Semiclassical versus Exact Quantization of the Sinh-Gordon Model

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Abstract

In this work we investigate the semiclassics of the Sinh-Gordon model. The Sinh-Gordon model is integrable, its explicit solutions of the classical and the quantum model are well known. This allows for a comprehensive investigation of the semiclassical quantization of the classical model as well as of the semiclassical limit of the exact quantum solution. Semiclassical means in this case that the key objects of quantum theory are constructed as formal power series. A quantity playing an important role in the quantum theory is the Q-function. The purpose of this work is to investigate to what extend the classical integrability of the model admits of a construction of the semiclassical expansion of the Q-function.
Therefore we used two conceptual independent approaches. In the one approach we start from the exact nonperturbative solution of the quantum model and calculate the semiclassical limit up to the next to leading order. Thereby we found the spectral curve, as well as the semiclassical expansion of the Q-function and of the eigenvalue of the monodromy matrix. In the other approach we constructed the first two orders of the semiclassical expansion of the Q-function, starting from the classical solution theory. The results of both approaches coincide.

Zusammenfassung

In dieser Arbeit untersuchen wir Semiklassik des Sinh-Gordon Modells. Das Sinh-Gordon Modell ist integrabel, die expliziten Lösungen der klassischen und der Quantentheorie sind bekannt. Das ermöglicht eine umfassende Untersuchung sowohl der semiklassischen Quantisierung des klassischen Modells als auch des semiklassischen Grenzwerts der exakten Quantenlösung. Semiklassisch bedeutet in dem Fall, dass die entscheidenden Grössen der Quantentheorie als formale Potenzreihe in $\hbar$ konstruiert werden. Eine Schlüsselrolle in der Lösung der Quantentheorie nimmt die Q-Funktion ein. Ziel der Arbeit ist es zu beantworten, inwieweit die klassische Integrabilität des Modells eine Konstruktion der semiklassischen Q-Funktion ermöglicht.
Dabei verfolgen wir zwei konzeptionell unabhängige Wege. Zum einen gehen wir von der exakten, nicht störungstheoretischen Lösung des Quantenmodells aus und berechnen den halbklassischen Grenzwert bis zur nächstführenden Ordnung in $\hbar$. Wir finden dabei die Spektralkurve, sowie die semiklassische Entwicklung der Q-Funktion und des Eigenwerts der Monodromiematrix. Zum anderen konstruieren wir die ersten beiden Ordnungen der halbklassischen Q-Funktion ausgehend von der klassischen Lösungstheorie des Modells. Die Resultate beider Herangehnsweisen stimmen überein.

Contents

Chapter 1

Introduction

This work deals with the semiclassical treatment of the Sinh-Gordon model in finite volume. For the definition of the Sinh-Gordon model, we take a field $\varphi(x,t)$, which lives on the cylinder $(x,t) \in [0,R] \times \mathbb{R}$ and satisfies periodic boundary conditions, $\varphi(x+R,t) = \varphi(x,t)$. Apart from that, it has to solve the so called Sinh-Gordon equation

$$\begin{cases} \left(\partial_t^2 - \partial_x^2\right) \varphi(x,t) = -8\pi\mu b \sinh(2b\varphi(x,t)) \\ \varphi(x+R,t) = \varphi(x,t) \end{cases} . \tag{1.1}$$

The special feature of the Sinh-Gordon model is the integrability of the classical and the quantum model. Integrability basically means that a model possesses as many commuting conserved quantities as degrees of freedom and can thus be solved explicitely in terms of first integrals. In the case of a field theory, infinitely many degrees of freedom are present as the field configuration has to be specified at every point in space-time.
In the Sinh-Gordon case, there are known solutions for the classical and the quantum model, which will be described later. In this thesis, we examined the transition of the classical solutions to the quantum ones and vice versa. In other words, we investigated the semiclassical quantization and the semiclassical limit of the exact solution of the model.

1.1 Motivation

In the following we will successively motivate why one studies integrable models at all, why a semiclassical analysis is interesting and why we are working with the Sinh-Gordon model.

1.1.1 Why integrable models

A general feature of integrable models is that they are, at least in principle, explicitely solvable. As this is a rare property in nature, but these models nevertheless appear in a wide range of areas like hydrodynamics, nonlinear optics, condensed matter, plasma physics, high energy physics, ..., these models are worth to be studied in more detail.
Apart from these concrete applications integrable models are interesting also from a more theoretical point of view: one of the main questions of modern theoretical and mathematical physics is the construction of nontrivial interacting quantum field theories. The quantum integrable models are one of the few examples where this has been done rigorously[1]. Although these models are usually defined in 1+1 dimensions and although there is no integrable quantum field theory in 3+1 Minkowski space, one can learn a lot about quantum field theory by studying the explicit solutions of these integrable QFT's. The knowledge of a complete non-perturbative solution of an interacting quantum field theory admits of a detailed study of perturbation theory and renormalization as well as of an investigation of non-perturbative effects[2].
From the mathematical point of view integrable models are also interesting. First of all they

[1] In fact, apart from the approach via integrable models, there is only the constructive field theory which deals with the clean construction of nontrivial interacting quantum field theories, see for example [1]. Since the 1970's, they achieved the construction of quiet some models in 2 and 3 dimensions, like the $P(\phi)_2$ models of Glimm and Jaffe or the Yukawa model. However, until now a direct link between integrable field theory and constructive field theory is missing.

[2] A famous example where nonperturbative effects become important is the up to now not understood problem of the quark confinement. In the regime where confinement takes place perturbation theory simply breaks down.

constitute examples of explicitely solvable partial differential equations. The second point is that there are many interesting structures connected with integrable models, like infinite dimensional Lie algebras and quantum groups.

1.1.2 Why semiclassics

Usually, if one has a classical field theory and is interested in its quantum counter part, it is absolutely not clear how to perform a rigorous quantization. This is also true in the case of integrable models. Therefore it is interesting to construct at least a semiclassical quantization of the model. Semiclassical means that the key objects characterizing the quantum field theory are constructed as formal power series in $\hbar$.
In this work we investigate how far the integrability at the classical level of the Sinh-Gordon model facilitates the construction of the semiclassical quantization. In order to appraise the results of the semiclassical quantization one should compare them to the non-perturbative solution of the quantum Sinh-Gordon model and to its semiclassical limit. As the semiclassical expansion is only a formal power series, it is interesting to see wether the behaviour of the full quantum solution differs from the one of the semiclassical expansion. If one has understood this, one can draw conclusions about the full quantum solution by studying the semimclassical quantization.
Another motivation for the calculation of the semiclassical limit is to get a mapping between the classical observables and their quantum counterparts. This mapping may not be obvious, as the exact quantization based on the integrability of a theory often produces somewhat abstract results.

1.1.3 Why the Sinh-Gordon model

The Sinh-Gordon model is one of the few models where both the classical and the quantum solution are known. Thus, if one wants to study the semiclassics of an integrable model, it is the perfect candidate.
Another motivation comes from the nonlinear σ models. In recent years, there has been a growing interest in nonlinear sigma models with non-compact target space, as there are possible applications in string theory on curved space time and to gauge theories via the AdS-CFT correspondence. Thereby a nonlinear sigma model describes a scalar field X, which takes values on a nonlinear manifold, the target space. The Lagrangian density is given as

$$\mathcal{L} = \frac{1}{2} g\left(\partial^{\mu} X_a, \partial_{\mu} X_a\right) - V(X), \tag{1.2}$$

where g is the Riemann metric of the target space and V is the potential.
In general, nonlinear sigma models are very hard to solve even if they are integrable. In case of a non-compact target space, this is worse . At this point, the Sinh-Gordon model becomes interesting as there is a qualitative similarity to some of those sigma models. So, by studying the Sinh-Gordon model, which is much easier to handle, one hope to understand certain features of the more difficult nonlinear sigma models.

In the followig two sections we will introduce the basics of the classical and the quantum Sinh-Gordon model. After that we will describe the main results of this thesis.

1.2 The classical Sinh-Gordon model

Classically, the Sinh-Gordon model has been solved by the inverse scattering method. Thereby, one attaches to the nonlinear partial differential equation a linear eigenvalue problem, whose potential is given by a solution of the original nonlinear equation. In the Sinh-Gordon case, the linear eigenvalue problem is given by

$$\begin{cases} \partial_x \Psi(x,t;\lambda) = U(x,t;\lambda)\Psi(x,t,\lambda), \\ \partial_t \Psi(x,t;\lambda) = V(x,t;\lambda)\Psi(x,t;\lambda), \end{cases} \tag{1.3}$$

where U and V are 2×2-matrices containing the Sinh-Gordon field $\varphi(x,t)$; λ is an auxiliary parameter, the spectral parameter. The key observation is that the spectrum of (1.3) does not depend on time if $\varphi(x,t)$ indeed solves the Sinh-Gordon equation. The idea is now to gain general information about the spectrum of (1.3) by doing spectral theory; then one can start from a fixed spectrum combined with initial conditions and determine the Sinh-Gordon field by an inverse transformation. Further information can be found in the section about the classical Sinh-Gordon theory.
An important quantity in the inverse scattering method is the monodromy matrix, which is defined as the path-ordered exponential

$$M(t_0;\lambda) = \mathcal{P}\exp\left(\int_0^R U(x',t_0;\lambda)dx'\right). \tag{1.4}$$

It is the matrix of parallel transport along the contour $t = t_0$, $0 \leq x \leq R$, namely it transfers the solutions of (1.3) across this contour,

$$M(t_0;\lambda)\Psi(0,t_0;\lambda) = \Psi(R,t_0;\lambda). \tag{1.5}$$

Other important quantities are the eigenvalues of the monodromy matrix $\Lambda(\lambda)$, $\Lambda^{-1}(\lambda)$, as well as its trace $T(\lambda) = \Lambda(\lambda) + \Lambda^{-1}(\lambda)$. The peculiar thing with the eigenvalues is that they are two-valued functions of $\lambda \in \mathbb{C}$. The Riemann surface Σ, on which they are single-valued, is called the spectral curve; it contains all the conserved quantities of the model. The curve is defined as

$$\Sigma : \mu^2 = T^2(\lambda) - 4. \tag{1.6}$$

It is a two-sheeted covering of the punctured plane $\mathbb{C}\setminus\{0,\infty\}$ with simple ramification points along the real axis. The Riemann surface has cuts between two adjacent branching points, respectively. We will restrict our considerations to the case of the so called finite zone solutions, i.e. to the case of finitely many cuts.
In order to describe the dynamics of the Sinh-Gordon model, it turned out to be useful to introduce a new set of canonical variables. These 'separated variables' live on the spectral curve. After having specified the initial conditions one finds that under time evolution they are moving around the cuts of the spectral curve.

1.3 The quantum Sinh-Gordon model

In order to get a rigorous quantization process and a solution of the quantum Sinh-Gordon model, A.Bytsko and J.Teschner made in [10] a lattice discretization of the model. Thereby they replaced the space by N lattice points x_n and the fields by $\varphi_n = \varphi(x_n)$, $\Pi_n = \Delta\Pi(x_n)$, where $\Delta = \frac{R}{N}$ is the lattice spacing. These fields get canonically quantized: They are considered as operators with commutation relations

$$[\varphi_n, \Pi_m] = 2\pi i \delta_{n,m}. \tag{1.7}$$

The commutation relations can be realized in the usual way on the Hilbert space $\mathcal{H} \equiv (L^2(\mathbb{R}))^{\otimes N}$.
In [10] Sklyanins separation of variables method (SOV) has been the basic tool for a solution of the lattice model. Thereby one introduces a new set of variables, which allow for a reduction of the infinite dimensional problem to an infinite set of decoupled one-dimensional ones.
A quantity that plays a key role in the SOV approach is the so called Q-function. In the setting of separated variables it plays the role of an one-dimensional wave function, as the wave function of the model can be written in the form

$$\langle y_1, y_2, \dots |\Psi\rangle = \prod_{j=1}^{N} Q(y_j) \tag{1.8}$$

where the $y_1, y_2, \dots$ are the separated variables and $|\Psi\rangle$ is an eigenstate of the model. The Q-function is a generating function of all the conserved integrals; it also encodes the energy eigenvalues of the theory. The Q-function is an analytic function in a certain stripe of the complex plane. Its asymptotics is fixed, so its only freedom is the number and the location of the zeros in the stripe. The position of these so called Bethe roots is not completely arbitrary; they have to satisfy a nonlinear system of equations and are parametrized by a set of integers, the so called Bethe numbers. All in all, the Bethe numbers determine an eigenstate completely.
In [byte06] they characterized the Q-function as a solution of a functional equation, Baxter's equation, with specified analytic properties. Thereby they found a representation of $Q(\lambda)$ via certain nonlinear integral equations. On the level of these integral equations it is possible to perform a continous limit; thus one arrives at the solution of the continuous Sinh-Gordon model. In the continuum Baxter's equation is given as

$$Q(\lambda)T(\lambda) = Q(q\lambda) + Q(q^{-1}\lambda). \tag{1.9}$$

Here $T(\lambda)$ is the trace of the quantum monodromy matrix; $q = e^{i\pi b^2}$ and b^2 corresponds to the Planck constant $\hbar$.

1.4 Results

This thesis is concerned with the semiclassical regime of the Sinh-Gordon model. On the one hand side we investigated the semiclassical limit of the full quantum theory. Here the main idea was that there is a classical limit, where the zeros of the Q-function condense into a finite number of real intervals. These intervals are then taken as the cuts of a Riemann surface, the classical spectral curve. We identified sets of Bethe numbers, whose corresponding Bethe roots are expected to condense in the limit that their number M goes to infinity and $M\hbar = const.$ Unfortunately, we were not able to proof the condensation of the Bethe roots rigorously, but we gave strong hints for the correctness of the statement[3]. Based on this conjecture we calculated the leading and the next to leading order of several quantities like the Q-function, the Bethe ansatz equation and the quantum eigenvalue of the monodromy matrix Λ_q. We could show, that the leading order of Λ_q indeed coincides with the classical eigenvalue of the monodromy matrix.

On the other hand we investigated the semiclassical quantization of the model. The aim was the determination of the semiclassical expansion of the Q-function, starting from the classical theory. Thereby each order of Q is assumed to live on the spectral curve.
In a first step we quantized classical separated variables on the spectral curve and found Baxter's equation (1.9). Having that we could easily calculate the leading order Q_0 of the Q-function as a solution of (1.9). We found, that Q_0, in order to be single valued has to satisfy the Bohr-Sommerfeld conditions

$$\oint_{a_j} d\log Q_0 = 2\pi i N_j \text{ for } 1 \leq j \leq n, N_j \in \mathbb{Z}. \tag{1.10}$$

Here the cycle a_j encircles the j^{th} cut of the spectral curve. The Bohr-Sommerfeld conditions are indeed quantization conditions of the spectral curve.
Baxter's equation allowed us to derive a functional equation for each order in $\hbar$, which should be satisfied by the corresponding order of Q. It is known that Q-functions of continuous field theories have certain analytical properties. Thus we investigated, how far the functional equations determine the semiclassical expansion of the Q-function if we demand it to have these analytical properties. We studied in particular the next to leading order and calculated the corresponding

[3]It is not too surprising that this proof is so hard; the statement that the Bethe roots condense in this limit is equivalent to the statement that the spectrum of the quantum theory becomes dense in the classical limit. In the whole physics, such a statement could be shown rigogously only for really simple models.

order of the Q-function, Q_1. Our result coincides with the one from the semiclassical limit, except that in the semiclassical quantization approach Q_1 is not completely fixed. An undetermined zero mode of the functional equation remains, which takes on a concrete value in the case of the semiclassical limit. In the concluding chapter we will make a proposal how to fix this zero mode. Apart from this zero mode, Q_1 is well defined without the need for further Bohr-Sommerfeld conditions.

1.5 Outline of the content

In this section we will give a short survey of the content.

Chapter 2: In this chapter we will introduce the necessary mathematical background, which will be needed for the understanding of the thesis. Thereby the focus lies on certain aspects of Riemann surface theory.

Chapter 3: Here we will describe the classical theory in more detail. Of special importance in the following will be the section about the eigenvalue $\Lambda(\lambda)$. There we analyze the analytical behaviour and determine an explicit form of $\Lambda(\lambda)$.

Chapter 4: This chapter is concerned with the semiclassical quantization of the model. To this aim we quantized the separated variables on the classical spectral curve and got as a result Baxter's equation. Baxter's equation, and in addition the demand that the Q-function exhibits certain analytical properties, allowed for a calculation of the leading order of the Q-function, and of the derivation of a functional equation for each higher order. It turned out, that in order to have a well defined leading order Q-function, one has to quantize the spectral curve by introducing Bohr-Sommerfeld quantization conditions.
We solved the next to leading order functional equation explicitely, using two different ways. Thereby, it turned out that the next to leading order of the Q-function, $Q_1(\lambda)$, could be calculated up to one zero mode of the functional equation, which couldn't be fixed at this stage. Apart from this, $Q_1(\lambda)$ is well defined without the need for further Bohr-Sommerfeld conditions.

Chapter 5: This chapter contains a description of the rigorous quantization of the Sinh-Gordon model. First, the model on the lattice gets quantized. Having that, one can take the continuous limit and ends up with a quantized continuous Sinh-Gordon model.

Chapter 6: This chapter is devoted to the semiclassical limit. The main idea is that there is a classical limit, where the roots of the Q-function condense into a finite number of real intervals. These intervals are then reintepreted as the cuts of a Riemann surface, the classical spectral curve. In this chapter, we identify sets of Bethe numbers, whose corresponding Bethe roots are expected to condense in the limit that their number M goes to infinity and $Mb^2 = const.$ We found strong hints in favour of this conjecture.
Based on this conjecture, we calculated the leading and next to leading order of several quantities, like the Bethe roots, the eigenvalue of the monodromy matrix and the Q-function. The results for the Q-function coincide with the ones from the semiclassical quantization, except that in this case the zero mode of the next to leading order takes on a concrete value.

Chapter 7: In the conclusions we first summarize the results of the preceding analysis. Then we will compare the results of the semiclassical limit to the ones of the semiclassical quantization.

Chapter 2

Mathematical basics

The aim of this section is to provide the necessary concepts of algebraic geometry, to the extend they are needed in this work. Further information and proofs can be found in [2].

Definition 2.0.1 *A Riemann surface is a connected, 1-dimensional complex manifold.*

Definition 2.0.2 *A Riemann surface that is homeomorphic to a sphere with g handles is called a closed Riemann surface of genus g.*

The important example for us are the hyperelliptic Riemann surfaces, which are defined by

$$\Sigma : \mu^2 = \prod_{j=1}^{2N+2} (\lambda - \lambda_j) \tag{2.1}$$

where $\lambda_i \neq \lambda_j \ \forall i \neq j$. The genus is N. The curve is a two sheeted covering of the Riemann sphere, i.e. there are two values of μ for each λ, depending on the sign of the root. At the so called branching points $\lambda_1, \ldots, \lambda_{2N+2}$, the two sheets of Σ touch each other. Close to a branching point λ_j, the curve is described by hyperelliptic coordinates

$$\mu = \sqrt{\lambda - \lambda_i}, \qquad \lambda \in \mathbb{C}. \tag{2.2}$$

This implies, that the differential $d\lambda = 2\mu d\mu$ has a simple zero at the branching point λ_i.
On Σ we introduce two families of closed curves $\{a_1, \ldots, a_N\}$ and $\{b_1, \ldots, b_N\}$, which form a basis for contour integration and are usually called 'canonical cycles'. We illustrate the general form and position of these cycles in figure 2.

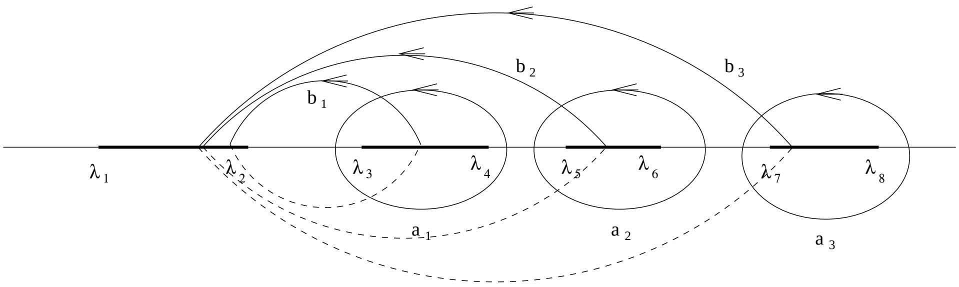

Figure 2.1: *A hyperelliptic Riemann surface of genus 3, canonical a- and b-cycles*

Definition 2.0.3 *A differential ω will be called holomorphic or abelian of the first kind if there exists a representation of the form*

$$w(\lambda) = f(\lambda) d\lambda \tag{2.3}$$

for every point $P = (\mu, \lambda) \in \Sigma$, where $f(\lambda)$ is an analytical function.

For example, the differentials

$$\omega_k(\lambda) = \frac{\lambda^{k-1}}{\sqrt{\prod_{j=1}^{2N+2}(\lambda - \lambda_j)}} d\lambda, \qquad k = 1, \ldots, N, \tag{2.4}$$

are holomorphic on the hyperelliptic curve (2.1).
Any holomorphic differential is closed. One defines the periods for the cycles $a_1, \ldots, a_N, b_1, \ldots, b_N$ of any closed differential ω as

$$A_i = \oint_{a_i} \omega, \qquad B_i = \oint_{b_i} \omega, \qquad i = 1, \ldots, N. \tag{2.5}$$

Proposition 2.0.1 *Let X be a compact Riemann surface of genus N. Then the space of holomorphic differentials is N-dimensional and, after a choice of the cycles $a_1, \ldots, a_N$, possesses a unique 'canonical' basis $\{\nu_1, \ldots, \nu_N\}$ with the property*

$$\oint_{a_k} \nu_j = \delta_{j,k}, \qquad \forall j, k = 1, \ldots, g. \tag{2.6}$$

One can express the new basis by the old one,

$$\nu_j = \sum_{k=1}^{N} C_{jk}\omega_k = \frac{\sum_{k=1}^{N} C_{jk}\lambda^{k-1}}{\sqrt{\prod_{l=1}^{2N+2}(\lambda - \lambda_l)}} d\lambda. \tag{2.7}$$

Further we will need the notion *abelian differential of the third kind.* A differential ω_{pq} is called abelian of the third kind if it is holomorphic except for the points p, q, where it has simple poles with residues $+1, -1$, respectively. In the case that all a-periods are vanishing, this differential is called normalized. For any points p, q on the hyperelliptic Riemann surface, the normalized abelian differential of the third kind, $\omega_{p,q}$, is uniquely determined.
For the canonical basis $\{\nu_1, \ldots, \nu_N\}$ of holomorphic differentials, we define the period matrix $B = (B_{ij})$ by

$$B_{ij} = \oint_{b_i} \nu_j. \tag{2.8}$$

The matrix B is symmetric and has a positive definite imaginary part. B allows for the definition of the Jacobi variety of the curve Σ:

Definition 2.0.4 *The abelian torus*

$$J(\Sigma) = \mathbb{C}^N / \{2\pi i \mathbb{Z}^N + 2\pi i B \mathbb{Z}^N\} \tag{2.9}$$

is called the Jacobi variety of Σ.

A mapping from Σ to $J(\Sigma)$ is given by the Abel map $\mathbf{A} : \lambda \to \mathbf{A}(\lambda)$ where $\mathbf{A}(\lambda) = (A_1(\lambda), \ldots, A_N(\lambda))$ with

$$A_j(\lambda) = 2\pi i \int_{\lambda_0}^{\lambda} \nu_j. \tag{2.10}$$

Here λ_0 is a fixed point on Σ; the path of integration from λ_0 to λ is chosen to be the same for all j. If another path is chosen, one had to add $\oint_\gamma \nu_j$ to the integral, where γ is a closed contour. Thanks to the special periods of the ν_j, this additional term is of the form

$$\oint_\gamma \nu_j = 2\pi i n_j + 2\pi i \sum_k B_{jk} m_k, \tag{2.11}$$

where n_j, $m_k \in \mathbb{Z}$. Thus the Abel map is well defined.
Now it is possible to construct the N-dimensional Θ-function of the Riemann surface,

$$\Theta(\lambda; B) = \sum_{\mathbf{k}\in\mathbb{Z}^N} \exp\left(i\pi(B\mathbf{k},\mathbf{k}) + 2\pi i(\mathbf{A}(\lambda),\mathbf{k})\right). \tag{2.12}$$

An important classical mathematical question was the problem of inverting the Abel map. This so called Jacobi inversion problem can be stated as follows: Find N points $P_1, \dots, P_N \in \Sigma$ such that

$$\sum_{k=1}^{N} \int_{P_0}^{P_k} \nu_j = \zeta_j \tag{2.13}$$

for a fixed point $\zeta = (\zeta_1, \dots, \zeta_N) \in J(\Sigma)$.
This problem can be solved in terms of zeros of the Θ-function, the concrete form will not be of interest in this work. Altogether, one has a map from the Riemann surface to its Jacobi variety, the Abel map, and its inverse, which maps the Jacobi variety to the Riemann surface.

Chapter 3

The classical Sinh-Gordon theory

In this section we will give an overview of the classical solution theory of the Sinh-Gordon model with periodic boundary conditions. The information has been taken out of [3], [4], [5]. Further information and proofs can be found in the literature specified above.

3.1 Definition of the model

The classical Sinh-Gordon model is a dynamical system whose degrees of freedom are encoded in the field $\varphi(x,t)$. The field $\varphi(x,t)$ is defined for
$(x,t) \in [0,R] \times \mathbb{R}$ with periodic boundary conditions $\varphi(x+R,t) = \varphi(x,t)$. The Hamiltonian is given as

$$H = \int_0^R \frac{dx}{4\pi} \left(\Pi^2(x,t) + (\partial_x \varphi(x,t)^2) + 8\pi\mu \cosh(2b\varphi(x,t))\right). \tag{3.1}$$

The dynamics of the system can be described in Hamiltonian form with variables $\varphi(x,t)$, $\Pi(x,t)$ and Poisson brackets

$$\{\Pi(x,t), \varphi(x',t)\} = 2\pi\delta(x-x'). \tag{3.2}$$

The time-evolution of an arbitrary observable $O(t)$ is then given as

$$\partial_t O(t) = \{H, O(t)\}. \tag{3.3}$$

As equation of motion we find

$$\begin{cases} \left(\partial_t^2 - \partial_x^2\right)\varphi(x,t) = -8\pi\mu b \sinh(2b\varphi(x,t)) \\ \varphi(x+R,t) = \varphi(x,t) \end{cases}. \tag{3.4}$$

3.2 Auxiliar linear problem

To solve all soliton equations one uses an approach called the inverse scattering method. Thereby one adjoints to the nonlinear partial differential equation a linear eigenvalue problem, whose potential is given by a solution of the soliton equation. In the Sinh-Gordon case the linear eigenvalue problem is given as

$$\begin{cases} \partial_x \Psi(x,t;\lambda) = U(x,t;\lambda)\Psi(x,t,\lambda), \\ \partial_t \Psi(x,t;\lambda) = V(x,t;\lambda)\Psi(x,t;\lambda), \end{cases} \tag{3.5}$$

where

$$U(x,t;\lambda) = \begin{pmatrix} \frac{b}{2}\left(\dot\varphi + \varphi'\right) & -\frac{m}{4\lambda}\exp(2b\varphi(x,t)) + \frac{m}{4}\lambda \\ \frac{m}{4\lambda}\exp(-2b\varphi(x,t)) - \frac{m}{4}\lambda & -\frac{b}{2}\left(\dot\varphi + \varphi'\right) \end{pmatrix} \tag{3.6}$$

and

$$V(x,t;\lambda) = \begin{pmatrix} \frac{b}{2}\left(\dot\varphi + \varphi'\right) & +\frac{m}{4\lambda}\exp(2b\varphi(x,t)) + \frac{m}{4}\lambda \\ -\frac{m}{4\lambda}\exp(-2b\varphi(x,t)) - \frac{m}{4}\lambda & -\frac{b}{2}\left(\dot\varphi + \varphi'\right) \end{pmatrix}. \tag{3.7}$$

λ is an auxiliary parameter, also called spectral parameter; $m = 4b\sqrt{\pi\mu}$ plays the role of a mass parameter.
The Sinh-Gordon equation is encoded in system (3.5) as a compatibility condition, i.e.

$$\partial_t \partial_x \Psi(x,t;\lambda) = \partial_x \partial_t \Psi(x,t;\lambda) \tag{3.8}$$

holds only for a solution $\Psi(x,t;\lambda)$ of (3.5) if $\varphi(x,t)$ solves the Sinh-Gordon equation.
Note, that we can rewrite the first equation of (3.5) as a four dimensional system, where λ plays the role of the spectral parameter,

$$Qf = \lambda f. \tag{3.9}$$

Therefore we introduce the following notation:

$$A = -\frac{2b}{m} \begin{pmatrix} 0 & (\dot{\varphi} + \varphi') \\ (\dot{\varphi} + \varphi') & 0 \end{pmatrix}, \tag{3.10}$$

$$B = \begin{pmatrix} \exp(-b\varphi) & 0 \\ 0 & \exp(b\varphi) \end{pmatrix}, \tag{3.11}$$

$$J = \frac{4}{m} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.12}$$

Now we set

$$Q = \begin{pmatrix} -J & 0 \\ 0 & 0 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} A & B \\ B & 0 \end{pmatrix}, \tag{3.13}$$

$$f = \begin{pmatrix} \Psi \\ \Phi \end{pmatrix}. \tag{3.14}$$

Inserting this into (3.9), we find it indeed equivalent to the first equation of (3.5),

$$Qf = \begin{pmatrix} -J\frac{\partial}{\partial x}\Psi + A\Psi + B\Phi \\ B\Psi \end{pmatrix} = \begin{pmatrix} \lambda\Psi \\ \lambda\Phi \end{pmatrix}. \tag{3.15}$$

So for all $\lambda \neq 0$ it holds that $\Phi = \frac{B}{\lambda}\Psi$, which directly implies the first equation of (3.5). If $\lambda = 0$, then $\Psi = 0$, which leads to $\Phi = 0$, i.e. there are no additional solutions in this case.
It is an important feature of (3.9) that the spectrum of Q does not depend on the time if and only if $\varphi(x,t)$ solves the Sinh-Gordon equation.
The idea is now to do some spectral theory for Q in order to gain general information about its spectrum. Then one will start from a fixed spectrum combined with some initial conditions and calculate $\varphi(x,t)$ by an inverse transformation.
First of all we have to specify the spectral problem, i.e. we have to choose a class of functions in that we want to solve (3.9). We choose the class of quasi-periodic functions $f : [0, L) \to \mathbb{C}^4$ with $f(x+L) = \Lambda f(x)$ for $0 \leq x < L$ and fixed multiplier $\Lambda \neq 0, \infty$.

