On the fusion in SL(2)-WZNW models and $6j$ symbols of $\mathcal{U}_q\mathbf{sl}(2) \times \mathcal{U}_{q'}\mathbf{osp}(1|2)$

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Abstract

We introduce a novel method to determine $6j$-symbols of quantum groups. This method is inspired by the methods used in the determination of fusing matrices of WZNW models. With this method we determine the $6j$-symbols of the quantum group $\mathcal{U}_q\mathbf{sl}(2)$ and the super quantum group $\mathcal{U}_q\mathbf{osp}(1|2)$.

We present the $6j$-symbols as a recurrence relation and its initial values. The $6j$-symbols transform between the s-channel and the u-channel decomposition of the invariants of the four-fold tensor product of modules of a quantum group. These invariants fulfil certain difference equations.
We set one of the representations in the invariant to the fundamental representation, and deduce a system of linear equations for the initial values of the recurrence relation determining the $6j$-symbols.

Zusammenfassung

Wir führen eine neue Methode ein, die $6j$-Symbole von Quantengruppen zu bestimmen. Inspiriert wird diese Methode durch Methoden, die in der Bestimmung der Fusionsmatrizen von WZNW-Modellen zur Anwendung kommen. Mit Hilfe dieser Methode bestimmen wir die $6j$-Symbole der Quantengruppe $\mathcal{U}_q\mathbf{sl}(2)$ und der Superquantengruppe $\mathcal{U}_q\mathbf{osp}(1|2)$.

Wir stellen die $6j$-Symbole als eine Rekursionsrelation samt Anfangswerten dar. Die $6j$-Symbole verbindet die s-Kanal- mit der u-Kanalzerlegung der Invarianten des vierfachen Tensorproduktes von Modulen der Quantengruppe. Diese Invarianten erfüllen bestimmte Differenzengleichungen.
Wenn eine der Darstellungen der Invarianten auf die fundamentale Darstellung eingeschränkt wird, können wir ein System linearer Gleichungen für die Anfangsbedingungen der Rekursionsrelationen der $6j$-Symbole herleiten.

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Chapter 1
Introduction

Two-dimensional conformal symmetry has been a highly successful tool of theoretical and mathematical physics for the past 25 years. In statistical models at the critical point we have conformal symmetry. This does not only help in the solution of models at the critical point, but makes possible the exploration of models in the vicinity of that point via conformal perturbation theory. String theory depends on conformal symmetry to make sure the world sheet description is independent of the parametrisation of the string. Vertex operators have cross fertilised into pure mathematics where they proliferate in representation theory and were used in the proof of the monstrous moonshine conjecture. Conformal symmetry has recently found a way into probability theory as the scaling limit of certain percolation models.

Despite these successes a classification of conformal field theory is far from complete. There are some well understood classes of conformal field theories. Among these are the celebrated minimal models and the Wess–Zumino–Novikov–Witten models (WZNW models in the following). In search of further classes of conformal field theories it is plausible to make the first steps by examining which generalisations of these well understood theories are possible. String theory on curved backgrounds for example demands a specific generalisation of WZNW models.
A further interesting feature is the connection to quantum groups. It has been shown that the superselection structure of WZNW models is equivalent to representation categories of corresponding quantum groups as a braided monoidal category. This correspondence can be formulated in terms of basic objects in both theories, in terms of the conformal blocks and invariant tensors.
We express these basic objects as a recurrence relation and its initial values simplifying this correspondence.

Before we state the original content of this work, we have to introduce the setting in which the main body of this work takes places. We give a quick overview of conformal field theory, especially the WZNW models and quantum groups. Quantum numbers and quantum calculus are introduced in appendix A. The introduction of super quantum numbers and a cursory overview of super linear algebra and super algebras have been relegated to appendix B.

With these things in place we state the content of this work in section 4 and give an overview of the main body.

1. Conformal field theory

The seminal work of Belavin, Polyakov and Zamolodchikov [**4**] describes a *conformal field theory* in the setting of Euclidean quantum field theory as the correlators of a large number of *local fields* $A(z,\overline{z})$. These fields are called *scaling fields* too. The set of local fields has the following properties.

(1) The derivative ∂A of a local field A is a local field again.
(2) There is a subset of local fields, called *quasi-primary fields* that transform under projective conformal transformations

$$z \mapsto w(z) = \frac{az+b}{cz+d} \quad \text{as}$$

$$\Phi(z,\overline{z}) \mapsto \left(\frac{\partial w}{\partial z}\right)^{h} \left(\frac{\partial \overline{w}}{\partial \overline{z}}\right)^{\overline{h}} \Phi(w(z),\overline{w}(\overline{z})) \ .$$

(3) Any local field A can be written as a linear combination of quasi-primary fields and their derivatives.
(4) The vacuum is invariant under projective conformal transformations.
(5) Since in two dimensions the set of conformal transformations extends to include the analytic transformations, we can identify a further subset of the local fields in two dimensional theories.
The subset of *primary fields* contains the quasi-primary fields that transform as

$$\Phi(z,\overline{z}) \mapsto \left(\frac{\partial w}{\partial z}\right)^{h} \left(\frac{\partial \overline{w}}{\partial \overline{z}}\right)^{\overline{h}} \Phi(w(z),\overline{w}(\overline{z})) \ .$$

under any conformal transformation.

All fields that are not primary are called *descendant* or *secondary fields.* The parameters $h,\bar{h}$ are called the *conformal weights* of the primary field Φ. They are connected to two parameters of statistical physics, the *scaling dimension* $d = h + \bar{h}$ and the *conformal spin* $s = h - \bar{h}$.

The space of states of a conformal field theories is a module for the Virasoro algebra. The Virasoro algebra is the algebra with generators $\{L_n \mid n \in \mathbf{Z}\}$ and the *central charge* c subject to the relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \ .$$

This is the basis for the second approach to conformal field theory. The space of states for a conformal field theory is a Hilbert space $\mathcal{H}$ decomposing into Virasoro modules

$$\mathcal{H} = \bigoplus_{h,\bar{h}} \mathcal{H}_h \otimes \bar{\mathcal{H}}_{\bar{h}} \ .$$

The Laurent modes L_n of the energy-momentum tensor T and the corresponding antiholomorphic objects are

$$T(z) = \sum_{n\in\mathbf{Z}} z^{-n-2} L_n, \quad \bar{T}(\bar{z}) = \sum_{n\in\mathbf{Z}} \bar{z}^{-n-2} \bar{L}_n \ .$$

Thus we have formally $L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)$. The Laurent modes fulfil the relations for two commuting copies of the Virasoro algebra.

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \ , \\
[\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \ , \\
[L_m, \bar{L}_n] &= 0 \ .
\end{aligned}$$

The halves are called the chiral algebras.

Note that the generators L_{-1}, L_0 and L_1 form a subalgebra isomorphic to $\mathbf{sl}(2)$. This algebra is often called the global or the projective conformal transformations.

The modes L_n of the energy-momentum tensor act on a primary field $\Phi(z,\bar{z})$ of conformal weight h by

$$[L_n, \Phi(z,\bar{z})] = z^n (z\partial_z + h(n+1))\Phi(z,\bar{z}) \ . \tag{1.1}$$

In the following we will mainly be considering the holomorphic chiral half of the theory.

Primary fields Φ of conformal weight h generate a Virasoro module by the *Verma module* construction[1]. The primary field itself corresponds to a highest weight vector of weight h. Its descendant fields are constructed in terms of the Virasoro generators $\Phi^{(n_l,\ldots,n_1)} = L_{-n_k} \cdots L_{-n_1} \Phi$

$$\left(L_n \Phi\right)(z,\bar{z}) = \frac{1}{2\pi i} \oint_z w^{n+1} T(w) \Phi(z,\bar{z}) \ .$$

The span of a primary field Φ and its descendants is called a *conformal family* and denoted by $[\Phi]$.

[1]Given a highest weight vector $\mathbf{v}_h$, $L_n \mathbf{v}_h = 0$, $n > 0$ the Verma module is the linear span of all elements $L_{-n_k} \cdots L_{-n_1} \mathbf{v}_h$. Virasoro elements act on such an elements by concatenation, if they are negative and if they are positive by $L_m L_{-n} \mathbf{v}_h = [L_m, L_{-n}] \mathbf{v}_h$. More on the Verma module follows below.

Conformal symmetry strongly constrains the correlation functions[2]. Correlation functions of primary fields Φ_i transform covariantly under conformal transformations $z \mapsto w = w(z)$

$$\langle \Phi_N(z_N) \cdots \Phi_1(z_1) \rangle = \prod_{i=1}^{N} \left(\frac{dw_i}{dz_i} \right)^{h_i} \langle \Phi_N(w_N) \cdots \Phi_1(w_1) \rangle \ .$$

This leads to the fact, that all n-point functions with $n \geq 2$ depend on the differences $z_{ij} = z_i - z_j$ only. It furthermore fixes the form of two- and three-point functions of primary fields

$$\langle \Phi_2(z_2)\Phi_1(z_1) \rangle = z_{12}^{-h_1} C_{12} \delta_{h_1,h_2} \ ,$$
$$\langle \Phi_3(z_3)\Phi_2(z_2)\Phi_1(z_1) \rangle = z_{12}^{h_{12}^3} z_{23}^{h_{23}^1} z_{13}^{h_{13}^2} C_{123} \ ,$$

with $h_{ij}^k = -h_i - h_j + h_k$. In the normalisation where $C_{ij} = \delta_{i,j}$ the coefficient C_{ijk} is just the coefficient of the operator product expansion of $\Phi_i\Phi_j$

$$\Phi_i(z)\Phi_j(w) = \sum_k \frac{C_{ijk}}{(z-w)^{h_i+h_j-h_k}} \Phi_k(w) + \cdots \ .$$

The four-point functions are determined up to a function of the *cross-ratios*

$$z = \frac{z_{43}z_{21}}{z_{42}z_{31}} \ .$$

With the parameter γ_{ij} such that $\sum_i \gamma_{ij} = 2h_j$ the four-point functions are of the form

$$\langle \Phi_4(z_4) \cdots \Phi_1(z_1) \rangle = \prod_{i<j} z_{ij}^{\gamma_{ij}} G_{41}^{32}(z) \ .$$

A correlation function of descendant fields can be expressed via the conformal symmetry as a string of differential operators acting on the correlation function of primary fields only[3]

$$\begin{aligned} &\left\langle \Phi^{(n_m,\ldots,n_1)}(z)\Phi_N(z_N) \cdots \Phi_1(z_1) \right\rangle \\ &\qquad = \mathcal{L}_{n_m}(z) \cdots \mathcal{L}_{n_1}(z) \langle \Phi(z)\Phi_N(z_N) \cdots \Phi_1(z_1) \rangle \ . \end{aligned}$$

The differential operators $\mathcal{L}$ are given by

$$\mathcal{L}_n(z) = \sum_{i=1}^{N} \left(\frac{(1-n)h_i}{(z-z_i)^n} - \frac{1}{(z-z_i)^{n-1}} \partial_{z_i} \right) \ .$$

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[2]Details can be found in any introductory work on conformal field theory, for example the book by Di Francesco, Mathieu and Senechal[**5**], the book by Henkel [**14**] or the article by Ginsparg [**11**].

[3]This fact is based on (1.1) and gives rises to the *conformal Ward identities*. The derivation is demonstrated for example in [**4**].

In some conformal field theories there are *null fields*. These are descendant fields $\chi^{(m_l,\dots,m_1)}$ from a primary field χ such that all correlation functions including this field vanish,

$$\left\langle \Phi_n \Phi_{n-1} \cdots \chi^{(m_l,\dots,m_1)} \cdots \Phi_1 \right\rangle = 0 \ .$$

Primary fields χ having null fields in their families are called *degenerate fields*. The families are also referred to as degenerate. The basic idea of work by Belavin, Polyakov and Zamolodchikov [**4**] is to use degenerate fields to find additional differential equations for correlation functions containing these fields. Analogously in a Virasoro module there can appear nullvectors, vectors orthogonal to all vectors of the module generating a highest weight sub-module.

The *state-operator correspondence* translates the problem of classifying the degenerate fields into a purely representation theoretical problem. In a well behaved conformal field theory primary fields are in one to one correspondence to vectors of highest weight in a representation of the Virasoro algebra. Let $|0\rangle$ be the vacuum of a given theory and $\mathbf{v}_h$ a vector of weight h. We then have for a field Φ of conformal weight h that

$$\mathbf{v}_h = \lim_{z\to 0} \Phi(z)|0\rangle$$

is a vector of highest weight h. The last equality is the limit of the more general

$$e^{zL_{-1}}\mathbf{v}_h = \Phi(z)|0\rangle \ .$$

When we let $\Phi(\mathbf{v}|z)$ denote the field corresponding to the vector $\mathbf{v}$ we can express the state-operator correspondence[4] by

$$L_n \Phi(\mathbf{v}|z) = \Phi(L_n \mathbf{v}|z) \ .$$

In purely representation theoretical terms of Virasoro modules we get the very same structure. Given a highest weight vector $\mathbf{v}_h$,

$$L_n \mathbf{v}_h = 0 \ , \quad \text{for } n > 0 \ ,$$

the negative modes L_{-n} generate the *Verma module* $\mathcal{M}_h$ spanned by vectors of the form

$$\mathbf{v}^{(h;n_l,\dots,n_1)} = L_{-n_l} \cdots L_{-n_1} \mathbf{v}_h \ .$$

The number $N = \sum_{i=1}^{l} n_i$ is called the level of the vector. The Verma module is not necessarily irreducible. It may contain *null vectors* $\mathbf{n}$, that is vectors of level $N > 0$ generating a submodule of their own with themselves as highest weight vector of weight $h + N$

$$L_n \mathbf{n} = 0 \ , \quad \text{for } n > 0 \ .$$

Such a Verma module is called *degenerate*. To get a irreducible module we have to divide all nullvectors out. Let $\mathcal{N}_h = \text{span}\{\mathbf{n} \in \mathcal{M}_h \mid L_n \mathbf{n} =$

[4]This can be seen by $\lim_{z\to 0} L_n \Phi(\mathbf{v}|z)|0\rangle = L_n \mathbf{v} = \lim_{z\to 0} \Phi(L_n \mathbf{v}|z)|0\rangle$.

$0,\ n > 0\}$ be the set of all submodules of the Verma module $\mathcal{M}_h$. Then we get an irreducible module $\mathcal{V}_h$ by setting

$$\mathcal{V}_h = \mathcal{M}_h / \mathcal{N}_h \ .$$

The first basic result of the representation theory of the Virasoro algebra[5] states that $\mathcal{M}_h$ is unitary if and only if $c = c(m), h = h(m)$, for some $m \in \mathbf{R} \setminus \{0, 1\}$ and some $r, s \in \mathbf{N}$ with $s < r$ where

$$c(m) = 1 - \frac{6}{m}(m+1) \quad \text{and} \quad h(m) = \frac{((m+1)r - ms)^2 - 1}{4m}(m+1) \ .$$

Conformal blocks. Consider now a correlation function $\langle \Phi_4 \Phi_3 \Phi_2 \Phi_1 \rangle$ of four primary fields Φ_i in the operator picture. There are two viable ways to decompose the correlation function into three-point functions. One way is to insert the operator product expansion for $\Phi_2 \Phi_1$ and determine the resulting three-point functions, the other is to insert the operator product expansion for $\Phi_3 \Phi_2$ and determine the three-point functions resulting from this insertion. Let Z denote the four-tuple (z_4, z_3, z_2, z_1).

$$\begin{aligned} \langle \Phi_4 \Phi_3 \Phi_2 \Phi_1 \rangle &= \sum_s \sum_{\mathbf{k}} \frac{C^{\mathbf{k}}_{12s}}{z_{21}^{h_1 + h_2 - h_s + K}} \left\langle \Phi_4 \Phi_3 \Phi_s^{\mathbf{k}} \right\rangle \\ &= \sum_s C_{12s} C_{s34} \mathcal{F}^{(s)}{}^{32}_{41}(s|Z) \ . \end{aligned}$$

The second insertion leads to

$$\langle \Phi_4 \Phi_3 \Phi_2 \Phi_1 \rangle = \sum_s C_{23s} C_{1s4} \mathcal{F}^{(u)}{}^{32}_{41}(s|Z) \ .$$

The functions $\mathcal{F}^{(s)}$ and $\mathcal{F}^{(u)}$ are called the s-channel and the u-channel conformal blocks. In general an explicit expression for the conformal block is not known.

Correlation functions of local fields do not depend on the order in which the fields appear in the correlation function. A requirement for physical consistency is thus that the s- and the u-channel decomposition of the four-point functions give the same result. This is called the *crossing symmetry*. From that it can be argued that the conformal blocks have to be related by a transformation F

$$\mathcal{F}^{(u)}{}^{32}_{41}(u|Z) = \sum_s \mathrm{F}_{us} \left[\begin{smallmatrix} 3 & 2 \\ 4 & 1 \end{smallmatrix} \right] \mathcal{F}^{(s)}{}^{32}_{41}(s|Z) \ .$$

The coefficients $\mathrm{F}_{us} \left[\begin{smallmatrix} 3 & 2 \\ 4 & 1 \end{smallmatrix} \right]$ form the *fusing matrix.*

[5] this was found by Kac in [**15**]. See [**16**] for a proof.

The minimal models. The *minimal models* introduced by Belavin, Polyakov and Zamolodchikov [**4**] correspond to degenerate Virasoro representations with $m = p/(q-p)$, where p and q are relatively prime positive integers and $r \leq p-1$, $s \leq q-1$ [**4**]. The conformal weights are periodic according to

$$h_{r,s}(p,q) = h_{r+p,s+q}(p,q) \ .$$

Graphically considering the lattice of allowed values (r,s) for a given pair (p,q) we get the *Kac table*. Each dot in the diagramme corresponds to a primary field of conformal weight $h_{r,s}(p,q)$.

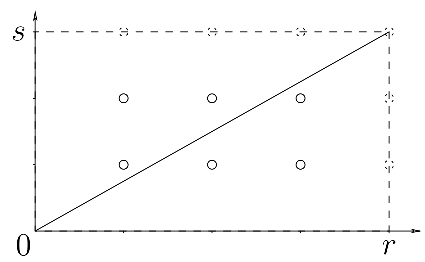

FIGURE 1: Kac table for $(p,q) = (4,3)$

Indeed the minimal models have become the prototype for certain classes of conformal field theories. The next step of generalisation are the rational conformal field theories. Rational conformal field theories emulate the structure of minimal models in so far as they have only a finite number of primary fields. The WZNW models[6] for compact groups are examples of rational conformal field theories.

The rational conformal field theories share as a common trait that all correlation functions decompose into a finite sum of products of holomorphic and antiholomorphic parts

$$\langle \Phi_n \cdots \Phi_1 \rangle = \sum_{i=1}^{N} |F_{n,\ldots,1}(j_i|z_n,\ldots,z_1)|^2 \ ,$$

with representation labels j_i. The space of states decomposes into a finite number of modules for the symmetry algebra of the theory.

$$\mathcal{H} = \bigoplus_{i=1}^{N} \mathcal{H}_{h_i} \otimes \bar{\mathcal{H}}_{\bar{h}_i} \ .$$

[6] named after Julius Wess, Bruno Zumino, Sergei Novikov and Edward Witten.

Because of the decomposition $\mathcal{H}_{h_i} \otimes \bar{\mathcal{H}}_{\bar{h}_i}$ of each sector into two parts, corresponding to the holomorphic and antiholomorphic parts of the fields, the symmetry algebra is called a *chiral algebra*.

WZNW models. WZNW models are a class of nonlinear sigma models taking values in the manifold of a compact simply-connected Lie group G. Let $\mathbf{g}$ be the Lie algebra of G. Then the Hilbert space of states is organised as sum of representations of the model's current algebra, the affine Kac–Moody algebra $\widehat{\mathbf{g}}_k$. Let $\mathbf{g}$ be a Lie algebra with a basis $\{J^a \mid a = 1, 2, \ldots, \dim \mathbf{g}\}$, structure constants f_c^{ab} and invariant bilinear form η^{ab}. The Kac–Moody algebra $\widehat{\mathbf{g}}_k$ is generated by the elements

$$\{\, J_n^a \mid a = 1, 2, \ldots, \dim \mathbf{g}, n \in \mathbf{Z}\}$$

with the product

$$[J_m^a, J_n^b] = \sum_c \mathrm{i} f_c^{ab} J_{m+n}^c + km\eta^{a,b}\delta_{m+n,0} \ .$$

Collecting the modes J_m^a into a generating function

$$J^a(z) = \sum_{n\in\mathbf{Z}} z^{-n-1} J_n^a$$

the algebraic properties of the product are encoded in the operator product expansion

$$J^a(z)J^b(w) = \frac{k\eta^{a,b}}{(z-w)^2} + \frac{if_c^{ab}}{z-w} + \cdots \ .$$

The Verma modules for the Kac–Moody algebra $\widehat{\mathbf{g}}_k$ are quite similar to the ones for the Virasoro algebra. Let R be a representation of the Lie algebra $\mathbf{g}$. The representation R, also called the *zero mode representation* now generates the Verma module. Given a vector $\mathbf{v}$ in the zero mode representation R the positive modes act as

$$\begin{aligned} J_n^a \mathbf{v} &= 0 \ , \quad \text{for } n > 0 \\ J_0^a \mathbf{v} &= R(J^a)\mathbf{v} \end{aligned}$$

and the negative modes J_{-n}^a generate the *Verma module* $\mathcal{M}_{R,k}$

$$\mathcal{M}_{R,k} = \mathrm{span}\left\{\, J_{-n_l}^{a_l} \cdots J_{-n_1}^{a_1}\mathbf{v} \;\middle|\; \mathbf{v} \in R, l \in \mathbf{N}\right\} \ .$$

The number $N = \sum_{i=1}^{l} n_i$ is called the level of the vector. The Verma module is not necessarily irreducible. It may contain *null vectors* $\mathbf{n}$, that is vectors of level $N > 0$ generating a submodule of their own

$$J_n^a \mathbf{n} = 0 \ , \quad \text{for } n > 0 \ .$$

Such a Verma module is called *degenerate*. To get a irreducible module we have to divide all nullvectors out. Let $\mathcal{N}_{R,k} = \mathrm{span}\{\mathbf{n} \in$

$\mathcal{M}_{R,k} \mid J^a_n \mathbf{n} = 0,\ n > 0\}$ be the set of all submodules of the Verma module $\mathcal{M}_{R,k}$. Then we get an irreducible module $\mathcal{V}_{R,k}$ by setting

$$\mathcal{V}_{R,k} = \mathcal{M}_{R,k}/\mathcal{N}_{R,k} \ .$$

Each $\widehat{\mathbf{g}}_k$-module is a Virasoro module by the Sugawara construction

$$L_n = -\frac{1}{2(k+h^\vee)} \sum_{l \in \mathbf{Z}, a} J^a_{-l} J^a_{l+n} \ ,$$

$$L_0 = -\frac{1}{2(k+h^\vee)} \sum_a \left(J^a_0 J^a_0 + \sum_{l \in \mathbf{Z}} J^a_{-l} J^a_{l+n} \right) \ .$$

Here $h^\vee$ is the dual Coxeter number. It depends on the structure constants via $h^\vee \delta_{a,b} = \sum_{c,d} f^{ac}_d f^{bc}_d$. The modes L_n satisfy the relations of the Virasoro algebra and couple to the modes J^a_m of the current algebra

$$[L_m, J^a_n] = -n J^a_{m+n} \ ,$$

$$[L_m, L_n] = (m-n) L_{m+n} + m(m^2-1) \frac{c_k}{k} \delta_{m+n,0} \ ,$$

with

$$c_k = \frac{k \dim \mathbf{g}}{k + h^\vee} \ .$$

The WZNW primary fields $\Phi^R(\mathbf{v}|z)$, labelled by a zero mode representation R and a vector $\mathbf{v} \in R$, are the fields transforming in a particularly simple way under the current algebra.

$$[J^a_n, \Phi^R(\mathbf{v}|z)] = z^n \Phi^R(J^a \mathbf{v}|z) \ .$$

2. Quantum groups

At an early point relations between conformal field theories and quantum groups were observed by Alvarez-Gaumé, Goméz and Sierra [**1**] as well as Moore and Seiberg [**29, 30**]. Subsequently these observed correspondences were rigorously proven by Kazhdan and Lusztig [**19, 20, 21**] and Finkelberg [**10**].

Quantum groups are generalisations or q-deformations of universal enveloping algebras of Lie algebras appearing, among other places, in soluble models of statistical physics and quantum field theory. Quantum groups are a particular kind of Hopf algebras. They were popularised by Drinfel'd in his report [**6**] to the International Congress of Mathematicians in 1986.

A *Hopf algebra* H is a unital algebra with product m and unit 1 inducing a map $\eta\colon \mathbf{C} \to A$ such that $\eta(\alpha) = \alpha 1$, together with additional linear maps $\Delta\colon A \to A \otimes A$, the *coproduct*, $\varepsilon\colon A \to \mathbf{C}$, the *counit* and $S\colon H \to H$, called the *antipode* or *coinverse* satisfying the equations

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \ , \qquad (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta \ , \tag{1.2}$$

$$\Delta(ab) = \Delta(a)\Delta(b) \ , \quad \Delta(1) = 1 \otimes 1 \ , \qquad \varepsilon(ab) = \varepsilon(a)\varepsilon(b) \ , \quad \varepsilon(1) = 1 \ , \tag{1.3}$$

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta \ . \tag{1.4}$$

The first two equations make H a coalgebra. The following equations ensure the compatibility of algebra and coalgebra structure of H, making H a bialgebra. The maps Δ and ε are algebra homomorphisms and m and η are coalgebra homomorphisms. The last equation characterising the antipode makes H a genuine Hopf algebra.
An element g of a Hopf algebra is called *group-like* if $\Delta(g) = g \otimes g$.
A Hopf algebra that is also a $*$-algebra is called a *Hopf $*$-algebra*.

It has been argued by Mack and Schomerus [**25**] that quantum groups are a natural extension of the concept of symmetry in the setting of quantum theory. Consider a quantum mechanical system with Hamiltonian H whose Hilbert space of states $\mathcal{H}$ is generated from a ground state $|0\rangle$ by field operators $\Psi^I_i(\mathbf{r}, t)$, with representation label I. A Hopf algebra $\mathcal{A}$ with a conjugation operation $*$, unit element e, coproduct Δ, counit ε and antipode S is called a symmetry of this system if $\mathcal{H}$ carries a unitary representation U of $\mathcal{A}$, the ground state $|0\rangle$ is invariant, all representation operators $U(\phi)$ commute with the Hamiltonian, and field operators transform covariantly. This means that for all $\phi \in \mathcal{A}$ we should have

$$\begin{aligned} U(\phi^*) &= U(\phi)^* \\ U(\phi)|0\rangle &= |0\rangle\varepsilon(\phi) \\ [U(\phi), H] &= 0 \\ U(\phi)\Psi^I_i &= \sum_{j,p} \Psi^I_j \tau^I_{ij}(\phi^{(1)}_p) U(\phi^{(2)}_p) \ . \end{aligned}$$

Here $\Delta(\phi) = \sum_p \phi^{(1)}_p \otimes \phi^{(2)}_p$ is the coproduct in Sweedler notation and τ^I_{ij} are matrix elements in a representation labelled by I.

3. The correspondence

Kazhdan and Lusztig showed that the superselection structure of WZNW models and the categories of finite dimensional representations of corresponding quantum groups are isomorphic as braided monoidal categories. The earliest observed example is that the SU(2)-WZNW model corresponds to the category of finite dimensional representations of $\mathcal{U}_q\mathbf{su}(2)$ for $q = \exp(2\pi i/(k+2))$ with the truncated tensor product for rational k.

A braided monoidal category is a category $\mathscr{C}$ together with an *associativity isomorphism*

$$\alpha\colon (V^3 \otimes V^2) \otimes V^1 \xrightarrow{\sim} V^3 \otimes (V^2 \otimes V^1)\ ,$$

satisfying the pentagon relation

$$\begin{array}{ccccc} ((V^4 \otimes V^3 \otimes)V^2) \otimes V^1 & \xrightarrow{\mathrm{id}\otimes\alpha} & (V^4 \otimes (V^3 \otimes V^2)) \otimes V^1 & \xrightarrow{\alpha} & V^4 \otimes ((V^3 \otimes V^2) \otimes V^1) \\ \Big\downarrow{\scriptstyle \alpha} & & & & \Big\downarrow{\scriptstyle \alpha\otimes\mathrm{id}} \\ (V^4 \otimes V^3) \otimes (V^2 \otimes V^1) & & \xrightarrow{\qquad\qquad \alpha\otimes\mathrm{id} \qquad\qquad} & & V^4 \otimes (V^3 \otimes (V^2 \otimes V^1)) \end{array}$$

and a *braiding isomorphism*

$$\beta\colon V^2 \otimes V^1 \xrightarrow{\sim} V^1 \otimes V^2\ ,$$

satisfying the hexagon relation

$$\begin{array}{ccccc} V^3 \otimes (V^2 \otimes V^1) & \xrightarrow{\alpha} & (V^3 \otimes V^2) \otimes V^1 & \xrightarrow{\beta} & (V^1 \otimes (V^3 \otimes V^2)) \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes\beta} & & & & \Big\downarrow{\scriptstyle \alpha} \\ V^3 \otimes (V^1 \otimes V^2) & \xrightarrow{\alpha} & (V^3 \otimes V^1) \otimes V^2 & \xrightarrow{\beta\otimes\mathrm{id}} & (V^1 \otimes V^3) \otimes V^2 \end{array}$$

In the case of the Lie algebra $\mathbf{sl}(2)$ the pentagon relation is related to the Biedenharn–Elliot equation.
The isomorphism of the superselection structure of a WZNW model and the category of finite dimensional representations of the corresponding quantum group is now expressible as follows. The $6j$-symbols correspond to the associativity isomorphism for the representations of the quantum group. The fusing matrices correspond to the associativity isomorphism for the superselection structure of the WZNW model. Since both categories are isomorphic the $6j$-symbols and the fusing matrices can only differ by a normalisation.

$$\frac{\nu^{j_3}_{j_4 j_s}\nu^{j_2}_{j_s j_1}}{\nu^{j_u}_{j_4 j_1}\nu^{j_3}_{j_u j_2}} \begin{Bmatrix} j_3 & j_2 & j_u \\ j_4 & j_1 & j_s \end{Bmatrix} = \mathrm{F}_{j_u j_s}\left[\begin{smallmatrix} j_3 & j_2 \\ j_4 & j_1 \end{smallmatrix}\right]\ . \tag{1.5}$$

What to expect in a broader setting is not so clear. There are a few examples, among them the following.
In the generalised setting of the Liouville model, a non-rational conformal field theory, Ponsot and Teschner found in [**34**] that the model's superselection structure corresponds to a category of infinite dimensional representations of the non-compact quantum group $\mathcal{U}_q\mathbf{sl}(2,\mathbf{R})$.
The point of interest of this work is an observation of Feigin and Malikov [**8**]. On basis of the fusion rules they conjectured a correspondence

of the SL(2)-WZNW model at rational k and a category of representations for the super quantum group $\mathcal{U}_q\mathbf{sl}(2) \times \mathcal{U}_{q'}\mathbf{osp}(1|2)$.

