Effective Action of Heterotic Compactification on $K3$ with non-trivial Gauge Bundles

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Abstract

In this thesis we study the heterotic string compactified on $K3$ with non-trivial gauge bundles. We focus on two backgrounds, the well-known standard embedding and abelian line bundles. Using a Kaluza-Klein reduction, the six-dimensional effective action is computed up to terms of order α'^2 with special attention on the hypermultiplet sector. We compute the moduli dependent couplings of the matter fields and analyze the geometry of the hyperscalar sigma model. Moreover, we prove the consistency with six-dimensional supergravity and derive the appropriate D-term scalar potential. For the line bundle backgrounds we show that the gauge flux stabilizes some geometrical moduli and renders some abelian vector multiplets massive.

Zusammenfassung

In dieser Dissertation untersuchen wir die Kompaktifizierung des heterotischen Strings auf $K3$ mit nicht-trivialen Eichbündeln. Wir konzentrieren uns auf zwei Hintergründe, die wohlbekannte Standard-Einbettung und abelsche Linienbündel. Durch Anwendung einer Kaluza-Klein Reduktion wird die sechsdimensionale effektive Wirkung bis zu Termen der Ordnung α'^2 berechnet, wobei besonderes Augenmerk auf den Hypermultiplet Sektor gelegt wird. Wir berechnen die Moduli abhängigen Kopplungen der Materiefelder and analysieren die Geometrie des Hyperskalaren Sigma Modells. Weiterhin beweisen wir die Konsistenz mit sechsdimensionaler Supergravitation und leiten das entsprechende D-Term skalare Potential her. Bezüglich der Linienbündelhintergründe zeigen wir, dass der Eichfluss einige der geometrischen Moduli stabilisiert und einige der abelschen Vektormultiplets massiv werden lässt.

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Contents

Chapter 1

Introduction

> The reason why particle physics appears to us more fundamental than other branches of physics is because it *is* more fundamental.
>
> - Steven Weinberg -

1.1 Road to the standard model

Over the course of the last century our understanding of the fundamental constituents of the physical world has evolved dramatically. From the structure of atoms to sub-nuclear particles and forces, nature has revealed its rich and beautiful microcosmic structure. The ongoing adventure of theoretical physics is to cast the experimental discoveries into a consistent picture and to predict future results. Understanding of the microcosm was driven by two major achievements of the twentieth century, relativity and quantum theory. Apart from well verified quantitative predictions, both theories changed our view of the world radically. Relativity presents to us a geometrized unified picture of space, time and gravitation [1]. At the same time, it introduces an observer dependence of various notions which before were believed to have objective meaning. Quantum theory, on the other hand, delivers a quasi-picture of the microscopic world. The discreteness of physical quantities at small scales solved many paradoxes of classical physics [2]. At the same time, the theory introduces new notions that appear to be paradoxical from a classical point of view, e.g. entanglement, non-separability and contextuality. The act of observation is so intimately linked to the very basics of the theory, that its theoretical terms seem to have only epistemological character or even that the notion of reality loses its objective meaning [3, 4].

Unfortunately, only the special theory of relativity could successfully be unified with quantum theory. This amounts, at first, to incorporate the symmetry of Minkowskian space-time into the quantum theory. Elementary (free) particles are understood and classified as irreducible representations of the Poincaré symmetry group [5]. To save the causal structure in the quantum regime, the interactions are required to be local, leading to a description in terms of local fields [6, 7]. In these fields the quantum non-commutative structure is realized, bearing the name quantum field theory [8, 9]. It was soon realized that electromagnetic interactions of electrons could be described in this framework and led to extraordinarily good predictions, e.g. the electron's magnetic moment. In particular, quantum field theory was, and is, powerful in application to high energy particle collisions. However, it was troubled by some technical issues. Calculations of scattering amplitudes were, and mostly are, limited to an asymptotic series expansion in a small coupling parameter, which was shown to be non-convergent. Even worse, also the individual series terms seemed to give meaningless answers at first sight. The infrared divergencies of massless particles could be solved heuristically by a redefinition of measurable asymptotic states and the ultra-violet infinities could be solved from first principles by clarifying the mathematical structure of quantum field theory [10]. However, the latter inevitably introduces an ambiguity of the scattering matrix encoded in undetermined parameters. If the number of these parameters stays finite in higher orders in perturbation theory, they can be eliminated by an (energy dependent) redefinition of the Lagrangian. Such a theory is called renormalizable and from this perspective, renormalization interpolates between perturbative expansions at different energy and parameter values.[1] Remarkably, a different approach to quantum field theory, coming from condensed matter physics, led to a more intuitive interpretation of renormalization and changed our view on the physical meaning of coupling constants. The concept of effective field theory gives up the ambition of renormalizability and fundamental fields, but gains applicability over a wide range of phenomena [11]. Such a theory has a finite range of validity bounded from above by a 'cutoff mass scale' with additional massive degrees of freedom appearing above this threshold.[2] Conversely, the effects of a theory in its low energy regime can be obtained by integrating out the massive degrees of freedom in the (Euclidean) path integral. The (infinite parameter) extension of an effective field theory to arbitrarily high mass scales is called an ultra-violet completion of the theory. The value of a coupling constant generically depends on the given energy and cutoff scale. Every theory that is not scale invariant contains renormalized running couplings which are interpreted as physically changing their strength. Despite this modern change of paradigm, the standard model of particle physics was originally constructed to match the old premise of a renormalizable quantum field theory.

In the mid twentieth century the plethora of baryons and mesons discovered in high energy collision experiments was a drawback for quantum field theory since a description in terms of hundreds of elementary fields was very unconvincing. Moreover, the experiments revealed that the sub-nuclear interactions are strongly coupled,

[1]Here the center-of-mass energy is meant which is a Lorentz-invariant quantity.

[2]Strictly speaking, a mass scale cutoff only works consistently in Euclidean momentum space where it defines a compact region.

rendering the coupling expansion series useless. Only at higher energies a weakly coupled behavior was observed. Both of these issues remained obscure until some ingenious symmetry arguments brought the chaotic particle zoo into order. With the application of representation theory of Lie algebras the particles were collected into multiplets. Their structure was deducible from the assumption of a set of elementary fields called quarks [12]. The symmetry group $SU(3)$ underlying the multiplets also led to the identification of the correct interactions. They are encoded by a Yang-Mills theory which is the natural non-abelian generalization of electromagnetism, considered as an abelian gauge theory. Classical Yang-Mills theory can be understood well in terms of geometrical quantities, e.g. vector bundles and curved connections on them. In this description there is even a strong similarity to gravity, considered as a gauge theory with local gauge group $SO(3,1)$. However, the quantized version of Yang-Mills theory became a much tougher challenge and a rigorous treatment remains unsolved to this day [13]. Nevertheless, its renormalizability could be proven and it was possible to derive the observed behavior of the running coupling of the strong interactions from the renormalization group [14]. The weak coupling at high energies is referred to as asymptotic freedom, whereas the strong coupling at low energies is referred to as the confining phase. In a parallel development the electromagnetic and weak interactions, e.g. governing the radioactive beta decay, were found to be governed by a Yang-Mills theory with gauge group $SU(2) \times U(1)$. However, in contrast to the strong interactions, the electro-weak sector exhibits properties which seemed quite unfamiliar in the first place. Most importantly, the interactions depend on particle helicities in such a way that observables are not invariant under spatial reflections, i.e. the theory is said to be parity violating. Accordingly, different gauge quantum numbers are assigned to left- and right-handed components of matter spinor fields, which is the characteristic chiral structure of the electro-weak interaction [15]. Second, the short range of the weak force could only be understood as being generated by massive gauge bosons. This phenomenon however seemed to be in conflict with the gauge principle, renormalizability and unitarity of the theory. It took another brilliant idea to realize that the masses could be dynamically generated by a non-symmetric vacuum state of an additional electro-weakly charged scalar field. This phenomenon is called the Higgs-effect, or spontaneous gauge symmetry breaking [16]. It guarantees renormalizability and generates masses for the weak gauge bosons, the coupled matter fields and for the scalar itself. As of this writing, the very existence of the massive Higgs boson has been verified at the Large Hadron Collider.

Despite of the great predictive success of the standard model of particle physics there are good reasons to believe that it only presents an incomplete picture of nature. It contains a number of about twenty dimensionless parameters which had to be determined by experiment. The reductionist principle, also called unification principle by physicists, suggests that one should search for a better theory that reduces this number. Unification has ever been a guiding principle in physics, aiming to explain a variety of phenomena as huge as possible via a number of parameters and equations as small as possible. Unfortunately, most modern unifying theories are not reductions but extensions of the standard model. In 'grand unified theories' the standard model gauge group is taken to be a low energy phenomenon of a unified (simple) gauge group at high energy scales. While the number of gauge couplings is reduced to one at best, the field content and the number of non-gauge (Yukawa-)couplings is larger than in the standard model. This is not a problem in principle since we do not know how many heavy particles still hide from our current observations. From this perspective, it is most natural to consider the standard model as an effective field theory. Its cutoff is to be defined at some point above the current experimentally accessible mass scale with an upper bound believed to be given by the Planck mass $m_{\rm PL} \sim 10^{19}$GeV. At this mass scale, thought experiments suggest that quantum effects get affected by black hole formation marking the regime of (the yet unknown) quantum gravity. In any case, the effective field theory philosophy brings in some problems for the standard model. Numerical parameter values, like the Higgs mass, are considered 'unnatural' as they receive quantum corrections which are large compared to their measured values. One could argue that the asymptotic convergence of the perturbative expansion simply fails in these cases. However, since no exact calculations apart from perturbation theory are available, one searches for extensions of the standard model which restore the power of perturbation theory.[3] Also, the large mass gap between the scale of electro-weak symmetry breaking and the Planck scale is often considered as an unnatural hierarchy.[4] The most popular standard model extensions refer to supersymmetry [17]. Supersymmetry is a conjectured extension of the Poincaré group, in fact a unique one circumventing the Coleman-Mandula theorem with fermionic symmetry generators [18, 19]. It follows that particles come in supermultiplets with equal number of bosonic and fermionic degrees of freedom and equal masses. This is empirically falsified but our world can still be realized as a non-supersymmetric vacuum state of a supersymmetric theory. Supersymmetry improves the power of perturbation theory due to loop cancellations of bosonic and fermionic super-partners and thus stabilizing quantum corrections of the Higgs mass to the moderate logarithmic behavior. Under the further assumption of the discrete R-parity symmetry, it also provides candidates for dark matter particles whose existence is suggested by astrophysical observations. Finally, supersymmetric grand unified theories suggest that, via renormalization group running, the three gauge couplings numerically merge at a mass scale of about 10^{16}GeV. However, at the other side of the coin, supersymmetric field theories introduce an additional large number of fields and parameters.

[3] Note that this reasoning is opposite to the reductionist principle. One should be aware of the danger of introducing a theory of epicycles to stay within possibly inappropriate calculative methods.

[4] The author does not share most concerns about naturalness of parameter values. They should be considered as environmental parameters which do not ask for a fundamental explanation. In a multiverse, as suggested by string theory and eternal inflation scenarios we can apply the weak anthropic principle.

The most serious handicap of the standard model of particle physics is the lack of containing gravity. There are, to this date, no empirical indications of a quantum theory of gravity, since the gravitational interaction strength is too small to influence particle scatterings at available energies. Even classical gravitational waves have not been observed yet. Nevertheless, it is believed that the quantum structure is a universal feature of our world, affecting gravity at the Planck scale as mentioned above. There exists a generalization of quantum field theory on curved space-times and some remarkable conclusions, e.g. Hawking radiation, Unruh radiation and black hole entropy, could be drawn from that setting [20–22]. However, it was shown that the coupling of classical gravity to quantum matter leads to inconsistencies, rendering a quantum version of gravity inevitable. But the very construction of a dynamical quantum gravity seems to be impossible in the canonical framework. From the particle physicist's point of view, a canonically quantized gravity, containing the graviton as an excitation of the linearized Einstein equations, yields a non-renormalizable quantum field theory. Therefore, it is believed that also general relativity, defined by the Einstein-Hilbert action, is an effective field theory which has to be supplemented by additional interactions which render the scattering amplitudes renormalizable or even finite.[5] However, simple modifications of the gravity action either yield non-renormalizability or non-unitarity. Hence, it was suggested that it may be inconsistent to consider gravity in isolation of matter. The problem of quantum gravity may be solved by finding the correct unification with the (standard model) matter fields. This unification is far from unique, however, if we again refer to symmetry arguments, there exists a preferred class of theories called supergravities [17, 23]. These field theories exhibit a local version of supersymmetry and always contain a graviton field together with a gravitino being the fermionic gauge field of the local supersymmetry. On the technical side, theories with extended supersymmetry gained interest because they are restrictive in the structure of couplings. The more supersymmetries are present the more rigid is the theory's interaction structure. For the maximal supergravities the interaction structure and couplings are uniquely fixed and scalar target spaces are symmetric (pseudo-)Riemannian manifolds.[6] From the phenomenological point of view, minimal supergravities ($\mathcal{N} = 1$ in four dimensions) are most relevant because only these allow chiral matter couplings. In this case, there is an infinite set of supergravities specified by the gauge group, the matter spectrum and couplings defined by a Kähler potential, a superpotential and a gauge kinetic function. Considered as quantum theories, the supergravities are non-renormalizable and hence have to be considered as effective field theories as well.[7] Remarkably, their

[5]There exists an alternative theory stating that gravity may have an ultra-violet fixed point of the renormalization group. It is called the 'asymptotic safety scenario'.

[6]Maximal supersymmetry comes with 32 real supercharges. Its smallest massless supermultiplet contains fields up to helicity two, which is the largest value for consistent gauge interactions.

[7]There are conjectures that maximal $\mathcal{N} = 8$ supergravity in four dimensions yields finite loop amplitudes at all orders in perturbation theory.

ultra-violet completion was found to be related to a totally different set of ideas and a more sophisticated framework called superstring theory.

1.2 String theory

The paradigm of string theory is the existence of one-dimensional extended objects constituting the fundamental degrees of freedom of our world. Their characteristic length l_s is assumed to be so small that current experiments are not able to resolve their extended structure directly. The classical dynamics of a string in a background spacetime M, sweeping out a two-dimensional world-sheet Σ, is defined by the Polyakov action [24, 25]

$$S = \tfrac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-h} h^{ab} \partial_a X^M \partial_b X^N G_{MN}(X) \ . \tag{1.2.1}$$

Here $X : \Sigma \to M$ is the embedding of the world-sheet into M, G_{MN} is the background metric of M and h_{ab} is the induced metric on Σ.[8] For a closed string the world-sheet has the topology of a cylinder and for an open string that of a strip. Classical solutions which minimize S correspond to world-sheets being minimal surfaces. The quantum structure of string theory is given by first quantizing the vibrational modes of the world-sheet which can be considered as a generalization of the 'first quantization' of a relativistic point particle. In flat Minkowskian space this quantization is straightforward but on a curved background (1.2.1) describes a non-linear sigma model. In the former case the spectrum of the system can be shown to consist of an infinite tower of modes of increasing mass. Therefore, in contrast to the point particle, a single string exhibits an unbounded number of degrees of freedom. Phenomenologically most relevant are the states of zero mass, which are degenerate and can be grouped into a finite number of irreducible representations of the ambient Lorentz group. The corresponding wave functions consist, amongst others, of a scalar Φ, the dilaton, an antisymmetric tensor field B_{MN}, also called Kalb-Ramond field, and a symmetric tensor field G_{MN}. It is one of string theory's celebrated results that G_{MN} can be identified with the graviton, the linearized Einstein equations arising from conditions of conformal symmetry. It has become clear that a 'realistic' spectrum of massless states, including fermions, can only be achieved from superstrings, i.e. supersymmetric extensions of the world-sheet field theory [26, 27]. However, the supersymmetric extensions are not unique. Instead, one finds five (apparently different) superstring theories, labeled type I, IIA, IIB and two heterotic superstrings [28].[9]

[8] Sometimes one allows also singular maps $X : \Sigma \to M$ which are no embeddings in the mathematical sense.

[9] Interestingly, the construction of the two heterotic theories was guided by the idea to find superstrings which reproduce the ten-dimensional $\mathcal{N} = 1$ supergravities not connected to string theory so far.

Moreover, the quantum consistency of these theories forces the ambient spacetime to be ten-dimensional. It can be shown that the spectra of massless states are also supersymmetric with respect to the ambient spacetime. The type IIA theory enjoys $\mathcal{N} = (1,1)$ supersymmetry, the type IIB theory $\mathcal{N} = (0,2)$ supersymmetry and the heterotic and type I theories have $\mathcal{N} = (0,1)$ supersymmetry in ten dimensions.

A second quantized theory, i.e. string field theory, is not fully established to this date [29]. Nevertheless, string scattering amplitudes have been constructed in a consistent way from the path integral approach. The loop order of these amplitudes is given by the topological genus of the world-sheet embedding X and every loop carries one power of the string coupling constant g_s. Together with the world-sheet sigma model we have a string perturbation expansion in two small parameters, $\alpha' = l_s^2$ and g_s. α' corrections encode the extended 'stringiness' and higher genus corrections encode the quantum properties of string theory. Remarkably, loop diagrams of the string perturbation theory are finite and it is believed that this holds for arbitrarily high orders. Technically, the extended world-sheet acts as a natural smearing function for interactions, yielding finite loop integrals. Due to this behavior, it is believed that string theory defines an ultra-violet complete scattering theory for gravitons, i.e. a consistent quantum gravity.[10] From the amplitudes and their Feynman rules not only the S-matrix can be calculated but also crucial information about classical background solutions can be gained. This is achieved by applying the background field method to the world-sheet theory, yielding the modified action

$$S = \tfrac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \Big(\sqrt{-h} h^{ab} G_{MN} + \varepsilon^{ab} B_{MN}\Big) \partial_a X^M \partial_b X^N + \tfrac{1}{2\pi} \int_\Sigma d^2\sigma \sqrt{-h} R \Phi + \dots\ , \tag{1.2.2}$$

where $\varepsilon_{12} = -\varepsilon_{21} = 1$ is the antisymmetric tensor and R is the Ricci-scalar of the world-sheet metric h. For a constant dilaton, the last term in (1.2.2) is proportional to a topological invariant, the Euler characteristic χ of Σ. As a consequence, for a fixed world-sheet topology the path integral is weighted by a factor $\exp(-\Phi\chi)$ such that one can identify

$$g_s = \exp(\Phi)\ . \tag{1.2.3}$$

Thus, the string coupling is actually a dynamical field leaving l_s to be the only undetermined fundamental parameter of string theory.

The absence of conformal anomalies of the world-sheet theory and the vanishing of tadpole amplitudes lead to constraints which turn out to be dynamical equations for the massless background fields G_{MN}, B_{MN}, Φ, etc. The low energy string effective action is defined as the field theory action reproducing these dynamical equations

[10] A common critics about string theory is the lack of background independence which should be manifest in quantum gravity. Indeed, string quantization is usually accomplished only relative to fixed backgrounds. However, from the string point of view, general relativity is just an emergent low energy theory with (active) diffeomorphisms as an accidental gauge symmetry.

[30, 31]. For all (closed) superstrings the string effective action contains a first sector which reads

$$S_1 = \frac{1}{2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\Big(R + 4\partial_M\Phi\partial^M\Phi - \frac{1}{12}H_{MNP}H^{MNP}\Big)\ . \tag{1.2.4}$$

Here R is the ten-dimensional Ricci-scalar and H is the field strength of the Kalb-Ramond field. Depending on the type of superstring, additional bosonic and fermionic fields appear in the effective action which can be interpreted as a point particle approximation of string theory. Remarkably, the effective action in each case describes a supergravity theory in ten dimensions. In the type II superstrings all massless modes fall into the $\mathcal{N}=2$ gravity-supermultiplet. In the type I and heterotic superstring the massless modes exhibit an $\mathcal{N}=1$ gravity-supermultiplet and a Yang-Mills supermultiplet. At first sight, it seemed that the latter superstrings were plagued by gravitational and gauge anomalies which would render the theories inconsistent [32]. Amazingly, it could be shown that the anomalies disappear via the Green-Schwarz mechanism only under a very specific restriction: The Yang-Mills gauge group must be chosen to be either $SO(32)$ or $E_8\times E_8$ [33].[11]

In the mid nineties an amazing conjecture emerged, stating that the different superstrings are connected by a net of duality relations [34–40]. A duality is not a mere symmetry of one theory but rather a map between apparently different theories which describe the same physics in (possibly) different parameter regimes by different 'fundamental' degrees of freedom. It extends the paradigm of effective field theory, where the notion of fundamental and composite fields also depends on the parameter regime. Examples of duality relations in field theory are the electro-magnetic duality of electromagnetism (with magnetic monopoles) and its supersymmetric, non-Abelian generalization called Seiberg duality. In string theory there exist strong-weak coupling dualities (S-dualities), large-small scale dualities (T-dualities) and certain mixtures called U-dualities. In string theory it can happen that the fundamental degrees of freedom of one framework are solitonic objects of the dual framework, e.g. D-branes [41]. Since the superstring theories are defined only perturbatively, the dualities cannot be proven rigorously. However, there is strong evidence in form of identical moduli spaces, global symmetry groups and BPS-spectra between the conjectured dual theories. The hidden nonperturbative physics behind the net of string theories is termed M-theory and its low energy approximation is known to be the unique eleven-dimensional supergravity [37, 42].

