomes smaller at high energy (asymptotic freedom).

The simplest way to introduce the running coupling, is to consider a dimensionless physical observable R which depends on a single energy scale $\sqrt{Q^2}$. This is the case, for example of the ratio of the annihilation cross section of electron-positron into hadron with the annihilation into muons where $Q^2 = S$ the center-of-mass energy. We assume that this scale is much bigger than the quark masses that can be therefore neglected. Now, dimensional analysis should implies that a dimensionless observable is independent of Q^2. However, higher order corrections produce divergences and so the perturbation series requires renormalization to remove ultraviolet divergences that in $d = 4 - 2\epsilon$ dimensions are regularized as $1/\epsilon$ poles. This poles can be removed defining a renormalized coupling constant at a certain renormalization scale μ_r^2. Consequently, in the finite $\epsilon \to 0$ limit, R depends in general on the ratio Q^2/μ_r^2 and the renormalized coupling α_s depends on μ_r^2; we call the renormalized coupling at the scale μ_r^2, $\alpha_{\mu_r^2}$. Since the renormalization scale is an arbitrary parameter introduced only to define the theory at the quantum level, we conclude that R has to be μ_r^2-independent. Formally this independence is expressed as follows:

$$\mu_r^2 \frac{d}{d\mu_r^2} R\left(\frac{Q^2}{\mu_r^2}, \alpha_{\mu_r^2}\right) = \left[\mu_r^2 \frac{\partial}{\partial \mu_r^2} + \beta(\alpha_{\mu_r^2})\frac{\partial}{\partial \alpha_{\mu_r^2}}\right] R = 0, \tag{30}$$

where

$$\beta(\alpha_{\mu_r^2}) = \mu_r^2 \frac{\partial \alpha_{\mu_r^2}}{\partial \mu_r^2} \tag{31}$$

Eq.(30) is a first order differential equation with the initial condition (at $Q^2 = \mu_r^2$) $R(1, \alpha_{\mu_r^2})$. This means that if we find a solution of Eq.(30), it is the only possible solution. This solution is easily found defining a function $\alpha_s(Q^2)$ such that

$$\alpha_s(\mu_r^2) = \alpha_{\mu_r^2} \tag{32}$$

and that

$$\ln\left(\frac{Q^2}{\mu_r^2}\right) = \int_{\alpha_{\mu_r^2}}^{\alpha_s(Q^2)} \frac{dx}{\beta(x)}. \tag{33}$$

In fact differentiating this equation, we find that

$$Q^2 \frac{\partial \alpha_s(Q^2)}{\partial Q^2} = \beta(\alpha_s(Q^2)) \tag{34}$$

$$\frac{\partial \alpha_s(Q^2)}{\partial \alpha_{\mu_r^2}} = \frac{\beta(\alpha_s(Q^2))}{\beta(\alpha_{\mu_r^2})} \tag{35}$$

and that $R(1, \alpha_s(Q^2))$ is the desired solution of Eq.(30), because

$$\frac{\partial}{\partial \alpha_{\mu_r^2}} = \frac{\partial \alpha_s(Q^2)}{\partial \alpha_{\mu_r^2}} \frac{\partial}{\partial \alpha_s(Q^2)}, \quad \mu_r^2 \frac{\partial}{\partial \mu_r^2} = -Q^2 \frac{\partial}{\partial Q^2} = -Q^2 \frac{\partial \alpha_s(Q^2)}{\partial Q^2} \frac{\partial}{\partial \alpha_s(Q^2)}. \quad (36)$$

This shows that all of the scale dependence in R enters through the running of the coupling constant $\alpha_s(Q^2)$. To find explicitly this function, we need to know the β-function so that we can solve Eq.(34). The β-function can be calculate perturbatively from the counterterms of the renormalization precedure and a knowledge to order α_s^{n+1} requires a n-loop computation. The perturbative expansion of the β-function is given by:

$$\beta(\alpha_s) = -\alpha_s \sum_{n=0}^{\infty} \beta_n \left(\frac{\alpha_s}{4\pi} \right)^{n+1}. \quad (37)$$

At the moment, the QCD β-function is known to order α_s^5 [23]: where in the $\overline{MS}$ scheme,

$$\beta_0 = 11 - \frac{2}{3} N_f, \quad \beta_1 = 102 - \frac{38}{3} N_f, \quad (38)$$

$$\beta_2 = \frac{2857}{2} - \frac{5033}{18} N_f + \frac{325}{54} N_f^2 \quad (39)$$

$$\beta_3 = \left(\frac{149753}{6} + 3564\xi_3 \right) - \left(\frac{1078361}{162} + \frac{6508}{27} \xi_3 \right) N_f$$

$$+ \left(\frac{50065}{162} + \frac{6472}{81} \xi_3 \right) N_f^2 + \frac{1093}{729} N_f^3, \quad (40)$$

with N_f the numbers of flavors and ξ is the Riemann zeta-function ($\xi_3 = 1.202056903\ldots$). The two loop solution of Eq.(34) is given by:

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu_r^2)}{1 + (\beta_0/4\pi)\alpha_s(\mu_r^2) \log \frac{Q^2}{\mu_r^2}} \left[1 - \frac{\beta_1}{4\pi\beta_0} \frac{\alpha_s(\mu_r^2) \log(1 + (\beta_0/4\pi)\alpha_s(\mu_r^2) \log \frac{Q^2}{\mu_r^2})}{1 + (\beta_0/4\pi)\alpha_s(\mu_r^2) \log \frac{Q^2}{\mu_r^2}} \right]$$

$$+ O(\alpha_s^{k+3} \log^k \frac{Q^2}{\mu_r^2}). \quad (41)$$

For simplicity, in many cases, we will use another parametrization of the coefficients β_n, which is obtained with the substitution:

$$\beta_n = b_n (4\pi)^{n+1}. \quad (42)$$

With this parametrization, the perturbative expansion of the β-function Eq.(37) becomes

$$\beta(\alpha_s) = -\sum_{n=0}^{\infty} b_n \alpha_s^{n+2}. \quad (43)$$

From Eq.(41), we see that $\alpha_s(Q^2)$ is a monotonically decreasing function of Q^2, because the coefficients β_0 and β_1 are positive (with $N_f \leq 6$). The running of $\alpha_s(Q^2)$ has been measured with great accuracy (see Figure 1.2). The fact that at high energy,
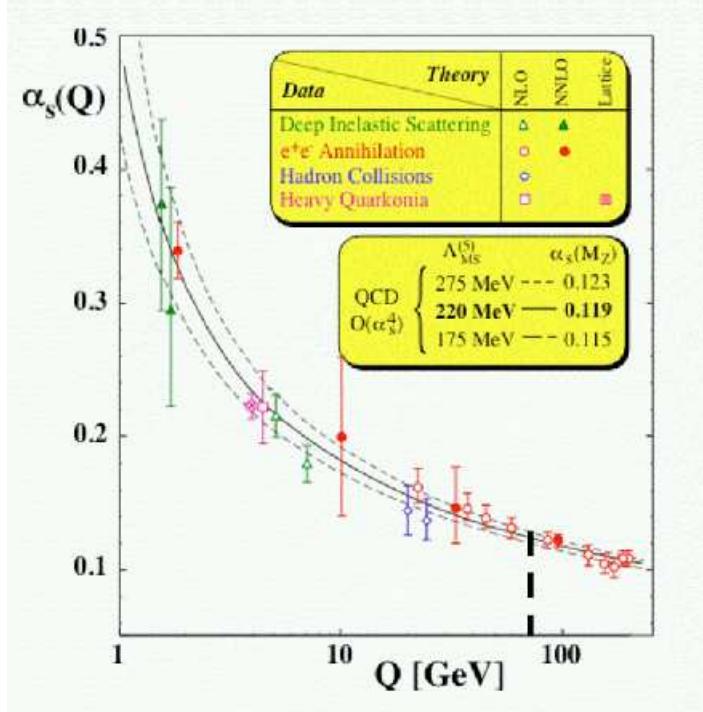
Figure 1.2: The running of the strong coupling constant. The asymptotic freedom is confirmed by the experiments

the running coupling becomes small is a peculiarity of non-abelian gauge theories and is called asymptotic freedom.

Hence, perturbative QCD can be applied when the relevant scale of a certain process is high enough such that the running coupling becomes small. A typical example is given by the annihilation of two high-energy electrons into hadrons. Perturbative QCD can also be applied to processes in which hadrons are present also in the initial state thanks to the factorization theorem. According to this theorem [11], the cross section for the production of some final state with high invariant mass Q^2 (the scale at which the running coupling constant is small) with two incoming hadrons is given by:

$$\sigma(P_1, P_2, Q^2) = \sum_{a,b} \int_0^1 dx_1 dx_2 F_a^{H_1}(x_1, \mu^2) F_b^{H_2}(x_2, \mu^2) \hat{\sigma}_{ab}(x_1 P_1, x_2 P_2, \alpha_s(\mu^2), Q^2, \mu^2).$$
(44)

For processes with a single incoming hadron the factorization theorem is simpler. For example for the deep inelastic scattering (DIS) of a lepton that exchanges a high square momentum Q^2 with the hadron, the cross section takes the form:

$$\sigma(P, Q^2) = \sum_a \int_0^1 dx F_a^H(x, \mu^2) \hat{\sigma}_a(xP, \alpha_s(\mu^2), Q^2, \mu^2).$$
(45)

In Eqs.(44,45), P_i is the momentum of the incoming hadron H_i. A beam of hadrons of type H_i is equivalent to a beam of constituents (or partons) which are quarks

or gluons. These constituents carry a longitudinal momentum $x_i P_i$ characterized by the parton densities $F_a^{H_i}(x_i, \mu^2)$. That is, given a hadron H_i with momentum P_i, the probability density to find in H_i the parton a with momentum $x_i P_i$ is given by $F_a^{H_i}(x_i, \mu^2)$. Furthermore, these functions are universal in the sense that they are process independent. The parton densities depends also on the so called factorization scale μ^2. This scale is introduced to separate off the non-perturbative part of the cross section (the parton densities) from the perturbative one $\hat{\sigma}_{a(b)}$. This is exactly the cross section where the incoming particles are the partons a (and b) and can be calculated as a perturbative expansion in $\alpha_s(\mu_r^2)$. The parton densities have a mild dependence on the scale μ^2 determined by the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations (see section 1.4). Here, we have chosen the renormalization scale μ_r^2 equal to the factorization scale μ^2 for simplicity. Anyway, in order to reintroduce the scale μ_r^2, we have only to rewrite $\alpha_s(\mu^2)$ in terms of μ_r^2 (see Eq.(41)) and expand it consistently with the order of the calculation. The μ^2 dependence in the parton densities is compensated by the μ^2 dependence in the partonic cross section $\hat{\sigma}$. However, with a fixed-order computaction of the partonic coefficient function at order α_s^k the hadronic cross section will still depend on μ^2 with a dependence which should be of order α_s^{k+1}. Hence, this dependence can be used to estimate the theoretical error of a fixed-order computation. A simple discussion about the dependence of the hadronic cross section on the factorization and renormalization scale is given in Ref.[24].

1.3 NLO DY and DIS cross sections

We consider for simplicity the classical Drell-Yan (DY) hadronic process for the production of a dimuon pair through a virtual photon γ^* (see Figure 1.3):

$$H_1(P_1) + H_2(P_2) = \gamma^*(Q) + X(K), \tag{46}$$

where H_1 and H_2 are the colliding hadrons with momementum P_1 and P_2 respectively, Q is the momentum of the virtual photon and X is any number of additional hadrons with total momentum K. For the process of Eq.(46), we define

$$x \equiv \frac{Q^2}{S}, \tag{47}$$

where $S = (P_1 + P_2)^2$ is the usual Mandelstam invariant, which can be viewed as the hadronic center-of-mass energy. It is clear that Eq.(47) represents the fraction of energy that the hadrons transfer to the photon and, hence, $0 \le x \le 1$.

According to the factorized expression of the QCD cross section Eq.(44), the LO Q^2 differential cross section is given by:

$$\frac{d\sigma}{dQ^2}(x, Q^2) = \sum_i \int_0^1 dx_1 dx_2 \left[q_i(x_1, \mu^2) \bar{q}_i(x_2, \mu^2) + \bar{q}_i(x_1) q_i(x_2, \mu^2) \right] \frac{d\hat{\sigma}_i}{dQ^2}, \tag{48}$$

$$\frac{d\hat{\sigma}_i}{dQ^2} = \sigma_0^{DY}(Q^2, x) Q_{q_i}^2 \delta(x_1 x_2 - x), \quad \sigma_0^{DY}(Q^2, x) = \frac{4\pi\alpha^2}{9Q^4} x \tag{49}$$
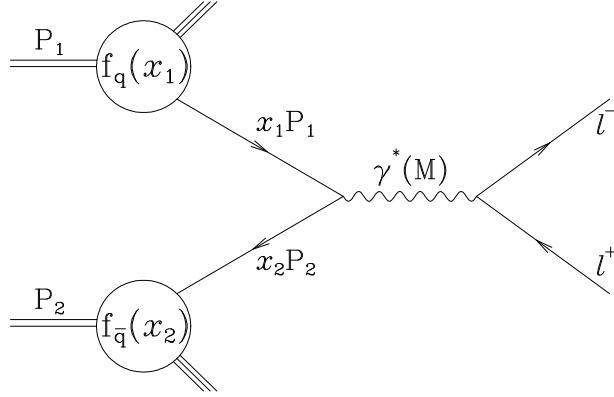
Figure 1.3: Drell-Yan pair production. Here $Q = M$.

where the functions $q_i(x_j, \mu^2)(\bar{q}_i(x_j, \mu^2))$ are the parton densities of a quark (or an anti-quark) of flavor i in the hadron $j = 1, 2$ at the scale μ^2, α is the fine-structure constant and Q_{q_i} is the fraction of electronic charge of the quark q_i. Now, if we define the dimensionaless cross section $\sigma(x, Q^2)$ as,

$$\sigma^{DY}(x, Q^2) \equiv \frac{1}{\sigma_0^{DY}} \frac{d\sigma}{dQ^2}(x, Q^2), \tag{50}$$

and use the identity,

$$\delta(x_1 x_2 - x) = \int_0^1 dz \delta(1 - z)\delta(x_1 x_2 z - x), \tag{51}$$

then Eqs.(48,49) become:

$$\sigma^{DY}(x, Q^2) =$$

$$= \sum_i \int_0^1 dx_1 dx_2 dz \, [q_i(x_1)\bar{q}_i(x_2) + \bar{q}_i(x_1)q_i(x_2)] \, Q_{q_i}^2 C_{qq}(z)\delta(x_1 x_2 z - x) \tag{52}$$

$$= \sum_i \int_x^1 \frac{dx_1}{x_1} \int_{x/x_1}^1 \frac{dx_2}{x_2} \, [q_i(x_1)\bar{q}_i(x_2) + \bar{q}_i(x_1)q_i(x_2))] \, Q_{q_i}^2 C_{qq}\left(\frac{x}{x_1 x_2}\right), \tag{53}$$

$$C_{qq}(z) = \delta(1 - z), \tag{54}$$

where $C_{qq}(z)$ is the LO Drell-Yan coefficient function. From Eq.(52), we see that the new variable z that we have introduced is in general given by

$$z = \frac{x}{x_1 x_2}. \tag{55}$$

This means that at the partonic level, z can be viewed as the fraction of energy that the colliding partons transfer to the virtual photon. At LO it is clear that $z = 1$ as can be explicitly seen from Eq.(54), because there is no emission but the virtual photon.

Beyond the LO the extra radiated partons in the final state can carry away some energy (so $z < 1$) and the gluon channel contributes. The NLO coefficient functions $C_{ab}(z)$ ($a, b = q, g$) recieves contributions that have infrared and ultraviolet. Infrared singularities cancel out (see e.g. [25]). The ultraviolet ones are reabsobed by renormalization of the bare parameters of the QCD Lagrangian, thus defining a renormalized strong coupling constant $\alpha_{\mu_r^2}$ at an arbitraty renormalization scale μ_r^2 (see section 1.2). Collinear divergences are cut off by infrared physics. They can be absorbed multiplicatively in redefinition of the parton densities [26], thus reabsorbing all dependence on soft physics in the parton distributions. The parton densities at a certain scale are determined by a reference process and their scale dependence is determined by the DGLAP equations (see section 1.4). However, there is an ambiguity on how to define the reference process, related to the fact that collinear divergences can always be factorized together with finite terms. The choice of these finite terms defines a factorization scheme. The most common factorization scheme is the $\overline{MS}$ scheme in which the collinear divergence (which is in $d = 4 - 2\epsilon$ dimensions a single pole $1/\epsilon$) is factorized together the finite terms $-\gamma_E + \log 4\pi$, where $\gamma_E = 0.5772...$ is the Euler gamma. Now, in order to avoid the perturbative expansion to receive large contributions, the factorization and the renormalization scales are expected to be chosen of the same order of the scale of the process Q^2. Here, for simplicity, we choose the factorization scale μ^2 equal to the renormalization scale μ_r^2. We report the NLO Drell-Yan cross section (see e.g. [27, 28]):

$$\sigma^{DY}(x, Q^2) = \sum_i Q_{q_i}^2 \int_x^1 \frac{dx_1}{x_1} \int_{x/x_1}^1 \frac{dx_2}{x_2}$$

$$\times \left\{ \left[q_i(x_1, \mu^2)\bar{q}_i(x_2, \mu^2) + (1 \leftrightarrow 2) \right] C_{qq}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) \right.$$

$$\left. + \left[g(x_1, \mu^2)\left(q_i(x_2, \mu^2) + \bar{q}_i(x_2, \mu^2) \right) + (1 \leftrightarrow 2) \right] C_{qg}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) \right\}, \quad (56)$$

where, in the $\overline{MS}$ scheme,

$$C_{qq}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = \delta(1 - z) + \frac{\alpha_s(\mu^2)}{2\pi}\left\{ \frac{4}{3}\left[\left(\frac{2\pi^2}{3} - 8\right)\delta(1 - z) \right.\right.$$

$$\left. + 4(1 + z^2)\left[\frac{\log(1 - z)}{1 - z} \right]_+ - 2\frac{1 + z^2}{1 - z}\log z \right]$$

$$\left. + \frac{8}{3}\left[\frac{1 + z^2}{[1 - z]_+} + \frac{3}{2}\delta(1 - z) \right]\log\left(\frac{Q^2}{\mu^2}\right) \right\}, \quad (57)$$

and

$$C_{qg}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = \frac{\alpha_s(\mu^2)}{2\pi}\left\{ \frac{1}{2}\left[(z^2 + (1 - z)^2)\log\frac{(1 - z)^2}{z} + \frac{1}{2} + 3z \right.\right.$$

$$\left. - \frac{7}{2}z^2 \right] + \frac{1}{2}[z^2 + (1 - z)^2]\log\left(\frac{Q^2}{\mu^2}\right) \right\}, \quad (58)$$
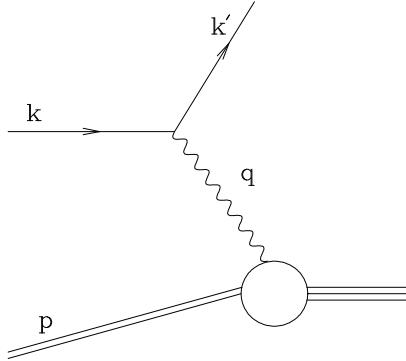
Figure 1.4: Deep inelastic electron-proton scattering

where the "+" distribution is defined as follows:

$$\int_0^1 dz f(z)[g(z)]_+ \equiv \int_0^1 dz[f(z) - f(1)]g(z). \tag{59}$$

Also for the case of the deep-inelastic scattering (DIS), we consider the simplest process in which a high energy electron scatters from a hadron exchanging with it a virtual photon γ^* (see Figure 1.4):

$$H(P) + e(k) \rightarrow e(k') + X(K), \tag{60}$$

where H is typically a proton with momentum P, e is the scattered electron and X is any collection of hadrons. The standard parametrization of DIS is done in terms of three relevant parameters:

$$Q^2 \equiv -q^2 \equiv -(k - k')^2 \tag{61}$$

$$y \equiv \frac{P \cdot q}{P \cdot k}; \quad 0 \leq y \leq 1 \tag{62}$$

$$x \equiv x_{Bj} = \frac{Q^2}{2P \cdot q} = \frac{Q^2}{(P + q)^2 + Q^2}; \quad 0 \leq x \leq 1, \tag{63}$$

where in the last line we have neglected the proton mass. Q^2 is the virtuality of the photon exchanged between the electron and the proton and y is the fraction of energy that the incoming electron transfer to the proton. The Bjorken scaling variable x has a simple physical interpretation: it is the fraction of longitudinal momentum of the LO incoming quark of the partonic subprocess.

Indeed, the most general parametrization of the Q^2 differential cross section is given by:

$$\frac{d\sigma}{dQ^2}(x, Q^2, y) = \frac{4\pi\alpha^2}{Q^4} \left[[1 + (1 - y)^2]F_1(x, Q^2) + \frac{(1 - y)}{x}(F_2(x, Q^2) - 2xF_1(x, Q^2)) \right], \tag{64}$$

The functions $F_{1(2)}$ are called structure functions and contains the information about the structure of the proton. In fact they are determined by the photon-proton sub-process in this way:

$$F_1(x, Q^2) = \frac{Q^2}{16\pi^2\alpha x}\left[\sigma_\Sigma(\gamma^* P) + \sigma_L(\gamma^* P)\right] \tag{65}$$

$$F_2(x, Q^2) = 2xF_1(x, Q^2) + F_L(x, Q^2) \tag{66}$$

$$F_L(x, Q^2) = \frac{Q^2}{4\pi^2\alpha}\sigma_L(\gamma^* P), \tag{67}$$

where $\sigma_\Sigma(\gamma^* P)$ and $\sigma_\Sigma(\gamma^* P)$ are the cross sections of the photon-proton process determined summing over all the virtual photon polarization and over only the longitudinal one respectively. At LO

$$F_1(x, Q^2) = \frac{1}{2}\sum_i Q_{q_i}^2[q_i(x, Q^2) + \bar{q}_i(x, Q^2)] \tag{68}$$

$$F_2(x, Q^2) = 2xF_1(x, \mu^2), \tag{69}$$

where q_i and $\bar{q}_i$ are the parton densities. In general the structure functions should depend on both x and Q^2, because these are the relevant kinematic variable of the photon-proton sub-process. Now we want, as we have done for the Drell-Yan case, rewrite the structure functions in terms of parton densities and of a coefficient function that can be computed in perturbative QCD. If we use the identity

$$\delta(y - x) = \int_0^1 dz\, \delta(1 - z)\delta(yz - x), \tag{70}$$

we have for the LO structure functions F_2 and F_L:

$$F_2(x, Q^2) = x\sum_i \int_0^1 dy dz\, \left[q_i(y, \mu^2) + \bar{q}_i(y, \mu^2)\right] Q_{q_i}^2 C_q(z)\delta(yz - x) \tag{71}$$

$$= x\sum_i \int_x^1 \frac{dy}{y}\left[q_i(y, \mu^2) + \bar{q}_i(y, \mu^2)\right] Q_{q_i}^2 C_q\left(\frac{x}{y}\right) \tag{72}$$

$$C_q(z) = \delta(1 - z) \tag{73}$$

$$F_L(x, Q^2) = 0, \tag{74}$$

where $C_q(z)$ is the LO DIS coefficient function for F_2. From Eq.(71), we see that the variable z is in general given by

$$z = \frac{x}{y}. \tag{75}$$

This means that at the partonic level, z can be viewed as the longitudinal momentum of the incoming parton before it scatters with the virtual photon. At LO it is clear that $z = 1$ as can be explicitly seen from Eq.(73), because there is no extra emissions.

Beyond the LO the extra radiated partons in the final state can carry some energy (so $z < 1$) and also the gluon channel contributes. At NLO we have ultraviolet, infrared and collinear singularities. They must be regularized and treated as in the

Drell-Yan case. We report the NLO structure functions (see e.g. [27, 28]) with the renormalization scale equal to the factorization scale:

$$
F_2(x, Q^2) = x \sum_i Q_{q_i}^2 \int_x^1 \frac{dy}{y} \left\{ \left[q_i(y, \mu^2) + \bar{q}_i(y, \mu^2) \right] C_q \left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) \right\}
$$
$$
+ x \sum_i Q_{q_i}^2 \int_x^1 \frac{dy}{y} \, g(y, \mu^2) C_g \left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) \right\} \tag{76}
$$

where, in the $\overline{MS}$ scheme,

$$
C_q \left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) = \delta(1-z) + \frac{\alpha_s(\mu^2)}{2\pi} \left\{ \frac{4}{3} \left[2 \left[\frac{\ln(1-z)}{1-z} \right]_+ - \frac{3}{2} \left[\frac{1}{1-z} \right]_+ \right. \right.
$$
$$
- (1+z) \ln(1-z) - \frac{1+z^2}{1-z} \ln z - \left(\frac{\pi^2}{3} + \frac{9}{2} \right) \delta(1-z)
$$
$$
\left. + 3 + 2z \right] + \frac{4}{3} \left[\frac{1+z^2}{[1-z]_+} + \frac{3}{2} \delta(1-z) \right] \log \left(\frac{Q^2}{\mu^2} \right) \right\}, \tag{77}
$$

$$
C_g \left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) = \frac{\alpha_s(\mu^2)}{2\pi} \left\{ \frac{1}{2} \left[\left((1-z)^2 + z^2 \right) \ln \left(\frac{1-z}{z} \right) - 8z^2 \right. \right.
$$
$$
\left. \left. + 8z - 1 + [z^2 + (1-z)^2] \log \left(\frac{Q^2}{\mu^2} \right) \right] \right\} \tag{78}
$$

and

$$
F_L(x, Q^2) = x \sum_i Q_{q_i}^2 \int_x^1 \frac{dy}{y} \left[q_i(y, \mu^2) + \bar{q}_i(y, \mu^2) \right] \frac{\alpha_s(\mu^2)}{2\pi} \left\{ \frac{8}{3} z \right.
$$
$$
+ \frac{4}{3} \left[\frac{1+z^2}{[1-z]_+} + \frac{3}{2} \delta(1-z) \right] \log \left(\frac{Q^2}{\mu^2} \right) \right\}
$$
$$
+ x \sum_i Q_{q_i}^2 \int_x^1 \frac{dy}{y} \, g(y, \mu^2) \frac{\alpha_s(\mu^2)}{2\pi} \left\{ 2z(1-z) \right.
$$
$$
\left. + \frac{1}{2} [z^2 + (1-z)^2] \log \left(\frac{Q^2}{\mu^2} \right) \right\}, \tag{79}
$$

is factorization scheme independent at the lowest non-vanishing order.

1.4 The DGLAP equations

The coefficient function and the parton densities depend on the factorization scale in such a way that the resulting hadronic cross section is μ^2-independent. The equations that fix the μ^2-dependence of parton densities (the so called Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations) can be found imposing the μ^2-independence of the DY cross section or of the DIS structure functions. For example,

imposing this condition to the explicit expression for the NLO F_2 (see Eqs.(76,77,78), we find the LO DGLAP evolution equations for the quark parton densities:

$$\mu^2 \frac{\partial q_i(x, \mu^2)}{\partial \mu^2} = \frac{\alpha_s(\mu^2)}{4\pi} \left\{ \sum_j \int_x^1 \frac{dy}{y} \left[P_{q_i q_j}^{(0)} \left(\frac{x}{y} \right) q_j(y, \mu^2) + P_{q_i \bar{q}_j}^{(0)} \left(\frac{x}{y} \right) \bar{q}_j(y, \mu^2) \right] \right.$$

$$\left. + \int_x^1 \frac{dy}{y} P_{q_i g}^{(0)} \left(\frac{x}{y} \right) g(y, \mu^2) \right\} + O(\alpha_s^2), \tag{80}$$

where

$$P_{q_i q_j}^{(0)}(z) = \delta_{ij} P_{qq}^{(0)}(z), \tag{81}$$

$$P_{q_i \bar{q}_j}^{(0)}(z) = 0 \tag{82}$$

$$P_{qq}^{(0)}(z) = \frac{8}{3} \left[\frac{1 + z^2}{[1 - z]_+} + \frac{3}{2} \delta(1 - z) \right], \tag{83}$$

and

$$P_{q_i g}^{(0)}(z) = \frac{1}{N_f} P_{qg}^{(0)}(z) = z^2 + (1 - z)^2, \tag{84}$$

with N_f the number of active flavors. The functions $P_{pp'}^{(0)}(z)$ are called LO splitting functions. They can be viewed as the probability per unit of $\ln(\mu^2/Q^2)$ to find a parton p in a parton p'. The LO evolution equation for the gluon can be calculated from the LO splitting diagrams for a quark into another quark and a gluon and for a gluon into two gluons. Furthermore, we simplify the notation introducing the convolution product $\otimes$, defined in this way:

$$(f_1 \otimes f_2 \otimes \cdots \otimes f_n)(x) = \int_0^1 dx_1 dx_2 \cdots dx_n f_1(x_1) f_2(x_2) \cdots f_n(x_n) \delta(x_1 x_2 \cdots x_n - x). \tag{85}$$

We report here the full result for the DGLAP evolution equations:

$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} q_i(z, \mu^2) \\ g(z, \mu^2) \end{pmatrix} = \sum_{q_j, \bar{q}_j} \begin{pmatrix} P_{q_i q_j}(z, \mu^2) & P_{q_i g}(z, \mu^2) \\ P_{g q_j}(z, \mu^2) & P_{gg}(z, \mu^2) \end{pmatrix} \otimes \begin{pmatrix} q_j(z, \mu^2) \\ g(z, \mu^2) \end{pmatrix}, \tag{86}$$

where q_i can be also a quark or anti-quark and where the splitting functions $P_{pp'}$ have the following perturbative expansion:

$$P_{q_i q_j}(z, \mu^2) = P_{\bar{q}_i \bar{q}_j}(z, \mu^2) = \frac{\alpha_s(\mu^2)}{4\pi} \delta_{ij} P_{qq}^{V\,(0)}(z)$$

$$+ \sum_{k=1}^{\infty} \left(\frac{\alpha_s(\mu^2)}{4\pi} \right)^{k+1} \left(\delta_{ij} P_{qq}^{V\,(k)}(z) + P_{qq}^{S\,(k)}(z) \right), \tag{87}$$

$$P_{q_i \bar{q}_j}(z, \mu^2) = P_{\bar{q}_i q_j}(z, \mu^2) = \left(\frac{\alpha_s(\mu^2)}{4\pi^2} \right)^2 \left(\delta_{ij} P_{q\bar{q}}^{V\,(1)} + P_{q\bar{q}}^{S\,(1)} \right)$$

$$+ \sum_{k=1}^{\infty} \left(\frac{\alpha_s(\mu^2)}{4\pi} \right)^{k+2} \left(\delta_{ij} P_{q\bar{q}}^{V\,(k+1)}(z) + P_{q\bar{q}}^{S\,(k+1)}(z) \right), \tag{88}$$

$$P_{q_i g}(z, \mu^2) \quad = \quad P_{\bar{q}_i g}(z, \mu^2) = \frac{1}{N_f} \sum_{k=1}^{\infty} \left(\frac{\alpha_s(\mu^2)}{4\pi} \right)^k P_{qg}^{(k-1)}, \tag{89}$$

$$P_{g q_i}(z, \mu^2) \quad = \quad P_{g \bar{q}_i} = \sum_{k=1}^{\infty} \left(\frac{\alpha_s(\mu^2)}{4\pi} \right)^k P_{gq}^{(k-1)}, \tag{90}$$

$$P_{gg}(z, \mu^2) \quad = \quad \sum_{k=1}^{\infty} \left(\frac{\alpha_s(\mu^2)}{4\pi} \right)^k P_{gg}^{(k-1)}, \tag{91}$$

where N_f is the number of active flavors. Eq.(86) represents a system of $2N_f + 1$ integro-differential equations equations. The solution to this system however can be calculated analytically for a certain fixed-order. In fact it can be translated into a system of ordinary differential equation performing a Mellin transform:

$$F_p(N, \mu^2) \quad = \quad \int_0^1 dz\, z^{N-1} F_p(z, \mu^2), \tag{92}$$

$$\mu^2 \frac{\partial F_p(N, \mu^2)}{\partial \mu^2} \quad = \quad \sum_{p'} \gamma_{pp'}^{AP}(N, \mu^2) F_{p'}(N, \mu^2), \tag{93}$$

where $p, p' = q_i, \bar{q}_j, g$ and

$$\gamma_{pp'}^{AP}(N, \mu^2) = \int_0^1 dz P_{pp'}(z, \mu^2). \tag{94}$$

After that, these equations can be decoupled searching linear combinations of parton densities that depends on the independent splitting functions of Eqs.(68-77) and that diagonalize the system. For example, at LO, there are 4 independent splitting functions which are $P_{qq}^{V\,(0)} = P_{qq}^{(0)}$, $P_{qg}^{(0)}$ (given in Eqs.(83,84) respectively) and

$$P_{gq}^{(0)}(z) \quad = \quad \frac{8}{3} \left[\frac{1 + (1-z)^2}{z} \right] \tag{95}$$

$$P_{gg}^{(0)}(z) \quad = \quad 12 \left[\frac{z}{[1-z]_+} + \frac{1-z}{z} + z(1-z) \right]$$
$$+ \left(\frac{44}{9} - \frac{2}{3} N_f \right) \delta(1-z). \tag{96}$$

At NLO there are 6 independent splitting functions which are for example $P_{qq}^{V}(z)$, $P_{qq}^{S}(z)$, $P_{q\bar{q}}^{V}(z)$, $P_{qg}(z)$, $P_{gq}(z)$ and $P_{gg}(z)$. They are given in Ref.[27] pages 111 and 112. The LO and the NLO solution to the DGLAP equations (93) in Mellin space is computed in the next Section.

1.5   NLO solution of the DGLAP evolution equations

In this Section, we want to solve the NLO DGLAP equations (Eq.(93) of section 1.4):

$$\mu^2 \frac{\partial F_p(N, \mu^2)}{\partial \mu^2} = \sum_{p'} \gamma_{pp'}(N, \mu^2) F_{p'}(N, \mu^2), \tag{97}$$

where all the splitting functions defined here (and in the following) have the same perturbative expansion:

$$\gamma_{pp'}(N, \mu^2) = \frac{\alpha_s(\mu^2)}{4\pi}\gamma_{pp'}^{(0)}(N) + \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2 \gamma_{pp'}^{(1)}(N) + O(\alpha_s^3). \tag{98}$$

This is a system of $2N_f + 1$ coupled equations with N_f the number of active flavors. At NLO, there are 6 independent splitting functions defined through the following equations:

$$\gamma_{gq_i} = \gamma_{g\bar{q}_i} \equiv \gamma_{gq} \tag{99}$$

$$\gamma_{q_i g} = \gamma_{\bar{q}_i g} \equiv 1/N_f \gamma_{qg} \tag{100}$$

$$\gamma_{q_i q_k} = \gamma_{\bar{q}_i \bar{q}_k} \equiv \delta_{ik}\gamma_{qq}^V + \gamma_{qq}^S \tag{101}$$

$$\gamma_{q_i \bar{q}_k} = \gamma_{\bar{q}_i q_k} \equiv \delta_{ik}\gamma_{q\bar{q}}^V + \gamma_{qq}^S \tag{102}$$

$$\gamma_{gg} \equiv \gamma_{gg}, \tag{103}$$

where i, k are a flavor index. We omit the dependence on N and μ^2 for brevity of notation. Note that beyond the NLO, there is one more independent splitting function. In fact in Eq.(102) we should substitute γ_{qq}^S with $\gamma_{q\bar{q}}^S$ which are different beyond the NLO [29, 30]. We note also that at LO $\gamma_{qq}^S = \gamma_{q\bar{q}}^S = \gamma_{q\bar{q}}^V = 0$ and hence at LO there are only 4 independent splitting functions.

Now, we define the $2N_f - 1$ so called non-singlet (NS) combinations

$$q_{(NS)k}^\pm = \sum_{i=1}^{k}(q_i \pm \bar{q}_i) - k(q_k \pm \bar{q}_k); \quad k = 2, \dots, N_f \tag{104}$$

$$q_{(NS)}^V = \sum_{i=1}^{N_f}(q_i - \bar{q}_i) \tag{105}$$

and the 2 so called singlet (S) combinations: g and

$$q_{(S)} = \sum_{i=1}^{N_f}(q_i + \bar{q}_i). \tag{106}$$

With this definitions, from Eq.(97) and Eqs.(99-103), we find that for the non-singlet combinations

$$\mu^2\frac{\partial q_{(NS)k}^\pm}{\partial\mu^2} = \gamma^\pm q_{(NS)k}^\pm \tag{107}$$

$$\mu^2\frac{\partial q_{(NS)}^V}{\partial\mu^2} = \gamma^- q_{(NS)}^V, \tag{108}$$

where

$$\gamma^\pm = \gamma_{qq}^V \pm \gamma_{q\bar{q}}^V. \tag{109}$$

For the 2 remaining singlet combinations, we find in the same way that

$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} q_{(S)} \\ g \end{pmatrix} = \begin{pmatrix} \gamma_{qq} & \gamma_{qg} \\ \gamma_{gq} & \gamma_{gg} \end{pmatrix} \begin{pmatrix} q_{(S)} \\ g \end{pmatrix}, \tag{110}$$

where

$$\gamma_{qq} = \gamma^+ + \gamma_{PS}, \qquad \gamma_{PS} \equiv 2N_f \gamma_{q\bar{q}}^S. \tag{111}$$

The NLO Mellin splitting functions can be found in Ref.[31] written in terms of harmonic sums. In many cases, however, their analytic continuation to all the complex plane is useful (see e.g. [32, 33]) For the NNLO solution of the DGLAP equations and the NNLO splitting functions we refer to [29, 30]. The techniques for the analytic continuations of the NNLO splitting functions can be found in Ref.[34].

For the NS combinations, the solution is easy to obtain. Indeed, making the change of variable

$$\frac{d\mu^2}{\mu^2} = \frac{d\alpha_s(\mu^2)}{\beta(\alpha_s(\mu^2))}, \tag{112}$$

where $\beta(\alpha_s)$ is the β function defined in section 1.2, we get:

$$\frac{q_{(NS)k}^\pm(\mu^2)}{q_{(NS)k}^\pm(\mu_0^2)} = \left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} \right)^{-\gamma^{(0)\pm}/\beta_0} \left[1 + \left(\frac{\gamma^{(1)\pm}}{\beta_0} - \frac{\beta_1 \gamma^{(0)\pm}}{\beta_0^2} \right) \left(\frac{\alpha_s(\mu_0^2)}{4\pi} - \frac{\alpha_s(\mu^2)}{4\pi} \right) \right] \tag{113}$$

and

$$\frac{q_{(NS)}^V(\mu^2)}{q_{(NS)}^V(\mu_0^2)} = \left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} \right)^{-\gamma^{(0)-}/\beta_0} \left[1 + \left(\frac{\gamma^{(1)-}}{\beta_0} - \frac{\beta_1 \gamma^{(0)-}}{\beta_0^2} \right) \left(\frac{\alpha_s(\mu_0^2)}{4\pi} - \frac{\alpha_s(\mu^2)}{4\pi} \right) \right], \tag{114}$$

where we have omitted the N-dependence of the splitting functions for brevity. For the S combinations, some linear algebra is needed. We, first, define the singlet vector and the splitting matrix:

$$\vec{q}_S \equiv \begin{pmatrix} q_{(S)} \\ g \end{pmatrix}, \qquad \tilde{\gamma}_S \equiv \begin{pmatrix} \gamma_{qq} & \gamma_{qg} \\ \gamma_{gq} & \gamma_{gg} \end{pmatrix}. \tag{115}$$

Using the NLO splitting matrix and the change of variable Eq.(112), we find immeditely the formal solution, which is

$$\vec{q}_S(\mu^2) = \exp \left\{ -R_0 \ln \frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} + R_1 \left(\frac{\alpha_s(\mu_0^2)}{4\pi} - \frac{\alpha_s(\mu^2)}{4\pi} \right) \right\} \vec{q}_S(\mu_0^2), \tag{116}$$

where

$$R_0 = \frac{\tilde{\gamma}_S^{(0)}}{\beta_0}, \qquad R_1 = \frac{\tilde{\gamma}_S^{(1)}}{\beta_0} - \frac{\beta_1 \tilde{\gamma}_S^{(0)}}{\beta_0^2}. \tag{117}$$

The two matrices R_0 and R_1 in Eq.(116) cannot be diagonalized simultaneously, as they do not commute. Hence, in order to extract the NLO solution from Eq.(116), we use the following Ansatz:

$$\vec{q}_S(\mu^2) = U(\alpha_s(\mu^2)) \left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} \right)^{-R_0} U^{-1}(\alpha_s(\mu_0^2)) \vec{q}_S(\mu_0^2), \tag{118}$$

where the matrix U has the perturbative expansion:

$$U(\alpha_s(\mu^2)) = 1 + \frac{\alpha_s(\mu^2)}{4\pi}U_1 + O(\alpha_s^2).\tag{119}$$

The condition that the matrix U_1 should satisfy can be easily obtained imposing that the derivative with respect to $\alpha_s(\mu^2)$ of Eq.(116) and of Eq.(118) are equal at NLO. Thus, we get

$$[U_1, R_0] = U_1 + R_1.\tag{120}$$

We write R_0 in terms of its 2 eigenvalues

$$R^\pm = \frac{1}{2\beta_0}\left[(\gamma_{qq}^{(0)} + \gamma_{gg}^{(0)}) \pm \sqrt{(\gamma_{qq}^{(0)} - \gamma_{gg}^{(0)})^2 + 4\gamma_{qg}^{(0)}\gamma_{gq}^{(0)}}\right]\tag{121}$$

and of the 2 corresponding eigenspaces projectors $P_\pm$:

$$R_0 = R^+ P_+ + R^- P_-.\tag{122}$$

The explicit expression for the projectors can be obtained using the completeness relation $P_+ + P_- = 1$. We find

$$P_\pm = \frac{1}{R^\pm - R^\mp}[R_0 - R^\mp].\tag{123}$$

Now, writing the matrices U_1 and R_1 in terms of these projectors

$$\begin{aligned}
U_1 &= P_-U_1P_- + P_-U_1P_+ + P_+U_1P_- + P_+U_1P_+ &(124)\\
R_1 &= P_-R_1P_- + P_-R_1P_+ + P_+R_1P_- + P_+R_1P_+, &(125)
\end{aligned}$$

substituting them and Eq.(122) into Eq.(120) and comparing each matrix element, we find

$$U_1 = -(P_-R_1P_- + P_+R_1P_+) + \frac{P_+R_1P_-}{R^- - R^+ - 1} + \frac{P_-R_1P_+}{R^+ - R^- - 1}.\tag{126}$$

Thanks to this result, we can now write the NLO solution of the singlet doublet in a form which is useful for practical calculations. Indeed, if we substitute Eq.(126) in Eq.(118), we get (at NLO)

$$\begin{aligned}
\vec{q}_S(\mu^2) &= \left\{\left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)^{-R_-}\left[P_- + \left(\frac{\alpha_s(\mu_0^2)}{4\pi} - \frac{\alpha_s(\mu^2)}{4\pi}\right)P_-R_1P_-\right.\right.\\
&\quad\left.\left. - \left(\frac{\alpha_s(\mu_0^2)}{4\pi} - \frac{\alpha_s(\mu^2)}{4\pi}\left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)^{R^- - R^+}\right)\frac{P_-R_1P_+}{R^+ - R^- - 1}\right]\right.\\
&\quad\left. + (+ \leftrightarrow -)\right\}\vec{q}_S(\mu_0^2).
\end{aligned}\tag{127}$$

After the evolution of the NS and S combinations has been performed from a certain scale μ_0^2 to the scale μ^2, we need to return to the parton distributions for all the quarks but the gluon. These are obtained straightforwardly with the following relations

$$
q_k + \bar{q}_k = \frac{1}{N_f} q_S - \frac{1}{k} q_{(NS)k}^+ + \sum_{i=k+1}^{N_f} \frac{1}{i(i-1)} q_{(NS)i}^+, \qquad k = 1, \ldots, N_f \qquad (128)
$$

$$
q_k - \bar{q}_k = \frac{1}{N_f} q_{(NS)}^V - \frac{1}{k} q_{(NS)k}^- + \sum_{i=k+1}^{N_f} \frac{1}{i(i-1)} q_{(NS)i}^-, \qquad k = 1, \ldots, N_f. \quad (129)
$$

However, Eqs.(113,114,127) represent the NLO solution of the DGLAP equations Eq.(97) in the case when the number of active flavors N_f has been kept fixed. This is the so called fixed flavor scheme solution. If we want to take into account the thresholds of the heavy quark flavors, we can evolve up the NS a S combinations from the scale μ_0^2 (with a certain number N_f of active flavors) to the scale of production of a new flavor. Then, we can take the result of this evolution as the starting point of a second evolution (with $N_f + 1$ active flavors this time) above the production scale of the new flavor, assuming that the new flavor vanishes at threshold. This is the most simple way to generate dynamically a new flavor.

Finally, we note that the procedure outlined in this appendix can be easily generalized beyond the NLO order [35]. Furthermore, in many cases, it is interesting to study the dependence on the renormalization scale, in order to estimate the theoretical error of the evolution. Here the renormalization scale μ_r^2 has been chosen equal to the factorization one μ^2 for simplicity. To restore the implicit μ_r^2-dependence in parton densities, we need only to rewrite the running coupling constant $\alpha_s(\mu^2)$ in terms of μ_r^2 (see Eq.(41) in section 1.2) in the splitting functions. Making this substitution, we have that the perturbative expansion of a generic splitting function Eq.(130) becomes

$$
\gamma_{pp'}(N, \mu^2, k') = \frac{\alpha_s(k'\mu^2)}{4\pi} \gamma_{pp'}^{(0)}(N) + \left(\frac{\alpha_s(k'\mu^2)}{4\pi} \right)^2 \left(\gamma_{pp'}^{(1)}(N) + \beta_0 \gamma_{pp'}^{(0)} \ln k' \right) + O(\alpha_s^3),
$$
$$(130)$$

where $k' = \mu_r^2/\mu^2$. Hence, in Eqs.(113,114,127), we should perform the following substitutions:

$$
\gamma^{(1)} \to \gamma^{(1)} + \beta_0 \gamma^{(0)} \ln k', \quad \alpha_s(\mu^2) \to \alpha_s(k'\mu^2), \quad \alpha_s(\mu_0^2) \to \alpha_s(k'\mu_0^2) \qquad (131)
$$

and use $k'\mu^2$ as reference scale for new flavors production.

Chapter 2

High order QCD and resummation

2.1 When is NLO not enough?

In section 1.3, we have discussed briefly the analytic NLO calculation of the full inclusive DIS and DY cross sections. However, in many cases, the NLO pQCD computation turns out not to be enough. This is, for example, often the case at LHC where the Higgs boson production has to be distinguished from the background. A computation beyond the NLO is needed also when the NLO corrections are large and higher-order calculation permit us to test the convergence of the perturbative expansion. In figure 2.1 the total cross section of the production of the Higgs boson at LHC [36] is plotted and we note convergence in going from LO to NLO and to NNLO.

This can also happen when a new parton level subprocess first appear at NLO. This is the case for example for the rapidity DY distributions at Tevatron (shown in
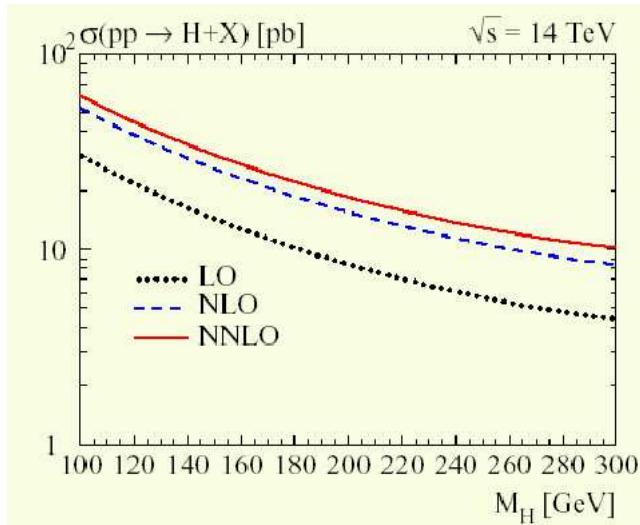
Figure 2.1: Total cross section for the Higgs boson production at LHC at (from bottom to top) at LO, NLO, NNLO in the gluon fusion channel [36].
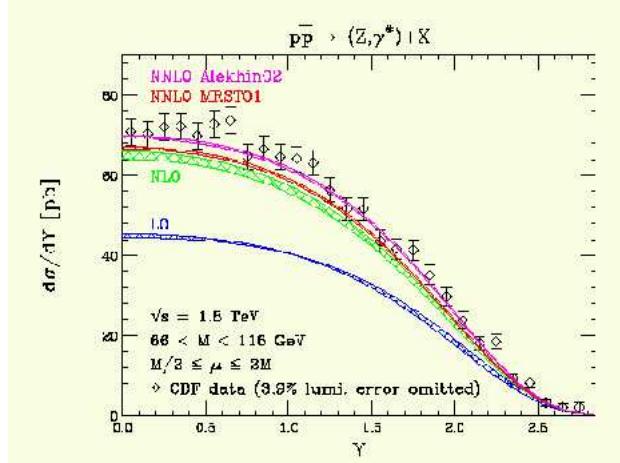
Figure 2.2: DY rapidity distribution for proton anti-proton collisions at Tevatron at (from bottom to top) LO, NLO, NNLO, together with the CDF data [37].
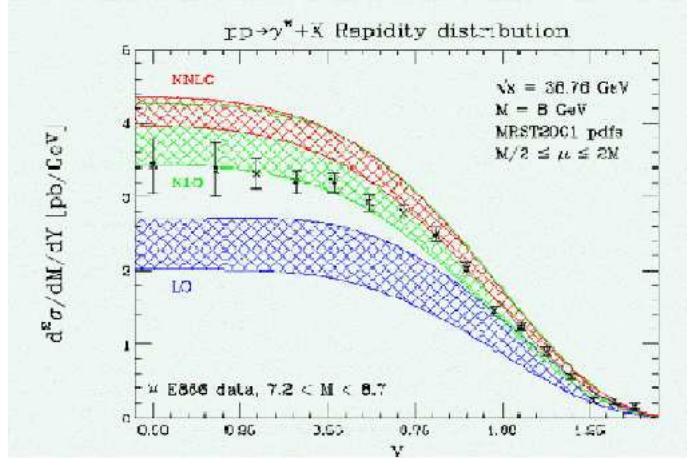
Figure 2.3: DY rapidity distribution for proton proton collisions at fixed-target experiment E866/NuSea at (from bottom to top) LO, NLO, NNLO, together with the data [19, 38].

figure 5.4) and at the fixed-target experiment E866/NuSea (shown in figure 2.3). The agreement with the data of figure 5.4 has represented an important test of the NNLO splitting functions [29, 30]. We note also that going from the LO to the NNLO the factorization scale dependence is significantly reduced.