3.3 Zero curvature condition

Condition (3.8) is equivalent to the famous zero curvature condition,

$$\partial_t U(x,t;\lambda) - \partial_x V(x,t;\lambda) + [U(x,t;\lambda), V(x,t;\lambda)] = 0. \tag{3.16}$$

The equations (3.5) and (3.16) have a natural geometric interpretation: The matrix functions $U(x,t;\lambda)$ and $V(x,t;\lambda)$ can be regarded as local connection coefficients in the trivial vector bundle

$\mathbb{R}^2 \times \mathbb{C}^2$. Here space-time $\mathbb{R}^2$ is the base; $\mathbb{C}^2$ is the fiber where the vector function $\Psi(x,t;\lambda)$ takes values on. The equations (3.5) imply that $\Psi(x,t;\lambda)$ is a covariantly constant vector, while (3.16) shows that the (U,V)-connection has zero curvature.
Now regard the 2×2 elementary solution $M_{x_0}(x,t_0;\lambda)$ of

$$\partial_x\Psi(x,t;\lambda) = U(x,t;\lambda)\Psi(x,t,\lambda) \tag{3.17}$$

with $M_{x_0}(x_0,t_0;\lambda) = \mathbb{1}$, $0 \le x_0 < L, x_0 \le x < x_0 + L$. The matrix $M_{x_0}(x,t_0;\lambda)$ can be written as path ordered exponential,

$$M_{x_0}(x,t_0;\lambda) = \mathcal{P}\exp\left(\int_{x_0}^{x} U(x',t_0;\lambda)dx'\right). \tag{3.18}$$

It is the matrix of parallel transport along the contour $t = t_0$, $x_0 \le x' \le x$, namely it transfers the solutions of (3.5) across this contour,

$$M_{x_0}(x,t_0;\lambda)\Psi(x_0,t_0;\lambda) = \Psi(x,t_0;\lambda). \tag{3.19}$$

Setting $x = x_0 + L$, we get the so called monodromy matrix $M_{x_0}(t_0,\lambda)$. We will denote the entries of the monodromy matrix as

$$M_{x_0}(t_0,\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda)) \end{pmatrix}. \tag{3.20}$$

Monodromy matrices for different values of t_0 or x_0 are conjugate to each other as can easily be shown with the zero curvature condition. Thus the trace
$T(\lambda) = A(\lambda) + D(\lambda)$ of $M_{x_0}(t_0,\lambda)$ depends neither on t_0 nor on x_0, which means that $T(\lambda)$ is a generating function for the integrals of the motion of the Sinh-Gordon model. As $U(x,t;\lambda)$ is analytical except for the points $\lambda = 0$ and $\lambda = \infty$, where it has simple poles, $T(\lambda)$ is analytical at $\mathbb{C}\setminus\{0,\infty\}$ and develops essentiel singularities at $\lambda = 0,\infty$. An asymptotic expansion of $T(\lambda)$ around these points gives two sets $\{I_k\}_k$, $\{J_k\}_k$, of local integrals of the motion,

$$\log T(\lambda) \underset{\lambda\to 0}{\sim} \frac{1}{\lambda} + \sum_{k=1}^{\infty}\lambda^k J_k, \qquad \log T(\lambda) \underset{\lambda\to\infty}{\sim} \lambda + \sum_{k=1}^{\infty}\frac{1}{\lambda^k}I_k. \tag{3.21}$$

It holds that $T \equiv T(\lambda^2)$.

3.4 Spectral curve

If Ψ is an eigenvector of $M_x(t,\lambda)$ with eigenvalue Λ, $M_x(t,\lambda)\Psi = \Lambda\Psi$, then

$$f(x,t;\lambda) = \begin{pmatrix} \Psi \\ \frac{B}{\lambda}\Psi \end{pmatrix} \tag{3.22}$$

will be a solution to the spectral problem for Q with spectral parameter λ and multiplier Λ,

$$Qf = \lambda f, \qquad f(x+L) = \Lambda f(x). \tag{3.23}$$

As $U(x,t;\lambda)$ is traceless it holds that $det\, M_x(t,\lambda) = 1$. So we find its eigenvalues $\Lambda_i(\lambda)$, $i = 1,2$ satisfying

$$\det\left(\Lambda_i(\lambda) - M_x(t,\lambda)\right) = \Lambda_i^2(\lambda) - T(\lambda)\Lambda_i(\lambda) + 1 = 0. \tag{3.24}$$

It follows, that λ belongs to the spectrum of Q with the multiplier $\Lambda(\lambda)$ if and only if $\Lambda(\lambda)$ solves (3.24).
It is convenient to define the discriminant $\Delta = \frac{T^2}{4} - 1$. In [McKean], it is shown that the zeros of

Δ are of multiplicity 1 or 2. Further it turned out to be useful to introduce the Riemann surface Σ, where $\Lambda(\lambda)$ is single valued. This so called spectral curve can be defined by

$$\Sigma : \mu^2 = \Delta(\lambda), \tag{3.25}$$

with points $\mathcal{P} = \left(\lambda^2, \sqrt{\Delta}\right)$. Σ is a two sheeted curve over the punctured plane $\mathbb{C} \setminus \{0, \infty\}$. It has simple ramification points at the simple roots of $\frac{T^2(\lambda^2)}{4} = 1$ and at $0, \infty$. We restrict ourself to Riemann surfaces of finite genus, i.e. we allow only for finitely many simple zeros of $\frac{T^2(\lambda^2)}{4} - 1$. At the ramifications $T(\lambda) = \pm 2$, i.e. $\Lambda(\lambda) = \pm 1$, which implies that the branching points correspond to periodic or antiperiodic solutions of the spectral problem. As Q is real and symmetric all the ramification points are real and positive, (we are working on the λ^2-plane). Thus we can order them, $0 < \lambda_1^2 < \lambda_2^2 < \cdots < \lambda_{2n}^2 < \infty$.
We represent the surface Σ as a double cover of the complex λ^2-plane, where the two sheets are connected along several cuts. One cut runs from $-\infty$ to 0 along the negative real axis, the other cuts run from λ_{2j-1}^2 to λ_{2j}^2, $j = 1, \ldots, n$, along the positive real axis, see figure 3.4.

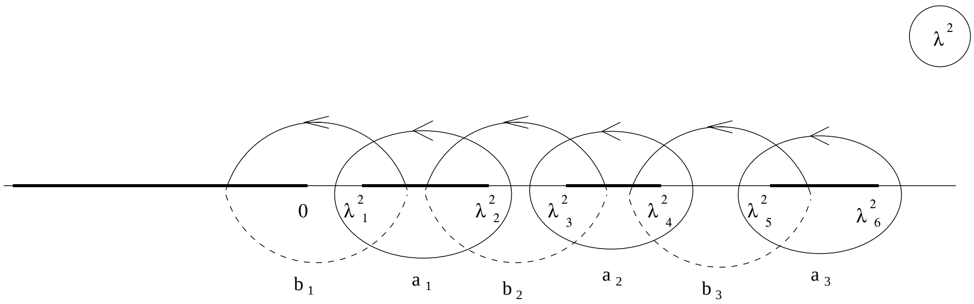

Figure 3.1: *The spectral curve for $n = 3$, represented in the λ^2-plane. Here we chose another set of b-cycles.*

Another convenient representation of Σ is on the λ-plane. Eliminating the double information due to $T = T(\lambda^2)$, we take the 'square root of the λ^2-plane'. Thereby the upper and lower sheet are mapped to the right and left half-plane of the λ-plane, respectively. The λ-plane is cut along the intervals $[-\lambda_{2n}, -\lambda_{2n-1}], \ldots, [\lambda_{2n-1}, \lambda_{2n}]$, where the upper and lower sides of the interval $[-\lambda_{2j}, -\lambda_{2j-1}]$ are identified with the upper and lower sides of the interval $[\lambda_{2j-1}, \lambda_{2j}]$, respectively, for all $j = 1, \ldots, n$.

3.5 The eigenvalue $\Lambda(\lambda)$

The eigenvalues of the monodromy matrix will play an important role in the following, so it seems reasonable to say a little bit more about them.
The eigenvalues $\Lambda_i(\lambda)$, $i = 1, 2$, satisfy equation (3.24); a solution that is analytic away from the cuts is

$$\Lambda_i(\lambda^2) \equiv \frac{1}{2}\left(T(\lambda^2) \pm i \prod_{r \in \mathbb{Z}} (\lambda^2 - \lambda_r^2) \sqrt{\prod_{j=1}^{n} (\lambda^2 - \lambda_{+,j}^2)(\lambda^2 - \lambda_{-,j}^2)}\right). \tag{3.26}$$

Here we introduced the notation

$$\Delta(\lambda^2) = 4 - T^2(\lambda^2) = \prod_{r \in \mathbb{Z}} (\lambda^2 - \lambda_r^2)^2 \prod_{j=1}^{n} (\lambda^2 - \lambda_{+,j}^2)(\lambda^2 - \lambda_{-,j}^2). \tag{3.27}$$

The square root is defined as follows: If we approach a cut from the left and continue on its upper side, we will pick a phase $-i$. If we continue on the upper side to its end and further, we will pick another phase $-i$. If we do the same on the lower side, we will pick phases $+i$ instead.
Note, that $\Lambda_1(\lambda^2)$ and $\Lambda_2(\lambda^2)$ are inverse to each other, as $det\, M_x(t,\lambda) = 1$.

Proposition 3.5.1 *The solutions $\Lambda_{\frac{1}{2}}(\lambda)$ in (3.26) are analytic away from the cuts. The $\Im \log \Lambda_1(\lambda)$ is monotonously increasing from $-\infty$ to ∞ as λ^2 is increasing from 0 to ∞, where $\Im \log \Lambda_2(\lambda)$ is monotonously decreasing from ∞ to $-\infty$ as λ^2 is increasing from 0 to ∞, or vice versa. The $\Im \log \Lambda_i(\lambda)$, $i = 1,2$, is a constant multiple of π on the cuts $\mathbb{F}_r = [\lambda^2_{+,r}, \lambda^2_{-,r+1}]$, respectivley. The values on the cuts $\mathbb{F}_r$ and $\mathbb{F}_{r+1}$ differ by π times the number of zeros of $T(\lambda)$ in the interval $[\lambda^2_{+,r}, \lambda^2_{-,r+1}]$, disregarding the multiplicities of the zeros.*

Proof It is clear that $\Lambda_1(\lambda), \Lambda_2(\lambda)$ are analytic, except for the cuts $\mathbb{F}_r$. It is also clear that $\Im \Lambda_{\frac{1}{2}}(\lambda)$ is zero on the cuts, i.e. $\Im \log \Lambda_{\frac{1}{2}}$ is a constant multiple of π on the cuts. We calculate for $\lambda \in \mathbb{R} \setminus \cup_{r=1}^n \mathbb{F}_r$

$$\frac{d}{d\lambda^2} Im \log \Lambda_{\frac{1}{2}}(\lambda^2) = \mp \frac{T'(\lambda^2)}{\prod_{r\in\mathbb{Z}}(\lambda^2-\lambda_r^2)\sqrt{\prod_{j=1}^n(\lambda^2-\lambda^2_{+,j})(\lambda^2-\lambda^2_{-,j})}}. \tag{3.28}$$

$T'(\lambda^2)$ has its simple zeros outside the cuts exactly at λ_r^2, $r \in \mathbb{Z}$. At $\lambda_{-,j}$ its sign is opposite to the one at $\lambda_{+,r}$, the same is true for
$\sqrt{\prod_{j=1}^n(\lambda^2-\lambda^2_{+,j})(\lambda^2-\lambda^2_{-,j})}$. So we find that $\Im \log \Lambda_{\frac{1}{2}}(\lambda^2)$ is monotonous on $\lambda \in \mathbb{R} \setminus \cup_{r=1}^n \mathbb{F}_r$.
Outside the cuts there are alternatingly zeros of $T(\lambda)$ and zeros of $\Im\Lambda(\lambda)$, which implies the last assertion.
Note that the last argument depends on the fact that $T^2(\lambda) - 4 = 0$ has only real solutions. This follows of the self-adjointness of the auxiliary linear problem. ∎

3.5.1 The explicit form

__Vacuum case:__
We will assume that the fields $\varphi(x,t) \equiv 0$ and $\Pi(x,t) \equiv 0$. In this case it is possible to calculate T and Λ explicitely.
The auxiliar linear problem is then given as

$$\left\{ \frac{d}{dx} + \begin{pmatrix} 0 & \frac{m}{4}\left(\frac{1}{\lambda}-\lambda\right) \\ -\frac{m}{4}\left(\frac{1}{\lambda}-\lambda\right) & 0 \end{pmatrix} \right\} \Psi(x,t) = 0. \tag{3.29}$$

This implies the monodromy matrix

$$M(R) = \begin{pmatrix} \cos\left(R\frac{m}{4}\left(\frac{1}{\lambda}-\lambda\right)\right) & \sin\left(R\frac{m}{4}\left(\frac{1}{\lambda}-\lambda\right)\right) \\ -\sin\left(R\frac{m}{4}\left(\frac{1}{\lambda}-\lambda\right)\right) & \cos\left(R\frac{m}{4}\left(\frac{1}{\lambda}-\lambda\right)\right) \end{pmatrix}. \tag{3.30}$$

Taking the trace we find

$$T(\lambda) = 2\cos\left(R\frac{m}{4}\left(\frac{1}{\lambda}-\lambda\right)\right), \tag{3.31}$$

or as function of λ^2

$$T(\lambda^2) = 2\cos\left(R\frac{m}{4}\left(\frac{1}{\sqrt{\lambda^2}}-\sqrt{\lambda^2}\right)\right). \tag{3.32}$$

As the cosine is an even function $T(\lambda^2)$ is single valued on the λ^2 plane. It develops essential singularities at $\lambda^2 = 0, \infty$. For $\Lambda_i(\lambda^2)$, $i = 1,2$, we find

$$\Lambda_i(\lambda^2) = \exp\left\{\pm i R\frac{m}{4}\left(\frac{1}{\sqrt{\lambda^2}}-\sqrt{\lambda^2}\right)\right\}. \tag{3.33}$$

Thus the $\Lambda_i(\lambda^2)$ are singlevalued on a double covering of the λ^2-plane, where both sheets are connected along the cut $(-\infty, 0)$.

General case:

In the general case it will not be possible to evaluate the path ordered exponential in order to calculate the monodromy matrix. But it is possible to determine $d\log\Lambda$ as a differential with specified properties. For that we only have to know its pole structure and its A-periods. As $\Lambda(\lambda)$ has neither poles nor zeros the differential $d\log\Lambda$ has its poles only at 0 and ∞. The character of the poles can be read out of the vacuum case: From (3.33) we calculate

$$d\log\Lambda_i^{vac} = \mp i\frac{Rm}{8}\left(\frac{1}{\sqrt{\lambda^2}^3} + \frac{1}{\sqrt{\lambda^2}}\right)d\lambda^2. \tag{3.34}$$

At 0, the local parameter is $\gamma = \sqrt{\lambda^2}$. There the differential takes the form

$$d\log\Lambda_i^{vac} = \mp i\frac{Rm}{4}\left(\frac{1}{\gamma^2} + 1\right)d\gamma. \tag{3.35}$$

At ∞ the local parameter is $\gamma = \frac{1}{\sqrt{\lambda^2}}$, the differential looks like

$$d\log\Lambda_i^{vac} = \pm i\frac{Rm}{4}\left(1 + \frac{1}{\gamma^2}\right)d\gamma. \tag{3.36}$$

Thus $d\log\Lambda_i$, $i = 1, 2$, should have double poles at 0 and ∞, with prefactors $\mp i\frac{Rm}{4}$ and $\pm i\frac{Rm}{4}$, respectively.

As $\Lambda(\lambda^2)$ is either real positive or real negative along the cuts it holds for $j = 1, \dots, n$ that

$$\oint_{a_j} d\log\Lambda_i = 0 \qquad \text{for } i = 1, 2. \tag{3.37}$$

Using this we find

$$d\log\Lambda_{\frac{1}{2}} = \mp i\frac{Rm}{8}\left(\frac{\prod_{j=1}^{2n}\lambda_j}{\sqrt{\lambda^2}^3} + \frac{\lambda^{2n}}{\sqrt{\lambda^2}}\right)\frac{d\lambda^2}{\sqrt{\prod_{j=1}^{2n}(\lambda^2 - \lambda_j^2)}} + \sum_{k=1}^{n} c_k\nu_k, \tag{3.38}$$

where the ν_k, $k = 1, \dots, n$ are the normalized holomorphic differentials (2.6). The parameters c_k can be determined numerically by condition (3.37).

The explicit representation of $d\log\Lambda$ allows for the calculation of $T(\lambda)$. This implies that all the information of the spectral curve Σ is already contained in the reduced spectral curve Σ', defined by

$$\Sigma' : \mu^2 = \prod_{j=1}^{n}(\lambda^2 - \lambda_{+,j}^2)(\lambda^2 - \lambda_{-,j}^2). \tag{3.39}$$

As a matter of fact, the specification of the branching points determines the spectral curve Σ uniquely.

3.6 Moduli space of the spectral curves

Above we have shown that the spectral curve is characterized by the $2n$ branching points $\lambda_{+,j}$ and $\lambda_{-,j}$, $j = 1, \dots, n$. These $2n$ parameters are not all independent as the Riemann surface is constrained by the demand that $\Lambda(\lambda)$ is single-valued, i.e.

$$\oint_{b_j} d\log\Lambda(\lambda) = 2\pi i M_j, \qquad M_j \in \mathbb{Z},\ j = 1, \dots, n-1. \tag{3.40}$$

The number M_j is the number of zeros of $T(\lambda)$ that are encircled by the cycle b_j. Further we have seen that

$$\mathcal{I}m \log \Lambda(\lambda) \in i\pi\mathbb{Z} \qquad \forall \lambda \in \mathbb{I}. \tag{3.41}$$

We fix the branch of $\log \Lambda$ by requiring that

$$\log \Lambda(\lambda) = -\frac{imR}{4}\left[\lambda - \frac{1}{\lambda}\right] + 2i\lambda \int_0^\infty \frac{\rho(\gamma)}{\lambda^2 - \gamma^2} d\gamma, \tag{3.42}$$

holds where the density $\rho(\lambda)$ is given as

$$\rho(\lambda) = \frac{\mathcal{R}e \log \Lambda(\lambda)}{\pi}. \tag{3.43}$$

A short cross-check shows that this $d\log\Lambda$ is indeed the same as (3.38), since it has the correct asymptotics, no poles apart from 0 and ∞, and vanishing a-periods.
Now we are ready to parametrize the moduli space of the finite zone solutions by the following $2n$ parameters:

1. the value of $\mathcal{I}m \log \Lambda(\lambda)$ on I_1,
2. $\oint_{b_j} d\log\Lambda(\lambda) = 2\pi i M_j$, $M_j \in \mathbb{Z}$, $j = 1, \ldots, n-1$ and
3. the n filling fractions $\varepsilon_j = \int_{I_j} \rho(\lambda)\frac{d\lambda}{\lambda} = \frac{1}{2\pi i}\oint_{a_j} \log\Lambda(\lambda)\frac{d\lambda}{\lambda} \in \mathbb{R}$.

3.7 The roots of Δ

We have mentioned before that the roots of Δ are of multiplicity 1 or 2.
We will need to know the general disposition of the zeros of $\Delta = \frac{\left(\Lambda(\lambda) - \Lambda^{-1}(\lambda)\right)^2}{4}$. Therefore we examine the vacuum where

$$\Delta = \sin^2\left(\frac{mR}{4}\left[\lambda - \frac{1}{\lambda}\right]\right). \tag{3.44}$$

Its zeros can be found at $\frac{mR}{4}\left[\lambda - \frac{1}{\lambda}\right] = \pi k$, $k \in \mathbb{Z}$, i.e. at $\lambda = \frac{2\pi k}{mR} \pm \sqrt{\left(\frac{2\pi k}{mR}\right)^2 + 1}$. Thus the zeros are coalescing at 0 and ∞ like

$$\lambda_k \sim \begin{cases} \frac{4\pi k}{mR}, & \lambda_k \to \pm\infty, \\ \\ \frac{mR}{4\pi k}, & \lambda_k \to \pm 0. \end{cases} \tag{3.45}$$

Now fix nontrivial $(\varphi(x,t), \Pi(x,t)) \in \mathcal{C}_1^\infty \times \mathcal{C}_1^\infty$. It has been shown in [4] that along the segment $(\varepsilon\varphi, \varepsilon\Pi)$, $0 \le \varepsilon \le 1$, the roots of Δ keep their multiplicity. This means that a double root λ_k^0 of the vacuum becomes a double root λ_k^1 of the excited state, or it splits into a pair of simple roots λ_k^-, λ_k^+. It further has been shown in [4] that the appraisals $\Delta \sim \sin^2\left(\frac{mR}{4}\frac{1}{\lambda}\right)$ and $\Delta \sim \sin^2\left(\frac{mR}{4}\lambda\right)$ remain valid for the excited states near $\lambda = 0$ and $\lambda = \infty$, respectively.

3.8 Fundamental Poisson Brackets and the r-Matrix

We suppose that $\varphi(x), \Pi(x)$ are defined on the interval $[0, L]$. We regard compactly supported functionals, i.e. functionals, that depend only on $\varphi(x), \Pi(x)$ for x inside the interval. For these functionals the poisson brackets are defined as

$$\{\mathcal{F}, \mathcal{G}\} = -2\pi \int_0^L \left(\frac{\delta\mathcal{F}}{\delta\varphi(x)}\frac{\delta\mathcal{G}}{\delta\Pi(x)} - \frac{\delta\mathcal{F}}{\delta\Pi(x)}\frac{\delta\mathcal{G}}{\delta\varphi(x)}\right) dx. \tag{3.46}$$

This directly leads to the basic Poisson brackets

$$\{\Pi(x,t),\varphi(y,t)\} = 2\pi\delta(x-y), \qquad \{\varphi(x,t),\varphi(y,t)\} = \{\Pi(x,t),\Pi(y,t)\} = 0. \tag{3.47}$$

Now we define $x' = \frac{1}{2}(t-x)$, $t' = \frac{1}{2}(t+x)$ and

$$\widetilde{U}(x',t';\tilde{\lambda}) = -RU(x'+t',t'-x';\lambda)R, \tag{3.48}$$

with

$$R = \begin{pmatrix} 0 & e^{\frac{b}{2}\varphi} \\ e^{-\frac{b}{2}\varphi} & 0 \end{pmatrix}. \tag{3.49}$$

This transformation is a kind of gauge transformation as it doesn't change the trace of the monodromy matrix and hence keeps the spectral curve unchanged.
Explicitely,

$$\widetilde{U}(x',t';\tilde{\lambda}) = \begin{pmatrix} -\frac{b}{2}\Pi(x',t') & -m\tilde{\lambda}e^{-b\varphi(x',t')} + \frac{m}{\tilde{\lambda}}e^{b\varphi(x',t')} \\ m\tilde{\lambda}e^{b\varphi(x',t')} - \frac{m}{\tilde{\lambda}}e^{-b\varphi(x',t')} & \frac{b}{2}\Pi(x',t') \end{pmatrix}, \tag{3.50}$$

where $m = \sqrt{\pi b^2\mu}$ and $\tilde{\lambda} = \frac{\lambda}{m}$.
We compute $\{\widetilde{U}(x',\tilde{\lambda}) \overset{\otimes}{,} \widetilde{U}(y',\tilde{\mu})\}$, where

$$\begin{aligned} &\{\widetilde{U}(x',\tilde{\lambda}) \overset{\otimes}{,} \widetilde{U}(y',\tilde{\mu})\} \\ &\qquad = -2\pi\int_0^L \left(\frac{\delta\widetilde{U}(x',\tilde{\lambda})}{\delta\varphi(z)} \otimes \frac{\delta\widetilde{U}(y',\tilde{\mu})}{\delta\Pi(z)} - \frac{\delta\widetilde{U}(x',\tilde{\lambda})}{\delta\Pi(z)} \otimes \frac{\delta\widetilde{U}(y',\tilde{\mu})}{\delta\varphi(z)}\right) dz, \end{aligned} \tag{3.51}$$

and find

$$\begin{aligned} \{\widetilde{U}(x',t';\tilde{\lambda}) \overset{\otimes}{,} \widetilde{U}(y',t';\tilde{\mu})\} = \frac{mb^2}{8}\bigg\{ &\left(\tilde{\lambda}+\frac{1}{\tilde{\lambda}}\right)\cosh b\varphi(x',t')\cdot\sigma_1\otimes\sigma_3 \\ &-\left(\tilde{\lambda}-\frac{1}{\tilde{\lambda}}\right)\sinh b\varphi(x',t')\cdot i\sigma_2\otimes\sigma_3 \\ &-\left(\tilde{\mu}+\frac{1}{\tilde{\mu}}\right)\cosh b\varphi(x',t')\cdot\sigma_3\otimes\sigma_1 \\ &+\left(\tilde{\mu}-\frac{1}{\tilde{\mu}}\right)\sinh b\varphi(x',t')\cdot\sigma_3\otimes i\sigma_2\bigg\}\,\delta(x'-y'). \end{aligned} \tag{3.52}$$

A lengthy calculation shows, that the right hand side of (3.52) is of the form

$$(3.52) = [r(\tilde{\lambda},\tilde{\mu}), \widetilde{U}(x',\tilde{\lambda})\otimes I + I\otimes\widetilde{U}(x',\tilde{\mu})]\delta(x'-y'), \tag{3.53}$$

where

$$r(\tilde{\lambda},\tilde{\mu}) = \frac{b^2}{2\sinh\left(\log\frac{\tilde{\lambda}}{\tilde{\mu}}\right)} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cosh\left(\log\frac{\tilde{\lambda}}{\tilde{\mu}}\right) & -1 & 0 \\ 0 & -1 & \cosh\left(\log\frac{\tilde{\lambda}}{\tilde{\mu}}\right) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.54}$$

is the classical r-matrix. Altogether,

$$\{\widetilde{U}(x',\tilde{\lambda}) \overset{\otimes}{,} \widetilde{U}(y',\tilde{\mu})\} = [r(\tilde{\lambda},\tilde{\mu}), \widetilde{U}(x',\tilde{\lambda})\otimes I + I\otimes\widetilde{U}(x',\tilde{\mu})]\delta(x'-y') \tag{3.55}$$

are the fundamental Poisson brackets of the Sinh Gordon model. In [3] it has been shown that this yields the Poisson brackets for the monodromy matrix

$$\left\{\widetilde{M}_{x_0'}(t_0',\tilde{\lambda}), \widetilde{M}_{x_0'}(t_0',\tilde{\mu})\right\} = \left[r(\tilde{\lambda},\tilde{\mu}), \widetilde{M}_{x_0'}(t_0',\tilde{\lambda})\otimes\widetilde{M}_{x_0'}(t_0',\tilde{\mu})\right]. \tag{3.56}$$