Besides the quest for understanding the true nature of what string theory actually is, there exists the branch of string phenomenology [45]. In this branch the connection of string theory to the empirically known particle physics and cosmology is studied. It is the ultimate goal to identify the (supersymmetric) standard model of particle physics inside the framework of string theory. In contrast to the early expectations,

[11] The gauge group $E_8\times E_8$ is possible only for the heterotic superstring.

this problem may not have a unique solution. In any case, it demands an explanation why out of the necessary ten spacetime dimensions only four are visible. There are two major possible scenarios: First, the visible world is embedded inside a ten-dimensional large ambient space and our ways of interaction are constrained to be inside this subspace [43, 44]. Second, the six invisible dimensions have taken a compact 'curled-up' vacuum state with length scales so tiny that our ways of interaction cannot resolve them. This idea was first taken up in the early twentieth century by Kaluza and Klein [46, 47]. In this thesis we will be concerned with this latter possibility known under the name string compactifications.

1.3 Calabi-Yau compactifications

Conformal invariance of the string world-sheet theory (to lowest order in α') restricts the compactified dimensions to constitute a Ricci-flat manifold Y. Additionally, supersymmetry of the world-sheet theory requires Y to be a complex Kähler manifold [49]. A necessary condition for the existence of a Kähler metric is that the holonomy group is reduced to $U(n)$ and Ricci-flatness further requires that the holonomy group is at least reduced to $SU(n)$, where n is the complex dimension of Y. These conditions together define the so-called Calabi-Yau manifolds [48]. Hence, the simplest space-time vacuum solutions are direct product manifolds of the form

$$\mathbb{M}^{9-d,1} \times Y^d \ . \tag{1.3.1}$$

For the heterotic string which is not fully supersymmetric on the world-sheet an analogue conclusion can be drawn by considering (spacetime-)supersymmetric solutions of the supergravity approximation [49]. In particular, the compactification (1.3.1) should be a solution of the gravitational equations of motion. Such background solutions are generically not invariant under the supersymmetry transformations of the theory. In this case, supersymmetry is spontaneously broken similar to the Higgs mechanism. A solution is (partially) supersymmetric if there exist (some set of) supersymmetry parameters which leave the solution invariant. There are phenomenological and technical reasons to consider compactifications which leave some supersymmetry unbroken at the TeV scale. For example, it can explain the existence of rather light scalar fields (like the Higgs boson) by protecting their mass from large quantum corrections.[12] Technically, the equations for unbroken supersymmetry are usually first order differential equations and hence, easier to solve than the full equations of motion.

[12]From a top down string theoretic point of view, field theory states are called light, if they have a mass much smaller than the Planck-scale or the compactification scale. Perturbatively, they arise from the massless string excitations.

A solution is invariant under supersymmetry if the infinitesimal transformations of the background fields vanish. They generically take the form

$$\delta_\epsilon \langle B \rangle \approx \langle F \rangle \epsilon \ , \qquad \delta_\epsilon \langle F \rangle \approx \langle B \rangle \epsilon \ , \tag{1.3.2}$$

where B stands for the bosonic fields, F for the fermionic fields of the theory and ϵ is a supersymmetry parameter spinor. Fermionic background solutions $\langle F \rangle \neq 0$ are not allowed, since coherent quantum states of fermionic degrees of freedom do not exist. Hence, the left equation in (1.3.2) gives no obstruction to supersymmetric solutions. The variation of the fermionic fields vanish only under specific conditions which, in particular, determine the shape of the compact manifold. For example, the variation of the gravitino generically reads

$$\delta_\epsilon \psi_M \approx \nabla_M \epsilon + \sum_i F^i_M \epsilon \ , \tag{1.3.3}$$

where F^i_M are certain gamma matrix contractions of the tensor field strengths F^i of the theory. In the type II theories these are the Kalb-Ramond field H and the various Ramond-Ramond fields. In the heterotic theories only the Kalb-Ramond field appears at this place. Clearly, Lorentz symmetry of the lower dimensional theory on $\mathbb{M}^{1,9-d}$ requires that non-zero tensor fields only exist along the compactified dimensions or are proportional to the volume form of the extended spacetime. In the simplest case where $\langle H \rangle = \langle F^i \rangle = 0$ along Y^d (1.3.3) requires the existence of a covariantly constant spinor on Y^d. This requires the holonomy group of the compact manifold to be maximally $SU(d/2)$, which corresponds to Calabi-Yau manifolds. If the holonomy group is even smaller, several constant spinors exist and, roughly speaking, each spinor gives rise to an unbroken supersymmetry on $\mathbb{M}^{1,9-d}$. Maximal unbroken supersymmetry can always be achieved by compactifications on tori.

In this thesis, we consider compactifications of the heterotic string which has historically been the first possibility to connect string theory with particle physics [49–52]. In contrast to the type II case, the perturbative heterotic string has an intrinsic gauge symmetry involving the gauge groups $SO(32)$ or $E_8 \times E_8$.[13] The unbroken gauge group in four dimensions is a subgroup thereof determined by the choice of a principal G-bundle over the compactified space. An $\mathcal{N} = 1$ supersymmetric theory with a chiral massless spectrum in four dimensions is obtained by compactifying on Calabi-Yau threefolds Y, appropriate $\mathbb{Z}_N$ orbifolds [55] or more abstractly, two-dimensional $(0, 2)$ superconformal field theories. The relation between orbifolds and smooth Calabi-Yau compactifications (with line bundles) has been studied in refs. [56–63]. Focusing on the smooth Calabi-Yau case, string phenomenological conclusions are derivable most reliably under the following assumptions: Characteristic radii of the Calabi-Yau be much larger than the string length l_s, and the curvature of Y be

[13] In the type II theories non-abelian gauge symmetries are only realized at localized solitonic objects as D-branes or at geometrical singularities of the compactified space [53, 54].

small compared to the string scale. In this case, a Kaluza-Klein reduction can be applied to the ten-dimensional heterotic supergravity approximation. Let Ψ be some ten-dimensional field satisfying the (free) equation of motion

$$D_{10}\Psi = 0 \ , \tag{1.3.4}$$

where D_{10} is some ten-dimensional differential operator. In the simplest case, this operator splits as $D_{10} = D_4 + D_6$ with D_6 being a differential operator on Y. Solutions are given by a product ansatz $\Psi(x, y) = \psi(x) \otimes f(y)$, where x are coordinates of $\mathbb{M}^{1,3}$ and y are coordinates on Y. Since Y is compact, the spectrum of D_6 is discrete with eigenfunctions f_n satisfying

$$D_6 f_n(y) = \lambda_n f_n(y) \ . \tag{1.3.5}$$

Expanding Ψ in these eigenfunctions, $\Psi(x, y) = \sum_n \psi_n(x) \otimes f_n(y)$, it follows that the eigenvalues $\lambda_n = m_n^2$ appear as mass terms in the four-dimensional equations of motion

$$(D_4 + \lambda_n)\psi_n(x) = 0 \ . \tag{1.3.6}$$

Hence, there exist massless fields in four dimensions if D_6 has zero-eigenvalues (with some multiplicity). For the Laplace operator $D_6 = \Delta_6$ and a simple circle compactification the eigenvalues scale like $m_n^2 = n^2/R^2$ where R is the circle radius. Experimental bounds imply that the characteristic radii are so small that even the lightest non-zero mass fields are out of current reach of observation. Therefore, in string phenomenology the Kaluza-Klein spectrum is truncated to the massless modes only. It turns out that in many cases the massless modes can be interpreted as deformations of a background structure, preserving some special condition. The four dimensional effective action then generically depends on topological invariants of Y and coupling functions which may depend on the geometric moduli [64–70]. The chiral spectrum, for example, is determined by the geometry of the chosen gauge bundle over the Calabi-Yau. In the simplest construction (the standard embedding) the net number of chiral families is given by an index theorem to be half of the Euler characteristic $\chi(Y)$. Hence, the solution of connecting string theory with particle physics is the problem of finding the correct vacuum solution of an a priori unique theory. The highly degenerate set of vacua is called the 'string landscape' and its full classification in terms of topological data of the Calabi-Yau, the gauge bundle and possibly flux parameters remains to be completed.[14]

Besides the above discrete degeneracy, it is well known that compact Calabi-Yau manifolds come in smooth families in which the metric varies, while preserving Ricci-flatness. As a consequence, these metric deformations show up in the lower dimensional theory as massless free scalar fields, the so called geometrical moduli. Though explicit metrics of compact Calabi-Yau manifolds are not known, the metric on the space of metric deformations can be determined (at least locally). For Calabi-Yau

[14] See for example [71] for a recent review.

manifolds the space of Ricci-flat metrics can be expressed, via algebraic geometry, as the space of complex structure deformations times the space of deformations of the Kähler class [72]. Phenomenologically the moduli are highly problematic since their number can be rather large. They would mediate fifth forces which are empirically strongly disfavored. Moreover, most coupling functions of other light fields depend on these moduli, in particular the overall volume modulus, spoiling the phenomenological predictions. To make contact with the empirical particle physics it is of great importance to find mechanisms which give mass to these moduli via generating a non-trivial scalar potential. This program is termed moduli stabilization. It was early on realized that quantum corrections of the (supersymmetrically broken) effective theory can in principle give mass to the moduli. However, these corrections are either trustful but insufficient or they are sufficiently big but lead away from the perturbative regime [73]. Therefore, most research has concentrated on tree-level and non-perturbative contributions to the scalar potential. In fact, it has been found that the geometrical moduli can partly be stabilized already at the classical level by so-called flux compactifications.[15] These are vacuum solutions where (some of) the tensor fields strengths H, F^i take non-zero values along Y^d. The equations of motion restrict the tensor fields to be harmonic, and hence, they can be described by cohomology classes

$$F_p \in H^p(Y, \mathbb{Z}) \ , \qquad \int_{\Gamma^I} F_p = m^I \ , \quad m^I \in \mathbb{Z} \ . \tag{1.3.7}$$

The fluxes F_p are quantized, i.e. taking integer values when integrated over integral p-cycles Γ^I. Whereas in type II theories the Ramond-Ramond fields offer a variety of flux configurations, in the heterotic theories only the Kalb-Ramond field is available. If there is a $U(1)$ gauge bundle involved, we can also speak of a gauge flux. In the general non-Abelian case, however, the Yang-Mills field strength plays a role quite different from the other tensor fields, as we will explain later. Unfortunately, most flux compactifications exclude Calabi-Yau manifolds as a base space due to the non-trivial back-reaction onto the metric. From (1.3.3) it follows that supersymmetry requires a global spinor constant with respect to a generalized, torsionful connection $\tilde{\nabla} = \nabla + \Sigma_i F^i$. Hence, the frame bundle of Y^d has reduced structure group $SU(d/2)$. However, the almost complex structure, constructed from spinor bilinears, is no longer ∇-constant and hence not compatible with the metric. Depending on the torsion class of the generalized connection one looses the Kähler property or even the complex structure of the compactified space. In the heterotic case this back-reaction is particularly strong and given by the Strominger equations [76]. On the other hand, it became clear recently that for heterotic compactifications the naïve counting of geometrical moduli has to be replaced by a more sophisticated analysis taking into account the gauge bundle. Roughly speaking, the equations of the gauge bundle depend on the complex structure in such a way that the moduli space of the latter is

[15] See for example [74] and [75].

smaller than expected and the Kähler moduli space gets bounded by domain walls [77, 78].

1.4 The heterotic string on $K3$

This thesis was partly motivated by the analysis of heterotic $T^6/\mathbb{Z}_6$ orbifolds which can be considered as (anisotropic) two-step compactifications, first on $T^4/\mathbb{Z}_3$ and then extending to $(T^4/\mathbb{Z}_3 \times T^2)/\mathbb{Z}_2$ [87–90]. In the intermediate step, $T^4/\mathbb{Z}_3$ can be identified with a singular orbifold limit of the four-dimensional Calabi-Yau manifold $K3$, i.e. a special point in the geometric moduli space of $K3$. While on the (flat) orbifold the massless spectrum and effective action can be derived from the string world-sheet theory, on the smooth Calabi-Yau we use the supergravity approximation from the beginning. Focusing on this intermediate step, we consider in this thesis the heterotic string theory compactified on

$$M = \mathbb{M}^{5,1} \times K3 \ , \tag{1.4.1}$$

where $\mathbb{M}^{5,1}$ is the six-dimensional Minkowski spacetime and $K3$ is the four-dimensional Calabi-Yau. The resulting six-dimensional theory has the minimal amount of eight supercharges, corresponding to minimal $\mathcal{N} = 1$ supersymmetry.[16] The main goal is to derive the effective action via a Kaluza-Klein reduction and to focus on the hypermultiplet sector, where the matter fields reside. For this purpose, we pay special attention to the non-trivial gauge bundle, forced upon us by the heterotic Bianchi identity (2.1.4) to have a consistent background. In particular, we derive in detail the matter fields and bundle moduli from deformations of the gauge connection, which has, to our knowledge, not been discussed in the physics literature. Since the effective action sensitively depends on the discrete choice of the gauge bundle topology we cannot give a model-independent answer. Instead, we focus on two prominent sub-classes of gauge bundles embedded in $E_8 \times E_8$: We discuss the well-known standard embedding of the gauge bundle into the tangent bundle and backgrounds with $U(1)$ line bundles. The dimensional reduction of the heterotic string effective action has been studied for Calabi-Yau three-folds in [65, 91, 92] and as truncations of torus compactifications in [93, 94]. The dimensional reduction on $K3$ respectively $K3 \times T^2$ was performed only at leading order in α' and excluding the matter fields in [79–85]. In this thesis we extend their analysis to terms up to order α'^2 including the matter couplings descending from the Chern-Simons three-forms. It is well known that the hypermultiplet sector stays unchanged when the theory compactified on $K3$ is further compactified on $K3 \times T^2$, yielding an $\mathcal{N} = 2$ supersymmetric theory in four

[16] In some references the minimal supersymmetry in six dimensions is denoted as $\mathcal{N} = (1,0)$ to emphasize its chiral structure, or even as $\mathcal{N} = 2$ due to its similarity to the corresponding extended supersymmetry in four dimensions.

dimensions.[17] Therefore, our results can be seen as a step to better understand the hypermultiplet sector of four-dimensional $\mathcal{N} = 2$ compactifications. Despite they are not ultimately relevant for phenomenology, $\mathcal{N} = 2$ supergravities have been investigated intensely in the last decade due to their rich mathematical structure.[18] A crucial feature are their moduli spaces, i.e. the target spaces of scalar fields. They exhibit an interesting geometrical structure, not as rigid as for ≥ 16 supercharges and not as arbitrary as for four supercharges. For $\mathcal{N} = 2$ theories in four dimensions the moduli space factorizes as

$$\mathcal{M} = \mathcal{M}_V \times \mathcal{M}_H \ , \tag{1.4.2}$$

where $\mathcal{M}_V$ is the moduli space of vector multiplets and $\mathcal{M}_H$ is the moduli space of hypermultiplets. The vector multiplet moduli space of heterotic compactifications on $K3 \times T^2$ is known to be (at leading order in g_s) [96]

$$\mathcal{M}_V = \frac{SU(1,1)}{U(1)} \times \frac{SO(2, n_V - 1)}{SO(2) \times SO(n_V - 1)} \ , \tag{1.4.3}$$

where the first factor is spanned by the axion-dilaton. We will solely be concerned in this thesis with $\mathcal{M}_H$ which is governed by quaternionic-Kähler geometry [97]. At order α'^0 the space $\mathcal{M}_H$ of neutral moduli is known to be [98]

$$\mathcal{M}_H = \frac{SO(4,20)}{SO(4) \times SO(20)} \ , \tag{1.4.4}$$

spanned by the $K3$ moduli and B-field moduli. The moduli spaces are of central importance in establishing the duality relations of string theory. Since the string effective actions are modified if higher genus (i.e. g_s) and higher order α' corrections are taken into account, also the compactified theory's moduli spaces are subject to these 'quantum corrections'. It is well known that we have the following perturbative quantum corrections

	$\mathcal{M}_V$	$\mathcal{M}_H$
IIA on CY_3	α'	g_s
IIB on CY_3	exact	α' and g_s
Het on $K3 \times T^2$	g_s	α'

Table 1.4.1: Quantum corrections of moduli spaces

This can be made plausible as g_s corrections only apply to the sector where the dilaton resides and α' corrections apply to the sector involving the Kähler moduli which incorporate sizes of radii. In particular, the moduli space $\mathcal{M}_H$ is extended at

[17] See for example the review [86].
[18] See [95] for a comprehensive review.

higher order in α' by the deformations of the gauge bundle and is unknown in general (see however [99–101, 103, 104]). The classically exact moduli space of type IIB vector multiplets can be taken as a reference point for possible dualities. In the type IIA case, the hyperscalar metric enjoys another special property, that is, for $g_s \to 0$ it is in the image of the c-map [105]. It describes a fibre bundle over a special-Kähler submanifold and can be characterized by a prepotential of this submanifold. This sigma model metric, including the Ramond-Ramond scalars, is known as the 'Ferrara-Sabharwal metric' [106]. In the heterotic case no such special condition holds. In this thesis we derive α' corrections to the heterotic hypermultiplet sector and it is our goal to prove the quaternionic-Kähler geometry for the case of included matter fields.

1.5 Outline of the thesis

In this thesis, we study the perturbative heterotic string compactified on $K3$ with nontrivial gauge bundles, e.g. the standard embedding and $U(1)$ line bundles. The effective action in six dimensions is derived via a Kaluza-Klein reduction up to terms of order α'^2, extending the analysis of [79–85]. In chapter 2, we set the stage and collect all necessary ingredients. We recall the heterotic supergravity in ten dimensions and the conditions for unbroken supersymmetry in section 2.1, i.e. the Calabi-Yau condition and the hermitean Yang-Mills equations for the gauge bundle. In section 2.2, some basic geometrical properties of $K3$ such as hyperkählerity and anti-selfduality are presented. Subsequently, we describe the moduli space of Ricci-flat metrics on $K3$ as well as the moduli space metric. As a reference for the later results, we recall in section 2.3 the generic six-dimensional action of minimal supergravity coupled to vector-multiplets, hypermultiplets and one tensor-multiplet. We pay special attention to the hypermultiplets which lie in the focus of this thesis. In section 2.4, we study in more detail the supersymmetric gauge bundle and its (massless) deformations. We apply the deformation theory of gauge connections and derive differential equations on $K3$ which, to our knowledge, have not been discussed in the literature. In particular, we derive the properties of zero modes relevant for the Kaluza-Klein reduction.

In chapter 3, we apply the Kaluza-Klein reduction to the gauge bundle background known as the standard embedding. We focus on the gauge sector and derive the six-dimensional effective action including the nontrivial moduli dependent matter field couplings which arise from the Chern-Simons couplings in ten dimensions. In section 3.3 our main result is presented where we pay special attention to the hyperscalar sigma model and the scalar potential. The latter is shown to be consistent with supergravity by applying a topological vanishing argument. The only surviving contribution is the D-term scalar potential involving the charged matter fields. We analyze the moduli space of the hyperscalar sigma model in more detail in section 3.4.

Due to the invariance of the hypermultiplet sector under a further compactification on T^2 our results can be interpreted as α' corrections to the (charged) moduli space of $\mathcal{N} = 2$ locally supersymmetric theories in four dimensions. Their kinetic couplings are governed by quaternionic-Kähler geometry which however we cannot prove in full generality. Subsequently, we focus on submanifolds like the charged scalar fibre and a certain truncation where the complex structure moduli are frozen. Finally, we compare a further truncation of our results to a known orbifold limit of $K3$.

In chapter 4 we consider abelian gauge bundles which are equivalent to quantized two-form fluxes. We derive the effective action and show its consistency with supergravity. However, most coupling functions are only given as abstract integrals such that the moduli dependence is hidden. Due to the rigidity of the fluxes some geometrical moduli get stabilized which we describe in section 4.3.1. The $U(1)$ subgroups defined by the line bundles first appear in the unbroken gauge group but their gauge bosons may acquire non-zero masses due to the Stückelberg mechanism involving the flux parameters, as we show in section 4.3.2.

In appendix A we provide additional mathematical details and calculations. In particular, we prove in appendix A.3 a Weitzenböck formula which is central for the derivation of zero mode properties in section 2.4. In appendix B we present a detailed derivation of the moduli dependent coupling functions in the standard embedding as well as kinetic terms from the proposed Kähler potential in section 3.4.3.