Calculations beyond the NLO can be important also in processes which involve large logaritms when different significant scales appear. In these cases, these large logarithms should be resummed and this is the topic of this thesis. A first example of these large logarithms has appeared in section 1.3. In fact, from Eqs.(57,77)of section 1.3, we see that there are contributions that become large when $z \to 1$ from the quark-antiquark channel in the DY case and from the quark channel for the structure

function F_2 in the DIS case. These are the terms proportional to

$$\alpha_s \left[\frac{\log(1-z)}{1-z}\right]_+ , \quad \alpha_s \left[\frac{1}{1-z}\right]_+ . \tag{1}$$

The terms of the type of Eq.(1) arise from the infrared cancellation between virtual and real emissions. It can be shown that enhanced contributions of the same type arise at all orders. In fact, at order $O(\alpha_s^n)$ there are contributions proportional to [1, 2, 3]:

$$\alpha_s^n \left[\frac{\log^m(1-z)}{1-z}\right]_+ , \quad m \le 2n - 1. \tag{2}$$

These terms become important in the limit $z \to 1$ spoiling the validity of the perturbative fixed-order QCD expansion and, thus, should be resummed to all-orders of pQCD.

The limit $z \to 1$ corresponds in general to the kinematic boundary where emitted partons are all soft (as it happens in the DY case) or collinear (as in the DIS and the prompt photon case as we shall see in section 4.3).

In fact, in the DY case, if we consider a contribution to the coefficient function with n radiated extra partons with momenta $k_1, \ldots, k_n$, the squaring of four-momentum conservation ($p_1 + p_2 = Q + k_1 + \cdots + k_n$) implies

$$x_1 x_2 S(1-z) = \sum_{i,j=1}^{n} k_i \cdot k_j + 2\sum_{i=1}^{n} Q \cdot k_i \tag{3}$$

$$= \sum_{i,j=1}^{n} k_i^0 k_j^0 (1 - \cos\theta_{ij}) + 2\sum_{i=1}^{n} k_i^0 (\sqrt{Q^2 + |\vec{Q}|^2} - |\vec{Q}|\cos\theta_i), \tag{4}$$

where θ_{ij} is the angle between $\vec{k}_i$ and $\vec{k}_j$ and θ_i is the angle between $\vec{k}_i$ and $\vec{Q}$. Since all the terms in the first sum of Eq.(4) are positive semi-definite and the terms $(\sqrt{Q^2 + |\vec{Q}|^2} - |\vec{Q}|\cos\theta_i)$ in the second sum are positive for all possible values of θ_i, we have that the limit $z = 1$ is achieve only for $k_i^0 = 0$ for all i. This means that when z approaches 1 all the emitted partons in the Drell-Yan processes are soft and that we have reached the threshold for the production of a virtual photon or a real vector boson.

In the DIS case, at the partonic level, we have

$$p + q = k_1 + \cdots + k_n + k_{n+1}, \tag{5}$$

where k_{n+1} is the LO outgoing parton. If we square this last equation, we get

$$\frac{Q^2(1-z)}{z} = \sum_{i,j=1}^{n+1} k_i^0 k_j^0 (1 - \cos\theta_{ij}), \tag{6}$$

where θ_{ij} is the angle between $\vec{k}_i$ and $\vec{k}_j$. Eq.(6) tells us that in the $z \to 1$ limit, there can be not only soft radiated partons in the final state, but there can be also a set
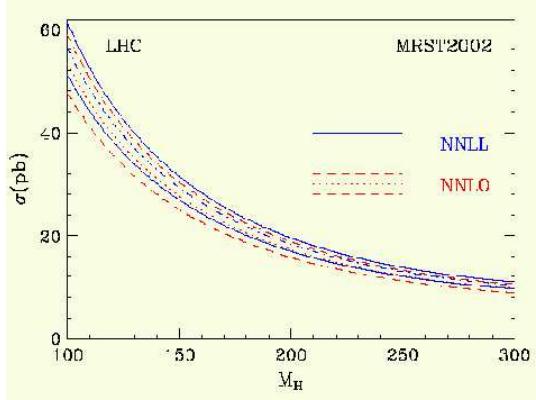
Figure 2.4: Total cross section for the Higgs boson production at LHC at (from bottom to top) at NNLO and NNLO improved with NNLL resummation in the gluon fusion channel [39].
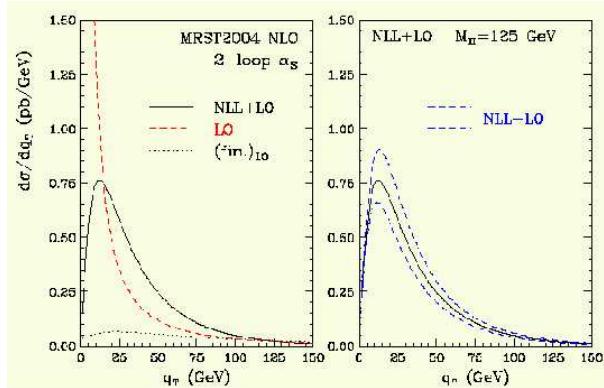
Figure 2.5: Total cross section for the transverse momentum Higgs boson production at LHC at LO and LO improved with NLL resummation in the gluon fusion channel [40].

of partons collinear to each other. However, in Section 3.2 we will show with a more detailed analysis of the DIS kinematics and phase space that the collinear partons are also soft in the $z \to$ limit for the deep-inelastic process.

An example of the impact of resummation can be seen in figure 2.4. There, the total cross section for the Higgs production at LHC is plotted at NNLO with its NNLL resummation improvement improvement [39]. The scale uncertainty reduced to about 15% at NNLO is further reduced to 10% by the NNLL resummation. The resummed large logaritms in this case are of the class of the DY-like soft emissions.

Resummation of another class of large logarithms plays a crucial role in transverse momentum distributions. Indeed, in figure 2.5, we observe that resummation changes substantially the behavior of the cross section for the production of the Higgs boson at small transverse momentum. In these case the large logarithms of $q_\perp^2/M_H^2$ with M_H the Higgs mass are resummed.

The state of art of QCD predictions for Higgs boson production at LHC is reported in figure 2.6 as it was summarized by Laura Reina at the CTEQ summer shool 2006

process	$\sigma_{NLO,NNLO}$ (by)
$gg \to H$ HIGLU MCFM MC@NLO	S.Dawson, NPB 359 (1991), A.Djouadi, M.Spira, P.Zerwas, PLB 264 (1991) C.J.Glosser et al., JHEP (2002); V.Ravindran et al., NPB 634 (2002) D. de Florian et al., PRL 82 (1999) R.Harlander, W.Kilgore, PRL 88 (2002) (NNLO) C.Anastasiou, K.Melnikov, NPB 646 (2002) (NNLO) V.Ravindran et al., NPB 665 (2003) (NNLO) S.Catani et al. JHEP 0307 (2003) (NNLL) G.Bozzi et al., PLB 564 (2003), NPB 737 (2006) (NNLL)
$q\bar{q} \to (W,Z)H$	T.Han, S.Willenbrock, PLB 273 (1991) O.Brien, A.Djouadi, R.Harlander, PLB 579 (2004) (NNLO)
$q\bar{q} \to q\bar{q}H$	T.Han, G.Valencia, S.Willenbrock, PRL 69 (1992) T.Figy, C.Oleari, D.Zeppenfeld, PRD 68 (2003)
$qq, gg \to t\bar{t}H$	W.Beenakker et al., PRL 87 (2001), NPB 653 (2003) S.Dawson et al., PRL 87 (2001), PRD 65 (2002), PRD 67,68 (2003)
$q\bar{q}, gg \to b\bar{b}H$	S.Dittmaier, M.Krämer, M.Spira, PRD 70 (2004) S.Dawson et al., PRD 69 (2004), PRL 94 (2005)
$gb(b) \to b(b)H$ MCFM	J.Cambell et al., PRD 67 (2003)
$bb \to (bb)H$ MCFM	D.A.Dicus et al., PRD 59 (1999); C.Balasz et al., PRD 60 (1999), R.Harlander, W.Kilgore, PRD 68 (2003) (NNLO)

Figure 2.6: State of art of QCD predictions for Higgs boson production at hadron colliders.

on QCD analysis and phenomenology, where also the Monte Carlo event generators are indicated.

Furthermore, at LHC, multi-particles/jet production will be the inescapable background to Higgs searches and searches for new physics. This means that we should have a precise knowledge of the QCD background. As seen previously, we know many QCD processes up to the NNLO. However, we have at the moment limited NLO knowledge of some important final states that will constitute background. They are

$$\to \quad W/Z + \text{jets} \quad (2j)$$
$$\to \quad WW/ZZ/WZ + \text{jets} \quad (0j)$$
$$\to \quad WWW/ZZZ/WZZ + \text{jets} \quad (0j)$$
$$\to \quad Q\bar{Q} + \text{jets} \quad (0j)$$
$$\to \quad \gamma + \text{jets} \quad (1j)$$
$$\to \quad \gamma\gamma + \text{jets}$$
$$\to \quad Z\gamma\gamma + \text{jets},$$

where in parenthesis is indicated the NLO knowledge.

Finally, we also note that in higher order contributions to the splitting functions (P_{gg}^1, P_{gq}^1 for example), it can be shown that there can appear also terms proportional to

$$\alpha_s^n \ln^m \frac{1}{z}; \qquad m \le n. \tag{7}$$

These contributions spoil the convergence when $z \to 0$ and, in order to have reliable predictions, must be resummed. The inclusion of the terms with $m = n$ defines a LL_z resummation, the inclusion of also the terms with $m = n - 1$ defines a NLL_z resummation. This resummation is realized by the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation. Anyway, in this thesis, we will not concentrate on this resummation. We

will give a briefly description of the various techniques to resum the large soft logs giving attention to the renormalization group approach and studying in detail its applications.

2.2   The renormalization group approach to resummation

The aim of resummation is to include all the logarithmic enhanced terms of the form of Eq.(2) of Section 2.1 with a certain hierarchy of logarithms that we shall define in the current section.

From Eqs.(56,76) of Section 1.3 we see that the QCD cross section (up to dimensional overall factors) can in general be written as a convolution of the parto densities $F_a^{H_i}(x_i, \mu^2)$ and of the dimensionless partonic cross section, i.e. the coefficient function $C(z, Q^2/\mu^2, \alpha_s(\mu^2))$:

$$\sigma_{\mathrm{DY}}(x, Q^2) = \sum_{a,b} \left[F_a^{H_1}(\mu^2) \otimes F_b^{H_2}(\mu^2) \otimes C_{ab}(Q^2/\mu^2, \alpha_s(\mu^2)) \right](x), \qquad (8)$$

for the DY case; and

$$\sigma_{\mathrm{DIS}}(x, Q^2) = \sum_{a} \left[F_a^{H}(\mu^2) \otimes C_a(Q^2/\mu^2, \alpha_s(\mu^2)) \right](x), \qquad (9)$$

for the DIS case. The convolution product $\otimes$ has been defined in Eq.(85) of Section 1.4. Performing the Mellin transformation

$$\sigma(N, Q^2) = \int_0^1 dx\, x^{N-1} \sigma(x, Q^2) \qquad (10)$$

we turn the convolution products of Eqs.(8,9) into ordinary products:

$$\begin{aligned}
\sigma_{\mathrm{DY}}(N, Q^2) &\equiv \sum_{a,b} \sigma_{ab}(N, Q^2) \\
&= \sum_{a,b} F_a^{H_1}(N, \mu^2) F_b^{H_2}(N, \mu^2) C_{ab}(N, Q^2/\mu^2, \alpha_s(\mu^2)), \qquad (11)
\end{aligned}$$

$$\sigma_{\mathrm{DIS}}(N, Q^2) \equiv \sum_{a} \sigma_a(N, Q^2) = \sum_{a} F_a^{H}(N, \mu^2) C_a(N, Q^2/\mu^2, \alpha_s(\mu^2)), \qquad (12)$$

where

$$C_{a(b)}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = \int_0^1 dz\, z^{N-1} C_{a(b)}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right), \qquad (13)$$

$$F_a^{H_j}(N, \mu^2) = \int_0^1 dx\, x^{N-1} F_a^{H_j}(x, \mu^2), \qquad (14)$$

and where the second index in brackets (b) is involved only when there are two hadrons in the initial state as the the DY case.

The large logs of $1 - z$ of Eq.(2) in Section 2.1 are mapped to the large logs of N by the Mellin transform. This fact and the relations between the large logs of $1 - z$ and the large logs of N are shown in detail in Appendix A.

Whereas the cross section $\sigma(N, Q^2)$ is clearly μ^2-independent, this is not the case for each contribution $\sigma_{a(b)}(N, Q^2)$. However, the μ^2 dependence of each contribution to the sum over $a, (b)$ in Eqs.(11,12) is proportional to the off-diagonal anomalous dimensions γ_{qg} and γ_{gq}. In the large N limit, these are suppressed by a power of $\frac{1}{N}$ in comparison to γ_{gg} and γ_{qq}, or, equivalently, the corresponding splitting functions are suppressed by a factor of $1 - z$ in the large z limit (see for example Eqs.(83,84) in Section 1.4). Hence, in the large N limit each parton subprocess can be treated independently, specifically, each $C_{a(b)}$ is separately renormalization-group invariant. Because we are interested in the behaviour of $C_{a(b)}(N, Q^2/\mu^2, \alpha_s(\mu^2))$ in the limit $N \to \infty$ we can treat each subprocess independently.

Because resummation takes the form of an exponentiation, we define a so-called physical anomalous dimension defined implicitly through the equation

$$Q^2 \frac{\partial \sigma_{a(b)}(N, Q^2)}{\partial Q^2} = \gamma_{a(b)}(N, \alpha_s(Q^2)) \, \sigma_{a(b)}(N, Q^2). \tag{15}$$

The physical anomalous dimensions $\gamma_{a(b)}$ Eq.(15) is independent of factorization scale, and it is related to the diagonal standard anomalous dimension γ_{cc}^{AP}, defined by

$$\mu^2 \frac{\partial F_c(N, \mu^2)}{\partial \mu^2} = \gamma_{cc}^{\text{AP}}(N, \alpha_s(\mu^2)) F_c(N, \mu^2), \tag{16}$$

according to

$$\gamma_{a(b)}(N, \alpha_s(Q^2)) = \frac{\partial \ln C_{a(b)}(N, Q^2/\mu^2, \alpha_s(\mu^2))}{\partial \ln Q^2} = \gamma_{aa}^{\text{AP}}(N, \alpha_s(Q^2)) \tag{17}$$

$$+ \gamma_{(bb)}^{\text{AP}}(N, \alpha_s(Q^2)) + \frac{\partial \ln C_{a(b)}(N, 1, \alpha_s(Q^2))}{\partial \ln Q^2}. \tag{18}$$

We recall that both the standard anomalous dimensions (Altarelli-Parisi splitting functions) and the coefficient function are computable in perturbation theory. Hence, the physical anomalous dimensions differs from the standard anomalous dimensions only beyond the LO in α_s as can be seen directly from Eq.(18). In terms of the physical anomalous dimensions, the cross section can be written as

$$\sigma(N, Q^2) = \sum_{a,(b)} K_{a(b)}(N; Q_0^2, Q^2) \, \sigma_{a(b)}(N, Q_0^2) \tag{19}$$

$$= \sum_{a,(b)} \exp\left[E_{a(b)}(N; Q_0^2, Q^2)\right] \sigma_{a(b)}(N, Q_0^2), \tag{20}$$

where

$$E_{a(b)}(N; Q_0^2, Q^2) = \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \gamma_{a(b)}(N, \alpha_s(k^2)) \tag{21}$$

$$= \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} [\gamma_{aa}^{\text{AP}}(N, \alpha_s(k^2)) + \gamma_{(bb)}^{\text{AP}}(N, \alpha_s(k^2))]$$

$$+ \ln C_{a(b)}(N, 1, \alpha_s(Q^2)) - \ln C_{a(b)}(N, 1, \alpha_s(Q_0^2)). \tag{22}$$

We now concentrate on the single subprocess with incoming partons $a, (b)$. Resummation of the large logs of N in the cross section is obtained performing their resummation in the physical anomalous dimension:

$$\sigma^{res}(N, Q^2) = \exp\left\{ \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \gamma^{res}(N, \alpha_s(k^2)) \right\} \sigma^{res}(N, Q_0^2). \tag{23}$$

This shows how in general the large logs of N can be exponentiated. For the DY case only the quark-anti-quark channel should be resummed and in the DIS case only the quark one. This is a consequence of the fact that the off-diagonal splitting functions are suppressed in the large N limit as discussed before.

The accuracy of resummation here depends on the accuracy at which the physical anomalous dimension γ is computed. We say that the physical anomalous dimension is resummed at the next^{k-1}-to-leading-logarithmic accuracy ($N^{k-1}LL$) when all the contributions of the form

$$\alpha_s^{n+m}(Q^2) \ln^m N; \qquad n = 0, \ldots, k-1 \tag{24}$$

are included in its determination. The goal of resummation is to determine the resummed physical anomalous dimension from at most a finite fixed-order computation of it. Clearly, once the resummed physical anomalous dimension is determined, it can then predict the leading, next-to-leading...logarithmic contributions to the cross section at all orders.

Here, in this Section, we shall only outline the key ideas of the renormalization group approach to resummation. However, throughout this thesis we shall show in detail how this method works and how the resummed physical anomalous dimensions obtained with such an approach can be fully determined expanding it to a certain finite fixed-order and comparing this expansion with Eq.(18) obtained from explicit computations.

The renormalization group approach to resummation is essentially divided in two steps. The first is to analyze the generic phase space measure in $d = 4 - 2\epsilon$ dimensions thus finding where the large logs are originated in the coefficient function and in the physical anomalous dimension. The second consists in resumming them imposing the renormalization group invariance of the physical anomalous dimension.

So, let's consider a generic phase space measure $d\phi_n$ for the emission of n massless partons with momenta $p_1, \ldots, p_n$. In Appendix B, we show that this phase space can be decomposed in terms of two-body phase space. Roughly speaking, the phase spaces measure for the emission of n partons can be viewed as the emission of two partons (one with momentum p_n and the other with momentum $P_n = p_1 + \cdots +_{n-1}$ and invariant mass P_n^2 times the phase space measure where the momentum P_n is incoming and the momenta $p_1, \ldots, p_n$ are outgoing. The price to pay for this is the introduction of an integration over the invariant mass P_n^2. Then using recursively this procedure, we obtain that the n-body phase space measure is decomposed in $n-1$ two-body phase space measures. This means that we have reduced the study of the soft emission of n-body phase space measure to the study of the soft emission of two-body phase spaces.

The two-body phase space with an incoming momentum P and two outgoing momenta Q and p in $d = 4 - 2\epsilon$ dimensions (derived explicitly in Eq.(16) of Section B) is given by

$$d\phi_2(P; Q, p) = N(\epsilon)(P^2)^{-\epsilon} \left(1 - \frac{Q^2}{P^2}\right)^{1-2\epsilon} d\Omega_{d-1}, \quad N(\epsilon) = \frac{1}{2(4\pi)^{2-2\epsilon}}, \tag{25}$$

where $d\Omega_{d-1}$ is the solid angle in $d - 1$ dimensions. For definiteness, we can think that this is the phase space measure for a single DIS-like or DY-like soft emission with with momentum p. Thus, we have

$$\text{DIS-like emission:} \quad P^2 \propto (1 - z_{DIS}); \quad Q^2 = 0 \tag{26}$$

$$\text{DY-like emission:} \quad \left(1 - \frac{Q^2}{P^2}\right) \propto (1 - z_{DY}); \quad P^2 = s_{DY}, \tag{27}$$

where z_{DIS}, z_{DY} are close to one for a soft emission. Hence, we have that the two-body phase space measure for a single soft emission contributes with a factor $(1 - z)^{-a\epsilon}$ with $a = 1$ for a DIS-like emission and $a = 2$ for DY-like emission. In the case of the prompt-photon process, we will see in Chapter 4 that there are both types of emission. The large logs of $1 - z$ are originated by the interference with the infrared poles in $\epsilon = 0$ in the square modulus amplitude in the $\epsilon \to 0$ limit. For example

$$\frac{1}{\epsilon}(1 - z)^{-a\epsilon} = \frac{1}{\epsilon} - \ln(1 - z)^a + O(\epsilon), \tag{28}$$

for the case of interference with a pole of order 1.

Now, since each factor of $(1 - z)^{-a\epsilon}$ that comes from the phase space measure is associated to a single real emission then it will appear in the coefficient function together with a power of the bare strong coupling constant α_0. In d-dimensions, the coupling constant is dimensionful, and thus on dimensional grounds each emission is accompanied by a factor

$$\alpha_0 \left[Q^2(1 - z)^a\right]^{-\epsilon}, \tag{29}$$

where Q^2 is now the typical perturbative scale of a certain process. Upon Mellin transformation, this becomes

$$\alpha_0 \left[\frac{Q^2}{N^a}\right]^{-\epsilon}. \tag{30}$$

Furthermore an analysis of the structure of diagrams shows that in the soft (large N) limit, all dependence on N appears through the variable Q^2/N^a also in the amplitude. Finally, a renormalization group argument shows that all this dependence can be reabsorbed in the running of the strong coupling. Indeed, the first order renormalization of the bare coupling constant at the renormalization scale μ

$$\alpha_0 = \mu^{2\epsilon}\alpha_s(\mu^2) + O(\alpha_s^2) \tag{31}$$

and the renormalization group invariance of the physical anomalous dimension imply that

$$\begin{aligned} \alpha_0 \left[\frac{Q^2}{N^a}\right]^{-\epsilon} &= \alpha_s(\mu^2) \left[\frac{Q^2}{\mu^2 N^a}\right]^{-\epsilon} + O(\alpha_s^2) \\ &= \alpha_s(Q^2/N^a) + O(\alpha_s^2), \end{aligned} \tag{32}$$

where α_0 is the bare coupling, $\alpha_s(\mu^2)$ the renormalized coupling and the higher order terms contain divergences which cancel those in the cross section. Following this line of argument one may show that the finite expression of the renormalized cross section in terms of the renormalized coupling is a function of $\alpha_s(Q^2)$ and $\alpha_s(Q^2/N^a)$ with numerical coefficients, up to $O(1/N)$ corrections. We shall see this in detail in Chapter 3.

2.3 Alternative approaches

The exponentiation of the large soft logs and their resummation has been demonstrated in QCD with the eikonal approximation [2] or thanks to strong non-standard factorization properties of the cross section in the soft limit [1]. Recently, also the effective field theoretic (EFT) approach has been applied to QCD resummation in Refs.[9, 10] for DIS and DY and in Ref.[41] for the B meson decay $B \to X_s \gamma$. In this Section, we shall only give a brief description of these alternative approaches to the resummation of the large perturbative logarithms.

2.3.1 Eikonal approach

We first consider the simpler case of QED. In QED the exponentiation of the large soft logs has been proved thanks to the eikonal approximation in Ref.[7]. We report the basic steps of the proof for the QED case and a brief description of the generalization to the QCD case.

Consider a final fermion line with momentum p' of a generic QED Feynman diagram. We attach n soft photons to this fermion line with momenta $k_1, \ldots, k_n$. For the moment we do not care whether these are external photons, virtual photons connected to each other, or virtual photons connected to vertices on other fermion lines. The amplitude for such a diagram has the following structure in the soft limit:

$$\bar{u}(p')(-ie\gamma^{\mu_1})\frac{i\slashed{p}'}{2p' \cdot k_1}(-ie\gamma^{\mu_2})\frac{i\slashed{p}'}{2p' \cdot (k_1 + k_2)} \cdots (-ie\gamma\mu_n)\frac{i\slashed{p}'}{2p' \cdot (k_1 + \cdots + k_n)}i\mathcal{M}_h, \quad (33)$$

where $e = -|e|$ is the electron charge and $i\mathcal{M}_h$ is the amplitude of the hard part of the process without the final fermion line we are considering. We note that here we have neglected the electron mass. Then we can push the factors of $\slashed{p}'$ to the left and use the Dirac equation $\bar{u}(p')\slashed{p}' = 0$:

$$\bar{u}(p')\gamma^{\mu_1}\slashed{p}'\gamma^{\mu_2}\slashed{p}' \cdots \gamma^{\mu_n}\slashed{p}' = \bar{u}(p')2p'^{\mu_1}\gamma^{\mu_2}\slashed{p}' \cdots \gamma^{\mu_n}\slashed{p}' = \bar{u}(p')2p'^{\mu_1}2p'^{\mu_2} \cdots 2p'^{\mu_n}. \quad (34)$$

Thus Eq.(33) becomes

$$e^n\bar{u}(p') \left(\frac{p'^{\mu_1}}{p' \cdot k_1}\right) \left(\frac{p'^{\mu_2}}{p' \cdot (k_1 + k_2)}\right) \cdots \left(\frac{p'^{\mu_n}}{p' \cdot (k_1 + \cdots + k_n)}\right) i\mathcal{M}_h. \quad (35)$$

Still working with only a final fermion line, we must now sum over all possible orderings of momenta $k_1, \ldots, k_n$. There are $n!$ different diagrams to sum, corresponding to

the $n!$ permutations of the n photon momenta. Let P denote one such permutation, so that $P(i)$ is the number between 1 and n that i is taken to. Now, using the identity

$$\sum_P \frac{1}{p \cdot k_{P(1)}} \frac{1}{p \cdot (k_{P(1)} + k_{P(2)})} \cdots \frac{1}{p \cdot (k_{P(1)}) + \cdots + k_{P(n)}} = \frac{1}{p \cdot k_1} \cdots \frac{1}{p \cdot k_n}, \quad (36)$$

the sum over all the permutations of the photons of Eq.(35) is:

$$e^n \bar{u}(p') \left(\frac{p'^{\mu_1}}{p' \cdot k_1} \right) \left(\frac{p'^{\mu_2}}{p' \cdot k_2} \right) \cdots \left(\frac{p'^{\mu_n}}{p' \cdot k_n} \right) i\mathcal{M}_h. \quad (37)$$

At this point, we consider an initial fermion line with momentum p. In this case the photon momenta in the denominators of the fermion propagators have an opposite sign. Therefore, if we sum over all the diagrams containing a total of n soft photons, connected in any possible order to an arbitrary number of initial and final fermion lines, Eq.(37) becomes:

$$e^n i\mathcal{M}_0 \prod_{r=1}^{n} \sum_i \frac{\eta_i p^{\mu_i}}{p_i \cdot k_r}, \quad (38)$$

where $i\mathcal{M}_0$ is the full amplitude of the hard part of the process and where the index r runs over the radiated photons and the index j runs over the initial and final fermion lines with

$$\eta_i = \begin{cases} 1 & \text{for a final fermion line} \\ -1 & \text{for an initial fermion line} \end{cases} \quad (39)$$

If only a real soft photon is radiated, we must multiply by its polarization vector, sum over polarizations, and integrate the squared matrix element over the photon phase space. In the Feynman gauge this gives a factor

$$Y = \int \frac{d^3 k}{(2\pi)^3 2k^0} e^2 \left(\sum_i \frac{\eta_i p_i}{p_i \cdot k} \right)^2 \quad (40)$$

in the final cross section. If n real photons are emitted, we get n such Y factors Eq.(40), and also a symmetry factor $1/n!$ since there are n identical bosons in the final state. The cross section resummed for the emission of any number of soft photons is therefore

$$\sigma^{res}(i \to f) = \sigma_0(i \to f) \sum_{n=0}^{\infty} \frac{Y^n}{n!} = \sigma_0(i \to f)e^Y, \quad (41)$$

where $\sigma_0(i \to f)$ is the cross section for the hard process without extra soft emissions. This result shows that all the possible soft real emissions exponentiate and that only the single emission contributes to the exponent. However, this is not the end of the story, because the exponent Y Eq.(40) is infrared divergent. Indeed, to obtain a reliable finite result, we must include also loop corrections to all orders. For a detailed analysis about the inclusion of loops see for example Ref.[42]. Here, we just give the final result which reads:

$$\sigma^{res}(i \to f) = \sigma_0(i \to f)e^{\sigma^{(1)}}, \quad (42)$$

where $\sigma^{(1)}$ is the cross section relative to the single soft emission from the hard process. Clearly, the accuracy of this resummation formula for soft photon emission Eq.(42) depends on the accuracy at which the exponent for the single emission is computed.

In Ref.[2] the exponentiation of the soft emissions, here outlined for QED, is generalized to the QCD case. Differently from QED, QCD is a non-abelian gauge theory and this implies that this generalization is highly non-trivial. Indeed, the gluons can interact with each other. This fact makes the exponentiation mechanism much more difficult since the three gluon vertex color factor is different from that of the quark-gluon vertex. In order to exponentiate the single emission cross section (as it happens in QED), one should prove that these gluon correlations cancels out order by order in perturbation theory. This is shown for example in Ref.[43]. According to this result, it has been shown in Ref.[2] how the the exponentiation of soft emission works in QCD resummation. We report here the result for the NLL resummed coefficient function in Mellin space for inclusive DIS and DY processes in the $\overline{MS}$ scheme in a compact form:

$$
\begin{aligned}
C_{\mathrm{NLL}}(N, Q^2/\mu^2, \alpha_s(\mu^2)) \;=\; & \exp\left\{ a \int_0^1 dx \, \frac{x^{N-1}-1}{1-x} \left[\int_{\mu^2}^{Q^2(1-x)^a} \frac{dk^2}{k^2} A(\alpha_s(k^2)) \right.\right. \\
& \left.\left. + B^{(a)}(\alpha_s(Q^2(1-x)^a)) \right] \right\},
\end{aligned} \tag{43}
$$

where

$$
A(\alpha_s) \;=\; A_1 \alpha_s + A_2 \alpha_s^2 + \dots \tag{44}
$$

$$
B^{(a)}(\alpha_s) \;=\; B_1^{(a)} \alpha_s + \dots \tag{45}
$$

with

$$
A_1 = \frac{C_F}{\pi}, \qquad A_2 = \frac{C_F}{2\pi^2}\left[C_A\left(\frac{67}{18} - \frac{\pi^2}{6}\right) - \frac{5}{9}N_f \right], \qquad B_1^{(a)} = -\frac{(2-a)3C_F}{4\pi}. \tag{46}
$$

Here $a = 1$ for the DIS structure functio F_2 and $a = 2$ for the DY case. How this result is strictly connected to the resummed results that can be obtained with the renormalization group approach will be discussed in Section 3.4.

2.3.2 Resummation from strong factorization properties

This is the approach of Ref.[1]. Also in this approach the results given in Eq.(43) of Section 2.3.1 are recovered. Here we give only a rough description of this method based on strong factorization properties of the QCD cross section.

It is essentially assumed that at the boundary of the phase space, the cross section is factorized in a hard and in a soft part and eventually in an other factor associated to final collinear jets as in the DIS case where there is an outgoing emitting quark. The final result is then obtained exponentiating the soft and collinear factors. This is done solving their evolution equations.

In Ref.[1] it is shown that the semi-inclusive cross section can be factorized in three factors relative to the three different regions in the momentum space of the process:

the off-shell partons that participate to the partonic hard process, the collinear and soft on-shell radiated partons. The cross section is given by

$$
\sigma(w) = H\left(\frac{p_1}{\mu}, \frac{p_2}{\mu}, \zeta_i\right) \int \frac{dw_1}{w_1} \frac{dw_2}{w_2} \frac{dw_3}{w_3} J_1\left(\frac{p_1 \cdot \zeta_1}{\mu}, w_1\left(\frac{Q}{\mu}\right)^a\right)
$$
$$
J_2\left(\frac{p_2 \cdot \zeta_2}{\mu}, w_2\left(\frac{Q}{\mu}\right)^a\right) S\left(w_s \frac{Q}{\mu}, \zeta_i\right) \delta(w - w_1 - w_2 - w_3), \quad (47)
$$

where a is the number of hadrons in the initial state, μ is the factorization scale and ζ_i are gauge-fixing parameters; the integration variables w_1, w_2 and w_3 are referred to the two collinear jets and to soft radiation respectively. Each factor of Eq.(47) is evaluated at the typical scale of the momentum space region which is associated to. The delta function imposes that

$$
w = w_1 + w_2 + w_3 = \begin{cases} 1 - x_{Bj}, & \text{for the DIS case} \\ 1 - Q^2/S & \text{for the DY case} \end{cases} \quad (48)
$$

The convolution of Eq.(47) is turned into an ordinary product performing the Mellin transform:

$$
\sigma(N) = \int_0^\infty dw\, e^{-Nw} \sigma(w) = H\left(\frac{p_1}{\mu}, \frac{p_2}{\mu}, \zeta_i\right) S\left(\frac{Q}{\mu N}, \zeta_i\right)
$$
$$
\times J_1\left(\frac{p_1 \cdot \zeta_1}{\mu}, \frac{Q}{\mu N^{1/a}}\right) J_2\left(\frac{p_2 \cdot \zeta_2}{\mu}, \frac{Q}{\mu N^{1/a}}\right). \quad (49)
$$

Each factor H, J_i, S satisfy the following evolution equations

$$
\mu^2 \frac{\partial}{\partial \mu^2} \ln H = -\gamma_H(\alpha_s(\mu^2)), \quad (50)
$$

$$
\mu^2 \frac{\partial}{\partial \mu^2} \ln S = -\gamma_S(\alpha_s(\mu^2)), \quad (51)
$$

$$
\mu^2 \frac{\partial}{\partial \mu^2} \ln J_i = -\gamma_{J_i}(\alpha_s(\mu^2)), \quad (52)
$$

where the physical anomalous dimensions $\gamma_H(\alpha_s)$, $\gamma_{J_i}(\alpha_s)$ and $\gamma_S(\alpha_s)$ are calculable in perturbation theory and must satisfy, according to the renormalization group invariance of the cross section the relation

$$
\gamma_H(\alpha_s) + \gamma_S(\alpha_S) + \sum_{i=1}^{2} \gamma_{J_i}(\alpha_s) = 0. \quad (53)
$$

Solving Eqs.(50-52) and imposing renormalization group invariance Eq.(53), the resummed section can be written in the form

$$
\sigma(N) = \exp\left\{ D_1(\alpha_s(Q^2)) + D_2\left(\alpha_s\left(\frac{Q^2}{N^2}\right)\right) - \frac{2}{a-1} \int_{\frac{Q}{N}}^{\frac{Q}{N^{1/a}}} \frac{d\xi}{\xi} \ln\left(\frac{\xi N}{Q}\right) A(\alpha_s \xi^2) \right.
$$
$$
\left. -2 \int_{\frac{Q}{N^{1/a}}}^{Q^2} \frac{d\xi}{\xi} \left[\ln\left(\frac{Q}{\xi}\right) A(\alpha_s(\xi^2)) - B(\alpha_s(\xi^2)) \right] \right\}, \quad (54)
$$

where the functions $A(\alpha_s)$, $B(\alpha_s)$, $D_i(\alpha_s)$ are determined in terms of the anomalous dimensions and the beta function. Finally, it can be shown that this result can be casted in the form of Eq.(43) in Section 2.3.1 for the resummed coefficient function.

2.3.3 Resummation from Effective Field Theory

This is the approach of Refs.[9, 10]. This EFT methodology to resum threshold logarithms is made concrete due to the recently developed "soft collinear effective theory" (SCET) [44, 45, 46, 47]. The SCET describes interactions between soft and collinear partons.

The starting point (considering the DY case as an example) is the collinearly factorized inclusive cross section in Mellin space:

$$\sigma(N, Q^2) = \sigma_0\, C(N, Q^2/\mu^2, \alpha_s(\mu^2)) F_1(N, \mu^2) F_2(N, \mu^2), \tag{55}$$

where σ_0 is the born level cross section and $F_i(N, \mu^2)$ are the parton densities at the factorization scale μ. Here, the basic idea is to write the coefficient function $C(N, Q^2/\mu^2, \alpha_s(\mu^2))$ in terms of an intermediate scale μ_I:

$$C(N, Q^2/\mu^2, \alpha_s(\mu^2)) = C(N, Q^2/\mu_I^2, \alpha_s(\mu_I^2)) \exp\left[-2\int_{\mu_I^2}^{\mu^2} \frac{dk^2}{k^2} \gamma_{qq}^{\mathrm{AP}}(N, \alpha_s(k^2))\right] \tag{56}$$

and then to compute $C(N, Q^2/\mu_I^2, \alpha_s(\mu_I^2))$ with the "soft collinear effective theory" with the intermediate scale μ_I^2 equal to the typical scale of the soft-collinear emission, i.e. $\mu_I = Q^2/N^2$ in the DY case and $\mu_I^2 = Q^2/N$ in the DIS case. In SCET, $C(N, Q^2/\mu_I^2, \alpha_s(\mu_I^2))$ has the following general structure:

$$C\left(N, \frac{Q^2}{\mu_I^2}, \alpha_s(\mu_I^2)\right) = \left|\tilde{C}\left(\frac{Q^2}{\mu_I^2}, \alpha_s(\mu_I^2)\right)\right|^2 \mathcal{M}(N, \alpha_s(\mu_I^2)), \tag{57}$$

Here $\tilde{C}(Q^2/\mu_I^2, \alpha_s(\mu_I^2))$ is the effective coupling that matches the full QCD theory currents

$$J_{QCD} = \tilde{C}(Q^2/\mu_I^2, \alpha_s(\mu_I^2)) J_{eff}(\mu_I^2). \tag{58}$$

We note that the effective coupling $\tilde{C}$ contains the perturbative contribution between the scale Q^2 and μ_I^2 and $J_{eff}(\mu_I^2)$ contains the soft and collinear contributions below the scale μ_I^2. Then $\mathcal{M}(N, \alpha_s(\mu_I^2))$ is the matching coefficient that guarantees that the EFT used generates the full QCD results in the appropriate kinematical limit. In SCET the matching coefficient $\mathcal{M}$ can be computed perturbativly and is free of any logarithms.

The effective coupling $\tilde{C}$ satisfy to a certain evolution equation

$$\mu^2 \frac{\partial}{\partial \mu^2} \ln \tilde{C}(Q^2/\mu^2, \alpha_s(\mu^2)) = -\frac{1}{2} \gamma_1(\alpha_s(\mu^2)) \tag{59}$$

where the physical anomalous dimension γ_1 is computable perturbativly in SCET. Finally, solving the evolution equation Eq.(59) one finds that Eq.(57) becomes

$$\begin{aligned}
C\left(N, \frac{Q^2}{\mu_I^2}, \alpha_s(\mu_I^2)\right) &= \left|\tilde{C}(1, \alpha_s(Q^2))\right|^2 e^{E_1(Q^2/\mu_I^2, alpha_s(Q^2))} \times \\
&\quad \times \mathcal{M}(N, \alpha_s(Q^2)) e^{E_2(Q^2/\mu_I^2, \alpha_s(Q^2))}, \tag{60}
\end{aligned}$$

where

$$E_1\left(\frac{Q^2}{\mu_I^2}, \alpha_s(Q^2)\right) = -\frac{1}{4}\int_{\mu_I^2}^{Q^2}\frac{dk^2}{k^2}\gamma_1(\alpha_s(k^2)), \tag{61}$$

$$E_2\left(\frac{Q^2}{\mu_I^2}, \alpha_s(Q^2)\right) = \int_{\mu_I^2}^{Q^2}\frac{dk^2}{k^2}\beta(\alpha_s(k^2))\frac{d\ln\mathcal{M}(N, \alpha_s(k^2))}{d\ln\alpha_s(k^2)}, \tag{62}$$

with $\beta(\alpha_s)$ the beta function of Eq.(43) of Section 1.2. $\tilde{C}(1, \alpha_s(Q^2))$ contains the non-logarithmic contribution of the purely virtual diagrams and the exponent E_1 contains all the logarithms originating from the same type of diagrams. E_2 encodes all the contributions due to the running of the coupling constant between the scale μ_I^2 and Q^2. All the logarithms appear only in the exponents (see Eqs.(56,60)) and the term $|\tilde{C}(1, \alpha_s(Q^2))|^2\mathcal{M}(N, \alpha_s(Q^2))$ is free of any large logarithms.

The various approaches can be related one to the other according to factorization properties of the QCD cross section in the soft limit. In this way, it is possible to show that all the approaches are equivalent except for the renormalization group approach that produces correct but less restrictive results.

Chapter 3

Renormalization group resummation of DIS and DY

In this chapter, we analyze in detail how the renormalization group approach to resummation works in the case of the all-inclusive Drell-Yan (DY) and deeep inelastic scattering (DIS). We recall that this is done only for the quark-anti-quark channel in the DY case and only for the quark channel in the DIS case as discussed in Section 2.2. First, we determine the N dependence of the regularized coefficient function in the large-N limit. Then we show that, given this form of the N-dependence of the regularized cross section, renormalization group invariance fixes the all-order dependence of the physical anomalous dimension in such a way that the infinite class of leading, next-to-leading etc. resummations can be found in terms of fixed order computations. This approach will lead us to resummation formulae valid to all logrithmic orders.

3.1 Kinematics of inclusive DIS in the soft limit

In the case of deep-inelastic scattering, the relevant parton subprocesses are:

$$\gamma^*(q) + \mathcal{Q}(p) \quad \rightarrow \quad \mathcal{Q}(p') + \mathcal{X}(K) \tag{1}$$

$$\gamma^*(q) + \mathcal{G}(p) \quad \rightarrow \quad \mathcal{Q}(p') + \mathcal{X}(K), \tag{2}$$

where $\mathcal{Q}$ is a quark or an anti-quark, $\mathcal{G}$ is a gluon and $\mathcal{X}$ is any collection of quarks and gluons. We are interested in the most singular parts in the limit $z \rightarrow 1$ of the phase space and of the amplitude for the generic processes Eqs.(1,2). We treat first the tree level processes and then we will introduce the loops.

Using Eq.(12) in Appendix B with $m = 1$ recursively, we can express the phase space for a generic process in terms of two-body phase space integrals. For the DIS processes Eqs.(1,2) with n extra emissions ($K = k_1 + \cdots + k_n$) we have

$$d\phi_{n+1}(p + q; k_1, \ldots, k_n, p')$$
$$= \int_0^s \frac{dM_n^2}{2\pi} d\phi_2(p + q; k_n, P_n) d\phi_n(P_n; k_1, \ldots, k_{n-1}, p')$$

$$= \int_0^s \frac{dM_n^2}{2\pi} d\phi_2(p+q; k_n, P_n)$$

$$\times \int_0^{M_n^2} \frac{dM_{n-1}^2}{2\pi} d\phi_2(P_n; k_{n-1}, P_{n-1}) d\phi_{n-1}(P_{n-1}; k_1, \dots, k_{n-2}, p')$$

$$= \int_0^s \frac{dM_n^2}{2\pi} d\phi_2(P_{n+1}; k_n, P_n) \int_0^{M_n^2} \frac{dM_{n-1}}{2\pi} d\phi_2(P_n; k_{n-1}, P_{n-1})$$

$$\times \cdots \times \int_0^{M_3^2} \frac{dM_2^2}{2\pi} d\phi_2(P_3; k_2, P_2) d\phi_2(P_2; k_1, P_1), \qquad (3)$$

where we have defined $M_n^2 \equiv P_n^2$, $P_1 \equiv p'$, $P_{n+1} \equiv p+q$ and

$$s \equiv M_{n+1}^2 = (p+q)^2 = Q^2 \frac{(1-z)}{z}; \qquad z = \frac{Q^2}{2p \cdot q}. \qquad (4)$$

Now, according to Eq.(16) in Appendix B we have for each two-body phase space

$$d\phi_2(P_{i+1}; k_i, P_i) = N(\epsilon)(M_{i+1})^{-2\epsilon} \left(1 - \frac{M_i^2}{M_{i+1}^2}\right)^{1-2\epsilon} d\Omega_i; \quad i = 1, \dots n, \qquad (5)$$

where

$$N(\epsilon) = \frac{1}{2(4\pi)^{2-2\epsilon}} \qquad (6)$$

and Ω_i is the solid angle in the center-of-mass frame of P_{i+1}. We perform the change of variables

$$z_i = \frac{M_i^2}{M_{i+1}}; \quad M_i^2 = s z_n \dots z_i; \quad i = 2, \dots, n. \qquad (7)$$

From the fact that $M_{i+1}^2 \geq M_i^2$ (we have one more real particle in P_{i+1} than in P_i), it follows that

$$0 \leq z_i \leq 1. \qquad (8)$$

From Eq.(7) we get

$$dM_n^2 dM_{n-1}^2 \cdots dM_2^2 = \det\left(\frac{\partial M_i^2}{\partial z_j}\right) dz_n dz_{n-1} \cdots dz_2, \qquad (9)$$

where

$$\det\left(\frac{\partial M_i^2}{\partial z_j}\right) = \frac{\partial M_n^2}{\partial z_n} \cdots \frac{\partial M_2^2}{\partial z_2} = s^{n-1} z_n^{n-2} z_{n-1}^{n-3} \cdots z_3. \qquad (10)$$

Furthermore, in these new variables the two-body phase space becomes

$$d\phi_2(P_{i+1}; k_i, P_i) = N(\epsilon) s^{-\epsilon}(z_n z_{n-1} \cdots z_{i+1})^{-\epsilon}(1 - z_i)^{1-2\epsilon} d\Omega_i. \qquad (11)$$

Substituting Eqs.(9,10,11) into the generic phase space Eq.(3), we finally get

$$d\phi_{n+1}(p+q; k_1, \dots, k_n, p') = 2\pi \left[\frac{N(\epsilon)}{2\pi}\right]^n s^{n-1-n\epsilon} d\Omega_n \cdots d\Omega_1$$

$$\times \int_0^1 dz_n z_n^{(n-2)-(n-1)\epsilon}(1 - z_n)^{1-2\epsilon} \cdots \int_0^1 dz_2 z_2^{-\epsilon}(1 - z_2)^{1-2\epsilon}. \qquad (12)$$

The dependence of the phase space on $1 - z$ comes entirely from the prefactor of $s^{n-1-n\epsilon}$ according to Eq.(4). Indeed the dependence on z and Q^2 has been entirely removed from the integration range thanks to the change of variables of Eq.(7).

Now, the amplitude whose square modulus is integrated with the phase space Eq.(12) is in general a function:

$$A_{n+1} = A_{n+1}(Q^2, s, z_2, \ldots, z_n, \Omega_1, \ldots, \Omega_n). \tag{13}$$

The number of independent variables for a process with 2 incoming particle (one on-shell and the other virtual) and $n + 1$ outgoing real particles is given by the number of parameters minus the on-shell conditions and the ten parameters of the Poincare' group:

$$4(n + 3) - (n + 2) - 10 = 3n. \tag{14}$$

These $3n$ variable correspond in this case to

$$Q^2, s, z_2, \ldots, z_n, \Omega_1, \ldots, \Omega_n, \tag{15}$$

where an azimutal angle is arbitrary. In the $z \to 1$ limit, $s \to 0$ and the dominant contribution is given by terms which are most singular as s vanishes. Because of cancellation of infrared singularities [48, 49], $|A_{n+1}|^2 \sim s^{-n+O(\epsilon)}$ when $s \to 0$. Indeed, a stronger singularity would lead to powerlike infrared divergences and a weaker singularity would lead to suppressed terms in the $z \to 1$ limit.

Hence only terms in the square amplitude which behave as $s^{-n+O(\epsilon)}$ contribute in the $z \to 1$ limit. In d dimensions, these pick up an $s^{-1-n\epsilon+O(\epsilon)}$ prefactor from the phase space Eq.(12). Let's consider the simplest case, that is the tree level case where we have only purely real soft emission. In this case $O(\epsilon) = 0$ and thus we get that the contribution to the coefficient function from the tree level diagrams with n extra radiated partons behaves as:

$$|A_{n+1}|^2 d\phi_{n+1} \sim s^{-1-n\epsilon} \int_0^1 dz_n \cdots dz_2 z_n^{(n-2)-(n-1)\epsilon} (1 - z_n)^{1-2\epsilon} \cdots z_2^{-\epsilon} (1 - z_2)^{1-2\epsilon}$$

$$\times \int d\Omega_1 \cdots d\Omega_n. \tag{16}$$

We note that each z integration can produce at most a $1/\epsilon$ pole from the soft region and that each angular integration can produce at most an additional $1/\epsilon$ pole from the collinear region (see Eq.(23) in Appendix B). Therefore, from the contribution of tree level diagrams with n extra radiated partons there come at most

$$\frac{1}{\epsilon^{n-1}} \frac{1}{\epsilon^n} = \frac{1}{\epsilon^{2n-1}} \tag{17}$$

poles in $\epsilon = 0$. All this means that we can write the $O(\alpha_s^n)$ to the bare coefficient function in d dimensions in the following form:

$$C_n'^{(0)}(z, Q^2, \epsilon) = (Q^2)^{-n\epsilon} \frac{C_{nn}^{(0)}(\epsilon)}{\Gamma(-n\epsilon)} (1 - z)^{-1-n\epsilon}, \tag{18}$$

where the factor $(Q^2)^{-n\epsilon}$ is due to elementary dimensional analysis, $C_{nn}^{(0)}(\epsilon)$ are coefficients with poles in $\epsilon = 0$ of order at most of $2n$ and the Γ function factor $\Gamma^{-1}(-n\epsilon)$ has been introduced for future convenience. For the LO tree level case (see Eq.(73) of Chapter 1) we have that

$$C_0^{\prime(0)}(z, Q^2, \epsilon) = \delta(1 - z). \tag{19}$$

Hence, the tree level coefficient function $C_{\text{tree}}^{(0)}$ in the $z \to 1$ limit has the form:

$$
\begin{aligned}
C_{\text{tree}}^{(0)}(z, Q^2, \alpha_0, \epsilon) &= \sum_{n=0}^{\infty} \alpha_0^n C_n^{\prime(0)}(z, Q^2, \epsilon) \tag{20} \\
&= \delta(1 - z) + \sum_{n=1}^{\infty} \alpha_0^n (Q^2)^{-n\epsilon} \frac{C_{nn}^{\prime(0)}(\epsilon)}{\Gamma(-n\epsilon)} (1 - z)^{-1-n\epsilon} \\
&\quad + O((1-z)^0), \tag{21}
\end{aligned}
$$

where α_0 is the bare i.e. the unrenormalized strong coupling constant.