3.9 Separation of variables

In order to describe the dynamics of the model, it turned out to be helpful to introduce a new set of variables, see for example [5], [3]. Therefore we regard the eigenvalues μ_j, $j \in \mathbb{Z}$, of Q, where the corresponding eigenfunction $f = (f_1, \dots, f_4)^T$ satisfies

$$f_1(L) = \Lambda(\mu_j) f_1(0) = 0. \tag{3.57}$$

This is equivalent to the condition

$$B(\mu_j) = 0, \tag{3.58}$$

scince

$$M_0(t, \mu_j) \begin{pmatrix} 0 \\ f_2(0) \end{pmatrix} = \begin{pmatrix} B(\mu_j) f_2(0) \\ D(\mu_j) f_2(0) \end{pmatrix}. \tag{3.59}$$

Thus the eigenfunction f has the multiplier $D(\mu_j) = \Lambda(\mu_j)$ if and only if $B(\mu_j) = 0$.
Self-adjointness of the operator Q implies that the roots of $B(\lambda)$ are simple and real. Moreover they are located on the points or intervalls where $T(\lambda) \geq 2$ because

$$1 = \det M_0(t, \mu_j) = A(\mu_j) D(\mu_j) \leq \frac{1}{2} |A(\mu_j) + D(\mu_j)| = \frac{1}{2} T(\mu_j). \tag{3.60}$$

For each zero μ_j of $B(\lambda)$ we define $f_j = -2 \log |D(\mu_j)|$. Using the fundamental Poisson brackets it could be shown that $\gamma_j = \log \mu_j^2$ and f_j are canonically conjugate variables $\forall j \in \mathbb{Z}$, i.e.

$$\{f_j, \gamma_k\} = \delta_{j,k}, \qquad \{f_j, f_k\} = \{\gamma_j, \gamma_k\} = 0 \qquad \forall j, k \tag{3.61}$$

with respect to the original Poisson bracket (3.46).
It also has been shown that the mapping between φ, Π and $\{\gamma_j, f_j\}_j$ is $1:1$ for any fixed spectral curve of finite genus. The reconstructed φ looks as follows,

$$\exp(\varphi(x,t)) = \prod_{j \in \mathbb{Z}} \frac{\mu_j^2(x,t)}{\mu_{j,0}^2} \qquad [4] \tag{3.62}$$

where the $\mu_{j,0}$ are the corresponding μ_j for the vacuum, or

$$\exp(2b\varphi(x,t)) = \prod_{l=1}^{N} \frac{\mu_l^2(x,t)}{\sqrt{\prod_{k=1}^{2N} \lambda_k^2}} \qquad [8]. \tag{3.63}$$

In the latter one, only finitely many cuts are present; the μ_j not belonging to a cut drop out.
In [8] they derived the following equation of motion for the μ_j,

$$\mu^2_{j, {t \atop x}} = 2 \left(1 \mp \pi \mu b^2 \frac{\prod_{k \neq j} \mu_k^2(x,t)}{\sqrt{\prod_{k=1}^{2N} \lambda_k^2}} \right) \frac{\left[\mu_j^2(x,t) \prod_{k=1}^{2N} \left(\mu_j^2(x,t) - \lambda_k^2 \right) \right]^{\frac{1}{2}}}{\prod_{k \neq j} \left(\mu_j^2(x,t) - \mu_k^2(x,t) \right)}. \tag{3.64}$$

3.10 Linearization

Equation (3.64) shows that the μ_j satisfy a relatively complicated dynamical system, but it turned out that this flow is linear on Jacobi variety of the spectral curve.
In order to show this one performs a change of variables $\tau_\pm = t \pm x$ that leads to the new form of (3.64)

$$(\mu_j^2)_{\tau_\pm} = 2 \left[\pi \mu b^2 \frac{\prod_{l \neq j} \mu_l^2}{(\prod_k \lambda_k^2)^{\frac{1}{2}}} \right]^{\binom{1}{0}} \left[\frac{\mu_j^2 \prod_{k=1}^{2N} (\mu_j^2 - \lambda_k^2)^{\frac{1}{2}}}{\prod_{l \neq j} (\mu_j^2 - \mu_l^2)} \right]. \tag{3.65}$$

Then one applies the Abel map to the points $\mu_1^2, \ldots, \mu_N^2$ and defines

$$l_j(\underline{\mu}) = -\sum_{k=1}^{N} \int_{\mu_{0,k}^2}^{\mu_k^2} \nu_j = -\sum_{l=1}^{N} C_{jl} \sum_{k=1}^{N} \int_{\mu_{k,0}^2}^{\mu_k^2} \frac{E^{l-1}}{\prod_{m=1}^{2N}(E-\lambda_m^2)} dE. \tag{3.66}$$

Now one computes the derivatives of $l_j(\underline{\mu})$ with respect to $\tau_\pm$,

$$\frac{\partial l_j(\underline{\mu})}{\partial \tau_\pm} = -\sum_{l=1^N} C_{jl} \sum_{k=1}^{N} \frac{\mu_k^{2(N-l)}}{\prod_{m=1}^{2N}(\mu_k^2-\lambda_m^2)} \frac{\partial \mu_k^2}{\partial \tau_\pm} \tag{3.67}$$

$$= -\sum_{l=1^N} 2C_{jl} \sum_{k=1}^{N} \left[\pi\mu b^2 \frac{\prod_{m\neq k}\mu_m^2}{(\prod_m \lambda_m^2)^{\frac{1}{2}}} \right]^{\binom{1}{0}} \frac{\mu_k^{2(N-l)}}{\prod_{n\neq k}(\mu_k^2-\mu_n^2)}. \tag{3.68}$$

It doesn't look like that, but this last term is independent of $\underline{\mu}$; thus $l_j(\underline{\mu})$ flows linearly with $\tau_\pm$. In [7] it has been proven that

$$l_j(\mu_1^2, \ldots, \mu_N^2) = -2\pi\mu b^2 \left[\left(C_{j1} + \frac{\pi b^2 \mu (-1)^N C_{jN}}{(\prod_k \lambda_k^2)^{\frac{1}{2}}} \right) x \right. \tag{3.69}$$

$$\left. + \left(C_{j1} + \frac{\pi b^2 \mu (-1)^{N+1} C_{jN}}{(\prod_k \lambda_k^2)^{\frac{1}{2}}} \right) t \right] + l_j^0. \tag{3.70}$$

The inverse transformation of the Abel map is done with the help of the Θ-function. The final solution is given as

$$\exp(b\varphi(x,t)) = \frac{\Theta(l(x,t)+\frac{1}{2}; B)}{\Theta(l(x,t); B)}, \tag{3.71}$$

where

$$\begin{aligned} l(x,t) &= (l_1, \ldots, l_N), \\ l_j(x,t) &= -2\pi\mu b^2 \left[\left(C_{j1} + \frac{\pi b^2 \mu (-1)^N C_{jN}}{(\prod_k \lambda_k^2)^{\frac{1}{2}}} \right) x \right. \\ &\quad \left. + \left(C_{j1} + \frac{\pi b^2 \mu (-1)^{N+1} C_{jN}}{(\prod_k \lambda_k^2)^{\frac{1}{2}}} \right) t \right] + l_j(0,0), \\ l + \frac{1}{2} &= (l_1 + \frac{1}{2}, \ldots, l_N + \frac{1}{2}). \end{aligned} \tag{3.72}$$

B is the period matrix of the curve.
In other words, first one maps the flow to the Jacobi variety. There it is completely characterized by the N phases $l_1(x,t), \ldots, l_N(x,t)$, each of which depends linearly on x and t. Then one maps it back to the spectral curve, which is done by equation (3.71)

Chapter 4

Semiclassical Quantization

In this chapter, we will develop a semiclassical quantization, based on the following input data:

1. the classical spectral curve and
2. the analytical properties of the Q-function.

The analytical properties of the Q-function are not a-priory given, but it seems sensible to take the following into account:

1. While working with other models, one has gained the experience that a quantum Q-function is an analytic function, i.e. it has no poles in the quantum case. In order to achieve this property for the semiclassical Q-function we demand the absence of its poles at least at the physical part of the spectral curve[1].
2. We will see in a moment that the eigenvalue of the monodromy matrix $\Lambda(\lambda)$ is somehow the derivative of the leading order $Q-$function. This fixes the asymptotics of the classical Q-function. We will demand that the higher orders share this asymptotics up to some prefactor.

The classical spectral curve will allow for the derivation of Baxter's equation and thus of a functional equation for each order of the Q-function. We will explore how far these functional equations, along with the demanded analytical behaviour of $Q(\lambda)$, determine the Q-function.
In the whole chapter we will work on the λ-plane.

4.1 Baxter's equation

In this section we will derive Baxter's equation. Therefore we will quantize the separated variables on the classical spectral curve.
In the classical model the separated variables (μ_k, ν_k) are defined as

$$B(\mu_k) = 0, \qquad \nu_k = A(\mu_k). \tag{4.1}$$

They satisfy the Poisson-brackets

$$\{\mu_k, \nu_l\} = \delta_{k,l}\mu_k\nu_k. \tag{4.2}$$

Scince $B(\mu_k) = 0$ it holds that $A(\mu_k) = \Lambda(\mu_k)$; we find that (μ_k, ν_k) are points on the spectral curve as

$$\nu_k + \nu_k^{-1} = T(\mu_k). \tag{4.3}$$

Now we introduce the quantum counterparts $\hat{\mu}_k, \hat{\nu}_k$ of the separated variables; they satisfy the commutation relations

$$\hat{\mu}_k\hat{\nu}_l = q^{2\delta_{k,l}}\hat{\nu}_l\hat{\mu}_k. \tag{4.4}$$

[1]The physical part of the spectral curve corresponds to the upper sheet of the curve. This name can be justified from the classical limit of the quantum model: the quantum model is defined on an infinite dimensional covering of the complex plane without cuts. In the classical limit this infinite dimensional covering is mapped to the upper sheet of the spectral curve, which contains now all the physical information. The lower sheet is more like a relict of the semiclassical limit, is has no physical meaning.

In the so called separation of variables representation all the $\hat{\mu}_k$ are diagonal. The commutator relations (4.4) imply that the $\hat{\nu}_k$ act as shift operators on the wavefunctions $\Psi(\mu_1, \mu_2, \dots) = \langle \mu_1, \mu_2, \dots | \Psi \rangle$,

$$\hat{\nu}_k \Psi(..., \mu_k, ...) = \Psi(..., q^2 \mu_k, ...). \tag{4.5}$$

This means that the quantized equation (4.3) applied to an eigenfunction $\Psi_t(..., \mu_k, ...)$ of $\hat{T}(\mu_k)$ becomes

$$\begin{aligned} \hat{T}(\mu_k)\Psi_t(..., \mu_k, ...) &= t(\mu_k)\Psi_t(\dots, \mu_k, \dots) && (4.6)\\ &= (\hat{\nu}_k + \hat{\nu}_k^{-1})\Psi_t(..., \mu_k, ...) && (4.7)\\ &= \Psi_t(..., q^2\mu_k, ...) + \Psi_t(..., q^{-2}\mu_k, ...). && (4.8) \end{aligned}$$

Remembering that in the setting of separated variables the wavefunctions can be written as infinite product

$$\Psi_t(\mu_1, \mu_2, \dots) = \prod_{j=1}^{\infty} Q_t(\mu_j) \tag{4.9}$$

we find Baxter's equation

$$T(\lambda)Q_t(\lambda) = Q_t(q\lambda) + Q_t(q^{-1}\lambda), \tag{4.10}$$

where $\lambda^2 = \mu$. In the following we will omit the index t of the Q-function.

4.2 Leading order

Baxter's equation implies, that

$$T(\lambda) = \frac{Q(\lambda q)}{Q(\lambda)} + \frac{Q(\lambda q^{-1})}{Q(\lambda)}, \tag{4.11}$$

where, as before, q is given as

$$q = e^{i\pi b^2}. \tag{4.12}$$

Since $T(\lambda) = \Lambda(\lambda) + \Lambda^{-1}(\lambda)$ in the classical theory, we assume for the moment, that

$$\Lambda(\lambda) = \lim_{b^2 \to 0} \frac{Q(\lambda q)}{Q(\lambda)}. \tag{4.13}$$

We are going to justify this assumption later.
Taking the logarithm of (4.13) we find

$$\begin{aligned} \log \Lambda(\lambda) &= \lim_{b^2 \to 0} \left(\log Q(\lambda + i\pi b^2 \lambda) - \log Q(\lambda) \right) \\ &= \lim_{b^2 \to 0} i\pi b^2 \lambda \frac{d}{d\lambda} \log Q_0(\lambda). \end{aligned} \tag{4.14}$$

Here $Q_0(\lambda)$ is the leading order contribution of $Q(\lambda)$.
In the semiclassical regime, where b^2 is small, but still finite, we get

$$Q_0(\lambda) = \exp\left\{ \frac{1}{i\pi b^2} \int^{\lambda} d\sigma \frac{\log \Lambda(\sigma)}{\sigma} \right\}. \tag{4.15}$$

This formula justifies the assumption above, because with that

$$\begin{aligned}\lim_{b^2\to 0} T(\lambda) &= \lim_{b^2\to 0}\left(\frac{Q_0\left(\lambda q\right)}{Q_0(\lambda)} + \frac{Q_0\left(\lambda q^{-1}\right)}{Q_0(\lambda)}\right) \\ &= \Lambda(\lambda) + \Lambda^{-1}(\lambda),\end{aligned} \tag{4.16}$$

which defines $\Lambda(\lambda)$ uniquely up to some sign convention.

We have to require, that $Q(\lambda)$ is a single valued function on the λ^2-plane with cuts. This leads to the Bohr-Sommerfeld quantization conditions,

$$\oint_{a_j} d\log Q_0(\lambda) = 2\pi i N_j \qquad \text{for } 1 \leq j \leq n, N_j \in \mathbb{Z}. \tag{4.17}$$

An equivalent representation is given by

$$\oint_{a_j} \log\Lambda(\sigma)\frac{d\sigma}{\sigma} = -2\pi^2 b^2 N_j \qquad \text{for } 1 \leq j \leq n. \tag{4.18}$$

4.3 Next to leading order

4.3.1 The functional equations

In this section, we sketch the derivation of the functional equations, that define the higher orders of the Q-function.
Therefore we need Baxter's equation and the periodicity of $T = T(\lambda^2)$, namely $T(\lambda) = T(-\lambda)$. This periodicity of $T(\lambda)$ implies that $Q(-\lambda)$ is a second solution of Baxter's equation,

$$T(\lambda)Q(-\lambda) = Q(-\lambda q) + Q(-\lambda q^{-1}). \tag{4.19}$$

The two Baxter equations imply

$$\frac{Q(\lambda q)}{Q(\lambda)} + \frac{Q(\lambda q^{-1})}{Q(\lambda)} = \frac{Q(-\lambda q)}{Q(-\lambda)} + \frac{Q(-\lambda q^{-1})}{Q(-\lambda)}. \tag{4.20}$$

This equation contains the functional equations, one for each order of Q, in a very condensed form. Expanding equation (4.20) in orders of b^2, we find that each order n of the equation is a functional equation for W_n. Introducing the following notation,

$$Q(\lambda) = \exp\left(\frac{i}{b^2}\sum_{k=0}^{\infty} b^{2k} W_k(\lambda)\right), \tag{4.21}$$

and expanding the terms in (4.20), we find after a lengthy calculation (see Appendix) the functional equation of the order b^{2n},

$$\begin{aligned}
&\Lambda(\lambda)\sum_{j=1}^{\infty}\frac{i^j}{j!}\sum_{\substack{\{\alpha_k\}_1^n\\ \sum\alpha_k k=k\\ \sum\alpha_k=j}}\binom{j}{\alpha_1\ldots\alpha_n}\prod_{k=1}^{n}(R_{k+1}(\lambda))^{\alpha_k}\\
&\qquad+\Lambda^{-1}(\lambda)\sum_{j=1}^{\infty}\frac{i^j}{j!}\sum_{\substack{\{\alpha_k\}_1^n\\ \sum\alpha_k k=k\\ \sum\alpha_k=j}}\binom{j}{\alpha_1\ldots\alpha_n}\prod_{k=1}^{n}(R^*_{k+1}(\lambda))^{\alpha_k}\\
&=\Lambda^{-1}(\lambda)\sum_{j=1}^{\infty}\frac{i^j}{j!}\sum_{\substack{\{\alpha_k\}_1^n\\ \sum\alpha_k k=k\\ \sum\alpha_k=j}}\binom{j}{\alpha_1\ldots\alpha_n}\prod_{k=1}^{n}(R_{k+1}(-\lambda))^{\alpha_k}\\
&\qquad+\Lambda(\lambda)\sum_{j=1}^{\infty}\frac{i^j}{j!}\sum_{\substack{\{\alpha_k\}_1^n\\ \sum\alpha_k k=k\\ \sum\alpha_k=j}}\binom{j}{\alpha_1\ldots\alpha_n}\prod_{k=1}^{n}(R^*_{k+1}(-\lambda))^{\alpha_k}.
\end{aligned}\tag{4.22}$$

Here

$$R_s(\lambda)=\sum_{k=0}^{s-1}\sum_{l=1}^{s-k}\frac{\lambda^l}{l!}\frac{d^l}{d\lambda^l}W_k(\lambda)g(s-k,l),\tag{4.23}$$

$$\Lambda(\lambda)=\exp\left(-\pi\lambda\frac{d}{d\lambda}W_0(\lambda)\right)\tag{4.24}$$

and $g(p,l)$ is a combinatorical factor,

$$g(p,l)=\sum_{\substack{\{\beta_j\}_1^p\\ \sum\beta_j j=p\\ \sum\beta_j=l}}\binom{l}{\beta_1\ldots\beta_p}\prod_{j=1}^{p}\left[\frac{(i\pi)^j}{j!}\right]^{\beta_j}.\tag{4.25}$$

$g(p,l)$ is replaced by its complex conjugate in $R^*_s(\lambda)$, λ is replaced by $-\lambda$ in $R_s(-\lambda)$, where $\lambda\frac{d}{d\lambda}$ is kept constant.

4.3.2 Examples

The first three functional equations have been calculated explicitely. We found

$$d(W_1(\lambda)+W_1(-\lambda))=id\log\left(\Lambda(\lambda)-\Lambda^{-1}(\lambda)\right),\tag{4.26}$$

$$\begin{aligned}
&d\left(W_2(\lambda)+W_2(-\lambda)\right)\\
&=-\frac{i\pi}{2}d\left\{\frac{\Lambda(\lambda)+\Lambda^{-1}(\lambda)}{\Lambda(\lambda)-\Lambda^{-1}(\lambda)}\lambda\frac{d}{d\lambda}\left(W_1(\lambda)-W_1(-\lambda)\right)\right\}
\end{aligned}\tag{4.27}$$

and

$$
\begin{aligned}
& dW_3(\lambda) + dW_3(-\lambda) = \\
& -\frac{i\pi}{2} d\left\{\frac{\Lambda(\lambda)+\Lambda^{-1}(\lambda)}{\Lambda(\lambda)-\Lambda^{-1}(\lambda)}\lambda\frac{d}{d\lambda}\left(W_2(\lambda)-W_2(-\lambda)\right)\right\} \\
& +\frac{i\pi^2}{8} d\left\{\left(\lambda\frac{d}{d\lambda}\left(W_1(\lambda)-W_1(-\lambda)\right)\right)^2\left(1-\left(\frac{\Lambda(\lambda)+\Lambda^{-1}(\lambda)}{\Lambda(\lambda)-\Lambda^{-1}(\lambda)}\right)^2\right)\right\} \\
& +\frac{i\pi^2}{24} d\left\{\left(\frac{\lambda\Lambda'(\lambda)}{\Lambda(\lambda)}\right)^2-\left(\lambda\frac{d}{d\lambda}\log\left(\Lambda(\lambda)-\Lambda^{-1}(\lambda)\right)\right)^2\right\} \\
& +\frac{i\pi^2}{12} d\left(\frac{\lambda d}{d\lambda}\right)^2\log\left(\Lambda(\lambda)-\Lambda^{-1}(\lambda)\right).
\end{aligned}
\tag{4.28}
$$

Regarding these examples, it is notable that the left and the right hand sides can be written as total differentials. We expect the higher orders to share this property, but the complexity of the problem prevented the proof up to now. Furthermore we see, that $W_2(\lambda)$ and $W_3(\lambda)$ have the same asymptotics as $W_1(\lambda)$. Again we expect this to hold for the higher orders, and again the problem is too intricate for a proof.

4.3.3 Solution of the next to leading order functional equation

We will solve the leading order functional equation (4.26). The solution should belong to the class of functions, whose members share the following analytical properties:

- They are analytic on the right halfplane, except for the cuts, where they are allowed to branch.
- Their leading asymptotical behaviour is proportional to λ and $\frac{1}{\lambda}$ for $\lambda \to \infty$ and $\lambda \to 0$, respectively.

We found two different ways of solving the functional equation. The first way is the more general one, which can be also applied to the other functional equations. Thereby we use an expansion of the right hand side into partial fractions and allocate the poles in such a way, that the solution is regular on the right half-plane. The other method is less constructive. It uses an integral representation, similar to the one Lukyanov has given for the vacuum.

Partial fraction decomposition

The idea is to decompose the right hand side of (4.26), $i d\log\left(\Lambda(\lambda)-\Lambda^{-1}(\lambda)\right)$, into its partial fractions and to attribute the singularities in such a way, that $dW_1(\lambda)$ is free of poles on the right half-plane. The poles of $i d\log\left(\Lambda(\lambda)-\Lambda^{-1}(\lambda)\right)$ are located at the zeros of $\Lambda(\lambda)-\Lambda(-\lambda)$. Unfortunately, except for the vacuum, it is not possible to calculate the location of the zeros of $\Lambda(\lambda)-\Lambda(-\lambda)$ analytically. So we will explicitely calculate $W_1(\lambda)$ for the vacuum and make some more general remarks for the excited states.

The vacuum case:

We start from the vacuum eigenvalue of the monodromy matrix

$$
\log\Lambda(\lambda) = -i\frac{mR}{4}\left(\lambda-\frac{1}{\lambda}\right). \tag{4.29}
$$

Plugging this in in (4.26) and using a product expansion of the sine, we can rewrite (4.26) as

$$
\begin{aligned}
& \left(W_1(\lambda)+W_1(-\lambda)\right) \\
& \quad = i\log\left\{-2\frac{mR}{4}\left(\lambda-\frac{1}{\lambda}\right)\prod_{k=1}^{\infty}\left(1-\left(\frac{mR}{4\pi k}\right)^2\left(\lambda-\frac{1}{\lambda}\right)^2\right)\right\}.
\end{aligned}
\tag{4.30}
$$

Now we will define a preliminary solution $\widetilde{W}_1(\lambda)$ of (4.26), which is regular on the right halfplane. Therefore it has to get all the poles of the left halfplane (as $\widetilde{W}_1(-\lambda)$ has to be regular at the left halfplane); we find the expression

$$\begin{aligned}\widetilde{W}_1(\lambda) &= i\log\left(\frac{\sqrt{2mR}}{2}\left(\frac{\lambda}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}\right)\right)\\ &+ i\sum_{k=1}^{\infty}\left\{\log\left(a_k\left(\lambda+\frac{1}{\lambda}\right)+\sqrt{1+4a_k^2}\right)-a_k\left(\lambda+\frac{1}{\lambda}\right)\right\}.\end{aligned} \tag{4.31}$$

The terms $a_k\left(\lambda+\frac{1}{\lambda}\right)$ have been introduced in order to ensure the existence of the sum in (4.31), $a_k = \frac{mR}{4\pi k}$. Up to now, this $\widetilde{W}_1(\lambda)$ doesn't have the correct asymptotical behaviour, as

$$\begin{aligned}&\lim_{\lambda\to 0,\infty} -\sum_{k=1}^{\infty}\left\{\log\left(a_k\left(\lambda+\frac{1}{\lambda}\right)+\sqrt{1+4a_k^2}\right)-a_k\left(\lambda+\frac{1}{\lambda}\right)\right\}\\ &= \left(\frac{mR}{4\pi}\left(\lambda+\frac{1}{\lambda}\right)\right)\log\left(\frac{mR}{4\pi}\left(\lambda+\frac{1}{\lambda}\right)\right)+\frac{mR}{4\pi}\left(\lambda+\frac{1}{\lambda}\right)(-1+\gamma_E),\end{aligned} \tag{4.32}$$

where $\gamma_E = 0.577216...$ is the Euler constant. Taking this into account, we find

Proposition 4.3.1 *The vacuum solution of the functional equation (4.26) is given by*

$$\begin{aligned}W_1(\lambda) &= i\log\left(\frac{\sqrt{2mR}}{2}\left(\frac{\lambda}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}\right)\right)\\ &+ i\sum_{k=1}^{\infty}\left\{\log\left(a_k\left(\lambda+\frac{1}{\lambda}\right)+\sqrt{1+4a_k^2}\right)-a_k\left(\lambda+\frac{1}{\lambda}\right)\right\}\\ &+ i\frac{mR}{4\pi}\left(\lambda+\frac{1}{\lambda}\right)\left(\log\frac{mR}{4\pi}+\gamma_E\right).\end{aligned} \tag{4.33}$$

Its asymptotical behaviour is described by

$$\lim_{\lambda\to 0,\infty} W_1(\lambda) = i\frac{mR}{4\pi}\left(\lambda+\frac{1}{\lambda}\right) - i\frac{mR}{4\pi}\left(\lambda+\frac{1}{\lambda}\right)\log\left(\lambda+\frac{1}{\lambda}\right). \tag{4.34}$$

Proof A detailed proof can be found in Appendix A.

Remarks:

1. We could modify the asymptotics of (4.31), because the term $i\frac{mR}{4\pi}\left(\lambda+\frac{1}{\lambda}\right)\left(\log\frac{mR}{4\pi}+\gamma_E\right)$ is a zero mode of the functional equation (4.26). By contrast, $\left(\frac{mR}{4\pi}\left(\lambda+\frac{1}{\lambda}\right)\right)\log\left(\frac{mR}{4\pi}\left(\lambda+\frac{1}{\lambda}\right)\right)$ is no zero mode of (4.26), which is the reason why we end up with the asymptotical behaviour (4.34).

2. The vacuum-$W_1(\lambda)$ is completely determined by the functional equation, the demanded analytical properties and the asymptotics (4.34). This asymptotics is indeed essential for the uniqueness of $W_1(\lambda)$: In the introduction of the $a_k\left(\lambda+\frac{1}{\lambda}\right)$ terms in (4.31) it is somewhat arbitrary, from which k on these terms are added, as the ones for k finite don't influence the existence of the sum. But they do contribute to the asymptotics and can thus be fixed.

3. Our result is the same, Lukyanov found in [6][2].

[2] In [6] Lukyanov guessed the explicit form of the vacuum Q-function and stated explicitely the semiclassical

Another representation of $W_1(\lambda)$ is given by

$$\begin{aligned} dW_1(\lambda) = {} & i\frac{1}{2}\Omega_{\infty,-1} + i\frac{1}{2}\Omega_{0,-1} + i\sum_{k=1}^{\infty}\left\{\Omega_{0,\beta_k^+} + \Omega_{\infty,\beta_k^-} - a_k\left(1-\frac{1}{\lambda^2}\right)d\lambda\right\} \\ & + i\frac{mR}{4\pi}\log\left(\frac{mR}{4\pi}e^{\gamma_E}\right)\left(1-\frac{1}{\lambda^2}\right)d\lambda. \end{aligned} \tag{4.37}$$

Here the $\Omega_{x,y}$ are the unique normalized abelian differentials of the third kind, with simple poles at x and y with residues -1 and $+1$, respectively. The β_k^+ and β_k^- are defined as

$$\beta_k^+ = \frac{1}{2a_k} - \sqrt{\frac{1}{4a_k^2}+1}, \qquad \beta_k^- = -\frac{1}{2a_k} - \sqrt{\frac{1}{4a_k^2}+1}. \tag{4.38}$$

The representation (4.37) is proven in the appendix A
It is interesting, to calculate the asymptotics of Ω_{0,β_k^+} and $\Omega_{\infty,\beta_k^-}$ for $k\to\infty$. Therefore we need the asymptotics

$$\beta_k^+ \to -a_k = -\frac{mR}{4\pi k}, \qquad \beta_k^- \to -\frac{1}{a_k} = -\frac{4\pi k}{mR}, \tag{4.39}$$

for $k\to\infty$. With that we find

$$\lim_{k\to\infty}\Omega_{0,\beta_k^+} = \lim_{k\to\infty}\frac{\beta_k^+}{\lambda(\lambda-\beta_k^+)}d\lambda = -\frac{a_k}{\lambda^2}d\lambda \tag{4.40}$$

and

$$\lim_{k\to\infty}\Omega_{\infty,\beta_k^-} = \lim_{k\to\infty}\frac{d\lambda}{\lambda-\beta_k^-} = a_k d\lambda. \tag{4.41}$$

We see, that $\sum_{k=1}^{\infty}\left(\Omega_{0,\beta_k^+} + \frac{a_k}{\lambda^2}d\lambda\right)$ and $\sum_{k=1}^{\infty}\left(\Omega_{\infty,\beta_k^-} - a_k d\lambda\right)$ are both well defined sums.