Chapter 2

Preliminaries

2.1 Heterotic supergravity

The string effective action governing the massless modes of the heterotic string in ten dimensions is given by the $\mathcal{N} = 1$ supergravity coupled to a super-Yang-Mills theory with gauge group $SO(32)$ or $E_8 \times E_8$; in this thesis we will only consider $E_8 \times E_8$. The corresponding supermultiplets have the following field content

	bosonic	fermionic
gravity multiplet	G_{MN}, B_{MN}, Φ	ψ_M, λ
Yang-Mills multiplet	$A_M^{\mathbf{496}}$	$\chi^{\mathbf{496}}$

Table 2.1.1: Massless fields of the heterotic string in ten dimensions

Here G_{MN} is the graviton, B_{MN} is the Kalb-Ramond two-form potential, Φ is the dilaton, ψ_M is the Majorana-Weyl gravitino and λ is the Majorana-Weyl dilatino. Furthermore, $A_M^{\mathbf{496}}$ is the Yang-Mills gauge potential and $\chi^{\mathbf{496}}$ is the gaugino, both valued in the adjoint representation of $E_8 \times E_8$. We will restrict our analysis to the bosonic part of the theory. The ten-dimensional bosonic Lagrangian, up to α'^2-terms, is given by [24]

$$\mathcal{L} = \tfrac{1}{2} e^{-2\Phi} \left(R * 1 + 4 d\Phi \wedge * d\Phi - \tfrac{1}{3} H \wedge * H + \alpha' (\mathrm{tr}\ F \wedge * F - \mathrm{tr}\ \tilde{R} \wedge * \tilde{R}) \right) \ . \quad (2.1.1)$$

Throughout this thesis we use the space-time metric signature $(-, +, +, +, ...)$ and anti-hermitean gauge generators with negative definite Killing form. $F = dA + A \wedge A$ is the Yang-Mills field strength and H is the field strength of the Kalb-Ramond field B defined as

$$H = dB + \alpha'(\omega^L - \omega^{YM}) \ , \quad (2.1.2)$$

where ω^L, ω^{YM} are the Lorentz- and Yang-Mills Chern-Simons three-forms

$$\begin{aligned}\omega^{YM} &= \mathrm{tr}(F\wedge A) - \tfrac{1}{3}\mathrm{tr}(A\wedge A\wedge A)\ , \\ \omega^{L} &= \mathrm{tr}(R\wedge \Theta) - \tfrac{1}{3}\mathrm{tr}(\Theta\wedge \Theta\wedge \Theta)\ .\end{aligned} \tag{2.1.3}$$

A is the gauge connection, R is the Riemann curvature two-form and Θ is the spin connection.[1] As a consequence, H satisfies the Bianchi identity (also called tadpole condition)

$$dH = \alpha'(\mathrm{tr}\ R\wedge R - \mathrm{tr}\ F\wedge F)\ . \tag{2.1.4}$$

Finally, the last term in (2.1.1) is the Gauss-Bonnet combination [107]

$$\mathrm{tr}\tilde{R}\wedge *\tilde{R} = R_{MNPQ}R^{MNPQ} - 4R_{MN}R^{MN} + R^2\ . \tag{2.1.5}$$

Besides (2.1.1) the ten-dimensional action contains the Green-Schwarz counter-term which is crucial for the anomaly freedom of the theory. We will however neglect this term in our analysis since it is a higher derivative term.

2.1.1 Supersymmetric string vacua

In the search for consistent string vacua one has to solve the equations of motion and certain constraint equations. For the heterotic string the crucial constraint arises from the integrated Bianchi identity (2.1.4)

$$\tfrac{1}{2}\int_Y \mathrm{tr}(F\wedge F) = \tfrac{1}{2}\int_Y \mathrm{tr}(R\wedge R)\ , \tag{2.1.6}$$

which is a topological equation enforcing a non-trivial gauge bundle H over Y. Instead of solving the equations of motion in full generality, one usually looks, for phenomenological and technical reasons, for compactifications which preserve some of the supersymmetries of the ten-dimensional theory. A heterotic compactification on $\mathbb{M}^{5,1}\times Y$ is supersymmetric (at tree level) if there exist nonzero supersymmetry parameters ϵ such that the variations of all fermionic fields vanish

$$\begin{aligned}\delta_\epsilon\lambda &= \Gamma^M(\nabla_M\Phi)\epsilon + \tfrac{1}{24}\mathcal{H}_{MNP}\Gamma^{MNP}\epsilon\ , \\ \delta_\epsilon\psi_M &= \nabla_M\epsilon - \tfrac{1}{4}\mathcal{H}_{MNP}\Gamma^{NP}\epsilon\ , \\ \delta_\epsilon\chi &= \mathcal{F}_{MN}\Gamma^{MN}\epsilon\ .\end{aligned} \tag{2.1.7}$$

Here we denote the background tensor fields by calligraphic letters, i.e. $\langle H\rangle = \mathcal{H}$ and $\langle F\rangle = \mathcal{F}$. The supersymmetry parameter in ten dimensions is a Majorana-Weyl spinor ϵ in the representation **16** of $SO(9,1)$. To preserve isotropy and Lorentz

[1] The trace $\mathrm{tr}R\wedge R$ is evaluated in the vector representation **10** of $SO(9,1)$ and $\mathrm{tr}F\wedge F = \frac{1}{30}\mathrm{Tr}F\wedge F$ is defined as $\frac{1}{30}$ of the trace in the adjoint representation of $E_8\times E_8$.

symmetry in $\mathbb{M}^{5,1}$ all background spinor fields must vanish and the tensor fields are allowed to take nontrivial values only along Y.[2]

We consider in this thesis the standard background ansatz given by[3]

$$\mathcal{H} = 0 \ , \quad \Phi = \text{const} \ . \tag{2.1.8}$$

Then the variation of the dilatino in (2.1.7) vanishes explicitly and the variation of the gravitino requires that there exists at least one covariantly constant spinor section, $\nabla\epsilon = 0$. This is a topological constraint which requires the holonomy group of Y to be some proper subgroup of $SO(4)$. The simplest example is a four-torus, $Y = T^4$, where the holonomy group is trivial. In this case, the local decomposition of the supersymmetry spinor

$$\mathbf{16} \to (\mathbf{4}, \mathbf{2}) \oplus (\mathbf{4}', \mathbf{2}') \tag{2.1.9}$$

can be globally extended, i.e. both internal spinors $\mathbf{2}$ and $\mathbf{2}'$ exist as global covariantly constant sections. One arrives at two six-dimensional spinors $\mathbf{4}$ and $\mathbf{4}'$ corresponding to $\mathcal{N} = 2$ supersymmetry. The minimal number of spinors can be obtained from Y having $SU(2)$ holonomy, which is one definition of a four-dimensional Calabi-Yau manifold. In this case there exists only one covariantly constant spinor, $\mathbf{2}$ or $\mathbf{2}'$, and one arrives at $\mathcal{N} = 1$ supersymmetry (eight real supercharges) in six dimensions.[4]

Finally, the vanishing of the gaugino variation restricts the gauge bundle to satisfy

$$\mathcal{F}_{\alpha\beta} = \mathcal{F}_{\bar{\alpha}\bar{\beta}} = 0 \ , \qquad g^{\alpha\bar{\beta}}\mathcal{F}_{\alpha\bar{\beta}} = 0 \ , \tag{2.1.10}$$

where the indices correspond to (anti-)holomorphic coordinates $z^\alpha, \bar{z}^{\bar{\alpha}}$ and $g_{\alpha\bar{\beta}}$ is the Kähler metric on Y. (2.1.10) are called Hermitean Yang-Mills (HYM) equations. The nontrivial gauge bundle is realized by switching on curvature in a sub-bundle of the $E_8 \times E_8$ principal bundle over Y. Let H be the subgroup of $E_8 \times E_8$ in which the curvature resides. Then, analogous to a Higgs mechanism, the gauge group breaks according to

$$E_8 \times E_8 \longrightarrow G \times \langle H \rangle \ , \tag{2.1.11}$$

where G is the maximal commutant of H. Summarizing, a minimally supersymmetric vacuum of the heterotic string consists of a Calabi-Yau manifold $Y = K3$ and a nontrivial H-bundle over Y satisfying the hermitean Yang-Mills equations and the topological constraint (2.1.4). For $K3$ the integral $\frac{1}{2}\int \text{tr}R \wedge R$ is known to be equal to the Euler characteristic $\chi = 24$. Hence, the gauge bundle is often said to be an instanton of charge 24. We will analyze $K3$ and the gauge bundle in more detail in the following sections.

[2]Hence, the SUSY variations of the bosonic fields automatically vanish and we are justified to leave out additional fermionic terms on the right hand sides of (2.1.7).

[3]Kalb-Ramond flux is not possible on $K3$ due to $H^3(K3, \mathbb{Z}) = 0$. However away from the standard embedding we have $\mathcal{H} \neq 0$ and the back-reaction to the geometry is mentioned in chapter 4.

[4]The details of the spinor decompositions are a bit more intricate than presented here due to the Majorana condition. They can be found in appendix A.1.

2.2 $K3$ surfaces

We would like to collect here some geometrical properties of $K3$ which we will need for our later analysis.[5] A $K3$ surface is a compact four-dimensional Calabi Yau manifold. As such, it is a Kähler manifold with vanishing first Chern class, $c_1(K3) = 0$. By Yau's theorem, there exists a unique Ricci-flat Kähler metric for every chosen Kähler class. In the following we will always refer to Ricci-flat $K3$'s, since only these constitute consistent backgrounds for the string world sheet, preserving the conformal symmetry. $K3$ surfaces are not only Calabi-Yau but also hyper-Kähler manifolds, a property which will affect the structure of the lower-dimensional theory.

A crucial observation is that in four dimensions the structure group of orthonormal frames is non-simple and decomposes as

$$SO(4) \cong SU(2)^+ \times SU(2)^- \ , \tag{2.2.1}$$

in the sense of complex Lie groups. This is related to the existence of the Weyl-spinor bundles $\mathcal{S}^+$ and $\mathcal{S}^-$ which are linked to the (complexified) tangent bundle as

$$T^{\mathbb{C}}_{K3} \cong \mathcal{S}^+ \otimes \mathcal{S}^- \ . \tag{2.2.2}$$

$\mathcal{S}^+$ and $\mathcal{S}^-$ are both complex two-dimensional bundles transforming irreducibly under the corresponding factors in (2.2.1). As a Calabi-Yau manifold, $K3$ has a reduced structure and holonomy group $G = SU(2)$ which can be identified with, say, the second factor in (2.2.1). As a consequence, there exist two non-vanishing and covariantly constant spinor sections. The Dirac-spinor $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ decomposes under the reduced structure group as

$$\mathbf{4} = \mathbf{2}^+ \oplus \mathbf{2}^- \longrightarrow \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{2}^- \ , \tag{2.2.3}$$

where the two singlets are non-vanishing spinors of the same chirality. Accordingly, vectors of the tangent bundle decompose under the reduced structure group as

$$\mathbf{4} = \mathbf{2}^+ \otimes \mathbf{2}^- \longrightarrow (\mathbf{1} \oplus \mathbf{1}) \otimes \mathbf{2}^- = \mathbf{2}^- \oplus \mathbf{2}^- \ . \tag{2.2.4}$$

As a Calabi-Yau manifold, there exists a complex structure $I \in \mathrm{End}(T_{K3})$ satisfying

$$I^2 = -\mathrm{id}_{T_{K3}} \ , \quad \nabla I = 0 \ , \tag{2.2.5}$$

where ∇ is the Levi-Civita connection of the Kähler metric g. Under this complex structure the complexified tangent bundle splits into the holomorphic and anti-holomorphic tangent bundle

$$T^{\mathbb{C}}_{K3} = \mathcal{T}_{K3} \oplus \overline{\mathcal{T}}_{K3} \ , \tag{2.2.6}$$

[5] For a comprehensive review see for example [108].

where $I(\mathcal{T}_{K3}) = i\mathcal{T}_{K3}$ and $I(\overline{\mathcal{T}}_{K3}) = -i\mathcal{T}_{K3}$, understood as acting on elements. Comparing to (2.2.4) we can identify the elements as the $\mathbf{2}^- \oplus \mathbf{2}^-$. The reduced structure group $SU(2)^-$ preserves the complex structure and acts irreducibly on each of the two terms. On the other hand, the first factor $SU(2)^+$ rotates the two singlet spinors and, accordingly, the tuple $(\mathcal{T}_{K3}, \overline{\mathcal{T}}_{K3})$ transforms as a doublet. In fact, it will turn out that $SU(2)^+$ becomes the R-symmetry group of the compactified theory.

Let us now deduce the hyperkähler structure from the fact that also the holonomy is reduced to $SU(2)^-$ due to the Kähler property and Ricci-flatness. Since the curvature of $K3$ takes values in the Lie algebra of the holonomy group, $SU(2)^+$ must be a flat sub-bundle of the frame bundle. Thus, it can be spanned by three global covariantly constant sections I_1, I_2, I_3, satisfying the $\mathfrak{su}(2)$ algebra. From $\nabla I = 0$ it follows that the complex structure must be a linear combination of I_1, I_2, I_3, which therefore square to -1, separately. These two conditions can be summarized in the quaternionic algebra

$$I_r I_s = -\mathrm{id}_{T_{K3}} \delta_{rs} + \varepsilon_{rst} I_t \ . \tag{2.2.7}$$

Hence, the chosen Kähler metric is compatible not only with the complex structure I, but for every complex structure constructed as

$$I = aI_1 + bI_2 + cI_3 \qquad \text{with} \qquad a^2 + b^2 + c^2 = 1 \ . \tag{2.2.8}$$

That is, $K3$ carries an integrable hypercomplex structure, i.e. it is a hyper-Kähler manifold.[6] The metric is hermitean with respect to all I_s,

$$g(I_s X, I_s Y) = g(X, Y) \ . \tag{2.2.9}$$

Moreover, we have a basis of three associated fundamental two-forms $J_s = g(I_s \cdot, \cdot)$, satisfying

$$\nabla J_s = dJ_s = 0 \ , \quad J_r \wedge J_s = 2\delta_{rs} \mathrm{vol} \ . \tag{2.2.10}$$

It can be shown that the above differential conditions define $K3$ as a topologically unique manifold with the Hodge numbers

$$\begin{matrix} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{matrix} \quad = \quad \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix} \ . \tag{2.2.11}$$

Here $h^{p,q}$ are the dimensions of the Dolbeault cohomology groups

$$H^{p,q}(K3, \mathbb{R}) = \frac{\ker\, \bar\partial : \Lambda^{p,q} \to \Lambda^{p,q+1}}{\mathrm{im}\, \bar\partial : \Lambda^{p,q-1} \to \Lambda^{p,q}} \ . \tag{2.2.12}$$

[6]This is consistent with the holonomy being $SU(2) \cong Sp(1)$, according to Berger's classification [109].

Let us consider the set of two-forms $\Lambda^2(K3)$ which carries an action of the Hodge star operator $\star$. There exists a positive scalar product on $\Lambda^2(K3)$ given by

$$\langle \eta_1, \eta_2 \rangle = \int_{K3} \eta_1 \wedge \star \eta_2 \ , \qquad (2.2.13)$$

which is volume independent. Since $\star^2 = \mathrm{id}_{\Lambda^2}$ on a four-dimensional Riemannian manifold, the two-forms decompose locally into two three-dimensional eigenspaces, $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, orthogonal with respect to (2.2.13). Moreover, $\star$ maps harmonic two-forms to harmonic two-forms, thus the middle cohomology group decomposes into the eigenspaces

$$H^2(K3, \mathbb{R}) = H_+ \oplus H_- \ , \qquad (2.2.14)$$

such that $\star H_+ = H_+$ and $\star H_- = -H_-$ (understood as acting on elements). Clearly, this decomposition is metric dependent, but the dimensions are fixed by the signature index of $K3$ to be [108]

$$\mathrm{dim} H_+ = 3 \ , \quad \mathrm{dim} H_- = 19 \ . \qquad (2.2.15)$$

By Poincaré duality, there is a bilinear form on the (integral) middle cohomology

$$\rho : H^2(K3, \mathbb{Z}) \times H^2(K3, \mathbb{Z}) \to 2\mathbb{Z} \ , \qquad (\eta_1, \eta_2) = \int_{K3} \eta_1 \wedge \eta_2 \ . \qquad (2.2.16)$$

of signature $(3, 19)$.[7] Clearly, selfdual two-forms have positive norm and anti-selfdual two-forms have negative norm with respect to ρ. For a given basis η_I, $I = 1, \ldots, 22$ of $H^2(K3, \mathbb{R})$ the bilinear form ρ is represented by the 'intersection' matrix [8]

$$\rho_{IJ} = \int_{K3} \eta_I \wedge \eta_J \ . \qquad (2.2.17)$$

Using that the three fundamental two-forms J_s square to the positive volume form (2.2.10), it follows that they must be a basis of H_+,

$$H_+ = \mathrm{span}(J_1, J_2, J_3) \ . \qquad (2.2.18)$$

From this basis not only the Kähler-form but also the holomorphic volume form Ω can be constructed. A convenient choice will be

$$J = \sqrt{\mathcal{V}} J_3 \ , \quad \Omega = \tfrac{1}{\sqrt{2}}(J_1 + i J_2) \ , \qquad (2.2.19)$$

[7] It can be shown that the bilinear form ρ promotes $H^2(K3, \mathbb{Z})$ to an even unimodular lattice. However we will only be concerned with the $\mathbb{R}$-valued cohomology as a vector space in the following.

[8] Strictly speaking, the term 'intersection matrix' refers only to an integral (co-)homology basis and contains information about the geometry of the Poincaré dual two-cycles. For $K3$ the integral intersection matrix takes the form $\rho = \sigma^1 \oplus \sigma^1 \oplus \sigma^1 \oplus (E_8) \oplus (E_8)$, where σ^1 is the first Pauli matrix and (E_8) is the Cartan matrix of the Lie algebra of E_8.

where the basis elements are now normalized as $(J_r, J_s) = 2\delta_{rs}$. With respect to the complex structure corresponding to J, Ω is a holomorphic (2,0)-form and J is a real (1,1)-form. They are normalized as

$$\int J \wedge J = 2\mathcal{V} \ , \qquad \int \Omega \wedge \bar{\Omega} = 2 \ , \qquad \|\Omega\|^2 = \tfrac{1}{2}\Omega_{\alpha\beta}\bar{\Omega}^{\alpha\beta} = \mathcal{V}^{-1} \ . \tag{2.2.20}$$

It follows from the Hodge diamond (2.2.11) that H_- is the subspace of $H^{1,1}(K3, \mathbb{R})$ orthogonal to J (with respect to ρ). By Yau's theorem [110] every embedding $H_+ \subset H^2(K3, \mathbb{R})$ defines a Ricci-flat Kähler metric, since it specifies the complex structure and the Kähler class, up to $SO(3)$ rotations. From (2.2.18) it follows that every selfdual two-form η_+ there exists a representative in the same cohomology class which is d-closed and covariantly constant with the following expansion

$$\eta_+ = \sum_s \Big(\int J_s \wedge \eta_+ \Big) J_s \ . \tag{2.2.21}$$

In the rest of this thesis we will often identify representatives of the same cohomology class.

Finally, let us show that the curvature two-form R of $K3$ takes values in

$$R \in \mathfrak{su}(2) \otimes \Lambda^2_-(K3) \ , \tag{2.2.22}$$

i.e. it is anti-selfdual itself (we follow [111]). On a four-dimensional manifold the Riemann curvature tensor R acts as an endomorphism on two-forms

$$R : \Lambda^2(M) \to \Lambda^2(M) \ , \qquad R(dx^m \wedge dx^n) = \tfrac{1}{2} R^{mn}{}_{pq} dx^p \wedge dx^q \ . \tag{2.2.23}$$

Moreover, if M is Ricci-flat, R acts block-diagonally on the decomposition $\Lambda^2_+ \oplus \Lambda^2_-$ of two-forms

$$R(\eta) = R_+(\eta_+) + R_-(\eta_-) \ , \tag{2.2.24}$$

where R_+ and R_- are related to the Weyl-tensor. On the other hand, using the metric, the local two-forms can be identified with skew-adjoint endomorphisms of the tangent bundle, $\Lambda^2(K3) \cong \mathrm{End}(T_{K3})$, $T^m{}_n = g^{mp}\eta_{pn}$. Under this isomorphism the selfdual and anti-selfdual two-forms give rise to the decomposition

$$\mathfrak{so}(4) = \mathfrak{so}(3)^+ \oplus \mathfrak{so}(3)^- \ , \tag{2.2.25}$$

which can be identified with the Lie algebra of the holonomy group decomposition (2.2.1). However, we know from (2.2.10), (2.2.18) that the bundle of selfdual two-forms is flat, that is, R_+ vanishes on $K3$. Hence, Λ^2_- is a non-trivial eigenspace of R and of the Hodge star operator, which therefore commute on this domain

$$R(\star\eta_-) = \star R(\eta_-) \ . \tag{2.2.26}$$

This implies anti-selfduality of the curvature tensor, $\frac{1}{2} R^{mn}{}_{pq}\varepsilon^{pq}{}_{rs} = -R^{mn}{}_{rs}$.