We will now study how the result of Eq.(21) is modified by the inclusion of loops. To this purpose, we notice that a generic amplitude with loops can be viewed as a tree-level amplitude formed with proper vertices. Contributions to the dimensionless coefficient function with powers of s^ϵ can only arise from loop integrations in the proper vertices. We thus consider only purely scalar loop integrals, since numerators of fermion or vector propagators and vertex factors cannot induce any dependence on s^ϵ. Let us therefore consider an arbitrary proper diagram G in a massless scalar theory with E external lines, I internal lines and V vertices. It can be shown (see e.g. section 6.2.3 of [50] and references therein) that, denoting with P the set of E external momenta and P_E the set of independent invariants, the corresponding amplitude $\tilde{A}_G(P_E)$ has the form

$$
\begin{aligned}
\tilde{A}_G(P_E) &= K(2\pi)^d \delta_d(P) A_G(P_E), \\
A_G(P_E) &= \frac{i^{I-L(d-1)}}{(4\pi)^{dL/2}} \Gamma(I - dL/2) \\
&\quad \times \prod_{l=1}^{I} \left[\int_0^1 d\beta_l \right] \frac{\delta\left(1 - \sum_{l=1}^{I} \beta_l\right)}{[P_G(\beta)]^{d(L+1)/2-I} [D_G(\beta, P_E)]^{I-dL/2}}. \tag{22}
\end{aligned}
$$

Here, β_l are the usual Feynman parameters, $P_G(\beta)$ is a homogeneous polynomial of degree L in the β_l, $D_G(\beta, P_E)$ is a homogeneous polynomial of degree $L + 1$ in the β_l with coefficients which are linear functions of the scalar products of the set P_E, i.e. with dimensions of $(\text{mass})^2$, and K collects all overall factors, such as couplings and symmetry factors.

The amplitude $\tilde{A}_G(P_E)$ Eq.(22) depends on s only through $D_G(\beta, P_E)$, which, in turn is linear in s. We can determine in general the dependence of $A_G(P_E)$ by considering two possible cases. The first possibility is that $D_G(\beta, P_E)$ is independent of all invariants except s, i.e. $D_G(\beta, P_E) = s d_G(\beta)$. In such case, $A_G(P_E)$ depends on s as

$$A_G(P_E) = \left(\frac{1}{s}\right)^{I-dL/2} a_G = \left(\frac{1}{s}\right)^{I-2L+L\epsilon} a_G, \tag{23}$$

where a_G is a numerical constant, obtained performing the Feynman parameters integrals. The second possibility is that $D_G(\beta, P_E)$ depends on some of the other invariants. In such case, $A_G(P_E)$ is manifestly an analytic function of s at $s = 0$, and thus it can be expanded in Taylor series around $s = 0$, with coefficients which depend on the other invariants. In the former case, Eq.(23) implies that the s dependence induced by loops integration in the square amplitude is given by integer powers of $s^{-\epsilon}$. In the latter case, the s dependence induced by loops integrations in the square amplitude is given by integer positive powers of s.

Therefore, we conclude that each loop integration can carry at most a factor of $s^{-\epsilon}$ and that Eq.(21), after the inclusion of loops, becomes:

$$C^{(0)}(z, Q^2, \alpha_0, \epsilon) = \sum_{n=0}^{\infty} \alpha_0^n C_n^{(0)}(z, Q^2, \epsilon), \tag{24}$$

$$C_n^{(0)}(z, Q^2, \epsilon) = (Q^2)^{-n\epsilon} \left[C_{n0}^{(0)}(\epsilon)\delta(1 - z) + \sum_{k=1}^{n} \frac{C_{nk}^{(0)}(\epsilon)}{\Gamma(-k\epsilon)}(1 - z)^{-1-k\epsilon} \right]$$
$$+ O((1 - z)^0), \tag{25}$$

where again for future convenience we have defined the coefficients of $(1 - z)^{-1-k\epsilon}$ in terms of $\Gamma^{-1}(-k\epsilon)$ and where $O((1 - z)^0)$ denotes terms which are not divergent as $z \to 1$ in the $\epsilon \to 0$ limit.

Using the identity

$$\int_0^1 dz\, z^{N-1}(1 - z)^{-1-k\epsilon} = \frac{\Gamma(N)\Gamma(-k\epsilon)}{\Gamma(N - k\epsilon)} \tag{26}$$

and the Stirling expansion Eq.(5) of Appendix A we get that the Mellin transform of Eqs.(55,25) in the large-N limit is given by

$$C^{(0)}(N, Q^2, \alpha_0, \epsilon) = \sum_{n=0}^{\infty} \alpha_0^n C_n^{(0)}(N, Q^2, \epsilon), \tag{27}$$

$$C_n^{(0)}(N, Q^2, \epsilon) = \sum_{k=0}^{n} C_{nk}^{(0)}(\epsilon)(Q^2)^{-(n-k)\epsilon} \left(\frac{Q^2}{N} \right)^{-k\epsilon} + O\left(\frac{1}{N} \right). \tag{28}$$

The content of this result is that, in the large-N limit, the dependence of the regularized cross section on N only goes through integer powers of the dimensionful variable $(Q^2/N)^{-\epsilon}$.

3.2 Kinematics of inclusive DY in the soft limit

In the Drell-Yan case the argument follows in an analogous way with minor modification which account for the different kinematics. In this case the relevant parton subprocesses are:

$$\mathcal{Q}(p) + \mathcal{Q}(p') \rightarrow \gamma^*(Q) + \mathcal{X} \tag{29}$$
$$\mathcal{Q}(p) + \mathcal{G}(p') \rightarrow \gamma^*(Q) + \mathcal{X} \tag{30}$$
$$\mathcal{G}(p) + \mathcal{G}(p') \rightarrow \gamma^*(Q) + \mathcal{X}. \tag{31}$$

The recursive application of Eq.(12) in Appendix B with $m = 1$, in this case gives:

$$
d\phi_{n+1}(p + p'; Q, k_1, \ldots, k_n)
$$
$$
= \int_0^s \frac{dM_n^2}{2\pi} d\phi_2(p + p'; k_n, P_n) d\phi_n(P_n; Q, k_1, \ldots, k_{n-1})
$$

$$
= \int_0^s \frac{dM_n^2}{2\pi} d\phi_2(p + p'; k_n, P_n)
$$
$$
\times \int_0^{M_n^2} \frac{dM_{n-1}^2}{2\pi} d\phi_2(P_n; k_{n-1}, P_{n-1}) d\phi_{n-1}(P_{n-1}; Q, k_1, \ldots, k_{n-2})
$$
$$
= \int_0^s \frac{dM_n^2}{2\pi} d\phi_2(P_{n+1}; k_n, P_n) \int_0^{M_n^2} \frac{dM_{n-1}}{2\pi} d\phi_2(P_n; k_{n-1}, P_{n-1})
$$
$$
\times \cdots \times \int_0^{M_3^2} \frac{dM_2^2}{2\pi} d\phi_2(P_3; k_2, P_2) d\phi_2(P_2; k_1, P_1), \tag{32}
$$

where now we have defined $P_{n+1} \equiv p + p'$, so $M_{n+1}^2 = s$ and $P_1 \equiv Q$. The change of variables which separates off the dependence on $(1 - z)$, where now $z = Q^2/s$, is

$$
z_i = \frac{M_i^2 - Q^2}{M_{i+1}^2 - Q^2}; \qquad i = 2, \ldots, n \tag{33}
$$
$$
M_i^2 - Q^2 = (s - Q^2) z_n \cdots z_i. \tag{34}
$$

Also here all z_i range between 0 and 1, because $M_i^2 \leq M_{i+1}^2 \leq Q^2$. From Eq.(33), we get:

$$
dM_n^2 dM_{n-1}^2 \cdots dM_2^2 = \det \left(\frac{\partial (M_i^2 - Q^2)}{\partial z_j} \right) dz_n dz_{n-1} \cdots dz_2, \tag{35}
$$

where

$$
\det \left(\frac{\partial (M_i^2 - Q^2)}{\partial z_j} \right) = \frac{\partial (M_n^2 - Q^2)}{\partial z_n} \cdots \frac{\partial (M_2^2 - Q^2)}{\partial z_2} = (s - Q^2)^{n-1} z_n^{n-2} z_{n-1}^{n-3} \cdots z_3. \tag{36}
$$

In this case in the new variables Eq.(33) the two-body phase space becomes

$$
d\phi_2(P_{i+1}; k_i, P_i)
$$
$$
= N(\epsilon)(M_{i+1}^2)^{-1+\epsilon} \left[(M_{i+1}^2 - Q^2) - (M_i^2 - Q^2) \right]^{1-2\epsilon} d\Omega_i
$$
$$
= N(\epsilon)(Q^2)^{-1+\epsilon}(s - Q^2)^{1-2\epsilon}(z_n \cdots z_{i+1})^{1-2\epsilon}(1 - z_i)^{1-2\epsilon} d\Omega_i, \tag{37}
$$

where in the last step we have replaced $(M_{i+1}^2)^{-1+\epsilon}$ by $(Q^2)^{-1+\epsilon}$ in the $z \to 1$ limit as can be seen comparing with Eq.(34). Now, substituting Eqs.(35,36,37) into Eq.(32), we finally get

$$
d\phi_{n+1}(p + p'; Q, k_1, \ldots, k_n) = 2\pi \left[\frac{N(\epsilon)}{2\pi} \right]^n (Q^2)^{-n(1-\epsilon)}(s - Q^2)^{2n-1-2n\epsilon} d\Omega_n \cdots d\Omega_1
$$
$$
\times \int_0^1 dz_n z_n^{(n-2)-(n-1)\epsilon}(1 - z_n)^{1-2\epsilon} \cdots \int_0^1 dz_2 z_2^{-\epsilon}(1 - z_2)^{1-2\epsilon}. \tag{38}
$$

The dependence on $1 - z$ is now entirely contained in the phase space prefactor

$$(Q^2)^{-n(1-\epsilon)}(s - Q^2)^{2n-1-2n\epsilon} = \frac{z^{1-2n+2n\epsilon}}{Q^2(1-z)} \left[Q^2(1-z)^2\right]^{n-n\epsilon}. \qquad (39)$$

As before, this proves that the coefficient function for real emission at tree level is given by

$$
\begin{aligned}
C_{\text{tree}}^{(0)}(z, Q^2, \alpha_0, \epsilon) &= \sum_{n=0}^{\infty} \alpha_0^n C_n'^{(0)}(z, Q^2, \epsilon) \qquad &(40) \\
&= \delta(1-z) + \sum_{n=1}^{\infty} \alpha_0^n (Q^2)^{-n\epsilon} \frac{C_{nn}'^{(0)}(\epsilon)}{\Gamma(-2n\epsilon)} (1-z)^{-1-2n\epsilon} \\
&\quad + O((1-z)^0), &(41)
\end{aligned}
$$

In this case the introduction of loops requires more care than for the deep-inelastic-scattering case. We shall now show the main difference between the DIS case and the DY case as far as the introduction of loops is concerned. The two-body kinematics (see Eq.(14) in Appendix B together with Eqs.(3,7) states that the radiated partons in the DIS case are all soft:

$$
\begin{aligned}
k_i^0 &= \frac{M_{i+1}}{2}\left(1 - \frac{M_i^2}{M_{i+1}^2}\right) = \frac{\sqrt{s}}{2}(z_n \cdots z_{i+1})^{1/2}(1 - z_i); \quad 1 \le i \le n-1 \quad &(42) \\
k_n^0 &= \frac{\sqrt{s}}{2}\left(1 - \frac{M_n^2}{s}\right) = \frac{\sqrt{s}}{2}(1 - z_n); \quad s = Q^2\frac{1-z}{z}. &(43)
\end{aligned}
$$

This confirms the validity of the argument of Section 3.1 for the introduction of loops, because all the invariants that can appear in the function $D_G(\beta, P_E)$ in Eq.(22) can be expresses in terms of the following ones

$$q^2 = -Q^2 \qquad (44)$$
$$p^2 = p'^2 = k_i^2 = 0 \qquad (45)$$
$$p \cdot p' \sim p \cdot k_i \sim Q^2 \qquad (46)$$
$$k_i \cdot k_j \sim Q^2(1-z), \qquad (47)$$

which are either constant or proportional to $1 - z$, i.e. to s. Here we have used Eqs.(42,43) of this Section, Eq.(6) in Chapter 2 and the fact that the definition of z in the DIS case Eq.(4) implies that

$$(p^0)^2 = \frac{Q^2}{4z(1-z)}. \qquad (48)$$

In the DY case things are quite different, because in this case two-body kinematics (see Eq.(14) in Appendix B together with Eqs.(32,33) gives

$$
\begin{aligned}
k_i^0 &= \frac{\sqrt{s}}{2}(1 - z)z_n \cdots z_{i+1}(1 - z_i) + O((1-z)^2); \quad 1 \le i \le n-1 \quad &(49) \\
k_n^0 &= \frac{\sqrt{s}}{2}(1 - z)(1 - z_n); \quad s = \frac{Q^2}{z} &(50)
\end{aligned}
$$

and this implies that all the invariants that can appear in the function $D_G(\beta, P_E)$ in Eq.(22) can be expresses in terms of the following ones

$$q^2 = Q^2 \tag{51}$$

$$p \cdot p' = s/2 \tag{52}$$

$$p \cdot k_i \sim p' \cdot k_i \sim s(1 - z) \tag{53}$$

$$k_i \cdot k_j \sim s(1 - z)^2. \tag{54}$$

Hence, we see that, in general, both odd and even powers of $(1 - z)^{-\epsilon}$ may arise adding the loops contribution to the tree level coefficient function Eq.(41). Here, in this thesis, we will assume that odd powers of $(1 - z)^{-\epsilon}$ do not arise, because it can be shown by explicit computations that it is the case up to order $O(\alpha_s^2)$. However, there are possible indications that at higher orders this assumption could not be true. Anyway the investigations of these aspects is beyond the aim of this thesis.

Thus, after the inclusions of loops and with our assumptions, Eq.(41) becomes

$$C^{(0)}(z, Q^2, \alpha_0, \epsilon) = \sum_{n=0}^{\infty} \alpha_0^n C_n^{(0)}(z, Q^2, \epsilon), \tag{55}$$

$$C_n^{(0)}(z, Q^2, \epsilon) = (Q^2)^{-n\epsilon} \left[C_{n0}^{(0)}(\epsilon)\delta(1 - z) + \sum_{k=1}^{n} \frac{C_{nk}^{(0)}(\epsilon)}{\Gamma(-2k\epsilon)} (1 - z)^{-1-2k\epsilon} \right]$$
$$+ O((1 - z)^0). \tag{56}$$

Its Mellin transform can be written in a compact way together with that of the DIS case Eq.(28):

$$C^{(0)}(N, Q^2, \alpha_0, \epsilon) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} C_{nk}^{(0)}(\epsilon)[\alpha_0(Q^2)^{-\epsilon}]^{(n-k)} \left[\alpha_0 \left(\frac{Q^2}{N^a} \right)^{-\epsilon} \right]^k + O\left(\frac{1}{N} \right), \tag{57}$$

where $a = 1$ for the DIS case and $a = 2$ for the DY case and where the coefficients are those that could be obtained from the parton-level cross sections for the partonic subprocesses that contribute to the given process.

3.3 Resummation from renormalization group improvement

In this section, we want to impose the restrictions that renormalization group invariance imposes on the cross section. Our only assumption is that the coefficient function can be multiplicatively renormalized. This means that all divergences can be removed from the bare coefficient function $C^{(0)}(N, Q^2, \alpha_0, \epsilon)$ Eq.(57) by defining a renormalized running coupling $\alpha_s(\mu^2)$ according to the implicit equation

$$\alpha_0(\mu^2, \alpha_s(\mu^2), \epsilon) = \mu^{2\epsilon} \alpha_s(\mu^2) Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon) \tag{58}$$

and a renormalized coefficient function

$$C\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2, \epsilon)\right) = Z^{(C)}(N, \alpha_s(\mu^2), \epsilon)C^{(0)}(N, Q^2, \alpha_0, \epsilon), \qquad (59)$$

where μ is the renormalization scale (here chosen equal to the factorization one) and $Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon)$ and $Z^C(N, \alpha_s(\mu^2), \epsilon)$ are computable in perturbation theory and have multiple poles in $\epsilon = 0$. The renormalized coefficient function $C(N, Q^2/\mu^2, \alpha_s(\mu^2, \epsilon))$ is finite in $\epsilon = 0$ and it can only depend on Q^2 through Q^2/μ^2, because $\alpha_s(\mu^2)$ is dimensionless.

The physical anomalous dimension is given by

$$\begin{aligned}
\gamma(N, \alpha_s(Q^2), \epsilon) &= Q^2 \frac{\partial}{\partial Q^2} \ln C\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = Q^2 \frac{\partial}{\partial Q^2} \ln C^{(0)}\left(N, Q^2, \alpha_0, \epsilon\right) \\
&= -\epsilon(\alpha_0 Q^{-2\epsilon}) \frac{\partial}{\partial(\alpha_0 Q^{-2\epsilon})} \ln C^{(0)}\left(N, Q^2, \alpha_0, \epsilon\right),
\end{aligned} \qquad (60)$$

where we have exploited the fact that $C^{(0)}$ Eq.(57) depends on Q^2 through the combination $\alpha_0 Q^{-2\epsilon}$. This implies that the physical anomalous dimension γ has the following perturbative expression:

$$\gamma(N, \alpha_s(Q^2), \epsilon) = \sum_{i=0}^{\infty} \sum_{j=0}^{n} \gamma_{ij}(\epsilon)[\alpha_0(Q^2)^{-\epsilon}]^{(i-j)} \left[\alpha_0 \left(\frac{Q^2}{N^a}\right)^{-\epsilon}\right]^j + O\left(\frac{1}{N}\right). \qquad (61)$$

The renormalized expression of the physical anomalous dimension is found expressing in this equation the bare coupling constant in terms of the renormalized one by means of Eq.(58). Now, the functions

$$(Q^2)^{-\epsilon}\alpha_0 = \left(\frac{Q^2}{\mu^2}\right)^{-\epsilon} \alpha_s(\mu^2)Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon) \qquad (62)$$

$$(Q^2/N^a)^{-\epsilon}\alpha_0 = \left(\frac{Q^2/N^a}{\mu^2}\right)^{-\epsilon} \alpha_s(\mu^2)Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon) \qquad (63)$$

are manifestly renormalization group invariant, i.e. μ^2-independent. Thus, it follows that

$$(Q^2)^{-\epsilon}\alpha_0 = \alpha_s(Q^2)Z^{(\alpha_s)}(\alpha_s(Q^2), \epsilon) \qquad (64)$$

$$(Q^2/N^a)^{-\epsilon}\alpha_0 = \alpha_s(Q^2/N^a)Z^{(\alpha_s)}(\alpha_s(Q^2/N^a), \epsilon). \qquad (65)$$

The renormalized physical anomalous dimension is then found by substituting Eqs.(64,65) into Eq.(61) and re-expanding $Z^{(\alpha_s)}$ in powers of the renormalized coupling. We obtain:

$$\gamma(N, \alpha_s(Q^2), \epsilon) = \sum_{m=1}^{\infty} \sum_{n=0}^{m} \gamma_{mn}^{\mathrm{R}}(\epsilon)\alpha_s^{m-n}(Q^2)\alpha_s^n(Q^2/N^a) + O\left(\frac{1}{N}\right). \qquad (66)$$

At this point, we cannot yet conclude that the four-dimensional physical anomalous dimension admits an expression of the form of Eq.(66), because the coefficients

$\gamma_{mn}^{\mathrm{R}}(\epsilon)$ are not necessarily finite as $\epsilon \to 0$. In order to understand this, it is convenient to separate off the N-independent terms in the renormalized physical anomalous dimension, i.e. the terms with $n = 0$ in the internal sum in Eq.(66). Namely, we write

$$\gamma(N, \alpha_s(Q^2), \epsilon) = \hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2/N^a), \epsilon) + \hat{\gamma}^{(c)}(\alpha_s(Q^2), \epsilon) + O\left(\frac{1}{N}\right), \qquad (67)$$

where we have defined

$$\hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2/N^a), \epsilon) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \gamma_{m+n\,n}^{\mathrm{R}}(\epsilon) \alpha_s^m(Q^2) \alpha_s^n(Q^2/N^a) \qquad (68)$$

$$\hat{\gamma}^{(c)}(\alpha_s(Q^2), \epsilon) = \sum_{m=1}^{\infty} \gamma_{m0}^{\mathrm{R}}(\epsilon) \alpha_s^m(Q^2). \qquad (69)$$

Whereas $\gamma(N, \alpha_s(Q^2), \epsilon)$ is finite in the limit $\epsilon \to 0$, where it coincides with the four-dimensional physical anomalous dimension, $\hat{\gamma}^{(l)}$ and $\hat{\gamma}^{(c)}$ are not necessarily finite as $\epsilon \to 0$. However, Eq.(67) implies that $\hat{\gamma}^{(l)}$ and $\hat{\gamma}^{(c)}$ can be made finite by adding and subtracting the counterterm

$$Z^{(\gamma)}(\alpha_s(Q^2), \epsilon) = \hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2), \epsilon). \qquad (70)$$

In this way the physical anomalous dimension Eq.(67) becomes

$$\gamma(N, \alpha_s(Q^2), \epsilon) = \gamma^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2/N^a), \epsilon) + \gamma^{(c)}(\alpha_s(Q^2), \epsilon) + O\left(\frac{1}{N}\right), \qquad (71)$$

where

$$\begin{aligned}
\gamma^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2/N^a), \epsilon) &= \hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2/N^a), \epsilon) + \\
&\quad - \hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2), \epsilon), \qquad (72) \\
\gamma^{(c)}(\alpha_s(Q^2), \epsilon) &= \hat{\gamma}^{(c)}(\alpha_s(Q^2), \epsilon) + \hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2), \epsilon). \qquad (73)
\end{aligned}$$

Now, $\gamma^{(c)}$ is clearly finite in $\epsilon = 0$, because at $N = 1$ $\gamma^{(l)}$ vanishes and it is N-independent. This also implies that $\gamma^{(l)}$ is finite for all N, because γ shoul be finite for all N. Therefore, $\gamma^{(l)}$ provides an expression of the resummed physical anomalous dimension in the large N limit, up to non-logarithmic terms:

$$\gamma(N, \alpha_s(Q^2), \epsilon) = \gamma^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2/N^a), \epsilon) + O(N^0). \qquad (74)$$

It is apparent from Eq.(73) that $\gamma^{(c)}$ is a power series in $\alpha_s(Q^2)$ with finite coefficients in the $\epsilon \to 0$ limit. In order to understand the perturbative structure of $\gamma^{(l)}$ as well, define implicitly the function $g(\alpha_s(Q^2), \alpha_s(Q^2/n), \epsilon)$ as

$$\gamma^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2/N^a), \epsilon) = \int_1^{N^a} \frac{dn}{n} \, g(\alpha_s(Q^2), \alpha_s(Q^2/n), \epsilon), \qquad (75)$$

where

$$g(\alpha_s(Q^2), \alpha_s(\mu^2), \epsilon) \equiv -\mu^2 \frac{\partial}{\partial \mu^2} \hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(\mu^2), \epsilon) \tag{76}$$

$$= -\beta^{(d)}(\alpha_s(\mu^2), \epsilon) \frac{\partial}{\partial \alpha_s(\mu^2)} \hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(\mu^2), \epsilon), \tag{77}$$

with $\beta^{(d)}(\alpha_s)$ is the d-dimensional beta function

$$\beta^{(d)}(\alpha_s(\mu^2), \epsilon) - \epsilon \alpha_s(\mu^2) + \beta(\alpha_s(\mu^2)) \tag{78}$$

and where we have performed the change of variable $n = Q^2/\mu^2$. It immediately follows from Eqs.(68-77) that g is a power series in $\alpha_s(Q^2)$ and $\alpha_s(\mu^2)$ with finite coefficients in the limit $\epsilon \to 0$:

$$\lim_{\epsilon \to 0} g(\alpha_s(Q^2), \alpha_s(\mu^2), \epsilon) \equiv g(\alpha_s(Q^2), \alpha_s(\mu^2))$$

$$= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} g_{mn} \alpha_s^m(Q^2) \alpha_s^n(\mu^2). \tag{79}$$

Hence, our final result for the four-dimensional all-rder resummed physical anomalous dimension is given by

$$\gamma^{\text{res}}(N, \alpha_s(Q^2)) = \int_1^{N^a} \frac{dn}{n} g(\alpha_s(Q^2), \alpha_s(Q^2/n)) + O(N^0). \tag{80}$$

This result can be compared to the all-order resummation formula derived in Ref.[51]. This resummation has the form of Eq.(80), but with g a function of $\alpha_s(\mu^2)$ only, i.e. with all $g_{mn} = 0$ when $m > 0$ and our result is thus less predictive in the sense that it requires a higher fixed-order computation of the physical anomalous dimension in order to extract the resummation coefficients g_{mn}. The predictive power of the resummation formulae is analyzed in detail in Chapter 7. According to the more restrictive result of Ref.[51], Eq.(80) becomes

$$\gamma^{\text{res}}(N, \alpha_s(Q^2)) = \int_1^{N^a} \frac{dn}{n} g(\alpha_s(Q^2/n)) + O(N^0), \tag{81}$$

where

$$g(\alpha_s(\mu^2)) = \sum_{n=1}^{\infty} g_{0n} \alpha_s^n(\mu^2). \tag{82}$$

The conditions under which the more restrictive result of Ref.[51] holds can be understood by comparing to our approach the derivation of that result. The approach of Ref.[51] is based on assuming the validity of the factorization formula Eq.(47) of Sec.2.3.2 which is more restrictive than the standard collinear factorization. This factorization was proven for a wide class of processes in Ref.[11], and implies that the perturbative coefficient function Eq.(57) in the large N limit can be factored as:

$$C^{(0)}(N, Q^2, \alpha_0, \epsilon) = C^{(0,l)}(Q^2/N^a, \alpha_0, \epsilon) C^{(0,c)}(Q^2, \alpha_0, \epsilon). \tag{83}$$

We notice that this can happen if and only if the coefficients $C_{nk}^{(0)}(\epsilon)$ in Eq.(57) can be written in the form

$$C_{nk}^{(0)}(\epsilon) = F_k(\epsilon)G_{n-k}(\epsilon). \tag{84}$$

The validity of factorization Eq.(47) of Sec.2.3.2 to all orders and for various processes is based on assumptions whose reliability will not be discussed here. Anyway, Eq.(83) implies that the physical anomalous dimension Eq.(60) becomes

$$\gamma(N, \alpha_s(Q^2), \epsilon) = \gamma^{(l)}(\alpha_s(Q^2/N^a), \epsilon) + \gamma^{(c)}(\alpha_s(Q^2), \epsilon). \tag{85}$$

Then, proceeding as before, one then ends up with the resummation formula Eq.(81).

3.4 NLL resummation

In this Section, we shall give explicit expressions of the reummation formulae at NLL for the deep-inelastic structure function F_2 and for the Drell-Yan cross section. These explicit expressions are useful for practical computations and we shall use them in Chapter 5.

The expression of the resummed physical anomalous dimension in Eq.(80) can be used to compute the resummed evolution factor $K^{res}(N; Q_0^2, Q^2)$ Eq.(19) in Section 2.2. At NLL, we get

$$K_{\mathrm{NLL}}(N; Q_0^2, Q^2) = \exp\left[E_{\mathrm{NLL}}(N; Q_0^2, Q^2)\right] = \exp\int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \gamma_{\mathrm{NLL}}^{\mathrm{res}}(N, \alpha_s(k^2)), \tag{86}$$

where

$$\gamma_{\mathrm{NLL}}^{\mathrm{res}}(N, \alpha_s(k^2)) = \int_1^{N^a} \frac{dn}{n} \left[g_{01}\alpha_s(k^2/n) + g_{02}\alpha_s^2(k^2/n)\right]; \qquad g_{11} = 0. \tag{87}$$

The fact that the resummation coefficient $g_{11} = 0$ for both the deep-inelastic and the Drell-Yan case can be shown by explicit computations of the fixed-order anomalous dimension (see Ref.[8] in Section 4.3). This shows that at NLL level Eq.(81) holds. However this does not mean that this is the case at all logarithmic orders.

Many times in literature, the resummed results are given in terms of the Mellin transform of a resummed physical anomalous dimension in z space. To rewrite Eq.(87) as the Mellin transform of a function of x, we can use the all-orders relations between the logs of N and the logs of $1 - z$ that are given in Appendix A. In particular, we can use Eqs.(23,24) in Appendix A to rewrite Eq.(87) at NLL in the following form:

$$\gamma_{\mathrm{NLL}}^{\mathrm{res}}(N, \alpha_s(k^2)) = a \int_0^1 dx \frac{x^{N-1} - 1}{1 - x} \left[\hat{g}_{01}\alpha_s(k^2(1-x)^a) + \hat{g}_{02}\alpha_s^2(k^2(1-x)^a)\right], \tag{88}$$

where

$$\hat{g}_{01} = -g_{01}, \qquad \hat{g}_{02} = -(g_{02} + a\gamma_E b_0 g_{01}). \tag{89}$$

and where we have used the definition of the beta function Eq.(43) in Section 1.2. As a consequence, the NLL resummed exponent in Eq.(86) can be rewritten in the following form

$$E_{\mathrm{NLL}}(N; Q_0^2, Q^2) = a \int_0^1 dx \frac{x^{N-1} - 1}{1 - x} \int_{Q_0^2(1-x)^a}^{Q^2(1-x)^a} \frac{dk^2}{k^2} \hat{g}(\alpha_s(k^2)), \tag{90}$$

where

$$\hat{g}(\alpha_s) = \hat{g}_{01}\alpha_s(k^2) + \hat{g}_{02}\alpha_s^2(k^2) \tag{91}$$

Beyond leading order the standard anomalous dimension differs from the physical one, so $\hat{g}_{02}$ receives a contribution both from the standard anomalous dimension and from the coefficient function. It is thus natural to rewrite the resummation formula Eq.(90) separating off the contribution the contribution which originates from the anomalous dimension γ^{AP} Eq.(18) of Section 2.2. This is done defining two functions of α_s, $A(\alpha_s)$ and $B^a(\alpha_s)$ in such a way that

$$\hat{g}(\alpha_s) = A(\alpha_s) + \frac{\partial B^{(a)}(\alpha_s(k^2))}{\partial \ln k^2}, \quad A(\alpha_s) = A_1\alpha_s + A_2\alpha_s^2, \quad B^{(a)}(\alpha_s) = B_1^{(a)}\alpha_s. \tag{92}$$

It is clear that the constant A_i are obtained directly form the coefficients of the $1/[1-x]_+$ terms of the i-loop quark-quark splitting functions and that the coefficients B_i^a depends on the particular process ($a = 1$ for DIS and $a = 2$ for DY). To the NLL order, we find that

$$\hat{g}_{01} = A_1, \qquad \hat{g}_{02} = A_2 - b_0 B_1^{(a)} \tag{93}$$

and that

$$E_{\mathrm{NLL}}(N; Q_0^2, Q^2) = a \int_0^1 dx \frac{x^{N-1} - 1}{1 - x} \left[\int_{Q_0^2(1-x)^a}^{Q^2(1-x)^a} \frac{dk^2}{k^2} A(\alpha_s(k^2)) \right.$$

$$\left. + B^{(a)}(\alpha_s(Q^2(1-x)^a)) - B^{(a)}(\alpha_s(Q_0^2(1-x)^a)) \right]. \tag{94}$$

We can then rewrite the resummed cross section

$$\sigma_{\mathrm{NLL}}(N, Q^2) = \exp \left[E_{\mathrm{NLL}}(N; Q_0^2, Q^2) \right] \sigma_{\mathrm{NLL}}(N, Q_0^2) \tag{95}$$

in a factorized form according to Eqs.(11,12) in Section 2.2 by collecting all Q^2-dependent contributions to the resummation Eq.(94) into a resummed perturbative coefficient function C_{NLL}:

$$\sigma_{\mathrm{NLL}}(N, Q^2) = C_{\mathrm{NLL}}(N, Q^2/\mu^2, \alpha_s(\mu^2))F(N, \mu^2), \tag{96}$$

where

$$C_{\mathrm{NLL}}(N, Q^2/\mu^2, \alpha_s(\mu^2)) = \exp \left\{ a \int_0^1 dx \frac{x^{N-1} - 1}{1 - x} \left[\int_{\mu^2}^{Q^2(1-x)^a} \frac{dk^2}{k^2} A(\alpha_s(k^2)) \right. \right.$$

$$\left. \left. + B^{(a)}(\alpha_s(Q^2(1-x)^a)) \right] \right\}, \tag{97}$$

which ha the same form of the resummed results discussed in Section 2.3.1. The precise definition of the parton distribution F and the factorization scale μ^2 will depend on the choice of factorization scheme: according to the choice of scheme, the resummed terms will be either part of the hard coefficient function C_{NLL}, or of the evolution of the parton distribution F. In the $\overline{MS}$ scheme the NLL coefficients A_1, A_2 and $B_1^{(a)}$ are given in Eq.(46) of Section 2.3.1 and the NNLL ones are given for example in Ref.[52]. These coefficients can also be obtained comparing a fixed order computation of the physical anomalous dimension with a fixed order expansion of Eq.(80) as is shown explicitly in Section 4.3 of Ref.[8].

To compute explicitly Eq.(97), we first exploit Eq.(20) in Appendix A at NLL level, thus finding

$$
C_{\mathrm{NLL}}(N, Q^2/\mu^2, \alpha_s(\mu^2)) = \exp\left\{ -\int_1^{N^a} \frac{dn}{n} \left[\int_{n\mu^2}^{Q^2} \frac{dk^2}{k^2} A(\alpha_s(k^2/n)) + \tilde{B}^{(a)}(\alpha_s(Q^2/n)) \right] \right\},
$$
(98)

where $\tilde{B}^{(a)}(\alpha_s) = B^{(a)}(\alpha_s) - a\gamma_E A_1 \alpha_s$. The explicit expression of Eq.(97) is then obtained performing the changes of variables,

$$
\frac{dk^2}{k^2} = \frac{d\alpha_s(k^2/n)}{\beta(\alpha_s(k^2/n))}, \quad \frac{dn}{n} = -\frac{d\alpha_s(Q^2/n)}{\beta(\alpha_s(Q^2/n))},
$$
(99)

to evaluate the integrals in Eq.(98) and using the two loop solution of the renormalitazion-group equation for the running of α_s given in Eq.(41) of Section 1.2, Now, after some algebra we find for the integral in Eq.(98):

$$
-\int_1^{N^a} \frac{dn}{n} \left[\int_{n\mu^2}^{Q^2} \frac{dk^2}{k^2} \left(A_1 \alpha_s\left(\frac{k^2}{n}\right) + A_2 \alpha_s^2\left(\frac{k^2}{n}\right) \right) + \tilde{B}_1^{(a)} \alpha_s\left(\frac{Q^2}{n}\right) \right]
$$
$$
= \log N g_1(\lambda, a) + g_2, (\lambda, a)
$$
(100)

where $\lambda = b_0 \alpha_s(\mu_r^2) \log N$ and

$$
g_1(\lambda, a) = \frac{A_1}{b_0 \lambda}[a\lambda + (1 - a\lambda)\log(1 - a\lambda)]
$$
(101)

$$
g_2(\lambda, a) = -\frac{A_1 a\gamma_E - B_1^{(a)}}{b_0} \log(1 - a\lambda) + \frac{A_1 b_1}{b_0^3}[a\lambda + \log(1 - a\lambda) + \frac{1}{2}\log^2(1 - a\lambda)]
$$
$$
-\frac{A_2}{b_0^2}[a\lambda + \log(1 - a\lambda)] + \log\left(\frac{Q^2}{\mu_r^2}\right) \frac{A_1}{b_0} \log(1 - a\lambda)
$$
$$
+ \log\left(\frac{\mu^2}{\mu_r^2}\right) \frac{A_1}{b_0} a\lambda,
$$
(102)

where $a = 1$ for the DIS case and $a = 2$ for the DY case. Evidently, in Eqs.(101,102) there is a dependence on the renormalization scale. To obtain the desired result, we simply have to keep the renormalization scale equal to the factorization scale $\mu_r^2 = \mu^2$. Thus, for the explicit analytic expression of Eq.(98), we get

$$
C_{\mathrm{NLL}}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = [\exp\{\log N g_1(\lambda, a) + g_2(\lambda, a)\}]_{\mu_r^2 = \mu^2},
$$
(103)

with the resummation coefficients given in Eq.(46) in Section 2.3.1 for the $\overline{MS}$ factorization scheme choice. However, a general analysis of the factorization scheme choices and changes for the resummation formulae is given for example in Section 6 of Ref.[8].

Chapter 4

Renormalization group resummation of prompt photon production

In this chapter, we prove the all-order exponentiation of soft logarithmic corrections to prompt photon production in hadronic collisions, by generalizing the renormalization group approach of chapter 3. Here, we will show that all large logs in the soft limit can be expressed in terms of two dimensionful variables. Then, we use the renormalization group to resum them. The resummation formulae that we obtain are more general though less predictive than those that can be obtained with other approaches discussed in chapter 2.

4.1   Kinematics and notation

We consider the process

$$H_1(P_1) + H_2(P_2) \to \gamma(p_\gamma) + X, \tag{1}$$

of two colliding hadrons H_1 and H_2 with momentum P_1 and P_2 respectively into a real photon with momentum p_γ and any collection of hadrons X. More specifically, we are interested in the differential cross section $p_\perp^3 \frac{d\sigma}{dp_\perp}(x_\perp, p_\perp^2)$, where $p_\perp$ is the transverse momentum of the photon with respect to the direction of the colliding hadrons H_1 and H_2, and

$$x_\perp = \frac{4p_\perp^2}{S}; \qquad S = (P_1 + P_2)^2. \tag{2}$$

The scaling variable x can be viewed as the squared fraction of transverse energy that the hadrons transfer to the outgoing particles (hence $0 \le x_\perp \le 1$) and S is the hadronic center-of-mass energy. We parametrize the momentum of the photon in terms of its partonic center-of-mass pseudorapidity and its transverse momentum $\vec{p}_\perp$. The pseudorapidiry of a massless particle is defined in terms its scattering angle θ in the center-of-mass frame as follows

$$\eta = -\ln(\tan(\theta/2)). \tag{3}$$

So, in the partonic center-of-mass frame, we can write:

$$p_\gamma = (p_\perp \cosh \hat{\eta}_\gamma, \vec{p}_\perp, p_\perp \sinh \hat{\eta}_\gamma). \tag{4}$$

In the same frame, the incoming partons' momenta can be written as

$$p_1 = x_1 P_1 = \frac{\sqrt{s}}{2}(1, \vec{0}_\perp, 1), \qquad p_2 = x_2 P_2 = \frac{\sqrt{s}}{2}(1, \vec{0}_\perp, -1), \tag{5}$$

where $x_{1(2)}$ are the longitudinal fraction of momentum of the parton $1(2)$ in the hadron $H_{1(2)}$ and $s = (p_1 + p_2)^2 = x_1 x_2 S$ is the center-of-mass energy of the partonic process. The relation between the hadronic center-of-mass pseudorapidity and the partonic one is obtained performing a boost along the collision axis:

$$\hat{\eta}_\gamma = \eta_\gamma - \frac{1}{2} \ln \frac{x_1}{x_2}. \tag{6}$$

The factorized expression for this cross section in perturbative QCD is

$$
\begin{aligned}
p_\perp^3 \frac{d\sigma}{dp_\perp}(x_\perp, p_\perp^2) &= \sum_{a,b} \int_0^1 dx_1\, dx_2\, dz\, x_1 F_a^{H_1}(x_1, \mu^2)\, x_2 F_b^{H_2}(x_2, \mu^2) \\
&\quad \times C_{ab}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) \delta\left(x_\perp - z x_1 x_2\right),
\end{aligned}
\tag{7}
$$

where $F_a^{H_1}(x_1, \mu^2)$, $F_b^{H_2}(x_2, \mu^2)$ are the distribution functions of partons a, b in the colliding hadrons. Here we have defined the perturbative scale Q^2 and the partonic scaling variable z as the squared fraction of transverse energy that the parton a, b transfer to the outgoing partons ($0 \leq z \leq 1$):

$$Q^2 = 4p_\perp^2, \tag{8}$$

$$z = \frac{Q^2}{s} = \frac{Q^2}{x_1 x_2 S}. \tag{9}$$

The coefficient function $C_{ab}(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2))$ is defined in terms of the partonic cross section for the process where partons a, b are incoming as

$$C_{ab}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = p_\perp^3 \frac{d\hat{\sigma}_{ab}}{dp_\perp}. \tag{10}$$

4.2   Leading order calculation

The hard-scattering subprocesses that contribute to the amplitude of the prompt-photon prodution at the leading order are:

$$q(p_1) + \bar{q}(p_2) \quad \rightarrow \quad g(p') + \gamma(p_\gamma) \tag{11}$$

$$q(p_1) + g(p_2) \quad \rightarrow \quad q(p') + \gamma(p_\gamma) \tag{12}$$

$$\bar{q}(p_1) + g(p_2) \quad \rightarrow \quad \bar{q}(p') + \gamma(p_\gamma) \tag{13}$$
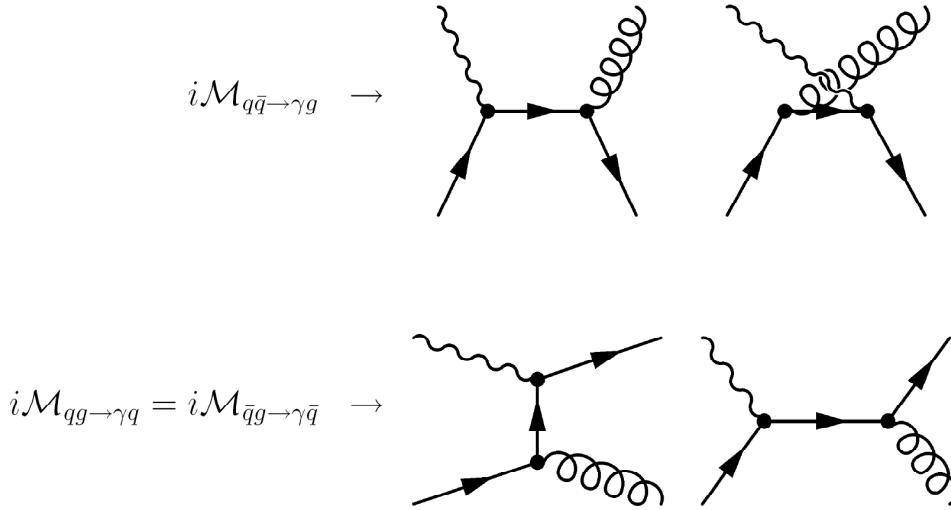
Figure 4.1: Feynman graphs that contribute to the amplitude of the partonic process at the leading order

The corresponding Feynman graphs are shown in figure 4.1.

We now want to obtain the coefficient function for these two elementary subprocesses at the leading order. Using the QCD Feynman rules to evaluate the first amplitude in figure 4.1, we have:

$$iM_{q\bar{q}\to\gamma g} = -i\bar{v}(p_2)Q_q e g \left[\frac{\gamma^\rho \not{p}_\gamma \gamma^\mu - 2\gamma^\rho p_1^\mu}{2p_1\cdot p_\gamma} + \frac{\gamma^\mu \not{p}'\gamma^\rho - 2\gamma^\mu p_1^\rho}{2p_1\cdot p'}\right]\epsilon_\mu^*(p_\gamma)\epsilon_\rho^*(p')t_{ji}^a u(p_1),$$

$$(14)$$

where $e = |e|$ is the electrical charge, g is the strong charge and Q_q is the electrical charge of the quark q in units of e. Taking the square modulus of equation (14) and averaging over the two quarks polarizations and colours we obtain:

$$\frac{1}{4}\frac{1}{N_C^2}\sum_{\text{pol,col}}|\mathcal{M}_{q\bar{q}\to\gamma g}|^2 = 2\frac{C_F}{N_C}Q_q^2 e^2 g^2\left[\frac{p_1\cdot p_\gamma}{p_1\cdot p'} + \frac{p_1\cdot p'}{p_1\cdot p_\gamma}\right],\qquad (15)$$

where N_C is the number of quark colours and $C_F \equiv (N_C^2 - 1)/2N_C$ is the Casimir operator with respect to the colour matrix t_{ji}^a.

Proceeding in the same way for the Feynman graphs contributing to the second amplitude in figure 4.1 we could obtain the averaged square modulus for this other subprocess, but, we note that it can be immediately obtained using crossing simmetry. More precisely, we only have to substitute p' with p_2 and take into account the fact that we must now average not over two quarks' colours but over the colours of a quark

and a gluon. Therefore we arrive at the following expression for the averaged square modulus of the second amplitude of figure 4.1:

$$
\frac{1}{4}\frac{1}{2N_CC_F}\frac{1}{N_C}\sum_{\text{pol,col}}|\mathcal{M}_{q(\bar{q})g\to\gamma q(\bar{q})}|^2 = \frac{1}{4}\frac{1}{2N_CC_F}\frac{1}{N_C}\sum_{\text{pol,col}}|\mathcal{M}_{q\bar{q}\to\gamma g}|^2\bigg|_{p'\to p_2} =
$$
$$
= \frac{1}{N_C}Q_q^2e^2g^2\left[\frac{p_1\cdot p_\gamma}{p_1\cdot p_2}+\frac{p_1\cdot p_2}{p_1\cdot p_\gamma}\right]. \tag{16}
$$

We will now rewrite equations (15) and (16) in terms of the kinematic parameters defined in section 4.1. For this purpose we first note that momentum conservation and the parametrizations defined in Eqs.(4,5) imply (in the centre of mass of the incident partons):

$$
p' = (p_\perp\cosh\hat{\eta}_\gamma, -\vec{p}_\perp, -p_\perp\sinh\hat{\eta}_\gamma) \tag{17}
$$
$$
s = (p_1+p_2)^2 = (p_\gamma+p')^2 = 4p_\perp^2\cosh^2\hat{\eta}_\gamma \tag{18}
$$
$$
z = \frac{4p_\perp^2}{s} = \frac{1}{\cosh^2\hat{\eta}_\gamma}. \tag{19}
$$

From the last equation we see that in the limit $z\to 1$, $\hat{\eta}_\gamma\to 0$. Physically this is because in this limit all the centre of mass energy is transverse and so the photon cannot have a non zero pseudorapidity. Now, using again Eqs.(4,5) and the equation for p' (17) we have:

$$
p_1\cdot p_2 = \frac{s}{2}
$$
$$
p_1\cdot p' = \frac{\sqrt{s}}{2}p_\perp e^{\hat{\eta}_\gamma} \tag{20}
$$
$$
p_1\cdot p_\gamma = \frac{\sqrt{s}}{2}p_\perp e^{-\hat{\eta}_\gamma}
$$

A combination of equations (19) and (20) yields our final result for the averaged square modulus of the amplitudes:

$$
\frac{1}{4}\frac{1}{N_C^2}\sum_{\text{pol,col}}|\mathcal{M}_{q\bar{q}\to\gamma g}|^2 = 4\frac{C_F}{N_C}Q_q^2e^2g^2\frac{(2-z)}{z} \tag{21}
$$
$$
\frac{1}{4}\frac{1}{2N_CC_F}\frac{1}{N_C}\sum_{\text{pol,col}}|\mathcal{M}_{q(\bar{q})g\to\gamma q(\bar{q})}|^2 = \frac{1}{2N_C}Q_q^2e^2g^2\left[1\pm\sqrt{1-z}\right.
$$
$$
\left.+\frac{4}{1\pm\sqrt{1-z}}\right], \tag{22}
$$

where the plus sign has to be chosen for positive values of the pseudorapidity $\hat{\eta}_\gamma$ and the minus sign for negative values. The two-body phase space is:

$$
d\phi(p_1+p_2;p_\gamma,p') = \frac{d^3p_\gamma}{(2\pi)^32E_{p_\gamma}}\frac{d^3p'}{(2\pi)^32E_{p'}}(2\pi)^4\delta_{(4)}(p'+p_\gamma-p_1-p_2)
$$
$$
= \frac{d^3p_\gamma}{4E_{p_\gamma}E_{p'}}\frac{1}{(2\pi)^2}\delta_{(1)}(E_{p'}+E_{p_\gamma}-E_{p_1}-E_{p_2}). \tag{23}
$$

Imposing the conservation of spatial momentum ($E_{p_\gamma} = E_{p'} = |\vec{p_\gamma}|$) the two-body phase space becomes:

$$
\begin{aligned}
d\phi(p_1 + p_2; p_\gamma, p') &= \frac{d^3 p_\gamma}{4|\vec{p_\gamma}|^2 (2\pi)^2} \delta_{(1)}(2|\vec{p_\gamma}| - \sqrt{\hat{s}}) \\
&= \frac{1}{16\pi} d\cos\theta d|\vec{p_\gamma}| \delta_{(1)}(|\vec{p_\gamma}| - \sqrt{\hat{s}}/2) \\
&= \frac{1}{16\pi} \delta_{(1)}(p_\perp \cosh\hat{\eta}_\gamma - \sqrt{\hat{s}}/2) d\cos\theta d|\vec{p_\gamma}|,
\end{aligned}
\tag{24}
$$

where θ is the scattering angle of the photon with respect to the collision axis. Because of the fact that we want to integrate over the pseudorapidity of the photon η_γ at fixed $p_\perp$, we must perform a change of variables. In particular we must rewrite the two-body phace space, expressed in terms of $\cos\theta$ and $|\vec{p_\gamma}|$, in terms of the new variables $\cosh\hat{\eta}_\gamma$ and $p_\perp$. This change of variables is given by the equations:

$$
\begin{cases}
\cos\theta = & \tanh\hat{\eta}_\gamma \\
|\vec{p_\gamma}| = & p_\perp \cosh\hat{\eta}_\gamma.
\end{cases}
$$

The determinant of the Jacobian matrix is easily obtained and is given by:

$$
|J| = \left\| \frac{\partial(\cos\theta, |\vec{p_\gamma}|)}{\partial(\cosh\hat{\eta}_\gamma, p_\perp)} \right\| = \frac{1}{\cosh\hat{\eta}_\gamma \sqrt{\cosh^2\hat{\eta}_\gamma - 1}}.
\tag{25}
$$

Thanks to equation (19), we obtain this determinant in term of z:

$$
|J| = \frac{z}{\sqrt{1-z}}
\tag{26}
$$

Using this last result, equation (24) becomes:

$$
\begin{aligned}
d\phi(p_1 + p_2; p_\gamma, p') &= \frac{1}{16\pi} \frac{z}{\sqrt{1-z}} \delta_{(1)}(\cosh\hat{\eta}_\gamma - \sqrt{s}/2p_\perp) \frac{dp_\perp}{p_\perp} d\cosh\hat{\eta}_\gamma \\
&= \frac{1}{16\pi} \frac{z}{\sqrt{1-z}} [\delta_{(1)}(\hat{\eta}_\gamma - \hat{\eta}_+) + \delta_{(1)}(\hat{\eta}_\gamma - \hat{\eta}_-)] \\
&\quad \times \frac{dp_\perp}{p_\perp} d\hat{\eta}_\gamma,
\end{aligned}
\tag{27}
$$

where $\hat{\eta}_+$ and $\hat{\eta}_-$ are the two solutions (one positive and one negative) of the equation imposed by the delta function $p_\perp \cosh\hat{\eta}_\gamma = \sqrt{s}/2$ which are:

$$
\hat{\eta}_\pm = \ln\left(\frac{\sqrt{s}}{2p_\perp} \pm \sqrt{\frac{s}{4p_\perp^2} - 1} \right).
\tag{28}
$$

The flux factor Φ is immediately obtained from equations (5) and (19):

$$
\Phi \equiv 4(p_1 \cdot p_2) = 2s = \frac{8p_\perp^2}{z}.
\tag{29}
$$

Remembering the definition of the QED and QCD coupling constants,

$$\alpha \equiv \frac{e^2}{4\pi} \qquad ; \qquad \alpha_s \equiv \frac{g^2}{4\pi},$$

and putting together expressions (21), (22), (27), (29) and performing the integration over $\hat{\eta}_\gamma$, we obtain our final result for the coefficient function at the leading order for the two subprocesses in figure 4.1:

$$C_{q\bar{q}\to\gamma g}^{(LO)}(z,\alpha_s) = \alpha\alpha_s Q_q^2\pi \frac{C_F}{N_C}\frac{z}{\sqrt{1-z}}(2-z) \qquad (30)$$

$$C_{q(\bar{q})g\to\gamma q(\bar{q})}^{(LO)}(z,\alpha_s) = \alpha\alpha_s Q_q^2\pi \frac{1}{2N_C}\frac{z}{\sqrt{1-z}}\left(1+\frac{z}{4}\right). \qquad (31)$$

4.3 The soft limit

We will study the cross section Eq. (7) in the threshold transverse limit, when the transverse transverse energy of outgoing particles is close to its maximal value ($x_\perp \to 1$ at the hadronic level or, equivalently, $z \to 1$ at the partonic level). The convolution in Eq. (7) is turned into an ordinary product by Mellin transformation:

$$\sigma(N,Q^2) = \sum_{a,b} \sigma_{ab}(N,Q^2) \qquad (32)$$

$$= \sum_{a,b} F_a^{H_1}(N+1,\mu^2)\, F_b^{H_2}(N+1,\mu^2)$$

$$\times C_{ab}\left(N,\frac{Q^2}{\mu^2},\alpha_s(\mu^2)\right). \qquad (33)$$

As discussed in Section 2.2, in the large N limit each parton subprocess can be treated independently, specifically, each C_{ab} is separately renormalization-group invariant.