<u>Exited states:</u>

An excited state corresponds to a field configuration (φ,Π). We will need the notion "vacuum limit" $\lim\limits_{vac}$. With that, we mean the transition of the state given by (φ,Π) to the vacuum, via the states $(\varepsilon\varphi,\varepsilon\Pi)$, $1\geq\varepsilon\geq 0$.
In the case of exited states, it holds, that

$$\Lambda(\lambda) - \Lambda^{-1}(\lambda) = i\prod_{\substack{k\in\mathbb{Z}\\ k\notin\{i_1,\dots,i_n\}}}(\lambda^2-\lambda_k^2)\sqrt{\prod_{j=1}^{n}(\lambda^2-{\lambda_{i_j}^+}^2)(\lambda^2-{\lambda_{i_j}^-}^2)}, \tag{4.42}$$

expansion

$$\begin{aligned} \log Q_0(\gamma) = {} & -\frac{mR\cosh\gamma}{2\pi b^2} - \frac{mR\cosh\gamma}{2\pi} + \frac{mR\gamma\sinh\gamma}{2\pi} \\ & - \int_{-\infty}^{\infty}\frac{d\tau}{2\pi}\frac{\log\left(1-e^{-mR\cosh\tau}\right)}{\cosh(\gamma-\tau)} + O(b^2). \end{aligned} \tag{4.35}$$

The representation of the Q-function in the form of an infinite product has been most important for us,

$$\begin{aligned} \frac{1}{Q_0(\gamma)} = {} & e^{\frac{mR\cosh\gamma}{2\pi b^2}}\left(\frac{mRe^{\gamma_E}}{4\pi}\right)^{\frac{mR\cosh\gamma}{2\pi}}\sqrt{2mR} \\ & \cdot\cosh\left(\frac{\gamma}{2}\right)\prod_{n=1}^{\infty}\left\{\sqrt{1+\left(\frac{mR}{2\pi n}\right)^2}+\frac{mR\cosh\gamma}{2\pi n}\right\}e^{-\frac{mR\cosh\gamma}{2\pi n}}, \end{aligned} \tag{4.36}$$

where $\gamma_E = 0.577216\ldots$ is the Euler constant. Making the change of variables $\lambda = e^{\gamma}$, we see that (4.36) indeed conincides with (4.33)

where $\lim_{vac} \lambda_k = \beta_k^{sign(k)}$.(Compare to the chapter "The roots of Δ".) We assume, that $i_1 < \cdots < i_l < 0$ and $0 \leq i_{l+1} < \ldots, < i_n$.
It is clear, that $d\log\left(\Lambda(\lambda) - \Lambda^{-1}(\lambda)\right)$ is a differential on the spectral curve, with simple poles of residue 1 at $\pm\lambda_k$ and $\pm\lambda_r^{\pm}$, where $k \in \mathbb{Z} \setminus \{i_1, \ldots, i_n\}$ and $r \in \{i_1, \ldots, i_n\}$. So in order to define $dW_1(\lambda)$, we have to split the poles up: $dW_1(\lambda)$ gets all the poles at the branching points, but only with residues $\frac{1}{2}$. (The branching points λ_k^+ and $-\lambda_k^+$, as well as the points λ_k^- and $-\lambda_k^-$ are identified. Thus we have to split the poles there symmetrically.) Also it gets all the other poles on the left half-plane, as it should be analytic on the right halfplane. (Compare to the vacuum!)
Inspired by the expression for the vacuum, we write

$$\begin{aligned} dW_1(\lambda) &= i\frac{1}{2}\Omega_{\infty,\lambda_0} + i\frac{1}{2}\Omega_{0,\lambda_0} \\ &+ i \sum_{\substack{k=1 \\ k\notin\{i_{l+1},\ldots,i_n\}}}^{\infty} \left(\Omega_{0,\lambda_k} + \frac{1}{\pi k} d\Gamma_0\right) + i \sum_{\substack{k=-1 \\ k\notin\{i_1,\ldots,i_l\}}}^{-\infty} \left(\Omega_{\infty,\lambda_{-k}} + \frac{1}{\pi k} d\Gamma_\infty\right) \\ &+ i \sum_{k\in\{i_{l+1},\ldots,i_n\}} \left(\frac{1}{2}\Omega_{0,\lambda_k^+} + \frac{1}{2}\Omega_{0,\lambda_k^-} + \frac{1}{2}\Omega_{0,-\lambda_k^+} + \frac{1}{2}\Omega_{0,-\lambda_k^-} + \frac{1}{\pi k} d\Gamma_0\right) \\ &+ i \sum_{k\in\{i_1,\ldots,i_l\}} \left(\frac{1}{2}\Omega_{\infty,\lambda_k^+} + \frac{1}{2}\Omega_{\infty,\lambda_k^-} + \frac{1}{2}\Omega_{\infty,-\lambda_k^+} + \frac{1}{2}\Omega_{\infty,-\lambda_k^-} + \frac{1}{\pi k} d\Gamma_\infty\right) \\ &+ i\frac{1}{\pi}\log\left(\frac{mR}{4\pi} e^{\gamma_E}\right)(d\Gamma_0 + d\Gamma_\infty). \end{aligned} \tag{4.43}$$

In the first line $\lim_{vac}\lambda_0 = -1$. The sums in the second line are well defined, as the distribution of the roots λ_k for $\lambda_k \to 0, \infty$ approximates the vacuum distribution. The differentials $d\Gamma_0$ and $d\Gamma_\infty$ are the unique normalized differentials with double poles at 0 and ∞, respectively, with coefficients $\frac{Rm}{4}$. We calculated the explicit forms analogously to $d\log\Lambda$,

$$d\Gamma_0 = \frac{Rm}{4}\left(\frac{\prod_{j=1}^{2n}\lambda_j}{\lambda^2}\right)\frac{d\lambda}{\sqrt{\prod_{j=1}^{2n}(\lambda^2 - \lambda_j^2)}} + \sum_{k=1}^{n} c_k\nu_k, \tag{4.44}$$

$$d\Gamma_\infty = \frac{Rm}{4}\lambda^{2n}\frac{d\lambda}{\sqrt{\prod_{j=1}^{2n}(\lambda^2 - \lambda_j^2)}} + \sum_{k=1}^{n} c_k\nu_k. \tag{4.45}$$

The zero modes $\sum_{k=1}^{n} c_k\nu_k$ ensure, that the a-periods of $d\Gamma_0$, $d\Gamma_\infty$ indeed vanish. The 3^{rd} and the 4^{th} line of (4.43) are due to the branching points.
Remarks:

1. Of course, (4.43) will hold only, if $i_{l+1} \neq 0$. In the other case, one had to replace the first line of (4.43) by

$$\begin{aligned} & i\frac{1}{4}\Omega_{0,\lambda_0^+} + i\frac{1}{4}\Omega_{0,-\lambda_0^+} + i\frac{1}{4}\Omega_{\infty,\lambda_0^+} + i\frac{1}{4}\Omega_{\infty,-\lambda_0^+} \\ & + i\frac{1}{4}\Omega_{0,\lambda_0^-} + i\frac{1}{4}\Omega_{0,-\lambda_0^-} + i\frac{1}{4}\Omega_{\infty,\lambda_0^-} + i\frac{1}{4}\Omega_{\infty,-\lambda_0^-}. \end{aligned} \tag{4.46}$$

2. Again the zero modes proportional to $d\Gamma_0$ and $d\Gamma_\infty$ are fixed only due to the demanded asymptotics of dW_1.

3. Nevertheless, the solution (4.43) is not unique: There are additional zero modes, which influence the branching behaviour of dW_1 at the cuts. This is described more detailed in the next section.

Integral representation

The following proposition gives a solution to (4.26). The proof can be found in Appendix A.

Proposition 4.3.2 *The function*

$$\begin{aligned} W_1(\lambda) = i\frac{mR}{4\pi}\left(\lambda+\lambda^{-1}\right) + \frac{\log\lambda}{\pi}\log\Lambda(\lambda) + \frac{i\lambda}{\pi}\int_0^\infty d\lambda' \frac{\log\left(1-\Lambda^2(-i\lambda')\right)}{\lambda^2+\lambda'^2} \\ + \lambda\int_{\mathbb{I}} d\lambda' \frac{f(\lambda')}{\lambda^2-\lambda'^2}, \qquad \lambda\notin\mathbb{I},\ \Re\lambda\geq 0, \end{aligned} \tag{4.47}$$

solves (4.26) and has the desired properties.
Here $f(\lambda)$ is an arbitrary continuous function along the cuts. The branch of $\log\lambda$ is chosen such that $\log\lambda$ is real along the positive real axis.

The function (4.47) can be defined on the negative half plane by analytical continuation. Examining $W_1(\lambda)$, we find the first two terms influencing its asymptotical behaviour, the third term containing all the poles and the last term determining the branching behaviour at the cuts. Note, that the last term is a zero mode of the functional equation (4.26), it branches as

$$\lambda\int_{\mathbb{I}} d\lambda' \frac{f(\lambda')}{(\lambda\pm i0)^2-\lambda'^2} = \lambda P.V.\int_{\mathbb{I}} d\lambda' \frac{f(\lambda')}{(\lambda)^2-\lambda'^2} \mp \frac{i\pi}{2}f(\lambda). \tag{4.48}$$

Thus the branching of the whole function is

$$W_1(\lambda+i0) = W_1(\lambda-i0) + \frac{2\lambda}{\pi}\log\Lambda(\lambda+i0) - i\pi f(\lambda) \text{ for } \lambda\in\mathbb{I}. \tag{4.49}$$

At this stage we can't fix the branching behaviour of $W_1(\lambda)$ at the cuts. Nevertheless, the analysis of the semiclassical limit will lead us to an assumption how one could pinpoint this zero mode.
The third term implies that the poles of $e^{(iW_1(\lambda))}$ are located on the negative real axis. In order to proof that we do an analytical continuation of the third term from $\lambda\in\mathbb{R}$ to $-\lambda$ along the path $\lambda(\varphi)=\lambda e^{i\varphi}$, $\varphi\in[0,\pi]$. We see, that the integrand develops a pole for $\lambda(\varphi)=i\lambda$ at $\lambda'=\lambda$, which leads to the extra term $i\log(1-\Lambda^2(-i\lambda(\varphi-\frac{\pi}{2})))$. Thus we find that for $\varphi=\pi$ we get the term $i\log\left(1-\Lambda^2(\lambda)\right)$. The remark that the only zeros of $1-\Lambda^2(\lambda)$ are located on the real axis, finishes the proof.

Chapter 5

The quantum Sinh-Gordon model

In this section we will describe the quantum Sinh-Gordon model in more detail. The information has been taken out of [10], [9], [11].

5.1 Lattice model

5.1.1 Lattice discretization

In order to avoid the ultraviolet divergencies which occur during the quantization of the continuous model, a lattice discretization is introduced. Thereby the space of radius R is replaced by N lattice sites $x_1, \ldots, x_N$ with lattice spacing $\Delta = \frac{R}{N}$. The field variables are discretized according to the standard recipe

$$\varphi_n \equiv \varphi(x_n), \qquad \Pi_n \equiv \Delta \Pi(x_n) \tag{5.1}$$

for $n = 1, \ldots, N$. These fields get canonically quantized: the variables φ_n and Π_n are considered as operators with commutation relations

$$[\varphi_n, \Pi_m] = 2\pi i \delta_{n,m}. \tag{5.2}$$

The commutation relations can be realized in the usual way on the Hilbert space $\mathcal{H} \equiv (L^2(\mathbb{R}))^{\otimes N}$. Another convenient set of variables is given by the operators f_k, defined as

$$f_{2n} \equiv e^{-2b\varphi_n}, \qquad f_{2n-1} \equiv e^{\frac{b}{2}(\Pi_n + \Pi_{n-1} - 2\varphi_n - 2\varphi_{n-1})}. \tag{5.3}$$

This change of variables is invertible for $N \equiv 2L+1$ odd, thus we restrict ourselves to this case in the following. The f_n satisfy

$$\begin{aligned} f_{2n\pm1} f_{2n} &= q^2 f_{2n} f_{2n\pm1} \text{ for } q = e^{i\pi b^2}, \\ f_n f_{n+m} &= f_{n+m} f_n \text{ for } m \geq 2. \end{aligned} \tag{5.4}$$

5.1.2 Lattice dynamics

The aim is the definition a suitable lattice dynamics. Therefore it turned out to be useful to replace space-time by the cylindric lattice

$$\mathcal{L} \equiv \{(\nu, \tau), \nu \in \mathbb{Z}/N\mathbb{Z}, \tau \in \mathbb{Z}, \nu + \tau = \text{even}\}. \tag{5.5}$$

The condition, that $\nu + \tau =$ even, implies that the lattice is rhombic; the next neighbours of the lattice point (ν, τ) are the points $(\nu \pm 1, \tau \pm 1)$.
The variables f_n are used to define a discret 'field' $f_{\nu,\tau}$: they play the role of the initial data for the time evolution of $f_{\nu,\tau}$, i.e.

$$f_{2r,0} \equiv f_{2r}, \qquad f_{2r-1,1} \equiv f_{2r-1}. \tag{5.6}$$

Time evolution is now defined as

$$f_{\nu,\tau+1} =\equiv f_{\nu,\tau-1}^{-\frac{1}{2}} \cdot g_\kappa(f_{\nu-1,\tau}) g_\kappa(f_{\nu+1,\tau}) \cdot f_{\nu,\tau-1}^{-\frac{1}{2}}, \tag{5.7}$$

where the function g is defined by

$$g_\kappa(z) = \frac{\kappa^2 + z}{1 + \kappa^2 z}. \tag{5.8}$$

Here κ is some scaling parameter of the theory.
In order to construct the time evolution operator U, acting as
$f_{\nu,\tau+1} = U^{-1} \cdot f_{\nu,\tau-1} \cdot U$, it is necessary to introduce the special functions $\omega_b(x)$ and $\phi(x)$,

$$\omega_b(x) = \frac{\zeta e^{\frac{i\pi}{2}x^2}}{\phi(x)}, \qquad \phi(x) = \exp\left(\int_{\mathbb{R}+i0} \frac{dt}{4t} \frac{e^{-2itx}}{\sinh(bt)\sinh(b^{-1}t)}\right), \tag{5.9}$$

where $\zeta = e^{\frac{i\pi}{24}(b^2+b^{-2})}$. The relevant analytic properties can be found in [10] . With these functions one can construct the function

$$G_v\left(e^{2\pi bx}\right) = \omega_b\left(\frac{v}{2} + x\right)\omega_b\left(\frac{v}{2} - x\right), \tag{5.10}$$

which satisfies the functional relations

$$\frac{G_{2s}(qz)}{G_{2s}(q^{-1}z)} = g_\kappa(z) \text{ for } \kappa = e^{-\pi bs}. \tag{5.11}$$

Now we define the operator U as

$$U = \prod_{n=1}^{N} G_{2s}(f_{2n}) \cdot U_0 \cdot \prod_{r=1}^{N} G_{2s}(f_{2r-1}). \tag{5.12}$$

Here U_0 is the parity operator acting as $U_0 \cdot f_k = f_k^{-1} \cdot U_0$. Thus U is indeed the time evolution operator. It has been proven in [11] , that this time evolution is integrable.

5.2 Solution of the lattice model

5.2.1 L- and T-operators

The monodromy matrix of the lattice model is of the form

$$M(u) \equiv L_N(u)L_{N-1}(u)\dots L_1(u), \tag{5.13}$$

where each Lax-matrix $L_n(u)$ is dedicated to the lattice site x_n. One possible choice for the L-Operator is given by

$$L_n(u) = \begin{pmatrix} e^{\frac{b}{2}\Pi_n}\left(1 + e^{-b(\varphi_n + 2\pi s)}\right)e^{\frac{b}{2}\Pi_n} & e^{-\pi bs}\sinh b\left(\pi u + \frac{\varphi_n}{2}\right) \\ e^{-\pi bs}\sinh b\left(\pi u - \frac{\varphi_n}{2}\right) & e^{-\frac{b}{2}\Pi_n}\left(1 + e^{b(\varphi_n - 2\pi s)}\right)e^{-\frac{b}{2}\Pi_n} \end{pmatrix}. \tag{5.14}$$

With this definition the commutation relations for the matrix elements of $L_n(u)$ can be written in Yang-Baxter form

$$R_{12}(u-v)L_{1n}(u)L_{2n}(v) = L_{2n}(u)L_{1n}(v)R_{12}(u-v), \tag{5.15}$$

where the $R-$matrix is

$$R(u) = \begin{pmatrix} \sinh \pi b(u+ib) & & & \\ & \sinh \pi bu & i\sin \pi b^2 & \\ & i\sin\pi b^2 & \sinh \pi bu & \\ & & & \sinh \pi b(u+ib) \end{pmatrix}. \tag{5.16}$$

The T-operator is defined as

$$T(u) = tr_{\mathbb{C}^2} M(u). \tag{5.17}$$

The Yang-Baxter equation above implies $[T(u), T(v)] = 0 \ \forall u, v \in \mathbb{R}$.
In [10], a Hamiltonian of the lattice Sinh-Gordon model has been introduced. Its exact form is not of interest here, but what is interesting is that it commutes with the trace of the monodromy matrix: $[\mathcal{H}, T(u)] = 0 \ \forall u \in \mathbb{C}$. As one is, as usual, interested in the point spectrum of the Hamiltonian we find the statement above to be very encouraging: Instead of solving the eigenvalue problem for the Hamiltonian, one tries to solve the so called 'auxiliary eigenvalue problem' first, namely the eigenvalue problem of $T(u)$.

5.2.2 Q-operator

It turned out, that, in order to solve the auxiliary eigenvalue problem, it is useful to introduce an operator $Q(u)$ satisfying the following properties:

1. $Q(u)$ is a normal operator: $Q(u)Q^*(v) = Q^*(v)Q(u)$;
2. $[Q(u), Q(v)] = 0$;
3. $[Q(u), T(u)] = 0$;
4. $Q(u)T(u) = (a(u))^N Q(u - ib) + (d(u))^N Q(u + ib)$.

The first two properties ensure, that all operators $Q(u)$, $u \in \mathbb{C}$ can be simultaneously diagonalized; their eigenvectors form a complete system of states in the Hilbert space. The last two properties imply that $T(u)$ and $Q(u)$ are simultaneously diagonal. Therefore the eigenvalue problem of $Q(u)$ might be considered as a refinement of the eigenvalue problem of $T(u)$.
An operator $Q(u)$, showing the above properties, has been constructed explicitely in [10]. Thereby they worked in the Schrödinger representation of the Hilbert space, i.e. in the representation, where the operators $x_n, n = 1, \ldots, N$ are diagonal. We will denote the integral kernel of $Q(u)$ in the Schrödinger representation by $Q_u(\mathbf{x}, \mathbf{x}')$, where $\mathbf{x} = (x_1, \ldots, x_N)$ and $\mathbf{x}' = (x'_1, \ldots, x'_N)$.
It is useful to introduce the special function $D_\alpha(x)$,

$$D_\alpha(x) = \frac{\omega_b(x + \alpha)}{\omega_b(x - \alpha)}, \tag{5.18}$$

as well as the notations

$$\sigma = s + \frac{i}{b}\left(b + b^{-1}\right), \qquad \bar{\sigma} = s - \frac{i}{b}\left(b + b^{-1}\right). \tag{5.19}$$

Definition 5.2.1 *Let the Q-operator Q(u) be defined in the Schrödinger representation by the following kernel*

$$\begin{aligned} &Q_u(\boldsymbol{x}, \boldsymbol{x}') = \\ &= (D_{-s}(u))^N \prod_{r=1}^{N} D_{\frac{1}{2}(\bar{\sigma}-u)}(x_r - x'_r) D_{\frac{1}{2}(\bar{\sigma}+u)}(x_{r-1} + x'_r) D_{-s}(x_r + x_{r-1}). \end{aligned} \tag{5.20}$$

The parameter s is defined by $\frac{1}{4}m\Delta = e^{-\pi b s}$.

Theorem 5.2.1 *The operator $Q(u)$ satisfies all relations 1.-4.. Baxter's equation holds for $Q(u)$ with the following coefficients*

$$d(u) = a(-u) = e^{\pi b\left(u + i\frac{b}{2}\right)} + \left(\frac{m\Delta}{4}\right)^2 e^{-\pi b\left(u + i\frac{b}{2}\right)}. \tag{5.21}$$

Proof The proof can be found in [10].

Now we consider an eigenstate Ψ_t for $T(u)$ with eigenvalue $t(u)$:

$$T(u)\Psi_t = t(u)\Psi_t. \tag{5.22}$$

As $Q(u)$ and $T(u)$ are simultaneously diagonal, it is an eigenstate of $Q(u)$, too:

$$Q(u)\Psi_t = q_t(u)\Psi_t. \tag{5.23}$$

It is clear from the property 4. above, that the eigenvalue $q_t(u)$ satisfies Baxter's equation:

$$t(u)q_t(u) = (a(u))^N q_t(u - ib) + (d(u))^N q_t(u + ib). \tag{5.24}$$

In [10] the analytical properties of $q_t(u)$ have been derived from the explicit form of the $Q-$operator (5.20). It has been found that

1. $q_t(u)$ is meromorphic in $\mathbb{C}$, with poles of maximal order N in $\mathcal{Y}_{-s} \cup \bar{\mathcal{Y}}_s$, where $\mathcal{Y}_s = \left\{s + i\left(\frac{b+b^{-1}}{2} + nb + mb^{-1}\right), \quad n, m \in \mathbb{N}_0\right\}$, $\bar{\mathcal{Y}}_s = (\mathcal{Y}_s)^*$;
2. $q_t(u) \sim \begin{cases} \exp\left(+i\pi N\left(s + \frac{i}{2}(b + b^{-1})\right)u\right) & \text{for } |u| \to \infty,\ |\arg(u)| < \frac{\pi}{2}, \\ \exp\left(-i\pi N\left(s + \frac{i}{2}(b + b^{-1})\right)u\right) & \text{for } |u| \to \infty,\ |\arg(u)| > \frac{\pi}{2}. \end{cases}$

Altogether this implies a necessary condition for $t(u)$ to be an eigenvalue of $T(u)$: $t(u)$ will only be an eigenvalue of $T(u)$, if there exists a meromorphic function $q_t(u)$, satisfying Baxter's equation, with the prescribed singular and asymptotical behaviour.

5.2.3 Separation of variables

The main idea of the separation of variables approach is to introduce a representation of the Hilbert space $\mathcal{H}$, such that the off-diagonal element of the monodromy matrix, $B(u)$, is diagonal. Here it is assumed, that the spectrum of $B(u)$ is simple. This implies, that the eigenstates of $B(u)$ are uniquely characterized by the corresponding eigenvalue $b(u)$. The function $e^{-2\pi bNu}b(u)$ is a polynomial of the order N in the variable $\lambda = e^{-2\pi bu}$, i.e. it has N unique zeros $y_1, \ldots, y_N$ in the stripe $\Im y_k \in \left(-\frac{1}{2b}, \frac{1}{2b}\right]$. This means, that the representation of the Hilbert space $\mathcal{H}$, in which $B(u)$ is diagonal, can be described by wave functions $\Psi(\mathbf{y})$ with $\mathbf{y} = (y_1, \ldots, y_N)$. This representation will be referred to as the SOV representation.
In [10] it has been shown, that, using this representation, the auxiliary eigenvalue problem (5.22) gets transformed into a system of Baxter equations:

$$t(y_k)\Psi(\mathbf{y}) = \left[(a(y_k))^N T_k^- + (d(y_k))^N T_k^+\right]\Psi(\mathbf{y}), \qquad k = 1, \ldots, N. \tag{5.25}$$

Here the operators $T_k^\pm$ are shift operators defined by
$T_k^\pm\Psi(\mathbf{y}) = \Psi(y_1, \ldots, y_k \pm ib, \ldots, y_N)$.
The first main advantage of the SOV representation is, that the coefficients $a(y_k)$, $d(y_k)$ depend only on one single variable. The second main advantage is, that the equations (5.25) are exactly the same as the equation (5.24). Thus any function $q_t(u)$ that solves (5.24) and fulfills the conditions above can be used to construct $\Psi(\mathbf{y})$:

$$\Psi_t(\mathbf{y}) = \prod_{k=1}^{N} q_t(y_k). \tag{5.26}$$

In [10] it has been shown, that the function $\Psi(\mathbf{y})$ in (5.26) represents indeed an element of the Hilbert space $\mathcal{H}$.
Altogether this means, that any solution of the necessary conditions (5.24), ... defines an eigenvector $|\Psi_t\rangle$ of $T(u)$ via (5.26); the conditions (5.24), .. are necessary and sufficient for $t(u)$ to be an eigenvalue of $T(u)$.

5.2.4 Integral equations

Summarizing the last two sections, we can say, that a function $t(u)$ will be the joint eigenvalue of the family of operators $T(u)$ if and only if there exists a function $q_t(u)$ that

1. satisfies baxters equation
$$t(u)q_t(u) = (a(u))^N q_t(u-ib) + (d(u))^N q_t(u+ib) \tag{5.27}$$
with $d(u) = a(-u) = e^{\pi b\left(u+i\frac{b}{2}\right)} + \left(\frac{m\Delta}{4}\right)^2 e^{-\pi b\left(u+i\frac{b}{2}\right)}$,

2. is meromorphic in $\mathbb{C}$, with poles of maximal order N in $\mathcal{Y}_{-s}\cup\bar{\mathcal{Y}}_s$, where $\mathcal{Y}_s = \left\{s + i\left(\frac{b+b^{-1}}{2} + nb + mb^{-1}\right), \quad n, m\right.$ $\bar{\mathcal{Y}}_s = (\mathcal{Y}_s)^*$;

3. has the asymptotics
$$q_t(u) \sim \begin{cases} \exp\left(+i\pi N\left(s + \frac{i}{2}(b+b^{-1})\right)u\right) & \text{for } |u| \to \infty,\ |\arg(u)| < \frac{\pi}{2}, \\ \exp\left(-i\pi N\left(s + \frac{i}{2}(b+b^{-1})\right)u\right) & \text{for } |u| \to \infty,\ |\arg(u)| > \frac{\pi}{2} \end{cases}$$

4. solves the quantum Wronskian relation
$q_t(u+i\delta)q_t(u-i\delta) - q_t(u+i\delta')q_t(u-i\delta') = W(u)$, where $2\delta = b + b^{-1}$, $\delta' = \delta - b$ and $W(u) = \left(e^{i\pi(\sigma^2+u^2)}D_\sigma(u)\right)^{-N}$.

The set of all solutions to this conditions specifies the point spectrum of the model and will be denoted $\mathcal{Q}$. In [9] this set of solutions has been characterized by a set of solutions to certain integral equations. We will describe this in the following.
First of all we will perform a change of variables and define
$$\vartheta = \frac{\pi}{2}\frac{u}{\delta}. \tag{5.28}$$
Now let $\mathbb{S}$ be the strip $\mathbb{S} = \left\{z \in \mathbb{C}; |\Im z| \le \frac{\pi}{2}\right\}$ and let $\partial\mathbb{S}$ be its boundary. Then we will split $\mathcal{Q}$ into subsets $\mathcal{Q}_M$, that contain all functions q_t with exactly M zeros in the strip $\mathbb{S}$:
$$\mathcal{Q}_M = \left\{ \begin{array}{l} q_t(\vartheta) : \exists \vartheta_1, \dots, \vartheta_M \in \mathbb{S} \text{ with } q_t(\vartheta_a) = 0 \text{ for } a = 1, \dots, M \\ \text{and } q_t \text{ does not vanish elsewhere in } \mathbb{S}. \end{array} \right\} \tag{5.29}$$
It has been shown, that the elements of Q_M are in one-to-one correspondence to the elements of a certain set $\mathcal{Y}_M$ which is defined in the following.