4. This work

We introduce a novel method to determine $6j$-symbols of quantum groups inspired by the determination of fusing matrices of WZNW models in the bootstrap approach to conformal field theory.

We represent the $6j$-symbols as a recurrence relation and a set of initial values, the *fundamental $6j$-symbols*. We find a representation of the quantum group in terms of multiplication operators $\mathsf{T}_x = q^{\mathrm{d}_x}$ and difference operators $[\mathrm{d}_x + a]$ acting on functions of a variable x, where d_x is the "power counting" operator $x\partial_x$. In a quantum group the symbol $[n]$ denotes the *quantum number* or *q-number*

$$[n] = \frac{q^n - q^{-1}}{q - q^{-1}} \ .$$

In a super quantum group it represents the *super quantum number*

$$\{n\} = \frac{q^{-\frac{n}{2}} - (-1)^n q^{\frac{n}{2}}}{q^{-\frac{1}{2}} + q^{\frac{1}{2}}} \ .$$

The invariants of the four-fold tensor product, the *four-point invariants* are intertwined by the $6j$-symbols. We find a subset of invariants, intertwined by the fundamental $6j$-symbols. In the considered representation the Casimir operator takes the form of a difference equation on the invariants.
In conformal field theory degenerate fields provide additional differential equations correlation functions containing these fields have to suffice. We observe that the fundamental representations[7] play a similar role for the quantum groups.
An ansatz for the invariants of this subset, motivated by this observation and properties of the hypergeometric series, finally yields a system of linear equations that determines the fundamental $6j$-symbols.

The great advantage of the proposed method is that it is easily adapted when one deals with super quantum groups. The determination of the $6j$-symbols of the super quantum group $\mathcal{U}_q\mathbf{osp}(1|2)$ bears a striking similarity to the analogue determination for the quantum group $\mathcal{U}_q\mathbf{sl}(2)$.

The proposed method is important for the study of non-rational conformal field theories because it states results of rational conformal

[7]The fundamental representation is the representation out of which by repeated tensor product all other representations can be generated.

field theory in a way that is generalisable to the non-rational case. In the same vein the proposed method would open an approach towards a version of the Kazhdan–Lusztig correspondence for generalised WZNW models.

A direct result is a different version of the $\mathbf{sl}(2)$-case of the Kazhdan–Lusztig correspondence. In the recursive presentation the proportionality (1.5) between the $6j$-symbols and the fusing matrices becomes the proportionality of the *fundamental* $6j$-symbols and fusing matrices

$$\frac{\nu^{j_3}_{j_4 j_s}\nu^{j^F}_{j_s j_1}}{\nu^{j_u}_{j_4 j_1}\nu^{j_3}_{j_u j^F}}\begin{Bmatrix} j_3 & j^F & j_u \\ j_4 & j_1 & j_s \end{Bmatrix} = \mathrm{F}_{j_u j_s}\begin{bmatrix} j_3 & j^F \\ j_4 & j_1 \end{bmatrix} \tag{1.6}$$

together with the equivalence of the corresponding recurrence relations. The representation with label j^F is the fundamental representation. This simplifies for example the calculatory effort needed to test for a proposed equality of a given WZNW model and a given quantum group.

As an application to these ideas, it would be possible to investigate the above mentioned conjecture by Feigin and Malikov [**8**]. Teschner determined the fundamental fusing matrices for a $\widehat{\mathbf{sl}}(2)_k$ current algebra in the context of the non-compact $\mathrm{SL}(2,\mathbf{C})/\mathrm{SU}(2)$-WZNW model.

In the following we will determine the fundamental $6j$-symbols of the quantum group $\mathcal{U}_q\mathbf{sl}(2)$ and of the super quantum group $\mathcal{U}_q\mathbf{osp}(1|2)$.

In chapter 2 we introduce the SL(2)-WZNW model. In chapter 3 we determine the fundamental $6j$-symbols of the quantum group $\mathcal{U}_q\mathbf{sl}(2)$. A rather strikingly similar deduction in chapter 4 will yield the fundamental $6j$-symbols of the super quantum group $\mathcal{U}_q\mathbf{osp}(1|2)$. We discuss our findings in chapter 5 and finish with an outlook. Appendix A contains a short introduction on quantum numbers and quantum calculus. It introduces the basic hypergeometric series and the Clebsch–Gordan coefficients of the quantum group $\mathcal{U}_q\mathbf{sl}(2)$. Super vector spaces and super algebra are quickly reviewed in appendix B. Furthermore there we introduce super quantum numbers, the super basic hypergeometric series and the Clebsch–Gordan coefficients of the quantum group $\mathcal{U}_q\mathbf{osp}(1|2)$.

Chapter 2
The SL(2)-WZNW model

In this chapter we present the method to derive fusing matrices based on the approach of Belavin, Polyakov and Zamolodchikov and the approach of Moore and Seiberg.

The four-point functions of a WZNW-model fulfil the Knizhnik–Zamolodchikov equations. These equations were introduced by Knizhnik and Zamolodchikov in [**23**].
Four-point functions containing a degenerate field satisfy yet a further differential equation. The additional differential equation for the degenerate field at $j = j_{2,1} = 1/2$ was determined and solved by Fateev and Zamolodchikov in [**7**]. Teschner determined and solved the differential equation for the degenerate field at $j = j_{1,2} = -t/2$.

The correlation functions fulfil the Knizhnik–Zamilodchikov equation. A correlation function containing a degenerate field satisfies an additional differential equation. We consider the degenerate fields Φ_{21} of spin $1/2$ and Φ_{12} of spin $-t/2$.
Correlation functions containing one of these fields are then of the form of a hypergeometric series ${}_2F_1$in the case of the degenerate field Φ_{21} and of a generalised hypergeometric series F_1 of two variables in the case of the degenerate field Φ_{12}.

The fusing matrices transform between different decompositions of the four-point function. They fulfil a recurrence relation. The initial values of this relation are the fundamental fusing matrices.
The fusing matrices linearly connect the conformal blocks in the s-channel decomposition with those in the u-channel decomposition. They fulfil a recurrence relation. The initial values of this recurrence relation are just the reduced fusing matrices connecting the reduced conformal blocks.
The connection coefficients for the different hypergeometric functions are well known. The identification of the reduced conformal blocks with certain hypergeometric functions thus determines the reduced fusing matrices. The recurrence relation determines the full fusing matrices. Thus the fusing matrices are determined.

The SL(2)-WZNW model is a WZNW model with current algebra $\widehat{\mathbf{sl}}(2)_k$. We call this model "SL(2)-model" because we do not want to choose a $*$-structure, because it is not used in the derivation of the fusing matrices. The derivation is in terms of the current algebra only. Specific instances of the SL-WZNW model are the SU(2)-WZNW model and the SL(2, $\mathbf{C}$)/SU(2)-WZNW model, also known as the H_3^+-model.
The SL(2)-model is a generalised WZNW model in two respects. Firstly the target space SL(2) is non-compact. Secondly the admissible representations when k is not an integer are non-unitary.

The model's chiral current algebra is generated by $J^a(z)$, $a = +, 0, -$ with modes

$$J^a(z) = \sum_{n\in\mathbf{Z}} z^{-n-1} J^a_n$$

subject to the relations

$$\begin{aligned}
[J^0_m, J^0_n] &= -\frac{k}{2} m\delta_{m+n,0} \\
[J^0_m, J^\pm_n] &= \pm J^\pm_{m+n} \\
[J^-_m, J^+_n] &= 2J^0_{m+n} + km\delta_{m+n,0} \ .
\end{aligned}$$

The zero mode algebra is realised as differential operators $\mathfrak{D}^a_x$, where

$$\mathfrak{D}^+_x = -x^2\partial_x + 2xj \ , \quad \mathfrak{D}^0_x = x\partial_x - j \ , \quad \mathfrak{D}^-_x = \partial_x \ .$$

The primary fields $\Phi^j(x|z)$ are defined by the action of the currents on them. The currents act on the primary fields as

$$J^a(z)\Phi^j(x,w) = \frac{1}{z-w}\mathfrak{D}^a_x\Phi^j(x,w) + \cdots \ .$$

The primary fields Φ^j have a conformal weight

$$h(j) = \frac{j(j+1)}{k+2} \ .$$

Chiral descendant fields are defined for each monomial $J^{a_l}_{-n_l}\cdots J^{a_1}_{-n_1}$ as the normal ordered product

$$\prod_{i=1}^{l} \frac{1}{(n_i-1)!} : \left(\partial_z^{n_l-1}J^{a_l}\right)\cdots\left(\partial_z^{n_1-1}J^{a_1}\right)\Phi^j(x|z) : \ .$$

We are interested in the case of $t = k+2$ non-integer. Feigin, Fuks and Malikov [**9**] showed that in this case, degenerate fields appear for spin $j = j_{r,s}$ with

$$\begin{aligned}
2j^+_{r,s} + 1 &= (r-1) - (s-1)t \ , \quad (r,s) \geq (1,0) \ , \\
2j^-_{r,s} + 1 &= -(r-1) + (s-1)t \ , \quad (r,s) \geq (1,1) \ .
\end{aligned}$$

Different approaches have been used to solve the SL(2)-model. The case of rational t is the one considered by Feigin and Malikov [**8**]. In his analysis of the H_3^+ or $\mathrm{SL}(2,\mathbf{C})/\mathrm{SU}(2)$ model Teschner [**38, 39**] used the bootstrap approach of Belavin, Polyakov and Zamolodchikov we will present in the following. See also Petersen, Rasmussen and Yu [**32, 33**] for a different approach.

The fusing matrices transform between different decompositions of the four-point function. They fulfil a recurrence relation. The initial values of this relation are the fundamental fusing matrices.
Independent of this the four-point functions fulfil the Knizhnik–Zamolodchikov equations. These equations were introduced by Knizhnik and Zamolodchikov in [**23**].
Four-point functions containing a degenerate field satisfy yet a further differential equation. The additional differential equation for the degenerate field at $j = j_{2,1} = 1/2$ was determined and solved by Fateev and Zamolodchikov in [**7**]. Teschner determined and solved the differential equation for the degenerate field at $j = j_{1,2} = -t/2$.
The following derivation of the differential equations for the four-point functions and of the fundamental fusing matrices in terms of vertex operators is given by Kanie and Tsuchiya in [**17**] in great detail.

1. Conformal blocks

Four-point functions of primary fields $\Phi^j(x|z)$ depend on two continuous variables. The variable x encodes the data of the Kac–Moody zero-mode representation. The Kac–Moody algebra $\widehat{\mathbf{sl}}(2)_k$ contains $\mathbf{sl}(2)$ as a subalgebra. The $\mathbf{sl}(2)$-invariance determines the x-dependence of the three-point function uniquely. The four-point function is determined only up to a dependency on the cross-ratio[1]

$$x = \frac{x_{41}x_{23}}{x_{43}x_{21}} \ .$$

The two ways to decompose the correlation function into three-point functions are called s- and u-channel decomposition. Denote the four-point function of primary fields by

$$\left\langle \Phi^{j_4}(x_4,z_4)\Phi^{j_3}(x_3,z_3)\Phi^{j_2}(x_2,z_2)\Phi^{j_1}(x_1,z_1)\right\rangle = G(J|X|Z) \ ,$$

where the uppercase letters J, X and Z collect the four-tuples of corresponding lowercase variables. The first way is to insert the operator product expansion for $\Phi^{j_4}\Phi^{j_3}$ and determine the resulting three-point functions,

$$G(J|X|Z) = \int_{j_s} C(j_1,j_2,j_s)C(j_s,j_3,j_4)\mathcal{F}^{(s)}_{j_s}(J|X|Z) \ ,$$

[1]Note the difference in the indices with respect to the cross-ratio of the z_i.

the second is to insert the operator product expansion for $\Phi^{j_3}\Phi^{j_2}$ and determine the three-point functions resulting from this insertion

$$G(J|X|Z) = \int_{j_u} C(j_2, j_3, j_u) C(j_1, j_u, j_4) \mathcal{F}^{(s)}_{j_u}(J|X|Z) \ .$$

Here the uppercase letters denote the four-tuples of corresponding variables, for example $X = (x_4, x_3, x_2, x_1)$. The functions $\mathcal{F}^{(s)}$ and $\mathcal{F}^{(u)}$ are called the s-channel and the u-channel conformal blocks.

An explicit form of the conformal blocks, which is especially handy in the following is

$$\mathcal{F}^{(i)}_{j_i}(J|X|Z) = \frac{x_{43}^{j_{21}^{43}} x_{42}^{2j_2} x_{41}^{j_{23}^{41}} x_{31}^{j_4^{123}}}{z_{43}^{h_{21}^{43}} z_{42}^{2h_2} z_{41}^{h_{23}^{41}} z_{31}^{h_4^{123}}} \mathcal{F}^{(i)}_{j_i}(J|x|z) \ .$$

The symbols j^{ab}_{cd} and j^{123}_4 are defined by the sums $j_a + j_b - j_c - j_d$ and $j_1 + j_2 + j_3 - j_4$ respectively. The symbols h^{ab}_{cd} and h^{123}_4 are defined analogously. Furthermore we introduce the label κ for the sum j^{123}_4.

The Knizhnik–Zamolodchikov equation. An additional constraint on the correlation functions of a WZNW model is the Knizhnik–Zamolodchikov equation. On a four-point function $G(J|X|Z)$ of primary fields the Knizhnik–Zamolodchikov equations take the form

$$t\partial_{z_i} G(J|X|Z) = \sum_{\substack{k=1 \\ k\neq i}}^{4} \frac{\mathfrak{D}_{ik}}{z_i - z_k} G(J|X|Z) \ .$$

The differential operators $\mathfrak{D}_{ik}$ are

$$\mathfrak{D}_{ik} = -\mathfrak{D}^0_{x_i}\mathfrak{D}^0_{x_k} + \tfrac{1}{2}\left(\mathfrak{D}^+_{x_i}\mathfrak{D}^-_{x_k} + \mathfrak{D}^-_{x_i}\mathfrak{D}^+_{x_k}\right) \ .$$

In the limit of $(x_4, x_3, x_2, x_1) \to (\infty, x, 0, 1)$ the equation for z_2 reduces to

$$\left(t\partial_{z_2} - \frac{P}{z_2 - z_1} - \frac{Q}{z_2 - z_3} \right) \mathcal{F}^{(s)}_{j_s}(J|x|Z) = 0 \ ,$$

with the second order differential operators

$$\begin{aligned} P &= x^2(1-x)\partial_x^2 + ((\kappa - 1)x^2 - 2j_1 x - 2j_2 x(1-x))\partial_x - 2j_2\kappa x + 2j_1 j_2 \\ Q &= (1-x)^2 x\partial_x^2 + (-(\kappa-1)(1-x)^2 + 2j_3(1-x) - 2j_2 x(1-x))\partial_x \\ &\quad - 2j_2\kappa(1-x) + 2j_3 j_2 \ . \end{aligned}$$

When we further let $(z_4, z_3, z_2, z_1) \to (\infty, 1, z, 0)$ we arrive at

(2.1)

$$\begin{aligned} &\big(- tz(z-1)\partial_z + x(1-x)(z-x)\partial_x^2 + \\ &+ \big((1-\kappa)(z - 2zx + x^2) + 2j_1 x(1-z) + 2j_2 x(1-x) + 2j_3 z(1-x)\big)\,\partial_x + \\ &+ 2j_2\kappa(x-z) + 2j_1 j_2(z-1) + 2j_3 j_2 z\big) \mathcal{F}^{(i)}_{j_i}(J|x|z) = 0 \end{aligned}$$

A recursive solution to this equation is found by using the ansatz

$$\mathcal{F}^{(s)}_{j_s}(J|x|z) = z^\lambda \sum_{n=0}^{\infty} z^n \Psi^{(s,n)}_{j_s}(J|x) \ . \tag{2.2}$$

For $n = 0$ this equation reduces to a hypergeometric differential equation $(P - t\lambda)\Psi^{(s,0)}_{j_s} = 0$

$$\Big(x(1-x)\partial_x^2 + ((1-x)2a + (\kappa-1)x - 2j_1 - 2j_2(1-x))\,\partial_x$$
$$+ a(\kappa-1) - 2j_2\kappa\Big)\Psi^{(0),j}_s(J|x) = 0 \ ,$$

provided $a = \frac{1}{2} + j_1 + j_2 \pm \left((\frac{1}{2} + j_1 + j_2)^2 - 2j_1j_2 - t\lambda\right)^{\frac{1}{2}} = j_{12}(j) + n$, $n = 0$.

The u-channel decomposition of the four point function gives a reduced Knizhnik–Zamolodchikov equation equal to (2.1) with x and $(1-x)$, z and $(1-z)$ as well as j_1 and j_3 exchanged. Using the same recursive ansatz for $n = 0$ we get $(Q - t\lambda)\Psi^{(u,0)}_{j_u} = 0$. We denote the u-channel conformal blocks by $\mathcal{F}^{(u)}_{j_u}(J|x|z)$.

Degenerate fields. Correlation functions including a degenerate field satisfy an additional differential equation. The degenerate fields we need in order to determine the fundamental fusing matrices are the degenerate primary fields Φ_{21} and Φ_{12} with spin equal to $j_{2,1} = \frac{1}{2}$ and $j_{1,2} = -\frac{t}{2}$ respectively.

The degenerate field Φ_{21}. The case of the field Φ_{21} transforming in the spin 1/2 representation has been among the first degenerate fields for the algebra $\widehat{\mathbf{sl}}(2)_k$ to be studied [**7**]. We will give a overview of the derivation of the conformal block in order to emphasise the approach we want to reuse in the quantum group case. The field Φ_{21} obeys the following equation

$$\partial_x^2 \Phi_{21}(x|z) = 0 \ .$$

This means that conformal block containing the degenerate field with spin $j_{2,1}$ satisfy the further equation

$$\partial_x^2 \mathcal{F}^{(i)}_{j_i} = 0 \ .$$

For definiteness we set the second spin $j_2 = 1/2$. The intermediate spins appearing in the s-channel decomposition are $j_1 + 1/2$ and $j_1 - 1/2$. We call these j_s^+ and j_s^- respectively. The intermediate spins appearing in the u-channel decomposition are $j_u^\pm = j_3 \pm 1/2$. Naturally the ansatz for the conformal blocks with such a degenerate field is $\mathcal{F}(J|x|z) = F_0(J|z) + xF_1(J|z)$. Because of the form (2.2) of the conformal blocks, the conformal blocks with $j_2 = 1/2$ can be expressed as

$$\mathcal{F}^{(s)}_{j_s^\pm}(J|x|z) = \left(a_s^\pm + b_s^\pm x\right) R^{(s)}(J|z) \ . \tag{2.3}$$

The term R is proportional to F_0 and F_1. A set of solutions is given by Teschner in terms of the hypergeometric series

$$ {}_2F_1(a,b;c;z) = \sum_{n\geq 0} \frac{(a)_n\,(b)_n}{(c)_n\,n!} z^n \ . $$

The symbol $(a)_n$ is the *shifted factorial* and defined as

$$ (a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \ . $$

The connection between the different values of the hypergeometric function are well know and determine the fundamental fusing matrices. Teschner [**38**] gives the formulae

(2.4)

$$ \mathcal{F}^{(s)}_{j_s^+} = \frac{\Gamma(w)\Gamma(w-u-v)}{\Gamma(w-u)\Gamma(w-v)}\mathcal{F}^{(u)}_{j_u^+} + \frac{\Gamma(w)\Gamma(u+v-w+1)}{\Gamma(u+1)\Gamma(v)}\mathcal{F}^{(u)}_{j_u^-} $$
$$ \mathcal{F}^{(s)}_{j_s^-} = \frac{\Gamma(1-w)\Gamma(w-u-v)}{\Gamma(-u)\Gamma(1-v)}\mathcal{F}^{(u)}_{j_u^+} + \frac{\Gamma(1-w)\Gamma(u+v+w+1)}{\Gamma(u-w+1)\Gamma(v-w+1)}\mathcal{F}^{(u)}_{j_u^-} \ . $$

The coefficients are

$$ u = -b^2(j^{134}+3/2)-1 \ , \quad v = -b^2 j_4^{13} \ , \quad w = -b^2(2j_1+1) \ . $$

The degenerate field Φ_{12}. The determination of the conformal blocks with one degenerate field Φ_{12} with spin $-t/2$ follows the same structure. The field Φ_{12} follows the equation of motion given by

$$:\big(J^+(x|z)\partial_x^2 - 2(1+t)J^0(x|z)\partial_x - t(1+t)J^-(x|z)\big)\,\Phi_{1,2}(x|z) := 0 \ , $$

with $J^a(x|z) = e^{xJ_0^-}J^a(z)e^{-xJ_0^-}$. This determines[2] a third order differential equation satisfied by the conformal blocks containing a degenerate field of spin $j_{1,2}$. It is of the form

$$ \begin{aligned} 0 = \big(&x(x-1)(x-z)\partial_x^3 + \\ &-((\kappa-2)(x^2-2zx+z)+2j_1x(z-1)-2(1+t)x(x-1)+2j_3x(x-1))\partial_x^2 + \\ &-(2(1+t)(j_1(z-1)+j_3z-(\kappa-1)(z-x))-t(1+t)(x+z+1))\partial_x + \\ &-t(1+t)\kappa\big)\mathcal{F}^{(i)}_{j_i}(J|x|z) \end{aligned} $$

This can be brought to the form satisfied by the generalised hypergeometric series of Appell[3]

$$ F_1(a,b_1,b_1;c;x,z) = \sum_{m,n\geq 0} \frac{(a)_{m+n}\,(b_1)_m\,(b_2)_n}{(c)_{m+n}\,m!n!} x^m z^n \ . $$

It can be shown that for non-integer values of $2j_1 - t$ and t there exist three linearly independent conformal blocks $\mathcal{F}^{(s)}_{j_s}$ of intermediate spin

[2]Details can be found in [**38**].

[3]Detail can be found in Slater's monograph [**37**].

j_s in the s-channel that can be identified with a generalised hypergeometric function F_1. The three possible spins are

$$j_s^+ = j_1 - \tfrac{t}{2} \ , \quad j_s^- = j_1 + \tfrac{t}{2} \text{ and } \quad j_s^\circ = -j_1 - 1 + \tfrac{t}{2} \ .$$

From the connection coefficients of F_1 Teschner determined the following relations between the s-channel and the u-channel conformal blocks

(2.5)

$$\begin{aligned}
\mathcal{F}^{1,2}_{s+} &= \frac{\Gamma(\gamma)\Gamma(\gamma-\beta-\beta'-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta-\beta')}\mathcal{F}^{1,2}_{t+} + \frac{\Gamma(\gamma)\Gamma(\alpha+\beta'-\gamma)}{\Gamma(\alpha)\Gamma(\beta')}\mathcal{F}^{1,2}_{t-} \\
&\quad + \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta')\Gamma(\alpha+\beta+\beta'-\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)\Gamma(\beta)}\mathcal{F}^{1,2}_{t\circ} \\
\mathcal{F}^{1,2}_{s-} &= \frac{\Gamma(2+\beta-\gamma)\Gamma(\gamma-\beta-\beta'-\alpha)}{\Gamma(1-\beta')\Gamma(1-\alpha)}\mathcal{F}^{1,2}_{t+} \\
&\quad + e^{\pi i\beta}\frac{\Gamma(2+\beta-\gamma)\Gamma(\alpha+\beta'-\gamma)}{\Gamma(1+\beta+\beta'-\gamma)\Gamma(1-\gamma+\alpha)}\mathcal{F}^{1,2}_{t-} \\
&\quad + e^{\pi i(\beta+\beta'+\alpha-\gamma)}\frac{\Gamma(2+\beta-\gamma)\Gamma(\gamma-\alpha-\beta')\Gamma(\alpha+\beta+\beta'-\gamma)}{\Gamma(1-\beta')\Gamma(1+\beta+\beta'-\gamma)\Gamma(\beta)}\mathcal{F}^{1,2}_{t\circ} \\
\mathcal{F}^{1,2}_{s\circ} &= \frac{\Gamma(\gamma-\beta)}{\Gamma(1-\beta)}\Bigg(\frac{\Gamma(2-\gamma)\Gamma(\gamma-\beta-\beta'-\alpha)}{\Gamma(\gamma-\beta-\beta')\Gamma(1-\alpha)}\mathcal{F}^{1,2}_{t+} \\
&\quad - e^{\pi i\gamma}\frac{\Gamma(2-\gamma)\Gamma(\alpha+\beta'-\gamma)}{\Gamma(1+\alpha-\gamma)\Gamma(\beta')}\mathcal{F}^{1,2}_{t-} \\
&\quad + e^{\pi i\gamma}\left(e^{\pi i(\beta+\beta'-\gamma)}\frac{\sin\pi\gamma}{\sin\pi\beta} - \frac{\sin\pi(\gamma-\alpha)}{\sin\pi(\gamma-\alpha-\beta')}\right)\frac{\Gamma(2-\gamma)\Gamma(\alpha+\beta+\beta'-\gamma)}{\Gamma(\beta)\Gamma(1+\alpha+\beta'-\gamma)}\mathcal{F}^{1,2}_{t\circ}\Bigg) \ .
\end{aligned}$$

The arguments expressed in spins j_i are

$$\begin{aligned}
\alpha &= j_4 - j_1 - j_3 + t/2 & \beta &= t \\
\beta' &= t/2 - j_1 - j_3 - j_4 - 1 & \gamma &= t - 2j_1 \ .
\end{aligned}$$

2. Fusing matrices

Crossing symmetry states that the two decompositions of the four-point function are in fact equal. In other words, the two decompositions correspond to different bases in the space of conformal blocks. The fusion matrices relate the two bases linearly.

$$\mathcal{F}^{(s)}_{j_s}(J|X|Z) = \int d\mu(j_u) \mathrm{F}_{j_s j_u}\left[\begin{smallmatrix} j_2 & j_3 \\ j_1 & j_4 \end{smallmatrix}\right] \mathcal{F}^{(u)}_{j_u}(J|X|Z) \ .$$

For the conformal blocks with degenerate representations of spin $1/2$ or $-t/2$ this linear relation becomes a sum of finitely many terms

$$\mathcal{F}^{(s)}_{j_s}(J|X|Z) = \sum_{j_u = j_{u,\min}}^{j_{u,\max}} \mathrm{F}_{j_s j_u}\left[\begin{smallmatrix} j_2 & j_3 \\ j_1 & j_4 \end{smallmatrix}\right] \mathcal{F}^{(u)}_{j_u}(J|X|Z) \ .$$

The recurrence relation. Moore and Seiberg derive a polynomial relation for the fusing matrices[4]

$$\sum_s \mathrm{F}_{p_2 s}\left[\begin{smallmatrix} j & k \\ p_1 & b \end{smallmatrix}\right] \mathrm{F}_{p_1 l}\left[\begin{smallmatrix} i & s \\ a & b \end{smallmatrix}\right] \mathrm{F}_{sr}\left[\begin{smallmatrix} i & j \\ l & k \end{smallmatrix}\right] = \mathrm{F}_{p_1 r}\left[\begin{smallmatrix} i & j \\ a & p_2 \end{smallmatrix}\right] \mathrm{F}_{p_2 l}\left[\begin{smallmatrix} r & k \\ a & b \end{smallmatrix}\right] \ .$$

They also show that the fusing matrices are symmetric under the following permutation of indices

$$\mathrm{F}_{pq}\left[\begin{smallmatrix} j & k \\ i & l \end{smallmatrix}\right] = \mathrm{F}_{pq}\left[\begin{smallmatrix} i & l \\ j & k \end{smallmatrix}\right] = \mathrm{F}_{pq}\left[\begin{smallmatrix} l & i \\ k & j \end{smallmatrix}\right]$$

and have the following orthogonality property

$$\sum_q \mathrm{F}_{pq}\left[\begin{smallmatrix} j & k \\ i & l \end{smallmatrix}\right] \mathrm{F}_{pq}\left[\begin{smallmatrix} l & k \\ i & j \end{smallmatrix}\right] = \delta_{p,q} \ .$$

From this it is possible to derive a recurrence relation for the fusing matrices.

The fundamental fusing matrices for conformal blocks with one degenerate field are identified with the connection coefficients of the hypergeometric functions ${}_2F_1$ and F_1 as already done in (2.4) and (2.5).

[4]The polynomial relation was first noted in [**28**]. More detail can be found in [**30**].

Chapter 3
The quantum group $\mathcal{U}_q\mathbf{sl}(2)$

In this chapter we determine the $6j$-symbols of the quantum group $\mathcal{U}_q\mathbf{sl}(2)$. We do this by the method introduced in the preceding chapter.

We represent the generators of $\mathcal{U}_q$ as difference operators. The invariant tensors play an analogue role to the correlation functions in the WZNW-model. The Casimir operator induces a difference equation all invariant tensors satisfy. This fixes the the four-point invariants to a q-hypergeometric form.
We set the representation label j_2 to the fundamental representation 1/2. This further restricts the fundamental four-point functions to a form from which it is possible to deduce a system of linear equations that determine the fundamental $6j$-symbols connecting the s-channel decomposition and the u-channel decomposition.

The quantum group $\mathcal{U}_q\mathbf{sl}(2)$ is the Hopf algebra generated by the elements E, F, K and K^{-1} and the relations

$$\begin{aligned} KEK^{-1} &= qE \qquad & [E,F] &= -\frac{K^2-K^{-2}}{q-q^{-1}} \\ KFK^{-1} &= q^{-1}F \qquad & KK^{-1} &= K^{-1}K = 1\ . \end{aligned} \tag{3.1}$$

When there is little cause for confusion, we will write $\mathcal{U}_q$ for short. The coproduct $\Delta : \mathcal{U}_q \to \mathcal{U}_q \otimes \mathcal{U}_q$ is given by

$$\begin{aligned} \Delta(E) &= E \otimes K + K^{-1} \otimes E \\ \Delta(F) &= F \otimes K + K^{-1} \otimes F \\ \Delta(K) &= K \otimes K \\ \Delta(K^{-1}) &= K^{-1} \otimes K^{-1}\ . \end{aligned} \tag{3.2}$$

Counit $\varepsilon : \mathcal{U}_q \to \mathbf{C}$ and antipode $S : \mathcal{U}_q \to \mathcal{U}_q$ are

$$\varepsilon(K) = \varepsilon(K^{-1}) = 1 \quad \varepsilon(E) = \varepsilon(F) = 0 \tag{3.3}$$

and

$$\begin{aligned} S(E) &= -qE \qquad & S(K) &= K^{-1} \\ S(F) &= -q^{-1}F \quad & S(K^{-1}) &= K\ . \end{aligned} \tag{3.4}$$

The quantum Casimir operator is the element

$$C_q = FE - \frac{qK^2 + q^{-1}K^{-2} - 2}{(q-q^{-1})^2} = EF - \frac{q^{-1}K^2 + qK^{-2} - 2}{(q-q^{-1})^2} \ .$$

When q is not a root of unity, the quantum Casimir operator generates the centre Z_q of $\mathcal{U}_q$.