At this point, it is interesting to discuss the differences in the large N behavior of the partonic subprocesses. The cross sections for the partonic channels with two quarks of different flavors ($ab = q\bar{q}'$, $\bar{q}q'$, qq, qq', $\bar{q}\bar{q}$, $\bar{q}\bar{q}'$) vanish at LO and are hence suppressed by a factor of α_s with respect to the subprocesses with $ab = q\bar{q}$, qg, $\bar{q}g$. Moreover, in the large N limit this relative suppression is further enhanced by a factor of $O(1/N)$ because the processes with two different quark flavors involve the off-diagonal Alatarelli-Parisi splitting functions. Therefore, these partonic channels do not contribute in the large N limit. The partonic channel $ab = gg$ has a different large N behavior. It begins to contribute at NLO via the partonic process $g + g \to \gamma + q + \bar{q}$, which again leads to a suppression effect of $O(1/N)$ with respect to the LO suprocesses. However, owing to the photon-gluon coupling through a fermion box, the partonic subprocess $g + g \to \gamma + g$ is also permitted. This subprocess is logarithmically-enhanced by multiple soft-gluon radiation in the final state, but it starts to contribute only at NNLO in perturbation theory. It follows that the partonic

channel $ab = gg$ is suppressed by a factor α_s^2 with respect to the LO partonic channels $ab = q\bar{q}$, qg, $\bar{q}g$ and it will enter the resummed cross secion only at NNLL order. In conclusion, the partonic channels that should be resummed are $ab = q\bar{q}$, qg, $\bar{q}g$, gg, where the last channel is that coupled to the gluon via a fermion box and eneters resummation only at NNLL.

Furthermore, on top of Eqs. (7, 33) the physical process Eq. (1) receives another factorized contribution, in which the final-state photon is produced by fragmentation of a primary parton produced in the partonic sub-process. However, the cross section for this process is also suppressed by a factor of $\frac{1}{N}$ in the large N limit. This is due to the fact that the fragmentation function carries this suppression, for the same reason why the anomalous dimensions γ_{qg} and γ_{gq} are suppressed. Therefore, we will disregard the fragmentation contribution.

According to Eqs.(19-22) of Section 2.2, the cross section can be written in terms of the physical anomalous dimensions:

$$\sigma(N, Q^2) \;=\; \sum_{a,b} K_{ab}(N; Q_0^2, Q^2)\, \sigma_{ab}(N, Q_0^2) \tag{34}$$

$$= \sum_{a,b} \exp\left[E_{ab}(N; Q_0^2, Q^2) \right] \sigma_{ab}(N, Q_0^2), \tag{35}$$

where

$$E_{ab}(N; Q_0^2, Q^2) \;=\; \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \gamma_{ab}(N, \alpha_s(k^2)) \tag{36}$$

$$= \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} [\gamma_{aa}^{\mathrm{AP}}(N, \alpha_s(k^2)) + \gamma_{bb}^{\mathrm{AP}}(N, \alpha_s(k^2))]$$

$$+ \ln C_{ab}(N, 1, \alpha_s(Q^2)) - \ln C_{ab}(N, 1, \alpha_s(Q_0^2)). \tag{37}$$

In the large-$x_\perp$ limit, the order-n coefficient of the perturbative expansion of the hadronic cross section is dominated by terms proportional to $\left[\frac{\ln^k(1-x_\perp)}{1-x_\perp} \right]_+$, with $k \leq 2n-1$, that must be resummed to all orders. Upon Mellin transformation, these lead to contributions proportional to powers of $\ln\frac{1}{N}$. In the sequel, we will consider the resummation of these contributions to all logarithmic orders, and disregard all contributions to the cross section which are suppressed by powers of $(1 - x_\perp)$, i.e., upon Mellin transformation, by powers of $\frac{1}{N}$.

The resummation is performed in two steps as in chapter 3. First, we show that the origin of the large logs is essentially kinematical: we identify the configurations which contribute in the soft limit, we show by explicit computation that large Sudakov logs are produced by the phase-space for real emission with the required kinematics as logs of two dimensionful variables, and we show that this conclusion is unaffected by virtual corrections. Second, we resum the logs of these variables using the renormalization group.

The l-th order correction to the leading $O(\alpha_s)$ partonic process receives contribution from the emission of up to $l + 1$ massless partons with momenta $k_1, \ldots, k_{l+1}$.

Four-momentum conservation implies:

$$p_1 + p_2 = p_\gamma + k_1 + \ldots k_{l+1}. \tag{38}$$

In the partonic center-of-mass frame, according to Eqs.(4,5), we have

$$(p_1 + p_2 - p_\gamma)^2 = \frac{Q^2}{z}(1 - \sqrt{z}\cosh\hat{\eta}_\gamma) = \sum_{i,j=1}^{l+1} k_i^0 k_j^0 (1 - \cos\theta_{ij}) \geq 0, \tag{39}$$

where θ_{ij} is the angle between $\vec{k}_i$ and $\vec{k}_j$. Hence,

$$1 \leq \cosh\hat{\eta}_\gamma \leq \frac{1}{\sqrt{z}}. \tag{40}$$

Therefore,

$$\sum_{i,j=1}^{l+1} k_i^0 k_j^0 (1 - \cos\theta_{ij}) = \frac{Q^2}{2}(1 - z) + O\left[(1-z)^2\right]. \tag{41}$$

Equation (41) implies that in the soft limit the sum of scalar products of momenta k_i of emitted partons Eq. (39) must vanish. However, contrary to the case of deep-inelastic scattering or Drell-Yan, not all momenta k_i of the emitted partons can be soft as $z \to 1$, because the three-momentum of the photon must be balanced. Assume thus that momenta $k_i, i = 1, \ldots, n; n < l + 1$ are soft in the $z \to 1$ limit, while momenta $k_i, i > n$ are non-soft. For the sake of simplicity, we relabel non-soft momenta as

$$k_j' = k_{n+j}; \qquad 1 \leq j \leq m+1; \qquad m = l - n. \tag{42}$$

The generic kinematic configuration in the $z = 1$ limit is then

$$\begin{aligned} k_i &= 0 \qquad 1 \leq i \leq n \\ \theta_{ij} &= 0; \qquad \sum_{j=1}^{m+1} k_j^{'0} = p_\perp \qquad 1 \leq i,j \leq m+1. \end{aligned} \tag{43}$$

for all n between 1 and l, namely, the configuration where at least one momentum is not soft, and the remaining momenta are either collinear to it, or soft.

With this labelling of the momenta, the phase space can be written, using twice the phase space decomposition of Eq.(12) in Appendix B, as

$$\begin{aligned} &d\phi_{n+m+2}(p_1 + p_2; p_\gamma, k_1, \ldots, k_n, k_1', \ldots, k_{m+1}') \\ &= \int_0^s \frac{dq^2}{2\pi} d\phi_{n+1}(p_1 + p_2; q, k_1, \ldots, k_n) \\ &\times \int_0^{q^2} \frac{dk'^2}{2\pi} d\phi_{m+1}(k'; k_1', \ldots, k_{m+1}') \, d\phi_2(q; p_\gamma, k'). \end{aligned} \tag{44}$$

We shall now compute the phase space in the $z \to 1$ limit in $d = 4 - 2\epsilon$ dimensions. Consider first the two-body phase space $d\phi_2$ in Eq. (44). In the rest frame of q we

have

$$
d\phi_2(q; p_\gamma, k') \;=\; \frac{d^{d-1}k'}{(2\pi)^{d-1}2k'^0} \frac{d^{d-1}p_\gamma}{(2\pi)^{d-1}2p_\gamma^0} (2\pi)^d \delta^{(d)}(q - k' - p_\gamma)
$$

$$
=\; \frac{(4\pi)^\epsilon}{8\pi\Gamma(1-\epsilon)} \frac{P^{1-2\epsilon}}{\sqrt{q^2}} \sin^{-2\epsilon}\theta_\gamma \, d\,|\vec{p}_\gamma| \, d\cos\theta_\gamma \, \delta(|\vec{p}_\gamma| - P),
$$

$$(45)$$

where

$$
P = \frac{\sqrt{q^2}}{2}\left(1 - \frac{k'^2}{q^2}\right). \tag{46}
$$

Because momenta k_i, $i \leq n$ are soft, up to terms suppressed by powers of $1 - z$, the rest frame of q is the same as the center-of-mass frame of the incoming partons, in which

$$
|\vec{p}_\gamma| = p_\perp \cosh\hat{\eta}_\gamma \tag{47}
$$
$$
\cos\theta_\gamma = \tanh\hat{\eta}_\gamma. \tag{48}
$$

Hence,

$$
d\phi_2(q; p_\gamma, k') = \frac{(4\pi)^\epsilon}{8\pi\Gamma(1-\epsilon)} \frac{(Q^2/4)^{-\epsilon}}{\sqrt{q^2}} \, dp_\perp \, d\hat{\eta}_\gamma \, \delta\left(\cosh\hat{\eta}_\gamma - \frac{2P}{\sqrt{Q^2}}\right). \tag{49}
$$

The conditions

$$
\cosh\hat{\eta}_\gamma = \frac{2P}{\sqrt{Q^2}} \geq 1; \qquad k'^2 \geq 0, \tag{50}
$$

together with Eq.(46), restrict the integration range to

$$
Q^2 \leq \quad q^2 \quad \leq s \tag{51}
$$
$$
0 \leq \quad k'^2 \quad \leq q^2 - \sqrt{Q^2 q^2}. \tag{52}
$$

It is now convenient to define new variables u, v

$$
q^2 = Q^2 + u(s - Q^2) \;=\; Q^2\left[1 + u(1-z)\right] + O((1-z)^2) \tag{53}
$$
$$
k'^2 = v(q^2 - \sqrt{Q^2 q^2}) \;=\; Q^2 \frac{1}{2}uv(1-z) + O((1-z)^2) \tag{54}
$$
$$
0 \leq u \leq 1 \quad ; \quad 0 \leq v \leq 1, \tag{55}
$$

in terms of which

$$
P = \frac{\sqrt{Q^2}}{2}\left[1 + \frac{1}{2}u(1-v)(1-z)\right] + O\left[(1-z)^2\right]. \tag{56}
$$

Thus, the two-body phase space Eq. (49) up to subleading terms is given by

$$
d\phi_2(q; p_\gamma, k') = \frac{(4\pi)^\epsilon}{8\pi\Gamma(1-\epsilon)} \frac{(Q^2/4)^{-\epsilon}}{\sqrt{Q^2}} \, dp_\perp \, d\hat{\eta}_\gamma \, \frac{\delta(\hat{\eta}_\gamma - \hat{\eta}_+) + \delta(\hat{\eta}_\gamma - \hat{\eta}_-)}{\sqrt{u(1-v)(1-z)}}, \tag{57}
$$

where

$$\hat{\eta}_{\pm} = \ln\left(\frac{2P}{\sqrt{Q^2}} \pm \sqrt{\frac{4P^2}{Q^2} - 1}\right) = \pm\sqrt{u(1-v)(1-z)}. \tag{58}$$

We now note that the phase-space element $d\phi_{n+1}(p_1 + p_2; q, k_1, \ldots, k_n)$ contains in the final state a system with large invariant mass $q^2 \geq Q^2$, plus a collection of n soft partons: this same configuration is encountered in the case of Drell-Yan pair production in the limit $z_{DY} = q^2/s \to 1$, discussed in Section 3.2 . Likewise, the phase space for the set of collinear partons $d\phi_{m+1}(k'; k'_1, \ldots, k'_{m+1})$ is the same as the phase space for deep-inelastic scattering (discussed in Section 3.1), where the invariant mass of the initial state k'^2 vanishes as $1 - z$ (see Eq. (54)). We may therefore use the results obtained in chapter 3, where, in the case of deep-inelastic scattering, one of the outgoing parton momenta (k'_{m+1}, say) was identified with the momentum of the leading-order outgoing quark p'. Hence Eq. (60) is obtained from the corresponding result in chapter 3 for deep-inelastic scattering by the replacement $p' \to k'_{m+1}$:

$$
\begin{aligned}
d\phi_{n+1}(p_1 + p_2; q, k_1, \ldots, k_n) &= 2\pi \left[\frac{N(\epsilon)}{2\pi}\right]^n (q^2)^{-n(1-\epsilon)}(s - q^2)^{2n-1-2n\epsilon} \\
&\quad \times d\Omega^{(n)}(\epsilon) \tag{59}
\end{aligned}
$$

$$
d\phi_{m+1}(k'; k'_1, \ldots, k'_{m+1}) = 2\pi \left[\frac{N(\epsilon)}{2\pi}\right]^m (k'^2)^{m-1-m\epsilon} d\Omega'^{(m)}(\epsilon), \tag{60}
$$

where $N(\epsilon) = 1/(2(4\pi)^{2-2\epsilon})$ and

$$
\begin{aligned}
d\Omega^{(n)}(\epsilon) &= d\Omega_1 \ldots d\Omega_n \int_0^1 dz_n z_n^{(n-2)+(n-1)(1-2\epsilon)}(1 - z_n)^{1-2\epsilon} \ldots \\
&\quad \times \int_0^1 dz_2 z_2^{1-2\epsilon}(1 - z_2)^{1-2\epsilon} \tag{61}
\end{aligned}
$$

$$
\begin{aligned}
d\Omega'^{(m)}(\epsilon) &= d\Omega'_1 \ldots d\Omega'_m \int_0^1 dz'_m z_m'^{(m-2)-(m-1)\epsilon}(1 - z'_m)^{1-2\epsilon} \ldots \\
&\quad \times \int_0^1 dz'_2 z_2'^{-\epsilon}(1 - z'_2)^{1-2\epsilon}. \tag{62}
\end{aligned}
$$

Here, the variables z_i, z'_i are defined as in chapter 3 for the Drell-Yan and deep-inelastic scattering respectively.

Equations (53,54) imply that the phase space depends on $(1 - z)^{-\epsilon}$ through the two variables

$$k'^2 \propto Q^2(1 - z) \tag{63}$$

$$\frac{(s - q^2)^2}{q^2} \propto Q^2(1 - z)^2, \tag{64}$$

where the coefficients of proportionality are dimensionless and z-independent. By explicitly combining the two-body phase space Eq. (57) and the phase spaces for soft

radiation Eq. (59) and for collinear radiation Eq. (60) we get

$$
\begin{aligned}
d\phi_{n+m+2}&(p_1 + p_2; p_\gamma, k_1, \ldots, k_n, k'_1, \ldots, k'_{m+1}) \\
&= (Q^2)^{n+m-(n+m+1)\epsilon} \frac{dp_\perp}{p_\perp} \frac{(1-z)^{2n+m-(2n+m)\epsilon}}{\sqrt{1-z}} \\
&\quad \times 2^{-m+m\epsilon} \frac{(16\pi)^{-1+\epsilon}}{\Gamma(1-\epsilon)} \left[\frac{N(\epsilon)}{2\pi}\right]^{n+m} d\hat{\eta}_\gamma \, d\Omega^{(n)}(\epsilon) \, d\Omega'^{(m)}(\epsilon) \\
&\quad \times \int_0^1 du \, \frac{u^{m-m\epsilon}(1-u)^{2n-1-2n\epsilon}}{\sqrt{u}} \int_0^1 dv \, \frac{v^{m-1-m\epsilon}}{\sqrt{1-v}} [\delta(\hat{\eta}_\gamma - \hat{\eta}_+) + \delta(\hat{\eta}_\gamma - \hat{\eta}_-)].
\end{aligned}
\tag{65}
$$

In the limiting cases $n = 0$ and $m = 0$ we have

$$
d\phi_1(p_1 + p_2; q) \;=\; 2\pi\delta(s - q^2) = \frac{2\pi}{Q^2(1-z)}\,\delta(1-u)
\tag{66}
$$

$$
d\phi_1(k'; p') \;=\; 2\pi\delta(k'^2) = \frac{4\pi}{Q^2 u(1-z)}\,\delta(v);
\tag{67}
$$

the corresponding expressions for the phase space are therefore obtained by simply replacing

$$
(1-u)^{-1} \, d\Omega^{(n)}(\epsilon) \to \delta(1-u); \qquad v^{-1} \, d\Omega'^{(m)}(\epsilon) \to \delta(v)
\tag{68}
$$

in Eq. (65) for $n = 0$, $m = 0$ respectively.

The logarithmic dependence of the four-dimensional cross section on $1 - z$ is due to interference between powers of $(1 - z)^{-\epsilon}$ and $\frac{1}{\epsilon}$ poles in the d-dimensional cross section. Hence, we must classify the dependence of the cross section on powers of $(1 - z)^{-\epsilon}$. We have established that in the phase space each real emission produces a factor of $[Q^2(1 - z)^2]^{1-\epsilon}$ if the emission is soft and a factor of $[Q^2(1 - z)]^{1-\epsilon}$ if the emission is collinear. The squared amplitude can only depend on $(1 - z)^{-\epsilon}$ because of loop integrations. This dependence for a generic proper Feynman diagram G will in general appear, as discussed in chapter 3, through the coefficient (see Eq.(22) of Section 3.1)

$$
[D_G(P_E)]^{dL/2-I},
\tag{69}
$$

where L and I are respectively the number of loops and internal lines in G, and $D_G(P_E)$ is a linear combination of all scalar products P_E of external momenta. In the soft limit all scalar products which vanish as $z \to 1$ are either proportional to $Q^2(1 - z)$ or to $Q^2(1 - z)^2$ as shown in Eqs.(47,53,54) of Section 3.2 . Equation (69) then implies that each loop integration can carry at most a factor of $[Q^2(1 - z)^2]^{-\epsilon}$ or $[Q^2(1 - z)]^{-\epsilon}$.

This then proves that the perturbative expansion of the bare coefficient function, for each sub-process which involves partons a, b, takes the form

$$
C^{(0)}(z, Q^2, \alpha_0, \epsilon) \;=\; \alpha\alpha_0(Q^2)^{-\epsilon} \sum_{l=0}^{\infty} \alpha_0^l \, C_l^{(0)}(z, Q^2, \epsilon)
\tag{70}
$$

$$
C_l^{(0)}(z, Q^2, \alpha_0, \epsilon) \;=\; \frac{(Q^2)^{-l\epsilon}}{\Gamma(1/2)\sqrt{1-z}} \sum_{k=0}^{l} \sum_{k'=0}^{l-k} C_{lkk'}^{(0)}(\epsilon)(1 - z)^{-2k\epsilon - k'\epsilon},
\tag{71}
$$

where the factor $1/\Gamma(1/2)$ was introduced for later convenience and terms $C_{lkk'}^{(0)}$ with $k + k' < l$ at order α_s^l are present in general because of loops. The coefficients $C_{lkk'}^{(0)}$ have poles in $\epsilon = 0$ up to order $2l$. To understand this, we have to count the independent variables for the prompt photon process. We have 2 incoming particles and $l + 2$ outgoing partons (a leading-order parton, the photon and l extra emissions). Therefore, imposing the on-shell conditions and the constraints due to Poincar invariance, we get

$$4(l + 4) - (l + 4) - 10 = 3l + 2, \tag{72}$$

independent variables. Now, we need to understand which are these independent variables: the general expression of the phase space in the threshold limit Eq.(65) is written in terms of $3l + 3$ variables which are

$$s, 4p_\perp^2, u, v, z_2, \ldots, z_n, z_2', \ldots, z_m', \Omega_1, \ldots, \Omega_n, \Omega_1', \ldots, \Omega_m', \hat{\eta}_\gamma, \tag{73}$$

where $n + m = l$ and each solid angle depends on two parameters. Clearly, one of them must be a function of some of the others because of Eq. 72. In fact, from Eq.(58), we know that $\hat{\eta}_\gamma$ depends on $u, v, 4p_\perp^2, s$. Thus, the $3l + 2$ independent variables on which depends the square modulus amplitude can be chosen as

$$s, 4p_\perp^2, u, v, z_2, \ldots, z_n, z_2', \ldots, z_m', \Omega_1, \ldots, \Omega_n, \Omega_1', \ldots, \Omega_m'. \tag{74}$$

Now, each of the l integrations over a solid angle can produce a pole $1/\epsilon$ from the collinear region. Furthermore, each of the l integrations over a dimensionless variable $u, v, z_2, \ldots, z_n, z_2', \ldots, z_m'$ can produce a pole $1/\epsilon$ from the soft region. This explains why the coefficients $C_{lkk'}^{(0)}$ can have poles in $\epsilon = 0$ up to order $2l$.

4.4 Resummation from renormalization group improvement

The Mellin transform of Eq. (70) can be performed using

$$\int_0^1 dz\, z^{N-1}(1 - z)^{-1/2 - 2k\epsilon - k'\epsilon} = \frac{\Gamma(1/2)}{\sqrt{N}} N^{2k\epsilon} N^{k'\epsilon} + O\left(\frac{1}{N}\right), \tag{75}$$

with the result

$$
\begin{aligned}
& C^{(0)}(N, Q^2, \alpha_0, \epsilon) \\
& = \frac{\alpha\alpha_0(Q^2)^{-\epsilon}}{\sqrt{N}} \sum_{l=0}^{\infty} \sum_{k=0}^{l} \sum_{k'=0}^{l-k} C_{lkk'}^{(0)}(\epsilon) \left[\left(\frac{Q^2}{N^2}\right)^{-\epsilon} \alpha_0\right]^k \left[\left(\frac{Q^2}{N}\right)^{-\epsilon} \alpha_0\right]^{k'} \\
& \times \left[(Q^2)^{-\epsilon}\alpha_0\right]^{l-k-k'} + O\left(\frac{1}{N}\right).
\end{aligned} \tag{76}
$$

Equation (76) shows that indeed as $N \to \infty$, up to $\frac{1}{N}$ corrections, the coefficient function depends on N through the two dimensionful variables $\frac{Q^2}{N^2}$ and $\frac{Q^2}{N}$. The argument henceforth follows in the same way as in chapter 3, in this more general situation.

The argument is based on the observation that, because of collinear factorization, the physical anomalous dimension

$$\gamma(N, \alpha_s(Q^2)) = Q^2 \frac{\partial}{\partial Q^2} \ln C(N, Q^2/\mu^2, \alpha_s(\mu^2)) \tag{77}$$

is renormalization-group invariant and finite when expressed in terms of the renormalized coupling $\alpha_s(\mu^2)$, related to α_0 by

$$\alpha_0(\mu^2, \alpha_s(\mu^2)) = \mu^{2\epsilon} \alpha_s(\mu^2) \, Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon), \tag{78}$$

where $Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon)$ is a power series in $\alpha_s(\mu^2)$. Because α_0 is manifestly independent of μ^2, Eq. (78) implies that the dimensionless combination $(Q^2)^{-\epsilon}\alpha_0(\alpha_s(\mu^2), \mu^2)$ can depend on Q^2 only through $\alpha_s(Q^2)$:

$$(Q^2)^{-\epsilon}\alpha_0(\alpha_s(\mu^2), \mu^2) = \alpha_s(Q^2) \, Z^{(\alpha_s)}(\alpha_s(Q^2), \epsilon). \tag{79}$$

Using Eq. (79) in Eq. (76), the coefficient function and consequently the physical anomalous dimension are seen to be given by a power series in $\alpha_s(Q^2)$, $\alpha_s(Q^2/N)$ and $\alpha_s(Q^2/N^2)$:

$$\gamma(N, \alpha_s(Q^2), \epsilon) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \gamma_{mnp}^R(\epsilon) \, \alpha_s^m(Q^2) \, \alpha_s^n(Q^2/N^2) \, \alpha_s^p(Q^2/N). \tag{80}$$

Even though the anomalous dimension is finite as $\epsilon \to 0$ for all N, the individual terms in the expansion Eq. (57) are not separately finite. However, if we separate N-dependent and N-independent terms in Eq. (57):

$$\gamma(N, \alpha_s(Q^2), \epsilon) = \hat{\gamma}^{(c)}(\alpha_s(Q^2), \epsilon) + \hat{\gamma}^{(l)}(N, \alpha_s(Q^2), \epsilon), \tag{81}$$

we note that the two functions

$$\gamma^{(c)}(\alpha_s(Q^2), \epsilon) \equiv \hat{\gamma}^{(c)}(\alpha_s(Q^2), \epsilon) + \hat{\gamma}^{(l)}(1, \alpha_s(Q^2), \epsilon) \tag{82}$$

$$\gamma^{(l)}(N, \alpha_s(Q^2), \epsilon) \equiv \hat{\gamma}^{(l)}(N, \alpha_s(Q^2), \epsilon) - \hat{\gamma}^{(l)}(1, \alpha_s(Q^2), \epsilon) \tag{83}$$

must be separately finite. In fact,

$$\gamma(N, \alpha_s(Q^2), \epsilon) = \gamma^{(c)}(\alpha_s(Q^2), \epsilon) + \gamma^{(l)}(N, \alpha_s(Q^2), \epsilon), \tag{84}$$

is finite for all N and $\gamma^{(l)}$ vanishes for $N = 1$. This implies that $\gamma^{(c)}(\alpha_s(Q^2), \epsilon)$ is finite in $\epsilon = 0$ and that $\gamma^{(l)}(N, \alpha_s(Q^2), \epsilon)$ is also finite in $\epsilon = 0$ for all N because of the N-independence of $\gamma^{(c)}(\alpha_s(Q^2), \epsilon)$.

We can rewrite conveniently

$$\gamma^{(l)}(N, \alpha_s(Q^2), \epsilon) = \int_1^N \frac{dn}{n} \, g(\alpha_s(Q^2), \alpha_s(Q^2/n^2), \alpha_s(Q^2/n), \epsilon), \tag{85}$$

where

$$g(\alpha_s(Q^2), \alpha_s(Q^2/n^2), \alpha_s(Q^2/n), \epsilon) = n\frac{\partial}{\partial n}\hat{\gamma}^{(l)}(n, \alpha_s(Q^2), \epsilon). \tag{86}$$

is a Taylor series in its arguments whose coefficients remain finite as $\epsilon \to 0$. In four dimension we have thus

$$
\begin{aligned}
\gamma(N, \alpha_s(Q^2)) &= \gamma^{(l)}(N, \alpha_s(Q^2), 0) + \gamma^{(c)}(N, \alpha_s(Q^2), 0) + O\left(\frac{1}{N}\right) \\
&= \gamma^{(l)}(N, \alpha_s(Q^2), 0) + O\left(N^0\right) \\
&= \int_1^N \frac{dn}{n}\, g(\alpha_s(Q^2), \alpha_s(Q^2/n^2), \alpha_s(Q^2/n)) + O\left(N^0\right),
\end{aligned}
\tag{87}
$$

where

$$
g(\alpha_s(Q^2), \alpha_s(Q^2/n^2), \alpha_s(Q^2/n)) \equiv \lim_{\epsilon \to 0} g(\alpha_s(Q^2), \alpha_s(Q^2/n^2), \alpha_s(Q^2/n), \epsilon) \tag{88}
$$

is a generic Taylor series of its arguments.

Renormalization group invariance thus implies that the physical anomalous dimension γ Eq. (77) depends on its three arguments Q^2, Q^2/N and Q^2/N^2 only through α_s. Clearly, any function of Q^2 and N can be expressed as a function of $\alpha_s(Q^2)$ and $\alpha_s(Q^2/N)$ or $\alpha_s(Q^2/N^2)$. The nontrivial statement, which endows Eq. (87) with predictive power, is that the log derivative of γ, $g(\alpha_s(Q^2), \alpha_s(Q^2/n^2), \alpha_s(Q^2/n))$ Eq. (86), is analytic in its three arguments. This immediately implies that when γ is computed at (fixed) order α_s^k, it is a polynomial in $\ln\frac{1}{N}$ of k-th order at most.

In order to discuss the factorization properties of our result we write the function g as

$$
\begin{aligned}
g(\alpha_s(Q^2), \alpha_s(Q^2/n^2), \alpha_s(Q^2/n)) &= g_1(\alpha_s(Q^2), \alpha_s(Q^2/n)) \\
&+ g_2(\alpha_s(Q^2), \alpha_s(Q^2/n^2)) \\
&+ g_3(\alpha_s(Q^2), \alpha_s(Q^2/n), \alpha_s(Q^2/n^2)),
\end{aligned}
\tag{89}
$$

where

$$
g_1(\alpha_s(Q^2), \alpha_s(Q^2/n)) = \sum_{m=0}^{\infty} \sum_{p=1}^{\infty} g_{m0p}\, \alpha_s^m(Q^2)\, \alpha_s^p(Q^2/n) \tag{90}
$$

$$
g_2(\alpha_s(Q^2), \alpha_s(Q^2/n^2)) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} g_{mn0}\, \alpha_s^m(Q^2)\, \alpha_s^n(Q^2/n^2) \tag{91}
$$

$$
\begin{aligned}
g_3(\alpha_s(Q^2), \alpha_s(Q^2/n), \alpha_s(Q^2/n^2)) &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} g_{mnp}\, \alpha_s^m(Q^2) \\
&\times \alpha_s^n(Q^2/n^2)\, \alpha_s^p(Q^2/n).
\end{aligned}
\tag{92}
$$

The dependence on the resummation variables Q^2, Q^2/N and Q^2/N^2 is fully factorized if the bare coefficient functions has the factorized structure

$$
C^{(0)}(N, Q^2, \alpha_0, \epsilon) = C^{(0,c)}(Q^2, \alpha_0, \epsilon)\, C_1^{(0,l)}(Q^2/N, \alpha_0, \epsilon)\, C_2^{(0,l)}(Q^2/N^2, \alpha_0, \epsilon). \tag{93}
$$

This is argued to be the case in the approach of Refs.[14, 12]. If this happens, the resummed anomalous dimension is given by Eq. (87) with all $g_{mnp} = 0$ except g_{0n0}, g_{00p}:

$$\gamma(N, \alpha_s(Q^2)) = \int_1^N \frac{dn}{n} g_1(0, \alpha_s(Q^2/n)) + \int_1^N \frac{dn}{n} g_2(0, \alpha_s(Q^2/n^2)). \qquad (94)$$

We recall that the coefficient function depends on the parton subprocess in which the incoming partons are a, b (compare Eq. (7)). So, the factorization Eq. (93) applies to the coefficient function corresponding to each subprocess, and the decomposition Eq. (94) to the physical anomalous dimension computed from each of these coefficient functions.

A weaker form of factorization is obtained assuming that in the soft limit the N-dependent and N-independent parts of the coefficient function factorize:

$$C^{(0)}(N, Q^2, \epsilon) = C^{(0,c)}(Q^2, \alpha_0, \epsilon) \, C^{(0,l)}(Q^2/N^2, Q^2/N, \alpha_0, \epsilon). \qquad (95)$$

This condition turns out to be satisfied [8] in Drell-Yan and deep-inelastic scattering to order α_s^2. It holds in QED to all orders [53] as a consequence of the fact that each emission in the soft limit can be described by universal (eikonal) factors, independent of the underlying diagram. This eikonal structure of Sudakov radiation has been argued in Refs. [2, 14] to apply also to QCD. If the factorized form Eq. (95) holds, the coefficients g_{mnp} Eqs. (90,91,92) vanish for all $m \neq 0$, and the physical anomalous dimension takes the form

$$\begin{aligned}
\gamma(N, \alpha_s(Q^2)) &= \int_1^N \frac{dn}{n} g_1(0, \alpha_s(Q^2/n)) + \int_1^N \frac{dn}{n} g_2(0, \alpha_s(Q^2/n^2)) \\
&+ \int_1^N \frac{dn}{n} g_3(0, \alpha_s(Q^2/n^2), \alpha_s(Q^2/n)).
\end{aligned} \qquad (96)$$

It is interesting to observe that in the approach of Refs. [14, 12] for processes where more than one colour structure contributes to the cross-section, the factorization Eq. (93) of the coefficient function is argued to take place separately for each colour structure. This means that in such case the exponentiation takes place for each colour structure independently, i.e. the resummed cross section for each parton subprocess is in turn expressed as a sum of factorized terms of the form of Eq. (93). This happens for instance in the case of heavy quark production [14, 54]. In prompt photon production different colour structures appear for the gluon-gluon subprocess which starts at next-to-next-to-leading order, hence their separated exponentiation would be relevant for next-to-next-to-leading log resummed results.

When several colour structures contribute to a given parton subprocess, the coefficients of the perturbative expansion Eq. (76) for that process take the form

$$C_{lkk'}^{(0)}(\epsilon) = C_{lkk'}^{(0)\mathbf{1}}(\epsilon) + C_{lkk'}^{(0)\mathbf{8}}(\epsilon), \qquad (97)$$

(assuming for definiteness that a colour singlet and octet contribution are present) so that the coefficient function can be written as a sum $C^{(0)} = C_{\mathbf{1}}^{(0)} + C_{\mathbf{8}}^{(0)}$. The

argument which leads from Eq. (76) to the resummed result Eq. (87) then implies that exponentiation takes place for each colour structure independently if and only if

$$\gamma_{\mathbf{1}} \equiv \partial \ln C_{\mathbf{1}}^{(0)}/\partial \ln Q^2, \qquad \gamma_{\mathbf{8}} \equiv \partial \ln C_{\mathbf{8}}^{(0)}/\partial \ln Q^2 \tag{98}$$

are separately finite.

This, however, is clearly a more restrictive assumption than that under which we have derived the result Eq. (87), namely that the full anomalous dimension γ is finite. It follows that exponentiation of each colour structure must be a special case of our result. However, this can only be true if the coefficients g_{ijk} of the expansion Eq. (90) of the physical anomalous dimension satisfy suitable relations. In particular, at the leading log level, it is easy to see that exponentiation of each colour structure is compatible with exponentiation of their sum only if the leading order coefficients are the same for the given colour structures: $g_{001}^{\mathbf{1}} = g_{001}^{\mathbf{8}}$ and $g_{010}^{\mathbf{1}} = g_{010}^{\mathbf{8}}$. This is indeed the case for heavy quark production (where $g_{001} = 0$).

Note that, however, if the factorization holds for each colour structure separately it will not apply to the sum of colour structures. For instance, the weaker form of factorization Eq. (95) requires that

$$C_{lkk'}^{(0)}(\epsilon) = F_{k+k'}(\epsilon)G_{l-k-k'}(\epsilon), \tag{99}$$

but

$$F_{k+k'}^{\mathbf{1}}(\epsilon)G_{l-k-k'}^{\mathbf{1}}(\epsilon) + F_{k+k'}^{\mathbf{8}}(\epsilon)G_{l-k-k'}^{\mathbf{8}}(\epsilon) \neq F_{k+k'}(\epsilon)G_{l-k-k'}(\epsilon). \tag{100}$$

Hence, our result Eq. (87) for the sum of colour structures is more general than the separate exponentiation of individual colour structures, but it leads to results which have weaker factorization properties.

4.5   Comparison with previous results

In this section, we want to make a comparison with the resummation formula for prompt photon production previously released. In order to do this, we need to rewrite the NLL result of Ref.[14] in our formalism. The physical anomalous dimension can be obtained performing the Q^2-logarithmic derivative of the NLL resummed exponent in the $\overline{MS}$ scheme of Ref.[14]. We obtain for a particular partonic sub-process:

$$\begin{aligned}
\gamma(N, \alpha_s(Q^2)) &= \int_0^1 dx \, \frac{x^{N-1} - 1}{1 - x} \big[\hat{g}_2 \alpha_s(Q^2(1-x)^2) + \hat{g}_2' \alpha_s^2(Q^2(1-x)^2) \\
&\quad + \hat{g}_1 \alpha_s(Q^2(1-x)) + \hat{g}_1' \alpha_s^2(Q^2(1-x-)) \big],
\end{aligned} \tag{101}$$

where

$$\hat{g}_2 \quad = \quad \frac{A_d^{(1)} + A_b^{(1)} - A_d^{(1)}}{\pi} \tag{102}$$

$$\hat{g}_2' \quad = \quad \frac{A_a^{(2)} + A_b^2 - A_d^{(2)}}{\pi^2} - \frac{\beta_0}{4\pi} \left[\frac{A_a^{(1)} + A_b^{(1)} - A_d^{(1)}}{\pi} \right] \ln 2 \tag{103}$$

$$\hat{g}_1 \quad = \quad \frac{A_d^{(1)}}{\pi} \tag{104}$$

$$\hat{g}_1' \quad = \quad \frac{A_d^{(2)}}{\pi^2} - \frac{\beta_0}{4\pi} \frac{B_d^{(1)}}{2\pi}. \tag{105}$$

Here, A_a^i is the coefficient of $\ln(1/N)$ in the Mellin transform of the P_{aa} Altarelli-Parisi splitting function at order α_s^i, β_0 is the α_s^2 coefficient of the β function (Eq.(38) in section 1.2) and $B_d^{(1)}$ is a constant to be determined from the comparison with the fixed-order calculation. In Eqs.(102-105) a, b are the incoming partons (on which the coefficient function implicitly depends) and d is the LO outgoing parton uniquely determined by the incoming ones. For completeness, we list explicitly these coefficients:

$$A_{a=q,\bar{q}}^{(1)} \quad = \quad \frac{4}{3}, \qquad\qquad A_{a=g}^1 = 3 \tag{106}$$

$$A_a^{(2)} \quad = \quad \frac{1}{2} A_a^{(1)} K, \qquad K = \frac{67}{6} - \frac{\pi^2}{6} - \frac{5}{9} N_f \tag{107}$$

$$B_{d=q,\bar{q}}^{(1)} \quad = \quad -2, \qquad\qquad B_{d=g}^{(1)} = -\frac{11}{2} + \frac{1}{3} N_f, \tag{108}$$

where N_f is the number of active flavors. Now, performing the change of variable

$$n = \frac{1}{1-x} \tag{109}$$

in the integral Eq.(101), we obtain at NLL

$$
\begin{aligned}
\gamma(N, \alpha_s(Q^2)) \quad = \quad & \int_0^1 \frac{dn}{n} \Big[g_{010} \alpha_s \left(\frac{Q^2}{n^2} \right) + g_{001} \alpha_s \left(\frac{Q^2}{n} \right) \\
& + g_{020} \alpha_s^2 \left(\frac{Q^2}{n^2} \right) + g_{002} \alpha_s^2 \left(\frac{Q^2}{n} \right) \Big],
\end{aligned}
\tag{110}
$$

where

$$g_{010} \quad = \quad -\hat{g}_2, \qquad\qquad g_{020} = - \left(\hat{g}_2' - \frac{\gamma_E \beta_0}{2\pi} \hat{g}_2 \right) \tag{111}$$

$$g_{001} \quad = \quad -\hat{g}_1, \qquad\qquad g_{002} = - \left(\hat{g}_1' - \frac{\gamma_E \beta_0}{4\pi} \hat{g}_1 \right), \tag{112}$$

with γ_E the Euler constant. Hence, according to Ref.[14], we know exactly the value of the coefficients g_{010}, g_{020}, g_{001} and g_{002}. This enables us to compute predictions of high-order logarithmic contributions to the physical anomalous dimension performing a fixed order expansion of γ.

We shall now show that the resummation formula of Ref.[14] predicts the coefficient of $\alpha_s^3 \ln^2(1/N)$ of the fixed order expansion of γ, while in our approach it is required in order to perform a NLL resummation. We need to expand Eq.(110) to order α_s^3 and this is obtained using the change of variable

$$\frac{dn}{n} = -\frac{d\alpha_s(Q^2/n^a)}{a\beta\alpha_s}, \qquad a = 1, 2 \tag{113}$$

to perform the integral and expanding the two loops running of α_s (see Eq.(43) in section 1.2). We find

$$
\begin{aligned}
\gamma \;=\;& [-(g_{001} + g_{010})]\alpha_s(Q^2)\ln\frac{1}{N} \\
&+[-(g_{002} + g_{020}) - (b_1/b_0)(g_{001} + g_{010})]\alpha_s^2(Q^2)\ln\frac{1}{N} \\
&+\left[\frac{b_0}{2}(g_{001} + 2g_{010})\right]\alpha_s^2(Q^2)\ln^2\frac{1}{N} + [-(g_{003} + g_{030})]\alpha_s^3(Q^2)\ln\frac{1}{N} \\
&+[(3b_1/2)(g_{001} + 2g_{010}) + b_0(g_{002} + 2g_{020})]\alpha_s^3(Q^2)\ln^2\frac{1}{N} \\
&+[-(b_0^2/3)(g_{001} + 4g_{010})]\alpha_s^3\ln^3\frac{1}{N} + O(\alpha_s^4).
\end{aligned}
\tag{114}
$$

In our approach, in order to determine the NLL resummation coefficients g_{010}, g_{020}, g_{001} and g_{002}, we must compare this expansion to a fixed order computation of the physical anomalous dimension, which in the general has the form

$$\gamma_{\mathrm{FO}}(N, \alpha_s) = \sum_{i=1}^{k} \alpha_s^i \sum_{j=1}^{i} \gamma_j^i \ln^j \frac{1}{N} + O(\alpha_s^{k+1}) + O(N^0), \tag{115}$$

where k is the fixed-order at which it has been computed (see Chapter 7 for a general discussion about the determination of the resummation coefficients). Hence, we determine the 4 NLL resummation coefficients through the following 4 independent conditions:

$$g_{001} + g_{010} = -\gamma_1^1 \tag{116}$$

$$g_{001} + 2g_{010} = \frac{2}{b_0}\gamma_2^2 \tag{117}$$

$$g_{002} + g_{020} = -\gamma_1^2 - \frac{b_1}{b_0}(g_{001} + g_{010}) \tag{118}$$

$$g_{002} + 2g_{020} = \frac{1}{b_0}\gamma_2^3 - \frac{3b_1}{2b_0}(g_{001} + 2g_{010}). \tag{119}$$

Thus, according to our formalism, all the coefficients γ_1^1, γ_2^2, γ_1^2 and γ_2^3 should be known. The first three coefficients are all known thanks to the explicit $O(\alpha_s^2)$ calculation of the prompt photon cross section [55, 56, 57]. The last one (γ_2^3), is not yet known from explicit $O(\alpha_s^3)$ calculation, but, according to the approach of Ref.[14], it

is predicted using Eqs.(119,111,112):

$$
\gamma_2^3 = -b_0 \left[\frac{2(A_a^{(2)} + A_b^{(2)}) - A_d^{(2)}}{\pi^2} - b_0(2\ln 2 + 4\gamma_E)\frac{A_a^{(1)} + A_b^{(1)}}{\pi} \right.
$$

$$
\left. + b_0(2\ln 2 + 3\gamma_E)\frac{A_d^{(1)}}{\pi} - \frac{b_0 B_d^{(1)}}{2\pi} \right] - b_1 \frac{3}{2} \left[\frac{2(A_a^{(1)} + A_b^{(1)}) - A_d^{(1)}}{\pi} \right] .
$$

(120)

The correctness of this result could be tested by an order α_s^3 calculation. If it were to fail, the more general resummation formula with g_{020} determined by Eq.(119) should be used, or one of the resummations which do not assume the factorization Eq.(93).

Chapter 5

Resummation of rapidity distributions

In this chapter, we present a derivation of the threshold resummation formula for the Drell-Yan and the prompt photon production rapidity distributions. Our arguments are valid for all values of rapidity and to all orders in perturbative QCD. For the case of the Drell-Yan process, resummation is realized in a universal way, i.e. both for the production of a virtual photon γ^* and the production of a vector boson $W^\pm$, Z^0. We will show that for the fixed-target proton-proton Drell-Yan experiment E866/NuSea used in current parton fits, the NLL resummation corrections are comparable to NLO fixed-order corrections and are crucial to obtain agreement with the data. This means that the NLL resummation of rapidity distributions is necessary and turns out to give better results than high-fixed-order calculations. We consider first the resummation of the DY case and its phenomenology and then the resummation of the prompt photon case.