Definition 5.2.2 *Let $\mathcal{Y}_M$ be the set of all functions $Y(\vartheta)$ that satisfy the following conditions:*

1. $\log Y(\vartheta) \sim -i\frac{2}{\pi}\delta^2 N\left(\left(\vartheta \pm \sigma \mp i\frac{\pi}{2}\right)^2 - \tau^2\right)$ *for* $|u| \to \infty$, $|\arg(\pm u)| < \frac{\pi}{2}$ *where* $\sigma = \frac{\pi s}{2\delta}$ *and* $\tau = \frac{\pi\delta'}{2\delta}$*;*

2. $Y(\vartheta)$ *is meromorphic with poles of maximal order N in* $\pm\frac{\pi}{2\delta}\left(\mathcal{Y}_{-s+i\tau} \cup \mathcal{Y}_{-s-i\tau}\right)$*;*

3. there are complex numbers $\vartheta_a \in \mathbb{S}$, $a = 1, \dots, M$, *with* $W(\vartheta) + Y(\vartheta) = 0$ *if* $\vartheta = \vartheta_a \pm i\frac{\pi}{2}$*;*

4. $Y(\vartheta)$ *satisfies the integral equation*
$$\begin{aligned} \log Y(\vartheta) = & \int_{\mathcal{C}} \frac{d\vartheta'}{4\pi}\sigma(\vartheta-\vartheta')\log(W(\vartheta') + Y(\vartheta')) \\ & - N\arctan\left(\frac{\cosh(\vartheta+i\tau)}{\sinh\sigma}\right) - N\arctan\left(\frac{\cosh(\vartheta-i\tau)}{\sinh\sigma}\right) \\ & - \sum_{a=1}^{M'}\log S(\vartheta-\vartheta_a-i\frac{\pi}{2}) - \frac{1}{2}\sum_{a=M'+1}^{M}\log S(\vartheta-\vartheta_a-i\frac{\pi}{2}). \end{aligned} \tag{5.30}$$

Here we have used $\sigma(\vartheta) = \frac{4\sin\vartheta_0 \cosh\vartheta}{\cosh 2\vartheta - \cos 2\vartheta_0}$ *with* $\vartheta_0 = \frac{\pi b}{2\delta}$. *Further we have assumed that* $\vartheta_1, \ldots, \vartheta_{M'} \in \mathbb{S} \setminus \partial\mathbb{S}$ *and* $\vartheta_{M'+1} \ldots, \vartheta_M \in \partial\mathbb{S}$. *The contour* $\mathcal{C}$ *is defined as* $\mathcal{C} = (\mathbb{R} + i0) \cup (\mathbb{R} - i0)$. *The branch of the* log *is the principal value with branch cuts starting at the zeros and poles of* $W(\vartheta) + Y(\vartheta)$ *and running to* $-\infty$.

Demanding consistency of the third and the fourth condition the parameters ϑ_a, $a = 1, \ldots, M$ sustain severe restrictions:

$$\begin{aligned} &\pi(2k_a + 1) - \int_{\mathcal{C}} \frac{d\vartheta}{4\pi} \tau(\vartheta_a - \vartheta) \log(W(\vartheta) + Y(\vartheta)) \\ &+ iN \arctan\left(\frac{\sinh(\vartheta_a + i\tau)}{i\sinh\sigma}\right) + iN \arctan\left(\frac{\sinh(\vartheta_a - i\tau)}{i\sinh\sigma}\right) = \\ &- \sum_{b=1}^{M'} \arg S(\vartheta_a - \vartheta_b) - \frac{1}{2} \sum_{M'+1}^{M} \arg S(\vartheta_a - \vartheta_b) \end{aligned} \tag{5.31}$$

where $\tau(\vartheta) = -\frac{4\sin\vartheta_0 \sinh\vartheta}{\cosh 2\vartheta + \cos 2\vartheta_0}$. Equations (5.31) are analogous to the famous Bethe ansatz equations. Now we are ready to quote the theorem of [9], that desrcibes the correspondence between the sets $\mathcal{Y}_M$ and $\mathcal{Q}_M$.

Theorem 5.2.2 *There is a one-to-one correspondence between the elements* $Y(\vartheta) \in \mathcal{Y}_M$ *and the elements* $q_t \in \mathcal{Q}_M$. *This correspondence can be desribed as follows. For a given element* $q_t \in \mathcal{Q}_M$ *one gets the corresponding function* $Y(\vartheta)$ *via*

$$W(\vartheta) + Y(\vartheta) = q_t\left(\vartheta + i\frac{\pi}{2}\right) q_t\left(\vartheta - i\frac{\pi}{2}\right). \tag{5.32}$$

The set $\{\vartheta_1, \ldots, \vartheta_M\}$ *is the set of zeros of* $q_t(\vartheta)$ *within* $\mathbb{S}$.
Conversely, given a solution $Y(\vartheta) \in \mathcal{Y}_M$ *to the equation (5.30) with* $\vartheta_a \in \mathbb{S} \setminus \partial\mathbb{S}$ *for* $a = 1, \ldots, M'$ *and* $\vartheta_a \in \partial\mathbb{S}$ *for* $a = M', \ldots, M$, *one defines the corresponding element* $q_t \in \mathcal{Q}$ *as*

$$\begin{aligned} \log q_t(\vartheta) = &\int_{\mathcal{C}} \frac{d\vartheta'}{4\pi} \frac{\log(W(\vartheta') + Y(\vartheta'))}{\cosh(\vartheta - \vartheta')} - N \arctan\left(\frac{\cosh\vartheta}{\sinh\sigma}\right) \\ &+ \sum_{a=1}^{M'} \int_{\mathcal{C}_a}^{\vartheta} d\vartheta' \frac{1}{\sinh(\vartheta' - \vartheta_a)} + \frac{1}{2} \sum_{a=M'+1}^{M} \int_{\mathcal{C}_a}^{\vartheta} d\vartheta' \frac{1}{\sinh(\vartheta' - \vartheta_a)}. \end{aligned} \tag{5.33}$$

The functions $q_t(\vartheta)$ *defined above are independent of the choice of contours* $\mathcal{C}_a$, *as long as they run from* $-\infty$ *to* ϑ, *avoiding the singular points* $\vartheta_a, a = 1, \ldots, M$.

Proof The proof can be found in [9].

5.3 Continuum limit

In [9] it has been shown, that there exists a well defined continuum limit on the level of the equations (5.30), (5.31) and (5.33). The continuum limit is defined such that $N \to \infty$ and $\sigma \to \infty$, where

$$\frac{mR}{2\sin\vartheta_0} = 2Ne^{-\sigma} \tag{5.34}$$

is kept constant. Now we are ready to give the main information about the quantum continuum Sinh-Gordon model. This information has been taken from [9].

Definition 5.3.1 *Let* $\mathcal{Y}_M$ *be the set of all functions* $Y(\vartheta)$ *such that*

1. $Y(\vartheta)$ *decays faster than exponetially for* $|Re(\vartheta)| \to \infty$, $|Im(\vartheta)| < \frac{\pi}{2}$,

2. $Y(\vartheta)$ is entirely analytic,

3. there is a tuple of complex numbers $\mathbf{t} = (\vartheta_1, \ldots, \vartheta_M)$ with $\vartheta_a \in \mathbb{S} \equiv \{z \in \mathbb{C} | |Imz| \leq \frac{\pi}{2}\}$, $a = 1, \ldots, M$, such that $1 + Y\left(\vartheta_a \pm i\frac{\pi}{2}\right) = 0$,

4. $Y(\vartheta)$ satisfies the nonlinear integral equation

$$\begin{aligned} \log Y(\vartheta) + mR \cosh\vartheta - \int_{\mathbb{R}} \frac{\vartheta'}{4\pi} \sigma(\vartheta - \vartheta') \log(1 + Y(\vartheta')) \\ - \sum_{a=1}^{M'} \log S(\vartheta - \vartheta_a - i\frac{\pi}{2}) - \frac{1}{2} \sum_{a=M'+1}^{M} S(\vartheta - \vartheta_a - i\frac{\pi}{2}) = 0. \end{aligned} \tag{5.35}$$

Here $S(\vartheta)$ is the infinite volume twoparticle scattering phase shift

$$S(\vartheta) = \frac{\sinh\vartheta - i\sin\vartheta_0}{\sinh\vartheta + i\sin\vartheta_0} \tag{5.36}$$

and the kernel σ is given as $\sigma(\vartheta) = \frac{d}{d\vartheta} \arg S(\vartheta)$.

Consistency of the conditions 3. and 4. requires, that the ϑ_a's obey the so called Bethe ansatz equations

$$\begin{aligned} \pi(2k_a + 1) + mR \sinh\vartheta_a - \int_{\mathbb{R}} \frac{d\vartheta}{2\pi} \tau(\vartheta_a - \vartheta) \log(1 + Y(\vartheta)) \\ + \sum_{b=1}^{M'} \log S(\vartheta_a - \vartheta_b) + \frac{1}{2} \sum_{b=M'+1}^{M} S(\vartheta_a - \vartheta_b) = 0, \end{aligned} \tag{5.37}$$

where $\tau(\vartheta) \equiv -i\sigma(\vartheta + i\frac{\pi}{2})$ and $\mathbf{k} = (k_1, \ldots, k_M) \in \mathbb{Z}^M$. Now we are ready to state the main claim of [9].

Claim 1 *The Hilbert space $\mathcal{H}_{SG}$ of the Sinh-Gordon model contains a sector $\mathcal{H}_{TBA}$ which exists for all $R > 0$ and coincides with $\mathcal{H}_{SG}$ both in the infrared limit $R \to \infty$ and the ultraviolet limit $R \to 0$, respectively.*
$\mathcal{H}_{TBA}$ decomposes into subspaces $\mathcal{H}_M$ as $\mathcal{H}_{TBA} = \oplus_{M=0}^{\infty} \mathcal{H}_M$. The sectors $\mathcal{H}_M$ have an orthonormal basis spanned by eigenvectors $e_{\boldsymbol{k}}$ to all the conserved quantities of the Sinh-Gordon model which are labeled by tuples $\boldsymbol{k} = (k_1, \ldots, k_M)$ of integers. The eigenvalue $E_{\boldsymbol{k}}$ of the Hamiltonian in the eigenstate $e_{\boldsymbol{k}}$ can be expressed as

$$E_{\boldsymbol{k}} = \sum_{a=1}^{M} m \cosh\vartheta_a - m \int_{\mathbb{R}} \frac{d\vartheta}{2\pi} \cosh\vartheta \log(1 + Y_{\boldsymbol{t}}), \tag{5.38}$$

where the tuple $\boldsymbol{t} = (\vartheta_1, \ldots, \vartheta_M)$ contains the solutions of the Bethe Ansatz equations corresponding to the tuple $\boldsymbol{k}$ and $Y_{\boldsymbol{t}} \in \mathcal{Y}_M$ corresponding to $\boldsymbol{t}$.

In order to make the whole game more accessible to semiclassical considerations, it is useful to reformulate the above description of the spectrum in terms of the Baxter equation. To each function $Y(\vartheta)$ one associates a function $Q(\vartheta)$ via

$$\begin{aligned} \log Q(\vartheta) = -mR \frac{\cosh\vartheta}{2\sin\vartheta_0} + \int_{\mathbb{R}} \frac{d\vartheta'}{2\pi} \frac{\log(1 + Y(\vartheta'))}{\cosh(\vartheta - \vartheta')} \\ + \sum_{a=1}^{M'} \int_{\mathcal{C}_a}^{\vartheta} d\vartheta' \frac{1}{\sinh(\vartheta' - \vartheta_a)} + \frac{1}{2} \sum_{a=M'+1}^{M} \int_{\mathcal{C}_a}^{\vartheta} d\vartheta' \frac{1}{\sinh(\vartheta' - \vartheta_a)}. \end{aligned} \tag{5.39}$$

Note that $Q(\vartheta) = 0$ for $\vartheta \in \mathbb{S}$ iff $\vartheta = \vartheta_a$, $a = 1, \ldots, M$.

Definition 5.3.2 *Let $\mathcal{Q}$ be the set of all solutions $Q(\vartheta)$ of the functional equation*

$$Q\left(\vartheta + i\delta\right) Q\left(\vartheta - i\delta\right) - Q\left(\vartheta + i\delta'\right) Q\left(\vartheta - i\delta'\right) = 1, \tag{5.40}$$

which satisfy the conditions

- *$Q(\vartheta)$ is entirely analytic*
- *$\log Q(\vartheta) \sim -\frac{mR}{2\sin\vartheta_0}\cosh\vartheta$ for $|Re(\vartheta)| \to \infty$, $|Im(\vartheta)| < \delta$.*

Here we introduced the notations $\delta = \frac{b+b^{-1}}{2}$ and $\delta' = \delta - b$.

Claim 2 *The elements of $\mathcal{Y}$ are in one-to-one correspondence with the elements of $\mathcal{Q}$.*

Note, that $\mathcal{Q}$ is a certain set of solutions of Baxter's T-Q-relation

$$T(\vartheta)Q(\vartheta) = Q(\vartheta + i\vartheta_0) + Q(\vartheta - i\vartheta_0). \tag{5.41}$$

Indeed, given $Q \in \mathcal{Q}$ and using the notation $Q_+(\vartheta) \equiv Q(\vartheta)$,
$Q_-(\vartheta) \equiv Q(\vartheta + i\pi - i\vartheta_0)$ one defines

$$T(\vartheta) \equiv Q_+(\vartheta + i\vartheta_0)Q_-(\vartheta - i\vartheta_0) - Q_+(\vartheta - i\vartheta_0)Q_-(\vartheta + i\vartheta_0). \tag{5.42}$$

Then one easily sees that T and Q satisfy the T-Q-relation with $T(\vartheta)$ beeing $i\pi - i\vartheta_0$ periodic, $T(\vartheta + i\pi - i\vartheta_0) = T(\vartheta)$.

Baxter's equation can be rewritten as

$$T(\vartheta) = \frac{Q(\vartheta + i\vartheta_0)}{Q(\vartheta)} + \frac{Q(\vartheta - i\vartheta_0)}{Q(\vartheta)}. \tag{5.43}$$

There is a striking similarity of the last equation with the classical
$T(\lambda) = \Lambda(\lambda) + \Lambda^{-1}(\lambda)$, so it is natural to introduce a quantum eigenvalue of the monodromy matrix,

$$\Lambda_q(\vartheta) = \frac{Q\left(\vartheta + i\frac{\vartheta_0}{2}\right)}{Q\left(\vartheta - i\frac{\vartheta_0}{2}\right)}. \tag{5.44}$$

With this definition, equation (5.43) becomes

$$T(\vartheta) = \Lambda_q\left(\vartheta + i\frac{\vartheta_0}{2}\right) + \Lambda_q^{-1}\left(\vartheta - i\frac{\vartheta_0}{2}\right). \tag{5.45}$$

Chapter 6

The semiclassical limit

Up to now we have studied the semiclassical quantization of the Sinh-Gordon model. It would be interesting to compare the results with the semiclassical limit of the theory, which we are determining in this chapter. Naturally we calculate the leading order first. Therefor we have to choose appropriate tuples $\mathbf{k} = (k_1, \ldots, k_M)$ of Bethe numbers and perform the coupled limit $b^2 \to 0$ and $M \to \infty$ with $b^2 M = const$. Afterwards we determine the next to leading order.

First of all we perform a change of variables. The quantum Sinh-Gordon model is defined on the $\vartheta-$plane, whereas the classical model lives on a double covering of the λ^2-plane. The connection between ϑ and λ is given as

$$\lambda = e^{\vartheta(1+b^2)}, \tag{6.1}$$

which is not suitable for semiclassical considerations. Hence we introduce a new variable

$$\gamma = \vartheta(1 + b^2), \tag{6.2}$$

which simplifies the map between the quantum plane and the classical spectral curve considerably. As we are interested in expansions in the parameter b^2, we also replace ϑ_0 by

$$\vartheta_0 = \frac{\pi b^2}{1 + b^2}. \tag{6.3}$$

6.1 The leading order

6.1.1 Bethe roots

The Bethe roots play a central role in the endeavour to understand the semiclassical limit of the Sinh-Gordon theory in finite volume. The correct choice of sets of Bethe numbers and the leading order distribution of the corresponding Bethe roots are the essential ingredients for the formation of the spectral curve in the classical limit. The next to leading order of the distribution of the Bethe roots is the basis for the calculation of all the other next to leading order quantities.

We act on the assumption, that there is a semiclassical expansion of the Bethe roots, i.e. we work with the ansatz

$$\gamma_a = \gamma_a^0 + \pi b^2 \gamma_a^1 + O(b^4). \tag{6.4}$$

Here the leading order γ_a^0 and next to leading order γ_a^1, $a = 1, \ldots, M$, are determined by the leading and the next to leading order Bethe ansatz equations, respectively. We assume, that for a fixed number of roots, both, the leading and the next to leading order, will not scale with b^2.
In the classical limit the Bethe ansatz equations reduce to

$$\pi(2k_a + 1) = mR \sinh \gamma_a + \sum_{b=1}^{M} \left\{ 2 \arctan \left(\frac{\sinh(\gamma_a - \gamma_b)}{\pi b^2} \right) + \pi \right\} \tag{6.5}$$

for $a = 1, \ldots, M$. Is has been proven in [9] that for each choice of integers $\mathbf{k} = (k_1, \ldots, k_M)$ an unique and real solution $\mathbf{t} = (\gamma_1, \ldots, \gamma_M)$ exists.

The right hand side of (6.5) is strictly monotonously increasing. Thus $k_a > k_b$ implies directly that $\gamma_a > \gamma_b$.

In the following conjecture we identify sets of Bethe numbers, which most likely lead to the classical finite zone solutions.

Conjecture 1 *Consider the tuple* $\boldsymbol{k} = (k_1^1, k_1^2, \ldots, k_1^{N_1}, k_2^1, \ldots, k_2^{N_2}, \ldots, k_n^1, \ldots, k_n^{N_n})$ *where*

- $k_1^1 = \Theta^1((b^2)^0)$,
- $k_i^j = k_i^1 + j - 1$ *for* $i = 1, \ldots, n$ *and* $j = 1, \ldots, N_j$,
- $\Delta k_{i+1} \equiv k_{i+1}^1 - k_i^{N_i} > 1$ *for* $i = 1, \ldots, n-1$,
- $N_i = \Theta(b^{-2})$, *for* $i = 1, \ldots, n$ *and*
- $\sum_{i=1}^n N_i = M$.

Then for $M \to \infty$, $M \cdot \pi b^2 = c$ *constant and* $N_i \cdot \pi b^2 \to c_i > 0$, *the Bethe roots in the tuple* $\boldsymbol{t}$ *corresponding to* $\boldsymbol{k}$ *condense into* n *finite intervals* I_i *on the real line.*

Strong hints for the correctness of this conjecture are provided in appendix B.

We denote the union of all the cuts by $\mathbb{I} = \cup_{i=1}^n I_i$.

The conjecture 1 allows for a concretion of the notion "classical limit".

Definition 6.1.1 *Let* $\mathcal{Q}_{M,c,k_1^1,\Delta k_2,\ldots,\Delta k_n,N_1,\ldots,N_n}$ *be a quantum state, which is characterized by* M, b^2 *and by the set of Bethe numbers as in conjecture 1. We refer to its limit* $M \to \infty$, *where* $M \cdot \pi b^2 = c$, $N_i \cdot \pi b^2 \to c_i > 0$ *and* $c, k_1^1, \Delta k_i$ *are kept constant* $\forall i$, *as its classical limit*

$$\mathcal{K}_{k_1^1,\Delta k_2,\ldots,\Delta k_n,c_1,\ldots,c_n} = \lim_{class} \mathcal{Q}_{M,c,k_1^1,\Delta k_2,\ldots,\Delta k_n,N_1,\ldots,N_n}. \tag{6.6}$$

Remarks:

1. The conjecture 1 above is equivalent to the statement, that $\mathcal{K}_{k_1^1,\Delta k_2,\ldots,\Delta k_n,c_1,\ldots,c_n}$ is a classical finite zone solution of the Sinh-Gordon equation, whose spectral curve is specified by the parameters $k_1^1, \Delta k_2, \ldots, \Delta k_n, c_1, \ldots, c_n$.
2. Since $E_{\mathbf{k}} = \sum_{a=1}^M m \cosh\vartheta_a - m \int_{\mathbb{R}} \frac{d\vartheta}{2\pi} \cosh\vartheta \log(1+Y_{\mathbf{t}})$ in the quantum theory, the conjecture above is also equivalent to the statement, that the spectrum becomes dense in the classical limit and confines to a finite set of real intervals.

As the Bethe roots are assumed to condense in the classical limit, it is sensible to introduce a density $\rho_c(\gamma)$ to describe their distribution. We define the density as

$$\rho_c(\gamma) = \lim_{\text{class}} \frac{\pi b^2}{\gamma_a - \gamma_{a-1}} \tag{6.7}$$

where $\gamma = \lim_{\text{class}} \gamma_a$. The density allows for the transformation of certain sums into integrals,

$$\pi b^2 \sum_{a=1}^M f(\gamma_a) \to \int_{\mathbb{I}} f(\gamma) \rho_c(\gamma) d\gamma. \tag{6.8}$$

Further information can be found in the appendix B.

[1] Θ is a Landau symbol.

6.1.2 Eigenvalue of the monodromy matrix

A quantity, which plays a key role in the classical model, is the eigenvalue of the monodromy matrix $\Lambda(\lambda)$. Here we calculate the classical limit $\Lambda_c(\gamma)$ of its quantum counterpart $\Lambda_q(\vartheta)$. Analyzing $\Lambda_c(\gamma)'$s analytical behaviour, we find that Λ_c and Λ coincide.

Determination of $\Lambda_c(\gamma)$

In order to determine $\Lambda_c(\gamma)$, we take the explicit form of $Q(\vartheta)$ and get

$$\log \Lambda_c(\vartheta) = \lim_{\substack{\vartheta_0 \to 0 \\ M\to\infty}} \left\{ \log Q\left(\vartheta + \frac{i\vartheta_0}{2}\right) - \log Q\left(\vartheta - \frac{i\vartheta_0}{2}\right)\right\} \tag{6.9}$$

$$= \lim_{\substack{\vartheta_0 \to 0 \\ M\to\infty}} \left\{ -\frac{mR}{2\sin(\vartheta_0)}\left[\cosh\left(\vartheta + i\frac{\vartheta_0}{2}\right) - \cosh\left(\vartheta - i\frac{\vartheta_0}{2}\right)\right] + \right.$$
$$\left. + \sum_{a=1}^{M}\left[\int_{C_a}^{\vartheta + i\frac{\vartheta_0}{2}} \frac{d\vartheta'}{\sinh(\vartheta' - \vartheta_a)} - \int_{C_a}^{\vartheta - i\frac{\vartheta_0}{2}} \frac{d\vartheta'}{\sinh(\vartheta' - \vartheta_a)}\right]\right\}. \tag{6.10}$$

This gives us the leading order term

$$\log\, \Lambda_c(\gamma) = -i\frac{mR}{2}\sinh(\gamma) + \mathop{lim}_{\substack{b^2 \to 0 \\ M\to\infty}} \sum_{a=1}^{M} \frac{i\pi b^2}{\sinh(\gamma - \gamma_a)} \tag{6.11}$$

for $\gamma \neq \gamma_a \forall a$. Going to the continuum as in App B we find for $\gamma \notin \mathbb{I}$

$$\log \Lambda_c(\gamma) = -i\frac{mR}{2}\sinh(\gamma) + i\sum_{k=1}^{n}\int_{I_k} \frac{\rho_c(\gamma')}{\sinh(\gamma - \gamma')}\, d\gamma'. \tag{6.12}$$

Properties of $\Lambda_c(\gamma)$

Proposition 6.1.1 *It holds, that*

$$\rho_c(\gamma) = \frac{\Re \log\, \Lambda_c(\gamma + i0)}{\pi}, \tag{6.13}$$

i.e. the density of the Bethe roots may be calculated from the classical eigenvalue of the monodromy matrix.

Proof This is a direct consequence of equation (6.12) and the identity

$$\frac{1}{\sinh(\gamma - \gamma' \mp i0)} = P.V.\frac{1}{\sinh(\gamma - \gamma')} \pm i\pi\delta(\gamma - \gamma'). \tag{6.14}$$

∎

By the same argument we see, that the real part of $\log \Lambda_c$ is discontinuous at the cuts,

$$\Re \log \Lambda_c(\gamma + i0) = -\Re \log \Lambda_c(\gamma - i0) \qquad \forall \gamma \in \mathbb{I}. \tag{6.15}$$

In contrast, the imaginary part of $\log \Lambda_c$ is continuous. In the following proposition we will show, that it is also related to the Bethe roots.

Proposition 6.1.2 *It holds that*

$$\pi\left(k_1^1 + \sum_{l=2}^{i}(\Delta k_l - 1)\right) = \Im\, \log\, \Lambda_c(\gamma)\ \ \textit{for } \gamma \in I_i \tag{6.16}$$

and tuples $\boldsymbol{k}$ *as in conjecture 1.*

Proof The Bethe ansatz equations imply to leading order

$$2\pi\left(k_1^1+\sum_{l=2}^{i}(\Delta k_l-1)\right)=mR\sinh\gamma_i^j-2\pi b^2\sum_{k=1}^{n}\sum_{\substack{l=1\\(l,k)\neq(j,i)}}^{N_k}\frac{1}{\sinh(\gamma_i^j-\gamma_k^l)} \tag{6.17}$$

for $i=1,\dots,n$ and $j=1,\dots,N_i$ and where $\Delta k_l=k_l^1-k_{l-1}^{N_{l-1}}$. Observe that the left hand side is independent of j, i.e. it is a characteristic of the interval I_i. Transformig the sums into integrals by using the leading order of the density (6.7) one finds for $\gamma\in I_i$

$$2\pi\left(k_1^1+\sum_{l=2}^{i}(\Delta k_l-1)\right)=mR\sinh\gamma-2\sum_{k=1}^{n}P.V.\int_{I_k}\frac{\rho_c(\gamma')}{\sinh(\gamma-\gamma')}d\gamma'. \tag{6.18}$$

Comparing (6.18) with (6.12) finishes the proof. ∎

Proposition 6.1.3 *We claim, that $\Lambda_c(\gamma)$ is an analytical function away from the cuts and away from $\pm\infty$. At $\pm\infty$ it exhibits essential singularities, at the cuts it branches like*

$$\Lambda_c(\gamma-i0)=\frac{1}{\Lambda_c(\gamma+i0)}\qquad\forall\gamma\in\mathbb{I}. \tag{6.19}$$

$\Lambda_c(\gamma)$ doesn't have neither poles nor zeros.