2.2.1 Moduli space of $K3$

In contrast to Calabi-Yau three-folds the moduli space of $K3$ does not factorize into a moduli space of the complex structure and the Kähler class. Instead, its global form is known. As shown above, a Ricci-flat Kähler metric is determined by the vector space embedding

$$H_+ \subset H^2(K3,\mathbb{R}) \quad \Leftrightarrow \quad \mathbb{R}^3_+ \subset \mathbb{R}^{3,19} \ . \tag{2.2.27}$$

The space of all such embeddings is given by the Grassmannian manifold

$$\mathrm{Gr}_{3,19} = \frac{SO(3,19)}{SO(3)\times SO(19)} \ , \tag{2.2.28}$$

where $SO(3)$ corresponds to basis rotations inside H_+ and $SO(19)$ corresponds to basis rotations in the orthogonal complement $H_+^\perp$. The overall volume of $K3$ is an additional independent parameter, such that the local moduli space of $K3$ takes the form[9]

$$\mathcal{M}_{K3} = \mathrm{Gr}_{3,19} \times \mathbb{R}^+ \ . \tag{2.2.29}$$

It has a dimension of 58. In the following we will use a particular parametrization of $\mathrm{Gr}_{3,19}$, taken from [113]. Let η_I, $I = 1,\dots,22$ be a basis of $H^2(K3,\mathbb{R})$ and J_s, $s = 1,2,3$ be an orthogonal basis of H_+, as in (2.2.18). Then we have the following expansion and normalization

$$J_s = t^I_s \eta_I \ , \qquad \rho_{IJ} t^I_r t^J_s = 2\delta_{rs} \ , \tag{2.2.30}$$

where t^I_s are the vector components, i.e. 66 parameters subject to six normalization equations and to the equivalence relation

$$t^I_s \sim \tilde{t}^I_s = R_s{}^r t_r \ , \quad R \in SO(3) \ . \tag{2.2.31}$$

These parameters will be used in the rest of this thesis to describe the moduli dependence of certain coupling functions which appear in the six-dimensional effective action. As a first example, let us state the metric on the moduli space (2.2.29) itself. Starting from the canonical line element [72]

$$\delta s^2_{K3} = -\tfrac{1}{4\mathcal{V}} \int_{K3} g^{mn} g^{pq} \delta g_{mp} \delta g_{nq} \ , \tag{2.2.32}$$

it was shown in [113] that the metric deformations can be expressed as deformations of the three fundamental two-forms, $\delta g_{mn} = -\sum_s (I_s)_m{}^p (\delta J_s)_{pn}$. Projecting out deformations which leave the three-plane H_+ invariant, one arrives at

$$\delta s^2_{K3} = \tfrac{1}{2}\big(\rho_{IJ} - \tfrac{1}{2}\rho_{IK}\rho_{JL} t^K_r t^L_r\big)\delta t^I_s \delta t^J_s - \tfrac{1}{4\mathcal{V}^2}\delta\mathcal{V}\delta\mathcal{V} \ , \tag{2.2.33}$$

[9] For the global moduli space of $K3$, orientation changes of the vector spaces are included and the automorphisms of the $H^2(K3,\mathbb{Z})$ lattice have to be modded out additionally. This detail will not be relevant for our analysis.

where summation over r, s is understood. It is well known that the geometrical moduli space is enhanced by the internal components of the Kalb-Ramond field [72]

$$\delta s_B^2 = -\tfrac{1}{4\mathcal{V}} \int_{K3} g^{mn} g^{pq} \delta B_{mp} \delta B_{nq} \ . \tag{2.2.34}$$

The massless deformations are given by harmonic two-forms, so there arise 22 real moduli scalars. We will see in the following chapter that, due to the Chern-Simons couplings (2.1.2), the metric (2.2.34) gets mixed with the gauge bundle moduli space at higher order in α'. If these corrections are neglected, the geometrical moduli space with torsion is locally isomorphic to

$$\tilde{\mathcal{M}}_{K3} = \frac{SO(4,20)}{SO(4) \times SO(20)} \ , \tag{2.2.35}$$

which is a quaternionic-Kähler manifold of real dimension 80. For a parametrization in terms of coset space matrices, see for example [114].

2.3 $\mathcal{N} = 1$ supergravity in six dimensions

In this section we review the generic form of $\mathcal{N} = 1$ locally supersymmetric field theories in six dimensions. This will be the reference for our later results on the compactified effective action. As shown in appendix A.1, the supercharges form a doublet of two Weyl spinors Q^A with the same chirality, satisfying a symplectic Majorana condition

$$\bar{Q}_A = \varepsilon_{AB} Q^B \ . \tag{2.3.1}$$

Their supersymmetry algebra (without central charges) reads

$$\{\bar{Q}_{A\bar{\alpha}}, Q^B_\beta\} = \delta^B_A (\sigma_\mu)_{\bar{\alpha}\beta} P^\mu \ , \quad A, B = 1, 2 \ , \tag{2.3.2}$$

which implies that the R-symmetry, i.e. the automorphism group of the algebra (2.3.2), is $Sp(1) \cong SU(2)$ with the supercharges in the fundamental representation. The component fields of the massless supermultiplets are given by [115]

	bosonic	fermionic
gravity multiplet	$g_{\mu\nu}, B^+_{\mu\nu}$	ψ^-_μ
tensor multiplet	$B^-_{\mu\nu}, \phi$	χ^+
vector multiplet	V_μ	χ^-
hypermultiplet	$4q$	$2\lambda^+$

Table 2.3.1: Supermultiplets of $\mathcal{N} = 1$ supergravity in six dimensions

Here $g_{\mu\nu}$ is the graviton of the six-dimensional space-time, $B^+_{\mu\nu}$ is an antisymmetric tensor with selfdual field strength and ψ^-_μ is the negative chirality gravitino. The tensor multiplet contains a tensor $B^-_{\mu\nu}$ with anti-selfdual field strength, the dilaton ϕ and the the dilatino χ^-. The vector multiplet contains a gauge boson V_μ and the gaugino χ^+. Finally, the hypermultiplet features four real scalars q and the hyperino λ^+.[10]

Clearly, the vector multiplet takes values in the adjoint representation of the gauge group G while the hypermultiplets take either values in some representation $\mathbf{R}$ of G or are neutral singlets. Note that all scalars, except the dilaton, reside in hypermultiplets. The absence of local anomalies constrains the massless spectrum to obey [116, 117]

$$29n_T + n_H - n_V = 273 \ , \tag{2.3.3}$$

where n_T denotes the number of tensor multiplets, n_H the number of hypermultiplets and n_V the number of vector multiplets. This condition is automatically satisfied in any $K3$ compactifications with supersymmetric bundle (2.1.10). In this paper we only consider perturbative $K3$-compactifications where $n_T = 1$, such that $n_H - n_V = 244$ holds.

For gauge groups of the form

$$G = \prod_\alpha G_\alpha \times \prod_m U(1)_m \ , \tag{2.3.4}$$

where G_α denotes any simple factor and $U(1)_m$ any abelian factor, the generic bosonic Lagrangian is given by [118, 119]

$$\begin{aligned} \mathcal{L}_6 = {} & \tfrac{1}{4}R * 1 - \tfrac{1}{2}e^{-2\phi}H \wedge *H + \tfrac{1}{4}d\phi \wedge *d\phi \\ & + \tfrac{1}{2}(c_\alpha e^{-\phi} + \tilde{c}_\alpha e^{\phi})\mathrm{tr}F^{\mathfrak{g}_\alpha} \wedge *F^{\mathfrak{g}_\alpha} - \tilde{c}_\alpha B \wedge \mathrm{tr}F^{\mathfrak{g}_\alpha} \wedge F^{\mathfrak{g}_\alpha} \\ & + \tfrac{1}{2}(c_{mn}\, e^{-\phi} + \tilde{c}_{mn}\, e^{\phi})F^m \wedge *F^n - \tilde{c}_{mn}\, B \wedge F^m \wedge F^n \\ & - \tfrac{1}{2}g_{uv}(q)\mathcal{D}q^u \wedge *\mathcal{D}q^v - V * 1 \ , \end{aligned} \tag{2.3.5}$$

where the non-abelian Yang-Mills field strengths are labeled as $F^{\mathfrak{g}_\alpha}$ and the abelian field strengths as F^m. Due to supersymmetry, the gauge kinetic functions only depend on the six-dimensional dilaton ϕ, with numerical factors $c_\alpha, \tilde{c}_\alpha, c_{mn}, \tilde{c}_{mn}$.[11] For the abelian factors kinetic mixing, parametrized by the off-diagonal part of $c_{mn}, \tilde{c}_{mn}$ is possible [121]. B is the sum of B^+ and B^-, and it is coupled to the vector multiplets via Chern-Simons forms appearing in its field strength $H = dB + \omega^L - c_\alpha \omega^{YM}_{\mathfrak{g}_\alpha} -$

[10]Note that χ^+ and χ^- are each doublets of $SU(2)_R$ while $2\lambda^+$ are two singlets. The number of on-shell degrees of freedom is always four.

[11]It was shown recently that these numerical factors are constrained to take values in a selfdual lattice [120].

$c_{mn}\omega^{YM}_{mn}$, where ω^L and $\omega^{YM}_{\mathfrak{g}_\alpha}$ are standard Chern-Simons forms while the 'mixed' abelian Chern-Simons form is given by

$$\omega^{YM}_{mn} = dV^m \wedge V^n \ . \tag{2.3.6}$$

The real hypermultiplet scalars are denoted by $q^u, u = 1, \dots, 4n_H$ and their kinetic terms are governed by the tensor $g_{uv}(q)$ defining a metric on the scalar field space. They are possibly charged under the gauge group, encoded in the covariant derivatives

$$\mathcal{D}q^u = dq^u - V^a K^{ua}(q) \ , \tag{2.3.7}$$

where a denotes the adjoint index of the gauge group and $K^{ua}(q)$ is a Killing vector on the hyperscalar field space. Gauging of hypermultiplets necessarily leads to a non-trivial scalar potential which takes the form of D-terms

$$V = \tfrac{1}{2}\sum_{a,s} \frac{D^{as}D^{as}}{c_\alpha e^{-\phi} + \tilde{c}_\alpha e^{\phi}} + \tfrac{1}{2}\sum_{m,n,s} \frac{D^{ms}D^{ns}}{c_{mn}e^{-\phi} + \tilde{c}_{mn}e^{\phi}} \ , \tag{2.3.8}$$

where $s = 1, 2, 3$ is an adjoint $SU(2)_R$-index. Moreover, we have

$$(D^{a,m})^A{}_B = \Gamma^A_{uB} K^{ua,m} \ , \qquad A, B = 1, 2 \ , \tag{2.3.9}$$

with Γ^A_{uB} being a composite $\mathfrak{su}(2)_R$-valued connection on the hyperscalar field space [118, 119]. Our main interest in the following will be to derive the six-dimensional couplings, i.e. the hyperscalar metric $g_{uv}(q)$ and the explicit form of the D-term.

2.3.1 Hypermultiplet sector

In this section we collect some basic facts about hypermultiplets which are important for the rest of this thesis. When constructing the smallest supermultiplet of $\mathcal{N} = 1$ in six dimensions (or $\mathcal{N} = 2$ in four dimensions) one encounters the half-hypermultiplet containing the following helicity degrees of freedom[12]

$$\text{half-hypermultiplet}: \quad \{-\tfrac{1}{2}, 0, 0, \tfrac{1}{2}\} \ . \tag{2.3.10}$$

It has the same helicity content as a chiral multiplet in four dimensions, however, it is generically not CPT-complete. All states in (2.3.10) have the same gauge quantum numbers, hence, for non-zero charge q the CPT-conjugate antiparticle states have to be added

$$\text{hypermultiplet}: \quad \underbrace{\{-\tfrac{1}{2}, 0, 0, \tfrac{1}{2}\}}_{q} \quad + \quad \underbrace{\{-\tfrac{1}{2}, 0, 0, \tfrac{1}{2}\}}_{-q} \ . \tag{2.3.11}$$

[12] Note that in six dimensions the notion of helicity is different from the four-dimensional one. The massless little group $SO(4) \cong SU(2)_+ \times SU(2)_-$ has rank two such that there are two helicity quantum numbers. The values $\pm\frac{1}{2}$ in (2.3.10) refer to the Cartan generator of $SU(2)_+$ corresponding to a chiral fermion λ_+

Thus, chiral fermions of a full hypermultiplet always occur in pairs of charge $\pm q$ or in vector-like representations $\mathbf{R} \oplus \overline{\mathbf{R}}$ in the non-abelian case. This is the reason why the chiral gauge spectrum of the standard model cannot be realized in an $\mathcal{N} = 2$ theory in four dimensions. For a neutral singlet half-hypermultiplet one might think that (2.3.10) can be its own anti-multiplet. However, this is impossible due to the following reason: Neutral spin-zero particles are described by real-valued scalar fields. On the other hand, the two spin-zero states form a doublet of $SU(2)_R$ which cannot be realized with two real scalar fields. Therefore, the CPT-conjugate states have to be added even in the case of zero charge. The full hypermultiplet contains an $SU(2)_R$-doublet of complex scalar fields $\Phi = (C, D)$ or more generally

$$\Phi^{\mathbf{R}} = \begin{pmatrix} C^{\mathbf{R}} \\ D^{\mathbf{R}} \end{pmatrix} \tag{2.3.12}$$

in the charged case. Finally, a half-hypermultiplet can be its own anti-multiplet only if its gauge representation $\mathbf{R}$ is pseudoreal. Its spin-zero degrees of freedom are described by the scalar fields of a full hypermultiplet, subject to the reality conditions $\bar{C}_x = \varepsilon_{xy} D^y$ and $\bar{D}_x = -\varepsilon_{xy} C^y$. A solution is easily found to be [122]

$$\text{half-hyperscalar}: \quad \Phi^x = \begin{pmatrix} C^x \\ \pm\varepsilon^{xy} \bar{C}_y \end{pmatrix} , \tag{2.3.13}$$

where ε_{xy} is the (antisymmetric) intertwiner between $\mathbf{R}$ and $\overline{\mathbf{R}}$.

In a supersymmetric field theory (2.3.5) the kinetic terms of the hyperscalars define a non-linear sigma model. The coupling function $g_{uv}(q)$ defines a (pseudo-) Riemannian metric on the scalar target space $\mathcal{M}$. Characteristically, supersymmetry restricts the geometry of $\mathcal{M}$ in the following way. By general arguments the supersymmetry parameter spinors have to be seen as a bundle over $\mathcal{M}$ and the algebra (2.3.2) holds at every point in $\mathcal{M}$. This implies that there exists a covariantly constant tensor field δ^{AB} such that the holonomy of the spinor bundle is reduced to the R-symmetry group $Sp(1) \cong SU(2)$. Furthermore, the supersymmetry variation of the hyperscalars reads

$$\delta_\varepsilon q^u = V^u_{aA} \varepsilon^A \lambda^{a+} , \quad a = 1, \ldots, 2n_H , \quad A = 1, 2 \tag{2.3.14}$$

where ε^A are the two supersymmetry spinors, λ^{a+} are the hyperinos and V^u_{aA} is a vielbein on $\mathcal{M}$. Non-degeneracy of the vielbein implies that the tangent bundle of the target space factorizes (at least locally) as

$$T\mathcal{M} = H \otimes E , \tag{2.3.15}$$

into a two-dimensional vector bundle H and a $2n_H$-dimensional vector bundle E which are both assumed to be pseudoreal. Covariant constancy of the vielbein further implies that the holonomy group of $\mathcal{M}$ is contained in $SU(2) \times Sp(n_H)$. In local supersymmetry the $SU(2)$ curvature is necessarily non-zero, hence the target space is a quaternionic-Kähler manifold. Moreover, the total scalar curvature is fixed by supersymmetry to the specific value [97]

$$R = -8n_H(n_H+2) \ . \tag{2.3.16}$$

Quaternionic-Kähler manifolds are generically not analytic in the quaternionic numbers, i.e. they cannot be covered by quaternionic coordinates. Nevertheless, the four scalars of one hypermultiplet can be formally grouped into a quaternion represented by a complex two-by-two matrix of the form [123]

$$Q = \begin{pmatrix} C & D \\ -\bar{D} & \bar{C} \end{pmatrix} \ . \tag{2.3.17}$$

Then $SU(2)_R$ acts by multiplication from the right and can be interpreted as multiplication with quaternion of unit norm. Clearly, our previous description in terms of the doublet $\Phi = (C, D)$ corresponds to the first row of Q. A half-hypermultiplet then can be written as a quaternion which is further constrained to the form

$$Q = \begin{pmatrix} C^x & \pm\varepsilon^{xy}\bar{C}_y \\ \mp\varepsilon_{xy}C^y & \bar{C}_x \end{pmatrix} \ . \tag{2.3.18}$$

In both cases, the first and second row transforms in $\mathbf{R}$ and $\overline{\mathbf{R}}$, respectively.

2.4 The gauge bundle and its deformations

In this chapter we set the stage for the dimensional reduction in the main part. As we saw in section 2.1 a heterotic compactification $M = \mathbb{M}^{1,5} \times Y$ consists of an internal manifold Y and a gauge bundle (i.e. a principal H-bundle) over Y. Group theoretically the gauge bundle is defined by an embedding $G \times H \subset E_8 \times E_8$, where the unbroken gauge group G is the commutant of H. For simplicity we assume that H is embedded inside one E_8 factor. For the dimensional reduction we have to study the properties of the background solution as well as its space of deformations. In particular, all bosonic matter fields arise from massless deformations of the background gauge connection. It is well known that the spectrum of matter fields can be derived from the chiral index of the appropriate Dirac operator and Dolbeault cohomology [79]. However, for the bosonic effective action the multiplicities are not sufficient, one also needs the precise properties of Kaluza-Klein zero modes. For a $K3$ compactification these zero modes have, to our knowledge, not been discussed in the literature. We therefore apply the local deformation theory of gauge connections, see for example chapter 4 of [112].

Let us focus on the hermitean Yang-Mills equations (2.1.10) which on $K3$ can be written in the more concise way

$$\mathcal{F} \wedge J = 0 \ , \qquad \mathcal{F} \in H^{1,1}(K3, \mathfrak{h}) \ , \tag{2.4.1}$$

where $\mathfrak{h}$ denotes the adjoint H-bundle. The first equation in (2.4.1) states that $\mathcal{F}$ is a primitive two-form. The second equation in (2.4.1) implies that the gauge bundle is a hermitean, holomorphic bundle with $\mathcal{F}$ being its curvature. The hermitean metric h is locally given by the Killing form of the Lie algebra $\mathfrak{h}$ and the hermitean connection $d_\mathcal{A} = d + \mathcal{A}$ is compatible with the metric

$$dh(\psi_1, \psi_2) = h(d_\mathcal{A}\psi_1, \psi_2) + h(\psi_1, d_\mathcal{A}\psi_2) \ , \tag{2.4.2}$$

where ψ_1, ψ_2 are local sections of the $\mathfrak{h}$-bundle. The curvature is given by

$$\mathcal{F} = d_\mathcal{A}^2 = d\mathcal{A} + \tfrac{1}{2}[\mathcal{A}, \mathcal{A}] \ . \tag{2.4.3}$$

Acting on differential forms, we always understand $d_\mathcal{A}$ as the exterior gauge-covariant derivative. The holomorphic structure is given by the split $d_\mathcal{A} = \partial_\mathcal{A} + \bar{\partial}_\mathcal{A}$, with the Dolbeault operator $\bar{\partial}_\mathcal{A} = \bar{\partial} + \mathcal{A}^{0,1}$ satisfying

$$\bar{\partial}_\mathcal{A}^2 = \mathcal{F}^{0,2} = 0 \ . \tag{2.4.4}$$

Hence, on a holomorphic bundle there exists a (twisted) $\bar{\partial}_\mathcal{A}$-Dolbeault complex and corresponding cohomology groups $H^{p,q}(K3, E)$, where E is the adjoint bundle or any associated vector bundle. Let us note that both Hermitean Yang-Mills equations in (2.4.1) together are equivalent to the anti-selfduality (ASD) condition [124]

$$\star \mathcal{F} = -\mathcal{F} \ . \tag{2.4.5}$$

In fact, we know from section 2.2 that the anti-selfdual two-forms are in the orthogonal complement of J inside $H^{1,1}(K3,\mathbb{R})$. There is a topological obstruction to the existence of a gauge bundle satisfying (2.4.5). In addition to the holomorphic structure (2.4.4) the bundle E must be slope-stable, i.e. every sub-bundle $E' \subset E$ satisfies[13]

$$\mu(E') = \int_{K3} c_1(E') \wedge J \quad < 0 \ . \tag{2.4.6}$$

Conversely, the Donaldson-Uhlenbeck-Yau theorem [125, 126] ensures that given a slope-stable bundle (2.4.6), there exists a unique solution of the hermitean Yang-Mills equations (2.4.1). In particular, (2.4.6) implies that there are no trivial line bundles (with $c_1(L) = 0$) inside E. It also implies that the bundle is irreducible, that is, the H-connection cannot be reduced to any proper subgroup $H' \subset H$. Let us note that slope-stability condition (2.4.6) depends on the Kähler-class of the compact manifold and therefore $\mu(E')$ can vary over the Kähler moduli space. As shown recently (for Calabi-Yau three-folds), this effect leads to domain walls in the Kähler moduli space of heterotic compactifications [77].