1. Finite dimensional representations

A *representation* (π, V) of the quantum group $\mathcal{U}_q$ is a linear space V together with a linear map $\pi : \mathcal{U}_q \to \mathrm{End}(V)$ such that $\pi(XY) = \pi(X)\pi(Y)$ for all $X, Y \in \mathcal{U}_q$.
Let (π, V) be a representation of $\mathcal{U}_q$. For every complex number λ define $V_\lambda := \{v \in V \mid \pi(K)v = \lambda v\}$. We call every nontrivial V_λ the *weight space* corresponding to the *weight* λ. Nonzero vectors in V_λ are called *weight vectors* or *vector of weight* λ[1]. A vector $v \in V$ is called a vector of *highest* weight λ' if $\pi(E)v = 0$ and $\pi(K)v = \lambda' v$. In this case λ' is called the *highest weight* of the representation (π, V).
If V is the direct sum of weight spaces of π, we call (π, V) a *weight representation*.

The value of the parameter q divides the representation theory of the quantum group $\mathcal{U}_q\mathbf{sl}(2)$ into two cases.

Generic q. If q is not a root of unity, the representation theory proceeds along the same paths as for the undeformed $\mathbf{sl}(2)$. The finite dimensional irreducible representations[2], the *spin* representations of $\mathcal{U}_q\mathbf{sl}(2)$ are labelled by a positive half integer spin j. It can be shown[3] that every finite dimensional representation of $\mathcal{U}_q\mathbf{sl}(2)$ is a weight representation and that every finite dimensional representation π of $\mathcal{U}_q\mathbf{sl}(2)$ is completely reducible, i.e., π is a direct sum of irreducible representations.

For generic q the center of $\mathcal{U}_q$ is generated by unit and Casimir operator C_q alone.

Rational q. When q is a root of unity, the center as well as the classes of representations extend. Let

$$q^n = 1 \ .$$

---

[1] In the following vectors of weight λ will be called eigenvectors of $\pi(K)$ with eigenvalue λ too. In addition whenever $\lambda = q^m$ we will call m weight or eigenvalue too.

[2] To be precise, the representations corresponding to representations of $\mathbf{sl}(2)$ depend on the spin j and a parameter w such that $w^4 = 1$. We will omit this parameter.

[3] see for example the books by Kassel [**18**] or Klimyk and Schmüdgen [**22**].

Then the generators E^n, F^n and K^n are elements of the center as well. In the class of irreducible representations *cyclic* and *semi-cyclic* representations appear[4]. These are representations with neither highest nor lowest weights and only highest or lowest weights respectively. A vastly complicating factor is that the tensor product of irreducible representations is not necessarily decomposable for rational q. A mitigating circumstance is that these unwieldy representations occur for spin $2j+1=n$ only. Below this bound the representation theory is basically as for generic q. The set of spin representations, semi-cyclic and indecomposable representations form a closed set under the tensor product[5]. Indecomposable representations appear[6] in the tensor product of the representations j_1 and j_2 if $j_1+j_2>n$.

We will confine ourselves to generic q in the following.

There are two realisations of finite dimensional representations of highest weight of $\mathcal{U}_q$ we will be considering in the following. We call these the m- and the x-representation. Since we are interested especially in quantum Clebsch–Gordan coefficients and $3j$ symbols of these, we will need the dual of each representations too.

The m-representation. Consider the irreducible representation (V^j,π_m) of highest weight q^j for $j\in\frac{1}{2}\mathbf{N}$ on the vector space V^j. There is a basis

$$\left\{\mathbf{e}^j_m \;\middle|\; m=-j,-j+1,\ldots,j\right\}$$

on which the generators E and F act as raising and lowering operators $X^\pm$, destroying the vector of highest weight $\mathbf{e}^j_j$ and of lowest weight $\mathbf{e}^j_{-j}$ respectively

$$\pi_m(X^\pm)\mathbf{e}^j_m=\mathcal{C}^\pm(j,m)\mathbf{e}^j_{m\pm1}\ ,\quad \pi_m(K)\mathbf{e}^j_m=q^m\mathbf{e}^j_m\ , \tag{3.5a}$$

$$\pi_m(X^+)\mathbf{e}^j_j=\pi_m(X^-)\mathbf{e}^j_{-j}=0\ . \tag{3.5b}$$

In the physics literature is conventional to choose a basis $\{\mathbf{f}^j_m \mid m=-j,-j+1,\ldots,j\}$ such that

$$\mathcal{C}^\pm(j,m)=\mp([j\mp m][j\pm m+1])^{\frac{1}{2}}$$

holds. Most results on quantum Clebsch–Gordan coefficients and quantum $3j$ symbols are stated with respect to this basis. We will however choose a basis $\{\mathbf{e}^j_m \mid m=-j,-j+1,\ldots,j\}$ such that

$$\mathcal{C}^\pm(j,m)=\mp[j\mp m]\ . \tag{3.5c}$$

[4] Arnaudon [**2**] gives a good classification.

[5] see Arnaudon [**2**].

[6] as demonstrated by Pasquier and Saleur in [**31**].

The basis $\{\mathbf{f}^j_m\}$ and the renormalised basis $\{\mathbf{e}^j_m\}$ are connected through a rescaling by

$$\mathbf{e}^j_m = \mathcal{N}(j,m)^{\frac{1}{2}}\,\mathbf{f}^j_m \qquad \mathcal{N}(j,m) = \frac{[j+m]!\,[j-m]!}{[2j]!} \ . \tag{3.6}$$

We will call the basis $\{\mathbf{e}^j_m\}$ *the m-basis* and the $\mathcal{U}_q$ representation π_m on it the *m-representation* in the following.

The quantum Casimir is identically $\left[j+\frac{1}{2}\right]^2$ on this representation.

The transposed m-representation. The space dual to V^j is the space $(V^j)^*$ of complex valued functions on V^j. It admits a basis

$$\left\{\check{\mathbf{e}}^j_m \;\middle|\; m=-j,-j+1,\ldots,j\right\}$$

such that

$$\check{\mathbf{e}}^j_m(\mathbf{e}^j_n) = \delta_{m,n} \ .$$

The basis $\{\check{\mathbf{e}}^j_m\}$ is called the basis *dual* to the basis $\{\mathbf{e}^j_m\}$. The bilinear form

$$\left(\mathbf{e}^j_m, \mathbf{e}^j_m\right) = \check{\mathbf{e}}^j_m(\mathbf{e}^j_n) = \delta_{m,n} \ . \tag{3.7}$$

induces the structure of a *right* $\mathcal{U}_q$-module on $(V^j)^*$, that is $\pi^t_m : \mathcal{U}_q \to \mathrm{End}((V^j)^*)$ is an antihomomorphism of algebras, inverting the sequence of all factors in products. We have

$$\pi^t_m(X^\pm)\check{\mathbf{e}}^j_m = \mathcal{C}^\pm(j,m\mp 1)\check{\mathbf{e}}^j_{m\mp 1} \ , \quad \pi^t_m(K)\check{\mathbf{e}}^j_m = q^m\check{\mathbf{e}}^j_m \ , \tag{3.8a}$$

$$\pi^t_m(X^+)\check{\mathbf{e}}^j_{-j} = \pi^t_m(X^-)\check{\mathbf{e}}^j_j = 0 \ . \tag{3.8b}$$

and

$$\pi^t_m(C_q) \equiv \left[j+\tfrac{1}{2}\right]^2 \ .$$

Invariant bilinear form. The invariant bilinear form B_q is the bilinear form on V^j invariant under $\mathcal{U}_q$. Let $\mathbf{v}$ be a vector in $V^j \otimes V^j$. The invariance under $\mathcal{U}_q$ then means

$$\begin{aligned} B_q\left(\Delta(K)\mathbf{v}\right) &= B_q\left(\mathbf{v}\right) \ , \\ B_q\left(\Delta(a)\mathbf{v}\right) &= 0 \ , \quad \text{for } a = E, F \ . \end{aligned}$$

Extend B_q to all finite dimensional $\mathcal{U}_q$-modules such that different modules are orthogonal. We have then, up to a j_2-dependent normalisation

$$B_q\left(\mathbf{e}^{j_2}_{m_2} \otimes \mathbf{e}^{j_1}_{m_1}\right) = \delta_{j_2,j_1}\delta_{m_2+m_1,0}(-1)^{j_2+m_2}q^{j_2+m_2}\mathcal{N}(j_2,m_2) \ .$$

The factor $\mathcal{N}$ is the same as the renormalisation in (3.6). We compress the coefficients of the invariant bilinear form into

$$\bar{\mathcal{N}}(j,m) = (-1)^{j+m}q^{j+m}\mathcal{N}(j,m) \ . \tag{3.9}$$

Decomposition of tensor products. The Clebsch–Gordan quantum coefficients intertwine tensor products of irreducible representations and irreducible representations of $\mathcal{U}_q$.

$$\mathbf{e}^j_m(j_2, j_1) = \sum_{m_1,m_2} \left[\begin{matrix} j \\ m \end{matrix}\middle|\begin{matrix} j_2 & j_1 \\ m_2 & m_1 \end{matrix}\right]_q \mathbf{e}^{j_2}_{m_2} \otimes \mathbf{e}^{j_1}_{m_1} \ . \tag{3.10}$$

The Clebsch–Gordan quantum coefficients are equal to zero unless several constraints are met.

$$\begin{aligned} |j_1 - j_2| \le j \le j_1 + j_2 \\ m = m_1 + m_2 \\ -j_i \le m_i \le j_i \ . \end{aligned} \tag{3.11}$$

We call the first two *horizontal* constraints and the remaining the *vertical* constraints. The Clebsch–Gordan-coefficients satisfy the recurrence relations

$$\begin{aligned} &\mathcal{C}^{\pm}(j,m) \left[\begin{smallmatrix} j \\ m\pm 1 \end{smallmatrix}\middle|\begin{smallmatrix} j_2 & j_1 \\ m_2 & m_1 \end{smallmatrix}\right]_q = \\ &q^{m_1}\mathcal{C}^{\pm}(j_2, m_2 \mp 1) \left[\begin{smallmatrix} j \\ m \end{smallmatrix}\middle|\begin{smallmatrix} j_2 & j_1 \\ m_2\mp 1 & m_1 \end{smallmatrix}\right]_q + q^{-m_2}\mathcal{C}^{\pm}(j_1, m_1 \mp 1) \left[\begin{smallmatrix} j \\ m \end{smallmatrix}\middle|\begin{smallmatrix} j_2 & j_1 \\ m_2 & m_1\mp 1 \end{smallmatrix}\right]_q \ . \end{aligned}$$

By these relations the Clebsch–Gordan coefficients are only determined up to a function of the representation labels j_i. We choose a normalisation for the Clebsch–Gordan coefficients that will ease computations later. We set

$$\left[\begin{matrix} j \\ -j \end{matrix}\middle|\begin{matrix} j_2 & j_1 \\ j_1 - j & -j_1 \end{matrix}\right]_q = 1 \ . \tag{3.12}$$

This normalisation differs from the Condon–Shortley convention.

The Clebsch–Gordan coefficients fulfil the following orthogonality and completeness relations.

$$\begin{aligned} &\sum_{m_1,m_2} \left[\begin{matrix} j \\ m \end{matrix}\middle|\begin{matrix} j_2 & j_1 \\ m_2 & m_1 \end{matrix}\right]_q \left[\begin{matrix} j' \\ m' \end{matrix}\middle|\begin{matrix} j_2 & j_1 \\ -m_2 & -m_1 \end{matrix}\right]_q \bar{\mathcal{N}}(j_2,m_2)\bar{\mathcal{N}}(j_1,m_1) \\ &\qquad = \delta_{j,j'}\delta_{m+m',0}\delta(j_1,j_2,j)\bar{\mathcal{N}}(j,m) \end{aligned} \tag{3.13a}$$

$$\begin{aligned} &\sum_{j,m} \left[\begin{matrix} j \\ m \end{matrix}\middle|\begin{matrix} j_2 & j_1 \\ m_2 & m_1 \end{matrix}\right]_q \left[\begin{matrix} j \\ -m \end{matrix}\middle|\begin{matrix} j_2 & j_1 \\ m_2' & m_1' \end{matrix}\right]_q \times \\ &\qquad \times \frac{\bar{\mathcal{N}}(j_2,m_2)\bar{\mathcal{N}}(j_2,m_2')\bar{\mathcal{N}}(j_1,m_1)\bar{\mathcal{N}}(j_1,m_1')}{\bar{\mathcal{N}}(j,m)} \\ &\qquad = \delta_{m_2+m_2',0}\delta_{m_1+m_1',0}\bar{\mathcal{N}}(j_2,m_2)\bar{\mathcal{N}}(j_1,m_1) \end{aligned} \tag{3.13b}$$

This can be seen as follows. The Clebsch–Gordan coefficients have inverses

$$\mathbf{e}^{j_2}_{m_2} \otimes \mathbf{e}^{j_1}_{m_1} = \sum_{j,m} \left[\begin{matrix} j \\ m \end{matrix}\middle|\begin{matrix} j_2 & j_1 \\ m_2 & m_1 \end{matrix}\right]^t_q \mathbf{e}^j_m(j_1, j_2) \ .$$

With these the Clebsch–Gordan coefficients satisfy a further set of relations

$$
\begin{aligned}
&\sum_{m_2,m_1} \left[\begin{array}{c|cc} j & j_2 & j_1 \\ m & m_2 & m_1 \end{array}\right] \left[\begin{array}{c|cc} j' & j_2 & j_1 \\ -m' & m_2 & m_1 \end{array}\right]^t \bar{\mathcal{N}}(j', -m') \\
&\qquad = \delta_{j,j'} \delta_{m+m',0} \bar{\mathcal{N}}(j, m) \\
&\sum_{j,m} \left[\begin{array}{c|cc} j & j_2 & j_1 \\ m & m_2 & m_1 \end{array}\right]^t \left[\begin{array}{c|cc} j & j_2 & j_1 \\ m & -m_2' & -m_1' \end{array}\right] \bar{\mathcal{N}}(j_2, -m_2') \bar{\mathcal{N}}(j_1, -m_1') \\
&\qquad = \delta_{m_2+m_2',0} \delta_{m_1+m_1',0} \bar{\mathcal{N}}(j_2, -m_2') \bar{\mathcal{N}}(j_1, -m_1') \\
&\sum_{j,m} \left[\begin{array}{c|cc} j & j_2 & j_1 \\ m & m_2 & m_1 \end{array}\right]^t \left[\begin{array}{c|cc} j & j_2 & j_1 \\ m & -m_2' & -m_1' \end{array}\right]^t \mathcal{N}(\bar{j}, m) \\
&\qquad = \delta_{m_2+m_2',0} \delta_{m_1+m_1',0} \mathcal{N}(\bar{j_2}, m_2) \mathcal{N}(\bar{j_1}, m_1)
\end{aligned}
$$

Comparing coefficients we see that

$$
\left[\begin{array}{c|cc} j & j_2 & j_1 \\ m & m_2 & m_1 \end{array}\right]_q^t = \left[\begin{array}{c|cc} j & j_2 & j_1 \\ -m & -m_2 & -m_1 \end{array}\right]_q \frac{\bar{\mathcal{N}}(j_2, m_2) \bar{\mathcal{N}}(j_1, m_1)}{\bar{\mathcal{N}}(j, m)} . \tag{3.14}
$$

From this it is possible to derive the orthogonality and completeness relations.

Invariants. Invariants in the n-fold tensor product can be constructed using Clebsch–Gordan decomposition in the $(n-1)$-fold tensor product together with an invariant in the two-fold tensor product. This invariant in the two-fold tensor product is the invariant bilinear form B_q. It is connected to certain Clebsch–Gordan coefficients. On the module $V^{j_2} \otimes V^{j_1}$ we have

$$
B_q \begin{pmatrix} j_2 & j_1 \\ m_2 & m_1 \end{pmatrix} := \left[\begin{array}{c|cc} 0 & j_2 & j_1 \\ 0 & m_2 & m_1 \end{array}\right]_q .
$$

In our normalisation the invariant 2-form is

$$
B_q \begin{pmatrix} j_2 & j_1 \\ m_2 & m_1 \end{pmatrix} = \delta_{j_2,j_1} \delta_{m_2+m_1,0} \; (-1)^{j_2+m_2} q^{-j_2-m_2} \mathcal{N}(j_2, m_2)^{-2} .
$$

For $\mathbf{sl}(2)$ the invariant of the three-fold tensor product is known as the Wigner $3j$ symbol. The $3j$ symbols are defined by lowering one index in the Clebsch–Gordan coefficients by means of the invariant 2-form

$$
\begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix}_q := \sum_{m=-j}^{j} B_q \begin{pmatrix} j_3 & j \\ m_3 & m \end{pmatrix} \left[\begin{array}{c|cc} j & j_2 & j_1 \\ m & m_2 & m_1 \end{array}\right]_q . \tag{3.15}
$$

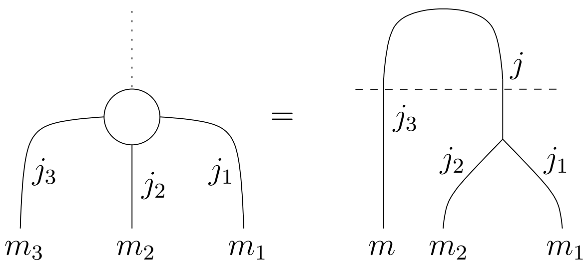

FIGURE 2: Wigner $3j$ symbol, invariant bilinear form and Clebsch–Gordan coefficient

The x-representation. The second realisation of π^j needed in out construction is "dual" to the m-basis in a different sense. Consider the generating polynomial[7] in the indeterminate x^{-1} for the m-basis

$$\mathbf{e}^j(x) = \sum_{m=-j}^{j} x^{-j-m}\mathbf{e}^j_m \ . \tag{3.16}$$

Let $Pol_{2j}(x^{-1})$ be the linear space of polynomials in the indeterminate x^{-1} of degree $2j$. Via the generating polynomial our choice of $\mathcal{C}^{\pm}$ then induces for every $j \in \frac{1}{2}\mathbf{N}$ from the m-representation a right $\mathcal{U}_q$-module structure on $Pol_{2j}(x^{-1})$. The elements of $\mathcal{U}_q$ act as finite difference operators T and $[\mathrm{d}_x + a]$ where $\mathsf{T} = \mathsf{T}_x = q^{x\partial_x} \in \mathrm{End}(Pol_{2j}(x))$ and $x\partial_x = \mathrm{d}_x$ such that

$$\mathsf{T}\, f(x) = f(qx) \quad \text{and} \quad [\mathrm{d}_x + a] = \frac{q^a\,\mathsf{T} - q^{-a}\,\mathsf{T}^{-1}}{q - q^{-1}} \ .$$

$$\begin{aligned} \pi_x(K) &= q^{-j}\,\mathsf{T}_x^{-1} \ , \\ \pi_x(E) &= -\,[\mathrm{d}_x + 2j]\,x \ , \\ \pi_x(F) &= -\,[\mathrm{d}_x]\,x^{-1} \ , \end{aligned} \tag{3.17}$$

and

$$\pi_x(C_q) \equiv -\left[j + \tfrac{1}{2}\right]^2 \ .$$

We call this the x-representation.

[7]Note that in a coordinate z such that $x = \mathrm{e}^{iz}$ this equivalent to the Fourier series. The inverse transformation is

$$\mathbf{e}^j_m = \oint_0 dx\ x^{j+m-1}\mathbf{e}^j(x) \ .$$

The transposed x-representation. The space $Pol_{2j}(x)$ of polynomials in the indeterminate x is dual to $Pol_{2j}(x^{-1})$ via the bilinear form $(\cdot,\cdot)_x : Pol_{2j}(x) \times Pol_{2j}(x^{-1})$

$$(q,p)_x = \oint_0 \frac{\mathrm{d}x}{x}\, q(x)p(x) \ .$$

Note that this bilinear form is just the residue of qp at $x=0$. The basis $\{x^{j+m} \mid m = -j, -j+1, \dots, j\}$ is the basis dual to $\{x^{-j-m}\}$ with respect to this pairing.

We find a left $\mathcal{U}_q$-module structure on $Pol_{2j}(x)$. $\mathcal{U}_q$ acts in the transposed of the x-representation.

$$\begin{aligned} \pi^t_x(K) &= q^{-j}\,\mathsf{T}_x \ , \\ \pi^t_x(E) &= x\,[\mathrm{d}_x - 2j] \ , \\ \pi^t_x(F) &= x^{-1}\,[\mathrm{d}_x] \end{aligned} \tag{3.18}$$

and

$$\pi^t_x(C_q) \equiv -\left[j+\tfrac{1}{2}\right]^2 \ .$$

This is the *transposed x-representation* and concludes the series of realisations we will need in the following.

Clebsch–Gordan coefficients and $3j$-symbols in the x basis. Analogously to the generating function of (3.16) we introduce quantum $3j$ symbols and Clebsch–Gordan coefficients in the x variable by demanding that every x_i appearing in such an expression should indicate a generating function of said expression in terms of the corresponding m_i. Consider the following example.

$$\begin{pmatrix} j_3 & j_2 & j_1 \\ x_3 & m_2 & m_1 \end{pmatrix}_q := \sum_{m_3=-j_3}^{j_3} x_3^{j_3+m_3} \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix}_q \ .$$

Clebsch–Gordan coefficients in the x basis intertwine tensor product and irreducible representations in the x basis by

$$\mathbf{e}^j(j_2,j_1;x) = \oint_0 \frac{dx_2}{x_2} \oint_0 \frac{dx_1}{x_1} \left[\begin{array}{c|cc} j & j_2 & j_1 \\ x & x_2 & x_1 \end{array}\right]_q \mathbf{e}^{j_2}(x_2) \otimes \mathbf{e}^{j_1}(x_1) \ .$$

Limits in x. We regard the indeterminate x as a point in the complex projective plane $\mathbb{P}\mathbf{C}$. The points 0 and ∞ then have a special meaning for polynomials in x. They single out the coefficients of the lowest, respectively highest power of x. Concretely, for $f^j(x) = \sum_{m=-j}^{j} f_m x^{j+m}$ we have

$$\begin{aligned} \lim_{x\to 0} f^j(x) &= f_{-j} \ , \\ \lim_{x\to\infty} x^{-2j} f^j(x) &= f_j \ . \end{aligned}$$

From this we see, that the endomorphism $q^{-j}\,\mathsf{T}_x$ representing K has the following effect in these limits

$$\pi_x^t(K)f(x) \to \begin{cases} q^{-j}f_{-j} & \text{for } x \to 0 \ , \\ q^j f_j & \text{for } x \to \infty \ . \end{cases}$$

2. Invariant tensors

Invariant tensors will be our key ingredient to the determination of the fundamental quantum $6j$-symbols. An invariant tensor t of $\mathcal{U}_q$ is an element of $\bigotimes_i V^{j_i}$ invariant under the action of $\mathcal{U}_q$. That is to say the relations

$$Kt = t \quad \text{and} \quad Et = Ft = 0 \tag{3.19}$$

hold. Invariant three-tensors and quantum $3j$ symbols[8] are in one-to-one correspondence.

Four-point invariants. An invariant tensor $\Psi \in V^{j_4} \otimes V^{j_3} \otimes V^{j_2} \otimes V^{j_1}$ will be called a four-point invariant. Graphically we can depict it as in figure 3.

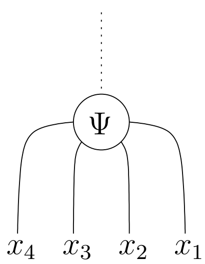

Figure 3: A four-point invariant

To shorten expressions in the four-fold tensor product, we introduce the notation $J = (j_4, j_3, j_2, j_1)$ and $X = (x_4, x_3, x_2, x_1)$. Thus we write for a function f in the four-fold tensor product

$$f(J|X) = f\begin{pmatrix} j_4 & j_3 & j_2 & j_1 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix} \ .$$

We can compose such an invariant using only the structures we already have at hand: the Clebsch–Gordan coefficients and the quantum $3j$-symbols.

$$\Psi(J|X) = \sum_{j_s} \Psi^{(s)}_{j_s}(J|X) \ ,$$

[8] and thus quantum Clebsch–Gordan coefficients

where

$$\Psi^{(s)}_{j_s}(J|X) = \sum_{m_s=-j_s}^{j_s} \begin{pmatrix} j_4 & j_3 & j_s \\ x_4 & x_3 & m_s \end{pmatrix}_q \left[\begin{matrix} j_s \\ m_s \end{matrix}\middle|\begin{matrix} j_2 & j_1 \\ x_2 & x_1 \end{matrix}\right]_q . \qquad (3.20)$$

This construction is depicted in figure 4.

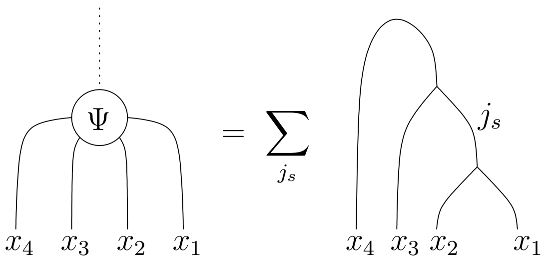

FIGURE 4: Decomposition of a four-point invariant — the s-channel

There is another way to arrive at a four-point invariant using Clebsch–Gordan coefficients and quantum $3j$-symbols, namely

$$\Psi^{(u)}_{j_u}(J|X) = \sum_{m_u=-j_u}^{j_u} \begin{pmatrix} j_4 & j_1 & j_u \\ x_4 & x_1 & m_u \end{pmatrix}_q \left[\begin{matrix} j_u \\ m_u \end{matrix}\middle|\begin{matrix} j_3 & j_2 \\ x_3 & x_2 \end{matrix}\right]_q . \qquad (3.21)$$

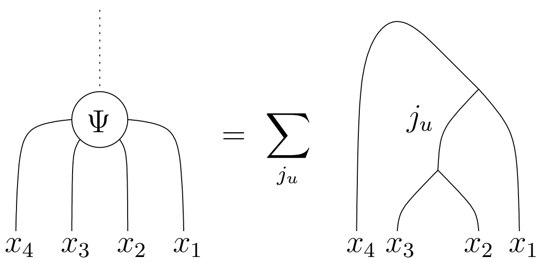

FIGURE 5: The u-channel decomposition

We refer to these two different decompositions as s- and u-channel.

3. Quantum $6j$-symbols

The quantum $6j$-symbols communicate the basis change between different reduced bases $\mathbf{e}^j_m(j_3, j_{12})$ and $\mathbf{e}^j_m(j_1, j_{23})$ of triple tensor products $V^{j_1} \otimes V^{j_2} \otimes V^{j_3}$ of $\mathcal{U}_q$ modules.

$$\mathbf{e}^{j_4}_{m_4}(j_u(j_3, j_2), j_1) = \sum_{j_s} \begin{Bmatrix} j_3 & j_2 & j_u \\ j_4 & j_1 & j_s \end{Bmatrix}_q \mathbf{e}^{j_4}_{m_4}(j_3, j_s(j_2, j_1)) \; .$$

$$\mathbf{e}^{j_4}_{m_4}(j_3, j_s(j_2, j_1)) = \sum_{j_u} \begin{Bmatrix} j_3 & j_2 & j_s \\ j_4 & j_1 & j_u \end{Bmatrix}^t_q \mathbf{e}^{j_4}_{m_4}(j_u(j_3, j_2), j_1) \; .$$

Thus they allow for the expression of the u-channel decomposition of the four-point invariant in terms of the s-channel decomposition and vice versa. When we express a vector of the reduced basis as string diagramme we get the graphical representation of the action of the $6j$-symbols in figure 6.

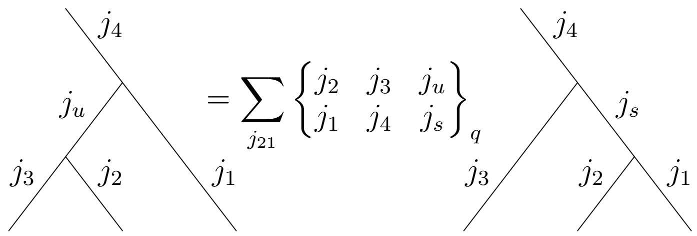

FIGURE 6: Connecting reduced bases in the triple tensor product

Consider the representation of the $6j$-symbols in terms of Clebsch–Gordan coefficients

$$\begin{Bmatrix} j_2 & j_3 & j_u \\ j_1 & j_4 & j_s \end{Bmatrix}_q = \sum_{m_3, m_2, m_1} \left[\begin{array}{c|cc} j_4 & j_u & j_1 \\ m_{123} & m_{23} & m_1 \end{array}\right]_q \left[\begin{array}{c|cc} j_u & j_3 & j_2 \\ m_{23} & m_3 & m_2 \end{array}\right]_q \times$$
$$\times \left[\begin{array}{c|cc} j_s & j_2 & j_1 \\ m_{12} & m_2 & m_1 \end{array}\right]^t_q \left[\begin{array}{c|cc} j_4 & j_3 & j_s \\ m_{123} & m_3 & m_{12} \end{array}\right]^t_q ,$$

$$\begin{Bmatrix} j_2 & j_3 & j_u \\ j_1 & j_4 & j_s \end{Bmatrix}^t_q = \sum_{m_3, m_2, m_1} \left[\begin{array}{c|cc} j_4 & j_3 & j_s \\ m_{123} & m_3 & m_{12} \end{array}\right]_q \left[\begin{array}{c|cc} j_s & j_2 & j_1 \\ m_{12} & m_2 & m_1 \end{array}\right]_q \times$$
$$\times \left[\begin{array}{c|cc} j_u & j_3 & j_2 \\ m_{23} & m_3 & m_2 \end{array}\right]^t_q \left[\begin{array}{c|cc} j_4 & j_u & j_1 \\ m_{123} & m_{23} & m_1 \end{array}\right]^t_q ,$$

where $m_{12} = m_1 + m_2$, $m_{23} = m_2 + m_3$ and $m_{123} = m_1 + m_2 + m_3$. Using the proportionality (3.14) between the Clebsch–Gordan coefficients and their transposes we quickly see that the $6j$-symbols fulfil the following orthogonality relations

$$\sum_{j_u} \begin{Bmatrix} j_2 & j_3 & j_u \\ j_1 & j_4 & j_s \end{Bmatrix}_q \begin{Bmatrix} j_2 & j_3 & j_u \\ j_1 & j_4 & j_s' \end{Bmatrix}_q = \delta_{j_s, j_s'} \,, \tag{3.22a}$$

$$\sum_{j_s} \begin{Bmatrix} j_2 & j_3 & j_u \\ j_1 & j_4 & j_s \end{Bmatrix}_q \begin{Bmatrix} j_2 & j_3 & j_u' \\ j_1 & j_4 & j_s \end{Bmatrix}_q = \delta_{j_u, j_u'} \,. \tag{3.22b}$$

Given a triple (j, j_2, j_1) we call the inequalities $|j_2 - j_1| \leq j \leq j_2 + j_1$ the triangle condition. From the representation in terms of Clebsch–Gordan coefficients we additionally draw, that the $6j$-symbols vanish whenever one of the triples (j_4, j_u, j_1), (j_u, j_3, j_2), (j_2, j_2, j_1) and (j_4, j_3, j_s) does not satisfy the triangle condition.