Proof As $\Lambda_c(\gamma)$ solves the functional equation $T(\gamma)=\Lambda(\gamma)+\Lambda^{-1}(\gamma)$ and $T(\gamma)$ is regular away from $\pm\infty$, $\Lambda_c(\gamma)$ clearly doesn't have any poles or zeros. In order to proof the remainder we regard

$$\begin{aligned}\Lambda_c(\gamma)=\cos&\left(-\frac{mR}{2}\sinh\gamma+\sum_{k=1}^{n}\int_{I_k}\frac{\rho(\gamma')}{\sinh(\gamma-\gamma')}d\gamma'\right)\\ +i\sin&\left(-\frac{mR}{2}\sinh\gamma+\sum_{k=1}^{n}\int_{I_k}\frac{\rho(\gamma')}{\sinh(\gamma-\gamma')}d\gamma'\right).\end{aligned} \tag{6.20}$$

This representation clearly implies the analyticity and the appearance of the essential singularities. In order to understand the branching behaviour, we make an analytical continuation of (6.20) from $\gamma-i0$ to $\gamma+i0$ for $\gamma\in\mathbb{I}$. Using once more the identity (6.14), we find

$$\begin{aligned}\Lambda_c(\gamma+i0)=\cos&\left(-\frac{mR}{2}\sinh\gamma+\sum_{k=1}^{n}P.V.\int_{I_k}\frac{\rho(\gamma')}{\sinh(\gamma-\gamma')}d\gamma'-i\pi\rho(\gamma)\right)\\ +i\sin&\left(-\frac{mR}{2}\sinh\gamma+\sum_{k=1}^{n}\int_{I_k}\frac{\rho(\gamma')}{\sinh(\gamma-\gamma')}d\gamma'-i\pi\rho(\gamma)\right).\end{aligned} \tag{6.21}$$

Because of proposition 6.1.2, we get

$$\Lambda_c(\gamma+i0)=\cos\left(\Im\log\Gamma(\gamma)\right)\left(\cosh(\pi\rho(\gamma))+\sinh(\pi\rho(\gamma))\right). \tag{6.22}$$

By the same arguments we find

$$\Lambda_c(\gamma-i0)=\cos\left(\Im\log\Gamma(\gamma)\right)\left(\cosh(\pi\rho(\gamma))-\sinh(\pi\rho(\gamma))\right). \tag{6.23}$$

The statement, that $\cos^2\left(\Im\log\Gamma(\gamma)\right)=1$ for $\gamma\in\mathbb{I}$ finishes the proof. ∎

Proposition 6.1.4 *$\Lambda_c(\gamma)$ is already welldefined and singlevalued on a truncated surface Σ', which is represented as the strip $\{\gamma\in\mathbb{C}||\Im\gamma|\leq\pi\}\setminus\mathbb{I}$ with identifications*

$$\begin{aligned}\mathbb{I}+i0&\equiv\mathbb{I}+i\pi-i0,\\ \mathbb{I}-i0&\equiv\mathbb{I}-i\pi+i0,\\ \mathbb{R}\setminus\mathbb{I}+i\pi-i0&\equiv\mathbb{R}\setminus\mathbb{I}-i\pi+i0.\end{aligned} \tag{6.24}$$

Proof From the representation (6.20) it follows that $\Lambda_c(\gamma + i\pi) = \Lambda_c^{-1}(\gamma)\ \forall\gamma \notin \mathbb{I}$. This is the case, as one never has to cross a cut while analytically continuing from γ to $\gamma + i\pi$. Together with the proposition above this implies that $\forall\gamma \in \mathbb{I}$

$$\Lambda_c(\gamma \pm i0) = \Lambda_c(\gamma \pm i\pi \mp i0), \tag{6.25}$$

which proves the assertion. ∎

This truncated Riemann surface Σ' is homeomorphic to the classical double covered λ^2-plane via the identification $\lambda^2 = \exp(2\gamma)$. Thus we can regard Λ_c as a function on the λ^2-plane

Corollary 1 *The classical limit $\Lambda_c(\lambda^2)$ of $\Lambda_q(\vartheta)$ and the classical eigenvalue of the monodromy matrix of the according spectral curve coincide.*

Proof As the analytical behaviour like analyticity, the asymptotics, the branching behaviour and the occurrence of poles or zeros coincide, the functions by themselves coincide. ∎

Corollary 2 *There are $\Delta k_{i+1} - 1$ simple zeros of $T(\gamma)$ between the intervals I_i and I_{i+1} $\forall i$.*

Proof From the chapter about the classical Sinh-Gordon model we know that $\Im \log \Lambda(\gamma)$ counts the zeros of $T(\gamma)$. As Λ_c and Λ coincide, the quotation of proposition 6.1.2 finishes the proof. ∎

6.1.3 Q-function

The leading order term of the $Q-$function is

$$\log Q(\gamma) = -mR\frac{\cosh\gamma}{2\pi b^2} + \sum_{a=1}^{M}\int_{C_a}^{\gamma}\frac{1}{\sinh(\gamma' - \gamma_a)}. \tag{6.26}$$

In the continuum limit this gives for $\gamma \notin \mathbb{I}$

$$\log Q(\gamma) = -mR\frac{\cosh\gamma}{2\pi b^2} + \frac{1}{\pi b^2}\int_{\mathbb{R}} d\gamma' \rho^0(\gamma')\int_{C_{\gamma'}}^{\gamma}\frac{d\gamma''}{\sinh(\gamma'' - \gamma')}. \tag{6.27}$$

Using (6.12) we find

$$\log Q(\gamma) = -\frac{i}{\pi b^2}\int_0^{\gamma}\log\Lambda_c(\gamma')d\gamma' + \frac{C}{\pi b^2} \tag{6.28}$$

where $C = -\frac{mR}{2} + \int_{\mathbb{I}} d\gamma' \rho^0(\gamma')\int_{C_{\gamma'}}^{0}\frac{d\gamma''}{\sinh(\gamma''-\gamma')}$.
We demand that $Q(\gamma)$ is single valued on the spectral curve. This leads us to the Bohr-Sommerfeld condition

$$\oint_{a_j} d\log Q(\gamma) = 2\pi i N_j, \qquad N_j \in \mathbb{Z},\ j = 1,\ldots,n. \tag{6.29}$$

The interpretation is the following. As long as $\pi b^2 = \frac{c}{M}$ is still finite, $Q(\gamma)$ is an analytic function, i.e. N_j is the number of zeros of $Q(\gamma)$ in I_j. In the classical limit $b^2 \to 0$, $Q(\gamma)$ develops essential singularities along the cuts and condition (6.29) becomes meaningless. We change equation (6.29) to

$$\oint_{a_j}\log\Lambda_c(\gamma)d\gamma = -2\pi^2 b^2 N_j = -2c\pi\frac{N_j}{M}, \tag{6.30}$$

which makes still sense in the classical limit. We see, that in the classical limit the right hand side of (6.30) might be any real number.

6.1.4 The spectral curves

Proposition 6.1.5 *If conjecture 1 is correct, our classical limit will reproduce the whole moduli space of the classical finite zone spectral curves.*

Proof We saw in the chapter about the classical Sinh-Gordon model, that a finite zone spectral curve is characterized by the following quantities:

1. the value of $\mathcal{I}m \log \Lambda(\gamma) \in \pi\mathbb{Z}$ on I_1,
2. $\oint_{b_j} d\log\Lambda(\gamma) = 2\pi i M_j$, $M_j \in \mathbb{Z}$, $j = 2, \dots, n$ and
3. the n filling fractions $\varepsilon_j = \int_{I_j} \rho(\gamma) d\gamma = -\frac{1}{2\pi} \oint_{a_j} \log\Lambda(\gamma) d\gamma$, which are real numbers with

$$\sum_{j=1}^{n} \varepsilon_j = c. \tag{6.31}$$

In the preceding we found, that

1. $\mathcal{I}m \log \Lambda(\gamma) = \pi k_1^1$ on I_1,
2. $\oint_{b_j} d\log\Lambda(\gamma) = $ #zeros of $T(\gamma)$ between I_j and I_{j+1}=$\Delta k_{j+1} - 1$,
3. $\varepsilon_j = \lim\limits_{class} \frac{cN_j}{M}$.

As long as we respect the conditions above, we are completely free to set k_1^1, Δk_j and $\lim\limits_{class} \frac{cN_j}{M}$ to whatever values we want. This finishes the proof. ∎

6.2 The next to leading order

6.2.1 Bethe roots

The next to leading order Bethe ansatz equations are found to be

$$\begin{aligned} & mR\left(\gamma_a^1 - \frac{\gamma_a^0}{\pi}\right)\cosh\gamma_a^0 - \frac{1}{\pi}\int_{\mathbb{R}} d\gamma \frac{\sinh(\gamma_a^0-\gamma)}{\cosh^2(\gamma_a^0-\gamma)} \log\left(1-\Lambda^2\left(\gamma - \frac{i\pi}{2}\right)\right) \\ & + 2b^2 \sum_{\substack{c=1\\c\neq a}}^{M} \left\{ \frac{1}{\sinh(\gamma_a^0-\gamma_c^0)} + \pi(\gamma_a^1-\gamma_c^1)\frac{\cosh(\gamma_a^0-\gamma_c^0)}{\sinh^2(\gamma_a^0-\gamma_c^0)} \right. \\ & \left. -(\gamma_a^0-\gamma_c^0)\frac{\cosh(\gamma_a^0-\gamma_c^0)}{\sinh^2(\gamma_a^0-\gamma_c^0)} \right\} = 0. \end{aligned} \tag{6.32}$$

In contrast to the leading order system, which is highly nonlinear, this is a linear system of equations for γ_a^1. We can reformulate system (6.32) as

$$\vec{\gamma}^1 = A\vec{\gamma}^1 + \vec{b} \tag{6.33}$$

where $\vec{\gamma}^1 = (\gamma_1^1, \gamma_1^1, \dots, \gamma_M^1)^T$, $\vec{b} = ...$ and A is the $M \times M$ matrix with

$$(A)_{cd} = \frac{2\pi b^2}{\cosh\gamma_c^0}\left(\frac{\cosh(\gamma_c^0-\gamma_d^0)}{\sinh^2(\gamma_c^0-\gamma_d^0)}(1-\delta_{cd}) - \delta_{cd}\sum_{\substack{a=1\\a\neq c}}^{M}\frac{\cosh(\gamma_c^0-\gamma_a^0)}{\sinh^2(\gamma_c^0-\gamma_a^0)}\right). \tag{6.34}$$

The form of the matrix A (6.34) implies directly that $1-A$ is diagonally dominant, i.e. that

$$|(1-A)_{aa}| > \sum_{\substack{b=1 \\ b\neq a}}^{M} |(1-A)_{ab}|. \tag{6.35}$$

A result from linear algebra states, that diagonally dominant matrices are invertible. This ensures the existence and uniqueness of the solution $\vec{\gamma}^1 = (1-A)^{-1}\vec{b}$.
Going to the continuum limit, system (6.32) will change into the linear integral equation

$$\begin{aligned} 2f(\gamma)\frac{d}{d\gamma}\Im\log\Lambda(\gamma) &= -\frac{1}{\pi}\int_{\mathbb{R}} d\gamma' \frac{\sinh(\gamma-\gamma')}{\cosh^2(\gamma-\gamma')}\log\left(1-\Lambda^2\left(\gamma'-\frac{i\pi}{2}\right)\right) \\ &+ \frac{2}{\pi}P.V.\int_{\mathbb{R}} d\gamma' \frac{\rho^0(\gamma')}{\sinh(\gamma-\gamma')} + 2\frac{d}{d\gamma}P.V.\int_{\mathbb{R}} d\gamma' \frac{f(\gamma')\rho^0(\gamma')}{\sinh(\gamma-\gamma')}, \end{aligned} \tag{6.36}$$

with

$$f(\gamma) = \gamma^1(\gamma) - \frac{\gamma}{\pi}. \tag{6.37}$$

The unique regular solution $f(\gamma)$ with $\gamma \in \mathbb{I}$ has been calculated in appendix B. We found

$$\begin{aligned} f(\gamma)\rho_c(\gamma) &= \frac{1}{\pi^2}\sqrt{\prod_{k=1}^{n}\tanh\left(\frac{\gamma-b_k+i0}{2}\right)\tanh\left(\frac{\gamma-a_k+i0}{2}\right)} \\ &\cdot \sum_k P.V.\int_{I_k}\frac{g(\gamma')}{\sinh(\gamma-\gamma')}\frac{d\gamma'}{\sqrt{\prod_{k=1}^{n}\tanh\left(\frac{\gamma'-b_k}{2}\right)\tanh\left(\frac{\gamma'-a_k}{2}\right)}} \end{aligned} \tag{6.38}$$

with

$$\begin{aligned} g(\gamma) = -\int_{-\infty}^{\gamma} d\gamma'' \Bigg\{ &\frac{1}{\pi}\int_{\mathbb{R}} d\gamma' \frac{\sinh(\gamma''-\gamma')}{\cosh^2(\gamma''-\gamma')}\log\left(1-\Lambda^2\left(\gamma'-\frac{i\pi}{2}\right)\right) \\ &+\frac{2}{\pi}P.V.\int_{\mathbb{R}} d\gamma' \frac{\rho_c(\gamma')}{\sinh(\gamma''-\gamma')}\Bigg\}. \end{aligned} \tag{6.39}$$

6.2.2 Eigenvalue of the monodromy matrix

The next to leading order of Λ_q is calculated in appendix B. We found

$$\begin{aligned} \log\Lambda_1(\gamma) &= i\frac{mR}{2}\gamma\cosh\gamma + \frac{i}{2}\int_{\mathbb{R}} d\gamma' \frac{\sinh(\gamma-\gamma')}{\cosh^2(\gamma-\gamma')}\log\left(1-\Lambda_c^2\left(\gamma'-\frac{i\pi}{2}\right)\right) \\ &- i\pi b^2\frac{d}{d\gamma}\sum_{a=1}^{M}\frac{\pi\gamma_a^1-\gamma_a^0}{\sinh(\gamma-\gamma_a^0)} + i\pi b^2\gamma\frac{d}{d\gamma}\sum_{a=1}^{M}\frac{1}{\sinh(\gamma-\gamma_a^0)} \\ &- i\pi b^2\sum_{a=1}^{M}\frac{1}{\sinh(\gamma-\gamma_a^0)}. \end{aligned} \tag{6.40}$$

Going to the continuum we get for $\gamma \notin \mathbb{I}$

$$\begin{aligned} \log\Lambda_1(\gamma) &= -\gamma\frac{d}{d\gamma}\log\Lambda_c(\gamma) + \frac{i}{2}\int_{\mathbb{R}} d\gamma' \frac{\sinh(\gamma-\gamma')}{\cosh^2(\gamma-\gamma')}\log\left(1-\Lambda_c^2\left(\gamma'-\frac{i\pi}{2}\right)\right) \\ &- i\frac{d}{d\gamma}\int_{\mathbb{I}}\frac{\rho_c(\gamma')}{\sinh(\gamma-\gamma')}(\pi\gamma^1(\gamma')-\gamma')d\gamma' - i\int_{\mathbb{I}}\frac{\rho_c(\gamma')}{\sinh(\gamma-\gamma')}d\gamma'. \end{aligned} \tag{6.41}$$

Having that, we find the imaginary part of log $\Lambda_1(\gamma)$,

$$\begin{aligned}\Im \log\, \Lambda_1(\gamma) &= \frac{1}{2}\int_{\mathbb{R}} d\gamma' \frac{\sinh(\gamma-\gamma')}{\cosh^2(\gamma-\gamma')}\log\left(1-\Lambda_c^2\left(\gamma'-\frac{i\pi}{2}\right)\right)\\ &- \frac{d}{d\gamma}P.V.\int_{\mathbb{I}}\frac{\rho_c(\gamma')}{\sinh(\gamma-\gamma')}(\pi\gamma^1(\gamma')-\gamma')d\gamma' - P.V.\int_{\mathbb{I}}\frac{\rho_c(\gamma')}{\sinh(\gamma-\gamma')}d\gamma'.\end{aligned} \tag{6.42}$$

It is interesting to compare this to (6.36). We see, that

$$2f(\gamma)\frac{d}{d\gamma}\Im\log\Lambda(\gamma) = -\frac{2}{\pi}\Im\log\Lambda_1(\gamma). \tag{6.43}$$

As $\Im\log\Lambda(\gamma)$ is constant at the cuts we get the condition that

$$\Im\log\Lambda_1(\gamma) = 0 \text{ for } \gamma\in\mathbb{I}. \tag{6.44}$$

We could take this as the defining equation for $\gamma^1(\gamma)$.

6.2.3 Q-function

The next to leading order of the $Q-$function is calculated in appendix B. We found for the first two orders of $\log Q$

$$\begin{aligned}&\log Q(\gamma) = -mR\frac{\cosh\gamma}{2\pi b^2} - mR\frac{\cosh\gamma}{2\pi} + \frac{mR}{2\pi}\gamma\sinh\gamma\\ &-\int_{\mathbb{R}}\frac{d\gamma'}{2\pi}\frac{\log(1-\Lambda^2(\gamma'-\frac{i\pi}{2}))}{\cosh(\gamma-\gamma')}\\ &+\sum_{a=1}^{M}\left\{\int_{\mathcal{C}_a}^{\gamma}\frac{1}{\sinh(\gamma'-\gamma_a)}d\gamma' + b^2\int_{\mathcal{C}_a}^{\gamma}\frac{\cosh(\gamma'-\gamma_a)}{\sinh^2(\gamma'-\gamma_a)}(\gamma'-\gamma_a)d\gamma'\right.\\ &\left.-b^2\int_{\mathcal{C}_a}^{\gamma}\frac{1}{\sinh(\gamma'-\gamma_a)}d\gamma'\right\} + O(b^4),\end{aligned} \tag{6.45}$$

where $\gamma\neq\gamma_a\forall a$. Going to the continuum, we find for $\gamma\notin\mathbb{I}$

$$\begin{aligned}\log Q(\gamma) &= -\frac{i\pi}{b^2}\int_0^{\gamma}\log\Lambda_c(\gamma')d\gamma' + \frac{C}{b^2}\\ &- mR\frac{\cosh\gamma}{2\pi} + i\frac{\gamma}{\pi}\log\Lambda(\gamma) - \int_{\mathbb{R}}\frac{d\gamma'}{2\pi}\frac{\log(1-\Lambda^2(\gamma'-\frac{i\pi}{2}))}{\cosh(\gamma-\gamma')}\\ &-\int_{\mathbb{I}}d\gamma'\frac{\rho^0(\gamma')}{\sinh(\gamma-\gamma')}\left(\gamma^1(\gamma')-\frac{\gamma'}{\pi}\right) + O(b^2).\end{aligned} \tag{6.46}$$

Note that the next to leading order of $Q(\gamma)$ is single valued on the spectral curve without the need for any further conditions.

Chapter 7

Conclusions

This work is concerned with the semiclassics of the Sinh-Gordon model. We investigated, how far the integrability of the model determines its semiclassical quantization. We tackled this problem following two conceptual independent approaches. In the one approach we studied the semiclassical limit of the full quantum solution, in the other one we semiclassically quantized the classical theory. In the following we will give a short summary of our results and afterwards compare the results of the semiclassical quantization to ones of the semiclassical limit.

7.1 Semiclassical quantization

The purpose was the determination of the semiclassical expansion of the Q-function, starting from the classical theory. This means, that the Q-function is constructed as a formal power series in $\hbar$, where each orders. of Q is defined on the spectral curve.

In a first step, we quantized the spectral curve and got Baxter's equation. Having Baxter's equation, we could easily calculate the leading order Q_0 of the Q-function,

$$Q_0(\lambda) = A \exp\left\{ \frac{1}{i\pi b^2} \int^{\lambda} d\sigma \frac{\log \Lambda(\sigma)}{\sigma} \right\}, \tag{7.1}$$

where Λ is the classical eigenvalue of the monodromy matrix and A is an integration constant. We found, that Q_0, in order to be single valued on the spectral curve, has to satisfy the Bohr-Sommerfeld conditions

$$\oint_{a_j} d \log Q_0(\lambda) = 2\pi i N_j \qquad \text{for } 1 \leq j \leq n, N_j \in \mathbb{Z}, \tag{7.2}$$

which are indeed quantization conditions of the spectral curve.

Baxter's equation allowed us further to derive a functional equation for each order in $\hbar$, which should be satisfied by the respective order of the Q-function.

While working with continuous quantum integrable models, one has gained the experience that Q-functions are analytic functions, whose asymptotical behaviour is dominated by the leading term in the semiclassical expansion.

Taking this into account, we investigated how far the functional equations determine the semiclassical expansion of the Q-function, if we demand that the higher orders of the Q-function are analytic on the physical part of the spectral curve, and that their asymptotical behaviour is given by the one of the classical Q_0. We laid particular emphasis on the next to leading order functional equation and calculated its solution, the next to leading order of the Q-function Q_1, explicitely, using two different ways. We found

$$\begin{aligned} \log Q_1(\lambda) = & -\frac{mR}{4\pi}\left(\lambda + \lambda^{-1}\right) + i\frac{\log\lambda}{\pi}\log\Lambda(\lambda) - \frac{\lambda}{\pi}\int_0^{\infty} d\lambda' \frac{\log\left(1 - \Lambda^2(-i\lambda')\right)}{\lambda^2 + \lambda'^2} \\ & + i\lambda \int_{\mathbb{I}} d\lambda' \frac{g(\lambda')}{\lambda^2 - \lambda'^2}, \qquad \lambda \notin \mathbb{I}, \Re\lambda \geq 0, \end{aligned} \tag{7.3}$$

where $g(\lambda)$ is an arbitrary continuous function along the cuts. We see, that Q_1 could be determined, except for the term containing $g(\lambda)$, which is a zero mode of the functional equation influencing the branching behaviour of Q_1. This zero mode will be discussed later.

7.2 Semiclassical limit

The aim was to calculate the classical limit and the first order corrections in $\hbar$ of the quantum Sinh-Gordon model's nonperturbative solution. The first step was to define a limit, where the roots of the Q-function condense into a finite number of real intervals. These intervals are taken as the cuts of a hyperelliptic Riemann surface, the spectral curve.
In order to define this limit, we had to identify sets of Bethe numbers whose dedicated Bethe roots show the designated behaviour in the case that their number $M \to \infty$ and $M \cdot \hbar = const.$ We proposed that the tuples
$\mathbf{k} = (k_1^1, k_1^2, \ldots, k_1^{N_1}, k_2^1, \ldots, k_2^{N_2}, \ldots, k_n^1, \ldots, k_n^{N_n})$, where

- $k_1^1 = O((b^2)^0)$,
- $k_i^j = k_i^1 + j - 1$ for $i = 1, \ldots, n$ and $j = 1, \ldots, N_j$,
- $\Delta k_{i+1} \equiv k_{i+1}^1 - k_i^{N_i} > 1$ for $i = 1, \ldots, n-1$,
- $N_i = O(b^{-2})$, for $i = 1, \ldots, n$ and
- $\sum_{i=1}^n N_i = M$,

do the job, but unfortunately we were not able to prove the condensation of the Bethe roots rigorously; we rather gave strong hints for the correctness of the statement.
Assuming that the Bethe roots indeed condense in the classical limit, it was sensible to introduce a density $\rho_c(\gamma)$, describing their distribution. Note, that the quantum variable γ is related to the classical λ by $\lambda = e^\gamma$.
Having that, we calculated the leading and the next to leading order of several quantities like the Bethe roots, the eigenvalue Λ_q of the monodromy matrix and the Q-function. We found the leading order of Λ_q to be specified by

$$\log \Lambda_c(\gamma) = -i\frac{mR}{2}\sinh(\gamma) + i\sum_{k=1}^{n}\int_{I_k}\frac{\rho_c(\gamma')}{\sinh(\gamma-\gamma')}\,d\gamma'. \tag{7.4}$$

and

$$\pi\left(k_1^1 + \sum_{l=2}^{i}(\Delta k_l - 1)\right) = \Im \log \Lambda_c(\gamma) \;\text{ for } \gamma \in I_i. \tag{7.5}$$

Note that the equations (7.5) are the leading order limit of the Bethe ansatz equations. We could show that $\Lambda_c(\gamma)$ corresponds indeed to the classical eigenvalue of the monodromy matrix $\Lambda(\lambda)$.
For the Q-function we found

$$\log Q_0(\gamma) = -\frac{i\pi}{b^2}\int_0^\gamma \log\Lambda_c(\gamma')d\gamma' + \frac{C}{b^2}, \tag{7.6}$$

where C is an integration constant. Again the leading order of the Q-function satisfies the Bohr-Sommerfeld condition (7.2). In this case the number N_j is specified in advance by the choice of a tupel of Bethe roots.
The next to leading order is given by

$$\begin{aligned}\log Q_1(\gamma) = &-mR\frac{\cosh\gamma}{2\pi} + i\frac{\gamma}{\pi}\log\Lambda(\gamma) - \int_{\mathbb{R}}\frac{d\gamma'}{2\pi}\frac{\log(1-\Lambda^2(\gamma'-\frac{i\pi}{2}))}{\cosh(\gamma-\gamma')}\\ &-\int_{\mathbb{I}} d\gamma'\frac{\rho^0(\gamma')}{\sinh(\gamma-\gamma')}\left(\gamma^1(\gamma') - \frac{\gamma'}{\pi}\right) + O(b^2).\end{aligned} \tag{7.7}$$

The unknown function $\gamma^1(\gamma)$ is fixed by the next to leading order of the Bethe ansatz equations,

$$\begin{aligned}2f(\gamma)\frac{d}{d\gamma}\Im\log\Lambda(\gamma) &= -\frac{1}{\pi}\int_{\mathbb{R}} d\gamma' \frac{\sinh(\gamma-\gamma')}{\cosh^2(\gamma-\gamma')}\log\left(1-\Lambda^2\left(\gamma'-\frac{i\pi}{2}\right)\right)\\ &+\frac{2}{\pi}P.V.\int_{\mathbb{R}} d\gamma'\frac{\rho^0(\gamma')}{\sinh(\gamma-\gamma')}+2\frac{d}{d\gamma}P.V.\int_{\mathbb{R}} d\gamma'\frac{f(\gamma')\rho^0(\gamma')}{\sinh(\gamma-\gamma')},\end{aligned} \tag{7.8}$$

with

$$f(\gamma) = \gamma^1(\gamma) - \frac{\gamma}{\pi}. \tag{7.9}$$

There is an interesting connection with the eigenvalue of the monodromy matrix: it holds that on the cuts of the spectral curve

$$2f(\gamma)\frac{d}{d\gamma}\Im\log\Lambda(\gamma) = -\frac{2}{\pi}\Im\log\Lambda_1(\gamma). \tag{7.10}$$

As $\Im\log\Lambda(\gamma)$ is constant at the cuts we find the condition

$$\Im\log\Lambda_1(\gamma) = 0 \text{ for } \gamma\in\mathbb{I}. \tag{7.11}$$

Thus the function $\gamma^1(\gamma)$ is determined by (7.11).

7.3 Semiclassical quantization versus semiclassical limit

In this section we will compare the results from the semiclassical quantization with the ones of the semiclassical limit.
It is sufficient to compare the Q-functions as the play the key role in the description of quantum integrable models. Regarding the leading order, we see that, after a change of variables from λ to γ, both Q_0 coincide up to a constant and both satisfy the Bohr-Sommerfeld conditions (7.2).
Going to the next to leading order we find the Q_1 from (7.3) and (7.7) almost coinciding: the first three terms are the same, respectively, which can be seen easily after a change of variables. The fourth terms also correspond to each other, except that in the case of (7.7) the integrand takes on a concrete value, thanks to the Bethe ansatz equations.
The equations (7.5) and (7.11), which both follow from the Bethe ansatz equations of the respective order, suggest the proposal that (7.5) holds for the whole quantum eigenvalue,

$$\Im\log\Lambda_q(\gamma) = \pi\left(k_1^1+\sum_{l=2}^{i}(\Delta k_l - 1)\right) \tag{7.12}$$

for $\gamma\in I_i$. Taking this condition in addition to the other ones in the semiclassical quantization approach, one could fix the zero modes that influence the branching behaviour of Q and thus determine at least Q_1 completely. However, we still have to proof (7.12) coming from the quantum theory and to find a semiclassical justification for it.

7.4 Outlook

Based on this work there are several natural open questions on could pursue next.
First of all on should investigate how far the identity $\Im\log\Lambda_q\in\pi\mathbb{Z}$ on the cuts holds true and how one could justify such a demand semiclassically.
Another thing one could analyze is how far out semiclassical approach can be used to determine the higher orders of the Q-functions. The question is, wether the functional equations, together with (7.12) and the demand that the Q-function is analytic on the physical branch of the spectral

curve and has a prescribed asymptotics, determine the higher orders of the Q-function completely. It would be also interesting to apply this semiclassical quantization method to other integrable models. On the one hand side one should apply this to models whose full quantum solution is known, in order to understand the general applicability of the method. On the other hand side it should be applied to models with unknown full quantum solution, as one could learn something about this solution.