There is an alternative perspective towards supersymmetric vacuum solutions. By general arguments, solutions with unbroken supersymmetry in Minkowski space are necessarily located at the zero locus of the effective scalar potential. A first approximation of the six-dimensional scalar potential can easily be found from integrating (2.1.1) over $K3$ with inserted background fields

$$\begin{aligned} V_6 &\sim -\tfrac{1}{2}\int_{K3} \mathrm{tr}(\mathcal{F}\wedge\star\mathcal{F}) + \tfrac{1}{2}\int_{K3}\mathrm{tr}(\mathcal{R}\wedge\mathcal{R}) \\ &= -\tfrac{1}{2}\int_{K3} \mathrm{tr}(\mathcal{F}\wedge\star\mathcal{F}) - \tfrac{1}{2}\int_{K3}\mathrm{tr}(\mathcal{F}\wedge\mathcal{F}) \\ &= -\int_{K3}\mathrm{tr}(\mathcal{F}_+\wedge\star\mathcal{F}_+)\ , \end{aligned} \tag{2.4.7}$$

where we used the integrated tadpole condition (2.1.4). $\mathcal{F}_+$ denotes selfdual fields, $\star\mathcal{F}_+ = \mathcal{F}_+$. Obviously, the zero locus of V_6 is just given by an anti-selfdual Yang-Mills background (2.4.5). Note that (2.4.7) defines a positive semidefinite scalar potential, because the trace is the negative definite Killing form of the Lie algebra (using antihermitean generators). In the following, we want to study the massless deformations of a background solution (2.4.5). First, there are deformations which strictly preserve (2.4.5), analogous to the geometrical Calabi-Yau moduli preserving Ricci-flatness. Second, there are deformations which violate (2.4.5) but nevertheless are massless.

[13]For simplicity we avoid the notion of sub-sheaves here.

2.4.1 Massless deformations

It is known that on a hermitean, holomorphic bundle there exists a unique connection compatible with both structures, called the Chern-connection. The physical fields arise from deformations of the connection which preserve the hermitean structure of the $\mathfrak{e}_8$-bundle but change its holomorphic structure. It is known that these form an affine space which can be parametrized by [112]

$$A = \mathcal{A} + a \ , \quad a \in \mathfrak{e}_8 \otimes \Lambda^1(M) \ , \tag{2.4.8}$$

with $\mathcal{A}$ a background solution of (2.4.5). Then the field strength deforms as

$$F = \mathcal{F} + f \ , \quad f = d_{\mathcal{A}} a + \tfrac{1}{2}[a, a] \ . \tag{2.4.9}$$

Gauge equivalent deformations are modded out by fixing the Lorenz gauge

$$d^*_{\mathcal{A}} a = 0 \ . \tag{2.4.10}$$

For a compactification $M = \mathbb{M}^{5,1} \times K3$ we decompose $a = a_1 + a_{\bar{1}}$, where a_1 denotes an external one-form on $\mathbb{M}^{5,1}$ and $a_{\bar{1}}$ denotes an internal one-form on $K3$. They deform the flat G- and the curved H-connection, respectively. In six dimensions, gauge bosons will arise from a_1 and (matter) scalars will arise from $a_{\bar{1}}$. Working with complex coordinates on $K3$ the internal one-form decomposes as $a_{\bar{1}} = a^{0,1} + a^{1,0}$. The hermitean structure is preserved if $a_{\bar{1}}$ satisfies the reality condition

$$(a^{1,0})^\dagger = -a^{0,1} \ , \tag{2.4.11}$$

with respect to the hermitean metric.

Let us consider the ten-dimensional Yang-Mills Lagrangian

$$\mathcal{L}_{YM} \sim \mathrm{tr}(F \wedge *F) \ . \tag{2.4.12}$$

Inserting (2.4.9) the six-dimensional effective mass terms are given by

$$\mathcal{L}_6^{\mathrm{mass}}[a_1] \sim \int_{K3} \mathrm{tr}(d_{\mathcal{A}} a_1 \wedge \star d_{\mathcal{A}} a_1) \ , \tag{2.4.13}$$

$$\mathcal{L}_6^{\mathrm{mass}}[a_{\bar{1}}] \sim \int_{K3} \mathrm{tr}(d_{\mathcal{A}} a_{\bar{1}} \wedge \star d_{\mathcal{A}} a_{\bar{1}}) + \int_{K3} \mathrm{tr}(a_{\bar{1}} \wedge \star [\mathcal{F}, a_{\bar{1}}]) \ . \tag{2.4.14}$$

From (2.4.13) it follows that massless gauge bosons V_i in six dimensions arise from the deformations a_1. The Kaluza-Klein expansion reads

$$a_1 = V_i \cdot \psi_i \ , \qquad d_{\mathcal{A}} \psi_i = 0 \ , \tag{2.4.15}$$

i.e. the zero modes in this case are covariantly constant functions (sections), and the multiplicity of gauge bosons depends on the number of independent sections ψ_i. In fact, for sections of a HYM-bundle we have the identity[14]

$$d^*_{\mathcal{A}} d_{\mathcal{A}} = 2\bar{\partial}^*_{\mathcal{A}} \bar{\partial}_{\mathcal{A}} \ , \tag{2.4.16}$$

It follows that $\ker(d_{\mathcal{A}}) = \ker(\bar{\partial}_{\mathcal{A}})$, such that the zero modes of (2.4.15) are counted by the appropriate Dolbeault cohomology. The mass operator for the scalars is identified from (2.4.14) as

$$\Delta_{YM} a_{\bar{1}} = d^*_{\mathcal{A}} d_{\mathcal{A}} a_{\bar{1}} + \star[\mathcal{F}_{\mathcal{A}}, a_{\bar{1}}] \ . \tag{2.4.17}$$

Since this is not a usual Laplacian, the zero modes are unknown and their connection to Dolbeault cohomology is obscure at first sight. (2.4.17) only implies that being $d_{\mathcal{A}}$-closed is a sufficient condition

$$\ker(d_{\mathcal{A}}) \subset \ker(\Delta_{YM}) \ . \tag{2.4.18}$$

We will now show that one-form zero modes of Δ_{YM} are in one-to-one correspondence with zero modes of the ordinary Laplacian $\Delta_{\bar{\partial}_{\mathcal{A}}} = \bar{\partial}^*_{\mathcal{A}}\bar{\partial}_{\mathcal{A}} + \bar{\partial}_{\mathcal{A}}\bar{\partial}^*_{\mathcal{A}}$. In other words, there exists a Weitzenböck formula between the two Laplacians. We find the following operator identity on one-forms[15]

$$d^*_{\mathcal{A}} d_{\mathcal{A}} a_{\bar{1}} = 2\Delta_{\bar{\partial}_{\mathcal{A}}} a_{\bar{1}} - d_{\mathcal{A}} d^*_{\mathcal{A}} a_{\bar{1}} + iJ \cdot [\mathcal{F}, a_{\bar{1}}] \ , \tag{2.4.19}$$

which we prove in the appendix A.3. Here $\cdot$ denotes the contraction of differential forms using the metric. The second term on the right hand side clearly vanishes in the Lorenz gauge $d^*_{\mathcal{A}} a_{\bar{1}} = 0$. Moreover, on a complex Kähler surface with HYM-bundle we have the second identity

$$\star [\mathcal{F}, a_{\bar{1}}] = -iJ \cdot [\mathcal{F}, a_{\bar{1}}] \ , \tag{2.4.20}$$

proved in appendix (A.3) as well. Inserting both identities into the mass operator (2.4.17), we are left with the (gauge fixed) identity on one-forms

$$\Delta_{YM} = 2\Delta_{\bar{\partial}_{\mathcal{A}}} = 2\Delta_{\partial_{\mathcal{A}}} \ . \tag{2.4.21}$$

Hence, we arrive at

$$\ker(\Delta_{YM}) = \big(\ker(\bar{\partial}_{\mathcal{A}}) \cup \ker(\partial_{\mathcal{A}})\big) \subset \ker(d_{\mathcal{A}}) \ , \tag{2.4.22}$$

which together with (2.4.18) implies the equality

$$\ker(\Delta_{YM}) = \ker(d_{\mathcal{A}}) \ . \tag{2.4.23}$$

[14] A proof can be found, for example, in appendix E of [128].

[15] There is an equivalent identity with $\Delta_{\partial_{\mathcal{A}}}$ instead of $\Delta_{\bar{\partial}_{\mathcal{A}}}$.

On the one hand, (2.4.21) establishes the link to Dolbeault cohomology, i.e. the zero modes are harmonic representatives of $H^{0,1}(K3, E)$. On the other hand, by (2.4.23) the zero modes are gauge-covariantly constant. This fact will be crucial for the derivation of the effective action, where it gives rise to a vanishing theorem. Finally, massless scalars C_j in six dimensions arise from the deformations $a_{\bar{1}}$. The Kaluza-Klein expansion reads

$$a_{\bar{1}} = C_j \cdot \omega_j \ , \qquad d_{\mathcal{A}}\omega_j = \Delta_{\bar{\partial}_{\mathcal{A}}}\omega_j = \Delta_{\partial_{\mathcal{A}}}\omega_j = 0 \ . \tag{2.4.24}$$

2.4.2 Matter fields and bundle moduli

In the previous section we studied the general properties of local massless deformations of the gauge connection (of one E_8 factor), parametrized by the one-form $a \in \mathfrak{e}_8 \otimes \Lambda^1(M)$. However, taking into account the nontrivial background solution (2.4.1) valued in a subgroup H, the gauge group is broken to the maximal commutant G. This is equivalent to the usual Higgs mechanism sourced by the gauge field itself. Due to the breaking, the deformations a are grouped into multiplets according to the decomposition of the adjoint representation $\mathfrak{e}_8 = \mathbf{248}$

$$E_8 \to G \times H \ : \quad \mathbf{248} \to \bigoplus_i (\mathbf{R}_i, \mathbf{S}_i) \oplus (\mathfrak{g}, \mathbf{1}) \oplus (\mathbf{1}, \mathfrak{h}) \ , \tag{2.4.25}$$

where $\mathfrak{g}$ and $\mathfrak{h}$ denote the adjoint representations of G and H, respectively. The $\mathbf{1}$ is the trivial representation and $(\mathbf{R}_i, \mathbf{S}_i)$ are group-specific representations. The right entries of (2.4.25) define vector bundles $E_{\mathbf{S}_i}$ with fibers $\mathbb{R}^{\mathbf{S}_i}$ or $\mathbb{C}^{\mathbf{S}_i}$, depending on the representation. They are associated with the principal H-bundle, i.e. they have the same topology and are HYM. The left entries define the representations of the six dimensional fields under the unbroken gauge group G. It is known from supersymmetry that in six dimensions massless hypermultiplets with representations $\mathbf{R}_i$ occur with multiplicities given by the chiral index[16] [79]

$$\chi(E_{\mathbf{S}_i}) = h^{0,0}(K3, E_{\mathbf{S}_i}) - h^{0,1}(K3, E_{\mathbf{S}_i}) + h^{0,2}(K3, E_{\mathbf{S}_i}) \ , \tag{2.4.26}$$

In fact, $h^{0,0}(K3, E)$ and $h^{0,2}(K3, E)$ vanish on every HYM-bundle E. This can be seen as follows: $H^{0,0}(K3, E)$ is the space of global sections of E, which are closed with respect to the Dolbeault operator $\bar{\partial}_{\mathcal{A}}$ on $K3$. Due to the identity (2.4.16) any such section is also covariantly constant. However, no constant sections exist, because E is nontrivial and irreducible. The vanishing of $H^{0,2}(K3, E)$ then follows by Serre duality [129]

$$H^{0,q}(K3, E) \cong \overline{H^{0,2-q}(K3, E^*)} \ . \tag{2.4.27}$$

[16] χ is called chiral index due to the equivalent definition $\chi(E) = n_E^+ - n_E^-$, where $n_E^\pm$ count the chiral zero modes of the Dirac operator. On $K3$ one has $\chi(E) = \chi(E^*)$, so complex conjugate representations always occur with equal multiplicities. Due to the definite chiralities in the vector- and hypermultiplets, $\chi(E)$ counts the difference of them.

Let us return to the different terms in (2.4.25), and ask which representations occur with gauge bosons and scalars in six dimensions. Gauge bosons only arise from the term $(\mathfrak{g}, \mathbf{1})$, because non-vanishing zero modes (2.4.15) only exist in the trivial bundle $E_{\mathbf{1}}$. The multiplicity is given by Dolbeault cohomology

$$h^{0,0}(K3, E_{\mathbf{1}}) = h^{0,0}(K3) = 1 \ , \tag{2.4.28}$$

yielding one gauge boson in the adjoint representation of the unbroken gauge group. In contrast, no scalars can occur in the adjoint representation $\mathfrak{g}$, because their zero modes take values in

$$H^{0,1}(K3, E_{\mathbf{1}}) = H^{0,1}(K3) = 0 \ . \tag{2.4.29}$$

Generically, one gets scalars from representations $(\mathbf{R}, \mathbf{S})$ with some multiplicity given by $h^{0,1}(K3, E_{\mathbf{S}})$. Here three cases can arise: First, if $\mathbf{R}$ and $\mathbf{S}$ are both real representations, the Kaluza-Klein expansion (2.4.24) yields real scalars

$$a_{\bar{1}} = B_j^{\mathbf{R}} \otimes \omega_j^{\mathbf{S}} \ . \tag{2.4.30}$$

Second, if $(\mathbf{R}, \mathbf{S})$ is a real representation but $\mathbf{R}$ and $\mathbf{S}$ are each pseudoreal, we are forced to use a real combination of complex scalars $C_j^{\mathbf{R}}$ and complex one-forms in the Kaluza-Klein expansion

$$a_{\bar{1}} = a^{0,1} + a^{1,0} = C_j^{\mathbf{R}} \otimes \omega_j^{\mathbf{S}} + \sigma(\bar{C}_j^{\overline{\mathbf{R}}} \otimes \bar{\omega}_j^{\overline{\mathbf{S}}}) \ . \tag{2.4.31}$$

Here σ is the intertwiner which maps $\overline{\mathbf{R}}$ to $\mathbf{R}$ and $\overline{\mathbf{S}}$ to $\mathbf{S}$. The complex scalars $C_j^{\mathbf{R}}$ align in half-hypermultiplets in six dimensions. Third, if $(\mathbf{R}, \mathbf{S})$ is a complex representation, (A.1.7) also restricts the Kaluza-Klein expansion to be a real combination of complex scalars. But a complex representation always occurs pairwise with its complex conjugate, $(\mathbf{R}, \mathbf{S}) \oplus (\overline{\mathbf{R}}, \overline{\mathbf{S}})$, because $\mathfrak{e}_8$ is real. This yields a pair of complex scalars $C_j^{\mathbf{R}}$, $\bar{D}_j^{\overline{\mathbf{R}}}$

$$a_{\bar{1}} = C_j^{\mathbf{R}} \otimes \omega_j^{\mathbf{S}} + \sigma(\bar{C}_j^{\overline{\mathbf{R}}} \otimes \bar{\omega}_j^{\overline{\mathbf{S}}}) + \bar{D}_j^{\overline{\mathbf{R}}} \otimes \bar{\varpi}_j^{\overline{\mathbf{S}}} + \sigma^{-1}(D_j^{\mathbf{R}} \otimes \varpi_j^{\mathbf{S}}) \ . \tag{2.4.32}$$

From (2.4.24) it follows that the zero modes are harmonic representatives of the cohomology groups

$$\begin{aligned} \omega_k &\in H^{0,1}(K3, E_{\mathbf{S}}) \ , \qquad & \overline{\omega}_k &\in H^{1,0}(K3, E_{\bar{\mathbf{S}}}) \ , \\ \varpi_k &\in H^{1,0}(K3, E_{\mathbf{S}}) \ , \qquad & \overline{\varpi}_k &\in H^{0,1}(K3, E_{\bar{\mathbf{S}}}) \ . \end{aligned} \tag{2.4.33}$$

Here $E_{\bar{\mathbf{S}}} = (E_{\mathbf{S}})^*$ is the dual vector bundle. On $K3$ the multiplicities of the scalars $C_j^{\mathbf{R}}$, $\bar{D}_j^{\overline{\mathbf{R}}}$ are the same due to Serre duality

$$\overline{H^{0,1}(K3, E_{\mathbf{S}})} \cong H^{0,1}(K3, E_{\bar{\mathbf{S}}}) \tag{2.4.34}$$

and can be computed via the chiral index (2.4.26).[17] Thus, in six dimensions the scalars align in hypermultiplets with scalar components $\Phi_k^{\mathbf{R}\oplus\overline{\mathbf{R}}} = (C_k^{\mathbf{R}}, \overline{D}_k^{\overline{\mathbf{R}}})$.

Finally, there may arise scalar fields from the term $(\mathbf{1}, \mathfrak{h})$ in (2.4.25), which play a special role. Applying the analysis of section 2.4.1 to this multiplet, there exist massless deformations if the cohomology group $H^{0,1}(K3, \mathfrak{h})$ is nontrivial. Let us now show that any such deformation is not only massless but a true modulus, i.e. a flat direction of the scalar potential

$$V_6 \sim -\int \mathrm{tr}(\mathcal{F}_+ \wedge \star\mathcal{F}_+) \ . \tag{2.4.35}$$

It is clear from (2.4.35) that flat deformations must preserve the anti-selfduality (or equivalently the HYM property) of the gauge bundle. In fact, it is known that the moduli space of ASD connections modulo gauge transformations is equivalent to the moduli space of holomorphic structures on the bundle (see for example [124]). A holomorphic structure is defined by a Dolbeault operator satisfying $\bar{\partial}_{\mathcal{A}}^2 = \mathcal{F}^{0,2} = 0$ and a deformation $A = \mathcal{A} + a$, with $a \in \Lambda^1(K3, \mathfrak{h})$ defines another holomorphic structure if $\mathcal{F}_A^{0,2} = 0$, i.e.

$$\bar{\partial}_{\mathcal{A}} a^{0,1} + \tfrac{1}{2}[a^{0,1}, a^{0,1}] = 0 \ . \tag{2.4.36}$$

Infinitesimally this yields $a^{0,1} \in \ker(\bar{\partial}_{\mathcal{A}})$. However, $a \in \ker(\bar{\partial}_{\mathcal{A}})$ contains directions which lead to gauge-equivalent holomorphic structures which have to be modded out. The gauge-equivalent Dolbeault operators are related by conjugation in H

$$\bar{\partial}_{\mathcal{A}}^h = h^{-1}\bar{\partial}_{\mathcal{A}} h \approx \bar{\partial}_{\mathcal{A}} + \bar{\partial}_{\mathcal{A}}\delta h \ , \tag{2.4.37}$$

where $h \in \Lambda^0(K3, H)$ and $h \approx \mathbf{1} + \delta h$, $\delta h \in \Lambda^0(K3, \mathfrak{h})$. Modding out the term $\bar{\partial}_{\mathcal{A}}\delta h \in \mathrm{Im}(\bar{\partial}_{\mathcal{A}})$, infinitesimal deformations of the holomorphic structure are given by

$$a^{0,1} \in \frac{\ker\, \bar{\partial}_{\mathcal{A}} :\ \Lambda^{0,1}(\mathfrak{h}) \to \Lambda^{0,2}(\mathfrak{h})}{\mathrm{Im}\, \bar{\partial}_{\mathcal{A}} :\ \Lambda^{0}(\mathfrak{h}) \to \Lambda^{0,1}(\mathfrak{h})} = H^{0,1}(K3, \mathfrak{h}) \ , \tag{2.4.38}$$

which is the same cohomology group as from the mass operator. By uniqueness, all massless deformations in the multiplet $(\mathbf{1}, \mathfrak{h})$ are actually complex structure moduli of the H-bundle. The Kaluza-Klein expansion yields complex singlet scalars ξ_k in six dimensions, which are termed bundle moduli

$$a_{\bar{1}} = \xi_k \otimes \alpha_k + \bar{\xi}_k \otimes \bar{\alpha}_k \ , \qquad \alpha_k \in H^{0,1}(K3, \mathfrak{h}) \ . \tag{2.4.39}$$

[17] On a Calabi Yau three-fold the scalars $C^{\mathbf{R}}$ and $\overline{D}^{\overline{\mathbf{R}}}$ occur with different multiplicities, yielding the four-dimensional chiral spectrum.

Chapter 3

Effective action from the standard embedding

In this chapter we will apply the local deformation theory of gauge connections from section 2.4 to a specific background solution. We derive the effective six-dimensional action with explicit moduli dependent coupling functions of the charged matter fields. Let us recall that heterotic compactifications are subject to the topological constraint

$$\tfrac{1}{2}\int_{K3} \mathrm{tr}(\mathcal{F}\wedge\mathcal{F}) = \tfrac{1}{2}\int_{K3} \mathrm{tr}(\mathcal{R}\wedge\mathcal{R}) = \chi_{K3} = 24 \ , \tag{3.0.1}$$

where χ_{K3} is the Euler characteristic of $K3$. This holds in the absence of branes and for vanishing H-flux, which is the only possibility on $K3$ due to $H^3(K3,\mathbb{R}) = 0$.[1] The standard embedding is defined as the solution of (3.0.1) with the integrands identified, i.e. $\mathcal{F}\equiv\mathcal{R}$ and $\mathcal{H}\equiv 0$ in (2.1.4) [49]. This is a valid solution of the hermitean Yang-Mills equations because $\mathcal{R}$ is anti-selfdual. Since the curvature two-form takes values in the Lie algebra of the holonomy group $SU(2)$, also the Yang-Mills field strength takes non-zero values in a $SU(2)$ sub-bundle inside one E_8. Hence, in the standard embedding we have the breaking pattern

$$E_8\times E_8 \longrightarrow E_8\times E_7\times\langle SU(2)\rangle \ , \tag{3.0.2}$$

where E_7 is the maximal commutant of $SU(2)$. Moreover, $SU(2)$ can be identified with structure group of the (anti-)holomorphic tangent bundle $\mathcal{T}_{K3}$, such that the Lie algebra can be identified as

$$\mathfrak{su}(2) = \mathrm{End}\ \mathcal{T}_{K3} \ , \tag{3.0.3}$$

i.e. with the bundle of linear transition functions acting on $\mathcal{T}_{K3}$. The hermitean Yang-Mills equations can be written as

$$\mathcal{F}\in H^{1,1}(\mathrm{End}\ \mathcal{T}_{K3}) \ , \quad \mathcal{F}\wedge J = 0 \ . \tag{3.0.4}$$

[1] Solutions $\mathcal{H}\neq 0$ are briefly discussed in chapter 4.