Recurrence relations for the quantum $6j$-symbols. We consider the four-fold tensor product $V^d \otimes V^c \otimes V^b \otimes V^a$. Let e denote the spin of the reduced basis. The pentagon relation for $\mathcal{U}_q$ encodes the way the different bases of the four-fold tensor product correspond to each other. We have

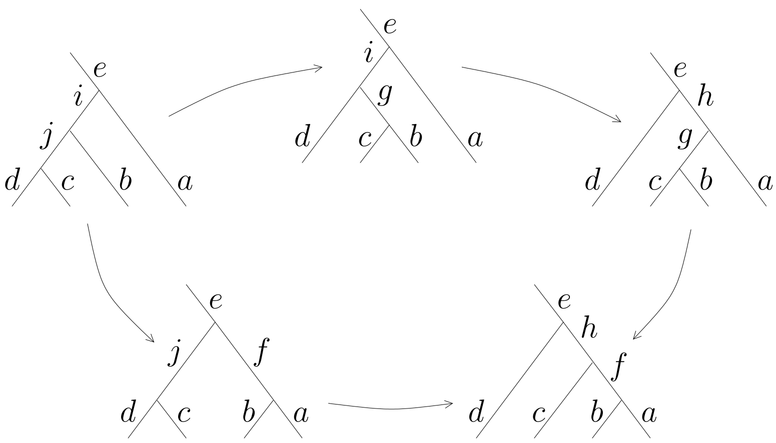

FIGURE 7: Pentagon relations

$$\begin{Bmatrix} d & c & h \\ e & f & j \end{Bmatrix}_q \begin{Bmatrix} j & b & f \\ e & a & i \end{Bmatrix}_q = \sum_g \begin{Bmatrix} c & d & f \\ h & a & g \end{Bmatrix}_q \begin{Bmatrix} d & g & h \\ e & a & i \end{Bmatrix}_q \begin{Bmatrix} d & c & g \\ i & b & j \end{Bmatrix}_q .$$

With the help of the orthogonality relation (3.22a) we get an equality with only one $6j$-symbol on the left-hand side.

$$\begin{Bmatrix} j & b & f \\ e & a & i \end{Bmatrix}_q = \sum_{g,h} \begin{Bmatrix} d & c & h \\ e & f & j \end{Bmatrix}_q \begin{Bmatrix} c & b & f \\ h & a & g \end{Bmatrix}_q \begin{Bmatrix} d & g & h \\ e & a & i \end{Bmatrix}_q \begin{Bmatrix} d & c & g \\ i & b & j \end{Bmatrix}_q .$$

We specialise this equation. We set $d = \frac{1}{2}$ and $c = j - \frac{1}{2}$.

$$\begin{Bmatrix} j & b & f \\ e & a & i \end{Bmatrix}_q = \sum_{g,h} \begin{Bmatrix} \frac{1}{2} & j-\frac{1}{2} & h \\ e & f & j \end{Bmatrix}_q \begin{Bmatrix} j-\frac{1}{2} & b & f \\ h & a & g \end{Bmatrix}_q \begin{Bmatrix} \frac{1}{2} & g & h \\ e & a & i \end{Bmatrix}_q \begin{Bmatrix} \frac{1}{2} & j-\frac{1}{2} & g \\ i & b & j \end{Bmatrix}_q .$$

The triangle condition constrains h and g by $|j-1| \leq g, h \leq j$ and $\left|g - \frac{1}{2}\right| \leq h \leq g + \frac{1}{2}$. This means we have an expression for the $6j$-symbol $\left\{\begin{smallmatrix} j & b & f \\ e & a & i \end{smallmatrix}\right\}$ in terms of one $6j$-symbol with $j-1/2$ and fundamental $6j$-symbols with one spin equal to $1/2$

$$\begin{Bmatrix} j & b & f \\ e & a & i \end{Bmatrix}_q = \sum_{g,h} S(a,b,f,i,j;h,g) \begin{Bmatrix} j-\frac{1}{2} & b & f \\ h & a & g \end{Bmatrix}_q . \tag{3.23}$$

Action on the four-point invariants. From the definition of the invariant tensors as product of Clebsch–Gordan coefficients and an invariant bilinear form the action of the $6j$-symbols on the four-point invariants can be deduced. In the following we will determine

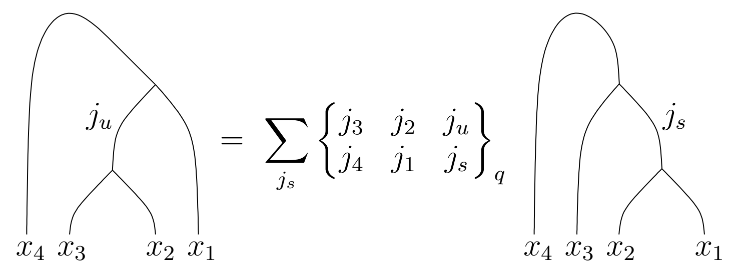

FIGURE 8: The action of the quantum $6j$-symbol

the fundamental four-point invariants that are connected by the fundamental $6j$-symbols. This gives a set of equations for the fundamental $6j$-symbols. The solutions to these are just the initial values to the recurrence relation above.

K **invariance.** We determine asymptotic properties of the four-point invariants Ψ through their invariance under K. We define one special limit $\check{\Psi}$ of the four-point invariant Ψ as

$$\Psi^{(i)}_{j_i}(J|X) \xrightarrow[x_4\to\infty]{x_2\to 0} \check{\Psi}^{(i)}_{j_i}(J|x_3,x_1) .$$

We call $\check{\Psi}$ the reduced blocks of $\mathcal{U}_q$.

Let T denote the multiplication operator on functions of x, such that

$$\mathsf{T}\, f(x) = f(qx) .$$

PROPOSITION 3.1. *The reduced blocks* $\check{\Psi}$ *have the structure of a function of the quotient* $x = x_3/x_1$ *times monomials in* x_1 *and* x_3

$$\check{\Psi}^{(i)}_{j_i}(J|x_3, x_1) = x_1^{\alpha_1^{(i)}} x_3^{\alpha_3^{(i)}} \mathcal{F}^{(i)}_{j_i}(J|x) \ , \tag{3.24}$$

such that $\alpha_1^{(i)} + \alpha_3^{(i)} = j_1 + j_2 + j_3 - j_4 =: \kappa$ *for* $i = s, u$.

PROOF. Let i be either s or u. K is a group-like element. Invariance under K thus means, we have for both, s- and u-channel

$$K\Psi^{(i)}_{j_i}(J|X) = \Psi^{(i)}_{j_i}(J|X) \ .$$

From the tensor representation of K we get

$$\begin{aligned} K\Psi^{(i)}_{j_i}(J|X) &= (K \otimes K \otimes K \otimes K)\, \Psi^{(i)}_{j_i}(J|X) \\ &\xrightarrow[x_4 \to \infty]{x_2 \to 0} q^{j_4 - j_2} K^{(3)} K^{(1)} \check{\Psi}^{(i)}_{j_i}(J|x_3, x_1) \\ &= q^{-j_4^{123}}\, \mathsf{T}_3\, \mathsf{T}_1\, \check{\Psi}^{(i)}_{j_i}(J|x_3, x_1) \ . \end{aligned}$$

This says, that $\check{\Psi}$ is a function of the quotient $x = x_1/x_3$ times monomials in x_1 and x_3 such that $\alpha_1^{(i)} + \alpha_3^{(i)} = j_4^{123}$ for $i = s, u$, as propositioned.

□

COROLLARY 3.1. *From that we gather an inverse proportionality between* T_1 *and* T_3

$$\mathsf{T}_1\, \check{\Psi}^{(i)}_{j_i}(J|x_3, x_1) = q^{\kappa}\, \mathsf{T}_3^{-1}\, \check{\Psi}^{(i)}_{j_i}(J|x_3, x_1) \ . \tag{3.25}$$

For the difference operator

$$[\mathrm{d}_x + a] = \frac{q^a\, \mathsf{T}_x - q^{-a}\, \mathsf{T}^{-1}}{q - q^{-1}}$$

this entails

$$[\mathrm{d}_1 + a]\, \check{\Psi}^{(i)}_{j_i} = -\, [\mathrm{d}_3 - a - \kappa]\, \check{\Psi}^{(i)}_{j_i} \ .$$

To determine the exponents $\alpha_1^{(i)}$ and $\alpha_3^{(i)}$, we look at the decompositions

$$\Psi^{(s)}_{j_s}(J|X) = \sum_{m_s} \begin{pmatrix} j_4 & j_3 & j_s \\ x_4 & x_3 & m_s \end{pmatrix}_q \left[\begin{array}{c|cc} j_s & j_2 & j_1 \\ m_s & x_2 & x_1 \end{array}\right]_q \ , \tag{3.26a}$$

$$\Psi^{(u)}_{j_u}(J|X) = \sum_{m_u} \begin{pmatrix} j_4 & j_u & j_1 \\ x_4 & m_u & x_1 \end{pmatrix}_q \left[\begin{array}{c|cc} j_u & j_3 & j_2 \\ m_u & x_3 & x_2 \end{array}\right]_q \tag{3.26b}$$

of the four-point invariants. Consider the s-channel. There we have

$$\Psi^{(s)}_{j}(J|X) = \sum_{m_i} x_4^{j_4+m_4} x_3^{j_3+m_3} x_2^{j_2+m_2} x_1^{j_1+m_1} \begin{pmatrix} j_4 & j_3 & j \\ m_4 & m_3 & m \end{pmatrix}_q \left[\begin{array}{c|cc} j & j_2 & j_1 \\ m & m_2 & m_1 \end{array}\right]_q$$

and in the limit $(x_4, x_2) \to (\infty, 0)$

$$\to \sum_{m_s, m_3, m_1} x_3^{j_3+m_3} x_1^{j_1+m_1} \begin{pmatrix} j_4 & j_3 & j \\ j_4 & m_3 & m \end{pmatrix}_q \left[\begin{array}{c|cc} j & j_2 & j_1 \\ m & -j_2 & m_1 \end{array}\right]_q$$

because of the horizontal constraints we have $m_3 = -j_4 - m_s$ and $m_1 = j_2 + m_s$

$$= \sum_{m_s} x_3^{j_3-j_4-m_s} x_1^{j_1+j_2+m_s} \begin{pmatrix} j_4 & j_3 & j \\ j_4 & -j_4-m_s & m \end{pmatrix}_q \left[\begin{array}{c|cc} j & j_2 & j_1 \\ m & -j_2 & j_2+m_s \end{array}\right]_q .$$

From (3.26b) and a similar derivation for the u-channel we get the following polynomial ansatz for the four-point invariants in the limit $(x_4, x_2) \to (\infty, 0)$

$$\text{(3.27a)} \qquad \begin{aligned} \check{\Psi}^{(s)}_{j_s}(J|x_3, x_1) &= x_3^{j_4^{123}} \sum_{m_s=-j_s}^{j_s} f^{(s)}_{m_s}(J|j_s) \left(\frac{x_1}{x_3}\right)^{j_1+j_2+m_s} \\ &=: x_3^{j_4^{123}} \mathcal{F}^{(s)}_j(J|x) , \end{aligned}$$

$$\text{(3.27b)} \qquad \begin{aligned} \check{\Psi}^{(u)}_{j_u}(J|x_3, x_1) &= x_3^{j_4^{123}} \sum_{m_u=-j_u}^{j_u} f^{(u)}_{m_u}(J|j_u) \left(\frac{x_1}{x_3}\right)^{j_1-j_4-m_u} \\ &=: x_3^{j_4^{123}} \mathcal{F}^{(u)}_j(J|x) , \end{aligned}$$

where $x = \frac{x_1}{x_3}$ and

$$\text{(3.27c)} \quad f^{(s)}_{m_s}(J|j_s) = \begin{pmatrix} j_4 & j_3 & j_s \\ j_4 & -j_4-m_s & m_s \end{pmatrix}_q \left[\begin{array}{c|cc} j_s & j_2 & j_1 \\ m_s & -j_2 & m_s+j_2 \end{array}\right]_q ,$$

$$\text{(3.27d)} \quad f^{(u)}_{m_u}(J|j_u) = \begin{pmatrix} j_4 & j_u & j_1 \\ j_4 & m_u & -j_4-m_u \end{pmatrix}_q \left[\begin{array}{c|cc} j_u & j_3 & j_2 \\ m_u & m_u+j_2 & -j_2 \end{array}\right]_q .$$

So in the *x-reduced blocks* $\mathcal{F}^{(i)}_j(J|x)$ we have at last functions in one variable only. For these we will find special cases that are linear functions and intertwined by the basic $6j$-symbols.

The quantum Casimir eigenfunction equation. We will find solutions for the functions $\mathcal{F}^{(i)}_j(J|x)$ based on q-hypergeometric functions.

PROPOSITION 3.2. *The functions $\mathcal{F}^{(i)}_j(J|x)$ fulfil the difference equations*

$$\text{(3.28a)} \quad \left(q^{j_2-j_4-1}\tfrac{x_1}{x_3}[\mathrm{d}_x-\kappa][\mathrm{d}_x-2j_1]-[\mathrm{d}_x-l][\mathrm{d}_x-l-2j_s-1]\right)\mathcal{F}^{(s)}_{j_s}(J|x)=0 ,$$

$$\text{(3.28b)} \quad \left(q^{j_2-j_4-1}\tfrac{x_1}{x_3}[\mathrm{d}_x+k-\kappa][\mathrm{d}_x+k+2j_u+1-\kappa]-[\mathrm{d}_x][\mathrm{d}_x+2j_3-\kappa]\right)\mathcal{F}^{(u)}_{j_u}(J|x)=0 .$$

PROOF. As the invariant $\Psi^{(i)}_{j_i}$ is defined by projecting onto the irreducible representation with spin j_i in the tensor product of the representations $V^{j_2} \otimes V^{j_1}$ we have that the q-Casimir operator is equal to the unit times $-\,[j_i+\frac{1}{2}]^2$.

$$\Delta_{21}(C_q)\Psi^{(s)}_{j_s}(J|X) = -\left[j_s+\tfrac{1}{2}\right]^2\Psi^{(s)}_{j_s}(J|X)\ ,$$
$$\Delta_{32}(C_q)\Psi^{(u)}_{j_u}(J|X) = -\left[j_u+\tfrac{1}{2}\right]^2\Psi^{(u)}_{j_u}(J|X)\ .$$

The indices 21 and 32 express the way the coproduct is embedded into the four-fold tensor product.

$$\Delta_{21} = \mathrm{id}\otimes\mathrm{id}\otimes\Delta \quad \text{and} \quad \Delta_{32} = \mathrm{id}\otimes\Delta\otimes\mathrm{id}\ .$$

In the transposed x-representation the quantum Casimir operator

$$C_q = EF - \frac{q^{-1}K^2+qK^{-2}-2}{(q-q^{-1})^2}\ .$$

takes the form of a difference operator. We determine this operator in two steps. First we consider the summand EF. In order to shorten expressions we introduce the following notation. Let a_n denote an element a in the n^{th} place of the tensor product. We assume that it is always clear from the context, which tensor product is meant.

$$a_n = \underset{1}{1}\otimes\cdots\otimes\underset{n-1}{1}\otimes\underset{n}{a}\otimes\underset{n+1}{1}\otimes\cdots\otimes 1\ .$$

We can express $\Delta_{21}(EF)$ now as

$$\begin{aligned}\Delta_{21}(EF)\Psi &= \left(E_2K_1+K_2^{-1}E_1\right)\left(F_2K_1+K_2^{-1}F_1\right)\Psi\\ &= \left(E_2F_2K_1^2+E_2K_2^{-1}K_1F_1+K_2^{-1}F_2E_1K_1+K_2^{-2}E_1F_1\right)\Psi\\ &\overset{a}{=} \left(E_2F_2K_1^2+E_2K_2^{-1}K_1F_1-K_4F_4K_3^2E_1K_1-K_3F_3E_1K_1\right)\Psi\ .\end{aligned}$$

Equality a uses the fact that the block Ψ is invariant under the algebra-like element F, which means that $\Delta^{(4)}(F)\Psi=0$ where the action of the generator F on the fourfold tensor product is

$$\Delta^{(4)}(F) = F_4K_3K_2K_1+K_4^{-1}F_3K_2K_1+K_4^{-1}K_3^{-1}F_2K_1+K_4^{-1}K_3^{-1}K_2^{-1}F_1\ .$$

The term $t=K_4^{-1}K_3^{-1}F_2K_1$ is replaced by $-\Delta^{(4)}(F)+t$.
We note that the term proportional to E_2 vanishes in the limit $x\to 0$. The same holds for the term proportional to F_4 in the limit $x_4\to\infty$. We notice that the term E_2F_2 is basically the Casimir operator.

$$E_2F_2 = (C_q)_2 + \frac{q^{-1}K_2^2+qK_2^{-2}-2}{(q-q^{-1})^2}\ .$$

This term vanishes in the limit $x_2\to 0$ because the second summand tends to $\left[j_2+\frac{1}{2}\right]^2$ and the Casimir operator is $-\left[j_2+\frac{1}{2}\right]^2$ on the irreducible representation with spin j_2. Now we are ready to state that on the reduced block $\check{\Psi}$ we have

$$\Delta_{21}(EF)\check{\Psi} = q^{j_2-j_4-1}\frac{x_1}{x_3}\,[\mathrm{d}_3-\kappa]\,[\mathrm{d}_1-2j_1]\,\check{\Psi}\ .$$

With the coproduct $\Delta K = K \otimes K$ of the group-like generator K it is a straightforward calculation to check that

$$\Delta_{21}\left(\frac{q^{-1}K^2 + qK^{-2} - 2}{(q-q^{-1})^2}\right)\check{\Psi} = \left[\mathrm{d}_1 - j_1 - j_2 - \tfrac{1}{2}\right]^2 \check{\Psi} \ .$$

A similar calculation for Δ_{32} yields the representation of the Casimir[9] operator on the reduced blocks as

$$q^{j_2-j_4-1}\frac{x_1}{x_3}\left[\mathrm{d}_1 - \kappa\right]\left[\mathrm{d}_1 - 2j_1\right] - \left[\mathrm{d}_1 - j_1 - j_2 - \tfrac{1}{2}\right]^2$$

in the s-channel and

$$q^{-j_2+j_4+1}\frac{x_3}{x_1}\left[\mathrm{d}_3 - \kappa\right]\left[\mathrm{d}_3 - 2j_3\right] - \left[\mathrm{d}_3 - j_2 - j_3 - \tfrac{1}{2}\right]^2$$

in the u-channel.

We introduce the indices l and k counting the difference of the intermediate representation's spin from the maximal spin. We set

$$\begin{aligned} j_s + l &= j_1 + j_2 \ , \qquad l = 0,1,2,\ldots,2\mu \text{ with } \mu = \min\{j_1, j_2\} \text{ and} \\ j_u + k &= j_2 + j_3 \ , \qquad k = 0,1,2,\ldots,2\nu \text{ with } \nu = \min\{j_2, j_3\} \ . \end{aligned}$$

A little q-number aerobics finds us

$$\begin{aligned} \left[j_s + \tfrac{1}{2}\right]^2 - \left[\mathrm{d}_1 - j_1 - j_2 - \tfrac{1}{2}\right]^2 &= \left[\mathrm{d}_1 - l\right]\left[\mathrm{d}_1 - 2j_s - l - 1\right] \ , \\ \left[j_u + \tfrac{1}{2}\right]^2 - \left[\mathrm{d}_3 - j_2 - j_3 - \tfrac{1}{2}\right]^2 &= \left[\mathrm{d}_3 - k\right]\left[\mathrm{d}_3 - 2j_u - k - 1\right] \ . \end{aligned}$$

Using the inverse proportionality (3.25) between T_1 and T_3 we finally arrive at

$$\left(q^{j_2-j_4-1}\frac{x_1}{x_3}\left[\mathrm{d}_x - \kappa\right]\left[\mathrm{d}_x - 2j_1\right] - \left[\mathrm{d}_x - l\right]\left[\mathrm{d}_x - l - 2j_s - 1\right]\right)\mathcal{F}^{(s)}_{j_s}(J|x) = 0 \ ,$$

$$\left(q^{j_2-j_4-1}\frac{x_1}{x_3}\left[\mathrm{d}_x + k - \kappa\right]\left[\mathrm{d}_x + k + 2j_u + 1 - \kappa\right] - \left[\mathrm{d}_x\right]\left[\mathrm{d}_x + 2j_3 - \kappa\right]\right)\mathcal{F}^{(u)}_{j_u}(J|x) = 0 \ .$$

$\square$

This is just nearly the q-hypergeometric difference equation[10]

$$\left(x\left[\mathrm{d}_x + a\right]\left[\mathrm{d}_x + b\right] - \left[\mathrm{d}_x\right]\left[\mathrm{d}_x + c - 1\right]\right)p(x) = 0$$

with the solution

$${}_2\Phi_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| x; q\right) = \sum_{n\geq 0}\frac{[a|n]\,[b|n]}{[c|n]\,[n]!}x^n \ .$$

---

[9] Note that the finite difference operators $\Delta_{21}(C_q)$ in the s-channel and $\Delta_{32}(C_q)$ in the u-channel are related by the exchanges

$$j_1 \leftrightarrow j_3 \quad \text{and} \quad \frac{x_1}{x_3} \leftrightarrow \frac{x_3}{x_1} \ .$$

[10] see appendix A.

Equations (3.28) can be brought to q-hypergeometric form by an ansatz

$$\mathcal{F}^{(s)}_{j_s}(J|x) = x^l g^{(s)}_{j_s}(q^{j_2-j_4-1}x)$$
$$\mathcal{F}^{(u)}_{j_u}(J|x) = g^{(u)}_{j_u}(q^{j_2-j_4-1}x) \ .$$

Thus for the functions $\mathcal{F}^{(i)}_{j_i}$ we determine the following q-hypergeometric form modulo a J-dependent normalisation

$$\text{(3.29a)} \qquad \mathcal{F}^{(s)}_{j_s}(J|x) = x^l {}_2\Phi_1\left(\begin{matrix} l-\kappa \quad l-2j_1 \\ -2j_s \end{matrix} \middle| q^{j_2-j_4-1}x; q \right) \ ,$$

$$\text{(3.29b)} \qquad \mathcal{F}^{(u)}_{j_u}(J|x) = {}_2\Phi_1\left(\begin{matrix} k-\kappa \quad k+2j_u+1-\kappa \\ 2j_3+1-\kappa \end{matrix} \middle| q^{j_2-j_4-1}x; q \right) \ .$$

We consider the case of one representation, say j_2, equal to $1/2$. Now only two intermediate spins will appear in both channels, $j_s^\pm = j_1 \pm 1/2$ and $j_u^\pm = j_3 \pm 1/2$. This is analogous to the insertion of a degenerate field of spin $1/2$ into the conformal blocks. This puts a further restriction on the functions $\mathcal{F}^{(i)}_{j_i}$ with $j_2 = 1/2$.

Proposition 3.3. *The functions $\mathcal{F}^{(i)}_{j_i^\pm}$ with $j_2 = 1/2$ are of the form*

$$\mathcal{F}^{(i)}_{j_i^\pm}(J|x) = \left(a_i^\pm + b_i^\pm x\right) R(J|x).$$

Proof. The invariance of the four-point invariant is expressed as

$$\Delta^{(4)}(F)\Psi^{(i)}_{j_i}(J|X) = 0 \ .$$

When we consider the limit $x_4 \to \infty$ an let $\Psi^{(i)}_{j_i} = \psi$ we have

$$\left(F_3K_2K_1 + K_3^{-1}F_2K_1 + K_3^{-1}K_2^{-1}F_1\right)\psi = 0 \ ,$$

or in terms of difference operators[11]

$$\left(q^{-j_1-j_2+\kappa}\frac{1}{x_3}\left(1-\mathsf{T}_3^{-2}\right) + q^{j_3-j_1-\kappa}\frac{1}{x_2}\mathsf{T}_3^{-2}\left(1-\mathsf{T}_2^{-2}\right) + q^{j_3+j_2-\kappa}\frac{1}{x_1}\left(\mathsf{T}_1^2-1\right)\right)\psi = 0 \ .$$

When j_2 is equal to $1/2$ we have[12] $F_2^2 \equiv 0$. This means that the polynomial solutions to the Casimir-eigenvalue equation are of the form

$$\psi = \psi_1 + x_2\psi_2 \ .$$

[11]Here we have used the relation $\mathsf{T}_3\,\mathsf{T}_2\,\mathsf{T}_1\,\psi = q^\kappa\psi$, which follows from the invariance of Ψ under the action of K, $\Delta^{(4)}(K)\Psi = \Psi$.

[12]This corresponds nicely to the additional equation $\partial_x^2\Psi_{21} = 0$ for the degenerate field of spin 1/2 in the WZNW-model.

The difference equation in ψ then becomes

$$\begin{aligned}
0 = Q^{-1} & \left(q^{-j_1-\frac{1}{2}+\kappa} \frac{1}{x_3} \left(1 - \mathsf{T}_3^{-2}\right) + q^{j_3+\frac{1}{2}-\kappa} \frac{1}{x_1} \left(\mathsf{T}_1^2 - 1\right) \right) \psi_1 + \\
& + q^{j_3-j_1+\kappa-1}\, \mathsf{T}_3^{-2}\, \psi_2 + \\
& + Q^{-1} x_2 \left(q^{-j_1-\frac{1}{2}+\kappa} \frac{1}{x_3} \left(1 - \mathsf{T}_3^{-2}\right) + q^{j_3+\frac{1}{2}-\kappa} \frac{1}{x_1} \left(\mathsf{T}_1^2 - 1\right) \right) \psi_2 \\
& =: \mathfrak{D}\psi_1 + q^{j_3-j_1+\kappa-1} Q\, \mathsf{T}_3^{-2}\, \psi_2 + x_2 \mathfrak{D}\psi_2 \; .
\end{aligned}$$

There are two linearly independent polynomial solutions ϕ' and ϕ to the equation $F_2^2\psi = 0$. A general solution ψ then is of the form

$$\psi = a\phi' + b\phi \; . \tag{3.30}$$

The solution ϕ'. We choose the first solution ϕ' to have a vanishing component ϕ'_2. That leaves the difference equation

$$0 = \mathfrak{D}\phi'_1 = \left(q^{-j_1-\frac{1}{2}+\kappa} \frac{1}{x_3} \left(1 - \mathsf{T}_3^{-2}\right) + q^{j_3-\frac{1}{2}-\kappa} \frac{1}{x_1} \left(\mathsf{T}_1^2 - 1\right) \right) \phi'_1 \; . \tag{3.31}$$

This leads to

$$\left(1 - q^{j_1+j_3+1-2\kappa} \frac{x_3}{x_1}\right) \phi'_1 = \left(1 - q^{j_1+j_3+1} \frac{x_3}{x_1}\right) \mathsf{T}_3^{-2}\, \phi'_1 \; .$$

We expand the quotient of $(\mathsf{T}_3^{-2}\, \phi'_1)$ and ϕ'_1

$$\begin{aligned}
\frac{(\mathsf{T}_3^{-2}\, \phi'_1)}{\phi'_1} &= \frac{\left(1 - q^{j_1+j_3+1-2\kappa} \frac{x_3}{x_1}\right)}{\left(1 - q^{j_1+j_3+1} \frac{x_3}{x_1}\right)} \\
&= \frac{\left(1 - q^{j_1+j_3+1-2\kappa} \frac{x_3}{x_1}\right) \left(1 - q^{j_1+j_3+3-2\kappa} \frac{x_3}{x_1}\right) \cdots \left(1 - q^{j_1+j_3-1} \frac{x_3}{x_1}\right)}{\left(1 - q^{j_1+j_3+3-2\kappa} \frac{x_3}{x_1}\right) \left(1 - q^{j_1+j_3+5-2\kappa} \frac{x_3}{x_1}\right) \cdots \left(1 - q^{j_1+j_3+1} \frac{x_3}{x_1}\right)} \; .
\end{aligned}$$

This determines ϕ'_1

$$\phi'_1 = \left(1 - q^{j_1+j_3+3-\kappa} \frac{x_3}{x_1}\right) \left(1 - q^{j_1+j_3+5-\kappa} \frac{x_3}{x_1}\right) \cdots \left(1 - q^{j_1+j_3+1+2\kappa} \frac{x_3}{x_1}\right) \; . \tag{3.32}$$

The first of the linear independent solutions to the equation $F_2^2\psi = 0$ is therefore

$$\phi' = \phi'_1 \; . \tag{3.33}$$

The solution ϕ. We see that the function ϕ_2 is determined by the homogeneous difference equation

$$\mathfrak{D}\phi_2 = 0 \tag{3.34}$$

and that the function ϕ_1 is determined by the inhomogeneous difference equation

$$\mathfrak{D}\phi_1 = -Q q^{j_3-j_1+\kappa-1}\, \mathsf{T}_3^{-2}\, \phi_2 \; . \tag{3.35}$$

First we determine ϕ_2. The equation (3.34) determining ϕ_2 is the same as the equation 3.31 determining ϕ_1'. We conclude that

$$\phi_2 = \phi_1' \ . \tag{3.36}$$

Now we determine ϕ_1. Because of (3.35) we have

$$q^{-j_1-\frac{1}{2}+\kappa}\frac{1}{x_3}\left(1-\mathsf{T}_3^{-2}\right)\phi_1 - q^{j_3+\frac{1}{2}-\kappa}\frac{1}{x_1}\left(1-\mathsf{T}_1^{2}\right)\phi_1 = -q^{j_3-\frac{1}{2}}Q\,\mathsf{T}_3^{-2}\,\phi_2 \ .$$

The difference operator in (3.35) is the same as in (3.34). Thus we make the ansatz $\phi_1 = \phi_2 g$. The operators $(1-\mathsf{T}_3^{-2})$ and $(1-\mathsf{T}_1^2)$ obey the following deformed product rule

$$(1-b\,\mathsf{T}^a)fg = ((1-b\,\mathsf{T}^a)f)\,g + (b\,\mathsf{T}^a\,f)\,((1-\mathsf{T}^a)g) \ .$$

Thus we have

$$\mathfrak{D}\phi_1 = (\mathfrak{D}\phi_2)g + (\mathsf{T}_3^{-2}\,\phi_2)q^\kappa\left(\frac{q^{-j_1-\frac{1}{2}}}{x_3}(1-\mathsf{T}_3^{-2}) - \frac{q^{j_3+\frac{1}{2}}}{x_1}(1-\mathsf{T}_1^2)\right)g \ .$$