Appendix A

Semiclassical quantization

A.1 Derivation of the functional equations

We will describe the derivation of the functional equations in more detail. What remains to be done is the expansion of the equation

$$\frac{Q(\lambda q)}{Q(\lambda)} + \frac{Q(\lambda q^{-1})}{Q(\lambda)} = \frac{Q(-\lambda q)}{Q(-\lambda)} + \frac{Q(-\lambda q^{-1})}{Q(-\lambda)}. \tag{A.1}$$

Starting from

$$Q(\lambda) = \exp\left(\frac{i}{b^2}\sum_{k=0}^{\infty} b^{2k} W_k(\lambda)\right) \tag{A.2}$$

we will expand $\frac{Q(\lambda q)}{Q(\lambda)}$ in powers of b^2. First we calculate

$$\begin{aligned} W_k(\lambda q) - W_k(\lambda) &= W_k\left(\lambda\left(\sum_{j=0}^{\infty}\frac{(i\pi b^2)^j}{j!}\right)\right) - W_k(\lambda) \\ &= \sum_{l=1}^{\infty}\frac{\lambda^l}{l!}\frac{d}{d\lambda}W_k(\lambda)\left(\sum_{j=1}^{\infty}\frac{(i\pi b^2)^j}{j!}\right)^l . \end{aligned} \tag{A.3}$$

Sorting this according to powers of b^2, we find

$$\begin{aligned} (A.3) &= \sum_{l=1}^{\infty}\frac{\lambda^l}{l!}\frac{d^l}{d\lambda^l}W_k(\lambda)\sum_{r=l}^{\infty} b^{2r} \sum_{\substack{\{\beta_j\}_1^r \\ \sum\beta_j j = r \\ \sum\beta_j = l}} \binom{l}{\beta_1\ldots\beta_r}\prod_{j=1}^{r}\left[\frac{(i\pi)^j}{j!}\right]^{\beta_j} \\ &= \sum_{p=1}^{\infty} b^{2p}\sum_{l=1}^{p}\frac{\lambda^l}{l!}\frac{d^l}{d\lambda^l}W_k(\lambda) g(p,l), \end{aligned} \tag{A.4}$$

where

$$g(p,l) = \sum_{\substack{\{\beta_j\}_1^p \\ \sum\beta_j j = p \\ \sum\beta_j = l}} \binom{l}{\beta_1\ldots\beta_p}\prod_{j=1}^{p}\left[\frac{(i\pi)^j}{j!}\right]^{\beta_j} \tag{A.5}$$

is a combinatorical factor. Now we calculate

$$\begin{aligned} \sum_{k=0}^{\infty} b^{2k}\left(W_k(\lambda q) - W_k(\lambda)\right) &= \sum_{s=1}^{\infty} b^{2s}\sum_{k=0}^{s-1}\sum_{l=1}^{s-k}\frac{\lambda^l}{l!}\frac{d^l}{d\lambda^l}W_k(\lambda) g(s-k,l) \\ &= \sum_{s=1}^{\infty} b^{2s} R_s(\lambda), \end{aligned} \tag{A.6}$$

where we introduced the abbreviation

$$R_s(\lambda) = \sum_{k=0}^{s-1} \sum_{l=1}^{s-k} \frac{\lambda^l}{l!} \frac{d^l}{d\lambda^l} W_k(\lambda) g(s-k, l). \tag{A.7}$$

Finally we arrive at

$$\begin{aligned}
\frac{Q(\lambda q)}{Q(\lambda)} &= \exp\left(\frac{i}{b^2}\sum_{s=1}^{\infty} b^{2s} R_s(\lambda)\right) \\
&= \exp\left(-\pi\lambda \frac{d}{d\lambda} W_0(\lambda)\right) \cdot \sum_{j=0}^{\infty} \frac{\left(i\sum_{s=1}^{\infty} b^{2s} R_{s+1}(\lambda)\right)^j}{j!} \\
&= \Lambda(\lambda) + \Lambda(\lambda)\sum_{r=1}^{\infty} b^{2r} \sum_{j=1}^{\infty} \frac{i^j}{j!} \sum_{\substack{\{\alpha_k\}_1^r \\ \sum \alpha_k k = k \\ \sum \alpha_k = j}} \binom{j}{\alpha_1 \ldots \alpha_r} \prod_{k=1}^{r} (R_{k+1}(\lambda))^{\alpha_k},
\end{aligned} \tag{A.8}$$

with

$$\Lambda(\lambda) = \exp\left(-\pi\lambda \frac{d}{d\lambda} W_0(\lambda)\right). \tag{A.9}$$

From (A.8) we can deduce the form of the other terms of (A.1). When q is replaced by q^{-1} in (A.8), one has to substitute the combinatorical factor $g(p, l)$ by its complex conjugate and $\Lambda(\lambda)$ by $\Lambda^{-1}(\lambda)$. When λ is replaced by $-\lambda$, one has to make the change in (A.8) too, where $\lambda\frac{d}{d\lambda}$ remains unchanged. Thus we get the functional equations (4.22).

A.2 Solution to the leading order functional equation

A.2.1 Partial fraction decomposition: The vacuum

For the vacuum we have an easy and explicit form of the eigenvalue of the monodromy matrix $\Lambda(\lambda)$,

$$\log \Lambda(\lambda) = -i\frac{mR}{4}\left(\lambda - \frac{1}{\lambda}\right), \tag{A.10}$$

which implies directly that

$$\Lambda(\lambda) - \frac{1}{\Lambda(\lambda)} = -2\sin\left(\frac{mR}{4}\left[\lambda - \frac{1}{\lambda}\right]\right). \tag{A.11}$$

The product expansion of the sine is given as

$$\sin x = x\prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2\pi^2}\right). \tag{A.12}$$

Thus we can rewrite (4.26) as

$$\begin{aligned}
&(W_1(\lambda) + W_1(-\lambda)) \\
&\quad = i\log\left\{-2\frac{mR}{4}\left(\lambda - \frac{1}{\lambda}\right)\prod_{k=1}^{\infty}\left(1 - \left(\frac{mR}{4\pi k}\right)^2\left(\lambda - \frac{1}{\lambda}\right)^2\right)\right\}.
\end{aligned} \tag{A.13}$$

Now we have to split the set of poles on the right hand side of the equation above into the poles on the left real real axis and the ones on the right real axis. Therefore we rewrite the first factor of the product as

$$-2\frac{mR}{4}\left(\lambda-\frac{1}{\lambda}\right)=-2mR\frac{1}{2}\left(\frac{\lambda}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}\right)\frac{1}{2}\left(\frac{\lambda}{\sqrt{\lambda}}-\frac{1}{\sqrt{\lambda}}\right). \tag{A.14}$$

Let's introduce the abbreviation

$$a_k=\frac{mR}{4\pi k}. \tag{A.15}$$

With that we can treat the other terms of the product and get

$$\begin{aligned}&\left(1-a_k\left(\lambda-\frac{1}{\lambda}\right)\right)\left(1+a_k\left(\lambda-\frac{1}{\lambda}\right)\right)\\&=-\frac{a_k^2}{\lambda^2}\left(\lambda-\beta_k^+\right)\left(\lambda-\beta_k^-\right)\left(\lambda+\beta_k^+\right)\left(\lambda+\beta_k^-\right),\end{aligned} \tag{A.16}$$

where we introduced the notation

$$\beta_k^+=\frac{1}{2a_k}-\sqrt{\frac{1}{4a_k^2}+1},\qquad \beta_k^-=-\frac{1}{2a_k}-\sqrt{\frac{1}{4a_k^2}+1}. \tag{A.17}$$

Dividing the set of poles we get

$$\begin{aligned}W_1(\lambda)&=i\log\left(\frac{\sqrt{2mR}}{2}\left(\frac{\lambda}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}\right)\right)\\&+i\sum_{k=1}^{\infty}\log\left\{a_k\left(\lambda+\frac{1}{\lambda}\right)+\sqrt{1+4a_k^2}\right\}\end{aligned} \tag{A.18}$$

Up to now, the sum in (A.18) is not well defined, we will regularize it by substracting the linear terms in $\frac{1}{k}$. As these terms are zero modes of (4.26), this is possible. We arrive at

$$\begin{aligned}W_1(\lambda)&=i\log\left(\frac{\sqrt{2mR}}{2}\left(\frac{\lambda}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}\right)\right)\\&+i\sum_{k=1}^{\infty}\left\{\log\left(a_k\left(\lambda+\frac{1}{\lambda}\right)+\sqrt{1+4a_k^2}\right)-a_k\left(\lambda+\frac{1}{\lambda}\right)\right\}.\end{aligned} \tag{A.19}$$

In principle this looks nice, but it still doesn't have the correct asymptotics. In order to fix the asymptotics, we will calculate the asymptotics of the sum first. Therefore we introduce the abbreviations $x=\frac{1}{2}(\lambda+\frac{1}{\lambda})$ and $a=\frac{mR}{2\pi}$. We rewrite the sum as

$$\sum_{k=1}^{\infty}\frac{ax}{k}-\log\left(\sqrt{1+\frac{a^2}{k^2}}+\frac{ax}{k}\right) \tag{A.20}$$

and calculate its asymptotics for $x\to\infty$. We calculate

$$\sum_{k=1}^{\infty}\frac{ax}{k}-\log\left(\sqrt{1+\frac{a^2}{k^2}}+\frac{ax}{k}\right)=\sum_{k=1}^{\infty}\int^x dx\frac{a}{k}\left(1-\frac{1}{1+\frac{ax}{k}}\right) \tag{A.21}$$

$$=\int^x dxa^2x\sum_{k=1}^{\infty}\frac{1}{k^2+axk} \tag{A.22}$$

The last sum can be found in the literature, for example in [12]:

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + axk} = \frac{1}{ax} \left(\frac{d}{d(ax)} \log \Gamma(ax+1) + \gamma_E \right), \tag{A.23}$$

where $\gamma_E = 0.577216...$ is the Euler constant. With that (A.22) becomes

$$(A.22) = \log \Gamma(ax+1) + ax\gamma_E. \tag{A.24}$$

An asymptotic expansion of the Γ-function is given by Stirlings formula,

$$\log \Gamma(ax+1) \sim (ax)\log(ax) - ax. \tag{A.25}$$

So all together the original sum has an asymptotical behaviour for $\lambda \to 0, \infty$ like

$$\begin{aligned} &- \sum_{k=1}^{\infty} \left\{ \log \left(a_k \left(\lambda + \frac{1}{\lambda} \right) + \sqrt{1 + 4a_k^2} \right) - a_k \left(\lambda + \frac{1}{\lambda} \right) \right\} \\ &\sim \left(\frac{mR}{4\pi} \left(\lambda + \frac{1}{\lambda} \right) \right) \log \left(\frac{mR}{4\pi} \left(\lambda + \frac{1}{\lambda} \right) \right) + \frac{mR}{4\pi} \left(\lambda + \frac{1}{\lambda} \right) (-1 + \gamma_E) \end{aligned} \tag{A.26}$$

Altogether we find for $W_1(\lambda)$

$$\begin{aligned} W_1(\lambda) = {} & i \log \left(\frac{\sqrt{2mR}}{2} \left(\frac{\lambda}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} \right) \right) \\ & + i \sum_{k=1}^{\infty} \left\{ \log \left(a_k \left(\lambda + \frac{1}{\lambda} \right) + \sqrt{1 + 4a_k^2} \right) - a_k \left(\lambda + \frac{1}{\lambda} \right) \right\} \\ & + i \frac{mR}{4\pi} \left(\lambda + \frac{1}{\lambda} \right) \left(\log \frac{mR}{4\pi} + \gamma_E \right). \end{aligned} \tag{A.27}$$

Its asymptotics for $\lambda \to 0, \infty$ is given by

$$W_1(\lambda) \sim i \frac{mR}{4\pi} \left(\lambda + \frac{1}{\lambda} \right) - i \frac{mR}{4\pi} \left(\lambda + \frac{1}{\lambda} \right) \log \left(\lambda + \frac{1}{\lambda} \right). \tag{A.28}$$

It is beneficial to represent W_1 in differential form,

$$\begin{aligned} dW_1(\lambda) = {} & i\frac{1}{2}\Omega_{\infty,-1} + i\frac{1}{2}\Omega_{0,-1} + i \sum_{k=1}^{\infty} \left\{ \Omega_{0,\beta_k^+} + \Omega_{\infty,\beta_k^-} - a_k \left(1 - \frac{1}{\lambda^2} \right) d\lambda \right\} \\ & + i \frac{mR}{4\pi} \log \left(\frac{mR}{4\pi} e^{\gamma_E} \right) \left(1 - \frac{1}{\lambda^2} \right) d\lambda. \end{aligned} \tag{A.29}$$

Here the $\Omega_{x,y}$ are the unique normalized abelian differentials of the third kind, with simple poles at x and y with residues -1 and $+1$, respectively. In order to prove (A.29), we calculate

$$\begin{aligned} d\log \left(\frac{\lambda}{\sqrt{\lambda}} + \sqrt{\lambda} \right) &= \frac{1}{2} \frac{1}{\lambda + 1} d\lambda - \frac{1}{2\lambda} \frac{1}{\lambda + 1} d\lambda \\ &= \frac{1}{2}\Omega_{\infty,-1} + \frac{1}{2}\Omega_{0,-1}. \end{aligned} \tag{A.30}$$

Further we calculate

$$\begin{aligned} d\log \left(a_k \left(\lambda + \frac{1}{\lambda} \right) + \sqrt{1 + 4a_k^2} \right) &= \left(-\frac{1}{\lambda} + \frac{1}{\lambda - \beta_k^+} + \frac{1}{\lambda - \beta_k^-} \right) d\lambda \\ &= \Omega_{0,\infty} + \Omega_{\infty,\beta_k^+} + \Omega_{\infty,\beta_k^-} \\ &= \Omega_{0,\beta_k^+} + \Omega_{\infty,\beta_k^-}, \end{aligned} \tag{A.31}$$

which proves the representation above.

A.2.2 Integral representation

We will proof the following proposition.

Proposition A.2.1 *The function*

$$\begin{aligned} W_1(\lambda) = i\frac{mR}{4\pi}\left(\lambda+\lambda^{-1}\right) + \frac{\log\lambda}{\pi}\log\Lambda(\lambda) + \frac{i\lambda}{\pi}\int_0^\infty d\lambda' \frac{\log\left(1-\Lambda^2(-i\lambda')\right)}{\lambda^2+\lambda'^2} \\ + \lambda\int_{\mathbb{I}} d\lambda' \frac{f(\lambda')}{\lambda^2-\lambda'^2}, \qquad \lambda\notin\mathbb{I},\ \Re\lambda\geq 0, \end{aligned} \tag{A.32}$$

solves the functional equation (4.26) and has the desired properties:

1. *It is analytic on the physical part of the* $\lambda-$*plane, i.e. on the positive halfplane, except for the cuts.*
2. *It has the leading asymptotical behaviour of* $W_0(\lambda)$*, i.e.* $\sim\lambda$ *and* $\sim\frac{1}{\lambda}$ *for* $\lambda\to\infty$ *and* $\lambda\to 0$*, respectively.*

Here $f(\lambda)$ *is an arbitrary continuous function along the cuts. The branch of* $\log\lambda$ *is chosen such that* $\log\lambda$ *is real along the positive real axis.*

Proof The first thing, we will verify, is that (A.32) indeed solves the functional equation (4.26). Therefore we will do an analytical continuation from λ to $-\lambda$ along a path $\mathcal{P}$, that crosses the imaginary axis only once on its positive side and avoids all the cuts. With this analytical con-

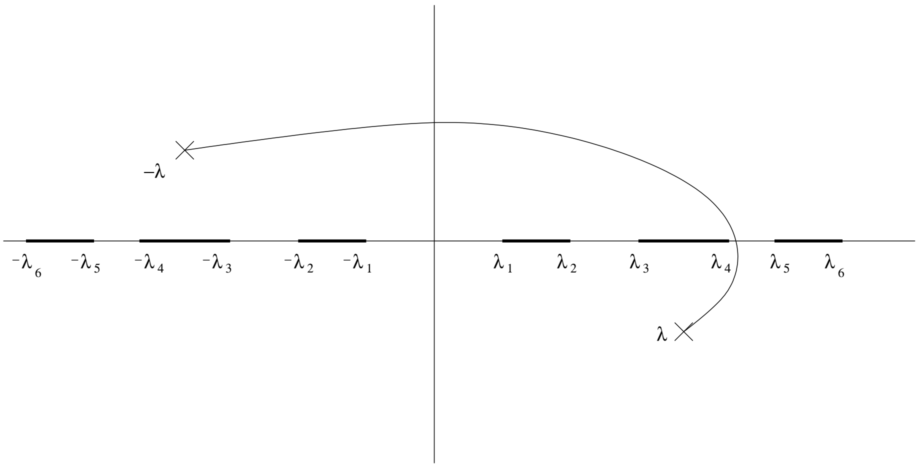

Figure A.1: *Path* $\mathcal{P}$ *from* λ *to* $-\lambda$

tinuation the first term of (A.32) is trivially a zero mode of equation (4.26). The second term gives

$$-\frac{\log\lambda}{\pi}\log\Lambda(\lambda) - i\log\Lambda(\lambda). \tag{A.33}$$

The third term behaves most interesting under analytical continuation. Its integrand has a pole as soos as λ is purely imaginary. This pole contributes to $W_1(-\lambda)$ with the extra term $i\log\left(1-\Lambda^2(\lambda)\right)$. As the path $\mathcal{P}$ avoids all the cuts, the fourth term doesn't make any problems and is also a zero mode. Altogether we find

$$\begin{aligned} W_1(-\lambda) = -i\frac{mR}{4\pi}\left(\lambda+\lambda^{-1}\right) - \frac{\log\lambda}{\pi}\log\Lambda(\lambda) - i\log\Lambda(\lambda) \\ -\frac{i\lambda}{\pi}\int_0^\infty d\lambda' \frac{\log\left(1-\Lambda^2(-i\lambda')\right)}{\lambda^2+\lambda'^2} + i\log\left(1-\Lambda^2(\lambda)\right) - \lambda\int_{\mathbb{I}} d\lambda' \frac{f(\lambda')}{\lambda^2-\lambda'^2}. \end{aligned} \tag{A.34}$$

Adding (A.32) and (A.34), we find the functional equation (4.26), which proves the assertion.
The next things we have to check are the desired properties of $W_1(\lambda)$.
The asymptotical behaviour is clearly OK.
It is also clear, that the first three terms are analytic in the positive half plane, except for the cuts, where the second term branches. The integrand of the fourth term has a pole for $\lambda \in \mathbb{I}$ and is regular elsewhere. But this pole doesn't imply poles of $W_1(\lambda)$ itself; it influences only the branching behaviour of $W_1(\lambda)$ as it branches as

$$\lambda \int_{\mathbb{I}} d\lambda' \frac{f(\lambda')}{(\lambda \pm i0)^2 - \lambda'^2} = \lambda P.V. \int_{\mathbb{I}} d\lambda' \frac{f(\lambda')}{(\lambda)^2 - \lambda'^2} \mp \frac{i\pi}{2} f(\lambda) \tag{A.35}$$

for $\lambda \in \mathbb{I}$. This finishes the proof. ∎

Appendix B

Semiclassical limit

B.1 The continuous limit

B.1.1 The density

In the chapter we will introduce the density $\rho(\gamma)$ of the Bethe roots. Therefor we will work with the ansatz

$$\gamma_a = \gamma_a^0 + \pi b^2 \gamma_a^1 + O(b^4). \tag{B.1}$$

Here the leading order γ_a^0 and next to leading order γ_a^1, $a = 1, \ldots, M$, are determined by the leading and the next to leading order Bethe ansatz equations, respectively. We assume, that for a fixed number of zeros both the leading and the next to leading order will not scale with b^2.
We introduce the spacing

$$\begin{aligned} \Delta_a &\equiv \gamma_a - \gamma_{a-1} \\ &= \Delta_a^0 + \pi b^2 \Delta_a^1 + O(b^4) \end{aligned} \tag{B.2}$$

and define the "discrete density"

$$\rho_d(\gamma_a) \equiv \frac{\pi b^2}{\Delta_a}. \tag{B.3}$$

Using the geometric series we find

$$\begin{aligned} \rho_d(\gamma_a) &= \frac{\pi b^2}{\Delta_a^0} - \left(\frac{\pi b^2}{\Delta_a^0}\right)^2 \Delta_a^1 + O(b^6) \\ &\equiv \rho_d^0(\gamma_a^0) + \pi b^2 \rho_d^1(\gamma_a^0) + O(b^4), \end{aligned} \tag{B.4}$$

i.e.

$$\rho_d^0(\gamma_a^0) = \frac{\pi b^2}{\Delta_a^0} \tag{B.5}$$

and

$$\rho_d^1(\gamma_a^0) = -\frac{\Delta_a^1}{\Delta_a^0} \rho_d^0(\gamma_a^0). \tag{B.6}$$

The main assumption is now, that the discrete density remains a well defined and regular object in the classical limit, where $\pi b^2 = M^{-1} \to 0$. In this case we define the classical density $\rho_c(\gamma)$ as

$$\rho_c(\gamma) = \lim_{b^2 \to 0} \rho_d(\gamma). \tag{B.7}$$

Now we regard the next to leading order, which is defined as

$$\rho^1(\gamma) = \lim_{class} \frac{\rho_d(\gamma) - \rho_c(\gamma)}{\pi b^2}. \tag{B.8}$$

We will assume, that the next to leading order corrections to the Bethe roots $\gamma_a^1 = \gamma^1(\gamma_a^0)$ build a continuous function $\gamma^1(\gamma)$ in the case of continuously distributed Bethe roots. If this function is also differentiable, then the density correction will be given as

$$\rho^1(\gamma) = -\rho_c^0(\gamma) \frac{d}{d\gamma} \gamma^1(\gamma). \tag{B.9}$$

B.1.2 The Riemannian sum

In this section we will calculate the semiclassical approximation of

$$\pi b^2 \sum_{a=1}^{M} f(\gamma_a) \tag{B.10}$$

up to the next to leading order. Here f is an arbitrary, at least one time continuously differentiable function. The definition of the density (B.3) implies directly

$$\pi b^2 f(\gamma_a) = f(\gamma_a)\rho_d(\gamma_a)\Delta_a. \tag{B.11}$$

Taylor expansion leads to

$$(B.11) = \left(f(\gamma_a^0) + \pi b^2 \gamma_a^1 f'(\gamma_a^0)\right)\left(\rho_d^0(\gamma_a^0) + \pi b^2 \rho_d^1(\gamma_a^0)\right)\left(\Delta_a^0 + \pi b^2 \Delta_a^1\right) + O(b^4) \tag{B.12}$$

and thus

$$(B.11) = f(\gamma_a^0)\rho_d^0(\gamma_a^0)\Delta_a^0 \tag{B.13}$$

$$+ \pi b^2 \left(\gamma_a^1 f'(\gamma_a^0)\rho_d^0(\gamma_a^0)\Delta_a^0 + f(\gamma_a^0)\left[\rho_d^1(\gamma_a^0)\Delta_a^0 + \rho_d^0(\gamma_a^1)\Delta_a^1\right]\right) + O(b^4). \tag{B.14}$$

Using (B.6) we see that the term in the square brackets vanishes. We find

$$\pi b^2 \sum_{a=1}^{M} f(\gamma_a) = \sum_{a=1}^{M} f(\gamma_a^0)\rho_d^0(\gamma_a^0)\Delta_a^0 + \pi b^2 \sum_{a=1}^{M} \gamma_a^1 f'(\gamma_a^0)\rho_d^0(\gamma_a^0)\Delta_a^0 + O(b^4), \tag{B.15}$$

i.e. in the continuous approximation we get

$$\pi b^2 \sum_{a=1}^{M} f(\gamma_a) \to \int_{\mathbb{I}} f(\gamma)\rho_c(\gamma)d\gamma + \pi b^2 \int_{\mathbb{I}} \gamma^1(\gamma) f'(\gamma)\rho_c(\gamma)d\gamma + O(b^4). \tag{B.16}$$

Here $\mathbb{I}$ is the support of the condensed Bethe roots.

B.2 Bethe ansatz equations

B.2.1 Formation of the spectral curve

In this section we will give strong hints, that conjecture 2 is true.

Conjecture 2 *Consider the tuple*
$\boldsymbol{k} = (k_1^1, k_1^2, \ldots, k_1^{N_1}, k_2^1, \ldots, k_2^{N_2}, \ldots, k_n^1, \ldots, k_n^{N_n})$ *where*

- $k_1^1 = O((b^2)^0)$,
- $k_i^j = k_i^1 + j - 1$ *for* $i = 1, \ldots, n$ *and* $j = 1, \ldots, N_j$,
- $\Delta k_{i+1} \equiv k_{i+1}^1 - k_i^{N_i} > 1$ *for* $i = 1, \ldots, n-1$,
- $N_i = O(b^{-2})$, *for* $i = 1, \ldots, n$ *and*
- $\sum_{i=1}^{n} N_i = M$.

Then for $M \to \infty$, $M \cdot \pi b^2 = c$ *constant, the Bethe roots in the tuple* $\boldsymbol{t}$ *corresponding to* $\boldsymbol{k}$ *will condense into* n *finite intervals* I_i *on the real line.*

Hints for the correctness: Unfortunately, we didn't find a rigorous proof of this conjecture. Nevertheless, we could proof parts of with additional assumptions, which are legitimated by a numerical analysis.
Remember, that the classical Bethe ansatz equations are of the form

$$\pi(2k_a+1) = mR\sinh\gamma_a + \sum_{b=1}^{M}\left\{2\arctan\left(\frac{\sinh(\gamma_a-\gamma_b)}{\pi b^2}\right)+\pi\right\}. \tag{B.17}$$

We will need the functional relation for the arctan,

$$\arctan\frac{1}{x} = sign(x)\frac{\pi}{2} - \arctan x, \tag{B.18}$$

and its expansion for $x < 1$,

$$\arctan x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}. \tag{B.19}$$

Indication, that the intervals I_j are at least of finite size:

We will work with the special case, that there is only one interval I. Assume, that the length of I scales with b^2 as $\gamma_M - \gamma_1 = xb^{2y}$, $y > 0$. Then it holds that $k_M - k_1 = M$ and thus

$$\begin{aligned} 2\pi M &= mR\left(\sinh(\gamma_1 + xb^{2y}) - \sinh\gamma_1\right) \\ &+ 2\sum_{c=1}^{M}\left(\arctan\left(\frac{\sinh(\gamma_1+xb^{2y}-\gamma_c)}{\pi b^2}\right) - \arctan\left(\frac{\sinh(\gamma_1-\gamma_c)}{\pi b^2}\right)\right). \end{aligned} \tag{B.20}$$

The first term in (B.20) is maximal if $\gamma_c = \gamma_1$; the second term is minimal if $\gamma_c = \gamma_M$. Altogether, it holds that

$$\begin{aligned} 2\pi M &\leq mR\left(\sinh(\gamma_1 + xb^{2y}) - \sinh\gamma_1\right) + 4M\arctan\left(\frac{xb^{2y}}{\pi b^2}\right) \\ &= mR\left(\sinh(\gamma_1 + xb^{2y}) - \sinh\gamma_1\right) + 2\pi M - 4M\arctan\left(\frac{\pi b^2}{xb^{2y}}\right). \end{aligned} \tag{B.21}$$

Using the expansion of the arctan and $M = \frac{c}{b^2}$, we find

$$4c\frac{\pi}{x}\frac{1}{b^{2y}} + O(b^{2(2-3y)}) \leq mR\left(\sinh(\gamma_1 + xb^{2y}) - \sinh\gamma_1\right), \tag{B.22}$$

which is a contradiction, at least as long γ_1 remains finite in the limit $b^2 \to 0$.
In order to check wether $\gamma_1 \to \pm\infty$ is possible we regard the Bethe ansatz equations. They directly imply that $\pi(2k_1 + 1) \geq \sinh\gamma_1$, which excludes the possibility of $\gamma_1 \to +\infty$. They also imply that $\pi(2k_1 + 1) \leq \sinh\gamma_M$, thus $\gamma_M \to -\infty$ is forbidden. Together with the assumption that the interval is of vanishing length in the limit $b^2 \to 0$, this contradicts $\gamma_1 \to \pm\infty$. ∎

Now we would like to prove that there the intervals I_j are also at most of finite size, but this turned out to be not so easy. The main problem is to show that the Bethe roots are homogeniously distributed, i.e. that all the distances $\gamma_{a+1} - \gamma_a$, for γ_a, γ_{a+1} in the same interval, scale with the same power of b^2.

Hints that the intervals I_j are of finite length under the additional assumption of homogeniously distributed Bethe roots:

Again, we will work with the special case where only one interval I is present.
The assumption that the Bethe roots are homogeniously distributed, together with the statement

that the intervals are at least of finite size, imply that the average distance of the Bethe roots scales with b^2 as b^{2z}, where $z \leq 1$.
We will assume that $\gamma_M - \gamma_1 = xb^y$ with $y \leq 0$. As before, equation (B.20) holds. Using the addition theorem for the arctan,

$$\arctan x - \arctan y = \pi + \arctan\left(\frac{x-y}{1+xy}\right) \text{ for } xy < -1, x > 0, \tag{B.23}$$

we find

$$mR\left(\sinh(\gamma_M) - \sinh\gamma_1\right) \tag{B.24}$$

$$= 4\pi + 2\sum_{c=1}^{M} \arctan\left(\frac{\sinh(\gamma_M - \gamma_c) + \sinh(\gamma_c - \gamma_1)}{\pi b^2\left(\frac{\sinh(\gamma_c-\gamma_1)\sinh(\gamma_1-\gamma_c+xb^y)}{(\pi b^2)^2} - 1\right)}\right) \tag{B.25}$$

$$\leq 4\pi + 2\sum_{c=1}^{M} \arctan\left(\frac{\sinh(xb^y)}{\pi b^2\left(\frac{\sinh(\gamma_c-\gamma_1)\sinh(\gamma_1-\gamma_c+xb^y)}{(\pi b^2)^2} - 1\right)}\right). \tag{B.26}$$

The 4π are there because (B.23) is true only for $x, y \neq 0$; the last inequality holds because $\sinh(a-b) + \sinh(b-c) \leq \sinh(a-c)$ for $c < b < a$; $a, b, c \in \mathbb{R}$.