The relation (3.0.3) will enable us to express the deformations of the gauge bundle in a particular simple way and to find the explicit moduli dependence of their coupling functions. This seems to be only possible in the standard embedding.

3.1 Yang-Mills sector

We now apply the analysis of section 2.4.2 to the standard embedding and derive the Kaluza-Klein expansion of the Yang-Mills field. The deformations a now come in multiplets determined by decomposition of the adjoint representation of E_8 under the breaking (3.0.2)

$$\begin{aligned} \mathbf{248} &\to (\mathbf{133},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{56},\mathbf{2}) \ , \\ T_A &\to (T_n, T_s, T_{xa}) \ . \end{aligned} \tag{3.1.1}$$

In the second line we labeled the corresponding generators. The second E_8 factor can be neglected. The right entries in (3.1.1) can be identified with the following vector bundles

$$\begin{aligned} E_{\mathbf{1}} &= \mathcal{O} \ , \\ E_{\mathbf{2}} &= \mathcal{T}_{K3} \ , \\ E_{\mathbf{3}} &= \mathfrak{su}(2) = \text{End } \mathcal{T}_{K3} \ , \end{aligned} \tag{3.1.2}$$

where $\mathcal{O}$ is the trivial line bundle over $K3$. As explained in (2.4.28) the term $(\mathbf{133},\mathbf{1})$ yields one gauge field in the adjoint of the unbroken gauge group E_7 in six dimensions

$$a_1^{(\mathbf{133},\mathbf{1})} = V^{\mathbf{133}} = V^n T_n \ . \tag{3.1.3}$$

The term $(\mathbf{1},\mathbf{3})$ corresponds to the bundle moduli as specified in (2.4.39)

$$a_{\bar{1}}^{(\mathbf{1},\mathbf{3})} = \xi_k \otimes \alpha_k^{\mathbf{3}} + \bar{\xi}_k \otimes \bar{\alpha}_k^{\mathbf{3}} \ = \xi_k \alpha_k^s T_s + \bar{\xi}_k \bar{\alpha}_k^s T_s \tag{3.1.4}$$

and the zero modes are harmonic representatives $\alpha_k \in H^{0,1}(\text{End } \mathcal{T}_{K3})$. Note that the up-type and down-type notation for the adjoint indices n and s is unnecessary but convenient when combined with the generators T_{xa}. The multiplicity of the bundle moduli (indexed by k) cannot be related to the Hodge numbers of $K3$ but can be computed via the chiral index (2.4.26)

$$h^{0,1}(\text{End } \mathcal{T}_{K3}) = -\chi(\text{End } \mathcal{T}_{K3}) \ . \tag{3.1.5}$$

The chiral index can be expressed with known characteristic classes via the Hirzebruch-Riemann-Roch theorem[2]

$$\chi(E_{\mathbf{S}}) = \int_{K3} \text{Td}(K3) \wedge ch(E_{\mathbf{S}}) = 2\text{rk}(E_{\mathbf{S}}) + ch_2(E_{\mathbf{S}}) \ . \tag{3.1.6}$$

[2]See for example chapter 5.1 of [112].

Here Td($K3$) is the Todd-class of $K3$, rk(E) is the rank of the vector bundle and $ch_2(E_{\mathbf{S}}) = -\frac{1}{2}\int \mathrm{tr}_{\mathbf{S}} \mathcal{F} \wedge \mathcal{F}$ is the second Chern-character known from the tadpole condition (3.0.1). Using rk(End $\mathcal{T}_{K3}) = 3$ and the fact that the trace in the adjoint of $SU(2)$ is four times the trace in the fundamental representation we get

$$\begin{aligned} h^{0,1}(\mathrm{End}\ \mathcal{T}_{K3}) &= -6 + \tfrac{1}{2}\int \mathrm{tr}_{\mathbf{3}}(\mathcal{F}\wedge\mathcal{F}) \\ &= -6 + \tfrac{4}{2}\int \mathrm{tr}_{\mathbf{2}}(\mathcal{F}\wedge\mathcal{F}) \\ &= -6 + 4\cdot 24 \\ &= 90\ . \end{aligned} \tag{3.1.7}$$

Since $(\mathbf{56},\mathbf{2})$ is a product of two pseudoreal representations, its massless Kaluza-Klein components are given according to (2.4.31) by

$$a_{\bar{1}}^{(\mathbf{56},\mathbf{2})} = C_j^{\mathbf{56}} \otimes \omega_j^{\mathbf{2}} + \sigma(\bar{C}_j^{\overline{\mathbf{56}}} \otimes \bar{\omega}_j^{\overline{\mathbf{2}}}) = C_j^x (\omega_j)^a T_{xa} + \varepsilon^{xy}\varepsilon^a{}_{\bar{b}} \bar{C}_{jy} (\bar{\omega}_j)^{\bar{b}} T_{xa}\ . \tag{3.1.8}$$

Here C_j^x are the massless charged matter scalars of the six-dimensional theory. The zero modes are harmonic representatives $\omega_j^{\mathbf{2}} \in H^{0,1}(\mathcal{T}_{K3})$. The intertwiner σ : $(\overline{\mathbf{56}},\overline{\mathbf{2}}) \mapsto (\mathbf{56},\mathbf{2})$ is given by the antisymmetric invariant tensors ε^{xy} and $\varepsilon^a{}_{\bar{b}}$ of E_7 and $SU(2)$, respectively. Note that $a, b = 1, 2$ are flat tangent indices of $\mathcal{T}_{K3}$ as well as $\bar{a}, \bar{b}$ are flat tangent indices of $\overline{\mathcal{T}}_{K3}$. Moreover, we use up-type and down-type indices to distinguish $\mathcal{T}_{K3}$ from its dual $\mathcal{T}^*_{K3}$ and to distinguish the $\mathbf{56}$ representation from its complex conjugate $\overline{\mathbf{56}}$.

Due to the standard embedding the zero modes $\omega_j^{\mathbf{2}}$ can be related to the usual harmonic $(1,1)$-forms η_j on $K3$ via the isomorphism

$$\begin{aligned} H^{0,1}(\mathcal{T}_{K3}) &\cong H^{1,1}(K3)\ , \\ H^{1,0}(\overline{\mathcal{T}_{K3}}) &\cong H^{1,1}(K3)\ . \end{aligned} \tag{3.1.9}$$

From the Hodge diamond (2.2.11) we see that the multiplicity (indexed by j) is 20. The isomorphisms are realized by first converting into curved indices, $(\omega_j)^a = e^a{}_\alpha(\omega_j)^\alpha$, via vielbeins and then using the holomorphic two-form Ω

$$\begin{aligned} (\omega_j)_{\bar{\alpha}}{}^{\beta} &= \tfrac{1}{||\Omega||^2}\bar{\Omega}^{\alpha\beta}(\eta_j)_{\alpha\bar{\alpha}}\ , \\ (\bar{\omega}_j)_{\alpha}{}^{\bar{\beta}} &= \tfrac{1}{||\Omega||^2}\Omega^{\bar{\alpha}\bar{\beta}}(\eta_j)_{\alpha\bar{\alpha}}\ , \end{aligned} \tag{3.1.10}$$

Here we wrote also the one-form index explicitly. The prefactor $\|\Omega\|^{-2}$ ensures that the zero mode is independent of the $K3$ volume. Interestingly, there exists an alternative isomorphism

$$H^{0,1}(\mathcal{T}_{K3}) \cong [\bar{\Omega}] \oplus H^{1,1}(K3)\backslash[J]\ , \tag{3.1.11}$$

that is, the zero modes are mapped to the anti-holomorphic two-form and the $(1,1)$-forms with the Kähler form excluded. This isomorphism is realized as

$$\omega_{\bar{\alpha}}{}^{\beta} = g^{\beta\bar{\beta}}(t_{(\bar{\alpha}\bar{\beta})} + t_{[\bar{\alpha}\bar{\beta}]}) = g^{\beta\bar{\beta}}(\bar{\Omega}^{\delta}{}_{(\bar{\alpha}}\eta_{\bar{\beta})\delta} + \bar{\Omega}_{\bar{\alpha}\bar{\beta}}) \ , \tag{3.1.12}$$

where $t_{(\bar{\alpha}\bar{\beta})}$ and $t_{[\bar{\alpha}\bar{\beta}]}$ is a symmetric and antisymmetric tensor, respectively. The Kähler form is excluded due to the vanishing symmetrized contraction $\bar{\Omega}^{\delta}_{(\bar{\alpha}}J_{\bar{\beta})\delta} = 0$. Both isomorphies are related by the rotation of the hypercomplex structure of $K3$ which rotates the triple (J_1, J_2, J_3). We see here that the zero modes of the charged matter fields are not independent of the geometric $K3$ moduli. They clearly depend on the very definition of $\mathcal{T}_{K3}$ which involves the metric and the complex structure of $K3$. Equivalently, the subspace $H^{1,1}(K3) \subset H^2(K3)$ is only defined by specifying Ω, i.e. the complex structure of $K3$. As a consequence, the coupling functions of the charged matter fields will be given by integrals which are not topological invariants but rather depend on the geometric $K3$ moduli. For definiteness we will use the first isomorphism (3.1.10) in the rest of this thesis.

3.1.1 Expansion of the field strength

We now insert the deformations of the gauge connection into the field strength. Recall that we generically have

$$F = \mathcal{F} + f \ , \qquad f = d_{\mathcal{A}}a + \tfrac{1}{2}[a,a] \tag{3.1.13}$$

We split the ten-dimensional derivative $d_{\mathcal{A}}$ into a derivative d along $\mathbb{M}^{5,1}$ which is just the exterior differential and a derivative $d_{\mathcal{A}}$ (by abuse of notation) along $K3$. Clearly, the generators in the representations $(\mathbf{133},\mathbf{1})$ and $(\mathbf{1},\mathbf{3})$ commute and $[\mathbf{133},\mathbf{56}] = \mathbf{56}$ is the action of E_7 in the fundamental representation. Together, we arrive at the following terms

$$f = f_2^{(\mathbf{133},\mathbf{1})} + f_{1,\bar{1}}^{(\mathbf{1},\mathbf{3})} + f_{1,\bar{1}}^{(\mathbf{56},\mathbf{2})} + f_{\bar{2}}^{(\mathbf{1},\mathbf{3})} + f_{\bar{2}}^{(\mathbf{133},\mathbf{1})} + f_{\bar{2}}^{(\mathbf{56},\mathbf{2})} \ . \tag{3.1.14}$$

Here we again denote by $f_{R,\bar{S}}$ a tensor with R indices along $\mathbb{M}^{5,1}$ and S indices along $K3$. The first term in (3.1.14) is the field strength (fluctuation) of the unbroken gauge group E_7

$$f_2^{(\mathbf{133},\mathbf{1})} = dV^{\mathbf{133}} + \tfrac{1}{2}[V^{\mathbf{133}}, V^{\mathbf{133}}] \ . \tag{3.1.15}$$

The second term in (3.1.14) reads

$$f_{1,\bar{1}}^{(\mathbf{1},\mathbf{3})} = d\xi_k \wedge \alpha_k^{\mathbf{3}} + d\bar{\xi}_k \wedge \bar{\alpha}_k^{\mathbf{3}} \ , \tag{3.1.16}$$

with no contribution from the commutator. The third term reads

$$\begin{aligned} f_{1,\bar{1}}^{(\mathbf{56},\mathbf{2})} &= da_{\bar{1}}^{(\mathbf{56},\mathbf{2})} + [a_1^{(\mathbf{133},\mathbf{1})}, a_{\bar{1}}^{(\mathbf{56},\mathbf{2})}] \\ &= \mathcal{D}C_j^{\mathbf{56}} \wedge \omega_j^{\mathbf{2}} + \sigma(\mathcal{D}\bar{C}_j^{\overline{\mathbf{56}}} \wedge \bar{\omega}_j^{\bar{\mathbf{2}}}) \ , \end{aligned} \tag{3.1.17}$$

where we introduced the E_7 gauge covariant derivative $\mathcal{D}$ defined by

$$\mathcal{D}C_j^x = dC_j^x + V^n(\tau_n)^x{}_y C_j^y \ . \tag{3.1.18}$$

Here $(\tau_n)^x{}_y$ is the E_7 generator in the **56** representation. Since **56** is pseudoreal the complex conjugate generators obey the reality condition

$$(\bar\tau_n)_x{}^y = \varepsilon_{xx'}\varepsilon^{yy'}(\tau_n)^{x'}{}_{y'} \ , \tag{3.1.19}$$

with $\varepsilon^{xx'}$ being the antisymmetric invariant tensor of E_7 and $\varepsilon_{xx'}$ its inverse. Let us now turn to the last two terms in (3.1.14). First recall from (2.4.24) that all zero modes are closed under the differential operator $d_{\mathcal{A}}$ along $K3$. Hence, the $f_{\bar 2}$ terms only arise from the commutator in (3.1.13). The relevant antisymmetric tensor products of representations are

$$\begin{aligned}(\mathbf{1},\mathbf{3}) \otimes_A (\mathbf{1},\mathbf{3}) &= (\mathbf{1},\mathbf{3}) \ ,\\ (\mathbf{56},\mathbf{2}) \otimes_A (\mathbf{56},\mathbf{2}) &= (\mathbf{1},\mathbf{3}) \oplus (\mathbf{133},\mathbf{1}) \ ,\\ (\mathbf{1},\mathbf{3}) \otimes_A (\mathbf{56},\mathbf{2}) &= (\mathbf{56},\mathbf{2}) \ .\end{aligned} \tag{3.1.20}$$

These are realized in terms of generators as

$$\begin{aligned}[][T_r, T_s] &= \varepsilon_{rs}{}^t T_t \ ,\\ [T_{xa}, T_{yb}] &= \varepsilon_{xy}(\sigma^s)_{ab}T_s + (\tau^n)_{xy}\varepsilon_{ab}T_n \ ,\\ [T_s, T_{xa}] &= (\sigma_s)_a{}^b T_{xb} \ ,\end{aligned} \tag{3.1.21}$$

where $\varepsilon_{rs}{}^t = \varepsilon_{rst}$ are the structure constants of $SU(2)$ and $(\sigma_s)_a{}^b$ are the Pauli matrices. We now put the terms together and use the zero mode properties. For the bundle moduli zero modes we not only know that they are $d_{\mathcal{A}}$-closed (infinitesimally) but also that they satisfy the nonlinear equation (2.4.36) (and its complex conjugate). This eliminates part of the commutator and we are left with

$$\begin{aligned} f_{\bar 2}^{(\mathbf{1},\mathbf{3})} &= \left[\xi_k\alpha_k^{\mathbf{3}},\ \bar\xi_l\bar\alpha_l^{\mathbf{3}}\right] + \tfrac{1}{2}\left[C_i^{\mathbf{56}}\omega_i^{\mathbf{2}} + \sigma(\bar C_i^{\overline{\mathbf{56}}}\bar\omega_i^{\bar{\mathbf{2}}}),\ C_j^{\mathbf{56}}\omega_j^{\mathbf{2}} + \sigma(\bar C_j^{\overline{\mathbf{56}}}\bar\omega_j^{\bar{\mathbf{2}}})\right]\\ &= \xi_k\bar\xi_l\alpha_k^r \wedge \bar\alpha_l^s\varepsilon_{rst}T_t + +\tfrac{1}{2}\big(\varepsilon_{xy}C_i^xC_j^y\omega_i^\alpha \wedge \omega_j^\beta R_{\alpha\beta}^s + \varepsilon^{xy}\bar C_{ix}\bar C_{jy}\bar\omega_i^{\bar\alpha} \wedge \bar\omega_j^{\bar\beta}\bar R_{\bar\alpha\bar\beta}^s\big)T_s \ ,\end{aligned} \tag{3.1.22}$$

where we abbreviated the tensor $R_{\alpha\beta}^s = e_\alpha{}^a e_\beta{}^b(\sigma^s)_{ab}$. Note that the mixed $\bar CC$-terms cancel out and that all terms in (3.1.22) are harmonic two-forms with values in the endomorphism bundle

$$f_{\bar 2}^{(\mathbf{1},\mathbf{3})} \in H^2(K3, \mathrm{End}\ \mathcal{T}_{K3}) \ . \tag{3.1.23}$$

This shows that (3.1.22) can be interpreted as a deformation of the background field strength $\mathcal{F}$. For the bundle moduli we know that they preserve the hermitean Yang-Mills equations, hence the first term in (3.1.22) is a $(1,1)$-form, i.e. a deformation

$\delta\mathcal{F}$ along a flat direction of the scalar potential by construction. The second term involving the matter fields is a $(2,0)$- and $(0,2)$-form, i.e. a deformation $\delta\mathcal{F}$ which clearly violates the hermitean Yang-Mills equations. Returning to (3.1.14), we have the next term

$$\begin{aligned} f_{\bar{2}}^{(\mathbf{56},\mathbf{2})} &= \left[\xi_k\alpha_k^{\mathbf{3}} + \bar{\xi}_k\bar{\alpha}_k^{\mathbf{3}},\ C_i^{\mathbf{56}}\omega_i^{\mathbf{2}} + \sigma(\bar{C}_i^{\overline{\mathbf{56}}}\bar{\omega}_i^{\bar{\mathbf{2}}})\right] \\ &= \left(\xi_k\alpha_k^s + \bar{\xi}_k\bar{\alpha}_k^s\right)\wedge\left(C_i^x\omega_i^a{(\sigma_s)_a}^b + \varepsilon^{xy}\bar{C}_{iy}\bar{\omega}_i^{\bar{b}}{\varepsilon^a}_{\bar{b}}{(\sigma_s)_a}^b\right)T_{xb}\ , \end{aligned} \tag{3.1.24}$$

which clearly belongs to the cohomology group

$$f_{\bar{2}}^{(\mathbf{56},\mathbf{2})} \in H^2(K3,\mathcal{T}_{K3})\ . \tag{3.1.25}$$

Finally, we have the term

$$\begin{aligned} f_{\bar{2}}^{(\mathbf{133},\mathbf{1})} =&\ \tfrac{1}{2}\left[C_i^{\mathbf{56}}\omega_i^{\mathbf{2}} + \sigma(\bar{C}_i^{\overline{\mathbf{56}}}\bar{\omega}_i^{\bar{\mathbf{2}}}),\ C_j^{\mathbf{56}}\omega_j^{\mathbf{2}} + \sigma(\bar{C}_j^{\overline{\mathbf{56}}}\bar{\omega}_j^{\bar{\mathbf{2}}})\right] \\ =&\ \tfrac{1}{2}\left(C_i^x(\tau^n)_{xy}C_j^y\right)T_n\ \omega_i^\alpha\wedge\omega_j^\beta {e_\alpha}^a {e_\beta}^b\varepsilon_{ab} \\ &+ \tfrac{1}{2}\left(\bar{C}_{ix}(\bar{\tau}^n)^{xy}\bar{C}_{jy}\right)T_n\ \bar{\omega}_i^{\bar{\alpha}}\wedge\bar{\omega}_j^{\bar{\beta}} {e_{\bar{\alpha}}}^{\bar{a}} {e_{\bar{\beta}}}^{\bar{b}} {\varepsilon^a}_{\bar{a}} {\varepsilon^b}_{\bar{b}}\varepsilon_{ab} \\ &+ \tfrac{1}{2}\left(C_i^x{(\tau^n)_x}^y\bar{C}_{jy} - C_i^x{(\tau^n)^y}_x\bar{C}_{jy}\right)T_n\ \omega_i^\alpha\wedge\bar{\omega}_j^{\bar{\beta}} {e_\alpha}^a {e_{\bar{\beta}}}^{\bar{b}} {\varepsilon^b}_{\bar{b}}\varepsilon_{ab}\ . \end{aligned} \tag{3.1.26}$$

Here we converted again to curved tangent indices for the zero modes ω_i. We can now identify the products of vielbeins and Levi-Civita symbols to be global tensors on $K3$

$$\begin{aligned} {e_\alpha}^a {e_\beta}^b\varepsilon_{ab} &= \Omega_{\alpha\beta}\ , \\ {e_{\bar{\alpha}}}^{\bar{a}} {e_{\bar{\beta}}}^{\bar{b}} {\varepsilon^a}_{\bar{a}} {\varepsilon^b}_{\bar{b}}\varepsilon_{ab} &= \bar{\Omega}_{\bar{\alpha}\bar{\beta}}\ , \\ {e_\alpha}^a {e_{\bar{\beta}}}^{\bar{b}} {\varepsilon^b}_{\bar{b}}\varepsilon_{ab} &= g_{\alpha\bar{\beta}}\ . \end{aligned} \tag{3.1.27}$$

Here, since we take ε_{ab} to be a constant tensor with respect to the spin connection, the converted tensor is constant with respect to the Levi-Civita connection, $\nabla\varepsilon_{\alpha\beta} = 0$. Hence, it must be equal to the holomorphic two-form $\Omega_{\alpha\beta}$, up to a constant which we omit. Clearly, $g_{\alpha\bar{\beta}}$ is the Kähler metric on $K3$ by construction. It will be convenient to write the terms in (3.1.26) as a quadratic (hermitean) form

$$f_{\bar{2}}^{(\mathbf{133},\mathbf{1})} = \frac{1}{2}\begin{pmatrix}\bar{C}_{ix}\\ \varepsilon_{xz}C_i^z\end{pmatrix}^T\begin{pmatrix}\bar{\omega}_i^{\bar{\alpha}}\wedge\omega_j^\beta\ g_{\bar{\alpha}\beta} & \bar{\omega}_i^{\bar{\alpha}}\wedge\bar{\omega}_j^{\bar{\beta}}\ \bar{\Omega}_{\bar{\alpha}\bar{\beta}}\\ \omega_i^\alpha\wedge\omega_j^\beta\ \Omega_{\alpha\beta} & \omega_i^\alpha\wedge\bar{\omega}_j^{\bar{\beta}}\ g_{\alpha\bar{\beta}}\end{pmatrix}{(\tau^n)^x}_y\begin{pmatrix}C_j^y\\ \varepsilon^{yz}\bar{C}_{jz}\end{pmatrix}T_n\ . \tag{3.1.28}$$

Due to the contractions these terms are harmonic two-forms on $K3$

$$f_{\bar{2}}^{(\mathbf{133},\mathbf{1})} \in H^2(K3,\mathbb{R})\ . \tag{3.1.29}$$

This completes the Kaluza-Klein reduction of the Yang-Mills sector. We now turn to the Kalb-Ramond field.