5.1 Threshold resummation of DY rapidity distributions

5.1.1 General kinematics of Drell-Yan rapidity distributions

We consider the general Drell-Yan process in which the collisions of two hadrons (H_1 and H_2) produce a virtual photon γ^* (or an on-shell vector boson V) and any collection of hadrons (X):

$$H_1(P_1) + H_2(P_2) \to \gamma^*(V)(Q) + X(K). \tag{1}$$

In particular, we are interested in the differential cross section $\frac{d\sigma}{dQ^2 dY}(x, Q^2, Y)$, where Q^2 is the invariant mass of the photon or of the vector boson, x is defined as usual as the fraction of invariant mass that the hadrons transfer to the photon (or to the vector boson) and Y is the rapidity of γ^* (V) in the hadronic centre-of-mass-frame:

$$x \equiv \frac{Q^2}{S}, \quad S = (P_1 + P_2)^2, \quad Y \equiv \frac{1}{2} \ln \left(\frac{E + Q_z}{E - Q_z} \right), \tag{2}$$

where E and p_z are the energy and the longitudinal momentum of $\gamma^*(V)$ respectively. In this frame, the four-vector Q of $\gamma^*(V)$ can be written in terms of its rapidity and its tranverse momentum

$$Q = (Q^0, \vec{Q}) = (\sqrt{Q^2 + Q_\perp^2} \cosh Y, \vec{Q}_\perp, \sqrt{Q^2 + Q_\perp^2} \sinh Y), \tag{3}$$

or in terms of the scattering angle θ

$$Q = (Q^0, \vec{Q}) = (\sqrt{Q^2 + |\vec{Q}|^2}, \vec{Q}_\perp, |\vec{Q}| \cos(\theta)). \tag{4}$$

For completeness, we recall that when $Q^2 = 0$ (which is not our case), the rapidity Y defined in Eq.(2) reduces to the pseudorapidity η according to Eq.(4):

$$\eta = -\ln(\tan(\theta/2)). \tag{5}$$

At the partonic level, a parton $1(2)$ in the hadron H_1 (H_2) carries a fraction of momentum x_1 (x_2):

$$p_1 = x_1 P_1 = x_1 \frac{\sqrt{S}}{2}(1, \vec{0}_\perp, 1)), \quad p_2 = x_2 P_2 = x_2 \frac{\sqrt{S}}{2}(1, \vec{0}_\perp, -1)). \tag{6}$$

It is clear that the hadronic center-of-mass frame does not coincide with the partonic one, because x_1 is, in general, different from x_2. Furthermore, from Eq.(2), we see that the rapidity is not an invariant. Hence, in order to define the rapidity in the partonic center-of-mass frame (y), we have to perform a boost of Y which connects the two frames. This provides us a relation between the rapidity in these two frames:

$$y = Y - \frac{1}{2}\ln(\frac{x_1}{x_2}). \tag{7}$$

In order to understand the kinematic configurations in terms of rapidity, it is convenient to define a new variable u,

$$u \equiv \frac{Q \cdot p_1}{Q \cdot p_2} = e^{-2y} = \frac{x_1}{x_2} e^{-2Y}. \tag{8}$$

With no partons radiated as in the case of the LO, the rapidity is obviously zero. Beyond the LO, one or more partons can be radiated. Now, if these partons are radiated collinear with the incoming parton 2, then the partonic rapidity reaches its maximum value and u its minimum one. Similarly the minimum value of y and the maximum value of u is achieved when the radiated partons are collinear with the incoming parton 1. To be more precise, suppose that in the first case the radiated partons (collinear with the parton 2) carry away a fraction of momentum equal to $(1-z)p_2$, so that by momentum conservation $Q = p_1 + zp_2$. In this case, we obtain immediately the lower bound for u, which is z. In the second case the collinear radiated partons have momentum $(1-z)p_1$, hence $Q = zp_1 + p_2$ and the upper bound of u is $1/z$. So, z can be interpreted as the fraction of invariant mass that incoming partons transfer to $\gamma^*(V)$. In fact:

$$z = \frac{Q^2}{2p_1 \cdot p_2} = \frac{Q^2}{(p_1 + p_2)^2} = \frac{x}{x_1 x_2}, \tag{9}$$

where we have neglected the quark masses. Therefore, we have that the kinematic constraints of u are:

$$z \leq u \leq \frac{1}{z}. \tag{10}$$

Then, since $x_1 < 1$ and $x_2 < 2$, the lower and upper bounds of z are:

$$x \leq z \leq 1. \tag{11}$$

Thanks to Eqs.(7,8), the first relation can be translated directly into a relation for the upper and lower limit of the partonic center-of-mass rapidity:

$$\frac{1}{2} \ln z \leq y \leq \frac{1}{2} \ln \frac{1}{z}. \tag{12}$$

Now, we need to obtain the boundaries of the hadronic center-of-mass rapidity. Substituting Eqs.(8, 9) into the two conditions $u \geq z$ and $u \leq 1/z$, we obtain the lower kinematical bound for x_1 and x_2:

$$x_1 \geq \sqrt{x} e^Y \equiv x_1^0, \quad x_2 \geq \sqrt{x} e^{-Y} \equiv x_2^0 \tag{13}$$

and the obvious requirment that $x_{1(2)}^0 \leq 1$ implies that the hadronic rapidity has a lower and an upper bound:

$$\frac{1}{2} \ln x \leq Y \leq \frac{1}{2} \ln \frac{1}{x}. \tag{14}$$

5.1.2 The universality of resummation in Drell-Yan processes

According to standard factorization of collinear singularities of perturbative QCD, the expression for the hadronic differential cross section in rapidity has the form,

$$
\begin{aligned}
\frac{d\sigma}{dQ^2 dY} &= \sum_{i,j} \int_{x_1^0}^1 dx_1 \int_{x_2^0}^1 dx_2 F_i^{H_1}(x_1, \mu^2) F_j^{H_2}(x_2, \mu^2) \\
&\times \frac{d\hat{\sigma}_{ij}}{dQ^2 dy} \left(x_1, x_2, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y \right),
\end{aligned}
\tag{15}
$$

where y depends on Y, x_1 and x_2 according to Eq.(7). The sum runs over all possible partonic subprocesses, $F_i^{(1)}$, $F_j^{(2)}$ are respectively the parton densities of the hadron H_1 and H_2, μ is the factorization scale (chosen equal to renormalization scale for simplicity) and $d\hat{\sigma}_{ij}/(dQ^2 dy)$ is the partonic cross section. Even if the cross section Eq.(15) is μ^2-independent, this is not the case for each parton subprocess. However, the μ^2-dependence of each contribution is proportional to the off-diagonal anomalous dimensions (or splitting functions), which in the threshold limit, $(z \to 1)$ are suppressed by factors of $1 - z$. Therefore, each partonic subprocess can be treated independently and is separately renormalization-group invariant. Furthermore, the suppression, in the threshold limit, of the off-diagonal splitting functions implies also

that only the gluon-quark channels are suppressed. So, in order to study resummation, we will consider only the quark- anti-quark channel, which can be related to the same dimensionless coefficient function $C(z, Q^2/\mu^2, \alpha_s(\mu^2), y)$ for both, the production of a virtual photon and the production of a on-shell vector boson. In fact, if for the production of a virtual photon, we define $C(z, Q^2/\mu^2, \alpha_s(\mu^2), y)$ through the equation,

$$x_1 x_2 \frac{d\hat{\sigma}_{q\bar{q}'}^{\gamma^*}}{dQ^2 dy}\left(x_1, x_2, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y\right) = \frac{4\pi\alpha^2 c_{q\bar{q}'}}{9Q^2 S} C\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y\right), \qquad (16)$$

where the prefactor $x_1 x_2$ has been introduced for future convenience, we find that for the case of the production of a real vector boson,

$$\begin{aligned} x_1 x_2 \frac{d\hat{\sigma}_{q\bar{q}'}^{V}}{dQ^2 dy}\left(x_1, x_2, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y\right) &= \frac{\pi G_F Q^2 \sqrt{2} c_{q\bar{q}'}}{3S}\delta(Q^2 - M_V^2) \\ &\times C\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y\right), \end{aligned} \qquad (17)$$

where G_F is the Fermi constant, M_V is the mass of the produced vector boson. The coefficients $c_{q\bar{q}'}$, for the different Drell-Yan processes, are given by:

$$\begin{aligned} c_{q\bar{q}'} &= Q_q^2 \delta_{q\bar{q}} \quad \text{for } \gamma^*, & (18) \\ c_{q\bar{q}'} &= |V_{qq'}|^2 \quad \text{for } W^{\pm}, & (19) \\ c_{q\bar{q}'} &= 4[(g_v^q)^2 + (g_a^q)^2]\delta_{q\bar{q}} \quad \text{for } Z^0. & (20) \end{aligned}$$

Here, Q_q^2 is the square charge of the quark q, $V_{qq'}$ are the CKM mixing factors for the quark flavors q, q' and

$$g_v^q = \frac{1}{2} - \frac{4}{3}\sin^2\theta_W, \quad g_a^q = \frac{1}{2} \quad \text{for an up-type quark,} \qquad (21)$$

$$g_v^q = -\frac{1}{2} + \frac{2}{3}\sin^2\theta_W, \quad g_a^q = -\frac{1}{2} \quad \text{for a down-type quark,} \qquad (22)$$

with θ_W the Weinberg weak mixing angle. As a consequence of these facts, resummation has to be performed only for the quark-anti-quark channels omitting the overall dimensional factors of $C(z, Q^2/\mu^2, \alpha_s(\mu^2), y)$ in the different Drell-Yan processes. Thus, we are left with the following dimensionless cross section, which has the form:

$$\begin{aligned} \sigma(x, Q^2, Y) &\equiv \int_{x_1^0}^{1} \frac{dx_1}{x_1} \int_{x_2^0}^{1} \frac{dx_2}{x_2} F_1^{H_1}(x_1, \mu^2) F_2^{H_2}(x_2, \mu^2) \\ &\times C\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y\right), \end{aligned} \qquad (23)$$

where F_1 and F_2 are quark or anti-quark parton densities in the hadron H_1 and H_2 respectively. This shows the universality of resummation in Drell-Yan processes in the sense that only the renormalization-group invariant quantity defined in Eq.(23) has to be resummed.

5.1.3 Factorization properties and the Mellin-Fourier transform

For the case of the rapidity-integrated cross section, resummation is usually done in Mellin space transforming the variable x into its conjugate variable N, because the Mellin transformation turns convolution products into ordinary products. Furthermore, the Mellin space is the natural space where to define resummation of leading, next-to-leading and so on logarithmic contribution, because in this space momentum conservation is respected as shown in [58]. In the case of the rapidity distribution, the Mellin transformation is not sufficient. In fact, rewriting Eq.(23) in this form

$$
\begin{aligned}
\sigma(x, Q^2, Y) \;=\; & \int_0^1 dx_1 dx_2 dz F_1^{H_1}(x_1, \mu^2) F_2^{H_2}(x_2, \mu^2) \\
\times \; & C\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y\right) \delta(x - x_1 x_2 z),
\end{aligned}
\tag{24}
$$

we see that the Mellin transform with respect to x,

$$
\sigma(N, Q^2, Y) \equiv \int_0^1 dx \, x^{N-1} \sigma(x, Q^2, Y),
\tag{25}
$$

does not diagonalize the triple integral in Eq.(24). This is due to the fact that the partonic center-of-mass rapidity y depends on x_1 and x_2 through Eq.(7). The ordinary product in Mellin space can be recovered performing the Mellin transform with respect to x of the Fourier transform of Eq.(24) with respect to Y. Calling the Fourier moments M, using Eq.(7) the relations (12,14) and the identity

$$
\begin{aligned}
C\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), Y - \frac{1}{2}\ln\frac{x_1}{x_2}\right) \;=\; & \int_{\ln\sqrt{z}}^{\ln 1/\sqrt{z}} dy \, C\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y\right) \\
\times \; & \delta\left(y - Y + \frac{1}{2}\ln\frac{x_1}{x_2}\right),
\end{aligned}
\tag{26}
$$

we find that

$$
\begin{aligned}
\sigma(N, Q^2, M) \;\equiv\; & \int_0^1 dx \, x^{N-1} \int_{\ln\sqrt{x}}^{\ln 1/\sqrt{x}} dY \, e^{iMY} \sigma(x, Q^2, Y) \tag{27} \\
=\; & F_1^{H_1}(N + iM/2, \mu^2) F_2^{H_2}(N - iM/2, \mu^2) \\
\times \; & C\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), M\right),
\end{aligned}
\tag{28}
$$

where

$$
F_i^{H_i}(N \pm iM/2, \mu^2) \;=\; \int_0^1 dx \, x^{N-1\pm iM/2} F_i^{H_i}(x, \mu^2),
\tag{29}
$$

$$
\begin{aligned}
C\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), M\right) \;=\; & \int_0^1 dz \, z^{N-1} \int_{\ln\sqrt{z}}^{\ln 1/\sqrt{z}} dy \, e^{iMy} \\
\times \; & C\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y\right).
\end{aligned}
\tag{30}
$$

Eq.(28) shows that performing the Mellin-Fourier moments of the hadronic dimensionless cross section Eq.(23), we recover an ordinary product of the Mellin-Fourier trnsform of the coefficient function and the Mellin moments of the parton densities translated outside the real axis by $\pm iM/2$. Because the coefficient function is symmetric in y, we can rewrite Eq.(30) in this way:

$$
\begin{aligned}
C\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), M\right) &= 2\int_0^1 dz\, z^{N-1} \int_0^{\ln 1/\sqrt{z}} dy\, \cos(My) \\
&\times\ C\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y\right).
\end{aligned}
\tag{31}
$$

From this last equation and Eq.(29), we see that the dependence on M, the Fourier conjugate of the rapidity y, originates from the parton densities, that depend on $N \mp iM/2$, and from the factor of $\cos(My)$ in the integrand of Eq.(31).

5.1.4   The all-order resummation formula and its NLL implementation

In this section, we show that the resummed expression of Eq.(28) is obtained by simply replacing the coefficient function $C\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), M\right)$ with its integral over y, resummed to the desired logarithmic accuracy. This is equivalent to saying that the factor of $\cos(My)$ in Eq.(31) is irrelevant in the large-N limit. Indeed, one can expand $\cos(My)$ in powers of y,

$$
\cos(My) = 1 - \frac{M^2 y^2}{2} + O(M^4 y^4).
\tag{32}
$$

and observe that the first term of this expansion leads to a convergent integral (the rapidity-integrated cross section), while the subsequent terms are suppressed by powers of $(1 - z)$, since the upper integration bound in Eq.(31) is

$$
\ln \frac{1}{\sqrt{z}} = \frac{1}{2}(1 - z) + O((1 - z)^2).
\tag{33}
$$

Hence, up to terms suppressed by factors $1/N$, Eq.(30) is equal to the Mellin transform of the rapidity-integrated Drell-Yan coefficient function that we call $C_I(N, Q^2/\mu^2, \alpha_s(\mu^2))$. This completes our proof. We get

$$
\begin{aligned}
\sigma^{res}(N, Q^2, M) &= F_1^{H_1}(N + iM/2, \mu^2) F_2^{H_2}(N - iM/2, \mu^2) \\
&\times\ C_I^{res}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right).
\end{aligned}
\tag{34}
$$

This theorethical result is very important: it shows that, near threshold, the Mellin-Fourier transform of the coefficient function does not depend on the Fourier moments and that this is valid to all orders of QCD perturbation theory. Furthermore this result remains valid for all values of hadronic center-of-mass rapidity, because we

have introduced a suitable integral transform over rapidity. The resummed rapidity-integrated Drell-Yan coefficient function to NLL order has been studied in Section 3.4. It is given by

$$C_I^{res}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = [\exp\{\ln N g_1(\lambda, 2) + g_2(\lambda, 2)\}]_{\mu_r^2 = \mu^2} \tag{35}$$

where $\lambda = b_0 \alpha_s(\mu_r^2) \ln N$ and where the resummation functions $g_1(\lambda, 2)$ and $g_2(\lambda, 2)$ are given in Eqs.(101,102) of Section 3.4 with the resummation coefficients in the $\overline{MS}$ scheme given in Eq.(46) of Section 2.3.1.

Now, we want to arrive to a NLO and NLL expression of the rapidity-dependent dimensional cross section. This is achieved firstly taking the Mellin and Fourier inverse transforms of $\sigma^{res}(N, Q^2, M)$ Eq.(34) in order to turn back to the variables x and Y:

$$\sigma^{res}(x, Q^2, Y) = \int_{-\infty}^{\infty} \frac{dM}{2\pi} e^{-iMY} \int_{C-i\infty}^{C+i\infty} \frac{dN}{2\pi i} x^{-N} \sigma^{res}(N, Q^2, M). \tag{36}$$

In principle the contour in the complex N-space of the inverse Mellin transform in Eq.(36) has to be chosen in such a way that the intersection of C with the real axis lies to the right of all the singularities of the integrand. In practice, this is not possible, because the resummed coefficient function Eqs.(101,102) of section 3.4 has a branch cut on the real positive axis for

$$N \geq N_L \equiv e^{\frac{1}{ab_0 \alpha_s(Q^2)}}, \tag{37}$$

which corresponds to the Landau singularity of $\alpha_s(Q^2/N^a)$ (see Eq.(41) in Appendix 1.2). This is due to the fact that if the N-space expression is expanded in powers of α_s, and the Mellin inversion is performed order by order, a divergent series is obtained. The "Minimal Prescription" proposed in [58] gives a well defined formula to obtain the resummed result in x-space to which the divergent series is asymptotic and is simply obtained choosing $C = C_{MP}$ in such a way that all the poles of the integrand are to the left, except the Landau pole Eq.(37). Recently, another method has been prposed in Ref.[59]. Here, we will adopt the "Minimal Prescription " formula, deforming the contour in order to improve numerical convergence and to avoid the singularities of the parton densities of Eq.(34) which are transated out of the real axis by $\pm iM/2$. Hence, we perform the N-integral in Eq.(36) over a curve Γ given by:

$$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 \tag{38}$$

$$\Gamma_1(t) = C_{MP} - i\frac{M}{2} + t(1+i), \quad t \in (-\infty, 0) \tag{39}$$

$$\Gamma_2(s) = C_{MP} + is\frac{M}{2}, \quad s \in (-1, 1) \tag{40}$$

$$\Gamma_3(t) = C_{MP} + i\frac{M}{2} - t(1-i), \quad t \in (0, +\infty) \tag{41}$$

The double inverse tranform of Eq.(36) over the curve Γ then becomes:

$$\sigma^{res}(x, Q^2, Y) = \frac{1}{\pi} \int_0^1 \frac{dm}{m} \cos(-Y \ln m) \sigma^{res}(x, Q^2, -\ln m), \tag{42}$$

where we have done the change of variable $M = -\ln m$. The factor $\sigma^{res}(x, Q^2, M)$ of the integrand in Eq.(42) is given by

$$\sigma^{res}(x, Q^2, M) = \tag{43}$$

$$\frac{1}{\pi} \int_0^1 \frac{ds}{s} \Re \left[x^{-C_{MP} - \ln s + i(M/2+1)} \sigma^{res}(C_{MP} + \ln s - i(M/2+1), Q^2, M) \right.$$

$$\left. \times (1-i) + \frac{sM}{2} x^{-C_{MP} - isM/2} \sigma^{res}(C_{MP} + isM/2, Q^2, M) \right],$$

where we have done another change of variables $(t = -\ln s)$. Eqs.(42,43) are the expressions that we use to evaluate numerically the resummed adimensional cross section in the variables x and Y Eq.(36). Furthermore, we need to know the analytic continuations to all the complex plane of the parton densities at the scale μ^2 in Eq.(34). Here, we need to evolve up a partonic fit taken at a certain scale solving the DGLAP evolution equations in Mellin space. The solution of the evolution equations is given in Section 1.5.

Finally, we want to obtain a NLO determination of the cross section improved with NLL resummation. In order to do this, we must keep the resummed dimensionaless part of the cross section Eq.(42), multiply it by the correct dimensional prefactors Eqs.(57,19,20) and parton densities, add to the resummed part the full NLO cross section and subtract the double-counted logarithmic enhanced contributions. Thus, we have

$$\frac{d\sigma}{dQ^2 dY} = \frac{d\sigma^{NLO}}{dQ^2 dY} + \frac{d\sigma^{res}}{dQ^2 dY} - \left[\frac{d\sigma^{res}}{dQ^2 dY} \right]_{\alpha_s=0} - \alpha_s \left[\frac{\partial}{\partial \alpha_s} \left(\frac{d\sigma^{res}}{dQ^2 dY} \right) \right]_{\alpha_s=0}. \tag{44}$$

The first term is the full NLO cross section given in [17, 60, 61, 62]. We report the complete expression in Appendix C.

The third and the fourth terms in Eq.(44) are obtained in the same way as the second one, but with the substitutions

$$C_I^{res}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) \rightarrow 1, \tag{45}$$

$$C_I^{res}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) \rightarrow \alpha_s(\mu^2) 2A_1 \left\{ \ln^2 N + \ln N \left[2\gamma_E - \ln\left(\frac{Q^2}{\mu^2}\right) \right] \right\},$$

respectively. The terms that appear in the second in the second line of Eqs.(45) are exactly the $O(\alpha_s)$ logarithmic enhanced contributions in the $\overline{MS}$ scheme.

We note that the resummed cross section Eq.(44) is relevant even when the variable x is not large. In fact, the cross section can get the dominant contributions from the integral in Eq.(23) for values of z Eq.(9) that are near the threshold even when x is not close to one, because of the strong suppression of parton densities $F_i(x_i, \mu^2)$ when x_i are large.

5.1.5 NLL impact of resummation at E866 experiment

To show the importance of this resummation, we have calculated the Drell-Yan rapidity distribution for proton-proton collisions at the Fermilab fixed-target experiment
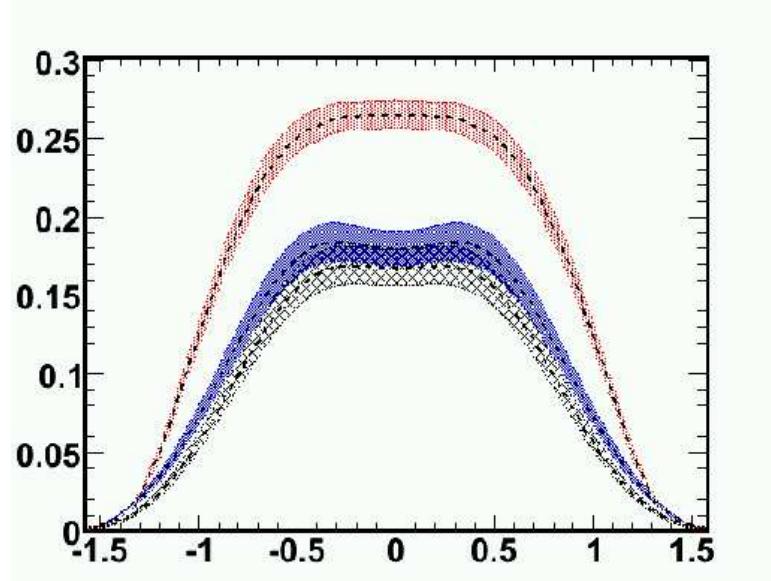
Figure 5.1: Y-dependence of $d^2\sigma/(dQ^2dY)$ in units of pb/GeV^2. The curves are, from top to bottom, the NLO result (red band), the LO+LL resummation (blue band) and the LO (black band). The bands are obtained varying the factorization scale between $\mu^2 = 2Q^2$ and $\mu^2 = 1/2Q^2$.

E866/NuSea [38]. The center-of-mass energy has been fixed at $\sqrt{S} = 38.76\,\mathrm{GeV}$ and the invariant mass of the virtual photon γ^* has been chosen to be $Q^2 = 64\,\mathrm{GeV}^2$ in analogy with [19]. Clearly the contribution of the virtual Z^0 can be neglected, because its mass is much bigger than Q^2. In this case $x = 0.04260$ and the upper and lower bound of the hadronic rapidity Y Eq.(14) are given by ± 1.57795. We have evolved up the MRST 2001 parton distributions (taken at $\mu^2 = 1\,\mathrm{GeV}^2$) in order to compare to Ref.[19], where the NNLO calculation is performed. However, results obtained using more modern parton sets should not be very different. The LO parton set is given in [63] with $\alpha_s^{LO}(m_Z) = 0.130$ and the NLO set is given in [64] with $\alpha_s^{NLO}(m_Z) = 0.119$. The evolution of parton densities at the scale μ^2 has been performed in the variable flavor number scheme. The quarks has been considered massless and, at the scale of the transition of the flavor number ($N_f \to N_f + 1$), the new flavor is generated dynamically. The resummation formula Eq.(34) together with Eqs.(101-35) has been used with the number of flavors $N_f = 4$.

In figure 5.1, we plot the rapidity-dependence of the cross section at LO, NLO and LO improved with LL resummation. The effect of LL resummation is small compared to the effect of the full NLO correction. We see that, at leading order, the impact of the resummation is negligible in comparison to the NLO fixed-order correction. This means that, at leading order, resummation is not necessary.

The LO, the NLO and its NLL improvement cross sections are shown in figure 5.2. The effect of the NLL resummation in the central rapidity region is almost as large as the NLO correction, but it reduces the cross section instead of enhancing it for not large values of rapidity. The origin of this suppression will be discussed in the next Section. Going from the LO result to the NLO with NLL resummation, we
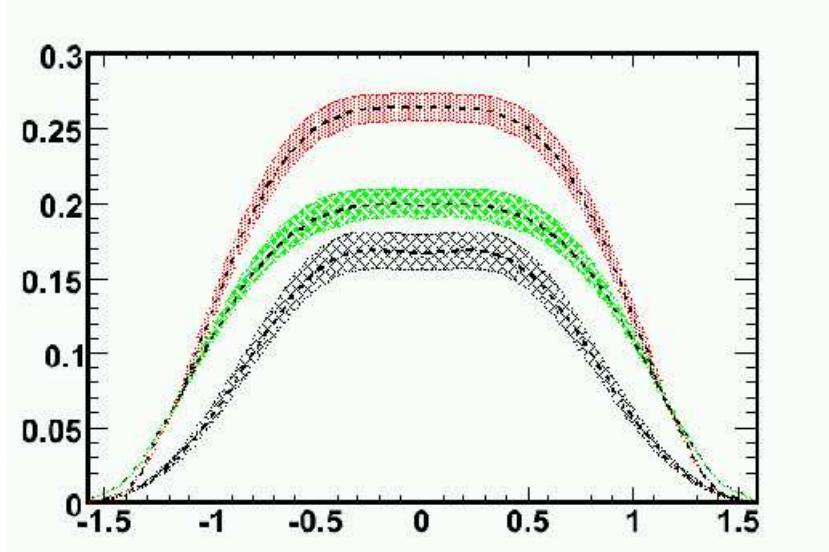
Figure 5.2: Y-dependence of $d^2\sigma/(dQ^2 dY)$ in units of pb/GeV^2. The curves are, from top to bottom, the NLO result (red band), the NLO+NLL resummation (green band) and the LO (black band). The bands are obtained as in figure 5.1.

note a reduction of the dependence on the factorization scale i.e. a reduction of the theoretical error.

Now, we want to establish if the leading logarithmic terms that are included in the resummed exptonent represent a good approximation to the exact fixed order computation. Only if this is the case, we can believe that our resummation is reliable in perturbative QCD. In order to do this, we compare the full NLO DY rapidity cross section with the one obtained including only the large-N leading terms of the coefficient function. For simplicity, we choose the factorization scale μ^2 to the scale of the process Q^2. The leading large-N coefficient function is given by:

$$C^{\text{lead}}(N, \alpha_s(Q^2)) = 1 + \alpha_s(Q^2) 2 A_1 \left(\ln^2 N + 2\gamma_E \ln N + \gamma_E^2 - 2 + \frac{\pi^2}{3} \right), \qquad (46)$$

where we have added the constant terms at $O(\alpha_s)$ which are not resummed. For an explicit derivation of these constant terms see for example Ref. [17] Section 3. We plot the result in Figure 5.3. We see that the leading terms Eq.(46) represent a good approximation to the exact NLO computation, because they account for more than 90% of the full NLO computation for all relevant rapidities.

In Figure 5.4, we plot only the $O(\alpha_s)$ correction. Here we see that the $O(\alpha_s)$ contribution of Eq.(46) represent a good approximation to the exact $O(\alpha_s)$ NLO contribution, because it accounts for more than 80% of the full $O(\alpha_s)$ NLO correction for all relevant rapidities.

In figure 5.5, we plot the experimental data of Ref.[38] converted to the Y variable together with our NLO and NLL resummed predictions.

The data in Ref.[38] are tabulated in invariant Drell-Yan pair mass $\sqrt{Q^2}$ and Feynman x_F bins. To convert the data to the hadronic rapidity Y, we have used the
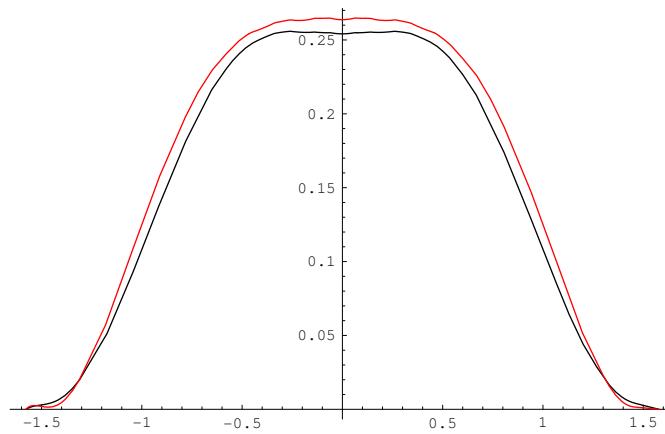
Figure 5.3: $d^2\sigma/(dQ^2 dY)$ in units of pb/GeV^2 for the full NLO computation (upper red line) and for the leading terms of Eq.(46) (lower black line). It has been calculated for one value of the factorization scale $\mu^2 = Q^2$.
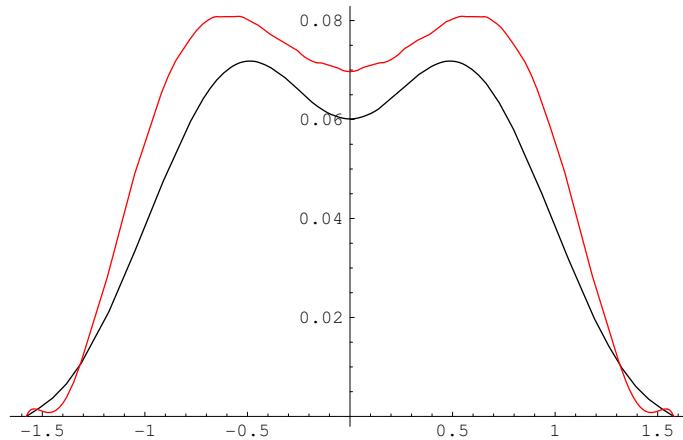

Figure 5.4: $d^2\sigma/(dQ^2 dY)$ in units of pb/GeV^2 for $O(\alpha_s)$ correction of the full NLO computation (upper red line) and for the leading $O(\alpha_s)$ term of Eq.(46) (lower black line).
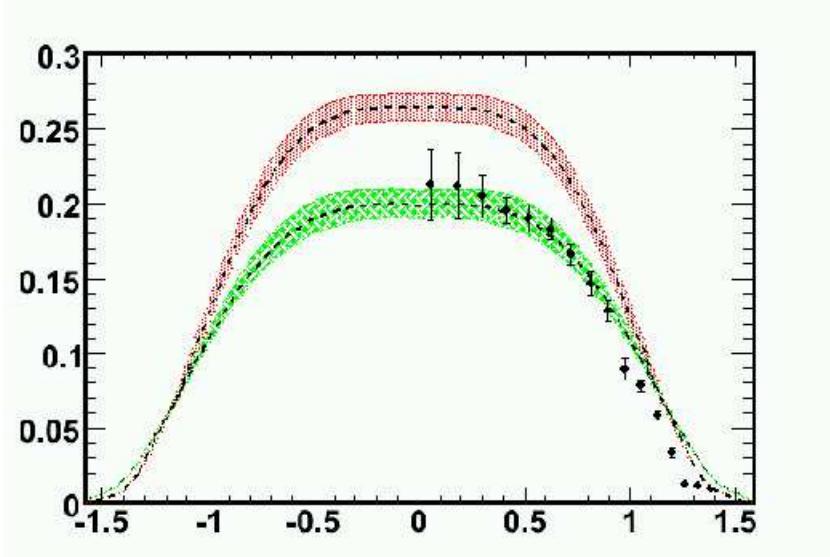
Figure 5.5: Y dependence of $d^2\sigma/(dQ^2 dY)$ in units of pb/GeV^2. The curves are, from top to bottom, the NLO result (red band) and the NLO+NLL resummation (green band) together with the E866/NuSea data. The bands are obtained as in figure 5.1.

definition of the Feynman x_F which is

$$x_F \equiv \frac{2Q_z}{\sqrt{S}} = \frac{2\sqrt{Q^2 + Q_\perp^2}\sinh Y}{\sqrt{S}}, \qquad (47)$$

where we have used Eq.(3). Solving Eq.(47) in Y we have

$$Y = \ln\left[H + \sqrt{H^2 + 1}\right], \qquad H = \frac{x_F\sqrt{S}}{2\sqrt{Q^2 + Q_\perp^2}}. \qquad (48)$$

With this equation and with the aid of the $Q_\perp$ distribution, which is also given in Ref.[38], we have converted the data from x_F to Y. Furthermore, for each x_F bin, we have done the weighted average of three $\sqrt{Q^2}$ bins ($7.2 \leq \sqrt{Q^2} \leq 7.7$; $7.7 \leq \sqrt{Q^2} \leq 8.2$ and $8.2 \leq \sqrt{Q^2} \leq 8.7$ with the energies in GeV).

The agreement with data is good and a great improvement for not large rapidity is obtained with respect to the NLO calculation. We note also that the NLL resummation gives better result than the NNLO calculation performed in [19]. The NNLO prediction has a worse agreement with data than the NLO one for not large values of rapidity. This result suggests that, for the case of rapidity distributions, NLL resummation is more important than high-fixed-order calculation and that it can be so even at higher center-of-mass energies.

5.2   The origin of suppression

In this section, we shall show that the suppression of the cross section of the NLL correction with the parameter choices of the experiment E866 is due to the shift in

the complex plane of the dominant contribution of the resummed exponent. We shall do it using a simplified toy-model.

Consider the collision of only two quarks with parton density

$$F(x) = (1 - x)^2. \tag{49}$$

Its Mellin transform is given by:

$$F(N) = \int_0^1 dx \, x^{N-1} (1-x)^2 = \frac{\Gamma(N)\Gamma(3)}{\Gamma(N+3)} = \frac{2}{N(N+1)(N+2)}. \tag{50}$$

Furthermore, we take the double-log approximation (DLA) which is obtained performing the limit $\lambda \to 0$ in the resummed exponent Eq.(35). Thus, in this simple model, the Mellin-Fourier transform of the NLL resummed cross section Eq.(44) can be written in the following form:

$$\sigma(N, M) = \sigma^{FO}(N, M) + |F(N + iM/2)|^2 \Delta\sigma^{\mathrm{DLA}}(N), \tag{51}$$

where $\sigma^{FO}(N, M)$ are the exact NLO Mellin-Fourier moments and where

$$\Delta\sigma^{\mathrm{DLA}}(N) = \left[e^{\alpha_s 2A_1 \ln^2 N} - 1 - \alpha_s 2A_1 \ln^2 N \right]. \tag{52}$$

If there is a suppression, this means that the quantity

$$\sigma(N, M) - \sigma^{FO}(N, M) = \frac{4\Delta\sigma^{\mathrm{DLA}}(N)}{\left[N^2 + \frac{M^2}{4}\right]\left[(N+1)^2 + \frac{M^2}{4}\right]\left[(N+2)^2 + \frac{M^2}{4}\right]}, \tag{53}$$

should produce a negative contribution in performing the inverse Mellin and Fourier transform. It is given by the integral

$$\int_{-\infty}^{\infty} \frac{dM}{2\pi} e^{-iMY} \int_{C-i\infty}^{C+i\infty} \frac{dN}{2\pi i} x^{-N} \frac{4\Delta\sigma^{\mathrm{DLA}}(N)}{\left[N^2 + \frac{M^2}{4}\right]\left[(N+1)^2 + \frac{M^2}{4}\right]\left[(N+2)^2 + \frac{M^2}{4}\right]}. \tag{54}$$

The integrand function of this expression has not only a cut on the negative real axis (as it happens in the inclusive case), but also poles that are shifted in the complex plane:

$$-n \pm \frac{iM}{2}; \qquad n = 0, 1, 2. \tag{55}$$

Because of the factor x^{-N} in the inverse Mellin integral in Eq.(54), its dominant contribution comes from the poles with $n = 0$ in Eq.(55). The contribution of the pole at $+iM/2$ is given by

$$\int_{-\infty}^{\infty} \frac{dM}{2\pi} e^{-iM(Y+\ln\sqrt{x})} \frac{4\Delta\sigma^{\mathrm{DLA}}(N = iM/2)}{iM(iM+1)(iM+2)}, \tag{56}$$

where

$$\Delta\sigma^{\mathrm{DLA}}(N = iM/2) = \exp\left[\alpha_s 2A_1 \left(\ln^2 \frac{|M|}{2} - \frac{\pi^2}{4} + i\pi \ln \frac{|M|}{2}\right)\right] + \tag{57}$$

$$-1 - \alpha_s 2A_1 \left(\ln^2 \frac{|M|}{2} - \frac{\pi^2}{4} + i\pi \ln \frac{|M|}{2}\right).$$

The important thing to notice of this contribution is the fact that the imaginary pole has produced an oscillating prefactor in front of the resummed exponent which, together with the oscillating factor of the Fourier inverse integral in Eq.(56) at zero hadronic rapidity Y, is given by

$$\exp\left[i\left(\alpha_s 2\pi A_1 \ln\frac{|M|}{2} - iM\ln\sqrt{x}\right)\right].\tag{58}$$

We note that, with the inclusion of the contribution of the other pole at $-iM/2$, the real and immaginary part of Eq.(57) contribute to the integral of Eq.(54). The real and immaginary part of Eq.(57) are given by:

$$\Re[\Delta\sigma^{\rm DLA}(iM/2)] = e^{\alpha_s 2A_1\left(\ln^2\frac{|M|}{2} - \frac{\pi^2}{4}\right)}\cos\left(\alpha_s 2\pi A_1\ln\frac{|M|}{2}\right)\tag{59}$$

$$-1 - \alpha_s 2A_1\left(\ln^2\frac{|M|}{2} - \frac{\pi^2}{4}\right)$$

$$\Im[\Delta\sigma^{\rm DLA}(iM/2)] = e^{\alpha_s 2A_1\left(\ln^2\frac{|M|}{2} - \frac{\pi^2}{4}\right)}\sin\left(\alpha_s 2\pi A_1\ln\frac{|M|}{2}\right)\tag{60}$$

$$-\alpha_s 2\pi A_1\ln\frac{|M|}{2}$$

Now, to roughly estimate the effect of the oscillating factor in Eq.(56), we use the value of $M = M_0$ where the phase of Eq.(58) is stationary and is given by:

$$|M_0| = \frac{\alpha_s 2\pi A_1}{|\ln\sqrt{x}|}.\tag{61}$$

Substituting this in Eqs.(59,60), we find a suppression of about 35% for the parameter choice of E866 experiment and a suppression of about 10% for the W boson production at RHIC with a center-of-mass energy of $\sqrt{S} = 500$GeV which has more or less the same value of x. We should now recall that this is a rough estimation and that there is also the contribution of the cut on the negative real axis which usually produces an enhancement. However, comparing this estimation with the result for the W boson production at RHIC (see e.g. figure 1 and 2 in reference [17]), where resummation produces an enhancement of about 4% we can believe that the ignored contribution in this section produce an enhancement of about 15%, thus giving a suppressio of about 20% at E866 experiment.

5.3 Resummation of prompt photon rapidity distribution

5.3.1 General kinematics of prompt photon rapidity distribution

Here, we consider the rapidity distribution of the prompt photon process discussed in chapter 4,

$$H_1(P_1) + H_2(P_2) \to \gamma(p_\gamma) + X.\tag{62}$$

Specifically, we are interested in the differential cross section $p_\perp^3 \frac{d\sigma}{dp_\perp dY}(x_\perp, p_\perp^2, \eta_\gamma)$, where as in section 4.1 of chapter 4 $p_\perp$ is the transverse momentum of the photon, η_γ is its hadronic center-of-mass pseudorapidity and

$$x_\perp = \frac{4p_\perp^2}{S}. \tag{63}$$

The pseudorapidity of the direct real photon in the partonic center-of-mass frame $\hat{\eta}_\gamma$ is related to η_γ through Eq.(6) in section 4.1:

$$\hat{\eta}_\gamma = \eta_\gamma - \frac{1}{2} \ln \frac{x_1}{x_2}. \tag{64}$$

Furthermore, as in chapter 4, we use the following parametrizations of the photon and of the incoming partons' momenta

$$p_\gamma = (p_\perp \cosh \hat{\eta}_\gamma, \vec{p}_\perp, p_\perp \sinh \hat{\eta}_\gamma), \tag{65}$$

$$p_1 = x_1 P_1 = \frac{\sqrt{s}}{2}(1, \vec{0}_\perp, 1), \tag{66}$$

$$p_2 = x_2 P_2 = \frac{\sqrt{s}}{2}(1, \vec{0}_\perp, -1). \tag{67}$$

The transverse energy that the partons can transfer to the outgoing partons must be less than the partonic center-of-mass energy $\sqrt{s} = \sqrt{x_1 x_2 S}$. This means that

$$z \cosh^2 \hat{\eta}_\gamma \leq 1, \tag{68}$$

where we have defined the parton scaling variable

$$z = \frac{Q^2}{s} = \frac{x_\perp}{x_1 x_2}, \tag{69}$$

with, as in chapter 4, $Q^2 = 4p_\perp^2$. Eq.(68) implies that the upper and lower boundaries for the partonic center-of-mass pseudorapidity are given by

$$\hat{\eta}_- \leq \hat{\eta}_\gamma \leq \hat{\eta}_+, \tag{70}$$

where

$$\hat{\eta}_\pm = \ln \left(\frac{1}{\sqrt{z}} \pm \sqrt{\frac{1}{z} - 1} \right) = \pm \ln \left(\frac{1}{\sqrt{z}} + \sqrt{\frac{1}{z} - 1} \right). \tag{71}$$

Using Eq.(64), we can rewrite the transverse energy condition Eq.(68) as a condition for the lower bound of x_2:

$$x_2 \geq \frac{x_1 \sqrt{x_\perp} e^{-\eta_\gamma}}{2x_1 - \sqrt{x_\perp} e^{\eta_\gamma}} \equiv x_2^0. \tag{72}$$

Now, the requirement that $x_2 \leq 1$ implies the lower bound for x_1:

$$x_1 \geq \frac{\sqrt{x_\perp} e^{\eta_\gamma}}{2 - \sqrt{x_\perp} e^{-\eta_\gamma}} \equiv x_1^0. \tag{73}$$

The upper and lower bounds of the hadronic center-of-mass pseudorapidity can be found with the obvious condition that $x_{1(2)}^0 \leq 1$. In this way, we find,

$$\eta_- \leq \eta_\gamma \leq \eta_+, \tag{74}$$

where

$$\eta_\pm = \ln\left(\frac{1}{\sqrt{x_\perp}} \pm \sqrt{\frac{1}{x_\perp} - 1}\right) = \pm\ln\left(\frac{1}{\sqrt{x_\perp}} + \sqrt{\frac{1}{x_\perp} - 1}\right). \tag{75}$$

5.3.2   Mellin-Fourier transform and all-order resummation

The expression with the factorization of collinear singularities of this cross section in perturbative QCD is

$$p_\perp^3 \frac{d\sigma}{dp_\perp d\eta_\gamma}(x_\perp, p_\perp^2, \eta_\gamma) = \sum_{a,b} \int_{x_1^0}^1 dx_1 \int_{x_2^0}^1 dx_2 x_1 F_a^{H_1}(x_1, \mu^2) x_2 F_b^{H_2}(x_2, \mu^2)$$

$$\times C_{ab}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \hat{\eta}_\gamma\right) \delta(x_\perp - zx_1x_2), \tag{76}$$

where $F_a^{H_1}(x_1, \mu^2)$, $F_b^{H_2}(x_2, \mu^2)$ are the distribution functions of partons a, b in the colliding hadrons and μ^2 is the factorization scale equal to the renormalization scale. $\hat{\eta}_\gamma$ is a function of η_γ, x_1 and x_2 as defined by Eq.(64). The coefficient function $C_{ab}(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \hat{\eta}_\gamma)$ is defined in terms of the partonic cross section for the process where partons a, b are incoming as

$$C_{ab}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \hat{\eta}_\gamma\right) = p_\perp^3 \frac{d\hat{\sigma}_{ab}}{dp_\perp d\eta_\gamma}. \tag{77}$$

To allow the Mellin transform to deconvolute Eq.(76), we first perform the Fourier transform with respect to η_γ, thus obtaining

$$\sigma(N, Q^2, M) = \int_0^1 dx_\perp x_\perp^{N-1} \int_{\eta_-}^{\eta_+} d\eta_\gamma p_\perp^3 \frac{d\sigma}{dp_\perp d\eta_\gamma}(x_\perp, p_\perp^2, \eta_\gamma) \tag{78}$$

$$= \sum_{a,b} F_a^{H_1}(N + 1 + iM/2, \mu^2) F_b^{H_2}(N + 1 - iM/2, \mu^2)$$

$$\times C_{ab}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), M\right). \tag{79}$$

where

$$F_c^{H_i}(N + 1 \pm iM/2, \mu^2) = \int_0^1 dx x^{N \pm iM/2} F_i^{H_i}(x, \mu^2), \tag{80}$$

$$C_{ab}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), M\right) = 2\int_0^1 dz z^{N-1} \int_0^{\hat{\eta}_+} d\hat{\eta}_\gamma \cos(M\hat{\eta}_\gamma)$$

$$\times \quad C\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \hat{\eta}_\gamma\right). \tag{81}$$

A resummed expression of Eq.(83) in the threshold limit for the transverse energy ($z \to 1$ or equivalently $N \to \infty$) is obtained in the same way as we have done at the beginning of section 5.1.4, since the upper integration bound of $\hat{\eta}_\gamma$ in Eq.(81) is

$$\hat{\eta}_+ = \ln \left(\frac{1}{\sqrt{z}} + \sqrt{\frac{1}{z} - 1} \right) = \sqrt{1 - z} - (1 - z) + O((1 - z)^{3/2}). \tag{82}$$

Thus, up to terms suppressed by factors $1/N$, the resummed expression of Eq.(83) is:

$$
\begin{aligned}
\sigma^{res}(N, Q^2, M) &= \sum_{a,b} F_a^{H_1}(N + 1 + iM/2, \mu^2) F_b^{H_2}(N + 1 - iM/2, \mu^2) \\
&\times C_{I\,ab}^{res}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right),
\end{aligned}
\tag{83}
$$

where $C_{I\,ab}^{res}$ is the resummed pseudorapidity-integrated coefficient function for the prompt photon production for the subprocess which involves the initial partons a, b. These resummed coefficient function has been studied in chapter 4 and in Refs. [14, 65]. This result is analogous to that of the Drell-Yan rapidity distributions case, in the sense that the resummed formula is obtained through a translation of the parton densities' moments by $1 \pm iM/2$ and the pseudorapidity-integrated coefficient functions.

Chapter 6

Renormalization group resummation of tranverse distributions

We prove the all-order exponentiation of soft logarithmic corrections at small transverse momentum to the distribution of Drell-Yan process. We apply the renormalization group approach developed in the context of integrated cross sections. We show that all large logs in the soft limit can be expressed in terms of a single dimensional variable, and we use the renormalization group to resum them. The resummed result that we obtain is, beyond the next-to-leading log accuracy, more general and less predictive than those previously released. The origin of this could be due to factorization properties of the cross section. The understanding of this point is a work in progress.

6.1 Drell-Yan distribution at small transverse momentum

We consider the Drell-Yan process

$$H_1(P_1) + H_2(P_2) \to \gamma^*(Q) + X(K), \tag{1}$$

and, in particular, the differential cross section $\frac{d\sigma}{dq_\perp^2 \, dY}(Q^2, q_\perp^2, x_1, x_2)$, where $q_\perp$ is the transverse momentum with respect to colliding axis of the hadrons H_1 and H_2, Q^2 is the virtuality of photon and x_1, x_2 are useful dimensionless variables, that, in terms of the hadronic center-of-mass squared energy $S = (P_1 + P_2)^2$ and the photon center-of-mass rapidity Y, are given by

$$x_1 = \sqrt{\frac{Q^2 + q_\perp^2}{S}} e^Y; \qquad x_2 = \sqrt{\frac{Q^2 + q_\perp^2}{S}} e^{-Y}. \tag{2}$$

The relation between these two variables and the fraction of energy carried by the virtual photon is

$$\frac{x_1 + x_2}{2} = \frac{E_{\gamma^*}}{\sqrt{S}}. \tag{3}$$

According to standard factorization of perturbative QCD, the expression for the differential cross section is

$$\frac{d\sigma}{dq_\perp^2 \, dY}(Q^2, q_\perp^2, x_1, x_2) = \int_{z_1^{min}}^1 dz_1 \int_{z_2^{min}}^1 dz_2 f_1(z_1, \mu^2) f_2(z_2, \mu^2)$$

$$\times \frac{d\hat{\sigma}}{dq_\perp^2 \, dy}(Q^2, q_\perp^2, s, y, \mu^2, \alpha_s(\mu^2)), \tag{4}$$

where $f_1(z_1, \mu^2), f_2(z_2, \mu^2)$ are the parton distribution functions of the colliding quark and anti-quark in the hadrons H_1 and H_2 respectively. The arbitrary scale μ^2 is the factorization scale, which, for simplicity, is chosen to be equal to the renormalization scale. The condition that the invariant mass of the emitted particles K^2 cannot be negative, imposes that $(z_1 - x_1)(z_2 - x_2) \geq \frac{q_\perp^2}{S}$ and, taking the small $q_\perp^2$ limit, we obtain that

$$z_1^{min} = x_1, \qquad z_2^{min} = x_2. \tag{5}$$

The partonic center-of-mass squared energy s and rapidity y are related to the hadronic ones by a scaling and a boost along the collision axis with respect to the longitudinal momentum fraction z_1, z_2 of the incoming partons:

$$s = z_1 z_2 S; \qquad y = Y - \frac{1}{2} \ln \frac{z_1}{z_2}. \tag{6}$$

We define analogous variables to that of Eqs.(2) at the partonic level

$$\xi_1 \equiv \frac{x_1}{z_1} = \sqrt{\frac{Q^2 + q_\perp^2}{s}} e^y; \qquad \xi_2 \equiv \frac{x_2}{z_2} = \sqrt{\frac{Q^2 + q_\perp^2}{s}} e^{-y}, \tag{7}$$

with inverse relations

$$s = \frac{Q^2 + q_\perp^2}{(x_1/z_1)(x_2/z_2)}; \qquad y = \frac{1}{2} \ln \frac{x_1/z_1}{x_2/z_2}. \tag{8}$$

Now, thanks to these equations, we can define a dimensionless differential cross section and coefficient function

$$W(q_\perp^2/Q^2, x_1, x_2) = \frac{Q^4}{x_1 x_2} \frac{d\sigma}{dq_\perp^2 \, dY}(Q^2, q_\perp^2, x_1, x_2), \tag{9}$$

$$\hat{W}\left(\frac{Q^2}{\mu^2}, \frac{q_\perp^2}{Q^2}, \frac{x_1}{z_1}, \frac{x_2}{z_2}, \alpha_s(\mu^2)\right) = \frac{Q^4}{(x_1/z_1)(x_2/z_2)} \frac{d\hat{\sigma}}{dq_\perp^2 \, dy}(Q^2, q_\perp^2, s, y, \mu^2, \alpha_s(\mu^2)), \tag{10}$$

in such a way that Eq.(4), together with the conditions Eqs.(5), takes the useful form of a convolution product

$$W(q_\perp^2/Q^2, x_1, x_2) = \int_{x_1}^1 \frac{dz_1}{z_1} \int_{x_2}^1 \frac{dz_2}{z_2} f_1(z_1, \mu^2) f_2(z_2, \mu^2)$$

$$\times \hat{W}(Q^2/\mu^2, q_\perp^2/Q^2, x_1/z_1, x_2/z_2, \alpha_s(\mu^2)), \tag{11}$$

which is valid only for small $q_\perp^2$.

6.2 The role of standard factorization

It is known that this expression (or equivalently Eq.(4)) is originated by the factorization of collinear divergences in the impact parameter ($\vec{b}$) which is conjugate upon Fourier transformation to the transverse momentum ($\vec{q}_\perp$):

$$W(Q^2 b^2, x_1, x_2) = \int d^2 q_\perp e^{i\vec{q}_\perp \vec{b}} W(q_\perp^2/Q^2, x_1, x_2). \tag{12}$$

In $d = 4 - 2\epsilon$ dimensions this factorization has the form

$$\hat{W}(Q^2/\mu^2, Q^2 b^2, x_1, x_2, \alpha_s(\mu^2)) =$$
$$= \int_{x_1}^1 \frac{dz_1}{z_1} \int_{x_2}^1 \frac{dz_2}{z_2} Z(z_1, \alpha_s(\mu^2), \epsilon) Z(z_2, \alpha_s(\mu^2), \epsilon) \hat{W}^{(0)}(Q^2, b^2, \frac{x_1}{z_1}, \frac{x_2}{z_2}, \alpha_0, \epsilon). \tag{13}$$

Note that the universal function Z that extracts the collinear divergences from the bare coefficient function doesn't depend on the Fourier conjugate (b) of the transverse momentum Ref.[11]. In Fourier space Eq.(11), becomes

$$\begin{aligned} W(Q^2 b^2, x_1, x_2) &= \int_{x_2}^1 \frac{dz_1}{z_1} \int_{x_2}^1 \frac{dz_2}{z_2} f_1(z_1, \mu^2) f_2(z_2, \mu^2) \\ &\times \hat{W}(Q^2/\mu^2, Q^2 b^2, x_1/z_1, x_2/z_2, \alpha_s(\mu^2)) \end{aligned} \tag{14}$$

Furthermore, Eq.(11) tells us that the differential cross section is a convolutional product which is diagonalized by a double Mellin transform. Thus, performing the double Mellin and Fourier transform, the coefficient function takes the simple factorized form:

$$W(Q^2 b^2, N_1, N_2) = f_1(N_1, \mu^2) f_2(N_2, \mu^2) \hat{W}(Q^2/\mu^2, Q^2 b^2, N_1, N_2, \alpha_s(\mu^2)). \tag{15}$$

Our goal is to resum the large logarithms $\ln Q^2 b^2$ to all logarithmic orders. These logs are present to all orders in the contributions to this differential cross section. They come from the kinematical region of soft and collinear emissions. However, we know from Eq.(13) that collinear divergences that arises in the limit $q_\perp \to 0$ are absorbed in parton distribution function evolution. Consequently, we will resum only the large logarithms $\ln Q^2 b^2$ that come from soft contributions.