It is plausible that the sum in (B.26) is finite:
regard the terms with $\gamma_c - \gamma_1 = O(b^z)$ or $\gamma_M - \gamma_c = O(b^z)$, $z \leq 2$. Each of these terms contributes to the sum with the order $max(O(b^2), O(b^{2-z}))$[1]. By assumption, the distance of two adjacent Bethe roots is at least of the order b^2; thus there are at most $min(O(b^{-2}), O(b^{z-2}))$[2] many Bethe roots with a distance of γ_1, γ_M of the order b^z. This implies that sum of their contributions is finite.
So, all we have to do is to split the line between γ_1 and γ_M into finitely many sectors S_z, each sector S_z having a distace $O(b^z)$ of the boundary points γ_1, γ_M. As the contribution of each sector is finite, the sum in (B.26) is also finite. This implies that the interval is of finite length.

Numerical analyses

Since a decent proof is lacking for conjecture 2, we studied the solution of the classical Bethe ansatz equations numerically, using Mathematica. We investigated the case where $n = 1$, $k_1^1 = 1$ and $M \cdot \pi b^2 = 1$; M varied from 1 to 800[3]. In the following scheme we list the results.

M	γ_1	γ_M	$\gamma_M - \gamma_1$	$\frac{\gamma_M - \gamma_1}{M}$
1	1.86230	1.86230	0	0
25	0.69527	2.60158	1.9063	0.076
50	0.62426	2.64105	2.0168	0.040
100	0.57891	2.66609	2.0872	0.021
150	0.56050	2.67624	2.1157	0.014
200	0.55012	2.68195	2.1318	0.011
400	0.53189	2.69199	2.1601	0.0054
512	0.52717	2.69458	2.1674	0.0042
600	0.52452	2.69604	2.1715	0.0036
700	0.52219	2.69732	2.1751	0.0031
800	0.52037	2.69832	2.1780	0.0027

It seems that the length of the interval, $\gamma_M - \gamma_1$, converges to some finite number.

[1] Terms for Bethe roots with $z \leq 0$ contribute with the order $O(b^2)$.
[2] There is only a total of $O(b^{-2})$ Bethe roots.
[3] 800 is the upper bound, up to which the available computers were able to solve the nonlinear system of Bethe ansatz equations.

It is instructive to investigate how the average spacing of the Bethe roots, $\frac{\gamma_M - \gamma_1}{M}$, depends on $\frac{1}{M}$ for $M \to \infty$. Therefore we regard figure B.2.1; we see a linear behaviour, the fitting line has the

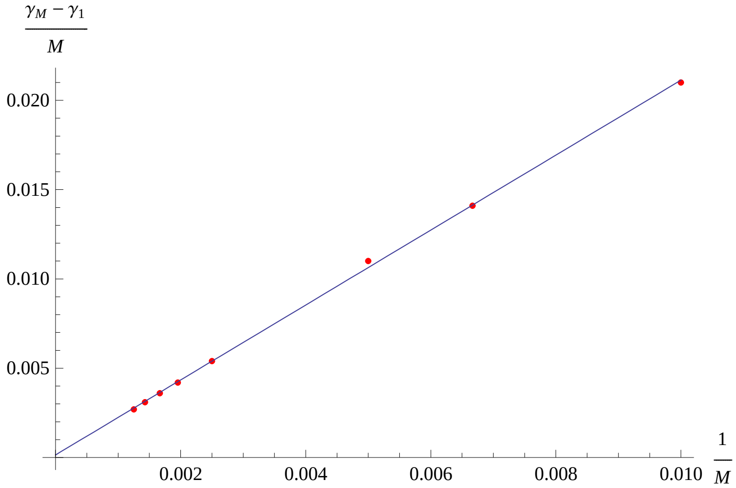

Figure B.1: *The average spacing of Bethe roots,* $\frac{\gamma_M - \gamma_1}{M}$*, over* $\frac{1}{M}$*; the dots are the data points, the line is the best linear fit.*

equation

$$f(x) = 2.09798\, x + 0.00015. \tag{B.27}$$

As the absolute term in (B.27) is near to 0 it will most likely vanish for $M \to \infty$; this would imply that the intervals are indeed of finite length.
Altogether, the numerical results support the claim of conjecture 2.

B.2.2 Next to leading order

In order to perform the semiclassical limit on the $\gamma-$plane we will need

$$\tau\left(\gamma_a - \gamma\right) = -\frac{2\pi b^2 \sinh\left(\gamma_a^0 - \gamma\right)}{\cosh^2(\gamma_a^0 - \gamma)} + O(b^4) \tag{B.28}$$

and

$$\begin{aligned} \arg S(\gamma_a - \gamma_b) = 2\pi\Theta(\gamma_a^0 - \gamma_b^0) - \frac{2\pi b^2}{\sinh(\gamma_a^0 - \gamma_b^0)} + \frac{2\pi b^4}{\sinh(\gamma_a^0 - \gamma_b^0)} \\ + \frac{2\pi^2 b^4 (\gamma_a^1 - \frac{\gamma_a^0}{\pi} - \gamma_b^1 - \frac{\gamma_b^0}{\pi}) \cosh(\gamma_a^0 - \gamma_b^0)}{\sinh^2(\gamma_a^0 - \gamma_b^0)} + O(b^6). \end{aligned} \tag{B.29}$$

With that we calculated the next to leading order of the Bethe ansatz equations and found

$$\begin{aligned} & mR\left(\gamma_a^1 - \frac{\gamma_a^0}{\pi}\right)\cosh\gamma_a^0 - \frac{1}{\pi}\int_{\mathbb{R}} d\gamma \frac{\sinh(\gamma_a^0 - \gamma)}{\cosh^2(\gamma_a^0 - \gamma)} \log\left(1 - \Lambda^2\left(\gamma - \frac{i\pi}{2}\right)\right) \\ & + 2b^2 \sum_{\substack{c=1 \\ c\neq a}}^{M} \left\{ \frac{1}{\sinh(\gamma_a^0 - \gamma_c^0)} + \pi(\gamma_a^1 - \gamma_c^1)\frac{\cosh(\gamma_a^0 - \gamma_c^0)}{\sinh^2(\gamma_a^0 - \gamma_c^0)} \right. \\ & \left. -(\gamma_a^0 - \gamma_c^0)\frac{\cosh(\gamma_a^0 - \gamma_c^0)}{\sinh^2(\gamma_a^0 - \gamma_c^0)} \right\} = 0. \end{aligned} \tag{B.30}$$

Another way of representing this is

$$
\begin{aligned}
&mR\left(\gamma_a^1 - \frac{\gamma_a^0}{\pi}\right)\cosh\gamma_a^0 - \frac{1}{\pi}\int_{\mathbb{R}} d\gamma \frac{\sinh(\gamma_a^0-\gamma)}{\cosh^2(\gamma_a^0-\gamma)}\log\left(1-\Lambda^2\left(\gamma-\frac{i\pi}{2}\right)\right)\\
&+2b^2\sum_{\substack{c=1\\c\neq a}}^{M}\frac{1}{\sinh(\gamma_a^0-\gamma_c^0)} - 2b^2(\pi\gamma_a^1-\gamma_a^0)\frac{d}{d\gamma_a^0}\sum_{\substack{c=1\\c\neq a}}^{M}\frac{1}{\sinh(\gamma_a^0-\gamma_c^0)}\\
&+2b^2\frac{d}{d\gamma_a^0}\sum_{\substack{c=1\\c\neq a}}^{M}(\pi\gamma_c^1-\gamma_c^0)\frac{1}{\sinh(\gamma_a^0-\gamma_c^0)} = 0.
\end{aligned}
\tag{B.31}
$$

Going to the continuum as usual we find for $\gamma \in \mathbb{I}$

$$
\begin{aligned}
2f(\gamma)\frac{d}{d\gamma}\Im\log\Lambda(\gamma) &= -\frac{1}{\pi}\int_{\mathbb{R}} d\gamma' \frac{\sinh(\gamma-\gamma')}{\cosh^2(\gamma-\gamma')}\log\left(1-\Lambda^2\left(\gamma'-\frac{i\pi}{2}\right)\right)\\
&+\frac{2}{\pi}P.V.\int_{\mathbb{R}} d\gamma'\frac{\rho^0(\gamma')}{\sinh(\gamma-\gamma')} + 2\frac{d}{d\gamma}P.V.\int_{\mathbb{R}} d\gamma'\frac{f(\gamma')\rho^0(\gamma')}{\sinh(\gamma-\gamma')},
\end{aligned}
\tag{B.32}
$$

with

$$
f(\gamma) = \gamma^1(\gamma) - \frac{\gamma}{\pi}. \tag{B.33}
$$

As the left hand side of (6.36) is zero on the cuts, equation (B.32) reduces to

$$
\begin{aligned}
0 = &-\frac{1}{\pi}\int_{\mathbb{R}} d\gamma' \frac{\sinh(\gamma-\gamma')}{\cosh^2(\gamma-\gamma')}\log\left(1-\Lambda^2\left(\gamma'-\frac{i\pi}{2}\right)\right)\\
&+\frac{2}{\pi}P.V.\int_{\mathbb{R}} d\gamma'\frac{\rho_c(\gamma')}{\sinh(\gamma-\gamma')} + 2\frac{d}{d\gamma}P.V.\int_{\mathbb{R}} d\gamma'\frac{f(\gamma')\rho_c(\gamma')}{\sinh(\gamma-\gamma')} \qquad \forall\gamma\in\mathbb{I}.
\end{aligned}
\tag{B.34}
$$

B.2.3 Solution of the integral equation

In this section we will solve equation (B.34). In principal we have to solve an equation of the type

$$
g(\gamma) = P.V.\int_{\mathbb{R}} d\gamma'\frac{f(\gamma')\rho_c(\gamma')}{\sinh(\gamma-\gamma')} \tag{B.35}
$$

with

$$
\begin{aligned}
g(\gamma) = -\int_{-\infty}^{\gamma} d\gamma''\Bigg\{&\frac{1}{\pi}\int_{\mathbb{R}} d\gamma'\frac{\sinh(\gamma''-\gamma')}{\cosh^2(\gamma''-\gamma')}\log\left(1-\Lambda^2\left(\gamma'-\frac{i\pi}{2}\right)\right)\\
&+\frac{2}{\pi}P.V.\int_{\mathbb{R}} d\gamma'\frac{\rho_c(\gamma')}{\sinh(\gamma''-\gamma')}\Bigg\}.
\end{aligned}
\tag{B.36}
$$

In order to solve such an equation, we define for $\gamma_1 \notin \mathbb{I}$

$$
U(\gamma_1) := -\frac{1}{2\pi i}\sum_{k=1}^{n}\int_{I_k} d\lambda\frac{f(\lambda)\rho^0(\lambda)}{\sinh(\gamma_1-\lambda)}. \tag{B.37}
$$

It holds trivially that

$$
U(\gamma\pm i0) = -\frac{1}{2\pi i}\sum_{k=1}^{n}P.V.\int_{I_k} d\lambda\frac{f(\lambda)\rho^0(\lambda)}{\sinh(\gamma-\lambda)} \pm \frac{1}{2}f(\gamma)\rho^0(\gamma) \tag{B.38}
$$

for $\gamma \in I_k$ for one k, i.e.

$$
U(\gamma+i0) - U(\gamma-i0) = f(\gamma)\rho^0(\gamma) \tag{B.39}
$$

and

$$\begin{aligned} U(\gamma+i0)+U(\gamma-i0) &= -\frac{1}{i\pi}\sum_{k=1}^{n} P.V. \int_{I_k} d\lambda \frac{f(\lambda)\rho^0(\lambda)}{\sinh(\gamma-\lambda)} \\ &= -\frac{g(\gamma)}{i\pi}. \end{aligned} \tag{B.40}$$

The idea is to solve the boundary problem (B.40) for $U(\gamma_1)$ and to determine $f(\gamma)$ then by (B.39). In order to solve (B.40) we define

$$V(\gamma_1) := \frac{U(\gamma_1)\exp\left(\frac{1}{2}\sum_{k=1}^{n}\int_{I_k}\frac{d\lambda}{\sinh(\gamma_1-\lambda)}\right)}{\prod_{k=1}^{n}\tanh\left(\frac{\gamma_1-a_k}{2}\right)}, \tag{B.41}$$

where the division by $\prod_{k=1}^{n}\tanh\left(\frac{\gamma_1-a_k}{2}\right)$ ensures the regularity of the final solution. So (B.40) is equivalent to

$$V(\gamma-i0) = V(\gamma+i0)+g_1(\gamma), \tag{B.42}$$

with

$$g_1(\gamma) = -\frac{g(\gamma)\exp\left(\frac{1}{2}\sum_{k=1}^{n}P.V.\int_{I_k}\frac{d\lambda}{\sinh(\gamma-\lambda)}\right)}{\pi\prod_{k=1}^{n}\tanh\left(\frac{\gamma_1-a_k}{2}\right)}. \tag{B.43}$$

Note that

$$\exp\left(\frac{1}{2}\sum_{k=1}^{n}P.V.\int_{a_k}^{b_k}\frac{d\lambda}{\sinh(\gamma-\lambda)}\right) = i\prod_{k=1}^{n}\sqrt{\frac{\tanh\left(\frac{\gamma-a_k}{2}\right)}{\tanh\left(\frac{\gamma-b_k}{2}\right)}}, \tag{B.44}$$

as

$$P.V.\int_{a_k}^{b_k}\frac{d\lambda}{\sinh(\gamma-\lambda)} = \log\left(\frac{\tanh\left(\frac{\gamma-a_k}{2}\right)}{\tanh\left(\frac{\gamma-b_k}{2}\right)}\right) \text{ for } \gamma\notin I_k \tag{B.45}$$

and

$$P.V.\int_{a_k}^{b_k}\frac{d\lambda}{\sinh(\gamma-\lambda)} = \log\left(\frac{\tanh\left(\frac{\gamma-a_k}{2}\right)}{\tanh\left(\frac{b_k-\gamma}{2}\right)}\right) \text{ for } \gamma\in I_k. \tag{B.46}$$

It follows that

$$g_1(\gamma) = -\frac{ig(\gamma)}{\pi\prod_{k=1}^{n}\sqrt{\tanh\left(\frac{\gamma-a_k}{2}\right)\tanh\left(\frac{\gamma-b_k}{2}\right)}}. \tag{B.47}$$

The solution of (B.42) is

$$V(\gamma_1) = \frac{1}{2\pi i}\sum_{k=1}^{n}\int_{I_k}\frac{g_1(\lambda)d\lambda}{\sinh(\lambda-\gamma_1)}. \tag{B.48}$$

The zero modes are given by the set of single valued functions. We demand the zero modes to be analytic and zero at ∞, so there are no zero modes at all.

Calculating $U(\gamma_1)$ we find

$$U(\gamma_1) = V(\gamma_1)\prod_{k=1}^{n}\tanh\left(\frac{\gamma_1 - a_k}{2}\right)\exp\left(-\frac{1}{2}\sum_k\int_{I_k}\frac{d\lambda}{\sinh(\gamma_1-\lambda)}\right) \tag{B.49}$$

$$\begin{aligned} &= \sqrt{\prod_{k=1}^{n}\tanh\left(\frac{\gamma_1 - b_k}{2}\right)\tanh\left(\frac{\gamma_1 - a_k}{2}\right)} \\ &\cdot \frac{1}{2\pi^2}\sum_k\int_{I_k}\frac{g(\lambda)}{\sinh(\gamma_1-\lambda)}\frac{d\lambda}{\sqrt{\prod_{k=1}^{n}\tanh\left(\frac{\lambda-b_k}{2}\right)\tanh\left(\frac{\lambda-a_k}{2}\right)}}. \end{aligned} \tag{B.50}$$

The square root is defined to be positive above the cuts . Now we find the solution

$$f(\gamma)\rho^0(\gamma) = U(\gamma + i0) - U(\gamma - i0) \tag{B.51}$$

$$\begin{aligned} &= \frac{1}{\pi^2}\sqrt{\prod_{k=1}^{n}\tanh\left(\frac{\gamma - b_k + i0}{2}\right)\tanh\left(\frac{\gamma - a_k + i0}{2}\right)} \\ &\cdot \sum_k P.V.\int_{I_k}\frac{g(\lambda)}{\sinh(\gamma-\lambda)}\frac{d\lambda}{\sqrt{\prod_{k=1}^{n}\tanh\left(\frac{\lambda-b_k}{2}\right)\tanh\left(\frac{\lambda-a_k}{2}\right)}}. \end{aligned} \tag{B.52}$$

for γ in the cuts. Note that this is the unique solution that is regular at the end of the cuts.

B.3 Eigenvalue of the monodromy matrix

B.3.1 The next to leading order

In order to calculate the next to leading order, we have to expand the difference $\log Q\left(\gamma + i\frac{\pi b^2}{2}\right) - \log Q\left(\gamma - i\frac{\pi b^2}{2}\right)$. We will start from the discrete semiclassical expansion of $\log Q$,

$$\begin{aligned} \log Q(\gamma) &= -mR\frac{\cosh\gamma}{2\pi b^2} - mR\frac{\cosh\gamma}{2\pi} + \frac{mR}{2\pi}\gamma\sinh\gamma \\ &- \int_{\mathbb{R}}\frac{d\gamma'}{2\pi}\frac{\log(1-\Lambda^2(\gamma' - \frac{i\pi}{2}))}{\cosh(\gamma-\gamma')} \\ &+ \sum_{a=1}^{M}\left\{\int_{\mathcal{C}_a}^{\gamma}\frac{1}{\sinh(\gamma'-\gamma_a)}d\gamma' + b^2\int_{\mathcal{C}_a}^{\gamma}\frac{\cosh(\gamma'-\gamma_a)}{\sinh^2(\gamma'-\gamma_a)}(\gamma'-\gamma_a)d\gamma' \right. \\ &\left. -b^2\int_{\mathcal{C}_a}^{\gamma}\frac{1}{\sinh(\gamma'-\gamma_a)}d\gamma'\right\} + O(b^4). \end{aligned} \tag{B.53}$$

For the calculation we will need

$$\cosh\left(\gamma + i\frac{\pi b^2}{2}\right) - \cosh\left(\gamma - i\frac{\pi b^2}{2}\right) = i\pi b^2\sinh\gamma + O(b^6), \tag{B.54}$$

$$\begin{aligned} &\left(\gamma + i\frac{\pi b^2}{2}\right)\sinh\left(\gamma + i\frac{\pi b^2}{2}\right) - \left(\gamma - i\frac{\pi b^2}{2}\right)\sinh\left(\gamma - i\frac{\pi b^2}{2}\right) \\ &= i\pi b^2\sinh\gamma + i\pi b^2\gamma\cosh\gamma + O(b^4) \end{aligned} \tag{B.55}$$

and

$$\cosh^{-1}\left(\gamma - \gamma' + i\frac{\pi b^2}{2}\right) - \cosh^{-1}\left(\gamma - \gamma' - i\frac{\pi b^2}{2}\right) \tag{B.56}$$

$$= -i\pi b^2\frac{\sinh(\gamma-\gamma')}{\cosh^2(\gamma-\gamma')} + O(b^6). \tag{B.57}$$

Further we will need

$$\left(\int_{C_a}^{\gamma+i\frac{\pi b^2}{2}} - \int_{C_a}^{\gamma-i\frac{\pi b^2}{2}}\right) d\gamma' f(\gamma') = i\pi b^2 f(\gamma) + O(b^6), \tag{B.58}$$

which holds as long as f is regular at γ. So we find for the next to leading order of $\log \Lambda_q$ for $\gamma \neq \gamma_a \forall a$

$$\begin{aligned} \log\,\Lambda_1(\gamma) &= i\frac{mR}{2}\gamma\cosh\gamma + \frac{i}{2}\int_{\mathbb{R}} d\gamma' \frac{\sinh(\gamma-\gamma')}{\cosh^2(\gamma-\gamma')}\log\left(1-\Lambda_c^2\left(\gamma'-\frac{i\pi}{2}\right)\right) \\ &- i\pi b^2 \frac{d}{d\gamma}\sum_{a=1}^{M}\frac{\pi\gamma_a^1-\gamma_a^0}{\sinh(\gamma-\gamma_a^0)} + i\pi b^2\gamma\frac{d}{d\gamma}\sum_{a=1}^{M}\frac{1}{\sinh(\gamma-\gamma_a^0)} \\ &- i\pi b^2\sum_{a=1}^{M}\frac{1}{\sinh(\gamma-\gamma_a^0)}. \end{aligned} \tag{B.59}$$

Going to the continuum we get for $\gamma \notin \mathbb{I}$

$$\begin{aligned} \log\,\Lambda_1(\gamma) &= -\gamma\frac{d}{d\gamma}\log\Lambda_c(\gamma) + \frac{i}{2}\int_{\mathbb{R}} d\gamma' \frac{\sinh(\gamma-\gamma')}{\cosh^2(\gamma-\gamma')}\log\left(1-\Lambda_c^2\left(\gamma'-\frac{i\pi}{2}\right)\right) \\ &- i\frac{d}{d\gamma}\int_{\mathbb{I}}\frac{\rho_c(\gamma')}{\sinh(\gamma-\gamma')}(\pi\gamma^1(\gamma')-\gamma')d\gamma' - i\int_{\mathbb{I}}\frac{\rho_c(\gamma')}{\sinh(\gamma-\gamma')}d\gamma'. \end{aligned} \tag{B.60}$$

B.4 Q-function

The $\vartheta-$plane $Q-$function is given as

$$\log Q(\vartheta) = -mR\frac{\cosh\vartheta}{2\sin\vartheta_0} + \int_{\mathbb{R}}\frac{d\vartheta'}{2\pi}\frac{\log(1+Y(\vartheta'))}{\cosh(\vartheta-\vartheta')} + \sum_{a=1}^{M}\int_{\mathcal{C}_a}^{\vartheta}\frac{d\vartheta'}{\sinh(\vartheta'-\vartheta_a)}. \tag{B.61}$$

We will calculate its semiclassical limit on the $\gamma-$plane. Therefor we will use the following expansions,

$$\cosh\frac{\gamma}{1+b^2} = \cosh\gamma - b^2\gamma\sinh\gamma + O(b^2), \tag{B.62}$$

$$\frac{1}{\sin\vartheta_0} = \frac{1}{\pi b^2} + \frac{1}{\pi} + O(b^2) \tag{B.63}$$

and

$$\frac{1}{\sinh(\vartheta'-\vartheta_a)} = \sinh^{-1}\left(\frac{\gamma'-\gamma_a}{1+b^2}\right) \tag{B.64}$$

$$= \frac{1}{\sinh(\gamma'-\gamma_a)} + b^2(\gamma'-\gamma_a)\frac{\cosh(\gamma'-\gamma_a)}{\sinh^2(\gamma'-\gamma_a)} + O(b^4). \tag{B.65}$$

Further we will need the integral

$$\begin{aligned} \int_{C_a}^{\vartheta}\frac{1}{\sinh(\vartheta'-\vartheta_a)}d\vartheta' &= \int_{C_a}^{\gamma}\sinh^{-1}\left(\frac{\gamma'-\gamma_a}{1+b^2}\right)\frac{d\gamma'}{1+b^2} \\ &= \int_{C_a}^{\gamma}\frac{1}{\sinh(\gamma'-\gamma_a)}d\gamma' + b^2\int_{C_a}^{\gamma}(\gamma'-\gamma_a)\frac{\cosh(\gamma'-\gamma_a)}{\sinh^2(\gamma'-\gamma_a)}d\gamma' \\ &- b^2\int_{C_a}^{\gamma}\frac{1}{\sinh(\gamma'-\gamma_a)}d\gamma' + O(b^4). \end{aligned} \tag{B.66}$$

It holds that

$$1+Y(\gamma)=\frac{1}{1-\Lambda^2\left(\gamma-\frac{i\pi}{2}\right)}+O(b^2). \tag{B.67}$$

Altogether we find as semiclassical expansion of (B.61)

$$\begin{aligned}\log Q(\gamma) &= -mR\frac{\cosh\gamma}{2\pi b^2}-mR\frac{\cosh\gamma}{2\pi}+\frac{mR}{2\pi}\gamma\sinh\gamma\\ &-\int_{\mathbb{R}}\frac{d\gamma'}{2\pi}\frac{\log(1-\Lambda^2(\gamma'-\frac{i\pi}{2}))}{\cosh(\gamma-\gamma')}\\ &+\sum_{a=1}^{M}\left\{\int_{\mathcal{C}_a}^{\gamma}\frac{1}{\sinh(\gamma'-\gamma_a)}d\gamma'+b^2\int_{\mathcal{C}_a}^{\gamma}\frac{\cosh(\gamma'-\gamma_a)}{\sinh^2(\gamma'-\gamma_a)}(\gamma'-\gamma_a)d\gamma'\right.\\ &\left.-b^2\int_{\mathcal{C}_a}^{\gamma}\frac{1}{\sinh(\gamma'-\gamma_a)}d\gamma'\right\}+O(b^4).\end{aligned} \tag{B.68}$$

Going to the continuous limit like in ... we find for $\gamma\notin\mathbb{I}$

$$\begin{aligned}\log Q(\gamma) &= -mR\frac{\cosh\gamma}{2\pi b^2}-mR\frac{\cosh\gamma}{2\pi}+\frac{mR}{2\pi}\gamma\sinh\gamma\\ &-\int_{\mathbb{R}}\frac{d\gamma'}{2\pi}\frac{\log(1-\Lambda^2(\gamma'-\frac{i\pi}{2}))}{\cosh(\gamma-\gamma')}+\frac{1}{\pi b^2}\int_{\mathbb{R}}d\gamma'\rho^0(\gamma')\int_{C_{\gamma'}}^{\gamma}\frac{d\gamma''}{\sinh(\gamma''-\gamma')}\\ &+\int_{\mathbb{R}}d\gamma'\gamma^1(\gamma')\rho^0(\gamma')\int_{\mathcal{C}_{\gamma'}}^{\gamma}\frac{\cosh(\gamma''-\gamma')}{\sinh^2(\gamma''-\gamma')}d\gamma''\\ &+\frac{1}{\pi}\int_{\mathbb{R}}d\gamma'\rho^0(\gamma')\int_{\mathcal{C}_{\gamma'}}^{\gamma}d\gamma''(\gamma''-\gamma')\frac{\cosh(\gamma''-\gamma')}{\sinh^2(\gamma''-\gamma')}\\ &-\frac{1}{\pi}\int_{\mathbb{R}}d\gamma'\rho^0(\gamma')\int_{C_{\gamma'}}^{\gamma}\frac{d\gamma''}{\sinh(\gamma''-\gamma')}+O(b^4).\end{aligned} \tag{B.69}$$

Using integration by parts we calculate

$$\int_{C_{\gamma'}}^{\gamma}\frac{\cosh(\gamma''-\gamma')}{\sinh^2(\gamma''-\gamma')}(\gamma''-\gamma')d\gamma''=-\frac{\gamma-\gamma'}{\sinh(\gamma-\gamma')}+\int_{C_{\gamma'}}^{\gamma}\frac{d\gamma''}{\sinh(\gamma''-\gamma')}. \tag{B.70}$$

Subsumed we find for $\gamma\notin\mathbb{I}$

$$\begin{aligned}\log Q(\gamma) &= -mR\frac{\cosh\gamma}{2\pi b^2}+\frac{1}{\pi b^2}\int_{\mathbb{R}}d\gamma'\rho^0(\gamma')\int_{C_{\gamma'}}^{\gamma}\frac{d\gamma''}{\sinh(\gamma''-\gamma')}\\ &-mR\frac{\cosh\gamma}{2\pi}+i\frac{\gamma}{\pi}\log\Lambda(\gamma)-\int_{\mathbb{R}}\frac{d\gamma'}{2\pi}\frac{\log(1-\Lambda^2(\gamma'-\frac{i\pi}{2}))}{\cosh(\gamma-\gamma')}\\ &-\int_{\mathbb{R}}d\gamma'\frac{\rho^0\gamma')}{\sinh(\gamma-\gamma')}\left(\gamma^1(\gamma')-\frac{\gamma'}{\pi}\right)+O(b^2).\end{aligned} \tag{B.71}$$

The function $\gamma^1(\gamma)$ is determined in the appendix about the integral equation.

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