3.2 Kalb-Ramond sector

In the heterotic string theory the B-field is not a usual two-form potential [102, 103]. In particular, it is not Yang-Mills gauge invariant, instead under an infinitesimal gauge transformation parametrized by Λ we have

$$\delta_\Lambda A = d\Lambda + [A, \Lambda] \ , \qquad \delta_\Lambda B = \mathrm{tr}(\Lambda dA) \ . \tag{3.2.1}$$

The gauge invariant field strength contains the Yang-Mills and Lorentzian Chern-Simons three-forms [24]

$$H = dB + \alpha'(\omega^L - \omega^{YM}) \ . \tag{3.2.2}$$

The Chern-Simons forms are local potential three-forms of the second Chern-characters

$$d\omega^{YM} = \mathrm{tr}F \wedge F \ , \qquad d\omega^L = \mathrm{tr}R \wedge R \ . \tag{3.2.3}$$

Since the gauge bundle and the tangent bundle of $K3$ are identified in the standard embedding with background values satisfying $\mathcal{F} = \mathcal{R}$, also the background values of the Chern-Simons forms can be taken to be equal, $\langle\omega^{YM}\rangle = \langle\omega^L\rangle$. However, it is clear that deformations of the $K3$ metric and of the gauge bundle are independent degrees of freedom, yielding non-trivial contributions to a Kaluza-Klein reduction of (3.2.2) at order α'. Starting from the ten-dimensional kinetic term

$$\mathcal{L}_{\mathrm{KR}} \sim H \wedge *H \ , \tag{3.2.4}$$

the equation motion $d^*H = 0$ and Bianchi identity $dH = 0$ imply that solutions are harmonic. Since we consider the trivial background solution $\mathcal{H} = 0$, massless (and in fact flat) deformations are given by an expansion into the harmonic zero- and two-forms on $K3$

$$H = H_3 + H_{1,\bar{2}} \ . \tag{3.2.5}$$

The first term is simply the six-dimensional field strength, analog of (3.2.2). On the other hand, $H_{1,\bar{2}}$ will yield non-trivial couplings of the six-dimensional scalar fields.

3.2.1 Kaluza-Klein expansion

Let us start by considering the Yang-Mills Chern-Simons form in ten dimensions which is defined by

$$\omega^{YM} = \mathrm{tr}(F \wedge A) - \tfrac{1}{3}\mathrm{tr}(A \wedge A \wedge A) \ . \tag{3.2.6}$$

Using our previous results from the Yang-Mills sector the tangent indices of $\omega^{YM}_{1,\bar{2}}$ allow the following combination of terms

$$\omega^{YM}_{1,\bar{2}} = \mathrm{tr}(f_{1,\bar{1}} \wedge A_{\bar{1}}) + \mathrm{tr}(F_{\bar{2}} \wedge a_1) - \mathrm{tr}(A_{\bar{1}} \wedge A_{\bar{1}} \wedge a_1) \ , \tag{3.2.7}$$

where $A_{\bar{1}} = \mathcal{A} + a_{\bar{1}}$ and $F_{\bar{2}} = \mathcal{F} + f_{\bar{2}}$ contain background plus fluctuations. Under the trace only terms in the same representations survive such that the only nonvanishing terms are

$$\omega^{YM}_{1,\bar{2}} = \mathrm{tr}\Big(f^{(\mathbf{56},\mathbf{2})}_{1,\bar{1}} \wedge a^{(\mathbf{56},\mathbf{2})}_{\bar{1}}\Big) + \mathrm{tr}\Big(f^{(\mathbf{1},\mathbf{3})}_{1,\bar{1}} \wedge (a^{(\mathbf{1},\mathbf{3})}_{\bar{1}} + \mathcal{A})\Big) \ . \tag{3.2.8}$$

The traces of the corresponding (antihermitean) generators read

$$\begin{aligned} \mathrm{tr}(T_s T_t) &= -\delta_{st} \ , \\ \mathrm{tr}(T_{xa} T_{yb}) &= -\varepsilon_{xy}\varepsilon_{ab} \ . \end{aligned} \tag{3.2.9}$$

where ε_{xy} and ε_{ab} are the antisymmetric invariant tensors of E_7 and $SU(2)$, respectively. Inserting the results of the previous section we obtain

$$\begin{aligned} \mathrm{tr}\Big(f^{(\mathbf{56},\mathbf{2})}_{1,\bar{1}} \wedge a^{(\mathbf{56},\mathbf{2})}_{\bar{1}}\Big) &= \varepsilon_{xy} C^x_i \mathcal{D} C^y_j \ \omega^\alpha_i \wedge \omega^\beta_j \Omega_{\alpha\beta} + \varepsilon^{xy} \bar{C}_{ix} \mathcal{D} \bar{C}_{jy} \ \bar{\omega}^{\bar{\alpha}}_i \wedge \bar{\omega}^{\bar{\beta}}_j \bar{\Omega}_{\bar{\alpha}\bar{\beta}} \\ &\quad + \Big(\bar{C}_{ix}\mathcal{D}C^x_j - C^x_j \mathcal{D}\bar{C}_{ix}\Big) \ \omega^\alpha_i \wedge \bar{\omega}^{\bar{\beta}}_j g_{\alpha\bar{\beta}} \ , \\ &= -\begin{pmatrix} \mathcal{D}\bar{C}_{ix} \\ \varepsilon_{xy}\mathcal{D}C^y_i \end{pmatrix}^T \begin{pmatrix} \bar{\omega}^{\bar{\alpha}}_i \wedge \omega^\beta_j \ g_{\bar{\alpha}\beta} & \bar{\omega}^{\bar{\alpha}}_i \wedge \bar{\omega}^{\bar{\beta}}_j \ \bar{\Omega}_{\bar{\alpha}\bar{\beta}} \\ \omega^\alpha_i \wedge \omega^\beta_j \ \Omega_{\alpha\beta} & \omega^\alpha_i \wedge \bar{\omega}^{\bar{\beta}}_j \ g_{\alpha\bar{\beta}} \end{pmatrix} \begin{pmatrix} C^x_j \\ \varepsilon^{xz}\bar{C}_{jz} \end{pmatrix} \ , \end{aligned} \tag{3.2.10}$$

where in the last line the term is written as a quadratic (hermitean) form, similar to (3.1.28). The second term in (3.2.8) reads

$$\begin{aligned} \mathrm{tr}\Big(f^{(\mathbf{1},\mathbf{3})}_{1,\bar{1}} \wedge (a^{(\mathbf{1},\mathbf{3})}_{\bar{1}} + \mathcal{A})\Big) &= -\,\xi_l d\xi_k \alpha^s_k \wedge \alpha^s_l - \bar{\xi}_l d\bar{\xi}_k \bar{\alpha}^s_k \wedge \bar{\alpha}^s_l - \Big(\bar{\xi}_l d\xi_k - \xi_k d\bar{\xi}_l\Big)\alpha^s_k \wedge \bar{\alpha}^s_l \\ &\quad - \Big(d\xi_k \alpha^s_k + d\bar{\xi}_k \bar{\alpha}^s_k\Big) \wedge \mathcal{A}^s \\ &= -\begin{pmatrix} d\bar{\xi}_k \\ d\xi_k \end{pmatrix}^T \begin{pmatrix} \bar{\alpha}^s_k \wedge \alpha^s_l & \bar{\alpha}^s_k \wedge \bar{\alpha}^s_l \\ \alpha^s_k \wedge \alpha^s_l & \alpha^s_k \wedge \bar{\alpha}^s_l \end{pmatrix} \begin{pmatrix} \xi_l \\ \bar{\xi}_l \end{pmatrix} \\ &\quad - \Big(d\xi_k \alpha^s_k + d\bar{\xi}_k \bar{\alpha}^s_k\Big) \wedge \mathcal{A}^s \ . \end{aligned} \tag{3.2.11}$$

Here the appearance of the backgroud gauge connection $\mathcal{A}$ seems problematic since a Kaluza-Klein expansion requires globally defined harmonic forms on $K3$. However, the corresponding term is a total derivative in six dimensions which can be removed by a redefinition of the B-field

$$\tilde{B}_{\bar{2}} = B_{\bar{2}} + \mathrm{tr}\Big((\xi_k \alpha^{\mathbf{3}}_k + \bar{\xi}_k \bar{\alpha}^{\mathbf{3}}_k) \wedge \mathcal{A}\Big) \ . \tag{3.2.12}$$

In fact, it is this combination which is invariant under $SU(2)$ gauge transformations of the curved Yang-Mills connection over $K3$. Writing $a = \xi_k \alpha^{\mathbf{3}}_k + \bar{\xi}_k \bar{\alpha}^{\mathbf{3}}_k$, the two terms transform in the opposite way (up to a total derivative)

$$\begin{aligned} \delta_\Lambda B_{\bar{2}} &= \mathrm{tr}\Big(\Lambda d(\mathcal{A} + a)\Big) = -\mathrm{tr}\Big(d\Lambda \wedge (\mathcal{A} + a)\Big) + d\text{-exact} \ , \\ \delta_\Lambda \mathrm{tr}(a \wedge \mathcal{A}) &= \mathrm{tr}\Big((d\Lambda + [a,\Lambda]) \wedge \mathcal{A}\Big) - \mathrm{tr}\Big(a \wedge (d\Lambda + [\mathcal{A},\Lambda])\Big) = \mathrm{tr}\Big(d\Lambda \wedge (\mathcal{A} + a)\Big) \ , \end{aligned} \tag{3.2.13}$$

where d is the exterior derivative on $K3$. Therefore, $\tilde{B}_{\bar{2}}$ can be expanded into globally defined, harmonic two-forms

$$\tilde{B}_{\bar{2}} = b^I \eta_I \ , \quad \eta_I \in H^2(K3, \mathbb{R}) \ , \tag{3.2.14}$$

where $b^I, I = 1, \ldots, 22$ are scalars in six dimensions.

3.2.2 Lorentz Chern-Simons form

We now turn to the Lorentzian Chern-Simons form. Recall that the background contribution is canceled exactly by the Yang-Mills Chern-Simons form, $\langle \omega^{YM} \rangle = \langle \omega^L \rangle$. However, fluctuations of the $K3$ metric yield fluctuations of the Levi-Civita connection, preserving Ricci-flatness and anti-selfduality of the Riemann tensor. Let us consider deformations of the Levi-Civita connection coming from the metric deformations of $K3$. Whereas in the Yang-Mills case the deformations were defined to preserve the hermitean metric of the E_8 bundle, here the deformations clearly change the metric on the tangent bundle. Since $K3$ has no isometries the ten-dimensional metric is decomposed as

$$(G_{MN}) = \begin{pmatrix} G_{\mu\nu} & 0 \\ 0 & \mathcal{G}_{mn} + g_{mn}(t) \end{pmatrix} \ , \tag{3.2.15}$$

where $\mathcal{G}_{mn}$ is the background $K3$ metric and $g_{mn}(t)$ are fluctuations which can be parametrized by the $K3$ moduli t. They are constrained to preserve the Ricci-flatness of $K3$, i.e. they satisfy the Lichnerowicz equations

$$\begin{aligned} \Delta_L g_{mn} &= \nabla^k \nabla_k g_{mn} + 2\mathcal{R}_{m\ n}^{\ p\ q} g_{pq} = 0 \ , \\ \nabla^m g_{mn} &= 0 \ , \qquad \mathcal{G}^{mn} g_{mn} = 0 \ . \end{aligned} \tag{3.2.16}$$

In ten dimensions we have an orthonormal frame field $e^A_M dx^M$ and the compatible Levi-Civita connection Θ locally given by

$$(\Theta_M)^A{}_B dx^M = e^A_N \partial_M e^N_B + e^A_N e^P_B \Gamma^N_{MP} \quad \in \Lambda^1(\mathbb{M}^{9,1}, \mathfrak{so}(9,1)) \ . \tag{3.2.17}$$

The deformation of the Christoffel symbol to linear order in g reads

$$\delta\Gamma^P_{MN} = \tfrac{1}{2} G^{PQ} (\nabla_M g_{NQ} + \nabla_N g_{MQ} - \nabla_Q g_{MN}) \ . \tag{3.2.18}$$

Restricting to internal deformations $g_{mn}(t)$ we get a deformation θ_1

$$\begin{aligned} \theta_1 = &\ e^a_m e^n_b \delta\Gamma^m_{\mu n} dx^\mu \\ = &\ \tfrac{1}{2} e^a_m e^n_b \mathcal{G}^{mp} \partial_\mu g_{np} dx^\mu \quad \in \Lambda^1(\mathbb{M}^{5,1}) \otimes \mathrm{End}(T_{K3}) \ . \end{aligned} \tag{3.2.19}$$

However, it cannot be considered as a deformation of the external flat connection which would be valued in $\mathrm{End}(T_{\mathbb{M}^{5,1}})$. In contrast to the background connection, this tensor need not be valued in $\mathfrak{so}(4)$. Using (3.2.16) we only know that it is in the traceless part of $\mathrm{End}(T_{K3})$, i.e. can be an arbitrary traceless matrix. Also, in contrast to the Yang-Mills case, this one-form need not be a global section of the $\mathrm{End}(T_{K3})$ bundle, because g_{mn} may be a deformation with zeros. Second, we get a deformation of the curved internal connection $\Theta_{\bar{1}} \to \Theta_{\bar{1}} + \theta_{\bar{1}}$

$$\begin{aligned} \theta_{\bar{1}} =& \; e^a_n e^p_b \delta\Gamma^n_{mp} dx^m \\ =& \; \tfrac{1}{2} e^a_n e^p_b \mathcal{G}^{nq}(\nabla_m g_{pq} + \nabla_p g_{mq} - \nabla_q g_{mp}) dx^m \quad \in \Lambda^1(K3, \mathrm{End}(T_{K3})) \; . \end{aligned} \tag{3.2.20}$$

The curvature deforms accordingly

$$R_{\nabla+\theta} = R_\nabla + \nabla\theta + \tfrac{1}{2}[\theta,\theta] \; , \tag{3.2.21}$$

such that we group the contributions according to external and internal spacetime indices.

$$\delta R_{1,\bar{1}} = d_1\theta_{\bar{1}} + \nabla_{\bar{1}}\theta_1 + [\theta_1, \theta_{\bar{1}}] \; , \tag{3.2.22}$$

$$\delta R_{\bar{2}} = \nabla_{\bar{1}}\theta_{\bar{1}} + \tfrac{1}{2}[\theta_{\bar{1}}, \theta_{\bar{1}}] \; . \tag{3.2.23}$$

Taking all possible combinations the deformation of the Lorentz Chern-Simons form $\omega^L_{1,\bar{2}}$, with one external and two internal spacetime indices, reads

$$\omega^L_{1,\bar{2}} = \mathrm{tr}(\delta R_{1,\bar{1}} \wedge \theta_{\bar{1}}) + \mathrm{tr}((\mathcal{R} + \delta R_{\bar{2}}) \wedge \theta_1) - \mathrm{tr}(\theta_1 \wedge (\Theta_{\bar{1}} + \theta_{\bar{1}}) \wedge (\Theta_{\bar{1}} + \theta_{\bar{1}})) \; , \tag{3.2.24}$$

and contains up to cubic terms in g_{mn}.

$$\begin{aligned} \omega^L_{1,\bar{2}} \text{ (linear)} &= \mathrm{tr}(\mathcal{R} \wedge \theta_1 - \Theta_{\bar{1}} \wedge \Theta_{\bar{1}} \wedge \theta_1) \\ &= \mathrm{tr}(d_{\bar{1}}\Theta_{\bar{1}} \wedge \theta_1) \; , \end{aligned} \tag{3.2.25}$$

$$\omega^L_{1,\bar{2}} \text{ (quadratic)} = \mathrm{tr}\big(\Theta_{\bar{1}} \wedge [\theta_1, \theta_{\bar{1}}] + (d_1\theta_{\bar{1}}) \wedge \theta_{\bar{1}} + (\nabla_{\bar{1}}\theta_1) \wedge \theta_{\bar{1}} + (\nabla_{\bar{1}}\theta_{\bar{1}}) \wedge \theta_1\big) \; , \tag{3.2.26}$$

$$\begin{aligned} \omega^L_{1,\bar{2}} \text{ (qubic)} &= \mathrm{tr}\big([\theta_1, \theta_{\bar{1}}] \wedge \theta_{\bar{1}} + \tfrac{1}{2}[\theta_{\bar{1}}, \theta_{\bar{1}}] \wedge \theta_1 - \theta_1 \wedge \theta_{\bar{1}} \wedge \theta_{\bar{1}}\big) \\ &= \mathrm{tr}\big([\theta_1, \theta_{\bar{1}}] \wedge \theta_{\bar{1}}\big) \; . \end{aligned} \tag{3.2.27}$$

The linear term is a total derivative in six dimensions, however the others may give non-trivial contributions to the geometrical moduli space metric. Unfortunately, we cannot translate the Lichnerowicz equations into a simple (first order) differential equation for θ_1 and $\theta_{\bar{1}}$ as we could do for the gauge connection (2.4.24). Therefore, the relevance of the Lorentz Chern-Simons forms for the six-dimensional effective action remains obscure and we leave this issue for future studies. As the final Kaluza-Klein expansion of $H_{1,\bar{2}}$ we take the result from the previous section

$$H_{1,\bar{2}} = db^I \wedge \eta_I + \alpha' \begin{pmatrix} d\bar{\xi}_k \\ d\xi_k \end{pmatrix}^T \begin{pmatrix} \bar{\alpha}^s_k \wedge \alpha^s_l & \bar{\alpha}^s_k \wedge \bar{\alpha}^s_l \\ \alpha^s_k \wedge \alpha^s_l & \alpha^s_k \wedge \bar{\alpha}^s_l \end{pmatrix} \begin{pmatrix} \xi_l \\ \bar{\xi}_l \end{pmatrix}$$
$$+ \alpha' \begin{pmatrix} \mathcal{D}\bar{C}_{ix} \\ \varepsilon_{xy}\mathcal{D}C^y_i \end{pmatrix}^T \begin{pmatrix} \bar{\omega}^{\bar{\alpha}}_i \wedge \omega^{\beta}_j \; g_{\bar{\alpha}\beta} & \bar{\omega}^{\bar{\alpha}}_i \wedge \bar{\omega}^{\bar{\beta}}_j \; \bar{\Omega}_{\bar{\alpha}\bar{\beta}} \\ \omega^{\alpha}_i \wedge \omega^{\beta}_j \; \Omega_{\alpha\beta} & \omega^{\alpha}_i \wedge \bar{\omega}^{\bar{\beta}}_j \; g_{\alpha\bar{\beta}} \end{pmatrix} \begin{pmatrix} C^x_j \\ \varepsilon^{xz}\bar{C}_{jz} \end{pmatrix} \; . \tag{3.2.28}$$

3.3 6D Effective action

Using the results from the previous sections we now derive the six-dimensional effective action up to order α'^2. This involves nontrivial integrals over $K3$ which yield moduli-dependent coupling functions. These have not been determined explicitly before. We here present the results and refer the reader to appendix B for a detailed computation. The effective action of the gravity-dilaton sector has been determined in ref. [113] and we include their result in the following. For the Einstein frame in six dimensions the dilaton has to be redefined as

$$\phi = \Phi - \tfrac{1}{2}\ln\mathcal{V} \; , \tag{3.3.1}$$

where Φ is the ten-dimensional dilaton and $\mathcal{V}$ is the $K3$ volume. The Einstein-frame metric is given by $g_{\mu\nu} = e^{-\phi}G_{\mu\nu}$. From this redefinition a factor of $\mathcal{V}^{-1}$ appears in front of all terms in the Lagrangian with nontrivial $K3$ integral. Altogether we arrive at the bosonic Lagrangian

$$\mathcal{L}_6 = \tfrac{1}{2}R * 1 \; - \; \tfrac{1}{6}e^{-2\phi}H \wedge *H \; + \; \tfrac{\alpha'}{2}e^{-\phi}\mathrm{tr}F^{\mathbf{133}} \wedge *F^{\mathbf{133}}$$
$$+ \; \tfrac{9}{2}d\phi \wedge *d\phi \; + \; \mathcal{L}_\sigma \; - \; V * 1 \; . \tag{3.3.2}$$

The first line contains the Einstein-Hilbert term, and kinetic terms of the Kalb-Ramond field H (including Chern-Simons couplings) and the Yang-Mills field strength $F^{\mathbf{133}}$ with gauge group E_7. The second line contains all scalars of the six-dimensional theory. $\mathcal{L}_\sigma$ contains kinetic terms of the scalars which belong to hypermultiplets and defines a non-linear sigma model. V is the corresponding scalar potential. The kinetic term of the dilaton is written separately because it belongs to the tensor multiplet. Note that we do not derive the six-dimensional Green-Schwarz counter-term since we only considered the ten-dimensional tree level action (2.1.1). In the following we concentrate on $\mathcal{L}_\sigma$ and V.