We define the usual physical anomalous dimension:

$$Q^2 \frac{\partial}{\partial Q^2} W(Q^2 b^2, N_1, N_2) = \gamma_{(W)}(Q^2 b^2, N_1, N_2, \alpha_s(Q^2)) W(Q^2 b^2, N_1, N_2). \tag{16}$$

It is clear that $\gamma_{(W)}(Q^2 b^2, N_1, N_2, \alpha_s(Q^2))$ is a renormalization group invariant and we will show that it is also independent of N_1 and N_2 when choosing the arbitrary scale μ^2 equal to $1/b^2$ and taking into account only soft contributions. Thus, in the soft limit and with the convenient choice $\mu^2 = 1/b^2$, we can write

$$\gamma_{(W)}^{SOFT}(Q^2 b^2, N_1, N_2, \alpha_s(Q^2)) = \gamma(1, Q^2 b^2, \alpha_s(Q^2)) \tag{17}$$

So, the resummed expression for the cross section Eq.(11) in Fourier space, in which the collinear contributions to the large $\ln Q^2 b^2$ are separated from the soft ones, has the general form:

$$
\begin{aligned}
W^{res}(Q^2 b^2, x_1, x_2) &= \int_{x_2}^1 \frac{dz_1}{z_1} \int_{x_2}^1 \frac{dz_2}{z_2} f_1(z_1, 1/b^2) f_2(z_2, 1/b^2) K^{res}(b^2, Q_0^2, Q^2) \\
&\times \hat{W}^{res}(Q_0^2 b^2, Q_0^2 b^2, x_1/z_1, x_2/z_2, \alpha_s(1/b^2)),
\end{aligned}
\tag{18}
$$

where

$$
K^{res}(Q^2 b^2, Q_0^2, Q^2) = \exp\left\{ \int_{Q_0^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \Gamma^{res}\left(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2)\right) \right\},
\tag{19}
$$

The scale Q_0^2, must be larger than the lower limit of the perturbative analysis ($Q_0^2 > \Lambda_{QCD}^2$). Hence, in order to absorb the possible large correction of the type $\ln Q_0^2 b^2$ we will always choose $Q_0^2 = 1/b^2$. Accordingly, the condition $Q_0^2 > \Lambda_{QCD}^2$ becomes $b^2 < 1/\Lambda_{QCD}^2$ and the resummed exponent of Eq.(19) is related to the resummed physical anomalous dimension γ^{res} through the logarithmic derivative:

$$
\gamma^{res}(1, Q^2 b^2, \alpha_s(Q^2)) = Q^2 \frac{\partial}{\partial Q^2} \int_{1/b^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \Gamma^{res}\left(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2)\right).
\tag{20}
$$

It is now clear that resummation of collinear emissions is realized by the parton distribution evolution thanks to the fact that the factorization scale μ^2 is arbitrary. Resummation of soft gluon emissions can be achieved by the resummation of the exponent that appears in this expression. This is the subject of the next section.

6.3 The $q_\perp^2$ singularities of soft gluon contributions

We now proceed through the calculation of the resummed exponent using kinematics analysis and renormalization group improvement. According to the Appendix B, the phase space measure in $d = 4 - 2\epsilon$ dimensions for n extra emissions of the partonic Drell-Yan subprocess can be written for $n = 0$ and $n \geq 1$ respectively as:

$$
\frac{d\phi_1(p_1 + p_2; q)}{dq_\perp^2 dy} = \frac{1}{Q^4} \delta(1 - \xi_1)\delta(1 - \xi_2)\delta(\hat{q}_\perp^2)
\tag{21}
$$

$$
\begin{aligned}
\frac{d\phi_{n+1}(p_1 + p_2; q, k_1, \ldots, k_n)}{dq_\perp^2 dy} &= N(\epsilon)(q_\perp^2)^{-\epsilon} \int_0^{(\sqrt{s} - \sqrt{Q^2})^2} \frac{dM^2}{2\pi} d\phi_n(k; k_1, \ldots, k_n) \\
&\times \delta(M^2 - M_0^2);
\end{aligned}
$$

$$
k^2 = M^2; \qquad M_0^2 = \frac{Q^2}{\xi_1 \xi_2}[(1 - \xi_1)(1 - \xi_2) + \hat{q}_\perp^2(1 - \xi_1 - \xi_2)],
\tag{22}
$$

where $N(\epsilon) = 1/(2(4\pi)^{2-2\epsilon})$, $\xi_i = x_i/z_i$, $\hat{q}_\perp^2 = q_\perp^2/Q^2$ and $d\Omega^{n-1}(\epsilon)$ stands for the integration of $n-1$ dimensionless variables $(z_i, i = 1, \ldots, n-1)$. ξ_1 and ξ_2 are related to the partonic center-of-mass rapidity (y) and energy (s) by the relations:

$$s = \frac{Q^2 + q_\perp^2}{\xi_1\xi_2}; \qquad y = \frac{1}{2}\ln\frac{\xi_1}{\xi_2}. \tag{23}$$

The phase space measure $d\phi_n(k; k_1, \ldots, k_n)$ is the same as the phase space measure of the DIS process with an incoming momentum with a nonzero invariant mass ($k^2 = M^2$) and n outgoing massless particles. This phase space has been analyzed in Section 3.1 and is given by,

$$d\phi_1(k; k_1) = 2\pi\delta(M^2), \qquad n = 0 \tag{24}$$

$$d\phi_n(k; k_1, \ldots, k_n) = 2\pi\left[\frac{N(\epsilon)}{2\pi}\right]^{n-1}(M^2)^{n-2-(n-1)\epsilon}d\Omega^{n-1}(\epsilon), \qquad n \geq 1. \tag{25}$$

According to this, we can rewrite Eqs.(21),(22) in this form:

$$\frac{d\phi_1(p_1 + p_2; q)}{dq_\perp^2 dy} = \frac{1}{Q^4}\delta(1-\xi_1)\delta(1-\xi_2)\delta(\hat{q}_\perp^2) \tag{26}$$

$$\frac{d\phi_2(p_1 + p_2; q, k_1)}{dq_\perp^2 dy} = (q_\perp^2)^{-\epsilon}N(\epsilon)\delta(M_0^2) \tag{27}$$

$$M_0^2 = \frac{Q^2}{\xi_1\xi_2}[(1-\xi_1)(1-\xi_2) + \hat{q}_\perp^2(1-\xi_1-\xi_2)]. \tag{28}$$

for $n = 0, 1$ respectively, and

$$\frac{d\phi_{n+1}(p_1 + p_2; q, k_1, \ldots, k_n)}{dq_\perp^2 dy} = (q_\perp^2)^{-\epsilon}2\pi\left[\frac{N(\epsilon)}{2\pi}\right]^n(M_0^2)^{n-2-(n-1)\epsilon}d\Omega^{n-1}(\epsilon) \tag{29}$$

for $n \geq 2$. The dependence of the phase space on $\hat{q}_\perp^2$ comes entirely from the factors:

$$(q_\perp^2)^{-\epsilon}\delta(M_0^2), \qquad n = 1 \tag{30}$$

$$(M_0^2)^{n-1}(q_\perp^2)^{-\epsilon}(M_0^2)^{-(n-1)\epsilon-1}, \qquad n \geq 1. \tag{31}$$

The phase space measure must be combined with the square modulus of the amplitude, in order to determine the logarithmic singularities in $q_\perp = 0$ which are regularized in $d = 4 - 2\epsilon$ dimensions. Studying the behavior of the invariants that can be constructed with the external momenta, we can establish in which kinematical region the square modulus of the amplitude can be singular in $\hat{q}_\perp^2 \to 0$. From the study of the DIS-like emissions (see Section 3.2) we know that

$$k_i^0 = \frac{\sqrt{M_0^2}}{2}(z_{n-1}\cdots z_{i+1})^{1/2}(1-z_i), \qquad 1 \leq i \leq n-2 \tag{32}$$

$$k_{n-1}^0 = \frac{\sqrt{M_0^2}}{2}(1-z_{n-1}) \tag{33}$$

$$k_n^0 = k_1^0. \tag{34}$$

This means that all the invariants that can appear in the function $D_G(\beta, P_E)$ in Eq.(22) of Section 3.1 can be expressed in terms of the following ones:

$$q^2 \;\; = \;\; Q^2, \qquad p_1^2 = p_2^2 = k_i^2 = 0, \qquad p_1 \cdot p_2 = \frac{s}{2} \tag{35}$$

$$k_i \cdot k_j \;\; \sim \;\; M_0^2 = \frac{Q^2}{\xi_1 \xi_2}[(1 - \xi_1)(1 - \xi_2) + \hat{q}_\perp^2(1 - \xi_1 - \xi_2)] \tag{36}$$

$$p_1 \cdot k_i \;\; \sim \;\; p_2 \cdot k_i \sim \sqrt{s M_0^2} \tag{37}$$

$$= \;\; Q^2 \left[\frac{(1 + \hat{q}_\perp^2)}{\xi_1 \xi_2}[(1 - \xi_1)(1 - \xi_2) + \hat{q}_\perp^2(1 - \xi_1 - \xi_2)] \right]^{1/2}. \tag{38}$$

In the case $n = 1$, the single emission squared amplitude at tree level has a $1/q_\perp^2$ singularity for $q_\perp \to 0$. This can be easily seen by an explicit $O(\alpha_s)$ computation (see for example Refs.[66, 67, 68]). Hence, in the general case, we expect that in the $q_\perp \to 0$ limit the squared amplitude has the following behavior

$$|A_{n+1}|^2 \sim \frac{1}{\hat{q}_\perp^2}(M_0^2)^{n_1}(M_0^2)^{n_2 \epsilon} g_{n_1 n_2}(\xi_1, \xi_2), \tag{39}$$

where N and k are integer or half-integer numbers (see Eqs.(36,37)). However, as discussed in Section 3.2, here we will assume that only the integer powers of M_0^2 contribute. Then, we know that phase space contributes the factors of Eqs.(30,31) and, hence, Eq.(39) implies that a generic contribution to the coefficient function $\hat{W}^{(0)}$ has the folowing structure:

$$(\hat{q}_\perp^2)^{-1-\epsilon}\delta(M_0^2)(M_0^2)^{n_1}(M_0^2)^{n_2 \epsilon} g_{n n_2}(\xi_1, \xi_2); \qquad n = 1, \tag{40}$$

$$(\hat{q}_\perp^2)^{-1-\epsilon}(M_0^2)^{-1-(n-n_2'-1)\epsilon}(M_0^2)^{n-1+n_1} g_{n n_2'}(\xi_1, \xi_2); \qquad n > 1, \tag{41}$$

where for the moment we do not care about the overall dimensional factor. Now, we are interested in taking into account only the $1/\hat{q}_\perp^2$ singularity, because more singular terms are forbidden and less singular ones are suppressed. As proven in Appendix D, in the limit $q_\perp^2 \to 0$,

$$\delta(M_0^2) = \frac{\xi_1 \xi_2}{Q^2} \left[\frac{\delta(1 - \xi_1)}{(1 - \xi_2)_+} + \frac{\delta(1 - \xi_2)}{(1 - \xi_1)_+} - \ln \hat{q}_\perp^2 \delta(1 - \xi_1)\delta(1 - \xi_2) \right] + O(\hat{q}_\perp^2), \tag{42}$$

and, for $\eta = -(n - n_2' - 1)\epsilon$ (with $\epsilon < 0$ $n - n_2' - 1 > 0$),

$$[(1 - \xi_1)(1 - \xi_2) + \hat{q}_\perp^2(1 - \xi_1 - \xi_2)]^{\eta - 1} = \tag{43}$$

$$= (1 - \xi_1)^{\eta - 1}(1 - \xi_2)^{\eta - 1} - \frac{(\hat{q}_\perp^2)^\eta}{\eta^2}\delta(1 - \xi_1)\delta(1 - \xi_2) + O(\hat{q}_\perp^2).$$

Therefore, $n_1 = -n + 1$, $n_2' < n - 1$. Furthermore, we note that the terms not proportional to $\delta(1 - \xi_1)\delta(1 - \xi_2)$ are divergent (in the $q_\perp \to 0$ limit) due to collinear emissions. These divergences are absorbed and resummed by the Altarelli-Parisi evolution of the parton distributions $f_1(z_1, \mu^2)$ and $f_2(z_2, \mu^2)$ when they are evaluated

at the scale $\mu^2 = 1/b^2$ in Fourier space (see Eq.(18)). As a first conclusion, we obtain that the contributions that must be resummed in the limit $\hat{q}_\perp^2 \to 0$ are those which are proportional to $\delta(1 - \xi_1)\delta(1 - \xi_2)$ and, thus, belong to the kinematical region of only the soft extra emissions. Hence, we obtain that the soft part of the coefficient function that must be resummed has, after the inclusion of loops, the following general form:

$$\hat{W}(Q^2, q_\perp^2, \xi_1, \xi_2, \alpha_0, \epsilon) = \sum_{n=0}^{\infty} \alpha_0^n \hat{W}_n(Q^2, q_\perp^2, \alpha_0, \epsilon)\delta(1 - \xi_1)\delta(1 - \xi_2), \qquad (44)$$

with

$$\hat{W}_n(Q^2, q_\perp^2, \alpha_0, \epsilon) = (Q^2)^{-n\epsilon}\Big[C_{n0}^{(0)}(\epsilon)\delta(\hat{q}_\perp^2) + \sum_{k=2}^{n} C_{nk}^{(0)}(\epsilon)(\hat{q}_\perp^2)^{-1-k\epsilon}$$

$$+ \sum_{k=1}^{n} C_{nk}'^{(0)}(\epsilon)(\hat{q}_\perp^2)^{-1-k\epsilon} \ln \hat{q}_\perp^2\Big], \qquad (45)$$

where the factor of $(Q^2)^{-n\epsilon}$ has been introduced for dimensional reasons. We, now, perform the double Mellin transform and the Fourier transform using the fact that:

$$\int d^2\hat{q}_\perp e^{i\hat{b}\cdot\hat{q}_\perp}\delta(\hat{q}_\perp^2) = \pi, \qquad (46)$$

$$\int d^2\hat{q}_\perp e^{i\hat{b}\cdot\hat{q}_\perp}(\hat{q}_\perp^2)^{-1-k\epsilon} = \pi F_k(\epsilon)(\hat{b}^2)^{k\epsilon}, \qquad (47)$$

$$\int d^2\hat{q}_\perp e^{i\hat{b}\cdot\hat{q}_\perp}(\hat{q}_\perp^2)^{-1-k\epsilon} \ln \hat{q}_\perp^2 = -\pi F_k(\epsilon)(\hat{b}^2)^{k\epsilon} \ln \hat{b}^2 - \pi\frac{F_k'(\epsilon)}{k}(\hat{b}^2)^{k\epsilon}, \qquad (48)$$

$$\hat{b}^2 \equiv Q^2 b^2; \qquad F_k(\epsilon) = -\frac{4^{-k\epsilon}}{k\epsilon}\frac{\Gamma(1 - k\epsilon)}{\Gamma(1 + k\epsilon)}. \qquad (49)$$

According to this, Eq.(44) has, after Mellin and Fourier transform, this structure:

$$\hat{W}^{(0)}(Q^2, b^2, \alpha_0, \epsilon) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \tilde{C}_{nk}^{(0)}(\epsilon)[(Q^2)^{-\epsilon}\alpha_0]^{n-k}[(1/b^2)^{-\epsilon}\alpha_0]^k + \qquad (50)$$

$$+ \ln Q^2 b^2 \sum_{n=1}^{\infty}\sum_{k=1}^{n} \tilde{C}_{nk}'^{(0)}(\epsilon)[(Q^2)^{-\epsilon}\alpha_0]^{n-k}[(1/b^2)^{-\epsilon}\alpha_0]^k$$

6.4 The resummed exponent in renormalization group approach

At this point, we calculate the resummed exponent that appears in Eq.(19):

$$
\int_{1/b^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \Gamma^{(0)}(Q^2, \bar{\mu}^2, b^2, \alpha_0, \epsilon) = \ln\left(\frac{\hat{W}^{(0)}(Q^2, b^2, \alpha_0, \epsilon)}{\hat{W}^{(0)}(1/b^2, b^2, \alpha_0, \epsilon)}\right) = \tag{51}
$$

$$
= \left[\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} E_{nk}^{(0)}(\epsilon)[(\bar{\mu}^2)^{-\epsilon}\alpha_0]^{n-k}[(1/b^2)^{-\epsilon}\alpha_0]^k\right]_{\bar{\mu}^2=1/b^2}^{\bar{\mu}^2=Q^2}
$$

$$
+ \ln\left(1 + \ln Q^2 b^2 \sum_{n=1}^{\infty}\sum_{k=1}^{n} \tilde{E}_{nk}^{(0)}(\epsilon)[(Q^2)^{-\epsilon}\alpha_0]^{n-k}\right.
$$

$$
\left.\times [(1/b^2)^{-\epsilon}\alpha_0]^k\right),
$$

where the last term has not been expanded because we must take into account that in the bare coefficient function Eq.(44) there is only one explicit logarithm. From the explicit calculation to order $O(\alpha_0)$, we find that

$$
E_{10}^{(0)}(\epsilon) = \frac{2\pi}{3}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + \pi^2 - 8\right), \tag{52}
$$

$$
\tilde{E}_{11}^{(0)}(\epsilon) = \frac{4\pi}{3}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}F_1(\epsilon). \tag{53}
$$

Now, we want to rewrite Eq.(51) in a renormalized form. To do this, we use, as expalined in Chapter 3, the fact that $(Q^2)^{-\epsilon}\alpha_0$ and $(1/b^2)^{-\epsilon}\alpha_0$ are renormalization group invariant. Consequently, we may write:

$$
(Q^2)^{-\epsilon}\alpha_0 = \alpha_s(Q^2)Z^{(\alpha_s)}(\alpha_s(Q^2), \epsilon), \tag{54}
$$

$$
(1/b^2)^{-\epsilon}\alpha_0 = \alpha_s(1/b^2)Z^{(\alpha_s)}(\alpha_s(1/b^2), \epsilon), \tag{55}
$$

where $Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon)$ has multiple poles at $\epsilon = 0$ and μ^2 is the renormalization scale which for simplicity has been chosen equal to the factorization scale. Furthermore we note that the universal functions $Z^{(\hat{W})}(N_1, \alpha_s(\mu^2), \epsilon)Z^{(\hat{W})}(N_2, \alpha_s(\mu^2), \epsilon)$ that extract the collinear poles from the coefficient function simplify in the first line of Eq.(51). Thus the renormalized expression of Eq.(51) has the form:

$$
\int_{1/b^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\Gamma(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2), \epsilon) = \left[\sum_{m=1}^{\infty}\sum_{n=0}^{n-1} E_{mn}^R(\epsilon)\alpha_s^{m-n}(\bar{\mu}^2)\alpha_s^n(1/b^2)\right]_{\bar{\mu}^2=1/b^2}^{\bar{\mu}^2=Q^2} \tag{56}
$$

$$
+ \ln\left(1 + \ln Q^2 b^2 \sum_{m=1}^{\infty}\sum_{n=1}^{m} \tilde{E}_{mn}^R(\epsilon)\right.
$$

$$
\left.\times \alpha_s(Q^2)^{m-n}\alpha_s^n(1/b^2)\right).
$$

The resummed exponent is clearly pole-free and so we can exploit the cancellation of the poles that could be present in the coefficients $E_{mn}^R(\epsilon)$ and $\tilde{E}_{mn}^R(\epsilon)$. Furthermore we want to perform a comparison with previously released resummation formulae given in [6, 67]. In order to do these two things we rewrite Eq.(56) in terms of the renormalized physical anomalous dimension $\Gamma(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2), \epsilon)$ and therefore, we calculate, according to Eq.(20), the logarithmic derivative of Eq.(56):

$$\gamma(1, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2), \epsilon) = \tag{57}$$

$$= \beta^{(d)}(\alpha_s(\bar{\mu}^2)) \frac{\partial}{\partial \alpha_s(\bar{\mu}^2)} \sum_{m=1}^{\infty} \sum_{n=0}^{n-1} E_{mn}^R(\epsilon) \alpha_s^{m-n}(\bar{\mu}^2) \alpha_s^n(1/b^2)$$

$$+ \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{m} \tilde{E}_{mn}^R(\epsilon) \alpha_s(\bar{\mu}^2)^{m-n} \alpha_s^n(1/b^2)}{1 + \ln \bar{\mu}^2 b^2 \sum_{m=1}^{\infty} \sum_{n=1}^{m} \tilde{E}_{mn}^R(\epsilon) \alpha_s(\bar{\mu}^2)^{m-n} \alpha_s^n(1/b^2)}$$

$$+ \frac{\ln \bar{\mu}^2 b^2 \beta^{(d)}(\alpha_s(\bar{\mu}^2)) \partial/\partial \alpha_s \sum_{m=1}^{\infty} \sum_{n=1}^{m} \tilde{E}_{mn}^R(\epsilon) \alpha_s(\bar{\mu}^2)^{m-n} \alpha_s^n(1/b^2)}{1 + \ln \bar{\mu}^2 b^2 \sum_{m=1}^{\infty} \sum_{n=1}^{m} \tilde{E}_{mn}^R(\epsilon) \alpha_s(\bar{\mu}^2)^{m-n} \alpha_s^n(1/b^2)},$$

where

$$\beta^{(d)}(\alpha_s(\bar{\mu}^2)) = -\epsilon \alpha_s(\bar{\mu}^2) + \beta(\alpha_s(\bar{\mu}^2)), \tag{58}$$

and $\beta(\alpha_s(\bar{\mu}^2)) = -\beta_0^2 \alpha_s(\bar{\mu}^2) + O(\alpha_s^3)$ is the usual four-dimensional β-function. Now, in order to isolate the terms which contain the explicit logs from the rest we add and subtract the term

$$\sum_{m=1}^{\infty} \sum_{n=1}^{m} \tilde{E}_{mn}^R(\epsilon) \alpha_s(\bar{\mu}^2)^{m-n} \alpha_s^n(1/b^2). \tag{59}$$

After this, we re-expand the various terms in powers of $\alpha_s(\bar{\mu}^2)$ and $\alpha_s(1/b^2)$, but not in powers of $\ln \bar{\mu}^2 b^2$. The result that we find in this way has the following structure:

$$\int_{1/b^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \Gamma(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2), \epsilon) = \int_{1/b^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left(\sum_{m=1}^{\infty} \sum_{n=0}^{m} \Gamma_{mn}^R(\epsilon) \alpha_s^{m-n}(\bar{\mu}^2) \alpha_s^n(1/b^2) \right)$$

$$+ \int_{1/b^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left(\frac{\ln \bar{\mu}^2 b^2 \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} \tilde{\Gamma}_{mn}^R(\epsilon) \alpha_s^{m-n}(\bar{\mu}^2) \alpha_s^n(1/b^2)}{1 + \ln \bar{\mu}^2 b^2 \sum_{m=1}^{\infty} \sum_{n=1}^{m} \tilde{E}_{mn}^R(\epsilon) \alpha_s^{m-n}(\bar{\mu}^2) \alpha_s^n(1/b^2)} \right) \tag{60}$$

To show the cancellation of divergences we rewrite the integrand separating off the b^2-independent terms as in Chapter 3 and Chapter 4:

$$\gamma(1, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2), \epsilon) = \hat{\Gamma}^{(c)}(\alpha_s(\bar{\mu}^2), \epsilon) + \hat{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(1/b^2), \epsilon)$$
$$+ \hat{\Gamma}'^{(l)}(\ln \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2), \alpha_s(1/b^2), \epsilon),$$

where

$$\hat{\Gamma}^{(c)}(\alpha_s(\bar{\mu}^2), \epsilon) = \sum_{m=1}^{\infty} \Gamma_m^R(\epsilon) \alpha_s^m(\bar{\mu}^2) \tag{61}$$

$$\hat{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(1/b^2), \epsilon) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \Gamma_{m+nn}^R(\epsilon) \alpha_s^m(\bar{\mu}^2) \alpha_s^n(1/b^2) \tag{62}$$

$$\hat{\Gamma}'^{(l)}\left(\ln\bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2), \alpha_s(1/b^2), \epsilon\right) = \frac{\ln\bar{\mu}^2 b^2 \sum_{m=2}^{\infty}\sum_{n=1}^{m-1}\tilde{\Gamma}_{mn}^R(\epsilon)\alpha_s^{m-n}(\bar{\mu}^2)\alpha_s^n(1/b^2)}{1+\ln\bar{\mu}^2 b^2 \sum_{m=1}^{\infty}\sum_{n=1}^{m}\tilde{E}_{mn}^R(\epsilon)\alpha_s^{m-n}(\bar{\mu}^2)\alpha_s^n(1/b^2)}$$

(63)

We know that

$$\alpha_s(1/b^2) = f(\alpha_s(\bar{\mu}^2), \ln\bar{\mu}^2 b^2),$$

(64)

so, in principle, it is possible to invert this relation in order to obtain:

$$\ln\bar{\mu}^2 b^2 = g(\alpha_s(\bar{\mu}^2), \alpha_s(1/b^2)).$$

(65)

The function g has the following perturbative expression:

$$g(\alpha_s(\bar{\mu}^2), \alpha_s(1/b^2)) = \frac{1}{\beta_0}\left(\frac{1}{\alpha_s(\bar{\mu}^2)} - \frac{1}{\alpha_s(1/b^2)}\right)\left(1 + \sum_{r=1}^{\infty} l_r \alpha_s^r(\bar{\mu}^2)\right).$$

(66)

Substituting this expression in Eq.(63) and re-expanding in powers of $\alpha_s(\bar{\mu}^2)$ and $\alpha_s(1/b^2)$, we obtain that:

$$\hat{\Gamma}'^{(l)}(\ln\bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2), \alpha_s(1/b^2), \epsilon) = \bar{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(1/b^2), \epsilon) - \bar{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(\bar{\mu}^2), \epsilon),$$

(67)

where

$$\bar{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(\mu^2), \epsilon) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\bar{\Gamma}_{m+nn}^R(\epsilon)\alpha_s^m(\bar{\mu}^2)\alpha_s^n(\mu^2).$$

(68)

We choose as counterterm,

$$Z^{(\Gamma)}(\alpha_s(\bar{\mu}^2), \epsilon) = \hat{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(\bar{\mu}^2), \epsilon).$$

(69)

With this choice we obtain:

$$\begin{aligned}
\gamma(1, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2), \epsilon) = & \hat{\Gamma}^{(c)}(\alpha_s(\bar{\mu}^2), \epsilon) + \hat{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(\bar{\mu}^2), \epsilon) + \\
& + \hat{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(1/b^2), \epsilon) - \hat{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(\bar{\mu}^2), \epsilon) + \\
& + \bar{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(1/b^2), \epsilon) - \bar{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(\bar{\mu}^2), \epsilon).
\end{aligned}$$

(70)

The first line is a power series with coefficients which are pole-free for each b^2, because the second and the third lines vanish when $b^2 = 1/\bar{\mu}^2$. Hence, the sum of the second and the third line must be finite at $\epsilon = 0$, but it is not necessarily analytic in $\alpha_s(\mu^2)$. To find its perturbative expression in powers of $\alpha_s(\mu^2)$ we rewrite the last two lines of Eq.(70) as

$$\int_{1/b^2}^{\bar{\mu}^2} \frac{d\mu^2}{\mu^2}\left(\frac{\partial}{\partial\ln\mu^2}\hat{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(\mu^2), \epsilon) + \frac{\partial}{\partial\ln\mu^2}\bar{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(1/b^2), \epsilon)\right).$$

(71)

There could be other residual cancellations of $\epsilon = 0$ poles between these two terms, but their sum must be finite at $\epsilon = 0$ and analytic in $\alpha_s(\mu^2)$ and $\alpha_s(1/b^2)$. Thus,

we get a perturbative expression of the physical anomalous dimension with finite coefficient:

$$\gamma(1, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2)) = -\int_{1/b^2}^{\bar{\mu}^2} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2)) - B(\alpha_s(\bar{\mu}^2)) -$$
$$-\int_{1/b^2}^{\bar{\mu}^2} \frac{d\mu^2}{\mu^2} C(\alpha_s(\bar{\mu}^2), \alpha_s(\mu^2)), \tag{72}$$

where

$$A(\alpha_s(\mu^2)) = \sum_{n=1}^{\infty} A_n \alpha_s^n(\mu^2) \tag{73}$$

$$B(\alpha_s(\bar{\mu}^2)) = \sum_{m=1}^{\infty} B_m \alpha_s^m(\bar{\mu}^2) \tag{74}$$

$$C(\alpha_s(\bar{\mu}^2), \alpha_s(\mu^2)) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \alpha_s^m(\bar{\mu}^2) \alpha_s^n(\mu^2) \tag{75}$$

$$\tag{76}$$

After an integration by parts of the first term we obtain an expression for the all-orders resummed exponent:

$$\int_{1/b^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \Gamma^{res}(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2)) = -\int_{1/b^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left[\ln \frac{Q^2}{\bar{\mu}^2} A(\alpha_s(\bar{\mu}^2)) + B(\alpha_s(\bar{\mu}^2)) \right. \tag{77}$$
$$\left. + \int_{1/b^2}^{\bar{\mu}^2} \frac{d\mu^2}{\mu^2} C(\alpha_s(\bar{\mu}^2), \alpha_s(\mu^2)) \right]$$

To obtain a LL resummation we need only a coefficient (A_1) and to obtain a NLL resummation we need four coefficients A_1, A_2, B_1, C_{11}. However it has been demonstrated by explicit calculations [67, 68] that

$$C_{11} = 0. \tag{78}$$

Therefore for the resummation at the NLL level we need only three coefficients A_1, A_2, B_1. Our general formula reduces to that of Ref.[6] when

$$C_{mn} = 0. \tag{79}$$

This restriction could be a consequence of the factorization of soft emissions from the hard part of the coefficient function, but this remains unproven.

The result for the anomalous dimension in Eq.(72) can be rewritten in our formalism performing the change of variable

$$n' = \frac{\bar{\mu}^2}{\mu^2}. \tag{80}$$

We get

$$\gamma(1, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2)) = -\int_1^{\bar{\mu}^2 b^2} \frac{dn'}{n'} G(\alpha_s(\bar{\mu}^2), \alpha_s(\bar{\mu}^2/n')) + \tilde{G}(\alpha_s(\bar{\mu}^2)), \qquad (81)$$

where

$$G(\alpha_s(\bar{\mu}^2), \alpha_s(\mu^2)) = \sum_{m=0}^{\infty} G_{mn} \alpha_s^m(\bar{\mu}^2) \alpha_s^n(\mu^2) \qquad (82)$$

$$\tilde{G}(\alpha_s(\bar{\mu}^2)) = \sum_{m=1}^{\infty} \tilde{G} \alpha_s^m(\bar{\mu}^2). \qquad (83)$$

The case of the resummation formula of Ref.[6] is obtained when G_{mn} is non-vanishing only when $m = 0$. In this case, we have

$$\gamma(1, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2)) = -\int_1^{\bar{\mu}^2 b^2} \frac{dn'}{n'} G(\alpha_s(\bar{\mu}^2/n')) + \tilde{G}(\alpha_s(\bar{\mu}^2)). \qquad (84)$$

6.5 Logs of $q_\perp^2$ vs. logs of b^2 to all logarithmic orders

Large logarithms of $q_\perp$ appear in the perturbative coefficients in the form of plus distributions. We define

$$\left[\frac{\ln^p(\hat{q}_\perp^2)}{\hat{q}_\perp^2} \right]_+ \qquad (85)$$

in such a way that

$$\int_0^1 d\hat{q}_\perp^2 \left[\frac{\ln^p(\hat{q}_\perp^2)}{\hat{q}_\perp^2} \right]_+ = 0. \qquad (86)$$

Let us consider the Fourier transforms

$$I_p(Q^2 b^2) = \frac{1}{Q^2} \int d^2 q_\perp e^{i\vec{q}_\perp \cdot \vec{b}} \left[\frac{\ln^p(\hat{q}_\perp^2)}{\hat{q}_\perp^2} \right]_+ = 2\pi \int_0^{\infty} \hat{q}_\perp d\hat{q}_\perp J_0(\hat{q}_\perp \hat{b}) \left[\frac{\ln^p(\hat{q}_\perp^2)}{\hat{q}_\perp^2} \right]_+, \qquad (87)$$

where we have used the definition of the 0-order Bessel function J_0:

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{iz\cos\theta}. \qquad (88)$$

We now exploit the definition of the plus distribution:

$$I_p(Q^2 b^2) = 2\pi \int_0^1 d\hat{q}_\perp [J_0(\hat{q}_\perp \hat{b}) - 1] \frac{\ln^p \hat{q}_\perp^2}{\hat{q}_\perp} + 2\pi \int_1^{\infty} d\hat{q}_\perp J_0(\hat{q}_\perp \hat{b}) \frac{\ln^p \hat{q}_\perp^2}{\hat{q}_\perp}. \qquad (89)$$

Writing $\ln^p \hat{q}_\perp^2$ as the pth α-derivative of $(\hat{q}_\perp^2)^\alpha$ at $\alpha = 0$, we get

$$
\begin{aligned}
I_p(Q^2 b^2) &= 2\pi \frac{\partial^p}{\partial \alpha^p} \left\{ \int_0^1 d\hat{q}_\perp [J_0(\hat{q}_\perp \hat{b}) - 1] \hat{q}_\perp^{2\alpha - 1} + \int_1^\infty d\hat{q}_\perp J_0(\hat{q}_\perp \hat{b}) \hat{q}_\perp^{2\alpha - 1} \right\} \\
&= 2\pi \frac{\partial^p}{\partial \alpha^p} \left[\int_0^\infty d\hat{q}_\perp J_0(\hat{q}_\perp \hat{b}) \hat{q}_\perp^{2\alpha - 1} - \int_0^1 d\hat{q}_\perp \hat{q}_\perp^{2\alpha - 1} \right] \\
&= \pi \frac{\partial^p}{\partial \alpha^p} \left[\left(\frac{Q^2 b^2}{4} \right)^{-\alpha} \frac{\Gamma(\alpha)}{\Gamma(1 - \alpha)} - \frac{1}{\alpha} \right],
\end{aligned}
\tag{90}
$$

where the last equality follows from the identity

$$
\int_0^\infty dx\, x^\mu J_\nu(ax) = 2^\mu a^{-\mu - 1} \frac{\Gamma(1/2 + \nu/2 + \mu/2)}{\Gamma(1/2 + \nu/2 - \mu/2)}
\tag{91}
$$

$$
a > 0; \qquad -\mathrm{Re}\,\nu - 1 < \mathrm{Re}\,\mu < 1/2.
\tag{92}
$$

From Eq.(90), we read off the generating function $G(\alpha)$ of I_p

$$
G(\alpha) = \frac{\pi}{\alpha} \left[\left(\frac{Q^2 b^2}{4} \right)^{-\alpha} \frac{\Gamma(1 + \alpha)}{\Gamma(1 - \alpha)} - 1 \right],
\tag{93}
$$

in the sense that

$$
I_p(Q^2 b^2) = \left[\frac{d^p}{d\alpha^p} G(\alpha) \right]_{\alpha = 0}.
\tag{94}
$$

Now, the generating function of logarithms of $Q^2 b^2$ is $(Q^2 b^2)^{-\alpha}$ in the sense that

$$
L_p \equiv \ln^p(1/(Q^2 b^2)) = \left[\frac{d^p}{d\alpha^p} (Q^2 b^2)^{-\alpha} \right]_{\alpha = 0}.
\tag{95}
$$

Inverting Eq.(93), we find the relation between the generating function of L_p and the generating function of I_p, which is

$$
(Q^2 b^2)^{-\alpha} = \frac{1}{\pi} S(\alpha)[\alpha G(\alpha) + \pi],
\tag{96}
$$

where

$$
S(\alpha) = \frac{1}{4^\alpha} \frac{\Gamma(1 - \alpha)}{\Gamma(1 + \alpha)}.
\tag{97}
$$

Performing the Taylor expansion of the r.h.s. of Eq.(96) around $\alpha = 0$ and using Eq.(94), we obtain:

$$
(Q^2 b^2)^{-\alpha} = \frac{1}{\pi} \sum_{m=0}^\infty \frac{\alpha^m}{m!} \sum_{i=0}^m \binom{m}{i} i I_{i-1} S^{(m-i)}(0),
\tag{98}
$$

where $S^{(j)}(0)$ is the j-th derivative of $S(\alpha)$ evaluated at $\alpha = 0$. Now, using Eq.(95), we the relation between L_p and I_p:

$$L_p = \frac{1}{\pi} \sum_{i=1}^{p} \binom{p}{i} i I_{i-1} S^{(p-1)}(0) = \frac{1}{\pi} \sum_{k=1}^{p} \binom{p-1}{k-1} p S^{(k-1)} I_{p-k}. \tag{99}$$

Thanks to the first equality in Eq.(87) and to the fact that

$$\ln^{p-k} \hat{q}_\perp^2 = \frac{1}{p(p-1)\cdots(p-k+1)} \frac{d^k}{d\ln^k \hat{q}_\perp^2} \ln^p \hat{q}_\perp^2, \tag{100}$$

we arrive at a relation to all logarithmic orders between the logs of b^2 and the logs of $q_\perp^2$:

$$L_p = \frac{1}{\pi} \sum_{k=1}^{p} \frac{S^{(k-1)}(0)}{(k-1)!} \int \frac{d^2 q_\perp}{Q^2} e^{i\vec{q}_\perp \cdot \vec{b}} \left[\frac{1}{\hat{q}_\perp^2} \frac{d^k}{d\ln^k \hat{q}_\perp^2} \ln^p \hat{q}_\perp^2 \right]_+. \tag{101}$$

This relation allows us to derive the relation between a generic function of $\ln(1/Q^2 b^2)$ and a function of $\ln \hat{q}_\perp^2$. Indeed, given a function

$$h\left(\ln \frac{1}{Q^2 b^2}\right) = \sum_{p=0}^{\infty} h_p \ln^p \frac{1}{Q^2 b^2}, \tag{102}$$

Eq.(101) implies:

$$h\left(\ln \frac{1}{Q^2 b^2}\right) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{S^{(k-1)}(0)}{(k-1)!} \int \frac{d^2 q_\perp}{Q^2} e^{i\vec{q}_\perp \cdot \vec{b}} \left[\frac{1}{\hat{q}_\perp^2} \frac{d^k}{d\ln^k \hat{q}_\perp^2} h(\ln \hat{q}_\perp^2) \right]_+. \tag{103}$$

The r.h.s. of Eq.(103) can be viewed as the Fourier transform of a function (more properly a distribution) $\hat{h}(\ln \hat{q}_\perp^2)$:

$$h\left(\ln \frac{1}{Q^2 b^2}\right) = \int \frac{d^2 q_\perp}{Q^2} e^{i\vec{q}_\perp \cdot \vec{b}} \hat{h}(\ln \hat{q}_\perp^2),$$

$$\hat{h}(\ln \hat{q}_\perp^2) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{S^{(k-1)}(0)}{(k-1)!} \left[\frac{1}{\hat{q}_\perp^2} \frac{d^k}{d\ln^k \hat{q}_\perp^2} h(\ln \hat{q}_\perp^2) \right]_+. \tag{104}$$

6.6   Resummation in $q_\perp$-space

In this section, we investigate the consequences of our general result Eq.(104) for the resummation at the NLL level of logarithmic accuracy. According to eq(19) and the discussion below and according to Eq.(77), we have that our resummation factor formula in Fourier space is:

$$K^{res}(Q^2 b^2, 1/b^2, Q^2) = \exp\left\{ E^{res}(Q^2 b^2, 1/b^2, Q^2) \right\}, \tag{105}$$

where

$$E^{res}(Q^2b^2, 1/b^2, Q^2) = \int_{1/b^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \Gamma^{res}(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2)), \tag{106}$$

and where at NLL level

$$\begin{aligned}
\Gamma_{NLL}^{res}(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2)) &= -\ln \frac{Q^2}{\bar{\mu}^2} \left[A_1 \alpha_s(\bar{\mu}^2) + A_2 \alpha_s^2(\bar{\mu}^2) \right] - B_1 \alpha_s(\bar{\mu}^2) \\
&\quad - \frac{C_{11}}{\beta_0} \alpha_s(\bar{\mu}^2) \ln \frac{\alpha_s(1/b^2)}{\alpha_s(\bar{\mu}^2)},
\end{aligned} \tag{107}$$

where we have used the definition of the β-function:

$$\mu^2 \frac{d}{d\mu^2} \alpha_s(\mu^2) = \beta \alpha_s = -\beta_0 \alpha_s^2 - \beta_1 \alpha_s^3 + O(\alpha_s^4) \tag{108}$$

and where we have used the change of variable

$$\frac{d\mu^2}{\mu^2} = \frac{d\alpha_s}{\beta(\alpha_s)} \tag{109}$$

to compute the integral that appears in the last term of Eq.(77). Now, thanks to Eq.(104), we can rewrite the resummed exponent in b-space Eq.(106) in terms of a resummed exponent defined in $q_\perp$-space. Thus, up to NNLL terms, we obtain:

$$E_{NLL}^{res}(Q^2b^2, 1/b^2, Q^2) = \int d^2 q_\perp e^{i\vec{q}_\perp \cdot \vec{b}} \left[\frac{\hat{\Gamma}_{NLL}^{res}(\hat{q}_\perp^2, q_\perp^2, Q^2)}{q_\perp^2} \right]_+, \tag{110}$$

where

$$\begin{aligned}
\hat{\Gamma}_{NLL}^{res}(\hat{q}_\perp^2, q_\perp^2, Q^2) &= = -\ln \hat{q}_\perp^2 \left[\hat{A}_1 \alpha_s(q_\perp^2) + \hat{A}_2 \alpha_s^2(q_\perp^2) \right] - \hat{B}_1 \alpha_s(q_\perp^2) + \\
&\quad - \frac{\hat{C}_{11}}{\beta_0} \alpha_s(q_\perp^2) \ln \frac{\alpha_s(Q^2)}{\alpha_s(q_\perp^2)}
\end{aligned} \tag{111}$$

and where the relation of the constant coefficients of this last equation and the of Eq.(107) is

$$\hat{A}_1 = \frac{A_1}{\pi} \tag{112}$$

$$\hat{A}_2 = -\left(\frac{A_2}{\pi} + \frac{A_1}{\pi} \beta_0 \ln \frac{e^{2\gamma_E}}{4} \right) \tag{113}$$

$$\hat{B}_1 = \frac{B_1}{\pi} - \frac{A_1}{\pi} \ln \frac{e^{2\gamma_E}}{4} \tag{114}$$

$$\hat{C}_{11} = \frac{C_{11}}{\pi}. \tag{115}$$

Here γ_E is the usual Euler gamma. Now, we want to define a resummation factor in $q_\perp$-space. Looking at Eq.(18), we note that large $\ln Q^2 b^2$ of collinear nature are

resummed by the parton distribution function. So, in order to define a resummation in $q_\perp$-space, we must take them into account. For simplicity, we consider the resummed part of non-singlet cross section, because the non-singlet parton distribution functions, which are defined as

$$f'_{a'}(N, \mu^2) = f_a(N, \mu^2) - f_b(N, \mu^2) \qquad a, b \neq g, \tag{116}$$

evolve independently. In particular, in Mellin moments N they satisfy the following evolution equations:

$$\mu^2 \frac{\partial}{\partial \mu^2} f'_{a'}(N, \mu^2) = \gamma'(N, \alpha_s(\mu^2)) f'_{a'}(N, \mu^2). \tag{117}$$

Hence, the non-singlet parton distribution functions evaluated at $\mu^2 = 1/b^2$ are related to the ones evaluated at $\mu^2 = Q^2$ by,

$$f'_{a'}(N, 1/b^2) = \exp\left\{ -\int_{1/b^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \gamma'(N, \alpha_s(\bar{\mu}^2)) \right\} f'_{a'}(N, Q^2). \tag{118}$$

Thus, the resummed part of the cross section with the non-singlet parton distribution functions evaluated at $\mu^2 = Q^2$ becomes

$$\exp\left\{ -\int_{1/b^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \sum_{j=1}^{2} \gamma'(N, \alpha_s(\bar{\mu}^2)) \right\} K^{res}(b^2, 1/b^2, Q^2) = \tag{119}$$

$$= \exp\left\{ \int_{1/b^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left[\Gamma^{res}(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2)) - \sum_{j=1}^{2} \gamma'(N_j, \alpha_s(\bar{\mu}^2)) \right] \right\}$$

The general relation between a function of $\ln Q^2 b^2$ and its Fourier anti-transform Eq.(104), immediately enables us to define a resummed exponent of the non-singlet part of the cross section in $q_\perp$-space, which is:

$$K^{res}(\hat{q}_\perp^2, q_\perp^2, Q^2) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{S^{(k-1)}(0)}{(k-1)!}$$

$$\times \left\{ \frac{1}{\hat{q}_\perp^2} \frac{d^k}{d \ln^k \hat{q}_\perp^2} \exp\left[\int_{q_\perp^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left(\Gamma^{res}(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2/q_\perp^2, \alpha_s(\bar{\mu}^2)) - \sum_{j=1}^{2} \gamma'(N_j, \alpha_s(\bar{\mu}^2)) \right) \right] \right\}_+$$

$$= \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{S^{(k-1)}(0)}{(k-1)!}$$

$$\times \left\{ \frac{d}{d\hat{q}_\perp^2} \frac{d^{k-1}}{d \ln^{k-1} \hat{q}_\perp^2} \exp\left[\int_{q_\perp^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left(\Gamma^{res}(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2/q_\perp^2, \alpha_s(\bar{\mu}^2)) - \sum_{j=1}^{2} \gamma'(N_j, \alpha_s(\bar{\mu}^2)) \right) \right] \right\}_+$$

$$= \lim_{\eta \to 0^+} \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{S^{(k)}(0)}{k!} \frac{d^k}{d \ln^k \hat{q}_\perp^2} \frac{d}{\hat{q}_\perp^2} \left\{ \theta(\hat{q}_\perp^2 - \eta) \exp\left[\int_{q_\perp^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \right. \right.$$

$$\times \left. \left. \left(\Gamma^{res}(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2/q_\perp^2, \alpha_s(\bar{\mu}^2)) - \sum_{j=1}^{2} \gamma'(N_j, \alpha_s(\bar{\mu}^2)) \right) \right] \right\}, \tag{120}$$

where the last equation defines implicity the $q_\perp$-space resummation exponent:

$$K^{res}(\hat{q}_\perp^2, q_\perp^2, Q^2) = \exp\left\{E^{res}(\hat{q}_\perp^2, q_\perp^2, Q^2)\right\}. \tag{121}$$

All the previously released expressions for this exponent given in [69, 70, 71] are particular cases of this general expression. They differ essentially in the criteria according to which the subleading terms are kept.

We want to calculate the NLL result in $q_\perp$-space. Thus, keeping only the terms up to NNLL in Eq.(120) we obtain

$$K^{res}_{NLL}(\hat{q}_\perp^2, q_\perp^2, Q^2) = \frac{1}{\pi}\frac{d}{\hat{q}_\perp^2}\left\{\theta(\hat{q}_\perp^2 - \eta)\exp\left[\int_{q_\perp^2}^{Q^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\left(\Gamma^{res}_{NLL}(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2/q_\perp^2, \alpha_s(\bar{\mu}^2))\right.\right.\right.$$

$$\left.\left.\left. - \sum_{j=1}^{2}\gamma'(N_j, \alpha_s(\bar{\mu}^2))\right)\right]\sum_{k=0}^{\infty}\frac{S^{(k)}(0)}{k!}[-\ln\hat{q}_\perp^2 A_1\alpha_s(q_\perp^2)]^k\right\}. \tag{122}$$

In order to compare this result at NLL to that of [70], we define a new variable h:

$$h \equiv 2\ln\hat{q}_\perp^2 A_1\alpha_s(q_\perp^2). \tag{123}$$

In terms of this variable and using Eq.(97) the series that appears in Eq.(122) can be computed:

$$\sum_{k=0}^{\infty}\frac{S^{(k)}(0)}{k!}[-\ln\hat{q}_\perp^2 A_1\alpha_s(q_\perp^2)]^k = S(-h/2) = 2^h\frac{\Gamma(1+h/2)}{\Gamma(1-h/2)}. \tag{124}$$

In conclusion, we obtain that the NLL resummation factor becomes

$$K^{res}_{NLL}(\hat{q}_\perp^2, q_\perp^2, Q^2) = \frac{1}{\pi}\frac{d}{\hat{q}_\perp^2}\left\{\theta(\hat{q}_\perp^2 - \eta)\exp\left[\int_{q_\perp^2}^{Q^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\left(\Gamma^{res}_{NLL}(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2/q_\perp^2, \alpha_s(\bar{\mu}^2))\right.\right.\right.$$

$$\left.\left.\left. - \sum_{j=1}^{2}\gamma'(N_j, \alpha_s(\bar{\mu}^2))\right)\right]2^h\frac{\Gamma(1+h/2)}{\Gamma(1-h/2)}\right\}, \tag{125}$$

which gives the same result given in Ref.[70] in the case that the coefficient C_{11} that appears in Eq.(107) is equal to zero and that the arbitrary constants c_1 and c_2 also defined in [70] are equal to one. It is clear that the last two terms of the exponential of our result let the non-singlet parton distribution densities, which enter the $\hat{q}_\perp^2$ derivative, evolve from the scale Q^2 to the scale $q_\perp^2$.