We note that $\mathfrak{D}\phi_2 = 0$ by necessity. This reduces equation (3.35) above to

$$q^\kappa\left(\frac{q^{-j_1-\frac{1}{2}}}{x_3}\left(1-\mathsf{T}_3^{-2}\right) - \frac{q^{j_3+\frac{1}{2}}}{x_1}\left(1-\mathsf{T}_1^2\right)\right)g = -Qq^{j_3-j_1+\kappa-1} \ .$$

The ansatz $g = \alpha x_3$ yields

$$q^{\kappa-j_1-\frac{1}{2}}(1-q^{-2})\alpha = -Qq^{j_3-j_1+\kappa-1}$$

and determines $\alpha = -q^{j_3-\frac{1}{2}}$. We conclude that

$$\phi = (\alpha x_3 + x_2)\phi_2 \ .$$

With this and the equality (3.36) of ϕ_2 and ϕ_1' the general form (3.30) of a solution is

$$\begin{aligned}\psi &= a\phi' + b\phi \\ &= a\phi_1' + b(\alpha x_3 + x_2)\phi_2 \\ &= (a + b(\alpha x_3 + x_2))\phi_1' \ .\end{aligned}$$

When x_2 tends to 0, ψ tends to $\check{\Psi}$. Thus we have proven that the reduced blocks with $j_2 = 1/2$ are of the form

$$\mathcal{F}^{(i)}_{j_i^\pm}(J|x) = \left(a_i^\pm + b_i^\pm x\right)R(J|x) \ .$$

□

Judicious inspection of the horizontal and vertical constraints (3.11) in the coefficients $f^{(i)}_{m_i}(J|j_{i^\pm})$ of (3.27) tells us that the polynomials $\mathcal{F}$

with $j_2 = 1/2$ have an appropriate number of terms. Indeed we have

$$\begin{aligned}
\mathcal{F}^{(s)}_{j_s^+}(J|x) &= f^{(s)}_{-j_1-\frac{1}{2}}(J|j_s^+) + f^{(s)}_{-j_1+\frac{1}{2}}(J|j_s^+)x + \cdots + f^{(s)}_{j_3-j_4}(J|j_s^+)x^\kappa \\
\mathcal{F}^{(s)}_{j_s^-}(J|x) &= f^{(s)}_{-j_1+\frac{1}{2}}(J|j_s^-)x + \cdots + f^{(s)}_{j_3-j_4}(J|j_s^-)x^\kappa
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}^{(u)}_{j_u^+}(J|x) &= f^{(u)}_{j_1-j_4}(J|j_u^+) + \cdots + f^{(u)}_{-j_3+\frac{1}{2}}(J|j_u^+)x^{\kappa-1} + f^{(u)}_{-j_3-\frac{1}{2}}(J|j_u^+)x^\kappa \\
\mathcal{F}^{(u)}_{j_u^-}(J|x) &= f^{(u)}_{j_1-j_4}(J|j_u^-) + \cdots + f^{(u)}_{-j_3+\frac{1}{2}}(J|j_u^-)x^{\kappa-1}
\end{aligned}$$

From this we see that the coefficients a_s^- and b_u^- vanish. Use of proposition 3.3 leaves us with the linear system of equalities for the fusing relations

$$\begin{aligned}
a_u^+ + b_u^+ x &= \begin{Bmatrix} j_3 & \frac{1}{2} & j_u^+ \\ j_4 & j_1 & j_s^+ \end{Bmatrix}_q \left(a_s^+ + b_s^+ x\right) + \begin{Bmatrix} j_3 & \frac{1}{2} & j_u^+ \\ j_4 & j_1 & j_s^- \end{Bmatrix}_q b_s^- x \\
a_u^- &= \begin{Bmatrix} j_3 & \frac{1}{2} & j_u^- \\ j_4 & j_1 & j_s^+ \end{Bmatrix}_q \left(a_s^+ + b_s^+ x\right) + \begin{Bmatrix} j_3 & \frac{1}{2} & j_u^- \\ j_4 & j_1 & j_s^- \end{Bmatrix}_q b_s^- x \; .
\end{aligned} \tag{3.37}$$

Comparison of the representations (3.27) and of proposition 3.3 for the reduced conformal blocks together with the expression

$$R(J|x) = r_0 + r_1 x + \cdots + r_{\kappa-1}x^{\kappa-1}$$

gives the equations

$$\begin{aligned}
r_0 a_s^+ &= 1 & r_0 a_u^+ &= q^{-\kappa/2}\frac{[\iota - 2j_1]}{[2j_3+1]} \\
r_0 b_s^+ &= -q^{-j_4-1/2}\frac{[\iota - 2j_3]}{[2j_1+1]} & r_0 b_u^+ &= -q^{-j_4-1/2-\kappa/2} \\
r_0 b_s^- &= -q^{-j_1-1/2} & r_0 a_u^- &= q^{-(\kappa-1)/2} \; .
\end{aligned}$$

With these we can solve the linear equations (3.37) and get

$$\begin{aligned}
\begin{Bmatrix} j_3 & \frac{1}{2} & j_u^+ \\ j_4 & j_1 & j_s^+ \end{Bmatrix}_q &= q^{-\frac{\kappa}{2}}\frac{[\iota - 2j_1]}{[2j_3+1]} \; , \\
\begin{Bmatrix} j_3 & \frac{1}{2} & j_u^- \\ j_4 & j_1 & j_s^+ \end{Bmatrix}_q &= q^{-\frac{\kappa-1}{2}} \; , \\
\begin{Bmatrix} j_3 & \frac{1}{2} & j_u^+ \\ j_4 & j_1 & j_s^- \end{Bmatrix}_q &= q^{-\frac{\iota-2j_1}{2}}\frac{[\kappa]\,[\iota+1]}{[2j_1+1]\,[2j_3+1]} \; , \\
\begin{Bmatrix} j_3 & \frac{1}{2} & j_u^- \\ j_4 & j_1 & j_s^- \end{Bmatrix}_q &= -q^{-\frac{\iota-j_1-1}{2}}\frac{[\iota - 2j_3]}{[2j_1+1]} \; ,
\end{aligned} \tag{3.38}$$

where $\iota = j_1 + j_2 + j_3 + j_4$.

We have thus determined the fundamental $6j$-symbols of $\mathcal{U}_q\mathbf{sl}(2)$. Together with the recurrence relation (3.23) these determine all $6j$-symbols of the quantum group $\mathcal{U}_q\mathbf{sl}(2)$.

Chapter 4

The quantum group $\mathcal{U}_q\mathbf{osp}(1|2)$

In this chapter we apply our method to the generalised setting of super quantum groups and determine the $6j$-symbols of the super quantum group $\mathcal{U}_q\mathbf{osp}(1|2)$.

Up to a certain point the derivation will be strikingly similar to the preceding chapter. We represent the generators of $\mathcal{U}_q$ as difference operators. All invariant tensors satisfy difference equations induced by the Casimir operator. This fixes the the four-point invariants to a super q-hypergeometric form.
The fundamental representation of $\mathcal{U}_q\mathbf{osp}(1|2)$ is three dimensional. This only gives a minor rise in complexity for the derivation of the $6j$-symbols. We set the representation label l_2 to the fundamental representation 1. This further restricts the fundamental four-point functions to a form from which it is possible to deduce a system of linear equations that determine the fundamental $6j$-symbols connecting the s-channel decomposition and the u-channel decomposition.

The graded quantum group $\mathcal{U}_q\mathbf{osp}(1|2)$ is generated by elements k, k^{-1}, e, f and the relations

$$\begin{aligned} kek^{-1} &= q^{\frac{1}{2}}e\ , \qquad ef + fe = -\frac{k^2-k^{-2}}{q-q^{-1}}\ ,\\ kfk^{-1} &= q^{-\frac{1}{2}}f\ , \qquad kk^{-1} = k^{-1}k = 1\ . \end{aligned} \tag{4.1}$$

It has the coproduct

$$\begin{aligned} \Delta(e) &= e\otimes k + k^{-1}\otimes e\ ,\\ \Delta(f) &= f\otimes k + k^{-1}\otimes f\ ,\\ \Delta(k) &= k\otimes k\ ,\\ \Delta(k^{-1}) &= k^{-1}\otimes k^{-1}\ , \end{aligned} \tag{4.2}$$

the counit

$$\varepsilon(k) = \varepsilon(k^{-1}) = \varepsilon(1) = 1\ , \quad \varepsilon(e) = \varepsilon(f) = 0\ , \tag{4.3}$$

and the antipode

$$S(e) = -q^{\frac{1}{2}}e \ , \qquad S(k) = k^{-1} \ ,$$
(4.4)
$$S(f) = -q^{-\frac{1}{2}}f \ , \quad S(k^{-1}) = k \ .$$

In addition to a Casimir operator C_q there is an operator S_q that commutes with the even part and anticommutes with the odd part of $\mathcal{U}_q\mathbf{osp}(1|2)$. S_q is called the Scasimir operator.

$$S_q = Q_{\mathfrak{s}}fe - \frac{q^{\frac{1}{2}}k^2 - q^{-\frac{1}{2}}k^{-2}}{Q} = -Q_{\mathfrak{s}}ef + \frac{q^{-\frac{1}{2}}k^2 - q^{\frac{1}{2}}k^{-2}}{Q} \ .$$

In order to reduce the clutter of qs we introduced the coefficients $Q = q + q^{-1}$ and $Q_{\mathfrak{s}} = q^{-1/2} + q^{1/2}$.

The Casimir operator simply is the square of the Scasimir operator modulo an additional constant.

$$C_q = S_q^2 + \text{const.} \ .$$

The super quantum group $\mathcal{U}_q\mathbf{osp}(1|2)$ can be seen as either the quantised universal enveloping algebra of the graded Lie algebra $\mathbf{osp}(1|2)$ or as the super analogue of the quantum group $\mathcal{U}_q\mathbf{sl}(2)$[1].

1. Finite dimensional representations

A representation of the quantum superalgebra[2] $\mathcal{U}_q\mathbf{osp}(1|2)$ is a super vector space V with an even subspace V_0 and an odd subspace V_1,

$$V = V_0 \oplus V_1 \ .$$

and a homomorphism of associative superalgebras

$$\rho : \mathcal{U}_q\mathbf{osp}(1|2) \to \mathbf{Hom}(V, V) \ .$$

Let (ρ, V) be a representation of $\mathcal{U}_q$. For every complex number ω define $V_\omega \colon = \{v \in V \mid \rho(k)v = \omega v\}$. We call every nontrivial V_ω the *weight space* corresponding to the *weight* ω, Nonzero vectors in V_ω are called *weight vectors* of *vectors of weight* ω[3]. A vector $v \in V_{\omega'}$ is called a vector of *highest* weight ω' if $\rho(e)v = 0$. In this case ω' is called the *highest weight* of the representation $\rho, V_{\omega'}$.

[1]The graded Lie algebra $\mathbf{osp}(1|2)$ has been closely examined by Nahm, Rittenberg and Scheunert in [**36**]. The super quantum group $\mathcal{U}_q\mathbf{osp}(1|2)$ has been introduced by Kulish and Reshetikhin in [**24**].

[2]This is one place where we break our naming convention. A "quantum superalgebra" naturally is a quantum group or Hopf algebra that is graded by $\mathbf{Z}/2\mathbf{Z}$.

[3]In the following vectors of weight ω will be called eigenvectors of $\rho(k)$ with eigenvalue ω too. In addition whenever $\omega = q^{m/2}$ we will call m weight or eigenvalue too.

If V is a direct sum of weight spaces of ρ we call ρ, V a *weight representation.*

The irreducible representations of $\mathcal{U}_q\mathbf{osp}(1|2)$ are grade star representations. That is to say, there is an involution $*$ on $\mathcal{U}_q$ and a Boolean parameter $\varepsilon \in \mathbf{Z}/2\mathbf{Z}$ such that in the representation

$$k^* = k \ , \quad e^* = (-1)^{\varepsilon} e \ , \quad f^* = (-1)^{\varepsilon+1} f$$

holds[4]. The parameter ε is called the *class* of the representation.

The representation theory of $\mathcal{U}_q$ falls into two different cases according to the value of the parameter q.

Generic q. The irreducible finite dimensional representations (ρ, V) of $\mathcal{U}_q\mathbf{osp}(1|2)$ are labelled by a positive integer $l \in \mathbf{Z}$ and an index $\lambda \in \mathbf{Z}_2 = \{0, 1\}$ and two further Boolean parameters $\varphi, \psi \in \mathbf{Z}_2$. The parameters φ and ψ are signature parameters of the Hermitean form on $V^{l,\lambda}$. The parity λ and the signature φ define the class ε of the representation. Since we have no use for the Hermitean form in the following, we drop the parameters φ, ψ and ε.

For each (l, λ) an irreducible finite dimensional representation is isomorphic to the following. The module $V^{l,\lambda}$ has a basis $\{\mathbf{e}^l_m \mid m = -l, -l+1, \cdots, l\}$ diagonalising the representation of k. The vector $\mathbf{e}^l_l(\lambda)$ is of highest weight with parity λ. The generators e and f of $\mathcal{U}_q\mathbf{osp}(1|2)$ act on this space as raising and lowering operators.

q a root of unity. Let q be a primitive p^{th} root of unity,

$$q^p = 1 \ .$$

The representations[5] of $\mathcal{U}_q\mathbf{osp}(1|2)$ in this case are plagued by problems similar to those of $\mathcal{U}_q\mathbf{sl}(2)$.

We consider generic q only in the following.

Tensor products of finite dimensional representations. The tensor product $X_1 \otimes X_2$ in the category of super vector spaces is graded by $\mathbf{Z}_2$. This affects super algebras and super modules. For super algebras we have

$$(X_1 \otimes X_2)(Y_1 \otimes Y_2) = (-1)^{\mathrm{p}(X_2)\,\mathrm{p}(Y_1)}(X_1 Y_1 \otimes X_2 Y_2) \ .$$

The effect for super modules is similar. Given two representations (ρ^1, V^1) and (ρ^2, V^2) of $\mathcal{U}_q\mathbf{osp}(1|2)$ the tensor product $((\rho^1 \otimes \rho^2) \circ$

[4]This was shown by Minnaert and Mozryzmas in [**27**] and goes back to results for $\mathbf{osp}(1|2)$ obtained by Nahm, Rittenberg and Scheunert in [**35**].

[5]Arnaudon gives a classification in [**3**].

$\Delta, V^1 \otimes V^2)$ again is a representation. Note that due to the graded nature of the tensor product the tensor representation acts on a vector $x_1 \otimes x_2$ as

$$\left((\rho^1 \otimes \rho^2) \circ \Delta\right)(X) x_1 \otimes x_2 = \left(\sum \rho^1(X_{(a)}) \otimes \rho^2(X_{(b)})\right) x_1 \otimes x_2$$
$$= (-1)^{\mathrm{p}\left(X_{(b)}\right)\mathrm{p}(x_1)} \sum \rho^1(X_{(a)}) x_1 \otimes \rho^2(X_{(b)}) x_2 \ .$$

Here we used the Sweedler notation for the coproduct.

$$\Delta(X) = \sum X_{(a)} \otimes X_{(b)} \ .$$

The m-representation. Let $\{\mathbf{e}^l_m(\lambda) \mid m = -l, -l+1, \ldots, l\}$ be a basis for the module $V^{l,\lambda}$ diagonalising k. Consider the following action of $\mathcal{U}_q$ on this basis.

$$\begin{aligned} \rho^{l,\lambda}(k)\mathbf{e}^l_m(\lambda) &= q^{\frac{m}{2}}\mathbf{e}^l_m(\lambda) \ , \\ \rho^{l,\lambda}(e)\mathbf{e}^l_m(\lambda) &= \mathcal{D}^+(l,m;\lambda)\mathbf{e}^l_{m+1}(\lambda) \ , \\ \rho^{l,\lambda}(f)\mathbf{e}^l_m(\lambda) &= \mathcal{D}^-(l,m;\lambda)\mathbf{e}^l_{m-1}(\lambda) \ , \end{aligned} \tag{4.5a}$$

$$\rho^{l,\lambda}(e)\mathbf{e}^l_l(\lambda) = \rho^{l,\lambda}(f)\mathbf{e}^l_{-l}(\lambda) = 0 \ . \tag{4.5b}$$

The parity of $\mathbf{e}^l_m(\lambda)$ is

$$\mathrm{p}\left(\mathbf{e}^l_m(\lambda)\right) = \lambda + l - m \mod 2 \ . \tag{4.6}$$

So for successive indices m the vectors $\mathbf{e}^l_m(\lambda)$ alternate in degree. The even and odd subspaces of $V^{l,\lambda}$ are interlaced by the action of e and f and we get the following picture[6].

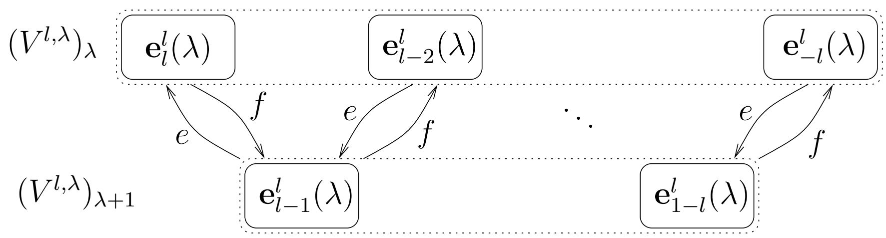

FIGURE 9: The interlacing of even and odd basis vectors in $V^{l,\lambda}$

A traditional approach would be to choose

$$\begin{aligned} \mathcal{D}^+(l,m;\lambda) &= (-1)^{l-m}\left(\{l-m\}\{l+m+1\}\gamma\right)^{1/2} \ , \\ \mathcal{D}^-(l,m;\lambda) &= \left(\{l+m\}\{l-m+1\}\gamma\right)^{1/2} \ . \end{aligned} \tag{4.7}$$

[6]For a super vector space V we denote by $(V)_0$ and $(V)_1$ its even part and its odd part, respectively.

We will however choose a different normalisation such that

$$\begin{aligned} \mathcal{D}^+(l,m;\lambda) &= (-1)^{l-m}\{l-m\}\beta \ , \\ \mathcal{D}^-(l,m;\lambda) &= \{l+m\}\beta \ , \end{aligned} \tag{4.8}$$

with $\beta^2 = Q_{\mathfrak{s}}/Q = (q^{-1/2}+q^{1/2})/(q-q^{-1})$. This said we call this the *m–representation* and denote it by ρ_m omitting the label (l,λ).

In the m–representation the Scasimir operator acts as

$$\rho_m(S_q)\mathbf{e}^l_m(\lambda) = (-1)^{l-m+1}\left[l+\tfrac{1}{2}\right]\mathbf{e}^l_m(\lambda) \ . \tag{4.9}$$

The basis $\{\mathbf{e}^l_m\}$ on which the action of e and f has coefficients $\mathcal{D}^\pm$ as in (4.8) is related to the basis $\{\mathbf{f}^l_m\}$ with respect to which the coefficients $\mathcal{D}^\pm$ of the action of e and f are of the form (4.7) by a rescaling by

$$\mathbf{e}^l_m(\lambda) = \mathcal{M}(l,m)^{\frac{1}{2}}\mathbf{f}^l_m(\lambda) \ , \quad \mathcal{M}(l,m) = \frac{\{l+m\}\{l-m\}}{\{2l\}} \ . \tag{4.10}$$

The transposed m–representation. The space dual to a super vector space V is the super vector space V^* of complex valued functions such that the even functionals vanish on V_1 and the odd functionals on V_0. Again a basis of $(V^l)^*$ is supplied by the functionals $\check{\mathbf{e}}^l_m(\lambda)$ such that

$$\check{\mathbf{e}}^l_m(\lambda)\left(\mathbf{e}^l_n(\lambda)\right) = \delta_{m,n} \ .$$

Without further ado we present the *transposed m–representation* ρ_m on the space $\left(V^l\right)^*$ dual to V^l.

$$\begin{aligned} \rho^t_m(k)\check{\mathbf{e}}^l_m(\lambda) &= q^{\frac{m}{2}}\check{\mathbf{e}}^l_m(\lambda) \ , \\ \rho^t_m(e)\check{\mathbf{e}}^l_m(\lambda) &= \mathcal{D}^+(l,m-1;\lambda)\check{\mathbf{e}}^l_{m-1}(\lambda) \ , \\ \rho^t_m(f)\check{\mathbf{e}}^l_m(\lambda) &= \mathcal{D}^-(l,m+1;\lambda)\check{\mathbf{e}}^l_{m+1}(\lambda) \ , \end{aligned} \tag{4.11a}$$

$$\rho^t_m(e)\check{\mathbf{e}}^l_{-l}(\lambda) = \rho^t_m(f)\check{\mathbf{e}}^l_l(\lambda) = 0 \ . \tag{4.11b}$$

The Scasimir acts as follows on this representation.

$$\rho^t_m(S_q)\check{\mathbf{e}}^l_m(\lambda) = (-1)^{l-m+1}\left[l+\tfrac{1}{2}\right]\check{\mathbf{e}}^l_m(\lambda) \ . \tag{4.12}$$

Invariant bilinear form. The invariant bilinear form $B_{q|\mathfrak{s}}$ on $\mathcal{U}_q$-modules is defined by the requirement that for vectors $\mathbf{v}$ in $V^{l,\lambda}\otimes V^{l,\lambda}$ it transformed according to

$$\begin{aligned} B_{q|\mathfrak{s}}(\Delta(k)\mathbf{v}) &= B_{q|\mathfrak{s}}(\mathbf{v}) \ , \\ B_{q|\mathfrak{s}}(\Delta(a)\mathbf{v}) &= 0 \ , \qquad \text{for } a = e, f \ . \end{aligned}$$

Extend $B_{q|\mathfrak{s}}$ to all finite dimensional $\mathcal{U}_q$-modules such that different modules are orthogonal. This determines $B_{q|\mathfrak{s}}$ up to a l_2-dependent factor and we have with the coefficient $\mathcal{M}$ from the relation (4.10)

$$\begin{aligned} B_{q|\mathfrak{s}}\left(\mathbf{e}^{l_2}_{m_2}(\lambda_2)\otimes\mathbf{e}^{l_1}_{m_1}(\lambda_1)\right) &= \delta_{l_2,l_1}\delta_{m_2+m_1,0}\delta_{\lambda_2,\lambda_1}\times \\ &\times(-1)^{l_2+m_2+\lambda_2+\frac{(l_2+m_2)(l_2+m_2+1)}{2}}q^{\frac{l_2+m_2}{2}}\mathcal{M}(j_2,m_2) \ . \end{aligned}$$

For future use, we compress the coefficient into symbol $\bar{\mathcal{M}}$

$$\bar{\mathcal{M}}(j,m;\lambda) = (-1)^{l+m+\lambda+\frac{(l+m)(l+m+1)}{2}} q^{\frac{l+m}{2}} \mathcal{M}(j,m) \ . \tag{4.13}$$

The invariant bilinear form extends to tensor products of modules as

$$B_{q|\mathfrak{s}}(a\otimes b, c\otimes d) = (-1)^{\mathrm{p}(b)\,\mathrm{p}(c)} B_{q|\mathfrak{s}}(a,c)\, B_{q|\mathfrak{s}}(b,d) \ .$$

Decomposition of tensor products. The tensor product representations and the irreducible representations are intertwined by the super quantum Clebsch–Gordan coefficients.

$$\mathbf{e}^l_m(l_2,l_1;\lambda) = \sum_{m_1,m_2} \begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m & m_2 & m_1 \end{bmatrix}_{q|\mathfrak{s}} \mathbf{e}^{l_2}_{m_2}(\lambda_2)\otimes \mathbf{e}^{l_1}_{m_1}(\lambda_1) \ .$$

The Clebsch–Gordan coefficients of $\mathcal{U}_q$ are constrained by the following inequalities[7]. Again we call them *horizontal* and *vertical* constraints.

$$\begin{aligned} |l_1 - l_2| \le l &\le l_1 + l_2 \ , \\ m &= m_1 + m_2 \ , \\ \lambda &= \lambda_1 + \lambda_2 + l + l_1 + l_2 \mod 2 \ , \\ -l_i \le m_i &\le l_i \ . \end{aligned} \tag{4.14}$$

The Clebsch–Gordan coefficients suffice the recurrence relations

(4.15)

$$\begin{aligned} \mathcal{D}^{\pm}(l,m) \begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m\pm 1 & m_2 & m_1 \end{bmatrix}_{q|\mathfrak{s}} &= q^{\frac{m_1}{2}} \mathcal{D}^{\pm}(l_2, m_2 \mp 1) \begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m & m_2\mp 1 & m_1 \end{bmatrix}_{q|\mathfrak{s}} + \\ &+ (-1)^{l_2+m_2+\lambda_2} q^{-\frac{m_2}{2}} \mathcal{D}^{\pm}(l_1, m_1\mp 1) \begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m & m_2 & m_1\mp 1 \end{bmatrix}_{q|\mathfrak{s}} \end{aligned}$$

These relations determine the Clebsch–Gordan coefficients up to a function of the representation labels l, l_1 and l_2. We choose the normalisation

$$\begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ -l & l_1 - l & -l_1 \end{bmatrix}_{q|\mathfrak{s}} = 1 \ . \tag{4.16}$$

[7] This was shown by Minnaert and Mozrzymas in [**26, 27**].

The Clebsch–Gordan coefficients satisfy the following orthogonality and completeness relations. Let $\mathcal{L} = (l_1+m_1+\lambda_1)(l_2+m_2+\lambda_2)$

(4.17a)

$$\sum_{m_1,m_2} \begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m & m_2 & m_1 \end{bmatrix}_{q|\mathfrak{s}} \begin{bmatrix} l'\lambda' & l_2\lambda_2 & l_1\lambda_1 \\ m' & -m_2 & -m_1 \end{bmatrix}_{q|\mathfrak{s}} (-1)^{\mathcal{L}} \times \\ \times \bar{\mathcal{M}}(l_2,m_2;\lambda_2)\bar{\mathcal{M}}(l_1,m_1;\lambda_1) \\ = \delta_{l,l'}\delta_{m,m'}\delta_{\lambda,\lambda'}\delta(l_1,l_2,l)\bar{\mathcal{M}}(l,m;\lambda)$$

(4.17b)

$$\sum_{l,m} \begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m & m_2 & m_1 \end{bmatrix}_{q|\mathfrak{s}} \begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ -m & m_2' & m_1' \end{bmatrix}_{q|\mathfrak{s}} \times \\ \times \frac{\bar{\mathcal{M}}(l_2,m_2;\lambda_2)\bar{\mathcal{M}}(l_2,m_2';\lambda_2)\bar{\mathcal{M}}(l_1,m_1;\lambda_1)\bar{\mathcal{M}}(l_1,m_1';\lambda_1)}{\bar{\mathcal{M}}(l,m;\lambda)} \\ = \delta_{m_1+m_1',0}\delta_{m_2+m_2',0}(-1)^{\mathcal{L}}\bar{\mathcal{M}}(l_2,m_2;\lambda_2)\bar{\mathcal{M}}(l_1,m_1;\lambda_1)$$

This can be seen as follows. The Clebsch–Gordan coefficients have inverses

$$\mathbf{e}^{l_2}_{m_2}(\lambda_2) \otimes \mathbf{e}^{l_1}_{m_1}(\lambda_1) = \sum_{l,m} \begin{bmatrix} l & l_2 & l_1 \\ m & m_2 & m_1 \end{bmatrix}^t_{q|\mathfrak{s}} \mathbf{e}^l_m(l_1,l_2;\lambda) \ .$$

With these the Clebsch–Gordan coefficients satisfy another set of relations. Let $\mathcal{L} = (l_2+m_2+\lambda_2)(l_1+m_1+\lambda_1)$, then we have

$$\sum_{m_2,m_1} \begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m & m_2 & m_1 \end{bmatrix} \begin{bmatrix} l'\lambda' & l_2\lambda_2 & l_1\lambda_1 \\ -m' & m_2 & m_1 \end{bmatrix}^t \bar{\mathcal{M}}(l',-m';\lambda) \\ = \delta_{l,l'}\delta_{m+m',0}\bar{\mathcal{M}}(l,m;\lambda)$$

$$\sum_{l,m} \begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m & m_2 & m_1 \end{bmatrix}^t \begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m & -m_2' & -m_1' \end{bmatrix} (-1)^{\mathcal{L}}\bar{\mathcal{M}}(l_2,-m_2';\lambda_2)\bar{\mathcal{M}}(l_1,-m_1';\lambda_1) \\ = \delta_{m_2+m_2',0}\delta_{m_1+m_1',0}(-1)^{\mathcal{L}}\bar{\mathcal{M}}(l_2,m_2;\lambda_2)\bar{\mathcal{M}}(l_1,m_1;\lambda_1)$$

$$\sum_{l,m} \begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m & m_2 & m_1 \end{bmatrix}^t \begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m & m_2' & m_1' \end{bmatrix}^t \bar{\mathcal{M}}(l,m;\lambda) \\ = \delta_{m_2+m_2',0}\delta_{m_1+m_1',0}(-1)^{\mathcal{L}}\bar{\mathcal{M}}(l_2,m_2;\lambda_2)\bar{\mathcal{M}}(l_1,m_1;\lambda_1)$$

Comparing coefficients we see that

$$\begin{bmatrix} l & l_2 & l_1 \\ m & m_2 & m_1 \end{bmatrix}^t_{q|\mathfrak{s}} = \begin{bmatrix} l & l_2 & l_1 \\ -m & -m_2 & -m_1 \end{bmatrix}_{q|\mathfrak{s}} \times \\ \times (-1)^{\mathcal{L}} \frac{\bar{\mathcal{M}}(l_2,m_2;\lambda_2)\bar{\mathcal{M}}(l_1,m_1;\lambda_1)}{\bar{\mathcal{M}}(l,m;\lambda)} \ .$$

From this it is possible to derive the orthogonality and completeness relations.