We conclude the chapter noting that also in this case the resummed results using the renormalization group approach are less predictive than results obtained with the approach of Ref.[1], as it is shown in Ref.[6]. Furthermore the conditions that reduce our results to those of Ref.[6] in terms of factorization properties is still an interesting open question.

Chapter 7

Predictive power of the resummation formulae

We have shown that the renormalization group resummed expressions are less predictive than those obtained with other approaches discussed in Sec.2.3. In this Chapter we shall compare the various approaches quantitatively. We will show that all the resummation coefficients can be determined by a fixed order computation. This can be useful, because the determination of the resummation coefficients from a fixed order computation represents a possible way to check the correctness of the resummation formulae with strong factorization properties.

In particular, in this chapter, we will show how the resummation coefficients g_{mnp} (for the prompt photon case), g_{mn} (for the DIS and DY cases) and $G_{mn}, \tilde{G}_m$ (for the DY transverse momentum distribution) can be determined. For the rapidity distributions of DY and DIS they are the same of the all-inclusive cases (see Chapter 5) The resummation coefficients are determined by comparing the expansion of the resummed anomalous dimension γ in powers of $\alpha_s(Q^2)$ with a fixed-order calculation, which in general has the form:

$$\gamma_{\mathrm{FO}}(N, \alpha_s) = \sum_{i=1}^{k_{\min}} \alpha_s^i \sum_{j=1}^{i} \gamma_j^i \ln^j \frac{1}{N} + O(\alpha_s^{k_{\min}+1}) + O(N^0), \tag{1}$$

where $\gamma_{\mathrm{FO}}(N, \alpha_s)$ is the physical anomalous dimension for each individual partonic subprocess for the promt photon case, for the qq channel in the DY case and for the q channel in the DIS case. For the case of the small transverse momentum DY distribution it ha the form:

$$\gamma_{\mathrm{FO}}(\bar{\mu}^2 b^2, \alpha_s) = \sum_{i=1}^{k_{\min}} \alpha_s^i \sum_{j=0}^{i} \tilde{\gamma}_j^i \ln^j \frac{1}{\bar{\mu} b^2} + O(\alpha_s^{k_{\min}+1}) + O(\frac{1}{\bar{\mu}^2 b^2}). \tag{2}$$

The number $k_{\min}$ is the minimum order at which the anomalous dimension must be calculated in order to determine its $N^{k-1}LL$ resummation.

For prompt photon production, the number of coefficients N_k that must be determined at each logarithmic order, and the minimum fixed order which is necessary in

	Prompt photon		
	Eq.(94) sec.4.4	Eq.(96) sec.4.4	Eq.(87) sec.4.4
N_k	$2k$	$\frac{k(k+3)}{2}$	$\frac{k(k+1)(k+5)}{6}$
$k_{\min}$	$k+1$	$2k$	$3k-1$

Table 7.1: Number of coefficients N_k and minimum order of the required perturbative calculation $k_{\min}$ for inclusive prompt photon $N^{k-1}LL$ resummation.

	DIS/DY	
	Eq.(81) sec.3.3	Eq.(80) sec.3.3
N_k	k	$\frac{k(k+1)}{2}$
$k_{\min}$	k	$2k-1$

Table 7.2: Number of coefficients N_k and minimum order of the required perturbative calculation $k_{\min}$ for inclusive DIS and DY $N^{k-1}LL$ resummation.

	DY transverse distribution	
	Eq.(84) sec.6.4	Eq.(81) sec.6.4
N_k	$2k-1$	$\frac{k^2+3k-2}{2}$
$k_{\min}$	k	$2k-1$

Table 7.3: Number of coefficients N_k and minimum order of the required perturbative calculation $k_{\min}$ for small transverse momentum DY $N^{k-1}LL$ resummation.

order to determine them are summarized in Table 7.1, according to whether the coefficient function is fully factorized [Eq.(94) sec.4.4], or has factorized N-dependent and N-independent terms [Eq.(96) sec.4.4], or not factorized at all [Eq. (87) sec4.4]. In the approach of Refs.[12, 14] the coefficient function is fully factorized, and furthermore some resummation coefficients are related to universal coefficients of Altarelli-Parisi splitting functions, so that $k_{\min} = k$. For prompt-photon production, available results do not allow to test factorization, and test relation of resummation coefficients to Altarelli-Parisi coefficients only to lowest $O(\alpha_s)$.

The results for DIS and Drell-Yan, according to whether the coefficient function has factorized N-dependent and N-independent terms as in Refs.[51, 2, 1] [Eq.(81) sec.3.3] or no factorization properties as in [(Eq.80) sec.3.3], are reported in table 7.2. Current fixed-order results support factorization for Drell-Yan and DIS only to the lowest nontrivial order $O(\alpha_s^2)$.

In table 7.3, we report also the results for the small transverse DY resummation. We list N_k and $k_{\min}$ for the approach of Ref.[6] [Eq.(84) sec.6.4] and for the renormalization group approach [Eq.(81) sec.6.4]. If the two cases are related by factorization properties of the cross section is not yet understood even if probable.

In the following, we present all the proofs of these results.

7.1 Prompt photon production in the strongest factorization case

This is the case of Eq.(94) in section 4.4. In this case there are $N_k = 2k$ non-vanishing coefficients g_{00i} and g_{0i0} $i = 1, 2, \ldots, k$. The resummed expression of the anomalous dimension at $N^{k-1}LL$ is given by:

$$\gamma(N, \alpha_s(Q^2)) = \int_1^N \frac{dn}{n} \left(\sum_{i=1}^k g_{0i0} \alpha_s^i(Q^2/n^2) \right) + \int_1^N \frac{dn}{n} \left(\sum_{i=1}^k g_{00i} \alpha_s^i(Q^2/n) \right). \quad (3)$$

We consider first the second integral in Eq.(3). Noting that:

$$\frac{dn}{n} = -\frac{d\alpha_s(Q^2/n)}{\beta(\alpha_s)}, \quad (4)$$

where

$$\beta(\alpha_s) = -b_0 \alpha_s^2 - b_1 \alpha_s^3 + O(\alpha_s^4) \quad (5)$$

$$b_0 \equiv \frac{\beta_0}{4\pi} \qquad b_1 \equiv \frac{\beta_1}{(4\pi)^2}, \quad (6)$$

with β_0 and β_1 given in Eq.(38) in section 1.2, we can rewrite it in the form:

$$\int_{\alpha_s(Q^2)}^{\alpha_s(Q^2/N)} d\alpha_s \frac{\sum_{i=1}^k g_{00i} \alpha_s^i}{\beta_0 \alpha_s^2 \left(1 + \frac{\beta_1}{\beta_0} \alpha_s + \frac{\beta_2}{\beta_0} \alpha_s^2 + \cdots \right)}. \quad (7)$$

Now, we expand up to order α_s^{k-2} each term that compares in the integrand of this last expression and collect all the coefficients that correspond to the same power of α_s. Doing this, we have that the integral (7) can be rewritten in the following form:

$$\frac{1}{b_0} \left\{ \int_{\alpha_s(Q^2)}^{\alpha_s(Q^2/N)} d\alpha_s \left[\frac{g_{001}}{\alpha_s} + (b_1^1 g_{001} + g_{002}) + (b_2^1 g_{001} + b_2^2 g_{002} + g_{003}) \alpha_s + \cdots + \right. \right.$$

$$\left. \left. + (b_{k-1}^1 g_{001} + \cdots + b_{k-1}^{k-1} g_{00k-1} + g_{00k}) \alpha_s^{k-2} \right] \right\}, \quad (8)$$

where $k > 1$ and the numbers b_i^j are build up with the coefficients of the β function. Now, we perform the integral over α_s. We get:

$$\int_1^N \frac{dn}{n} \left(\sum_{i=1}^k g_{00i} \alpha_s^i(Q^2/n) \right) =$$

$$= \frac{1}{b_0} \left\{ g_{001} \ln \left(\frac{\alpha_s(Q^2/N)}{\alpha_s(Q^2)} \right) + (b_1^1 g_{001} + g_{002})[\alpha_s(Q^2/N) - \alpha_s(Q^2)] + \right.$$

$$+ \frac{1}{2} (b_2^1 g_{001} + b_2^2 g_{002} + g_{003})[\alpha_s^2(Q^2/N) - \alpha_s^2(Q^2)] + \cdots +$$

$$\left. + \frac{1}{k-1} (b_{k-1}^1 g_{001} + \cdots + b_{k-1}^{k-1} g_{00k-1} + g_{00k})[\alpha_s^{k-1}(Q^2/N) - \alpha_s^{k-1}(Q^2)] \right\} \quad (9)$$

To perform the first integral of Eq.(3), we it is sufficient to note that in this case

$$\frac{dn}{n} = -\frac{d\alpha_s(Q^2/n^2)}{2\beta(\alpha_s)}.$$ (10)

and proceed as before. The result that we obtain is:

$$\int_1^N \frac{dn}{n}\left(\sum_{i=1}^k g_{0i0}\alpha_s^i(Q^2/n^2)\right) =$$

$$= \frac{1}{2b_0}\left\{g_{010}\ln\left(\frac{\alpha_s(Q^2/N^2)}{\alpha_s(Q^2)}\right) + (b_1^1 g_{010} + g_{020})[\alpha_s(Q^2/N^2) - \alpha_s(Q^2)] + \right.$$

$$+\frac{1}{2}(b_2^1 g_{010} + b_2^2 g_{020} + g_{030})[\alpha_s^2(Q^2/N^2) - \alpha_s^2(Q^2)] + \cdots +$$

$$\left. +\frac{1}{k-1}(b_{k-1}^1 g_{010} + \cdots + b_{k-1}^{k-1}g_{0k-10} + g_{0k0})[\alpha_s^{k-1}(Q^2/N^2) - \alpha_s^{k-1}(Q^2)]\right\}$$ (11)

At this point, we take the first term of Eq.(9) together with the first term of Eq.(11) in order to isolate the first contributions of Eq.(3). We have:

$$\frac{1}{2b_0}\left[g_{010}\ln\left(\frac{\alpha_s(Q^2/N^2)}{\alpha_s(Q^2)}\right) + 2g_{001}\ln\left(\frac{\alpha_s(Q^2/N)}{\alpha_s(Q^2)}\right)\right]$$ (12)

From this contribution, we want to extract the firs two LL terms. Hence, using the one loop running of $\alpha_s(Q^2/N^a), a = 1, 2$ (Eq.34 of section 1.2), we obtain for this contribution:

$$\alpha_s(Q^2)\ln\frac{1}{N}[-(g_{001} + g_{010})] \quad + \quad \alpha_s^2(Q^2)\ln^2\frac{1}{N}[b_0/2(g_{001} + 2g_{010})]$$
$$+O(\alpha_s^2\ln(N)) + O(\alpha_s^{3+i}\ln^j(N)),$$ (13)

where $i \geq 0, 1 \leq j \leq i + 3$. Now, we take the second term of Eq.(9) together with the second term of Eq.(11) in order to keep the second the second contributions of Eq.(3) and we have:

$$\frac{1}{2b_0}\left[(b_1^1 g_{010} + g_{020})(\alpha_s(Q^2/N^2) - \alpha_s(Q^2)) + 2(b_1^1 g_{001} + g_{002})(\alpha_s(Q^2/N) - \alpha_s(Q^2))\right].$$ (14)

From this contribution, we want to extract the first two NLL terms. In order to do this, we use the one loop running of $\alpha_s(Q^2/N^a), a = 1, 2$ and observe that the coefficients of g_{001} and g_{010} are modified by the last two terms of Eq.(13). We get:

$$\alpha_s^2(Q^2)\ln(1/N)[-(c_1^1 g_{010} + d_1^1 g_{001} + g_{020} + g_{002})] +$$
$$\alpha_s^3(Q^2)\ln^2(1/N)[b_0(2\tilde{c}_1^1 g_{010} + \tilde{d}_1^1 g_{001} + 2g_{020} + g_{002})]$$
$$+O(\alpha_s^3\ln(N)) + O(\alpha_s^{4+i}\ln^j(N)),$$ (15)

where $c_1^1, d_1^1, \tilde{c}_1^1, \tilde{d}_1^1$ are coefficients (that are of no concern to us) and $i \geq 0, 1 \leq j \leq i + 3$. This procedure can be repeated for all the other contributions. So, we take the

k-th term of Eq.(9) together with the k-th term of Eq.(11) in order to keep the k-th contributions of Eq.(3). For the general k-th contribution, we get:

$$\frac{1}{2b_0}\left[\frac{1}{k-1}(b_{k-1}^1 g_{010} + \cdots + b_{k-1}^{k-1} g_{0k-10} + g_{0k0})(\alpha_s^{k-1}(Q^2/N^2) - \alpha_s^{k-1}(Q^2)) + \right.$$
$$\left. +\frac{2}{k-1}(b_{k-1}^1 g_{001} + \cdots + b_{k-1}^{k-1} g_{00k-1} + g_{00k})(\alpha_s^{k-1}(Q^2/N) - \alpha_s^{k-1}(Q^2))\right]. \quad (16)$$

From this contribution, we want to extract the first two $N^{k-1}LL$ terms. In order to do this, again, we use the one loop running of $\alpha_s(Q^2/N^a)$, $a = 1, 2$ and note that the coefficients g_{00i} and g_{0i0} with $i = 1, 2, \ldots, k-1$ are modified by the previous terms. For this generic term, we get:

$$\alpha_s^k \ln\frac{1}{N}[-(c_{k-1}^1 g_{010} + \cdots + c_{k-1}^{k-1} g_{0k-10} + d_{k-1}^1 g_{001} + \cdots + d_{k-1}^{k-1} g_{00k-1} + g_{0k0} + g_{00k})] +$$
$$+\alpha_s^{k+1} \ln^2\frac{1}{N}[b_0 k/2(2\tilde{c}_{k-1}^1 g_{010} + \cdots + 2\tilde{c}_{k-1}^{k-1} g_{0k-10} + \tilde{d}_{k-1}^1 g_{001} + \cdots + \tilde{d}_{k-1}^{k-1} g_{00k-1}$$
$$+2g_{0k0} + g_{00k})] + O(\alpha_s^{k+1}\ln(N)) + O(\alpha_s^{k+2+i}\ln^j(N)), \quad (17)$$

where $i \geq 0, 1 \leq j \leq i + 3$.

To summarize, Eqs. (13,15,17) tell us that from the expression of the physical anomalous dimension Eq.(3), we can extract the following linear combinations of the coefficients $g_{010}, \ldots, g_{0k0}, g_{001}, \ldots, g_{00k}$:

$$l_1 = -(g_{001} + g_{010})$$
$$l_2 = \frac{b_0}{2}(g_{001} + 2g_{010})$$
$$l_3 = -(c_1^1 g_{010} + d_1^1 g_{001} + g_{020} + g_{002})$$
$$l_4 = b_0(2\tilde{c}_1^1 g_{010} + \tilde{d}_1^1 g_{001} + 2g_{020} + g_{002})$$
$$\cdots$$
$$l_{2k-1} = -(c_{k-1}^1 g_{010} + \cdots + c_{k-1}^{k-1} g_{0k-10} + d_{k-1}^1 g_{001} + \cdots + d_{k-1}^{k-1} g_{00k-1} + g_{0k0} + g_{00k})$$
$$l_{2k} = \frac{kb_0}{2}(2\tilde{c}_{k-1}^1 g_{010} + \cdots + 2\tilde{c}_{k-1}^{k-1} g_{0k-10} + \tilde{d}_{k-1}^1 g_{001} + \cdots + \tilde{d}_{k-1}^{k-1} g_{00k-1} + 2g_{0k0} + g_{00k}),$$

with $k \geq 2$ and l_i, $i = 1, \ldots, 2k$ the known terms. These are $2k$ independent linear combinations that determine the $2k$ coefficients $g_{010}, \ldots, g_{0k0}, g_{001}, \ldots, g_{00k}$ of a $N^{k-1}LL$ resummation comparing them with the correspondent terms of Eq.(2) up to order α_s^{k+1}. This is a direct consequence of the fact that the two vectors $(1,1)$ and $(1,2)$ are independent. This shows that, in order to obtain a $N^{k-1}LL$ resummation in the case of the strongest factorization (Eq.(94) in section 4.4) , we need to know a $N^{k+1}LO$ fixed order calculation of the physical anomalous dimension. Hence, in this case $k_{\min} = k + 1$.

7.2   Prompt photon production in the weaker factorization case

This is the case of Eq.(96) in section 4.4. In this case in order to perform a LL resummation, we need two coefficients (g_{010}, g_{001}); to perform a NLL resummation three more coefficients are added $(g_{020}, g_{002}, g_{011})$; in general to perform a $N^{k-1}LL$ resummation $k + 1$ coefficients are added $(g_{0ij}, i + j = k)$ to those of the $N^{k-2}LL$ resummation. Hence, in order to perform a $N^{k-1}LL$ resummation, we need to determine

$$N_k = \sum_{p=1}^{k} (p + 1) = \frac{k(k + 3)}{2}, \tag{18}$$

coefficients. We want to determine $k_{\min}$ in Eq.(2) so that all the $k(k + 3)/2$ are fixed by the same number of independent conditions obtained from the fixed order expansion of the resummed physical anomalous dimension. We note, first of all, that this happens if we can extract 2 independent conditions from the LL contributions, 3 from the NLL one and $k + 1$ from the $N^{k-1}LL$. The $N^{k-1}LL$ expression of the physical anomalous dimension in this case is given by:

$$\gamma(N, \alpha_s(Q^2)) \;=\; \int_1^N \frac{dn}{n} \left(\sum_{i=1}^{k} g_{0i0} \alpha_s^i(Q^2/n^2) \right) + \int_1^N \frac{dn}{n} \left(\sum_{i=1}^{k} g_{00i} \alpha_s^i(Q^2/n) \right) +$$

$$+ \int_1^N \frac{dn}{n} \sum_{s=2}^{k} \sum_{i=1}^{s-1} g_{0is-i} \alpha_s^i(Q^2/n^2) \alpha_s^{s-i}(Q^2/n). \tag{19}$$

The first two LL contributions have been already computed and are given in Eq.(13).

Now, we extract the first 3 NLL contributions of Eq.(19). Recalling the derivation of Eq.(14), we obtain that these contributions are contained in the following expression:

$$\frac{1}{2b_0} \left[(b_1^1 g_{010} + g_{020})(\alpha_s(Q^2/N^2) - \alpha_s(Q^2)) + 2(b_1^1 g_{001} + g_{002})(\alpha_s(Q^2/N) - \alpha_s(Q^2)) \right]$$

$$+ \int_1^N \frac{dn}{n} g_{011} \alpha_s(Q^2/n^2) \alpha_s(Q^2/n). \tag{20}$$

Since

$$\alpha_s(Q^2/p^a) = \frac{\alpha_s(Q^2)}{1 + a\beta_0 \alpha_s(Q^2) \ln \frac{1}{p}} + O(\alpha_s^{2+i} \ln^j \frac{1}{p}), \quad i \geq 0, \quad 1 \leq j \leq i$$

$$\frac{1}{1 + ab_0 \alpha_s(Q^2) \ln \frac{1}{p}} = \sum_{j=0}^{\infty} (-)^j a^j b_0^j \alpha_s^j(Q^2) \ln^j \frac{1}{p}, \tag{21}$$

and keeping in mind that corrections to the NLL come from Eq.(13), we have that

Eq.(20) become:

$$
\frac{1}{2b_0} \sum_{j=1}^{\infty} (-)^j [2^j (c_{1j}^1 g_{010} + g_{020}) + 2(d_{1j}^1 g_{001} + g_{002})] b_0^j \alpha_s^{j+1}(Q^2) \ln^j \frac{1}{N} + \tag{22}
$$

$$
+ \int_1^N \frac{dn}{n} g_{011} \alpha_s^2(Q^2) \left(\sum_{j=0}^{\infty} (-)^j b_0^j \alpha_s^j(Q^2) \ln^j \frac{1}{n} \right) \left(\sum_{i=0}^{\infty} (-)^i 2^i b_0^i \alpha_s^i(Q^2) \ln^i \frac{1}{n} \right).
$$

The Cauchy product of the two series in Eq.(22) is given by

$$
(\sum_{j=0}^{\infty} (-)^j b_0^j \alpha_s^j(Q^2) \ln^j \frac{1}{n})(\sum_{i=0}^{\infty} (-)^i 2^i b_0^i \alpha_s^i(Q^2) \ln^i \frac{1}{n}) =
$$

$$
= \frac{1}{2b_0} \sum_{j=1}^{\infty} (-)^{j-1} \left(\sum_{i=1}^{j} 2^i \right) b_0^j \alpha_s^{j-1}(Q^2) \ln^{j-1} \frac{1}{n} \tag{23}
$$

and

$$
\sum_{i=1}^{j} 2^i = 2(2^j - 1). \tag{24}
$$

Now, because

$$
\frac{dn}{n} = -d \ln \frac{1}{n}, \tag{25}
$$

we can perform the integration. We get

$$
\frac{1}{2b_0} \sum_{j=1}^{3} (-)^j \left[2^j (c_{1j}^1 g_{010} + g_{020}) + 2(d_{1j}^1 g_{001} + g_{002}) + \frac{2(2^j - 1)}{j} g_{011} \right] b_0^j \alpha_s^{j+1}(Q^2) \ln^j \frac{1}{n}, \tag{26}
$$

where c_1^{1j}, d_1^{1j} are certain coefficients we don not need to worry about. The first three NLL contributions are given by $j = 1, 2, 3$.

The last step is to extract the first $k + 1$ $N^{k-1}LL$ contributions that come from Eq.(19). Recalling how Eq.((16)) was computed, we have that the desired contributions are contained in the following expression:

$$
\frac{1}{2b_0} \left[\frac{1}{k-1} (b_{k-1}^1 g_{010} + \cdots + b_{k-1}^{k-1} g_{0k-10} + g_{0k0})(\alpha_s^{k-1}(Q^2/N^2) - \alpha_s^{k-1}(Q^2)) + \right.
$$

$$
\left. + \frac{2}{k-1} (b_{k-1}^1 g_{001} + \cdots + b_{k-1}^{k-1} g_{00k-1} + g_{00k})(\alpha_s^{k-1}(Q^2/N) - \alpha_s^{k-1}(Q^2)) \right] +
$$

$$
+ \int_1^N \frac{dn}{n} \sum_{i=1}^{k-1} g_{0ik-i} \alpha_s^i(Q^2/n^2) \alpha_s^{k-i}(Q^2/n) \tag{27}
$$

We use the following relations

$$
\alpha_s^r(Q^2/p^a) = \frac{\alpha_s^r(Q^2)}{(1 + ab_0 \alpha_s(Q^2) \ln \frac{1}{p})^r} + O(\alpha_s^{2+i} \ln^j \frac{1}{p}), \quad i \geq 0, 1 \leq j \leq i + 1,
$$

$$
\frac{1}{(1 + ab_0 \alpha_s(Q^2) \ln \frac{1}{p})^r} = \sum_{m=0}^{\infty} (-)^m \binom{r+m-1}{m} a^m b_0^m \alpha_s^m(Q^2) \ln^m \frac{1}{p}, \tag{28}
$$

where $\binom{r}{m}$ are the usual binomial coefficients. With this, we can compute the integral in Eq.(27) performing the Cauchy product of the two series expansion of $\alpha_s(Q^2/n^2)$ and of $\alpha_s(Q^2/n)$ and performing explicitly the integral using the change of variable Eq.(3). We get:

$$
\int_1^N \frac{dn}{n} \sum_{i=1}^{k-1} g_{0ik-i} \alpha_s^i(Q^2/n^2) \alpha_s^{k-i}(Q^2/n) = \sum_{i=1}^{k-1} g_{0ik-i} \sum_{m=0}^{\infty} C_m^{(i,k-i)} b_0^m \alpha_s^{k+m}(Q^2) \ln^{m+1} \frac{1}{N},
\tag{29}
$$

where

$$
C_m^{(i,j)} = \frac{(-1)^{m+1}}{m+1} \sum_{l=0}^{m} 2^l \binom{l+i-1}{i-1} \binom{m-l+j-1}{j-1},
\tag{30}
$$

and where

$$
\binom{n}{-1} = \frac{\Gamma(n+1)}{\Gamma(0)\Gamma(n+2)}
\tag{31}
$$

is equal to 1 for $n = -1$ and 0 otherwise. Thefore, keeping in mind the calculation of Eq.(16) and taking the first $k+1$ $N^{k-1}LL$ contributions of Eq.(27), we have

$$
\sum_{m=0}^{k} \left\{ \frac{(-)^{m+1}}{k-1} \binom{k+m-1}{m+1} [2^m (c_{k-1m}^1 g_{010} + \cdots + c_{k-1m}^{k-1} g_{0k-10} + g_{0k0}) + \right.
$$

$$
+ (d_{k-1m}^1 g_{001} + \cdots + d_{k-1j}^{k-1} g_{00k-1} + g_{00k}) + \sum_{t=2}^{k-1} \sum_{i=1}^{t-1} g_{0it-i} f_{itm}^{(k-1)}] +
$$

$$
\left. + \sum_{i=1}^{k-1} g_{0ik-i} C_m^{(i,k-i)} \right\} b_0^m \alpha_s^{k+m}(Q^2) \ln^{m+1} \frac{1}{N},
\tag{32}
$$

where $c_{k-1m}^i, d_{k-1m}^i, f_{itm}^{(k-1)}$ are certain coefficients we do not have to worry about. At this point, we can make some simplifications. In fact, since

$$
C_m^{(0,k)} = \frac{(-1)^{m+1}}{m+1} \binom{m+k-1}{k-1} = \frac{(-1)^{m+1}}{k-1} \binom{m+k-1}{m+1}
\tag{33}
$$

and (see Appendix E)

$$
C_m^{(k,0)} = 2^m C_m^{(0,k)},
\tag{34}
$$

we can write Eq.(32) in the following form:

$$
\sum_{m=0}^{k} \left\{ [C_m^{(k,0)} (c_{k-1m}^1 g_{010} + \cdots + c_{k-1m}^{k-1} g_{0k-10}) + \right.
$$

$$
+ C_m^{(0,k)} (d_{k-1m}^1 g_{001} + \cdots + d_{k-1j}^{k-1} g_{00k-1}) + C_m^{(0,k)} \sum_{t=2}^{k-1} \sum_{i=1}^{t-1} g_{0it-i} f_{itm}^{(k-1)}] +
$$

$$
\left. + \sum_{i=0}^{k} g_{0ik-i} C_m^{(i,k-i)} \right\} b_0^m \alpha_s^{k+m}(Q^2) \ln^{m+1} \frac{1}{N},
\tag{35}
$$

What this result tells us, is that passing from the $N^{k-2}LL$ to the $N^{k-1}LL$ resummation, $k + 1$ new resummation coefficients are added. In Eq.(35), we have $k + 1$ conditions for this coefficients (one for each m) to be set equal to the corresponding fixed order contribution of Eq.(2) . We shall now show that this conditions are independent. This is equivalent to showing that the $k + 1$ linear combinations

$$\sum_{i=0}^{k} g_{0ik-i} \tilde{C}_m^{(i,k-i)} \qquad m = 0, 1, \dots, k \tag{36}$$

with

$$\tilde{C}_m^{(i,j)} \equiv (-)^{m+1}(m+1)C_m^{(i,j)} = \sum_{l=0}^{m} 2^l \binom{l+i-1}{i-1} \binom{m-l+j-1}{j-1} \tag{37}$$

are independent. Moreover, this is equivalent to showing that for each k the columns of the $(k+1) \times (k+1)$ matrix $A_{mi}^k \equiv \tilde{C}_m^{(i,k-1)}$ are independent vectors. To show this, we need to use two identities proved in Appendix E:

$$\tilde{C}_m^{(i,0)} = 2^m \tilde{C}_m^{(0,i)} \tag{38}$$

$$\tilde{C}_m^{(i,j)} = 2\tilde{C}_m^{(i,j-1)} - \tilde{C}_m^{(i-1,j)}; \qquad i,j \geq 1, \tag{39}$$

$$\tilde{C}_m^{(0,i)} = \binom{m+i-1}{i-1} \tag{40}$$

which allows us to to compute the columns of the matrix $A_{mj}^{(k)}$ explicitly for all k. To show their independence, we use induction on k, i.e. we demonstrate the independence of the columns for $k = 1$ and then we assume that the property is valid for $k - 1$ to prove that it remains valid for k. In the case $k = 1$, we have a 2×2 matrix:

$$(\tilde{C}_m^{(0,1)} \quad \tilde{C}_m^{(1,0)}) = (1 \quad 2^m), \tag{41}$$

and now it is clear the two columns are independent, because 1 and 2^m are independent functions of m. Now, using the induction hypothesis

$$\sum_{i=0}^{k-1} \alpha_i \tilde{C}_m^{(i,k-1-i)} = 0 \Longleftrightarrow \alpha_i = 0, \quad i = 0, \dots, k-1, \tag{42}$$

we want to show that it is sufficient to prove that:

$$\sum_{i=0}^{k} \beta_i \tilde{C}_m^{(i,k-i)} = 0 \Longleftrightarrow \beta_i = 0, \quad i = 0, \dots, k. \tag{43}$$

Using the relations (38,39), we have:

$$
\begin{aligned}
\sum_{j=0}^{k} \beta_j \tilde{C}_m^{(j,k-j)} &= (\beta_0 + 2^m \beta_k)\tilde{C}_m^{(0,k)} + \sum_{j=1}^{k-1} \beta_j \tilde{C}_m^{(j,k-j)} \\
&= (\beta_0 + 2^m \beta_k)\tilde{C}_m^{(0,k)} + 2\sum_{j=1}^{k-1} \beta_j \tilde{C}_m^{(j,k-1-j)} - \sum_{j=1}^{k-1} \beta_j \tilde{C}_m^{(j-1,k-j)} \\
&= (\beta_0 + 2^m \beta_k)\tilde{C}_m^{(0,k)} + 2\sum_{j=1}^{k-1} \beta_j \tilde{C}_m^{(j,k-1-j)} - \sum_{j'=0}^{k-2} \beta_{j'+1} \tilde{C}_m^{(j',k-1-j')} \\
&= (\beta_0 + 2^m \beta_k)\tilde{C}_m^{(0,k)} - \beta_1 \tilde{C}_m^{(0,k-1)} + \sum_{j=1}^{k-2} (2\beta_j - \beta_{j+1})\tilde{C}_m^{(j,k-1-j)} \\
&\quad + 2\beta_{k-1}\tilde{C}_m^{(k-1,0)} = 0
\end{aligned}
\tag{44}
$$

Now, from Eq.(40), we know that $\tilde{C}_m^{(0,k)}$ is a degree-$(k-1)$ polynomial in m. Furthermore, from Eq.(39), we know that the vectors $\tilde{C}_m^{(j,k-1-j)}$ with $j = 0, \ldots, k-1$ are at most polynomials of degree $k-2$ in m. Consequently, Eq.(44) can be satisfied if and only if

$$
(\beta_0 + 2^m \beta_k)\tilde{C}_m^{(0,k)} = 0,
\tag{45}
$$

$$
-\beta_1 \tilde{C}_m^{(0,k-1)} + \sum_{j=1}^{k-2} (2\beta_j - \beta_{j+1})\tilde{C}_m^{(j,k-1-j)} + 2\beta_{k-1}\tilde{C}_m^{(k-1,0)} = 0.
\tag{46}
$$

From the first, it follows that:

$$
\beta_0 = \beta_k = 0,
\tag{47}
$$

while from the second, thanks to the induction hypothesis Eq.(42), we have that

$$
\beta_1 = \beta_{k-1} = 0,
\tag{48}
$$

and that

$$
\beta_1 = \frac{1}{2}\beta_2 = \frac{1}{4}\beta_3 = \cdots = \frac{1}{2^{k-2}}\beta_{k-1}.
\tag{49}
$$

In conclusion from Eqs.(47,48,49) it follows that

$$
\beta_j = 0, \qquad\qquad j = 0, \ldots, k.
\tag{50}
$$

This completes the proof that the columns of the squared $(k+1) \times (k+1)$ matrices $A_{mj}^{(k)} \equiv \tilde{C}_m^{(j,k-j)}$ are independent for all k. This shows that, in order to obtain a $N^{k-1}LL$ resummation in the case of the weaker factorization (Eq.(96) in section 4.4), we need to know a $N^{2k}LO$ fixed order calculation of the physical anomalous dimension. Hence, in this case $k_{\min} = 2k$.

7.3 Prompt photon in the general case

Let us now consider the most general case, in which the coefficient function does not satisfy any factorization property. This is the case of Eq.(87) in section 4.4. In this case, in order to perform a LL resummation we need 2 coefficients (g_{001}, g_{010}); to perform a NLL resummation 5 coefficients are added $(g_{002}, g_{101}, g_{020}, g_{110}, g_{011})$ and to perform a $N^{k-1}LL$ resummation $k(k + 3)/2$ coefficients are added $(g_{mnp}$ with $m + n + p = k$ but without $n = p = 0)$. Thus, in order to perform a $N^{k-1}LL$ resummation, we need to determine

$$N_k = \sum_{p=1}^{k} \frac{p(p + 3)}{2} = \frac{k(k + 1)(k + 5)}{6} \tag{51}$$

coefficients. The $N^{k-1}LL$ expression of the physical anomalous dimension is given, in this case, by

$$\gamma(N, \alpha_s(Q^2)) = \int_1^N \frac{dn}{n} \sum_{s=1}^{k} \sum_{i=0}^{s-1} \sum_{j=0}^{s-i} g_{ijs-i-j} \alpha_s^i(Q^2) \alpha_s^j(Q^2/n^2) \alpha_s^{s-i-j}(Q^2/n). \tag{52}$$

Now, we proceed in the same way as we have done in section 7.2 and we find that the $k(k + 3)/2$ new coefficients that are added passing from the $N^{k-2}LL$ to the $N^{k-1}LL$ contributions appear only in the following combinations:

$$\sum_{i=0}^{k-1} \sum_{j=0}^{k-i} g_{ijk-i-j} \sum_{m=0}^{\infty} C_m^{(j,k-i-j)} b_0^m \alpha_s(Q^2)^{k+m} \ln^{m+1} \frac{1}{N}. \tag{53}$$

Each term with fixed m in the expansion Eq. (66) provides a new condition on these coefficients. However, these conditions are not linearly independent for all choices of m. Indeed, let us define the matrix $C_m^{(j,k-i-j)} \equiv D_{m(i,j)}^{(k)}$, where the lines are labelled by the index m and the columns by the multi-index (i, j). This matrix gives the linear combination of the coefficients g_{mnp} in Eq.(66) to be determined and it turns out to be of rank

$$rg(D_{m(i,j)}^{(k)}) = 2k \leq \frac{k(k + 3)}{2}. \tag{54}$$

We shall now prove this statement:

$D_{m(i,j)}^{(k)}$ is a $M \times \frac{k(k+3)}{2}$ matrix , whose columns are the M-component vectors

$$D_m^{(k)} = C_m^{(j,k-i-j)}; \qquad 0 \leq i \leq k - 1; \quad 0 \leq j \leq k - i; \qquad 0 \leq m \leq M. \tag{55}$$

We use induction on k. For $k = 1$, $D^{(1)}$ is a 2×2 matrix with columns

$$D_m^{(1)} = \left(C_m^{(0,1)}, \quad C_m^{(1,0)} \right) = \frac{(-1)^{m+1}}{m + 1} \left(1, \quad 2^m \right), \tag{56}$$

that are linearly independent; the rank of $D^{(1)}$ is 2. Let us check explicitly also the case $k = 2$. In this case

$$D_m^{(2)} = \left(C_m^{(0,1)}, \quad C_m^{(1,0)}, \quad C_m^{(0,2)}, \quad C_m^{(1,1)}, \quad C_m^{(2,0)} \right). \tag{57}$$

The first two columns are the same as in the case $k = 1$: they span a 2-dimensional subspace. The last three columns are independent as a consequence of Eq.(43) with $k = 1$. Furthermore, $C_m^{(0,2)}$ and $C_m^{(2,0)} = 2^m C_m^{(0,2)}$ are independent of all other columns, because they are the only ones that are proportional to a degree-1 polynomial in m. Finally, $C_m^{(1,1)}$ is a linear combination of the first two columns, as a consequence of Eqs.(37,39) with $i = j = 1$. Thus, the rank of $D^{(2)}$ is $2 + 2 = 4$.

We now assume that $D^{(k-1)}$ has rank $2(k-1)$, and we write the columns of $D^{(k)}$ as

$$D_m^{(k)} = \left(C_m^{(j,k-1-i-j)}, \quad C_m^{(l,k-l)} \right) \tag{58}$$

$$0 \le i \le k - 2, \qquad 0 \le j \le k - 1 - i \qquad\qquad 0 \le l \le k. \tag{59}$$

By the induction hypothesis, only $2(k-1)$ of the columns $C_m^{(j,k-1-i-j)}$ are independent. The columns $C_m^{(l,k-l)}$ are all independent as a consequence of Eq.(43); among them, those with $1 \le l \le k - 1$ can be expressed as linear combinations of $C_m^{(j,k-1-i-j)}$ by Eq. (39). Only $C_m^{(0,k)}$ and $C_m^{(k,0)}$ are independent of all other columns because they are proportional to a degree-$(k-1)$ polynomial in m, while all others are at most of degree $(k - 2)$. Hence, only two independent vectors are added to the $2(k - 1)$-dimensional subspace spanned by $C_m^{(j,k-1-i-j)}$, and the rank of $D^{(k)}$ is

$$2(k - 1) + 2 = 2k. \tag{60}$$

It follows that each individual terms in the sum over m in Eq. (66) depends only on $2k$ independent linear combinations of the coefficients $g_{ijk-i-j}$, $0, \le i \le k - 1$, $0 \le j \le k - i$.

This means that the $N^{k-1}LL$ order resummed result depends only on $2k$ independent linear combinations of the $k(k+3)/2$ new coefficients that are added passing from the $N^{k-2}LL$ resummation to the $N^{k-1}LL$ one and that the remaining coefficients are arbitrary. Because a term with fixed m in Eq.(66) is of order α_s^{k+m}, this implies that a computation of the anomalous dimension up to fixed order $k_{\min} = 3k - 1$ is sufficient for the $N^{k-1}LL$ resummation, because $m = 0, 1, \ldots, k - 1$. Note that when going from $N^{k-1}LL$ to $N^k LL$, at this higher order, in general some new linear combinations of the $k(k + 3)/2$ coefficients, that we have added from the $N^{k-2}LL$ to the $N^{k-1}LL$, will appear through terms depending on β_1. Hence, some of the combinations of coefficients that were left undetermined in the $N^{k-1}LL$ resummation will now become determined. However, this does not affect the value $k_{\min}$ of the fixed-order accuracy needed to push the resummed accuracy at one extra order. In conclusion, even in the absence of any factorization, despite the fact that now the number of coefficients which must be determined grows cubically according to Eq. (51), the required order in α_s of the computation which determines them grows only linearly.

7.4   DIS and DY in all cases

This is the case of Eq.(80) in section 3.3. Here we discuss the case without assuming any factorization property, because the factorized case will be recovered as a particular

case. So, in the general case, at LL we need to determine 1 coefficient (g_{01}); at NLL 2 coefficients are added (g_{02}, g_{11}) and at $N^{k-1}LL$, k coefficients (g_{ik-i} with $i = 0, 1, \ldots, k-1$) are added to the $N^{k-2}LL$ ones . Thus, in order to perform a $N^{k-1}LL$ resummation, we need to determine

$$N_k = \sum_{p=1}^{k} p = \frac{k(k+1)}{2} \tag{61}$$

coefficients. The $N^{k-1}LL$ expression of the physical anomalous dimension is given by

$$\gamma(N, \alpha_s(Q^2)) = a \int_1^N \frac{dn}{n} \sum_{s=1}^{k} \sum_{i=0}^{s-1} g_{is-i} \alpha_s^i(Q^2) \alpha_s^{s-i}(Q^2/n^a), \tag{62}$$

where $a = 1$ for DIS and $a = 2$ for DY. Now, we proceed in the same way as we have done in the previous sections and we find that the k new coefficients that are added passing from the $N^{k-2}LL$ to the $N^{k-1}LL$ contributions appear only in the following combinations:

$$\sum_{i=0}^{k-1} g_{ik-i} \sum_{m=0}^{\infty} a^{m+1} C_m^{(0,k-i)} b_0^m \alpha_s(Q^2)^{k+m} \ln^{m+1} \frac{1}{N}. \tag{63}$$

Again, each term with fixed m in this expansion provides a new condition on these coefficients. These conditions are all linearly independent for any choice of m, because the $k \times k$ matrix $A_{mi}^{(k)} \equiv C_m^{0,k-i}$ with $m, i = 0, 1, \ldots, k-1$ has all independent columns. This is a direct consequence of the fact that each column is a polynomial with different degree in m. This implies that, in order to determine a $N^{k-1}LL$ resummation, a computation of the physical anomalous dimension up to order $k_{\min} = 2k - 1$ is sufficient. In the more restrictive case Eq.(81) in Section 3.3, the only non-vanishing coefficients are $g_{0,k}$. This means that going from the $N^{k-2}LL$ to the $N^{k-1}LL$ we need to take only one combination of the expansion Eq.(63). Hence, in this more restrictive case, $N_k = k$ and $k_{\min} = k$.

7.5 DY small transverse momentum distribution

This is the case of Eq.(81) in section 6.4 obtained with the renormalization group approach. The result of the approach of Ref.[6] (reported in Eq.(81) of section 6.4) will be recovered as a particular case. In the most general case, at LL we need to determine 1 coefficient (G_{01}); at NLL 3 more coefficients are added ($G_{11}, G_{02}, \tilde{G}_1$) and at $N^{k-1}LL$, $k + 1$ coefficients (G_{ik-i} with $i = 0, 1, \ldots, k-1$ and $\tilde{G}_{k-1}$) are added. Therefore, in order to perform a $N^{k-1}LL$ resummation, we need to determine

$$N_k = \sum_{p=1}^{k}(p+1) - 1 = \frac{k^2 + 3k - 2}{2} \tag{64}$$

coefficients. The $N^{k-1}LL$ expression of the physical anomalous dimension is given by

$$\gamma(1, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2)) = -\int_0^{\bar{\mu}^2 b^2} \frac{dn'}{n'} \sum_{s=1}^{k} \sum_{i=0}^{s-1} G_{i\,s-i}\alpha_s^i(\bar{\mu}^2)\alpha_s^{s-i}(\bar{\mu}^2/n') + \sum_{i=1}^{k-1} \tilde{G}_i\alpha_s^i(\bar{\mu}^2). \quad (65)$$

Now, we proceed as before and we find that the k new coefficients that are added passing from the $N^{k-2}LL$ to the $N^{k-1}LL$ contributions appear only in the following combinations:

$$-\sum_{i=0}^{k-1} G_{i\,k-i} \sum_{m=0}^{\infty} C_m^{(0,k-i)} b_0^m \alpha_s(\bar{\mu}^2)^{k+m} \ln^{m+1}\left(\frac{1}{\bar{\mu}^2 b^2}\right) + \tilde{G}_{k-1}\alpha_s^{k-1}(\bar{\mu}^2). \quad (66)$$

As before, each term with fixed m in this expansion provides a new independent condition on this coefficients. This implies that, in order to determine a $N^{k-1}LL$ resummation, a computation of the physical anomalous dimension up to order $k_{\min} = 2k - 1$ is sufficient. In the more restrictive case of Eq.(84 sec.6.4), the only non-vanishing coefficients are $G_{0,k}$ and $\tilde{G}_{k-1}$. This means that going from the $N^{k-2}LL$ to the $N^{k-1}LL$ only 2 coefficients are added but the LL where we have only one coefficient. Hence, in this more restrictive case, $N_k = 2k - 1$ and $k_{\min} = k$.

Chapter 8

Conclusions

In this thesis, we have studied the renormalization group approach to resummation for all inclusive deep-inelastic and Drell-Yan processes. The advantage of this approach is that it does not rely on factorization of the physical cross section, and in fact it simply follows from general kinematic properties of the phase space. Then we have analyzed some of its generalizations.

In particular, we have presented a generalization to prompt photon production of the approach to Sudakov resummation which has been described in Chapter 3 for deep-inelastic scattering and Drell-Yan production. It is interesting to see that also with the more intricate two-scale kinematics that characterizes prompt photon production in the soft limit, it remains true that resummation simply follows from general kinematic properties of the phase space. Also, this approach does not require a separate treatment of individual colour structures when more than one colour structure contributes to fixed order results.

The resummation formulae obtained here turn out to be less predictive than previous results: a higher fixed-order computation is required in order to determine the resummed result. This depends on the fact that here neither specific factorization properties of the cross section in the soft limit is assumed, nor that soft emission satisfies eikonal-like relations which allow one to determine some of the resummation coefficients in terms of universal properties of collinear radiation. Currently, fixed-order results are only available up to $O(\alpha_s^2)$ for prompt photon production. An order α_s^3 computation is required to check nontrivial properties of the structure of resummation: for example, factorization, whose effects only appear at the next-to-leading log level, can only be tested at $O(\alpha_s^3)$. The greater flexibility of the approach presented here would turn out to be necessary if the prediction obtained using the more restrictive resummation were to fail at order α_s^3.

We have also proved a resumation formula for the Drell-Yan rapidity distributions to all logarithmic accuracy and valid for all values of rapidity. Isolating a universal dimensionless coefficient function, which is exactly that ones of the Drell-Yan rapidity-integrated, we have shown a general procedure to obtain resummed results to NLL for the rapidity distributions of a virtual photon γ^* or of a real vector boson $W^\pm, Z^0$. Furthermore, we have outlined a general method to calculate numerical predictions and analyzed the impact of resummation for the fixed-target experiment E866/NuSea.

123

This shows that NLL resummation has an important effects on predictions of differential rapidity cross sections giving an agreement with data that is better than NNLO full calculations. We have found a suppression of the cross section for not large values of hadronic rapidity instead of enhancing it. This suppression arises due to the shift in the complex plane of the dominant contribution of resummed exponent. These leaves open questions for future studies about possible suppression of the rapidity integrated cross sections at small x.

The study of the renormalization group resummation applied to the case of small transverse momentum distribution of Drell-Yan pairs has opened further interestingly aspects about the relation between factorization properties of the cross section and the final structure of the resummed results which has been not yet well understood.

Furthermore, because of its generality, renormalization group resummation lends itself naturally to some important future applications. They are the factorization of resummation of rare meson decay processes (like $B \to X_s \gamma$) and the resummation of generalized parton densities in deeply virtual proton Compton scattering.