Invariants. Invariants in the n-fold tensor product can be constructed using Clebsch–Gordan coefficients in the $(n-1)$-fold tensor product together with an invariant in the two-fold tensor product. This invariant in the two-fold tensor product is the invariant bilinear form $B_{q|\mathfrak{s}}$. It is connected to the Clebsch–Gordan coefficients intertwining a tensor product with the trivial representation. On the module $V^{l_2,\lambda_2} \otimes V^{l_1,\lambda_1}$ we have

$$B_{q|\mathfrak{s}} \begin{pmatrix} l_2\lambda_2 & l_1\lambda_1 \\ m_2 & m_1 \end{pmatrix} := \left[\begin{matrix} 0\lambda \\ 0 \end{matrix} \middle| \begin{matrix} l_2\lambda_2 & l_1\lambda_1 \\ m_2 & m_1 \end{matrix} \right]_{q|\mathfrak{s}} .$$

Thus we have because of (4.15), (4.16) and (4.14)

$$B_{q|\mathfrak{s}} \begin{pmatrix} l_2\lambda_2 & l_1\lambda_1 \\ m_2 & m_1 \end{pmatrix} = \delta_{l_1,l_2}\delta_{m_1+m_2,0}\delta_{\lambda_2,\lambda_1} \times$$
$$\times (-1)^{(l_1+m_1)\lambda_2 + \frac{(l_1+m_1)(l_1+m_1+1)}{2}} q^{-\frac{l_1+m_1}{2}} \mathcal{M}(l_1,m_1)^{-1} .$$

The invariant of the three-fold tensor product is the super quantum $3j$ symbol. These $3j$ symbols are defined by lowering one index in the Clebsch–Gordan coefficients by means of the invariant 2-form

$$\begin{pmatrix} l_3\lambda_3 & l_2\lambda_2 & l_1\lambda_1 \\ m_3 & m_2 & m_1 \end{pmatrix}_{q|\mathfrak{s}} := \sum_{m=-j}^{j} B_{q|\mathfrak{s}} \begin{pmatrix} l_3\lambda_3 & l\lambda \\ m_3 & m \end{pmatrix} \left[\begin{matrix} l \\ m \end{matrix} \middle| \begin{matrix} l_2 & l_1 \\ m_2 & m_1 \end{matrix} \right]_{q|\mathfrak{s}} .$$

The (x,θ)–Representation. The representation dual to the m-representation acts on a non-supercommutative super vectorspace. We introduce the idempotent variable θ commuting with the c-number variable x

$$\theta^2 = 1 ,$$
$$x\theta = \theta x .$$

The variables x and θ algebraically generate $Pol(x,\theta) = \mathbf{C}[x,\theta]$ the complex polynomials in x and θ. Let $Pol^n(x,\theta)$ denote the super vector space of polynomials of degree n or less. The super vector space $Pol_{2l}(x,\theta)$ decomposes into the two sub super vector spaces $\tilde{V}^{l,\lambda}$, with $\lambda \in \mathbf{Z}_2$ the parity of the highest weight vector of the corresponding m-representation

$$Pol_{2l}(x,\theta) = \tilde{V}^{l,0} \oplus \tilde{V}^{l,1}$$

where

$$\tilde{V}^{l,0} = \mathrm{span}\left\{1, x^{-1}\theta, x^{-2}, x^{-3}\theta, \ldots, x^{-2l+1}\theta, x^{-2l}\right\} ,$$
$$\tilde{V}^{l,1} = \mathrm{span}\left\{\theta, x^{-1}, x^{-2}\theta, x^{-3}, \ldots, x^{-2l+1}, x^{-2l}\theta\right\} .$$

Note that the spaces $\tilde{V}^{l,\lambda}$ do *not* correspond to the even and odd parts of $Pol_{2l}(x,\theta)$. The spaces $\tilde{V}^{l,\lambda}$ are super vector spaces in their own right.

Now the $\mathcal{U}_q\mathbf{osp}(1|2)$ module $V^{l,\lambda}$ is equivalent as super vector space to $\tilde{V}^{l,\lambda}$ via the transformation

$$\mathbf{e}^{l,\lambda}(x,\theta) = \sum_{m=-l}^{l} x^{-l-m}\theta^{\lambda+l+m}\mathbf{e}^{l}_{m}(\lambda) \ . \tag{4.18}$$

The super vector space $\tilde{V}^{l,\lambda}$ is a right $\mathcal{U}_q\mathbf{osp}(1|2)$-module and the generators act as

$$\rho_x(k) = q^{-\frac{l}{2}}\,\mathsf{T}^{-\frac{1}{2}} \ , \tag{4.19a}$$

$$\rho_x(e) = (-1)^{\mathrm{d}_x}\{\mathrm{d}_x + 2l\}x\theta\beta \ . \tag{4.19b}$$

$$\rho_x(f) = (-1)^{\mathrm{d}_x+1}\{\mathrm{d}_x\}x^{-1}\theta\beta \ , \tag{4.19c}$$

We call $(\rho_x, \tilde{V}^{l,\lambda})$ the *(x,θ)-representation.* The Scasimir acts as

$$\rho_x(S_q) = (-1)^{\mathrm{d}_x+1}\left[l+\tfrac{1}{2}\right] \ .$$

The transposed (x,θ)–representations. The space $Pol_{2l}(x^{-1},\theta)$ is dual to the space $Pol_{2l}(x,\theta)$ via the pairing

$$(g,f)_{x,\theta} = \oint_0 \frac{\mathrm{d}x}{x} g(x,\theta)f(x,\theta) \ .$$

This induces a representation ρ^t_x on the space $(\tilde{V}^{l,\lambda})^* \subset Pol_{2l}(x,\theta)$

$$(\tilde{V}^{l,0})^* = \mathrm{span}\left\{1, x\theta, x^2, x^3\theta, \ldots, x^{2l-1}\theta, x^{2l}\right\}$$
$$(\tilde{V}^{l,1})^* = \mathrm{span}\left\{\theta, x, x^2\theta, x^3, \ldots, x^{2l-1}, x^{2l}\theta\right\}$$

that acts as

$$\rho^t_x(k) = q^{-\frac{l}{2}}\,\mathsf{T}^{\frac{1}{2}} \ , \tag{4.20a}$$

$$\rho^t_x(e) = -\beta\theta x\{\mathrm{d}_x - 2l\} \ , \tag{4.20b}$$

$$\rho^t_x(f) = \beta\theta x^{-1}\{\mathrm{d}_x\} \ . \tag{4.20c}$$

We call $(\rho^t_x, (\tilde{V}^{l,\lambda})^*)$ the *transposed (x,θ)-representation.* The Scasimir acts as

$$\rho^t_x(S_q) = (-1)^{\mathrm{d}_x+1}\left[l+\tfrac{1}{2}\right] \ .$$

A remark on the sign of the Scasimir. The Scasimir's coefficient is $\left[l+\frac{1}{2}\right]$ in all four representations. A unified representation for the sign operators $(-1)^{l+m}$ and $(-1)^{\mathrm{d}_x}$ is given by the operator F_V. The operator F_V maps every element of a super vector space V to its parity.

$$\mathsf{F}_V \colon V \to \mathbf{Z}/2\mathbf{Z}$$
$$v \mapsto \mathrm{p}\,(v) \ .$$

In the following we will suppress the index V.
Using this operator we can unify the representations of S_q. In all four representations considered we have

$$S_q \simeq (-1)^{\mathsf{F}+\lambda+1} \left[l+\tfrac{1}{2}\right] \ .$$

Representations in block form. A common representation of the super vector space $V = V_0 \oplus V_1$ is the m-representation deinterlaced. There we have a basis $\{v_i \mid i = 1, 2, \ldots, \dim V_0 + \dim V_1\}$ such that the first $\dim V_0$ basis vectors are even and the remaining are odd. A linear transformations f in $\mathrm{Hom}(V, W)$ in this basis is

$$f \simeq \begin{pmatrix} A & B \\ C & D \end{pmatrix} \ .$$

Here $A \oplus D \in \mathrm{Hom}(V, W)_0$ are the even and $B \oplus C \in \mathrm{Hom}(V, W)_1$ the odd transformations.
The super vector space $Pol_{2l}(x, \theta)$ is isomorphic to $\mathbf{C}^2 \otimes Pol_{2l}(x)$. For a given polynomial

$$f(x, \theta) = f_0(x) + \theta f_1(x)$$

we identify

$$f(x) = \begin{pmatrix} f_0(x) \\ f_1(x) \end{pmatrix} \ .$$

Under this identification the representation of the generators $x = e, f$ of $\mathcal{U}_q\mathbf{osp}(1|2)$ becomes off-diagonal and for the generator k diagonal

$$x = \begin{pmatrix} 0 & X \\ X' & 0 \end{pmatrix} \quad \text{and} \quad k = \begin{pmatrix} K & 0 \\ 0 & K' \end{pmatrix} \ .$$

This emphasises the odd parity of e and f and the even parity of k.

Super limits. Consider functions in the variables x and θ. We regard the pair (x, θ) as a point x in the complex projective plane $\mathbb{P}\mathbf{C}$ with an adjoined superline. The points 0 and ∞ then have a special meaning for polynomials in (x, θ). They single out the coefficients of the lowest, respectively highest power of x. For a given graded polynomial

$$f^l(x, \theta) = \sum_{m=-l}^{l} f_m x^{l+m} \theta^{\lambda+l+m}$$

we get the limits

$$\lim_{x \to 0} f^l(x, \theta) = f_{-l} \theta^{\lambda} \ ,$$
$$\lim_{x \to \infty} x^{-2l} f^l(x, \theta) = f_l \theta^{\lambda} \ .$$

The representations of k basically correspond to numerical factors

$$\lim_{x\to 0} \rho_x^t(k) f^l(x,\theta) = q^{-\frac{l}{2}} f_{-l}\theta^\lambda \ ,$$
$$\lim_{x\to\infty} x^{-2l}\rho_x^t(k) f^l(x,\theta) = q^{\frac{l}{2}} f_l\theta^\lambda \ .$$

2. Invariant Tensors

An invariant tensor of $\mathcal{U}_q\mathbf{osp}(1|2)$ is an element t of $\bigotimes_i V^{l_i,\lambda_i}$ that is invariant under the action of the super quantum group.

$$et = ft = 0 \quad \text{and} \quad kt = t \ .$$

To avoid clutter we collect variables of the same type into uppercase variables, $L = (l_4, l_3, l_2, l_1)$, $\Lambda = (\lambda_4, \lambda_3, \lambda_2, \lambda_1)$, $X = (x_4, x_3, x_2, x_1)$ and $\Theta = (\theta_4, \theta_3, \theta_2, \theta_1)$. Additionally we introduce $\mathfrak{x}^n$ as a shorter expression for the pair (x, θ). The pair of X and Θ is collected into $\mathfrak{X}$. A function f in the four-fold tensor product would thus be written

$$f(L,\Lambda|X,\Theta) = f(L,\Lambda|\mathfrak{X}) = f\begin{pmatrix} l_4\lambda_4 & l_3\lambda_3 & l_2\lambda_2 & l_1\lambda_1 \\ \mathfrak{x}_4 & \mathfrak{x}_3 & \mathfrak{x}_2 & \mathfrak{x}_1 \end{pmatrix} \ .$$

3. Super quantum $6j$-symbols

The super quantum $6j$-symbols communicate the basis change between different reduced bases $\mathbf{e}^l_m(l_3, l_{12}; \lambda)$ and $\mathbf{e}^l_m(l_1, l_{23}, \lambda)$ of triple tensor products $V^{l_1,\lambda_1} \otimes V^{l_2,\lambda_2} \otimes V^{l_3,\lambda_3}$ of $\mathcal{U}_q$ modules.

$$\mathbf{e}^l_m(l_{32}, l_1; \lambda) = \sum_{l_{21}} \begin{Bmatrix} l_3\lambda_3 & l_2\lambda_2 & l_{32\lambda_{32}} \\ l_4\lambda_4 & l_1\lambda_1 & l_{21\lambda_{21}} \end{Bmatrix}_q \mathbf{e}^l_m(l_3, l_{21}, \lambda) \ .$$

We define the s- and the u-channel decomposition of the four-point invariant as the following.

$$\text{(4.21a)}\quad \Psi^{(s)}_{l_s}(L,\Lambda|\mathfrak{X}) = \sum_{m_s} \begin{pmatrix} l_4\lambda_4 & l_3\lambda_3 & l_s\lambda_s \\ \mathfrak{x}_4 & \mathfrak{x}_3 & m_s \end{pmatrix}_{q|\mathfrak{s}} \left[\begin{array}{c|cc} l_s\lambda_s & l_2\lambda_2 & l_1\lambda_1 \\ m_s & \mathfrak{x}_2 & \mathfrak{x}_1 \end{array}\right]_{q|\mathfrak{s}}$$

$$\text{(4.21b)}\quad \Psi^{(u)}_{l_u}(L,\Lambda|\mathfrak{X}) = \sum_{m_u} \begin{pmatrix} l_4\lambda_4 & l_s\lambda_s & l_1\lambda_1 \\ \mathfrak{x}_4 & m_s & \mathfrak{x}_1 \end{pmatrix}_{q|\mathfrak{s}} \left[\begin{array}{c|cc} l_s\lambda_s & l_3\lambda_3 & l_2\lambda_2 \\ m_s & \mathfrak{x}_3 & \mathfrak{x}_2 \end{array}\right]_{q|\mathfrak{s}}$$

The parity λ_s of the intermediate representation (l_s, λ_s) is

$$\lambda_s = l_1 + l_2 + l + \lambda_1 + \lambda_2 \mod 2 \ .$$

The super quantum $6j$-symbols relate the s- and the u-channel decomposition of the four-point invariant.

$$\Psi^{(u)}_{l_u}(L,\Lambda|\mathfrak{X}) = \sum_{l_s} \begin{Bmatrix} l_2\lambda_2 & l_3\lambda_3 & l_u\lambda_u \\ l_1\lambda_1 & l_4\lambda_4 & l_s\lambda_s \end{Bmatrix}_{q|\mathfrak{s}} \Psi^{(s)}_{l_s}(L,\Lambda|\mathfrak{X}) \ .$$

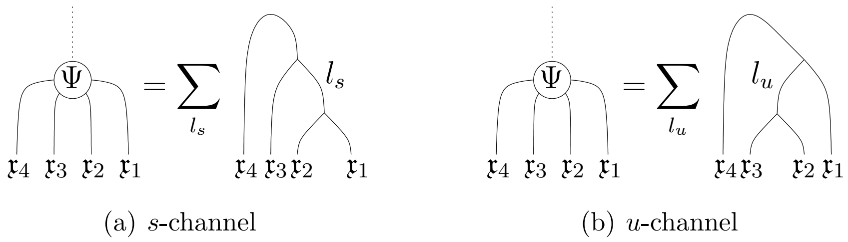

FIGURE 10: Decompositions of the four-point invariants

Recurrence relation. The derivation of the recurrence relations for $\mathcal{U}_q\mathbf{osp}(1|2)$ is word for word as for $\mathcal{U}_q\mathbf{sl}(2)$. The difference is in the initial values and in the spin of the fundamental representation.
Let a, b, c and d each denote a pair (j, λ) of representation label j and parity λ. We consider the four-fold tensor product $V^d \otimes V^c \otimes V^b \otimes V^a$. Let e denote the spin of the reduced basis. The pentagon relation for $\mathcal{U}_q$ encodes the way the different bases of the four-fold tensor product correspond to each other. We have

$$\begin{Bmatrix} d & c & h \\ e & f & j \end{Bmatrix}_q \begin{Bmatrix} j & b & f \\ e & a & i \end{Bmatrix}_q = \sum_g \begin{Bmatrix} c & d & f \\ h & a & g \end{Bmatrix}_q \begin{Bmatrix} d & g & h \\ e & a & i \end{Bmatrix}_q \begin{Bmatrix} d & c & g \\ i & b & j \end{Bmatrix}_q .$$

This is illustrated in figure 7 on page 34. The recurrence relation is of the form

$$\begin{Bmatrix} j & b & f \\ e & a & i \end{Bmatrix}_{q|\mathfrak{s}} = \sum_{g,h} S_{\mathfrak{s}}(a, b, f, i, j; h, g) \begin{Bmatrix} j-1 & b & f \\ h & a & g \end{Bmatrix}_{q|\mathfrak{s}} . \tag{4.22}$$

The triangle condition constrains h and g by $|j-2| \leq g, h \leq j$ and $|g-1| \leq h \leq g+1$. This means we have an expression for the $6j$-symbol $\left\{ \begin{smallmatrix} j & b & f \\ e & a & i \end{smallmatrix} \right\}$ in terms of one $6j$-symbol with $j-1$ and fundamental $6j$-symbols with one spin equal to $1/2$

k **invariance.** We introduce the *reduced blocks* of $\mathcal{U}_q$ via the limit $\check{\Psi}^{(i)}$ of $\Psi^{(i)}$.

$$\Psi^{(i)}_{l_i}(L, \Lambda|\mathfrak{X}) \xrightarrow[x_4\to\infty]{x_2\to 0} \theta_2^{\lambda_2}\theta_4^{\lambda_4}\check{\Psi}^{(i)}_{j_i}(J, \Lambda|\mathfrak{x}_3, \mathfrak{x}_1) .$$

Note that in order to extract the factor $\theta_2^{\lambda_2}\theta_4^{\lambda_4}$ from the polynomial $\check{\Psi}^{(i)}$, we had to introduce some signs.

$$\check{\Psi}^{(i)}_{l_i}(L, \Lambda|\mathfrak{x}_3, \mathfrak{x}_1) = \sum_{m_i} (-1)^{\mathcal{P}^{(i)}_{m_i}} x_3^{l_3+m_3}\theta_3^{\lambda_3+l_3+m_3} x_1^{l_1+m_1}\theta_1^{\lambda_1+l_1+m_1} f^{(i)}_{m_i}(L, \Lambda|l_i) ,$$

where m_3 and m_1 are linear combinations of l_1, l_2, l_3, l_4 and the intermediate values m_s or m_u in the respective channels, by virtue of the horizontal constraints

$$\mathcal{P}^{(s)}_{m_s} = (\lambda_3 + l_3 - l_4 - m_s)\,\lambda_2 \ ,$$
$$\mathcal{P}^{(u)}_{m_u} = (\lambda_1 + l_1 - l_4 - m_u)\,(\lambda_3 + l_3 + l_2 + m_u) + (\lambda_3 + \lambda_1 + \kappa)\,\lambda_2 \ .$$

The operator T is the multiplication operator on functions of x, such that

$$\mathsf{T}\, f(x) = f(qx) \ .$$

Proposition 4.1. *The reduced blocks $\check{\Psi}$ have the structure of a function of the quotient $x = x_1/x_3$ times monomials in x_1 and x_3*

$$\check{\Psi}^{(i)}_{l_i}(L.\Lambda|\mathfrak{x}_3, \mathfrak{x}_1) = x_1^{\alpha_1^{(i)}}\, x_3^{\alpha_3^{(i)}}\, \tilde{\mathcal{F}}^{(i)}_{l_i}(L, \Lambda|x, \theta_3, \theta_1) \ , \tag{4.23}$$

such that $\alpha_1^{(i)} + \alpha_3^{(i)} = l_1 + l_2 + l_3 - l_4 =: \kappa$ for $i = s, u$.

Proof. Because of the invariance of s- and u-channel blocks under the action of k, we have

$$k\Psi^{(i)}_{l_i}(J, \Lambda|\mathfrak{X}) = \Psi^{(i)}_{j_i}(J, \Lambda|\mathfrak{X}) \ , \quad i = s, u$$

and in the limit

$$\begin{aligned} k\Psi^{(i)}_{j_i}(L, \Lambda|\mathfrak{X}) &= (k \otimes k \otimes k \otimes k)\Psi^{(i)}_{j_i}(L, \Lambda|\mathfrak{X}) \\ &\xrightarrow[x_4\to\infty]{x_2\to 0} q^{\frac{l_4 - l_2}{2}} k_3 k_1 \theta_4^{\lambda_4}\theta_2^{\lambda_2}\check{\Psi}^{(i)}_{j_i}(L, \Lambda|\mathfrak{x}_3, \mathfrak{x}_1) \\ &= q^{-\kappa}\,\mathsf{T}_3\,\mathsf{T}_1\,\theta_4^{\lambda_4}\theta_2^{\lambda_2}\check{\Psi}^{(i)}_{j_i}(L, \Lambda|\mathfrak{x}_3, \mathfrak{x}_1) \end{aligned}$$

Thus we find that

$$\mathsf{T}_3^{\frac{1}{2}}\,\mathsf{T}_1^{\frac{1}{2}}\,\check{\Psi}^{(i)}_{l_i}(L, \Lambda|\mathfrak{x}_3, \mathfrak{x}_1) = q^{\kappa/2}\check{\Psi}^{(i)}_{l_i}(L, \Lambda|\mathfrak{x}_3, \mathfrak{x}_1) \ , \tag{4.24}$$

with $\kappa = l_1 + l_2 + l_3 - l_4$, as proposed. □

Corollary 4.1. *From that we gather an inverse proportionality between T_1 and T_3*

$$\mathsf{T}_1\,\check{\Psi}^{(i)}_{l_i}(L.\Lambda|\mathfrak{x}_3, \mathfrak{x}_1) = q^{\kappa}\,\mathsf{T}_3^{-1}\,\check{\Psi}^{(i)}_{l_i}(L, \Lambda|\mathfrak{x}_3, \mathfrak{x}_1) \ . \tag{4.25}$$

For the difference operator

$$\{\mathrm{d}_x + a\} = \frac{q^{-\frac{a}{2}}\,\mathsf{T}_x^{-\frac{1}{2}} - (-1)^{\mathrm{d}_x + a} q^{\frac{a}{2}}\,\mathsf{T}_x^{\frac{1}{2}}}{Q_{\mathfrak{s}}}$$

this entails

$$\{\mathrm{d}_1 + a\}\check{\Psi}^{(i)}_{l_i} = -\{\mathrm{d}_3 - a - \kappa\}\check{\Psi}^{(i)}_{l_i} \ .$$

We introduce the $\mathfrak{x}$-*reduced blocks* with $\mathfrak{x} = (x, \mathrm{i}\theta) = (\frac{x_1}{x_3}, \theta_3\theta_1)$ as

$$\mathcal{F}^{(i)}_{l_i}(L, \Lambda|\mathfrak{x}) = x_3^{-\kappa}\theta_3^{\kappa+\lambda_1+\lambda_3}\check{\Psi}^{(i)}_{l_i}(L, \Lambda|\mathfrak{x}_3, \mathfrak{x}_1) \ .$$

Merging θ_3 and θ_1 into $\mathrm{i}\theta$ we get additional signs

$$\mathcal{S}^{(s)}_{m_s} = \mathcal{P}^{(s)}_{m_s} + (l_1 + l_2 + \lambda_1 + m_s - 1)(l_1 + l_2 + \lambda_1 + m_s)/2 \ ,$$
$$\mathcal{S}^{(u)}_{m_u} = \mathcal{P}^{(u)}_{m_u} + (l_1 - l_4 + \lambda_1 - m_u - 1)(l_1 - l_4 + \lambda_1 - m_u)/2 \ ,$$

Note that $\mathcal{S}^{(i)}_{m+2} = \mathcal{S}^{(i)}_m + 1 \mod 2$. Together with what we know from above we get

$$\mathcal{F}^{(s)}_{l_s}(L,\Lambda|\mathfrak{x}) = \sum_{m_s=-l_s}^{l_s} (-1)^{\mathcal{S}^{(s)}_{m_s}} f^{(s)}_{m_s}(L,\Lambda|l_s) x^{l_1+l_2+m_s} (\mathrm{i}\theta)^{l_1+l_2+m_s+\lambda_1} \tag{4.26a}$$

$$\mathcal{F}^{(u)}_{l_u}(L,\Lambda|\mathfrak{x}) = \sum_{m_u=-l_u}^{l_u} (-1)^{\mathcal{S}^{(u)}_{m_u}} f^{(u)}_{m_u}(L,\Lambda|l_u) x^{l_3-l_4+m_u} (\mathrm{i}\theta)^{l_3-l_4+m_u+\lambda_1} \tag{4.26b}$$

The Scasimir eigenvalue equation. The blocks $\Psi^{(i)}$, $i = s, u$ are super eigenfunctions[8] of the Scasimir with the eigenvalue $(-1)^{\mathsf{F}+1}\left[l_i + \frac{1}{2}\right]$. We introduce ℓ and k such that

$$l_s + \ell = l_1 + l_2 \quad \text{and} \quad l_u + k = l_2 + l_3 \ .$$

In the following we find a difference equation for the $\mathfrak{x}$-reduced blocks, that is remarkably similar to the one we found for $\mathcal{U}_q\mathbf{sl}(2)$.

PROPOSITION 4.2. *The $\mathfrak{x}$-reduced blocks $\mathcal{F}^{(i)}_{l_i}$, $i = s, u$ fulfil the difference equations*

$$\begin{aligned}\Big((-1)^{\kappa+\lambda_3+\lambda_1} q^{\frac{-l_4+l_2-1}{2}} x_3^{-1} x_1 \theta_3\theta_1 \{\mathrm{d}_x - \kappa\}\{\mathrm{d}_x - 2l_1\} \\ -\{\mathrm{d}_x - \ell\}\{\mathrm{d}_x - 2l_s - \ell - 1\}\Big)\mathcal{F}^{(s)}_{l_s} = 0 \ ,\end{aligned} \tag{4.27a}$$

$$\begin{aligned}\Big((-1)^{\kappa+\lambda_3+\lambda_1} q^{\frac{l_4-l_2+1}{2}} x_3 x_1^{-1} \theta_3\theta_1 \{\mathrm{d}_x + 2l_3 - \kappa\}\{\mathrm{d}_x\} \\ -\{\mathrm{d}_x + k - \kappa\}\{\mathrm{d}_x + 2l_u + k + 1 - \kappa\}\Big)\mathcal{F}^{(u)}_{l_u} = 0 \ .\end{aligned} \tag{4.27b}$$

PROOF. Consider the eigenvalue equations

$$\begin{aligned}\Delta_{21}\left(S_q - (-1)^{\mathsf{F}_s+1}\left[l_s + \tfrac{1}{2}\right]\right)\Psi^{(s)} = 0 \ , \\ \Delta_{32}\left(S_q - (-1)^{\mathsf{F}_u+1}\left[l_u + \tfrac{1}{2}\right]\right)\Psi^{(u)} = 0 \ .\end{aligned}$$

The indices 21 and 32 express the way the coproduct is embedded into the four-fold tensor product.

$$\Delta_{21} = \mathrm{id} \otimes \mathrm{id} \otimes \Delta \quad \text{and} \quad \Delta_{32} = \mathrm{id} \otimes \Delta \otimes \mathrm{id} \ .$$

In the transposed x-representation the Scasimir operator

$$S_q = -Q_{\mathfrak{s}} ef + \frac{q^{-\frac{1}{2}}k^2 - q^{\frac{1}{2}}k^{-2}}{Q} \ .$$

takes the form of a difference operator. We determine this operator in two steps. First we consider the summand ef. We use the leg notation

[8] that is the even and odd parts are eigenfunctions, not necessarily to the same eigenvalue.

again. Let a_n act as an element a on the n^{th} place of the tensor product and as identity at all other places. We can express $\Delta_{21}(ef)$ now as

$$\begin{aligned}\Delta_{21}(ef)\Psi &= \left(e_2k_1 + k_2^{-1}e_1\right)\left(f_2k_1 + k_2^{-1}f_1\right)\Psi \\ &= \left(e_2f_2k_1^2 + e_2k_2^{-1}k_1f_1 - k_2^{-1}f_2e_1k_1 + k_2^{-2}e_1f_1\right)\Psi \\ &\stackrel{a}{=} \left(e_2f_2k_1^2 + e_2k_2^{-1}k_1f_1 + k_4f_4k_3^2e_1k_1 + k_3f_3e_1k_1\right)\Psi\ .\end{aligned}$$

Equality a uses the fact that the block Ψ is invariant under the algebra-like element f, which means that $\Delta^{(4)}(f)\Psi = 0$ where the action of the generator f on the four-fold tensor product is

$$\Delta^{(4)}(f) = f_4k_3k_2k_1 + k_4^{-1}f_3k_2k_1 + k_4^{-1}k_3^{-1}f_2k_1 + k_4^{-1}k_3^{-1}k_2^{-1}f_1\ .$$

The term $t = k_4^{-1}k_3^{-1}f_2k_1$ is replaced by $-\Delta^{(4)}(f) + t$. We note that the term proportional to e_2 vanishes in the limit $x \to 0$. The same goes for the term proportional to f_4 in the limit $x_4 \to \infty$.
We notice that the term e_2f_2 is basically the Scasimir operator.

$$e_2f_2 = \left(-(S_q)_2 + \frac{q^{-\frac{1}{2}}k_2^2 - q^{\frac{1}{2}}k_2^{-2}}{q - q^{-1}}\right)Q_{\mathfrak{s}}^{-1}\ .$$

This term vanishes in the limit $x_2 \to 0$ because the second summand tends to $-\left[l_2 + \frac{1}{2}\right]$ and the Casimir operator is $(-1)^{l_2+m_2+1}\left[l_2 + \frac{1}{2}\right]$ on the irreducible representation with spin l_2. Now we are ready to state that on the reduced block $\check{\Psi}$ we have

$$\Delta_{21}(ef)\check{\Psi} = q^{\frac{l_2-l_4-1}{2}}\frac{x_1}{x_3}\{\mathrm{d}_3 - \kappa\}\{\mathrm{d}_1 - 2l_1\}\check{\Psi}\ .$$

With the coproduct $\Delta k = k \otimes k$ of the group-like generator k it is a straightforward calculation to check that

$$\Delta_{21}\left(\frac{q^{-\frac{1}{2}}k^2 + q^{\frac{1}{2}}k^{-2}}{q - q^{-1}}\right)\check{\Psi} = \left[\mathrm{d}_1 - l_1 - l_2 - \tfrac{1}{2}\right]\check{\Psi}\ .$$

A similar calculation for Δ_{32} yields the representation of the Scasimir in the limit $(x_4, x_2) \to (\infty, 0)$ as

$$q^{\frac{-l_4+l_2-1}{2}}\beta^2 Q_{\mathfrak{s}}x_3^{-1}x_1\theta_3\theta_1(-1)^{\mathsf{F}_3}\{\mathrm{d}_1 - 2l_1\}\{\mathrm{d}_3\} + \left[\mathrm{d}_1 - l_2 - l_1 - \tfrac{1}{2}\right]\ ,$$

in the s-channel and

$$q^{\frac{l_4-l_2+1}{2}}\beta^2 Q_{\mathfrak{s}}x_3x_1^{-1}\theta_3\theta_1(-1)^{\mathsf{F}_?}\{\mathrm{d}_3 - 2l_3\}\{\mathrm{d}_1\} + \left[\mathrm{d}_3 - l_3 - l_2 - \tfrac{1}{2}\right]\ .$$

in the u-channel.

We introduce the indices ℓ and k counting the difference of the intermediate representation's spin from the maximal spin. We set

$$\begin{aligned} l_s + \ell &= l_1 + l_2\ , \qquad && \ell = 0, 1, 2, \ldots, 2\mu \text{ with } \mu = \min\{l_1, l_2\} \text{ and} \\ l_u + k &= l_2 + l_3\ , && k = 0, 1, 2, \ldots, 2\nu \text{ with } \nu = \min\{l_2, l_3\}\ . \end{aligned}$$

Straightforward calculation leads to

$$-\{\mathrm{d}_1-\ell\}\{\mathrm{d}_1-2l_s-\ell-1\}\beta^2 Q_{\mathfrak{s}} = \left[\mathrm{d}_1-l_1-l_2-\tfrac{1}{2}\right]+(-1)^{\mathsf{F}_s}\left[l_s+\tfrac{1}{2}\right] ,$$
$$-\{\mathrm{d}_3-k\}\{\mathrm{d}_3-2l_u-k-1\}\beta^2 Q_{\mathfrak{s}} = \left[\mathrm{d}_3-l_2-l_3-\tfrac{1}{2}\right]+(-1)^{\mathsf{F}_u}\left[l_u+\tfrac{1}{2}\right] .$$

Thus we let $x=x_1/x_3$ and come up with the proposed equations

$$\Big((-1)^{\kappa+\lambda_3+\lambda_1} q^{\frac{-l_4+l_2-1}{2}} x_3^{-1}x_1\theta_3\theta_1\{\mathrm{d}_x-\kappa\}\{\mathrm{d}_x-2l_1\}$$
$$-\{\mathrm{d}_x-\ell\}\{\mathrm{d}_x-2l_s-\ell-1\}\Big)\mathcal{F}^{(s)}_{l_s}=0 ,$$
$$\Big((-1)^{\kappa+\lambda_3+\lambda_1} q^{\frac{l_4-l_2+1}{2}} x_3x_1^{-1}\theta_3\theta_1\{\mathrm{d}_x+2l_3-\kappa\}\{\mathrm{d}_x\}$$
$$-\{\mathrm{d}_x+k-\kappa\}\{\mathrm{d}_x+2l_u+k+1-\kappa\}\Big)\mathcal{F}^{(u)}_{l_u}=0 .$$

□

Up to now there has been no structural difference to what we found for $\mathcal{U}_q\mathbf{sl}(2)$. We will now consider the last structural equivalence before we find the point where the two derivations depart from one another.

Consider the *super basic hypergeometric* difference equation

(4.28) $$\left(x\theta\{\mathrm{d}_x+a\}\{\mathrm{d}_x+b\}-\{\mathrm{d}_x\}\{d_x+c-1\}\right)P(x)=0 .$$

This equation has two solutions parametrised by a Boolean number λ. We call the parametrised solution the *super basic hypergeometric series* and denote it by ${}_2\Pi_1(a,b;c|x,\theta;\lambda;q)$. A series representation is given by

(4.29) $${}_2\Pi_1\left(\begin{matrix} a & b \\ & c\end{matrix}\,\middle|\,\begin{matrix} x,\theta \\ \lambda,q\end{matrix}\right)=\sum_{n\geq 0}\frac{\{a|n\}\{b|n\}}{\{c|n\}\{n\}!}x^n\theta^{\lambda+n} .$$

We observe that the difference equations (4.27) are nearly of super basic hypergeometric type. With an ansatz

$$\mathcal{F}^{(s)}_{l_s}(L,\Lambda|\mathfrak{x})=x^{\ell}{}_2\Pi_1\left(\begin{matrix}\ell-\kappa & \ell-2l_1 \\ & -2l_s-\ell\end{matrix}\,\middle|\,\begin{matrix}(-1)^{\mu}q^{\nu}\frac{x_1}{x_3},\theta_3\theta_1 \\ \lambda_1,q\end{matrix}\right) ,$$
$$\mathcal{F}^{(u)}_{l_u}(L,\Lambda|\mathfrak{x})={}_2\Pi_1\left(\begin{matrix}k-\kappa & 2l_u+k+1-\kappa \\ & 2l_3-\kappa+1\end{matrix}\,\middle|\,\begin{matrix}(-1)^{\mu}q^{\nu}\frac{x_1}{x_3},\theta_3\theta_1 \\ \lambda_1,q\end{matrix}\right)$$

with $\mu=\kappa+\lambda_3+\lambda_1$ and $2\nu=-l_4+l_2-1$ it can be made such.

We now come to the point where the derivation of the 6j-symbols differs from that in the quantum group $\mathcal{U}_q\mathbf{sl}(2)$. The fundamental representation $j=\frac{1}{2}$ for $\mathcal{U}_q\mathbf{sl}(2)$ is two-dimensional. Therefore we had to examine two intermediate representation and to determine two factors $a_i^{\pm}$ and $b_i^{\pm}$ in each channel.
For $\mathcal{U}_q\mathbf{osp}(1|2)$ the situation is similar but slightly more complex. The fundamental representation $l=1$ is three-dimensional. So we have to deal with three intermediate representations and with three coefficients

a_i^ν, b_i^ν and c_i^ν in both the channels. We call the $\mathfrak{x}$-reduced blocks $\mathcal{F}$ for $l_2 = 1$ the *fundamental $\mathfrak{x}$-reduced blocks.*

PROPOSITION 4.3. *The fundamental $\mathfrak{x}$-reduced blocks $\mathcal{F}$ are of the form*

$$\mathcal{F}^{(i)}_{l_i^\nu} = \left(a_i^\nu + b_i^\nu x \mathrm{i}\theta + c_i^\nu x^2(\mathrm{i}\theta)^2\right) R(L,\Lambda|\mathfrak{x}) \ , \tag{4.30}$$

for $\nu = -, \circ, +$.

PROOF. The invariance of the four-point invariant is expressed as

$$\Delta^{(4)}(f)\Psi^{(i)}_{l_i}(L,\Lambda|\mathfrak{X}) = 0 \ .$$

When we consider the limit $x_4 \to \infty$ an let $\Psi^{(i)}_{j_i} = \psi$ we have

$$\left(f_3 k_2 k_1 + k_3^{-1} f_2 k_1 + k_3^{-1} k_2^{-1} f_1\right)\psi = 0 \ ,$$

or in terms of difference operators[9]

$$\left(\frac{q^{\frac{-l_1-l_2+\kappa}{2}}}{x_3\theta_3}\left(\mathsf{T}_3^{-1} - (-1)^{\mathsf{F}_3}\right) + \frac{q^{\frac{l_3-l_1-\kappa}{2}}}{x_2\theta_2}\mathsf{T}_3^{-1}\left(\mathsf{T}_2^{-1} - (-1)^{\mathsf{F}_2}\right) + \right.$$
$$\left. + \frac{q^{\frac{l_3+l_2-\kappa}{2}}}{x_1\theta_1}\left(1 - (-1)^{\mathsf{F}_1}\mathsf{T}_1\right)\right)\psi = 0 \ .$$

When $l_2 = 1$ is equal to 1 we have $f_2^3 \equiv 0$. The polynomial solutions to the Scasimir-eigenvalue equation are of the form

$$\psi = \psi_1 + x_2\theta_2\psi_2 + x_2^2\psi_3 \ .$$

Define the difference operator $\mathfrak{D}$ by

$$\mathfrak{D} = \frac{q^{\frac{-l_1-1+\kappa}{2}}}{x_3\theta_3}\left(\mathsf{T}_3^{-1} - (-1)^{\mathsf{F}_3}\right) + \frac{q^{\frac{l_3+1-\kappa}{2}}}{x_1\theta_1}\left(1 - (-1)^{\mathsf{F}_1}\mathsf{T}_1\right) \ .$$

The difference equation for ψ then becomes

$$\begin{aligned} 0 &= \mathfrak{D}\psi_1 + q^{\frac{l_1-l_3+\kappa-1}{2}} Q_{\mathfrak{s}}\,\mathsf{T}_3^{-1}\,\psi_2 + \\ &+ x_2\theta_2\mathfrak{D}\psi_2 + q^{\frac{l_1-l_3+\kappa-2}{2}} Q\,\mathsf{T}_3^{-1}\,\psi_3 + \\ &+ x_2^2\mathfrak{D}\psi_3 \ . \end{aligned}$$

There are three linearly independent polynomial solutions to the equation $f_2^3\psi = 0$. We denote them by ϕ'', ϕ' and ϕ. A general solution is of the form

$$\psi = a\phi'' + b\phi' + c\phi \ . \tag{4.31}$$

[9]Here we have used the relation $\mathsf{T}_3\,\mathsf{T}_2\,\mathsf{T}_1\,\psi = q^\kappa\psi$, which follows from the invariance of Ψ under the action of k, $\Delta^{(4)}(k)\Psi = \Psi$.

The solution ϕ''. We choose solution ϕ'' to have vanishing components ϕ''_2 and ϕ''_3. Component ϕ''_1 then is determined by the equation

$$\mathfrak{D}\phi''_1 = 0 \ . \tag{4.32}$$

From this we get

$$\left((-1)^{\mathsf{F}_3} - q^{\frac{l_1+l_3+2-2\kappa}{2}} \frac{x_3\theta_3}{x_1\theta_1}\right) \mathsf{T}_3^{-1}\,\phi''_1 = \left(1 - q^{\frac{l_1+l_3+2}{2}} \frac{x_3\theta_3}{x_1\theta_1}(-1)^{\mathsf{F}_1}\right)\phi''_1 \ .$$

We take the quotient of $\mathsf{T}_3^{-1}\,\phi''_1$ and ϕ''_1 and expand,

$$\frac{\mathsf{T}_3^{-1}\,\phi''_1}{\phi''_1} = \frac{\left((-1)^{\mathsf{F}_3} - q^{\frac{l_1+l_3+2-2\kappa}{2}} \frac{x_3\theta_3}{x_1\theta_1}\right)\left((-1)^{\mathsf{F}_3} - q^{\frac{l_1+l_3+4-2\kappa}{2}} \frac{x_3\theta_3}{x_1\theta_1}\right)\cdots\left((-1)^{\mathsf{F}_3} - q^{\frac{l_1+l_3}{2}} \frac{x_3\theta_3}{x_1\theta_1}\right)}{\left(1 - q^{\frac{l_1+l_3+4-2\kappa}{2}} \frac{x_3\theta_3}{x_1\theta_1}(-1)^{\mathsf{F}_1}\right)\cdots\left(1 - q^{\frac{l_1+l_3}{2}} \frac{x_3\theta_3}{x_1\theta_1}(-1)^{\mathsf{F}_1}\right)\left(1 - q^{\frac{l_1+l_3+2}{2}} \frac{x_3\theta_3}{x_1\theta_1}(-1)^{\mathsf{F}_1}\right)} \ . \tag{4.33}$$

This is the quotient of two polynomials in $x_3/x_1\theta_3\theta_1$. The parity operators F_3 and F_1 introduce relative signs into the polynomials and a single global sign each. These global signs derive from constants in θ_3 and θ_1. We denote these constants with $\theta_3^{\lambda_3}$ and $\theta_1^{\lambda_1}$ respectively. This yields the form of ϕ''_1 as

$$\phi''_1 = \left(1 - f_1 \frac{x_3\theta_3}{x_1\theta_1} + f_2\left(\frac{x_3\theta_3}{x_1\theta_1}\right)^2 - \cdots + f_{\kappa-2}\left(\frac{x_3\theta_3}{x_1\theta_1}\right)^{\kappa-2}\right)\theta_3^{\lambda_3}\theta_1^{\lambda_1} \ . \tag{4.34}$$

This determines ϕ''.

The solution ϕ'. We choose the solution ϕ' such that the component ϕ'_3 vanishes. The remaining components ϕ'_1 and ϕ'_2 are determined by the equations

$$\mathfrak{D}\phi'_1 = -q^{\frac{l_1-l_3+\kappa-1}{2}} Q_{\mathfrak{s}}\,\mathsf{T}_3^{-1}\,\phi_2 \tag{4.35}$$

$$\mathfrak{D}\phi'_2 = 0 \ . \tag{4.36}$$

The equation determining ϕ'_2 (4.36) is the same as (4.32) determining ϕ''_1. We conclude that

$$\phi'_2 = \phi''_1 \ . \tag{4.37}$$

We note that (4.35) which determines ϕ'_1 basically is (4.36) with an inhomogeneity. Thus we make the ansatz $\phi'_1 = \phi'_2 g$. The operators $((-1)^{\mathsf{F}_3} - \mathsf{T}_3^{-1})$ and $(1 - (-1)^{\mathsf{F}_1}\,\mathsf{T}_1)$ obey the deformed product rule

$$\left((-1)^{\mathsf{F}} - \mathsf{T}^a\right) fg = \left(\left((-1)^{\mathsf{F}} - \mathsf{T}^a\right) f\right) g + \left(\mathsf{T}^a\, f\right)\left(1 - \mathsf{T}^a\right) g \ .$$

Thus (4.35) becomes

$$\left(\mathfrak{D}\phi'_2\right) g + \left(\mathsf{T}_3^{-1}\,\phi'_2\right)\left(-\frac{q^{\frac{-l_1-1+\kappa}{2}}}{x_3\theta_3}\left(1 - \mathsf{T}_3^{-1}\right) + \frac{q^{\frac{l_3+1+\kappa}{2}}}{x_1\theta_1}(-1)^{\mathsf{F}_1}\left(1 - \mathsf{T}_1\right)\right) g$$
$$= -q^{\frac{l_3-l_1+\kappa-1}{2}} Q_{\mathfrak{s}}\,\mathsf{T}_3^{-1}\,\phi'_2 \ .$$

We note that $\mathfrak{D}\phi_2' = 0$. The ansatz $g = \alpha x_3\theta_3$ yields

$$q^{\frac{-l_1-2+\kappa}{2}}\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)\alpha = q^{\frac{l_3-l_1+\kappa-1}{2}}Q_{\mathfrak{s}}$$

and determines $\alpha = q^{l_3+\frac{1}{2}}Q_{\mathfrak{s}}/(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$. We conclude that

$$\phi' = (\alpha x_3\theta_3 + x_2\theta_2)\phi_2' \ .$$

The solution ϕ. The solution $\phi = \phi_1 + x_2\theta_2\phi_2 + x_2\phi_3$ is determined by the following equations

$$\mathfrak{D}\phi_1 = -q^{\frac{l_1-l_3+\kappa-1}{2}}Q_{\mathfrak{s}}\,\mathsf{T}_3^{-1}\,\phi_2 \tag{4.38}$$

$$\mathfrak{D}\phi_2 = -q^{\frac{l_1-l_3+\kappa-2}{2}}Q\,\mathsf{T}_3^{-1}\,\phi_3 \tag{4.39}$$

$$\mathfrak{D}\phi_3 = 0 \ . \tag{4.40}$$

We proceed as before. We note that the difference operators in (4.40) and (4.32) are the same and conclude that

$$\phi_3 = \phi_1'' \ .$$

For ϕ_2 we make the ansatz $\phi_2 = \phi_3 u$ leading to

$$\begin{aligned}(\mathfrak{D}\phi_3)\,u + \left(\mathsf{T}_3^{-1}\,\phi_3\right)\left(-\frac{q^{\frac{-l_1-1+\kappa}{2}}}{x_3\theta_3}\left(1-\mathsf{T}_3^{-1}\right) + \frac{q^{\frac{l_3+1+\kappa}{2}}}{x_1\theta_1}(-1)^{\mathsf{F}_1}\left(1-\mathsf{T}_1\right)\right)u \\ = -q^{\frac{l_3-l_1+\kappa-2}{2}}Q_{\mathfrak{s}}\,\mathsf{T}_3^{-1}\,\phi_3\end{aligned}$$

We note that $\mathfrak{D}\phi_3$ vanishes and make the ansatz $u = \beta x_3\theta_3$. This yields

$$q^{\frac{-l_1-2+\kappa}{2}}\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)\beta = q^{\frac{l_3-l_1+\kappa-2}{2}}Q_{\mathfrak{s}}$$

and determines $\beta = q^{\frac{l_3+1}{2}}Q_{\mathfrak{s}}/(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$.
For ϕ_1 we make the ansatz $\phi_1 = \phi_3 v$ leading to

$$\begin{aligned}(\mathfrak{D}\phi_3)\,v + \left(\mathsf{T}_3^{-1}\,\phi_3\right)\left(-\frac{q^{\frac{-l_1-1+\kappa}{2}}}{x_3\theta_3}\left(1-\mathsf{T}_3^{-1}\right) + \frac{q^{\frac{l_3+1+\kappa}{2}}}{x_1\theta_1}(-1)^{\mathsf{F}_1}\left(1-\mathsf{T}_1\right)\right)v \\ = -q^{\frac{l_3-l_1+\kappa-1}{2}}Q_{\mathfrak{s}}\,\mathsf{T}_3^{-1}\,\phi_2\end{aligned}$$

We have already determined ϕ_2 to be equal to $\phi_3\beta x_3\theta_3$. Inserting

$$\mathsf{T}_3^{-1}(\phi_3\beta x_3\theta_3) = (\mathsf{T}_3^{-1}\,\phi_3)q^{-1}\beta x_3\theta_3$$

together with the ansatz $v = \gamma x_3^2$ into the equation above, leads to

$$q^{\frac{-l_1-3+\kappa}{2}}\gamma x_3\theta_3\left(q - q^{-1}\right) = q^{\frac{l_3-l_1+\kappa-3}{2}}Q_{\mathfrak{s}}\beta x_3\theta_3\left(\mathsf{T}_3^{-1}\,\phi_3\right) \ .$$

This determines $\gamma = \beta Q_{\mathfrak{s}}/Qq^{\frac{l_3}{2}} = q^{\frac{2l_3+1}{2}}Q_{\mathfrak{s}}^2/(Q(q^{\frac{1}{2}} - q^{-\frac{1}{2}}))$. We conclude that

$$\begin{aligned}\phi &= \phi_1 + x_2\theta_2\phi_2 + x_2^2\phi_3 \\ &= \left(\gamma x_3^2 + \beta x_2x_3\theta_2\theta_3 + x_2^2\right)\phi_3 \ .\end{aligned}$$

Note that $\phi_3 = \phi_2' = \phi_1''$. The general solution (4.31) is thus

$$\begin{aligned}\psi &= a\phi'' + b\phi' + c\phi \\ &= a\phi_1'' + b\left(\alpha x_3\theta_3 + x_2\theta_2\right)\phi_2' + c\left(\gamma x_3^2 + \beta x_2 x_3\theta_2\theta_3 + x_2^2\right)\phi_3 \\ &= \left(a + b\left(\alpha x_3\theta_3 + x_2\theta_2\right) + c\left(\gamma x_3^2 + \beta x_2 x_3\theta_2\theta_3 + x_2^2\right)\right)\phi_1'' \,.\end{aligned}$$

When x_2 tends to 0, ψ tends to $\check{\Psi}$. Thus we have proven that for $j_2 = 1$ the $\mathfrak{x}$-reduced blocks $\mathcal{F}$ are of the form

$$\mathcal{F}^{(i)}_{l_i^\nu} = \left(a_i^\nu + b_i^\nu x\mathrm{i}\theta + c_i^\nu x^2(\mathrm{i}\theta)^2\right) R(L,\Lambda|\mathfrak{x}) \;,$$

for $\nu = -, \circ, +$. □

Looking at the $\mathfrak{x}$-reduced blocks $\mathcal{F}$ in a little more detail we see from the horizontal and vertical constraints that for the appropriate coefficients r_m^ν

$$\begin{aligned}\mathcal{F}^{(s)}_{l_s^-} &= r^-_{-l_1+1}x^2(\mathrm{i}\theta)^{2+\lambda_1} + \cdots + r^-_{l_3-l_4}x^\kappa(\mathrm{i}\theta)^{\kappa+\lambda_1} \\ \mathcal{F}^{(s)}_{l_s^\circ} &= r^\circ_{-l_1}x^1(\mathrm{i}\theta)^{1+\lambda_1} + r^\circ_{-l_1+1}x^2(\mathrm{i}\theta)^{2+\lambda_1} + \cdots + r^\circ_{l_3-l_4}x^\kappa(\mathrm{i}\theta)^{\kappa+\lambda_1} \\ \mathcal{F}^{(s)}_{l_s^+} &= r^+_{-l_1-1}(\mathrm{i}\theta)^{\lambda_1} + r^+_{-l_1}x^1(\mathrm{i}\theta)^{1+\lambda_1} + r^+_{-l_1+1}x^2(\mathrm{i}\theta)^{2+\lambda_1} + \cdots + r^+_{l_3-l_4}x^\kappa(\mathrm{i}\theta)^{\kappa+\lambda_1}\end{aligned}$$

in the s-channel and in the u-channel

$$\begin{aligned}\mathcal{F}^{(u)}_{l_u^-} &= r^-_{l_1-l_4}(\mathrm{i}\theta)^{\lambda_1} + \cdots + r^-_{-l_3+1}x^{\kappa-2}(\mathrm{i}\theta)^{\kappa-2+\lambda_1} \\ \mathcal{F}^{(u)}_{l_u^\circ} &= r^\circ_{l_1-l_4}(\mathrm{i}\theta)^{\lambda_1} + \cdots + r^\circ_{-l_3+1}x^{\kappa-2}(\mathrm{i}\theta)^{\kappa-2+\lambda_1} + r^\circ_{-l_3}x^{\kappa-1}(\mathrm{i}\theta)^{\kappa-1+\lambda_1} \\ \mathcal{F}^{(u)}_{l_u^+} &= r^+_{l_1-l_4}(\mathrm{i}\theta)^{\lambda_1} + \cdots + r^+_{-l_3+1}x^{\kappa-2}(\mathrm{i}\theta)^{\kappa-2+\lambda_1} + r^+_{-l_3}x^{\kappa-1}(\mathrm{i}\theta)^{\kappa-1+\lambda_1} + r^+_{-l_3-1}x^\kappa(\mathrm{i}\theta)^{\kappa+\lambda_1} \,.\end{aligned}$$

So we have already that the coefficients $a_s^-, a_s^\circ, b_s^-, b_i^-, c_u^-$ and c_u° vanish. The remaining coefficients we determine from the Clebsch–Gordan coefficients using the formulae (B.8) in the appendix.

$$\begin{aligned}r_0 a_s^+ &= (-1)^{\mathcal{S}^{(s)}_{-l_1-1}} \\ r_0 b_s^+ &= (-1)^{\mathcal{S}^{(s)}_{-l_1}} q^{-\frac{l_4}{2}} \frac{\{l_3^{14}+1\}\{2\}}{\{2l_1+2\}} \\ r_0 c_s^+ &= (-1)^{\mathcal{S}^{(s)}_{-l_1+1}} q^{-l_4}\left(\frac{\{\kappa-1\}\{\kappa\}}{\{2l_1+1\}\{2l_1+2\}} - \frac{\{\kappa-2\}\{\kappa\}}{\{2l_1+2l\}} - \frac{\{\kappa-3\}\{\kappa-2\}}{\{2\}}\right) \\ r_0 b_s^\circ &= (-1)^{\mathcal{S}^{(s)}_{-l_1}+1+\lambda_2} q^{\frac{-l_1-1}{2}} \\ r_0 c_s^\circ &= (-1)^{\mathcal{S}^{(s)}_{-l_1-1}+\lambda_2} q^{\frac{-l_1-l_4-1}{2}} \frac{\{l_3^{14}\}}{\{2l_1\}} \\ r_0 c_s^- &= (-1)^{\mathcal{S}^{(s)}_{l_1+1}+1} q^{-l_1}\end{aligned}$$

$$r_0 a_u^+ = (-1)^{\mathcal{S}^{(u)}_{l_1-l_4}} q^{\frac{\kappa}{2}} \frac{\{l_1^{34}\}\{l_1^{34}+1\}}{\{2l_3+1\}\{2l_3+2\}}$$
$$r_0 b_u^+ = (-1)^{\mathcal{S}^{(u)}_{l_1-l_4-1}+\kappa+\lambda_2} q^{\frac{\kappa-l_4}{2}} \frac{\{l_1^{34}+1\}\{2\}}{\{2l_3+2\}}$$
$$r_0 c_u^+ = (-1)^{\mathcal{S}^{(u)}_{l_1-l_4}} q^{-l_4} \left(\frac{\{l_1^{34}+3\}\{l_1^{34}+2\}\{\kappa-1\}\{\kappa\}}{\{2l_3+2\}\{2l_3+1\}} - \frac{\{2\}\{l_1^{34}+1\}^2\{l_1^{34}\}}{\{2l_3+2\}^2\{2l_3+1\}} - \frac{\{\kappa-3\}\{\kappa-2\}}{\{2\}\{2l_3+2\}\{2l_3+1\}} \right)$$
$$r_0 a_u^\circ = (-1)^{\mathcal{S}^{(u)}_{l_1-l_4}+\kappa-1} q^{\frac{\kappa-1}{2}} \frac{\{l_1^{34}\}}{\{2l_3\}}$$
$$r_0 b_u^\circ = (-1)^{\mathcal{S}^{(s)}_{l_1-l_4-1}-1+\lambda_2} q^{-\frac{l_4}{2}} \frac{\{l_1^{34}\}}{\{2l_3\}}$$
$$r_0 a_u^- = (-1)^{\mathcal{S}^{(u)}_{l_1-l_4}} q^{\frac{\kappa-2}{2}}$$

The expressions for c_s^+ and c_u^+ unfortunately are rather bulky. There might well be a way to algebraically bring them to a form more manageable. This would simplify the expressions for the 6j-symbols that depend on these coefficients.
Using the ansatz $\mathcal{F} = (a + bx\mathrm{i}\theta + cx^2)R$ of (4.30) the fusing relations

$$\mathcal{F}^{(u)}_{j_u^\mu}(L,\Lambda|\mathfrak{x}) = \sum_{j_s^\nu \in \{j_1, j_1\pm 1\}} \begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^\mu \\ l_4\lambda_4 & l_1\lambda_1 & l_s^\nu \end{Bmatrix}_{q|\mathfrak{s}} \mathcal{F}^{(s)}_{j_s^\nu}(L,\Lambda|\mathfrak{x})$$

simplify to the relations

$$\begin{aligned} a_u^+\mathrm{i}\theta + b_u^+ x\mathrm{i}\theta + c_u^+ x^2(\mathrm{i}\theta)^2 = &\begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^+ \\ l_4\lambda_4 & l_1\lambda_1 & l_s^- \end{Bmatrix}_{q|\mathfrak{s}} c_s^- x^2(\mathrm{i}\theta)^2 + \\ &+ \begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^+ \\ l_4\lambda_4 & l_1\lambda_1 & l_s^\circ \end{Bmatrix}_{q|\mathfrak{s}} \left(b_s^\circ x\mathrm{i}\theta + c_s^\circ x^2(\mathrm{i}\theta)^2\right) + \\ &+ \begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^+ \\ l_4\lambda_4 & l_1\lambda_1 & l_s^+ \end{Bmatrix}_{q|\mathfrak{s}} \left(a_s^+\mathrm{i}\theta + b_s^+ x\mathrm{i}\theta + c_s^+ x^2(\mathrm{i}\theta)^2\right) \ , \end{aligned}$$

$$\begin{aligned} a_u^\circ + b_u^\circ x\mathrm{i}\theta = &\begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^\circ \\ l_4\lambda_4 & l_1\lambda_1 & l_s^- \end{Bmatrix}_{q|\mathfrak{s}} c_s^- x^2(\mathrm{i}\theta)^2 + \\ &+ \begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^\circ \\ l_4\lambda_4 & l_1\lambda_1 & l_s^\circ \end{Bmatrix}_{q|\mathfrak{s}} \left(b_s^\circ x\mathrm{i}\theta + c_s^\circ x^2(\mathrm{i}\theta)^2\right) + \\ &+ \begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^\circ \\ l_4\lambda_4 & l_1\lambda_1 & l_s^+ \end{Bmatrix}_{q|\mathfrak{s}} \left(a_s^+ + b_s^+ x\mathrm{i}\theta + c_s^+ x^2(\mathrm{i}\theta)^2\right) \ , \end{aligned}$$

$$\begin{aligned}
a_u^- = &\begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^- \\ l_4\lambda_4 & l_1\lambda_1 & l_s^- \end{Bmatrix}_{q|\mathfrak{s}} c_s^- x^2(\mathrm{i}\theta)^2 + \\
&+ \begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^- \\ l_4\lambda_4 & l_1\lambda_1 & l_s^\circ \end{Bmatrix}_{q|\mathfrak{s}} \left(b_s^\circ x\mathrm{i}\theta + c_s^\circ x^2(\mathrm{i}\theta)^2\right) + \\
&+ \begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^- \\ l_4\lambda_4 & l_1\lambda_1 & l_s^+ \end{Bmatrix}_{q|\mathfrak{s}} \left(a_s^+ + b_s^+ x\mathrm{i}\theta + c_s^+ x^2(\mathrm{i}\theta)^2\right) \ .
\end{aligned}$$

Solving these relations for the $6j$-symbols we arrive at the fundamental $6j$-symbols of the super quantum group $\mathcal{U}_q\mathbf{osp}(1|2)$. The bulky form of c_s^+ and c_u^+ makes it unappealing to give the $6j$-symbols depending on these in terms of spins L and parities Λ. For said $6j$-symbols we note the construction in terms of the coefficients a_i^ν, b_i^ν and c_i^ν. The remaining $6j$-symbols we give in terms of L and Λ.
Let $\mathcal{L} = \mathcal{S}^{(u)}_{l_1-l_4} + \mathcal{S}^{(s)}_{-l_1-1}$. The fundamental $6j$-symbols then are

(4.41a)

$$\begin{aligned}
\begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^+ \\ l_4\lambda_4 & l_1\lambda_1 & l_s^+ \end{Bmatrix}_{q|\mathfrak{s}} &= (-1)^{\mathcal{L}} q^{\frac{\kappa}{2}} \frac{\{l_1^{34}\}\{l_1^{34}+1\}}{\{2l_3+1\}\{2l_3+2\}} \\
\begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^\circ \\ l_4\lambda_4 & l_1\lambda_1 & l_s^+ \end{Bmatrix}_{q|\mathfrak{s}} &= (-1)^{\mathcal{L}+\kappa-1} q^{\frac{\kappa-1}{2}} \frac{\{l_1^{34}\}}{\{2l_3\}} \\
\begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^- \\ l_4\lambda_4 & l_1\lambda_1 & l_s^+ \end{Bmatrix}_{q|\mathfrak{s}} &= (-1)^{\mathcal{L}} q^{\frac{\kappa-2}{2}} \\
\begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^+ \\ l_4\lambda_4 & l_1\lambda_1 & l_s^\circ \end{Bmatrix}_{q|\mathfrak{s}} &= (-1)^{\mathcal{L}+1} q^{\frac{\kappa+l_1-l_4+1}{2}} \frac{\{l_1^{34}\}\{2\}}{\{2l_3+2\}} \left(1 - \frac{\{l_1^{34}+1\}\{l_3^{14}+1\}}{\{2l_3+1\}\{2l_1+2\}}\right) \\
\begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^\circ \\ l_4\lambda_4 & l_1\lambda_1 & l_s^\circ \end{Bmatrix}_{q|\mathfrak{s}} &= (-1)^{\mathcal{L}} q^{\frac{\kappa+l_1-l_4+1}{2}} \frac{\{l_1^{34}\}}{\{2l_3\}} \left(1 - \frac{\{l_3^{14}+1\}\{2\}}{\{2l_1+2\}}\right) \\
\begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^- \\ l_4\lambda_4 & l_1\lambda_1 & l_s^\circ \end{Bmatrix}_{q|\mathfrak{s}} &= (-1)^{\mathcal{L}+\lambda_2} q^{\frac{\kappa-l_4+l_1-1}{2}} \frac{\{l_3^{14}+1\}\{2\}}{\{2l_1+2\}} \ .
\end{aligned}$$

(4.41b)

$$\begin{aligned}
\begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^+ \\ l_4\lambda_4 & l_1\lambda_1 & l_s^- \end{Bmatrix}_{q|\mathfrak{s}} &= \frac{c_u^+}{c_s^-} - \frac{c_s^+ a_u^+}{c_s^- a_s^+} - \frac{c_s^\circ}{c_s^-}\left(\frac{b_u^+}{b_s^\circ} - \frac{b_s^+ a_u^+}{b_s^\circ a_s^+}\right) \\
\begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^\circ \\ l_4\lambda_4 & l_1\lambda_1 & l_s^- \end{Bmatrix}_{q|\mathfrak{s}} &= -\frac{c_s^+ a_u^\circ}{c_s^- a_s^+} - \frac{c_s^\circ}{c_s^-}\left(\frac{b_u^\circ}{b_s^\circ} - \frac{b_s^+ a_u^\circ}{b_s^\circ a_s^+}\right) \\
\begin{Bmatrix} l_3\lambda_3 & 1\lambda_2 & l_u^- \\ l_4\lambda_4 & l_1\lambda_1 & l_s^- \end{Bmatrix}_{q|\mathfrak{s}} &= -\frac{c_s^+ a_u^-}{c_s^- a_s^+} + \frac{c_s^\circ b_s^+ a_u^-}{c_s^- b_s^\circ a_s^+} \ .
\end{aligned}$$

Thus we have determined the fundamental $6j$-symbols of $\mathcal{U}_q\mathbf{osp}(1|2)$. Together with the recurrence relation (4.22) these determine all $6j$-symbols of the super quantum group $\mathcal{U}_q\mathbf{osp}(1|2)$.