Appendix A

Relations between logarithms of N and logarithms of $(1 - z)$

In this Appendix we want to find the general relations between the logarithms of N and the logarithms of $(1 - z)$. Let's consider a generic logarithmical enhanced term in z space

$$\left[\frac{\ln^p(1 - z)}{1 - z} \right]_+ \tag{1}$$

and take its Mellin transform:

$$I_p \equiv \int_0^1 dz \frac{z^{N-1} - 1}{1 - z} \ln^p(1 - z). \tag{2}$$

To find a general relation between the terms of the type of Eq.(1) and the logs of N, we first notice that all integrals I_p Eq.(2) can be obtained from a generating function $G(\eta)$:

$$I_p \; = \; \frac{d^p}{d\eta^p} G(\eta)|_{\eta=0}, \tag{3}$$

$$G(\eta) \; = \; \int_0^1 dz (z^{N-1} - 1) e^{(\eta-1)\ln(1-z)} = \frac{\Gamma(N)\Gamma(\eta)}{\Gamma(N + \eta)} - \frac{1}{\eta}. \tag{4}$$

Now, using the Stirging expansion of the Γ function at large N

$$\Gamma(N + 1) = \sqrt{2\pi N} e^{N \ln N - N} + O\left(\frac{1}{N}\right), \tag{5}$$

we get

$$G(\eta) = \frac{1}{\eta} \left[\frac{\Gamma(1 + \eta)}{N^\eta} - 1 \right] + O\left(\frac{1}{N}\right). \tag{6}$$

At this point, we notice that $N^{-\eta}$ is just the generating function of the log of N:

$$L_p \equiv \ln^p \frac{1}{N} = \frac{d^p}{d\eta^p} e^{\eta \ln 1/N}|_{\eta=0}. \tag{7}$$

Hence, Eq.(6) can be viewed as a relation between the generating function for I_p and for L_p. In particular, Taylor-expanding $\Gamma(1+\eta)$ in Eq.(6) leads to leading, next-to-leading,... $\ln N$ relations:

$$
\begin{aligned}
G(\eta) &= \frac{1}{\eta}\left[\frac{1}{N^\eta}\sum_{k=0}^\infty \frac{\Gamma^{(k)}(1)}{k!}\eta^k - 1\right] \\
&= \sum_{k=0}^\infty \frac{\Gamma^{(k)}(1)}{k!}\frac{1}{\eta}\frac{d^k}{d\ln^k 1/N}[e^{\eta \ln 1/N} - 1] \\
&= -\sum_{k=0}^\infty \frac{\Gamma^{(k)}(1)}{k!}\frac{d^k}{d\ln^k 1/N}\int_0^{1-\frac{1}{N}} dz\, e^{(\eta-1)\ln(1-z)}.
\end{aligned}
\tag{8}
$$

Now, if we put this last equation in Eq.(3), we obtain:

$$
\begin{aligned}
I_p &= -\sum_{k=0}^\infty \frac{\Gamma^{(k)}(1)}{k!}\frac{d^k}{d\ln^k 1/N}\int_0^{1-\frac{1}{N}} dz\frac{\ln^p(1-z)}{1-z} + O\left(\frac{1}{N}\right) \tag{9} \\
&= \frac{1}{p+1}\sum_{k=0}^{p+1}\binom{p+1}{k}\Gamma^{(k)}(1)\left(\ln\frac{1}{N}\right)^{p+1-k} + O\left(\frac{1}{N}\right), \tag{10}
\end{aligned}
$$

where in the last equality we have used the identity

$$
\frac{d^k}{d\ln^k 1/N}\int_0^{1-\frac{1}{N}} dz\frac{\ln^p(1-z)}{1-z} = -\frac{1}{p+1}k!\binom{p+1}{k}\left(\ln\frac{1}{N}\right)^{p+1-k}. \tag{11}
$$

This last expression is equal to zero when $k > p+1$. This result can be expressed in terms of derivatives with respect to $\ln(1-z)$ with the identity

$$
\frac{d^k}{d\ln^k 1/N}\int_0^{1-\frac{1}{N}} dz\frac{\ln^p(1-z)}{1-z} = \int_0^{1-\frac{1}{N}}\frac{dz}{1-z}\frac{d^k\ln^p(1-z)}{d\ln^k(1-z)} \tag{12}
$$

thus obtaining for I_p Eq.(9) the following all-logarithmic-order relation:

$$
\begin{aligned}
I_p &= -\sum_{k=0}^p \frac{\Gamma^{(k)}(1)}{k!}\int_0^{1-\frac{1}{N}}\frac{dz}{1-z}\frac{d^k\ln^p(1-z)}{d\ln^k(1-z)} \\
&\quad +\frac{\Gamma^{(p+1)}(1)}{p+1} + O\left(\frac{1}{N}\right).
\end{aligned}
\tag{13}
$$

The inverse result, expressing L_p in terms of I_p, can be analogously found inverting the relation between the generating functions Eq.(6):

$$
N^{-\eta} = \frac{\eta G(\eta) + 1}{\Gamma(1+\eta)}. \tag{14}
$$

Proceeding as before, we get

$$
\ln^n \frac{1}{N} = \sum_{i=1}^{n} \binom{n}{i} \Delta^{(n-i)}(1) i I_{i-1} + \Delta^{(n)}(1) + O\left(\frac{1}{N}\right) \tag{15}
$$

$$
= \sum_{k=1}^{n} \frac{\Delta^{(k-1)}(1)}{(k-1)!} \int_0^1 dz \frac{z^{N-1}-1}{1-z} \frac{d^k \ln^n(1-z)}{d\ln^k(1-z)}
$$

$$
+ \Delta^{(n)}(1) + O\left(\frac{1}{N}\right), \tag{16}
$$

where $\Delta^{(k)}(\eta)$ is the kth derivative of

$$
\Delta(\eta) \equiv \frac{1}{\Gamma(\eta)}. \tag{17}
$$

Because of the p-independence of the coefficients of the expansion Eq.(13), we can determine explicitly the Mellin transform of a generic logarithmical enhanced function

$$
\left[\frac{\hat{g}(\ln(1-z)^a)}{1-z}\right]_+ = \sum_{p=0}^{\infty} \hat{g}_p \left[\frac{\ln^p(1-z)^a}{1-z}\right]_+. \tag{18}
$$

Indeed, its Mellin transform up to non-logarithmical terms is given by

$$
\int_0^1 dz \frac{z^{N-1}-1}{1-z} \hat{g}(\ln(1-z)^a) = \sum_{p=0}^{\infty} \hat{g}_p a^p I_p
$$

$$
= \sum_{p=0}^{\infty} \frac{\hat{g}_p a^p}{p+1} \sum_{k=0}^{p+1} \binom{p+1}{k} \Gamma^{(k)}(1) \left(\ln\frac{1}{N}\right)^{p+1-k} \tag{19}
$$

$$
= -\sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(1)}{k!} \int_0^{1-\frac{1}{N}} \frac{dz}{1-z} \frac{d^k}{d\ln^k(1-z)^a} \hat{g}(\ln(1-z)^a)
$$

$$
= \int_0^{1-\frac{1}{N}} \frac{dz}{1-z} g(\ln(1-z)^a) = \frac{1}{a} \int_1^{N^a} \frac{dn}{n} g(\ln\frac{1}{n}), \tag{20}
$$

where in the last equality we have done the change of variable $n = (1-z)^{-a}$ and where

$$
g(\ln\mathcal{K}) \equiv -\sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(1)a^k}{k!} \frac{d^k}{d\ln^k\mathcal{K}} \hat{g}(\ln\mathcal{K}). \tag{21}
$$

The inverse relation can be analogously derive. Namely, we can cast the integral of any function

$$
g(\ln(1-z)^a) = \sum_{p=0}^{\infty} g_p \ln^p(1-z)^a \tag{22}
$$

as a Mellin transform, up to non-logarithmic terms:

$$
\frac{1}{a} \int_1^{N^a} \frac{dn}{n} g(\ln\frac{1}{n}) = \int_0^1 dz \frac{z^{N-1}-1}{1-z} \hat{g}(\ln(1-z)^a), \tag{23}
$$

where

$$\hat{g}(\ln \mathcal{K}) \equiv - \sum_{k=0}^{\infty} \frac{\Delta^{(k)}(1) a^k}{k!} \frac{d^k}{d \ln^k \mathcal{K}} g(\ln \mathcal{K}). \tag{24}$$

For completeness, we recall that $\Gamma'(1) = -\gamma_E = -0.5772$ and that in some cases it can be useful to perform the change of variable $n = n'^a$ so that we have

$$\frac{1}{a} \int_1^{N^a} \frac{dn}{n} g(\ln \frac{1}{n}) = \int_1^N \frac{dn'}{n'} g(\ln \frac{1}{n'^a}) \tag{25}$$

Appendix B

Phase space

The n-body phase space can be expressed in terms of $(n-m)$-body and $(m+1)$-body phase spaces. To prove this statement, let us consider the definition of the phase space in $d = 4 - 2\epsilon$ dimensions for a generic process with incoming momentum P and n on-shell particles in the final state with outgoing momenta $p_1, p_2, \ldots, p_n$:

$$d\phi_n(P; p_1, \ldots, p_n) = \frac{d^{d-1}p_1}{(2\pi)^{d-1}2p_1^0} \cdots \frac{d^{d-1}p_n}{(2\pi)^{d-1}2p_n^0} (2\pi)^d \delta_d(P - p_1 - \cdots - p_n). \quad (1)$$

The momentum-conservation delta function can be rewritten as

$$\delta_d(P - p_1 - \cdots - p_n) = \int d^d Q \, \delta_d(P - Q - p_1 - \cdots - p_m)$$
$$\times \delta_d(Q - p_{m+1} - \cdots - p_n). \quad (2)$$

The integration measure of this equation can be rewritten in this way:

$$d^d Q = (2\pi)^d \frac{d^{d-1}Q}{(2\pi)^{d-1}2Q^0} \frac{d(Q^0)^2}{2\pi} = (2\pi)^d \frac{d^{d-1}Q}{(2\pi)^{d-1}2Q^0} \frac{dQ^2}{2\pi}, \quad (3)$$

where

$$Q^2 = (Q^0)^2 - |\vec{Q}|^2 \quad (4)$$
$$= (p_{m+1} + \cdots + p_n)^2 \quad (5)$$
$$= (P - p_1 - \cdots - p_m)^2. \quad (6)$$

We shall now find the minimum and the maximum value of the variable Q^2. If we use Eq.(5) in the center-of-mass frame of the momentum $p_{m+1} + \cdots + p_n$, we have

$$Q^2 = (p_{m+1}^0 + \cdots + p_n^0)^2. \quad (7)$$

The minimum value of Q^2 is achieved when all the energies $p_{m+1}^0, \ldots, p_n^0$ are equal to their invariant masses ($m_i = \sqrt{p_i^2}$). Hence

$$Q_{min}^2 = (\sqrt{p_{m+1}^2} + \cdots + \sqrt{p_n^2})^2, \quad (8)$$

because Q^2 is a Lorentz ivariant. From Eq.(6), we have that in the center-of-mass frame of the momentum $P - p_1 - \cdots - p_m$,

$$Q^2 = (P^0 - p_1^0 - \cdots - p_m^0)^2. \tag{9}$$

Now, the minimum value of $p_1^0 + \cdots + p_m^0$ is achieved when all these terms reduces to their invariant masses ($m_i = \sqrt{p_i^2}$). In this case

$$\vec{p}_1 = \cdots = \vec{p}_m = 0 \tag{10}$$

and this implies that $\vec{P} = 0$ and that $P^0 = \sqrt{P^2}$. Therefore, we obtain

$$Q^2_{max} = (\sqrt{P^2} - \sqrt{p_1^2} - \cdots - \sqrt{p_m^2})^2, \tag{11}$$

again as a consequence of the Lorentz invariance of Q^2. We obtain immediately the general phase space decomposition formula using Eqs.(1,2,3) together:

$$
\begin{aligned}
d\phi_n(P; p_1, \ldots, p_n) &= \int_{Q^2_{min}}^{Q^2_{max}} \frac{dQ^2}{2\pi} d\phi_{m+1}(P; Q, p_1, \ldots, p_m) \\
&\times d\phi_{n-m}(Q; p_{m+1}, \ldots, p_n),
\end{aligned}
\tag{12}
$$

where Q^2_{min} and Q^2_{max} are given by Eq.(8) and Eq.(11) respectively. Using recursively Eq.(12) with $m = 1$, it is possible to rewrite a n-body phase space in terms of n two-body phase spaces. We shall now compute the two-body phase space in the center-of-mass frame. We have

$$
\begin{aligned}
d\phi_2(P; Q, p) &= \frac{d^{d-1}Q}{(2\pi)^{d-1} 2Q^0} \frac{d^{d-1}p}{(2\pi)^{d-1} 2p^0} (2\pi)^d \delta_d(P - Q - p) \\
&= \frac{(2\pi)^{2\epsilon-2}}{4} \frac{d^{d-1}p}{Q^0 p^0} \delta(P^0 - Q^0 - p^0).
\end{aligned}
\tag{13}
$$

In the center-of-mass frame $\vec{P} = 0$ and thus $P^0 = \sqrt{P^2}$. If we can neglect the invariant mass of p, we have that in this frame $|\vec{Q}| = |\vec{p}| = p^0$ and that

$$
\begin{aligned}
\delta(P^0 - Q^0 - p^0) &= \frac{\sqrt{(p^0)^2 + Q^2}}{\sqrt{P^2}} \delta\left(p^0 - \frac{P^2 - Q^2}{2\sqrt{P^2}}\right) \\
&= \frac{Q^0}{\sqrt{P^2}} \delta\left(p^0 - \frac{P^2 - Q^2}{2\sqrt{P^2}}\right).
\end{aligned}
\tag{14}
$$

Now,

$$d^{d-1}p = |\vec{p}|^{d-2} d|\vec{p}| d\Omega_{d-1} = (p^0)^{d-2} dp^0 d\Omega_{d-1}., \tag{15}$$

where $d\Omega_{d-1}$ is the solid angle in $d-1$ dimensions. Substituting Eqs.(14,15) in Eq.(13) and integrating over p^0, we obtain

$$
\begin{aligned}
d\phi_2(P; Q, p) &= \frac{(2\pi)^{2\epsilon-2}}{4} \frac{(p^0)^{1-2\epsilon}}{\sqrt{P^2}} d\Omega_{d-1} \\
&= N(\epsilon)(P^2)^{-\epsilon} \left(1 - \frac{Q^2}{P^2}\right)^{1-2\epsilon} d\Omega_{d-1},
\end{aligned}
\tag{16}
$$

where

$$N(\epsilon) = \frac{1}{2(4\pi)^{2-2\epsilon}}. \tag{17}$$

Finally, we want to calculate $d\Omega_{d-1}$. To do this, we use its definition

$$d\Omega_{d-1} = d\theta_{d-1} \sin^{d-3}\theta_{d-1} d\Omega_{d-2}, \tag{18}$$

which can be applied recursively i times till $d - 2 - i > 0$ with $0 \le \theta_{d-i} \le \pi$. The normalization of the solid angles can be obtained performing the gaussian integral in spherical coordinates thus giving

$$\Omega_{d-1} = \int d\Omega_{d-1} = \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)}, \tag{19}$$

where Γ is the usual gamma function

$$\Gamma(\alpha) = \int_0^\infty dt\, e^{-t} t^{\alpha-1} \tag{20}$$

In our case $(d = 4 - 2\epsilon)$ we can use Eq.(18) two times and we odtain

$$d\Omega_{d-1} = d\Omega_{3-2\epsilon} = \sin^{1-2\epsilon}\theta\, d\theta \sin^{-2\epsilon}\phi\, d\phi\, d\Omega_{1-2\epsilon}. \tag{21}$$

In many cases it is useful to rewrite this equation in terms of other variables variables

$$y_1 = \frac{1 + \cos\theta}{2}, \qquad y_2 = \frac{1 + \cos\phi}{2}, \qquad 0 \le y_i \le 1. \tag{22}$$

Doing this change of variables and recalling that the two-body squared amplitude cannot depend on more than two angles, we obtain

$$d\Omega_{3-2\epsilon} = \frac{4^{1-2\epsilon}\pi^{1/2-\epsilon}}{\Gamma(1/2-\epsilon)} \int_0^1 dy_1 [y_1(1-y_1)]^{-\epsilon} \int_0^1 dy_2 [y_2(1-y_2)]^{-1/2-\epsilon}. \tag{23}$$

We recall for completeness that the integrals in Eqs.(12,23) are indicated only to remind the integration range. They can be performed without the matching with the square amplitude of the corrispondig process only for the determination of their normalization. We can check explicitly that Eq.(23) is correct performing its integral. This is easily done using the definition of the B function

$$B(z, w) \equiv \int_0^1 dt\, t^{z-1}(1-t)^{w-1} = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \tag{24}$$

and the Legendre duplication formula

$$\Gamma(2z) = (2\pi)^{-1/2} 2^{2z-1/2} \Gamma(z)\Gamma(z+1/2). \tag{25}$$

We find

$$\Omega_{3-2\epsilon} = \frac{2\pi^{3/2-\epsilon}}{\Gamma(3/2-\epsilon)}, \tag{26}$$

which is in agreement with Eq.(19).

Appendix C

Full NLO expression for DY rapidity distributions

We report here the complete expression of the NLO DY distributions given in [17, 60, 61, 62] with the factorization scale equal to the renormalization scale:

$$
\begin{aligned}
\frac{d\sigma^{NLO}}{dQ^2 dY} = {}& N(Q^2) \sum_{q,q'} c_{qq'} \int_{x_1^0}^1 \frac{dx_1}{x_1} \int_{x_2^0}^1 \frac{dx_2}{x_2} \\
& \times \left\{ \left[C_{q\bar{q}}^{(0)}(x_1, x_2, Y) + \frac{\alpha_s(\mu^2)}{2\pi} C_{q\bar{q}}^{(1)}\left(x_1, x_2, Y, \frac{Q^2}{\mu^2}\right) \right] \right. \\
& \times \left\{ q(x_1, \mu^2)\bar{q}'(x_2, \mu^2) + \bar{q}(x_1, \mu^2)q'(x_2, \mu^2) \right\} \\
& + \frac{\alpha_s(\mu^2)}{2\pi} C_{gq}^{(1)}(x_1, x_2, Y) g(x_1, \mu^2) \left\{ q'(x_2, \mu^2) + \bar{q}'(x_2, \mu^2) \right\} \\
& \left. + \frac{\alpha_s(\mu^2)}{2\pi} C_{qg}^{(1)}(x_1, x_2, Y) \left\{ q(x_1, \mu^2) + \bar{q}(x_1, \mu^2) \right\} g(x_2, \mu^2) \right\},
\end{aligned}
\tag{1}
$$

where

$$
N(Q^2) = \frac{4\pi\alpha^2}{9Q^2 S} \quad \text{for } \gamma^*,
\tag{2}
$$

$$
N(Q^2) = \frac{\pi G_F Q^2 \sqrt{2}}{3S} \delta(Q^2 - M_V^2) \quad \text{for } Z^0 \text{ and } W^\pm,
\tag{3}
$$

133

and where

$$C_{q\bar{q}}^{(0)}(x_1, x_2, Y) = x_1 x_2 \delta(x_1 - x_1^0)\delta(x_2 - x_2^0), \quad x_{1(2)}^0 = \sqrt{x}\, e^{\pm Y}, \tag{4}$$

$$C_{q\bar{q}}^{(1)}\left(x_1, x_2, Y, \frac{Q^2}{\mu^2}\right) = x_1 x_2 C_F \Bigg\{ \delta(x_1 - x_1^0)\delta(x_2 - x_2^0)\left[\frac{\pi^2}{3} - 8\right.$$

$$+ 2\mathrm{Li}_2(x_1^0) + 2\mathrm{Li}_2(x_2^0) + \ln^2(1 - x_1^0) + \ln^2(1 - x_2^0) + 2\ln\frac{x_1^0}{1 - x_1^0}$$

$$\times 2\ln\frac{x_2^0}{1 - x_2^0}\Bigg] + \left(\delta(x_1 - x_1^0)\left[\frac{1}{x_2} - \frac{x_2^0}{x_2^2} - \frac{x_2^{0\,2} + x_2^2}{x_2^2(x_2 - x_2^0)}\right.\right.$$

$$\times \ln\frac{x_2^0}{x_2} + \frac{x_2^{0\,2} + x_2^2}{x_2^2}\left(\frac{\ln(1 - x_2^0/x_2)}{x_2 - x_2^0}\right)_+ + \frac{x_2^{0\,2} + x_2^2}{x_2^2}$$

$$\times \frac{1}{(x_2 - x_2^0)_+}\ln\frac{2x_2^0(1 - x_1^0)}{x_1^0(x_2 + x_2^0)}\Bigg] + (1 \leftrightarrow 2)\Bigg)$$

$$+ \frac{G^A(x_1, x_2, x_1^0, x_2^0)}{[(x_1 - x_1^0)(x_2 - x_2^0)]_+} + H^A(x_1, x_2, x_1^0, x_2^0) + \ln\frac{Q^2}{\mu^2}$$

$$\times \Bigg\{ \delta(x_1 - x_1^0)\delta(x_2 - x_2^0)\left[3 + 2\ln\frac{1 - x_1^0}{x_1^0} + 2\ln\frac{1 - x_2^0}{x_2^0}\right]$$

$$+ \left(\delta(x_1 - x_1^0)\frac{x_2^{0\,2} + x_2^2}{x_2^2}\frac{1}{(x_2 - x_2^0)_+} + (1 \leftrightarrow 2)\right)\Bigg\}\Bigg\}, \tag{5}$$

$$C_{gq}^{(1)}(x_1, x_2, Y) = x_1 x_2 T_F \Bigg\{ \frac{\delta(x_2 - x_2^0)}{x_1^3}\left[(x_1^{0\,2} + (x_1 - x_1^0)^2)\right.$$

$$\times \ln\frac{2(x_1 - x_1^0)(1 - x_2^0)}{(x_1 + x_1^0)x_2^0} + 2x_1^0(x_1 - x_1^0)\Bigg] + \frac{G^C(x_1, x_2, Y)}{(x_2 - x_2^0)_+}$$

$$+ H^C(x_1, x_2, Y) + \ln\frac{Q^2}{\mu^2}\left\{\frac{\delta(x_2 - x_2^0)}{x_1^3}(x_1^{0\,2} + (x_1 - x_1^0)^2)\right\}\Bigg\}, \tag{6}$$

$$C_{qg}^{(1)}(x_1, x_2, Y) = C_{gq}^{(1)}(x_2, x_1, -Y), \tag{7}$$

with

$$\mathrm{Li}_2(x) = -\int_0^x dt\, \frac{\ln(1 - t)}{t}, \tag{8}$$

$$G^A(x_1, x_2, Y) = \frac{2(x_1 x_2 + x_1^0 x_2^0)(x_1^2 x_2^2 + x_1^{0\,2} x_2^{0\,2})}{x_1^2 x_2^2 (x_1 + x_1^0)(x_2 + x_2^0)}, \tag{9}$$

$$H^A(x_1, x_2, Y) = -\frac{4x_1^0 x_2^0(x_1 x_2 + x_1^0 x_2^0)}{x_1 x_2 (x_1 x_2^0 + x_2 x_1^0)^2}, \tag{10}$$

$$G^C(x_1, x_2, Y) = \frac{2x_2^0(x_1^{0\,2} x_2^{0\,2} + (x_1 x_2 - x_1^0 x_2^0)^2)(x_1 x_2 + x_1^0 x_2^0)}{x_1^3 x_2^2 (x_1 x_2^0 + x_2 x_1^0)(x_2 + x_2^0)}, \tag{11}$$

and

$$H^C(x_1, x_2, Y) = \frac{2x_1^0 x_2^0(x_1^0 x_2^0 + x_1 x_2)(x_1 x_1^0 x_2^2 + x_1^0 x_2^0(x_1 x_2^0 + 2x_1^0 x_2))}{x_1^2 x_2^2 (x_1 x_2^0 + x_2 x_1^0)^3}. \tag{12}$$

Appendix D

Proof of some identities of chapter 6

D.1   Proof of Eq.(42)

We compute in the limit $\hat{q}_\perp^2 \to 0$ the distribution $\delta(M_0^2)$ where M_0^2 is given by Eq.(28). Firstly we have

$$\delta(M_0^2) = \frac{\xi_1 \xi_2}{Q^2} \delta((1-\xi_1)(1-\xi_2) + \hat{q}_\perp^2(1-\xi_1-\xi_2)). \tag{1}$$

Then taking a generic test function $f(\xi_1, \xi_2)$,

$$\int d\xi_1 \int d\xi_2 f(\xi_1, \xi_2) \delta((1-\xi_1)(1-\xi_2) + \hat{q}_\perp^2(1-\xi_1-\xi_2))$$

$$= \int d\xi_1 \int d\xi_2 \frac{f(\xi_1, \xi_2)}{1-\xi_2+\hat{q}_\perp^2} \delta\left(1-\xi_1 - \frac{\xi_2 \hat{q}_\perp^2}{1-\xi_2+\hat{q}_\perp^2}\right)$$

$$= \int d\xi_1 \int d\xi_2 \frac{f(\xi_1, \xi_2) - f(\xi_1, 1)}{1-\xi_2+\hat{q}_\perp^2} \delta\left(1-\xi_1 - \frac{\xi_2 \hat{q}_\perp^2}{1-\xi_2+\hat{q}_\perp^2}\right)$$

$$+ \int d\xi_1 f(\xi_1, 1) \int d\xi_2 \frac{1}{1-\xi_2+\hat{q}_\perp^2} \delta\left(1-\xi_1 - \frac{\xi_2 \hat{q}_\perp^2}{1-\xi_2+\hat{q}_\perp^2}\right)$$

$$= \int d\xi_1 \int d\xi_2 \frac{f(\xi_1, \xi_2) - f(\xi_1, 1)}{1-\xi_2+\hat{q}_\perp^2} \delta\left(1-\xi_1 - \frac{\xi_2 \hat{q}_\perp^2}{1-\xi_2+\hat{q}_\perp^2}\right)$$

$$+ \int d\xi_1 \frac{f(\xi_1, 1)}{1-\xi_1+\hat{q}_\perp^2} \int d\xi_2 \delta\left(1-\xi_2 - \frac{\xi_1 \hat{q}_\perp^2}{1-\xi_1+\hat{q}_\perp^2}\right)$$

$$= \int d\xi_1 \int d\xi_2 \frac{f(\xi_1, \xi_2) - f(\xi_1, 1)}{1-\xi_2+\hat{q}_\perp^2} \delta\left(1-\xi_1 - \frac{\xi_2 \hat{q}_\perp^2}{1-\xi_2+\hat{q}_\perp^2}\right)$$

$$+ \int d\xi_1 \frac{f(\xi_1, 1)}{1-\xi_1+\hat{q}_\perp^2}$$

$$= \int d\xi_1 \int d\xi_2 \frac{f(\xi_1, \xi_2) - f(\xi_1, 1)}{1-\xi_2+\hat{q}_\perp^2} \delta\left(1-\xi_1 - \frac{\xi_2 \hat{q}_\perp^2}{1-\xi_2+\hat{q}_\perp^2}\right)$$

$$+ \int d\xi_1 \frac{f(\xi_1, 1) - f(1, 1)}{1-\xi_1+\hat{q}_\perp^2} + f(1, 1) \ln \frac{1+\hat{q}_\perp^2}{\hat{q}_\perp^2}. \tag{2}$$

Now, taking the limit $\hat{q}_\perp^2 \to 0$ and using Eq.(1), we obtain the following identity in the distribution sense

$$\delta(M_0^2) = \frac{\xi_1 \xi_2}{Q^2}\left[\frac{\delta(1-\xi_1)}{(1-\xi_2)_+} + \frac{\delta(1-\xi_2)}{(1-\xi_1)_+} - \ln\hat{q}_\perp^2\delta(1-\xi_1)\delta(1-\xi_2)\right] + O(\hat{q}_\perp^2), \qquad (3)$$

which is exactly Eq.(42).

D.2   Proof of Eq.(43)

Let us compute for a generic test function $f(\xi_1, \xi_2)$ the distribution

$$T_{\hat{q}_\perp^2} = \int d\xi_1 \int d\xi_2 [(1-\xi_1)(1-\xi_2) + \hat{q}_\perp^2(1-\xi_1-\xi_2)]^{\eta-1} f(\xi_1, \xi_2) \qquad (4)$$

in the small-$\hat{q}_\perp^2$ limit with $\eta = a|\epsilon|$, $a > 0$ and $\epsilon \neq 0$. The integration range is fixed by the requirement

$$M_0^2 \;=\; \frac{Q^2}{\xi_1 \xi_2}[(1-\xi_1)(1-\xi_2) + \hat{q}_\perp^2(1-\xi_1-\xi_2)] \geq 0, \qquad (5)$$

which gives

$$0 \leq \xi_1 \leq 1 \qquad 0 \leq \xi_2 \leq \frac{(1-\xi_1)(1+\hat{q}_\perp^2)}{1-\xi_1+\hat{q}_\perp^2} = \bar{\xi}_2 \qquad (6)$$

or equivalently

$$0 \leq \xi_2 \leq 1 \qquad 0 \leq \xi_1 \leq \frac{(1-\xi_2)(1+\hat{q}_\perp^2)}{1-\xi_2+\hat{q}_\perp^2} = \bar{\xi}_1. \qquad (7)$$

We observe also that

$$\begin{aligned}(1-\xi_1)(1-\xi_2) + \hat{q}_\perp^2(1-\xi_1-\xi_2) \;&=\; (1-\xi_2+\hat{q}_\perp^2)\left(1-\xi_1-\frac{\xi_2\hat{q}_\perp^2}{1-\xi_2+\hat{q}_\perp^2}\right) \\ &=\; (1-\xi_1+\hat{q}_\perp^2)\left(1-\xi_2-\frac{\xi_1\hat{q}_\perp^2}{1-\xi_1+\hat{q}_\perp^2}\right). \quad (8)\end{aligned}$$

Next, we decompose

$$T_{\hat{q}_\perp^2} = T_{\hat{q}_\perp^2}^1 + T_{\hat{q}_\perp^2}^2 \qquad (9)$$

where

$$\begin{aligned}T_{\hat{q}_\perp^2}^1 \;&=\; \int_0^1 d\xi_1(1-\xi_1+\hat{q}_\perp^2)^{\eta-1}\int_0^{\bar{\xi}_2} d\xi_2\left(1-\xi_2-\frac{\xi_1\hat{q}_\perp^2}{1-\xi_1+\hat{q}_\perp^2}\right)^{\eta-1}[f(\xi_1,\xi_2)-f(\xi_1,1)] \\ T_{\hat{q}_\perp^2}^2 \;&=\; \int_0^1 d\xi_1(1-\xi_1+\hat{q}_\perp^2)^{\eta-1}\int_0^{\bar{\xi}_2} d\xi_2\left(1-\xi_2-\frac{\xi_1\hat{q}_\perp^2}{1-\xi_1+\hat{q}_\perp^2}\right)^{\eta-1}f(\xi_1,1). \quad (10)\end{aligned}$$

The ξ_2 integral in $T^2_{\hat{q}^2_\perp}$ is immediately performed:

$$\int_0^{\bar{\xi}_2} d\xi_2 \left(1 - \xi_2 - \frac{\xi_1 \hat{q}^2_\perp}{1 - \xi_1 + \hat{q}^2_\perp}\right)^{\eta-1} = \frac{1}{\eta}\left[\frac{(1-\xi_1)(1+\hat{q}^2_\perp)}{1-\xi_1+\hat{q}^2_\perp}\right]^\eta . \tag{11}$$

Therefore,

$$T^2_{\hat{q}^2_\perp} = \frac{(1+\hat{q}^2_\perp)^\eta}{\eta}\int_0^1 d\xi_1 f(\xi_1,1)\frac{(1-\xi_1)^\eta}{1-\xi_1+\hat{q}^2_\perp}. \tag{12}$$

Proceeding as above, we regularize the ξ_1 integral:

$$T^2_{\hat{q}^2_\perp} = T^{21}_{\hat{q}^2_\perp} + T^{22}_{\hat{q}^2_\perp}, \tag{13}$$

where

$$T^{21}_{\hat{q}^2_\perp} = \frac{(1+\hat{q}^2_\perp)^\eta}{\eta}\int_0^1 d\xi_1 \frac{(1-\xi_1)^\eta}{1-\xi_1+\hat{q}^2_\perp}[f(\xi_1,1)-f(1,1)] \tag{14}$$

$$T^{22}_{\hat{q}^2_\perp} = \frac{(1+\hat{q}^2_\perp)^\eta}{\eta}f(1,1)\int_0^1 d\xi_1 \frac{(1-\xi_1)^\eta}{1-\xi_1+\hat{q}^2_\perp}$$

$$= f(1,1)\frac{(1+\hat{q}^2_\perp)^\eta}{\eta}\frac{(1+\hat{q}^2_\perp)^\eta - (\hat{q}^2_\perp)^\eta}{\eta}. \tag{15}$$

The two distributions $T^{21}_{\hat{q}^2_\perp}$ and $T^{22}_{\hat{q}^2_\perp}$ are now well defined as $\hat{q}^2_\perp \to 0$. Similarly,

$$T^1_{\hat{q}^2_\perp} = T^{11}_{\hat{q}^2_\perp} + T^{12}_{\hat{q}^2_\perp}, \tag{16}$$

where

$$T^{11}_{\hat{q}^2_\perp} = \int_0^1 d\xi_1(1-\xi_1+\hat{q}^2_\perp)^{\eta-1}\int_0^{\bar{\xi}_2} d\xi_2\left(1-\xi_2-\frac{\xi_1\hat{q}^2_\perp}{1-\xi_1+\hat{q}^2_\perp}\right)^{\eta-1} \tag{17}$$

$$\{[f(\xi_1,\xi_2)-f(\xi_1,1)]-[f(1,\xi_2)-f(11)]\}$$

$$T^{12}_{\hat{q}^2_\perp} = \int_0^1 d\xi_1(1-\xi_1+\hat{q}^2_\perp)^{\eta-1}\int_0^{\bar{\xi}_2} d\xi_2\left(1-\xi_2-\frac{\xi_1\hat{q}^2_\perp}{1-\xi_1+\hat{q}^2_\perp}\right)^{\eta-1}$$

$$[f(1,\xi_2)-f(1,1)] \tag{18}$$

The term $T^{12}_{\hat{q}^2_\perp}$ can be computed by changing the order of integration and exploiting Eq.(8) and using eq(11), thus obtaining

$$T^{12}_{\hat{q}^2_\perp} = \frac{(1+\hat{q}^2_\perp)^\eta}{\eta}\int_0^1 d\xi_2 \frac{(1-\xi_2)^\eta}{1-\xi_2+\hat{q}^2_\perp}[f(1,\xi_2)-f(1,1)]. \tag{19}$$

Collecting Eqs.(14,15,17,19), taking the limit $\hat{q}^2_\perp \to 0$ and using the identity

$$(1-\xi)^{\eta-1} = (1-\xi)^{\eta-1}_+ + \frac{1}{\eta}\delta(1-\xi), \tag{20}$$

we obtain the following identity in sense of distributions:

$$[(1-\xi_1)(1-\xi_2)+\hat{q}^2_\perp(1-\xi_1-\xi_2)]^{\eta-1} = \tag{21}$$

$$= (1-\xi_1)^{\eta-1}(1-\xi_2)^{\eta-1} - \frac{(\hat{q}^2_\perp)^\eta}{\eta^2}\delta(1-\xi_1)\delta(1-\xi_2) + O(\hat{q}^2_\perp)$$

which is exactly Eq.(43).

Appendix E

Proof of combinatoric properties of chapter 7

Let us consider the coefficients defined by

$$C_m^{(i,j)} = \frac{(-1)^{m+1}}{m+1} \sum_{l=0}^{m} 2^l \binom{l+i-1}{i-1} \binom{m-l+j-1}{j-1}. \tag{1}$$

We shall now prove that

$$C_m^{(i,0)} = 2^m C_m^{(0,i)} \tag{2}$$

$$C_m^{(i,j)} = 2C_m^{(i,j-1)} - C_m^{(i-1,j)}; \qquad i,j \geq 1, \tag{3}$$

or, equivalently, that

$$\tilde{C}_m^{(i,0)} = 2^m \tilde{C}_m^{(0,i)} \tag{4}$$

$$\tilde{C}_m^{(i,j)} = 2\tilde{C}_m^{(i,j-1)} - \tilde{C}_m^{(i-1,j)}; \qquad i,j \geq 1,, \tag{5}$$

where

$$\tilde{C}_m^{(i,j)} \equiv (-)^{m+1}(m+1)C_m^{(i,j)} = \sum_{l=0}^{m} 2^l \binom{l+i-1}{i-1} \binom{m-l+j-1}{j-1}. \tag{6}$$

Eq.(4) follows immediately. Indeed:

$$\tilde{C}_m^{(i,0)} = 2^m \binom{m+i-1}{i-1} \tag{7}$$

$$\tilde{C}_m^{(0,i)} = \binom{m+i-1}{i-1}. \tag{8}$$

We turn now to the proof of Eq.(5). In the case $i = j = 1$ it follows again immediately. In fact we have:

$$\tilde{C}_m^{(0,1)} = 1, \tilde{C}_m^{(1,0)} = 2^m, \tilde{C}_m^{(1,1)} = \sum_{l=0}^{m} 2^l = 2^{m+1} - 1. \tag{9}$$

In the cases $i = 1, j = 2$ e $i = 2, j = 1$ it is quite easy:

$$\tilde{C}_m^{(1,1)} = 2^{m+1} - 1, \quad \tilde{C}_m^{(0,2)} = m + 1, \quad \tilde{C}_m^{(2,0)} = 2^m(m+1) \tag{10}$$

$$\tilde{C}_m^{(1,2)} = 2(2^{m+1} - 1) - (m+1), \quad \tilde{C}_m^{(2,1)} = 2^{m+1}(m+1) - (2^{m+1} - 1), \tag{11}$$

where we have use the equality:

$$\sum_{l=0}^{m} l 2^l = \lim_{\alpha \to 1} \frac{1}{\ln 2} \frac{d}{d\alpha} \sum_{l=0}^{m} (e^{\alpha \ln 2})^l = 2^{m+1}(m+1) - 2(2^{m+1} - 1). \tag{12}$$

In the other cases, i.e. for $i, j \geq 2$, it follows in a straightforward way:

$$2\tilde{C}_m^{(i,j-1)} - \tilde{C}_m^{(i-1,j)} =$$

$$= 2 \sum_{l=0}^{m} 2^l \binom{l+i-1}{i-1} \binom{m-l+j-2}{j-2} - \sum_{l=0}^{m} 2^l \binom{l+i-2}{i-2} \binom{m-l+j-1}{j-1}$$

$$= \tilde{C}_m^{(i,j)} - \sum_{l=0}^{m} 2^{l+1} \binom{l+i-1}{i-1} \binom{m-l+j-2}{j-1} + \sum_{l=0}^{m} 2^l \binom{l+i-2}{i-1} \binom{m-l+j-1}{j-1}$$

$$= \tilde{C}_m^{(i,j)} - \sum_{l'=1}^{m+1} 2^{l'} \binom{l'+i-2}{i-1} \binom{m-l'+j-1}{j-1} + \sum_{l=0}^{m} 2^l \binom{l+i-2}{i-1} \binom{m-l+j-1}{j-1}$$

$$= \tilde{C}_m^{(i,j)} - 2^{m+1} \binom{m+i-1}{i-1} \binom{j-2}{j-1} + \binom{i-2}{i-1} \binom{m+j-1}{j-1}$$

$$= \tilde{C}_m^{(i,j)}, \tag{13}$$

where $l' = l + 1$ and where in the second, the third and the last line we have used the following identities:

$$\binom{n+r-2}{r-2} = \binom{n+r-1}{r-1} - \binom{n+r-2}{r-1}, \tag{14}$$

$$\binom{r-2}{r-1} = \frac{\Gamma(r-1)}{\Gamma(r)\Gamma(0)} = 0. \tag{15}$$

We note that these last two properties are valid only for $r \geq 2$ and this is the reason why the cases $i = j = 1$ e $i = 1, j = 2$ and $i = 2, j = 1$ have been treated separately.

Bibliography

[1] George Sterman. Summation of large corrections to short distance hadronic cross-sections. *Nucl. Phys.*, B281:310, 1987.

[2] S. Catani and L. Trentadue. Resummation of the QCD perturbative series for hard processes. *Nucl. Phys.*, B327:323, 1989.

[3] S. Catani and L. Trentadue. Comment on QCD exponentiation at large x. *Nucl. Phys.*, B353:183–186, 1991.

[4] Gennaro Corcella and Lorenzo Magnea. Soft-gluon resummation effects on parton distributions. *Phys. Rev.*, D72:074017, 2005, hep-ph/0506278.

[5] Gennaro Corcella and Lorenzo Magnea. Impact of large-x resummation on parton distribution functions. *AIP Conf. Proc.*, 792:303–306, 2005, hep-ph/0507042.

[6] John C. Collins, Davison E. Soper, and George Sterman. Transverse momentum distribution in Drell-Yan pair and W and Z boson production. *Nucl. Phys.*, B250:199, 1985.

[7] V. V. Sudakov. Vertex parts at very high-energies in quantum electrodynamics. *Sov. Phys. JETP*, 3:65–71, 1956.

[8] Stefano Forte and Giovanni Ridolfi. Renormalization group approach to soft gluon resummation. *Nucl. Phys.*, B650:229–270, 2003, hep-ph/0209154.

[9] Aneesh V. Manohar. Deep inelastic scattering as $x \to 1$ using soft-collinear effective theory. *Phys. Rev.*, D68:114019, 2003, hep-ph/0309176.

[10] Ahmad Idilbi and Xiang-dong Ji. Threshold resummation for drell-yan process in soft- collinear effective theory. *Phys. Rev.*, D72:054016, 2005, hep-ph/0501006.

[11] John C. Collins, Davison E. Soper, and George Sterman. Factorization of hard processes in QCD. *Adv. Ser. Direct. High Energy Phys.*, 5:1–91, 1988, hep-ph/0409313.

[12] Eric Laenen, Gianluca Oderda, and George Sterman. Resummation of threshold corrections for single-particle inclusive cross sections. *Phys. Lett.*, B438:173–183, 1998, hep-ph/9806467.

[13] Nikolaos Kidonakis, Gianluca Oderda, and George Sterman. Evolution of color exchange in QCD hard scattering. *Nucl. Phys.*, B531:365–402, 1998, hep-ph/9803241.

[14] Stefano Catani, Michelangelo L. Mangano, and Paolo Nason. Sudakov resummation for prompt photon production in hadron collisions. *JHEP*, 07:024, 1998, hep-ph/9806484.

[15] Roberto Bonciani, Stefano Catani, Michelangelo L. Mangano, and Paolo Nason. Sudakov resummation of multiparton QCD cross sections. *Phys. Lett.*, B575:268–278, 2003, hep-ph/0307035.

[16] Eric Laenen and George Sterman. Resummation for Drell-Yan differential distributions. Presented at Particles & Fields 92: 7th Meeting of the Division of Particles Fields of the APS (DPF 92) (NOTE: Dates changed from Oct 13-17), Batavia, IL, 10-14 Nov 1992.

[17] Asmita Mukherjee and Werner Vogelsang. Threshold resummation for W-boson production at RHIC. *Phys. Rev.*, D73:074005, 2006, hep-ph/0601162.

[18] George Sterman and Werner Vogelsang. Threshold resummation and rapidity dependence. *JHEP*, 02:016, 2001, hep-ph/0011289.

[19] Charalampos Anastasiou, Lance J. Dixon, Kirill Melnikov, and Frank Petriello. Dilepton rapidity distribution in the Drell-Yan process at NNLO in QCD. *Phys. Rev. Lett.*, 91:182002, 2003, hep-ph/0306192.

[20] V. Ravindran, J. Smith and W. L. van Neerven QCD threshold corrections to di-lepton and Higgs rapidity distributions beyond N^2LO, hep-ph/0608308.

[21] V. Ravindran and J. Smith. Threshold corrections to rapidity distributions of Z and $W^{\pm}$ bosons beyond N^2LO at hadron colliders, arXiv:0708.1689 [hep-ph]

[22] Michael E. Peskin, Daniel V. Schroeder. *An introduction to Quantum Field Theory.* PERSEUS BOOKS, 1995.

[23] T. van Ritbergen, J. A. M. Vermaseren, and S. A. Larin. The four-loop beta function in quantum chromodynamics. *Phys. Lett.*, B400:379–384, 1997, hep-ph/9701390.

[24] G. Ridolfi and S. Forte. Renormalization and factorization scale dependence of observables in QCD. *J. Phys.*, G25:1555–1556, 1999.

[25] Steven Weinberg. *The Quantum Theory of Fields*, volume 2. 1996.

[26] G. Curci, W. Furmanski, and R. Petronzio. Evolution of parton densities beyond leading order: The nonsinglet case. *Nucl. Phys.*, B175:27, 1980.

[27] R.K. Ellis, W.J. Stirling and B.R. Webber. *QCD and Collider Physics.* Cambridge University Press, 1989.

[28] Richard. D. Field. *Applications of Perturbative QCD*. Addison-Wesley Publishing Company, 1989.

[29] A. Vogt, S. Moch, and J. A. M. Vermaseren. The three-loop splitting functions in qcd: The singlet case. *Nucl. Phys.*, B691:129–181, 2004, hep-ph/0404111.

[30] S. Moch, J. A. M. Vermaseren, and A. Vogt. The three-loop splitting functions in qcd: The non-singlet case. *Nucl. Phys.*, B688:101–134, 2004, hep-ph/0403192.

[31] E. G. Floratos, C. Kounnas, and R. Lacaze. Higher order QCD effects in inclusive annihilation and deep inelastic scattering. *Nucl. Phys.*, B192:417, 1981.

[32] M. Gluck, E. Reya, and A. Vogt. Radiatively generated parton distributions for high-energy collisions. *Z. Phys.*, C48:471–482, 1990.

[33] A. Gonzalez-Arroyo, C. Lopez, and F. J. Yndurain. Second order contributions to the structure functions in deep inelastic scattering. i. theoretical calculations. *Nucl. Phys.*, B153:161–186, 1979.

[34] Johannes Blumlein and Stefan Kurth. Harmonic sums and Mellin transforms up to two-loop order. *Phys. Rev.*, D60:014018, 1999, hep-ph/9810241.

[35] A. Vogt. Efficient evolution of unpolarized and polarized parton distributions with QCD-PEGASUS. *Comput. Phys. Commun.*, 170:65–92, 2005, hep-ph/0408244.

[36] Robert V. Harlander and William B. Kilgore. Next-to-next-to-leading order higgs production at hadron colliders. *Phys. Rev. Lett.*, 88:201801, 2002, hep-ph/0201206.

[37] Charalampos Anastasiou, Lance J. Dixon, Kirill Melnikov, and Frank Petriello. High-precision qcd at hadron colliders: Electroweak gauge boson rapidity distributions at nnlo. *Phys. Rev.*, D69:094008, 2004, hep-ph/0312266.

[38] J. C. Webb et al. Absolute Drell-Yan dimuon cross sections in 800-gev/c p p and p d collisions. 2003, hep-ex/0302019.

[39] Stefano Catani, Daniel de Florian, Massimiliano Grazzini, and Paolo Nason. Soft-gluon resummation for higgs boson production at hadron colliders. *JHEP*, 07:028, 2003, hep-ph/0306211.

[40] Giuseppe Bozzi, Stefano Catani, Daniel de Florian, and Massimiliano Grazzini. Transverse-momentum resummation and the spectrum of the higgs boson at the lhc. *Nucl. Phys.*, B737:73–120, 2006, hep-ph/0508068.

[41] Christian W. Bauer, Sean Fleming, and Michael E. Luke. Summing sudakov logarithms in $B \to X_s \gamma$ gamma in effective field theory. *Phys. Rev.*, D63:014006, 2001, hep-ph/0005275.

[42] D. R. Yennie, Steven C. Frautschi, and H. Suura. The infrared divergence phenomena and high-energy processes. *Ann. Phys.*, 13:379–452, 1961.

[43] D. Amati, A. Bassetto, M. Ciafaloni, G. Marchesini, and G. Veneziano. A treatment of hard processes sensitive to the infrared structure of qcd. *Nucl. Phys.*, B173:429, 1980.

[44] Christian W. Bauer, Sean Fleming, Dan Pirjol, and Iain W. Stewart. An effective field theory for collinear and soft gluons: Heavy to light decays. *Phys. Rev.*, D63:114020, 2001, hep-ph/0011336.

[45] Christian W. Bauer, Dan Pirjol, and Iain W. Stewart. Soft-collinear factorization in effective field theory. *Phys. Rev.*, D65:054022, 2002, hep-ph/0109045.

[46] Junegone Chay and Chul Kim. Collinear effective theory at subleading order and its application to heavy-light currents. *Phys. Rev.*, D65:114016, 2002, hep-ph/0201197.

[47] Christian W. Bauer, Sean Fleming, Dan Pirjol, Ira Z. Rothstein, and Iain W. Stewart. Hard scattering factorization from effective field theory. *Phys. Rev.*, D66:014017, 2002, hep-ph/0202088.

[48] T. Kinoshita. Mass singularities of Feynman amplitudes. *J. Math. Phys.*, 3:650–677, 1962.

[49] T. D. Lee and M. Nauenberg. Degenerate systems and mass singularities. *Phys. Rev.*, 133:B1549–B1562, 1964.

[50] J.B. Zuber C. Itzykson. *Quantum Field Theory.* 1980.

[51] Harry Contopanagos, Eric Laenen, and George Sterman. Sudakov factorization and resummation. *Nucl. Phys.*, B484:303–330, 1997, hep-ph/9604313.

[52] A. Vogt. Next-to-next-to-leading logarithmic threshold resummation for deep-inelastic scattering and the Drell-Yan process. *Phys. Lett.*, B497:228–234, 2001, hep-ph/0010146.

[53] Steven Weinberg. *The Quantum Theory of Fields*, volume 1. 1996.

[54] Roberto Bonciani, Stefano Catani, Michelangelo L. Mangano, and Paolo Nason. NLL resummation of the heavy-quark hadroproduction cross- section. *Nucl. Phys.*, B529:424–450, 1998, hep-ph/9801375.

[55] P. Aurenche, A. Douiri, R. Baier, M. Fontannaz, and D. Schiff. Prompt photon production at large p(t) in QCD beyond the leading order. *Phys. Lett.*, B140:87, 1984.

[56] P. Aurenche, R. Baier, M. Fontannaz, and D. Schiff. Prompt photon production at large p(t) scheme invariant QCD predictions and comparison with experiment. *Nucl. Phys.*, B297:661, 1988.

[57] L. E. Gordon and W. Vogelsang. Polarized and unpolarized prompt photon production beyond the leading order. *Phys. Rev.*, D48:3136–3159, 1993.

[58] Stefano Catani, Michelangelo L. Mangano, Paolo Nason, and Luca Trentadue. The resummation of soft gluon in hadronic collisions. *Nucl. Phys.*, B478:273–310, 1996, hep-ph/9604351.

[59] Stefano Forte, Giovanni Ridolfi, Joan Rojo, and Maria Ubiali. Borel resummation of soft gluon radiation and higher twists. *Phys. Lett.*, B635:313–319, 2006, hep-ph/0601048.

[60] T. Gehrmann. QCD corrections to double and single spin asymmetries in vector boson production at polarized hadron colliders. *Nucl. Phys.*, B534:21–39, 1998, hep-ph/9710508.

[61] P. J. Sutton, Alan D. Martin, R. G. Roberts, and W. James Stirling. Parton distributions for the pion extracted from Drell-Yan and prompt photon experiments. *Phys. Rev.*, D45:2349–2359, 1992.

[62] Guido Altarelli, R. K. Ellis, and G. Martinelli. Large Perturbative corrections to the Drell-Yan process in QCD. *Nucl. Phys.*, B157:461, 1979.

[63] A. D. Martin, R. G. Roberts, W. J. Stirling, and R. S. Thorne. NNLO global parton analysis. *Phys. Lett.*, B531:216–224, 2002, hep-ph/0201127.

[64] Alan D. Martin, R. G. Roberts, W. J. Stirling, and R. S. Thorne. MRST2001: Partons and alpha(s) from precise deep inelastic scattering and Tevatron jet data. *Eur. Phys. J.*, C23:73–87, 2002, hep-ph/0110215.

[65] Paolo Bolzoni. Threshold resummation of Drell-Yan rapidity distributions. 2006, hep-ph/0609073.

[66] Guido Altarelli, R. K. Ellis, M. Greco, and G. Martinelli. Vector boson production at colliders: A theoretical reappraisal. *Nucl. Phys.*, B246:12, 1984.

[67] R. K. Ellis, G. Martinelli, and R. Petronzio. Lepton pair production at large transverse momentum in second order QCD. *Nucl. Phys.*, B211:106, 1983.

[68] R. K. Ellis, G. Martinelli, and R. Petronzio. Second order corrections to the Drell-Yan process at large transverse momentum. *Phys. Lett.*, B104:45, 1981.

[69] R. K. Ellis and Sinisa Veseli. W and Z transverse momentum distributions: Resummation in q(t)-space. *Nucl. Phys.*, B511:649–669, 1998, hep-ph/9706526.

[70] Stefano Frixione, Paolo Nason, and Giovanni Ridolfi. Problems in the resummation of soft-gluon effects in the transverse-momentum distributions of massive vector bosons in hadronic collisions. *Nucl. Phys.*, B542:311–328, 1999, hep-ph/9809367.

[71] Anna Kulesza and W. James Stirling. Soft gluon resummation in transverse momentum space for electroweak boson production at hadron colliders. *Eur. Phys. J.*, C20:349–356, 2001, hep-ph/0103089.

Acknowledgements

I want to thank S. Forte and G. Ridolfi for giving me the opportunity to work with them, for all the things that I have learnt from them and for their friendship.

I thank my parents, because they wanted me to watch how much beautiful is the world. Furthermore, I want to thank them for their supporting and for all the times that they had reason.

I want to thank all the students of the Modern Physics, Quantum Mechanics, General Physics and Mathematical Analysis courses, because with their questions they have helped me not to forget that I study physics because I love reality.

Finally, I want to thank all my friends, because I am not yet able to give them back what they gave to me: the reason for not to quit in any circumstance of life.