Chapter 5
Conclusion and outlook

We determined the $6j$ symbols of the quantum groups $\mathcal{U}_q\mathbf{sl}(2)$ and $\mathcal{U}_q\mathbf{osp}(1|2)$ in a way inspired by the determination of the fusing matrices in WZNW models in the approach of Belavin, Polyakov and Zamolodchikov to conformal field theory.

The four-point invariants Ψ of $\mathcal{U}_q\mathbf{sl}(2)$ can be decomposed into blocks $\Psi^{(s)}$ and $\Psi^{(u)}$

$$\Psi^{(s)}_{j_s}(J|X) = \sum_{m_s} \begin{pmatrix} j_4 & j_3 & j_s \\ x_4 & x_3 & m_s \end{pmatrix}_q \left[\begin{array}{c|cc} j_s & j_2 & j_1 \\ m_s & x_2 & x_1 \end{array}\right]_q , \tag{5.1a}$$

$$\Psi^{(u)}_{j_u}(J|X) = \sum_{m_u} \begin{pmatrix} j_4 & j_u & j_1 \\ x_4 & m_u & x_1 \end{pmatrix}_q \left[\begin{array}{c|cc} j_u & j_3 & j_2 \\ m_u & x_3 & x_2 \end{array}\right]_q . \tag{5.1b}$$

For $\mathcal{U}_q\mathbf{osp}(1|2)$ we have an analogue decomposition. These blocks are connected linearly by the $6j$-symbols

$$\Psi^{(u)}_{j_u} = \sum_{j_s} \begin{Bmatrix} j_3 & j_2 & j_u \\ j_4 & j_1 & j_s \end{Bmatrix} \Psi^{(s)}_{j_s} . \tag{5.2}$$

We considered representations of $\mathcal{U}_q\mathbf{sl}(2)$ and $\mathcal{U}_q\mathbf{osp}(1|2)$ as difference operators on the spaces $Pol_{2j}(x)$ and $Pol_{2l}(x,\theta)$. In these representations the Casimir operator and the Scasimir operator induce difference equations the invariants fulfil. From these equations we deduced that the invariants are of a generalised hypergeometric form. The blocks of $\mathcal{U}_q\mathbf{sl}(2)$ have the form of basic hypergeometric functions

$${}_2\Phi_1\left(\begin{array}{cc|c} a & b & \\ & c & x;q \end{array}\right) = \sum_{n\geq 0} \frac{[a|n]\,[b|n]}{[c|n]\,[n]!} x^n .$$

The blocks of $\mathcal{U}_q\mathbf{osp}(1|2)$ have the form of a natural generalisation thereof. They are of a super basic hypergeometric type

$${}_2\Pi_1\left(\begin{array}{cc|c} a & b & x,\theta \\ & c & \lambda, q \end{array}\right) = \sum_{n\geq 0} \frac{\{a|n\}\{b|n\}}{\{c|n\}\{n\}!} x^n \theta^{\lambda+n} .$$

Inspired by the role the degenerate fields of WZNW-models play in the determination of the fusing matrices we specialised one of the representation to the fundamental representation, 1/2 for $\mathcal{U}_q\mathbf{sl}(2)$ and 1 for $\mathcal{U}_q\mathbf{osp}(1|2)$.
This reduces the amount of intermediate representations considerably.

The remaining spins are $j_s\pm = j_1 \pm 1/2$ and $j_u^\pm = j_3 \pm 1/2$ for $\mathcal{U}_q\mathbf{sl}(2)$ and $l_s^\nu = l_1 + \nu$ and $l_u^\nu = l_3 + \nu$ with $\nu = -1, 0, 1$ for $\mathcal{U}_q\mathbf{osp}(1|2)$. Furthermore the specialisation to the fundamental representation puts further constraints on the form of the reduced blocks. We found that for $\mathcal{F} = x_3^\kappa \Psi.$ in $\mathcal{U}_q\mathbf{sl}(2)$ we have

$$\mathcal{F}^{(i)}_{j_i^\pm}(J|x) = \left(a_i^\pm + b_i^\pm x\right) R(J|x) \ .$$

For $\mathcal{U}_q\mathbf{osp}(1|2)$ we found

$$\mathcal{F}^{(i)}_{l_i^\nu}(L, \Lambda|\mathfrak{x}) = \left(a_i^\nu + b_i^\nu x i\theta + c_i^\nu x^2\right) R(L, \Lambda|\mathfrak{x}) \quad \text{for } \nu = -, \circ, + \ .$$

We determined the coefficients a, b and c from the definition (5.1) of the blocks $\Psi^{(s)}$ and $\Psi^{(i)}$ by direct evaluation of the Clebsch–Gordan coefficients concerned.
Together with the fusing relations (5.2) this lead us to a system of linear equations for the fundamental $6j$-symbols. These fundamental $6j$-symbols, together with the recurrence relation the $6j$-symbols obey, determine the $6j$-symbols.

Outlook

The determination of the coefficients a, b and c by direct evaluation of the Clebsch–Gordan coeffiencts is tedious and not fully in harmony with the rest of the method. It would be satisfying to determine these coefficients by considering further difference equations. An approach in this direction showed much promise, but met subtle difficulties.

As an application to these ideas, it is possible to investigate the conjecture by Feigin and Malikov [**8**] mentioned in the introduction. We sketch the steps that have to be taken in this investigation.
When we consider the case of rational t as a limit of generic t we have to give an argument as to why the tensor product is replaced with the truncated tensor product. Similarily some arguments of Teschner's derivation in the SL(2)/SU(2)-model have to be generalised to the case of rational t.
With the help of the expressions Teschner determined for the fusing matrices in the context of the SL(2)/SU(2)-model the normalisation coefficients ν^j_{lk} in the equality

$$\frac{\nu^{j_3}_{j_4 j_s}\nu^{j^F}_{j_s j_1}}{\nu^{j_u}_{j_4 j_1}\nu^{j_3}_{j_u j^F}} \begin{Bmatrix} j_3 & j^F & j_u \\ j_4 & j_1 & j_s \end{Bmatrix} = \mathrm{F}_{j_u j_s} \begin{bmatrix} j_3 & j^F \\ j_4 & j_1 \end{bmatrix}$$

could be determined. Thus proving or disproving the conjecture.

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Appendix A
Quantum calculus

We introduce the *quantum* generalisation of c-numbers and a difference calculus on these. We introduce the basic hypergeometric function. Finally we collect recurrence relations for the quantum Clebsch–Gordan coefficients.

1. Quantum numbers

Quantum numbers or q-numbers by themselves have nothing to do with quantum theory. They date back to the 19^{th} century when there was the first gold-rush for q-analogues[1]. An extensive introduction can be found in the book by Klimyk and Schmüdgen [**22**].

$$[n]_q = [n] = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{A.1}$$

The *q-factorial* $[n]_q! = [n]!$ is defined as

$$\begin{aligned} &[0]! = 1 \\ &[n]! = [1]\,[2]\cdots[n] \quad \text{for } n \geq 1 \ . \end{aligned}$$

The *shifted q-factorial* $[a|n]_q = [a|n]$ is defined as

$$\begin{aligned} [a|0] &= 1 \\ [a|n] &= [a]\,[a+1]\,[a+2]\cdots[a+n-1] \quad \text{for } n \geq 1 \\ [a|-n] &= \frac{1}{[a-1]\,[a-2]\cdots[a-n]} = \frac{1}{[a-n|n]} \quad \text{for } n \geq 1 \end{aligned} \tag{A.2}$$

2. Difference operators

The multiplication operator $\mathsf{T} = q^{\mathrm{d}_x} = q^{x\partial_x}$ on functions of x is defined as

$$\mathsf{T}\, f(x) = f(qx) \ .$$

[1] A q-analogue of a given object is a parametrised family of objects which includes the original object for a special value of the parameter.

When considering q-deformation, people often study variables z such that $x = q^z$. In that case T is also known as a shift operator.
From T we construct the difference operator as

$$[\mathrm{d}_x + a] = \frac{q^a\,\mathsf{T} - q^{-a}\,\mathsf{T}^{-1}}{q - q^{-1}} \ .$$

Properties. The multiplication and difference operators commute as follows with x

$$\mathsf{T}\,x = qx\,\mathsf{T} \quad \text{and} \quad [\mathrm{d}_x]\,x = x\,[\mathrm{d}_x + 1] \ .$$

When we let q tend to 1 we reproduce the identity from T and an operator proportional to the differential quotient from the difference operator.

3. Basic hypergeometric series

The basic hypergeometric series was first considered[2] by Eduard Heine in the 1840s as a generalisation of Gauss's hypergeometric series. In modern parlance it is also called the q-hypergeometric series. The basic hypergeometric series ${}_2\Phi_1(a, b; c|x; q)$ is the solution of the hypergeometric difference equation

$$(x\,[\mathrm{d}_x + a]\,[\mathrm{d}_x + b] - [\mathrm{d}_x]\,[d_x + c - 1])\,P(x) = 0 \ . \tag{A.3}$$

We use the following series representation for ${}_2\Phi_1$

$${}_2\Phi_1\!\left(\begin{matrix} a & b \\ & c \end{matrix}\,\middle|\, x; q\right) = \sum_{n \geq 0} \frac{[a|n]\,[b|n]}{[c|n]\,[n]!} x^n \ . \tag{A.4}$$

4. Quantum Clebsch–Gordan coefficients

The quantum Clebsch–Gordan-coefficients communicate the base change between the tensor basis and the reduced basis in a tensor product of $\mathcal{U}_q$-modules.

$$\mathbf{e}^j_m(j_2, j_1) = \sum_{m_1, m_2} \begin{bmatrix} j & j_2 & j_2 \\ m & m_2 & m_2 \end{bmatrix}_q \mathbf{e}^{j_2}_{m_2} \otimes \mathbf{e}^{j_1}_{m_1}$$

For a representation of $\mathcal{U}_q$ acting as

$$\pi_m(X^{\pm})\mathbf{e}^j_m = \mathcal{C}^{\pm}(j, m)\mathbf{e}^j_{m\pm1} \ , \quad \pi_m(K)\mathbf{e}^j_m = q^m \mathbf{e}^j_m \ ,$$

[2]It is introduced in [**12**] and further properties are explored in [**13**]. To be precise Heine considered the analogue series on quantum numbers

$$[\![a]\!] = \frac{1 - q^a}{1 - q} \ .$$

the Clebsch–Gordan-coefficients satisfy the recurrence relation

$$\mathcal{C}^{\pm}(j,m)\left[\begin{smallmatrix} j \\ m\pm 1\end{smallmatrix}\middle|\begin{smallmatrix} j_2 & j_1 \\ m_2 & m_1\end{smallmatrix}\right]_q =$$

$$q^{m_1}\mathcal{C}^{\pm}(j_2,m_2\mp 1)\left[\begin{smallmatrix} j \\ m\end{smallmatrix}\middle|\begin{smallmatrix} j_2 & j_1 \\ m_2\mp 1 & m_1\end{smallmatrix}\right]_q + q^{-m_2}\mathcal{C}^{\pm}(j_1,m_1\mp 1)\left[\begin{smallmatrix} j \\ m\end{smallmatrix}\middle|\begin{smallmatrix} j_2 & j_1 \\ m_2 & m_1\mp 1\end{smallmatrix}\right]_q .$$

For minimal m we get

$$\begin{aligned}\left[\begin{matrix} j \\ -j\end{matrix}\middle|\begin{matrix} j_2 & j_1 \\ m_2 & m_1\end{matrix}\right]_q &= -q^{j+1}\frac{\mathcal{C}^-(j_1,m_1+1)}{\mathcal{C}^-(j_2,m_2)}\left[\begin{matrix} j \\ -j\end{matrix}\middle|\begin{matrix} j_2 & j_1 \\ m_2-1 & m_1+1\end{matrix}\right]_q \\ &= -q^{-j-1}\frac{\mathcal{C}^-(j_2,m_2+1)}{\mathcal{C}^-(j_1,m_1)}\left[\begin{matrix} j \\ -j\end{matrix}\middle|\begin{matrix} j_2 & j_1 \\ m_2+1 & m_1-1\end{matrix}\right]_q .\end{aligned}$$

Analogously we get for minimal m_2

$$\begin{aligned}\left[\begin{matrix} j \\ m\end{matrix}\middle|\begin{matrix} j_2 & j_1 \\ -j_2 & m_1\end{matrix}\right]_q &= q^{j_2}\frac{\mathcal{C}^+(j_1,m_1-1)}{\mathcal{C}^+(j,m-1)}\left[\begin{matrix} j \\ m-1\end{matrix}\middle|\begin{matrix} j_2 & j_1 \\ -j_2 & m_1-1\end{matrix}\right]_q \\ &= q^{-j_2}\frac{\mathcal{C}^+(j,m)}{\mathcal{C}^+(j_1,m_1)}\left[\begin{matrix} j & j_2 \\ m+1 & -j_2\end{matrix}\middle|\begin{matrix} j_1 \\ m_1+1\end{matrix}\right]_q\end{aligned}$$

and for minimal m_1

$$\begin{aligned}\left[\begin{matrix} j \\ m\end{matrix}\middle|\begin{matrix} j_2 & j_1 \\ -j_2 & m_1\end{matrix}\right]_q &= q^{-j_1}\frac{\mathcal{C}^+(j_2,m_2-1)}{\mathcal{C}^+(j,m-1)}\left[\begin{matrix} j \\ m-1\end{matrix}\middle|\begin{matrix} j_2 & j_1 \\ m_2-1 & -j_1\end{matrix}\right]_q \\ &= q^{j_1}\frac{\mathcal{C}^+(j,m)}{\mathcal{C}^+(j_2,m_2)}\left[\begin{matrix} j \\ m+1\end{matrix}\middle|\begin{matrix} j_2 & j_1 \\ m_2+1 & -j_1\end{matrix}\right]_q .\end{aligned}$$

Appendix B
A few things super

We introduce the *super* prefix as a $\mathbf{Z}_2$ grading. We introduce super vector spaces and super algebras. The main topic of this chapter is the combination of the *super* and *quantum* structures.

A short remark on terminology. We will name a super generalisation of a given object, say a commutator, be prefixing “super” to the objects name. In the given case this would result in a super commutator. Since we apply the same naming convention for quantum generalisations, some primacy has to be established. We choose to apply “quantum” first, then “super”, resulting in, say super quantum commutator.

A highly worthwhile introduction to the mathematical side of super symmetry is the monograph by Varadarajan [**40**] from which the first two sections of this chapter draw.

1. Super linear algebra

Definition B.1. A *k-super vector space* V is a vector space over the field k graded by $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$. That means it decomposes into the direct sum of two vector spaces

$$V = V_0 \oplus V_1 \ .$$

The elements of V_0 are called *even* and V_0 the *even subspace*, elements of V_1 are called *odd* and V_1 the *odd subspace*. Elements of either subspace only are called *homogeneous*.
We introduce the *parity* p, a binary function on V such that

$$\mathrm{p}\,|_{V_i} \equiv i \ , \quad i = 0, 1 \ .$$

Example 1. For a field k the super coordinate space $k^{m|n}$ is as a set isomorphic to the coordinate space k^{m+n} spanned by the vectors e_i, $i = 1, 2, \ldots, m+n$. The even subspace is spanned by the vectors e_i, $i = 1, 2, \ldots, m$ and the odd subspace is spanned by the vectors e_i, $i = m+1, m+2, \ldots, n$.

DEFINITION B.2. A linear map $f : V \to W$ between super vector spaces is called *parity preserving* if

$$f(V_i) \subseteq W_i \ , \quad i = 0, 1$$

and *parity reversing* if

$$f(V_i) \subseteq W_{i+1} \ , \quad i = 0, 1 \ .$$

Every linear map between super vector spaces can be decomposed into a parity preserving and an parity reversing map. Thus the space of linear maps between super vector spaces, say V and W, is itself a super vector space. We denote it by $\mathbf{Hom}(V, W)$.
The homomorphisms of super vector spaces are the parity preserving linear maps between them. We denote this space by $\mathrm{Hom}(V, W)$.

DEFINITION B.3. The super vector spaces taken as objects and the parity preserving linear maps taken as morphisms constitute a category. We call this category of (k-) super vector spaces $\mathscr{S}\mathscr{V}$.

The category of super vector spaces has a lot of additional structure. We will list the ones most important to us.

Monoidal category. The category $\mathscr{S}\mathscr{V}$ is a *monoidal category* with the super tensor product as monoidal product and the even super vector space $k^{1|0}$ as unit object.

Symmetric monoidal category. With respect to the braiding

$$\tau_{V,W} \colon V \otimes W \to W \otimes V$$
$$\tau(x \otimes y) = (-1)^{\mathrm{p}(x)\,\mathrm{p}(y)} y \otimes x$$

$\mathscr{S}\mathscr{V}$ is *symmetric monoidal category.*

Closed monoidal category. Super vector spaces constitute a *closed monoidal category* with internal Hom object $\mathbf{Hom}(V, W)$, the super vector space of all linear maps $V \to W$. This means that the funtor $- \otimes V$ left adjoint to the functor $\mathbf{Hom}(V, -)$. We have

$$\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(U, \mathbf{Hom}(V, W)) \ .$$

This is just as in the non-super case where we have that, for example linear functions on the tensor product $X \otimes Y$ of two vector spaces are equivalent to bilinear functions on the product $X \times Y$.

2. Super algebra

DEFINITION B.4. A super algebra A over the field k is a $\mathbf{Z}/2\mathbf{Z}$-graded k-module

$$A = A_0 \oplus A_1 \ ,$$

in such a way that the grading is compatible with the module structure

$$A_i A_j \subseteq A_{i+j} \ .$$

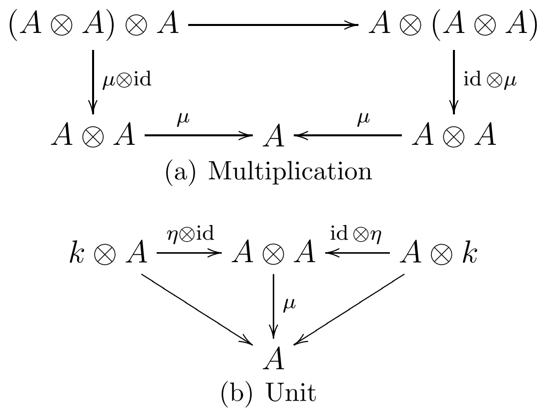
Expressed in terms of the parity we have

$$\mathrm{p}\,(xy) = \mathrm{p}\,(x) + \mathrm{p}\,(y) \ .$$

Similarly the parity extends to the (super) tensor product of super algebras.

$$\mathrm{p}\,(x \otimes y) = \mathrm{p}\,(x) + \mathrm{p}\,(y) \ .$$

Categorially, a super algebra A is a super vector space with maps $\mu : A \otimes A \to A$ and $\eta : k \to A$ such that (A, μ, η) is a monoid in the category of super vector spaces. This expresses the commuting of the two diagrammes in figure 11.



(a) Multiplication

(b) Unit

FIGURE 11: A categorial monoid

EXAMPLE 2. The polynomials $Pol(x, \theta) = \mathbf{C}[x, \theta]$ in the two variables x and θ with the relations

$$x\theta = \theta x$$
$$\theta^2 = 1$$

is a superalgebra. The variable x is even and θ is odd.

Remark on idempotent odd numbers. Let A be a supergalgebra with an odd part of dimension 2 or higher. If the odd elements are idempotent A naturally includes the complex numbers. Consider the product $\theta_i\theta_j$ with $i \neq j$. We then have

$$\begin{aligned}(\theta_a\theta_b)^1 &= \theta_a\theta_b\\ (\theta_a\theta_b)^2 &= -1\\ (\theta_a\theta_b)^3 &= -\theta_a\theta_b\\ (\theta_a\theta_b)^4 &= 1\end{aligned}$$

3. Super quantum numbers

Super quantum numbers are a natural extension of the construction of the quantum numbers.

$$\{n\}_q = \{n\} = \frac{q^{-n/2} - (-1)^n q^{n/2}}{q^{-1/2} + q^{1/2}} \tag{B.1}$$

The super quantum numbers enjoy a large number of useful algebraic properties in analogy the quantum numbers

$$\begin{aligned}\{m\} &= q^{-(n-1)/2} - q^{-(n-1)/2+1} + \cdots (-1)^{m-1} q^{(n-1)/2}\\ \{m+n\} &= q^{-m/2}\{n\} + (-1)^n q^{n/2}\{m\}\\ \{m-n\} &= q^{n/2}\{m\} - (-1)^{m-n} q^{m/2}\{n\}\\ 0 &= \{a\}\{b-c\} + (-1)^{a-c}\{b\}\{c-a\} + (-1)^{b-c}\{c\}\{a-b\}\\ \{n\} &= \frac{q^{-1}+q}{q^{-1/2}+q^{1/2}}\{n-2\} - \{n-4\}\end{aligned}$$

$$\{a\}\{b\} - \{a-1\}\{b+1\} = (-1)^{a+1}\{-a+b+1\}$$

For $n \in \mathbf{N}$ we introduce the *sq-factorial* $\{n\}_q! = \{n\}!$ by setting

$$\{n\}! = \{1\}\{2\}\cdots\{n\}\ , \quad \{0\}! := 1\ . \tag{B.3}$$

The *shifted super q-factorial* $\{a|n\}_q = \{a|n\}$ is defined as

$$\begin{aligned}\{a|0\} &= 1\\ \{a|n\} &= \{a\}\{a+1\}\{a+2\}\cdots\{a+n-1\} \quad \text{for } n \geq 1\\ \{a|-n\} &= \{a-1\}\{a-2\}\cdots\{a-n\} \quad \text{for } n \geq 1\end{aligned} \tag{B.4}$$

4. Super difference operators

The multiplication operator $\mathsf{T} = q^{\mathrm{d}_x} = q^{x\partial_x}$ is defined as in appendix A on functions of x

$$\mathsf{T}\, f(x) = f(qx)\ .$$

We define the *super difference operator* as

$$\text{(B.5)}\qquad \{\mathrm{d}_x + c\}_q = \frac{q^{-\frac{c}{2}}\,\mathsf{T}^{-\frac{1}{2}} - (-1)^{c+x\partial_x} q^{\frac{c}{2}}\,\mathsf{T}^{\frac{1}{2}}}{q^{-\frac{1}{2}} + q^{\frac{1}{2}}}$$

where $\mathrm{d}_x = x\partial_x$. The difference operator commutes with x in the following way.

$$\{\mathrm{d}_x + c\}x = x\{\mathrm{d}_x + c + 1\}\ .$$

5. Super basic hypergeometric series

In the super algebra $Pol(x,\theta) = \mathbf{C}[x,\theta]$ with $\theta^2 = 1$ consider the *super q-hypergeometric difference equation*

$$\text{(B.6)}\qquad \left(x\theta\{\mathrm{d}_x + a\}\{\mathrm{d}_x + b\} - \{\mathrm{d}_x\}\{d_x + c - 1\}\right)P(x) = 0\ .$$

This equation has two solutions parametrised by a Boolean number λ. We call the parametrised solution the *super basic hypergeometric series* and denote it by ${}_2\Pi_1(a,b;c|x,\theta;\lambda;q)$. A series representation is given by

$$\text{(B.7)}\qquad {}_2\Pi_1\left(\begin{matrix} a & b \\ & c \end{matrix}\,\middle|\, \begin{matrix} x,\theta \\ \lambda, q \end{matrix}\right) = \sum_{n\geq 0} \frac{\{a|n\}\{b|n\}}{\{c|n\}\{n\}!} x^n \theta^{\lambda+n}\ .$$

6. Super quantum Clebsch–Gordan coefficients

The tensor product representations and the irreducible representations are intertwined by the super quantum Clebsch–Gordan coefficients.

$$\mathbf{e}^l_m(l_2,l_1;\lambda) = \sum_{m_1,m_2} \left[\begin{matrix} l\lambda \\ m \end{matrix}\middle| \begin{matrix} l_2\lambda_2 & l_1\lambda_1 \\ m_2 & m_1 \end{matrix}\right]_{q|\mathfrak{s}} \mathbf{e}^{l_2}_{m_2}(\lambda_2)\otimes \mathbf{e}^{l_1}_{m_1}(\lambda_1)\ .$$

The recurrence relations are up to a sign mutatis mutandis the same as for $\mathcal{U}_q\mathbf{sl}(2)$.

$$\mathcal{D}^{\pm}(l,m)\left[\begin{matrix} l\lambda \\ m\pm 1 \end{matrix}\middle| \begin{matrix} l_2\lambda_2 & l_1\lambda_1 \\ m_2 & m_1 \end{matrix}\right]_{q|\mathfrak{s}} = q^{\frac{m_1}{2}}\mathcal{D}^{\pm}(l_2,m_2\mp 1)\left[\begin{matrix} l\lambda \\ m \end{matrix}\middle| \begin{matrix} l_2\lambda_2 & l_1\lambda_1 \\ m_2\mp 1 & m_1 \end{matrix}\right]_{q|\mathfrak{s}} +$$
$$+(-1)^{l_2+m_2+\lambda_2} q^{-\frac{m_2}{2}}\mathcal{D}^{\pm}(l_1,m_1\mp 1)\left[\begin{matrix} l\lambda \\ m \end{matrix}\middle| \begin{matrix} l_2\lambda_2 & l_1\lambda_1 \\ m_2 & m_1\mp 1 \end{matrix}\right]_{q|\mathfrak{s}}$$

For minimal m we have

$$\left[\begin{matrix} l\lambda \\ -l \end{matrix}\middle| \begin{matrix} l_2\lambda_2 & l_1\lambda_1 \\ m_2 & m_1 \end{matrix}\right]_{q|\mathfrak{s}} = (-1)^{l_2+m_2+\lambda_2} q^{\frac{l+1}{2}} \frac{\mathcal{D}^-(l_1,m_1+1)}{\mathcal{D}^-(l_2,m_2)} \left[\begin{matrix} l\lambda \\ -l \end{matrix}\middle| \begin{matrix} l_2\lambda_2 & l_1\lambda_1 \\ m_2-1 & m_1+1 \end{matrix}\right]_{q|\mathfrak{s}},$$
$$= (-1)^{l_2+m_2+1+\lambda_2} q^{-\frac{l+1}{2}} \frac{\mathcal{D}^-(l_2,m_2+1)}{\mathcal{D}^-(l_1,m_1)} \left[\begin{matrix} l\lambda \\ -l \end{matrix}\middle| \begin{matrix} l_2\lambda_2 & l_1\lambda_1 \\ m_2+1 & m_1-1 \end{matrix}\right]_{q|\mathfrak{s}}.$$

Similarly for minimal m_2

$$\begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m & -l_2 & m_1 \end{bmatrix}_{q|\mathfrak{s}} = (-1)^{\lambda_2} q^{\frac{l_2}{2}} \frac{\mathcal{D}^+(l_1, m_1-1)}{\mathcal{D}^+(l, m-1)} \begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m-1 & -l_2 & m_1-1 \end{bmatrix}_{q|\mathfrak{s}} ,$$

$$= (-1)^{\lambda_2} q^{-\frac{l_2}{2}} \frac{\mathcal{D}^+(l, m)}{\mathcal{D}^+(l_1, m_1)} \begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m+1 & -l_2 & m_1+1 \end{bmatrix}_{q|\mathfrak{s}}$$

and minimal m_1

$$\begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m & m_2 & -l_1 \end{bmatrix}_{q|\mathfrak{s}} = q^{-\frac{l_1}{2}} \frac{\mathcal{D}^+(l_2, m_2-1)}{\mathcal{D}^+(l, m-1)} \begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m-1 & m_2-1 & -l_1 \end{bmatrix}_{q|\mathfrak{s}} ,$$

$$= q^{\frac{l_1}{2}} \frac{\mathcal{D}^+(l, m)}{\mathcal{D}^+(l_2, m_2)} \begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m+1 & m_2+1 & -l_1 \end{bmatrix}_{q|\mathfrak{s}}$$

With the normalisation

$$\begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ -l & l_1-l & -l_1 \end{bmatrix}_{q|\mathfrak{s}} = 1$$

and our choice of

$$\mathcal{D}^+(l, m; \lambda) = (-1)^{l-m}\{l-m\}\beta \quad \text{and} \quad \mathcal{D}^-(l, m; \lambda) = \{l+m\}\beta ,$$

we find for the Clebsch–Gordan coefficients with one lowest vector (B.8a)

$$\begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ -l & m_2 & m_1 \end{bmatrix}_{q|\mathfrak{s}} = (-1)^{\mathcal{L}} q^{-\frac{(l_1+m_1)(l+1)}{2}} \frac{\{l_2+m_2+l_1+m_1\}!}{\{l_2+m_2\}!\{l_1+m_1\}!} ,$$

with $\mathcal{L} = (l_2+m_2+\lambda_2)(l_1+m_1) + (l_1+m_1)(l_1+m_1+1)/2$

(B.8b)

$$\begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m & -l_2 & m_1 \end{bmatrix}_{q|\mathfrak{s}} = (-1)^{\mathcal{L}'} q^{\frac{l_2(m-1)+(l-l_1)(l+1)}{2}} \frac{\{l-m\}!\{l_1-m_1+l+m\}!}{\{l_1-m_1\}!\{2l\}!} ,$$

with $\mathcal{L}' = (l_1+l_2+m)\lambda_2 + (l_1+l_2-l)(l_1+l_2-l+1)/2$

(B.8c)

$$\begin{bmatrix} l\lambda & l_2\lambda_2 & l_1\lambda_1 \\ m & m_2 & -l_1 \end{bmatrix}_{q|\mathfrak{s}} = (-1)^{(l_2+m_2+1)(l+m)} q^{\frac{(l+m)l_1}{2}} \frac{\{l-m\}!\{l_2-m_2+l+m\}!}{\{l_2-m_2\}!\{2l\}!} .$$

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