3.3.1 Kinetic terms

As explained in section 2.3 minimal local supersymmetry in six dimensions (equivalent to $\mathcal{N} = 2$ supersymmetry in four dimensions) puts severe constraints on the geometry of the non-linear sigma model of hypermultiplet scalars. Their target space is a quaternionic-Kähler manifold, consisting of a (pseudo-)Riemannian metric, three almost complex structures and a certain compatibility between them. The sigma model containing the scalar fields belonging to hypermultiplets is given by the Lagrangian

$$\begin{aligned}\mathcal{L}_\sigma = & \tfrac{1}{4} h_{IJ} dt_s^I \wedge *dt_s^J \; - \; \tfrac{1}{8\mathcal{V}^2} d\mathcal{V} \wedge *d\mathcal{V} \\ & - \; \alpha' g_{ij} \mathcal{D}\bar{C}_{ix} \wedge *\mathcal{D}C_j^x \; - \; \tfrac{\alpha'}{\mathcal{V}} \mathcal{G}_{kl} d\bar{\xi}_k \wedge *d\xi_l \\ & - \; \tfrac{1}{6\mathcal{V}} g_{IJ} \mathcal{D}_C b^I \wedge *\mathcal{D}_C b^J \ .\end{aligned} \tag{3.3.3}$$

In the first line we have the kinetic terms defining the metric on the geometric $K3$ moduli space $\mathcal{M}_{K3}$ with

$$h_{IJ} = \rho_{IJ} - \tfrac{1}{2} \rho_{IK} \rho_{JL} t_s^K t_s^L \ , \tag{3.3.4}$$

(see (2.2.29) and (2.2.33)). In the second line of (3.3.3) there are the kinetic terms of the scalars from the Yang-Mills sector, i.e. the matter fields C_i^x and the bundle moduli ξ_k. The matter fields are charged under the unbroken gauge group E_7 and exhibit multiplets in the **56** representation. Their covariant derivatives are given by

$$\mathcal{D}C_i^x = dC_i^x + V^n (\tau_n)^x{}_y C_i^y \ , \tag{3.3.5}$$

where $V^n \ , n = 1, \ldots, 133$ are the gauge bosons. From (3.3.5) we easily read off the linear Killing vectors

$$(K_n)_i^x = (\tau_n)^x{}_y C_i^y \ . \tag{3.3.6}$$

The leading kinetic metric of the matter fields is derived in (B.1.7) and reads

$$g_{ij} = \int_{K3} \eta_i \wedge \star \eta_j \ , \tag{3.3.7}$$

where $\eta_i \ , i = 3, \ldots, 22$ is a basis of harmonic two-forms spanning $H^{1,1}(K3, \mathbb{R})$. g_{ij} depends on the $K3$ moduli as it is a projection of g_{IJ} defined below. If we fix $H^{1,1}(K3, \mathbb{R})$ by fixing a complex structure on $K3$ via $\Omega = \frac{1}{\sqrt{2}}(J_1 + iJ_2)$, the moduli t_1^I, t_2^I are frozen. Then g_{ij} only depends on the remaining Kähler moduli t_3^i according to

$$g_{ij} = -\rho_{ij} + \rho_{ik} \rho_{jl} t_3^i t_3^j \ . \tag{3.3.8}$$

We see here already that the couplings of the charged matter fields require the moduli space of $K3$ to be considered as a fibre bundle of Kähler moduli over the base of complex structure moduli. This division certainly spoils the hypercomplex structure and hence, the manifest $SU(2)_R$ symmetry of the effective action. The bundle moduli ξ_k are complex fields which are singlets under the gauge group E_7. Their leading kinetic metric of the bundle moduli is given by

$$\mathcal{G}_{kl} = \int_{K3} \bar{\alpha}_k^s \wedge \star \alpha_l^s \ . \tag{3.3.9}$$

It certainly depends on the $K3$ moduli but we cannot derive the explicit form.[3] In the third line of (3.3.3) we find the kinetic term of the B-field scalars coupled to the Yang-Mills scalars at order α'. The leading kinetic metric in front is derived in (B.1.5) and reads

$$g_{IJ} = -\rho_{IJ} + \rho_{IK}\rho_{JL}t_s^K t_s^L \ . \tag{3.3.10}$$

The further couplings are encoded in the expression

$$\begin{aligned}\mathcal{D}_C b^I = db^I &+ \alpha' M^I_{ij}(C^x_j \mathcal{D}\bar{C}_{ix} - \bar{C}_{ix}\mathcal{D}C^x_j) + \alpha'(N^I_{ij}\varepsilon_{xy}C^x_i\mathcal{D}C^y_j + \bar{N}^I_{ij}\varepsilon^{xy}\bar{C}_{ix}\mathcal{D}\bar{C}_{jy}) \\ &+ \alpha'\mathcal{M}^I_{kl}(\bar{\xi}_k d\xi_l - \xi_l d\bar{\xi}_k) + \alpha'(\mathcal{N}^I_{kl}\xi_k d\xi_l + \bar{\mathcal{N}}^I_{kl}\bar{\xi}_k d\bar{\xi}_l)\end{aligned} \tag{3.3.11}$$

Here the coupling functions M^I_{ij} and N^I_{ij} are derived in appendix B.1 and read

$$\begin{aligned}M^I_{ij} &= -i\mathcal{V}^{\frac{1}{2}}\left(\rho_{ij}t^I_3 - \delta^I_j\rho_{Ki}t^K_3 - \delta^I_i\rho_{Kj}t^K_3\right) \ , \\ N^I_{ij} &= \tfrac{1}{\sqrt{2}}\rho_{ij}(t^I_1 - it^I_2) \ .\end{aligned} \tag{3.3.12}$$

Both, M^I_{ij} and N^I_{ij}, depend on the $K3$ moduli but for a fixed complex structure we have the following simplification. In a basis (η_1, η_2, η_i) of $H^2(K3, \mathbb{R})$, where η_1, η_2 span the complex structure two-plane, we have $\langle J_{1,2}, \eta_I\rangle = 0$ for $I = i$ and $\langle J_3, \eta_I\rangle = 0$ for $I = 1, 2$. This implies

$$\begin{aligned}N^I_{ij} &\neq 0 \quad \text{only for } I = 1, 2 \ , \\ M^I_{ij} &\neq 0 \quad \text{only for } I = 3, \ldots, 22 \ .\end{aligned} \tag{3.3.13}$$

Moreover, since t^I_1, t^I_2 are frozen, we have

$$N^{1,2}_{ij} = z^{1,2}\rho_{ij} \ , \quad z^{1,2} = \tfrac{1}{\sqrt{2}}(\langle t^I_1\rangle - i\langle t^I_2\rangle) \ , \tag{3.3.14}$$

with two fixed complex numbers $z^{1,2}$. In this case the couplings (3.3.11) factorize as

$$\mathcal{D}_C b^I = \begin{pmatrix} db^{1,2} + \alpha'\rho_{kl}\left(z^{1,2}\varepsilon_{xy}C^x_k\mathcal{D}C^y_l + \bar{z}^{1,2}\varepsilon^{xy}\bar{C}_{kx}\mathcal{D}\bar{C}_{ly}\right) + \ldots \\ db^i + \alpha' M^i_{kl}\left(\bar{C}_{kx}\mathcal{D}C^x_l - C^x_l\mathcal{D}\bar{C}_{kx}\right) + \ldots \end{pmatrix} \ , \tag{3.3.15}$$

where the dots stand for the $\bar{\xi}d\xi$ terms. Since ρ_{ij} is symmetric and $z^{1,2}$ is constant, the matter couplings in the first line of (3.3.15) are total derivatives and they can be absorbed by a redefinition of b^1, b^2

$$\tilde{b}^{1,2} = b^{1,2} + \tfrac{\alpha'}{2}\rho_{kl}\left(z^{1,2}\varepsilon_{xy}C^x_k C^y_l + \bar{z}^{1,2}\varepsilon^{xy}\bar{C}_{kx}\bar{C}_{ly}\right) \ . \tag{3.3.16}$$

Finally, the coupling functions $\mathcal{M}^I_{kl}$ and $\mathcal{N}^I_{kl}$ of the bundle moduli can only be given as integrals

$$\mathcal{N}^I_{kl} = \rho^{IJ}\int_{K3}\alpha^s_k \wedge \alpha^s_l \wedge \eta_J \ , \quad \mathcal{M}^I_{kl} = \rho^{IJ}\int_{K3}\alpha^s_k \wedge \bar{\alpha}^s_l \wedge \eta_J \ . \tag{3.3.17}$$

[3] It would be helpful to find out how the Hodge star operator acts on the bundle valued one-forms α^s_k.

whose moduli dependence is implicit but is expected to be non-trivial. Summarizing, we see from the results that at order α' the charged matter fields and bundle moduli have non-trivial coupling functions and kinetically mix with the B-field scalars.

3.3.2 The scalar potential

We now turn to the scalar potential which consists of the terms descending from (2.1.1) with all space-time indices tangent to $K3$. A priori, we should expect that the geometric moduli preserving the Ricci-flatness of $K3$, as well as the bundle moduli preserving the ASD condition of the Yang-Mills field strength, are flat directions of the scalar potential, i.e. V should be independent of them. Conversely, the charged matter fields are massless but we saw in (3.1.22) that they correspond to deformations of the Yang-Mills field strength which violate the hermitean Yang-Mills equations. Therefore the scalar potential should depend on them in a non-trivial way. From (2.4.7) we also know that only selfdual components of the Yang-Mills field strength can appear in the scalar potential. This reasoning is generalized here to the relevant fluctuations in (3.1.14).

Recall from (2.4.7) that the scalar potential in six dimensions arises from dimensional reduction of the Yang-Mills and Gauss-Bonnet term in (2.1.1). Including the numerical factors, the scalar potential in the Einstein frame reads

$$V = -\tfrac{\alpha'}{\mathcal{V}} e^{\phi} \int_{K3} \mathrm{tr}(F_+ \wedge \star F_+) \ , \tag{3.3.18}$$

where F_+ is the selfdual component of the Yang-Mills field strength with all indices tangent to $K3$.[4] Instead of $\mathcal{F}$ we include here all relevant fluctuations $F = \mathcal{F} + f$ which were derived in section 3.1.1. Terms like F_+ can arise in two different ways. First, the $K3$ metric deforms in such a way that a previously ASD background solution $\mathcal{F}$ acquires selfdual components. If the Yang-Mills bundle is not able to adjust accordingly to another ASD solution, the corresponding $K3$ moduli are stabilized. From the functional dependence of the projector (B.1.3) it follows that a quadratic mass term of the corresponding $K3$ moduli is generated.

$$\begin{aligned} V &\supset -\tfrac{\alpha'}{\mathcal{V}} e^{\phi} \int_{K3} \mathrm{tr}(\mathcal{F} \wedge P_+ \mathcal{F}) \\ &= -\tfrac{\alpha'}{\mathcal{V}} e^{\phi} \rho_{JL} t_s^L t_s^K \int_{K3} \mathrm{tr}(\psi^I \psi^J \eta_I \wedge \eta_K) \ , \end{aligned} \tag{3.3.19}$$

where we locally expanded the background field strength as $\mathcal{F} = \psi^I \otimes \eta_I$ with local sections $\psi^I \in \Gamma(\mathrm{End}\ \mathcal{T}_{K3})$. Second, for a fixed $K3$ metric, the charged matter fields,

[4] For brevity we leave out the label $\bar{2}$ in the expressions $F_{\bar{2}}, f_{\bar{2}}$, etc. which indicated that two space-time indices are tangent to $K3$.

despite being massless, correspond to deformations of the Yang-Mills field strength which violate the hermitean Yang-Mills equations according to (3.1.22). Therefore, the scalar potential should explicitly depend on the charged matter fields. In the standard embedding the first mechanism does not apply because, due to $\mathcal{F} = \mathcal{R}$, one always has an ASD curvature at each point in the $K3$ moduli space. In any case, we consider continuous deformations of the gauge bundle curvature f, resulting from the restricted set of deformations of the connection a as described in section 2.4.1. By assumption, these do not change the topology of the bundle, especially its second Chern character

$$0 = \int_{K3} \mathrm{tr}(f \wedge f) = \int_{K3} \mathrm{tr}(f_+ \wedge f_+) + \int_{K3} \mathrm{tr}(f_- \wedge f_-) \ . \tag{3.3.20}$$

Due to the signature $(3, 19)$ of the $K3$ intersection matrix, the two terms are actually of opposite sign. For consistency the deformations either satisfy (in cohomology)

$$f_+ = f_- = 0 \qquad \text{or} \qquad f_+ \neq 0,\ f_- \neq 0 \ . \tag{3.3.21}$$

For a non-trivial scalar potential (3.3.29) it is necessary to have the second case. In the following we will show that the above conditions, together with a vanishing theorem, rule out all terms except one from the list (3.1.14) to appear in the scalar potential. Let us check the conditions (3.3.21) for the bundle moduli (and vanishing matter fields) where we know a priori that no potential exists. The only contribution then comes from (3.1.22) and reads

$$f^{(\mathbf{1},\mathbf{3})}(\xi) = \xi_k \bar{\xi}_l \alpha_k^r \wedge \bar{\alpha}_l^s \varepsilon_{rst} T_t \quad \in H^{1,1}(K3, \mathrm{End}\ \mathcal{T}_{K3}) \ . \tag{3.3.22}$$

This is a non-zero deformation of the background, which can possess selfdual and anti-selfdual components. We now show by contradiction that the selfdual component is trivial in cohomology. Any two-form in $H^2(\mathrm{End}\ \mathcal{T}_{K3})$ can be locally trivialized as

$$f^{(\mathbf{1},\mathbf{3})} = f^i \otimes \omega_i \ \in \ \Gamma(\mathrm{End}\ \mathcal{T}_{K3}) \otimes \Lambda^2(K3) \ , \tag{3.3.23}$$

where ω_i, $i = 1, ..., 6$, is a local basis of two-forms (on a four-dimensional space). Since the zero modes in α_k^r are $d_{\mathcal{A}}$-closed so is their product, $d_{\mathcal{A}} f^{(\mathbf{1},\mathbf{3})} = 0$. This is a local property which implies

$$0 = d_{\mathcal{A}}(f^i \otimes \omega_i) = (d_{\mathcal{A}} f^i) \wedge \omega_i + f^i (d\omega_i) \ . \tag{3.3.24}$$

Assume now that there exists a non-trivial selfdual component $f_+^{(\mathbf{1},\mathbf{3})} \in H_+^2(\mathrm{End}\ \mathcal{T}_{K3})$. Then (3.3.24) is satisfied for some selfdual two-forms ω_{i+}. In fact, $\Lambda_+^2(K3)$ is a three-dimensional vector space and we know from section 2.2 that there is a basis consisting of J_1, J_2, J_3. These two-forms ar d-closed and exist globally, hence, (3.3.24) implies

$$0 = d_{\mathcal{A}} f_+^{(\mathbf{1},\mathbf{3})} = (d_{\mathcal{A}} f^i) \wedge \omega_{i+} \ . \tag{3.3.25}$$

That is, there exists at least one global covariantly constant section $f_i \in \Gamma(\text{End}\,\mathcal{T}_{K3})$. However, since End $\mathcal{T}_{K3}$ is a nontrivial, irreducible bundle, no such section (except the zero section) exists. Equivalently, global sections are counted by the cohomology group $H^0(\text{End}\,\mathcal{T}_{K3})$ which we showed to be zero in section 2.4.2. This argument can be generalized to all terms $f_{\bar{2}}$ in (3.1.14) which take values in a non-trivial bundle.

$$f_+^{(\mathbf{1},\mathbf{3})} = f_+^{(\mathbf{56},\mathbf{2})} = 0 \ , \tag{3.3.26}$$

where all terms with the bundle moduli and/or the charged matter fields are included. It follows that $f_+^{(\mathbf{133},\mathbf{1})}$ is the only contribution to the scalar potential. This is consistent with the constraints of supergravity in six dimensions, where only a D-term potential exists containing the Killing vectors of the corresponding charged fields. We compute the selfdual component of (3.1.28) in appendix B.2. The result reads

$$f_+^{(\mathbf{133},\mathbf{1})} = \begin{pmatrix} \bar{C}_{ix} \\ \varepsilon_{xz} C_i^z \end{pmatrix}^T \begin{pmatrix} -ig_{ij}J & \frac{1}{2}\rho_{ij}\Omega \\ \frac{1}{2}\rho_{ij}\bar{\Omega} & ig_{ij}J \end{pmatrix} (\tau^n)^x{}_y \begin{pmatrix} C_j^y \\ \varepsilon^{yz}\bar{C}_{jz} \end{pmatrix} T_n \ . \tag{3.3.27}$$

In six-dimensional supergravity the D-term is valued in $\mathfrak{su}(2)_R$ as shown in (2.3.9). Therefore, from the matrix in (3.3.27) we can deduce that the R-symmetry of the six-dimensional theory is just given by the $SU(2) \cong SO(3)$ group which rotates the three complex structures (J_1, J_2, J_3) on $K3$. In fact, the D-term with values in the Pauli matrices $\sigma^{(s)}$ can be derived by taking the projection integrals

$$f_+^n = \tilde{D}^{n(s)} J_s \ , \qquad \tilde{D}^{n(s)} = \tfrac{1}{2} \int_{K3} f_+^n \wedge J_s \ . \tag{3.3.28}$$

Inserting this into (3.3.18) the full scalar potential takes the form[5]

$$\begin{aligned} V &= -\tfrac{\alpha'}{4\mathcal{V}} e^{\phi} \tilde{D}^{n(s)} \tilde{D}^{m(t)} \text{tr}(T_n T_m) \int_{K3} J_s \wedge J_t \\ &= \tfrac{\alpha'}{2\mathcal{V}} e^{\phi} \tilde{D}^{n(s)} \tilde{D}^{n(s)} \ . \end{aligned} \tag{3.3.29}$$

The actual D-term is then identified by including the volume factor as

$$\begin{aligned} D^{n(s)} &= \tfrac{1}{\sqrt{\mathcal{V}}} \int_{K3} f_+^n \wedge J_s \\ &= \bar{\Phi}_{ix}^T \big(\tfrac{1}{\sqrt{2\mathcal{V}}} \rho_{ij} \sigma^{(1)}, -i \tfrac{1}{\sqrt{2\mathcal{V}}} \rho_{ij} \sigma^{(2)}, -2ig_{ij}\sigma^{(3)} \big) (\tau^n)^x{}_y \Phi_j^y \ , \end{aligned} \tag{3.3.30}$$

where we abbreviated the half-hypermultiplet doublet as $\Phi_i^x = (C_i^x, \varepsilon^{xy}\bar{C}_{iy})^T$. Hence, our result for the scalar potential is consistent with the generic six-dimensional supergravity (2.4.35). It is a quartic potential for the charged matter fields which may contain flat directions (which depend on the $K3$ moduli). In these directions the moduli space of vacua has a Higgs branch, where the gauge group is broken further.

[5] The sign in V changes here because for anti-hermitean generators we have $\text{tr}(T_n T_m) = -\delta_{nm}$.

3.4 Geometry of the hyperscalar sigma model

In this section we interpret the hyperscalar sector of our derived six-dimensional Lagrangian $\mathcal{L}_\sigma$ as a non-linear sigma model. The scalar fields take values in a target space $\mathcal{M}$ whose metric is determined by their kinetic terms. Since compactification on a further torus, i.e. on $K3 \times T^2$, leaves the hypermultiplet sector unchanged, we can interpret the results as part of an $\mathcal{N} = 2$ locally supersymmetric theory in four dimensions. In both cases, supersymmetry restricts the target space of hyperscalars to be a quaternionic-Kähler (QK) manifold with a constant negative scalar curvature.[6]

The simplest non-trivial QK sigma models are given by the quaternionic projective space $\mathbb{HP}^n$ and its pseudo-Riemannian analogues. Using homogeneous quaternionic coordinates as in (2.3.17), the quaternionic Fubini-Study metric on $\mathbb{HP}^n$ for example reads [132]

$$ds^2 = \frac{\mathrm{tr}(dQ^\dagger \cdot dQ)}{\mathrm{tr}(Q^\dagger \cdot Q)} - \frac{\mathrm{tr}(Q^\dagger \cdot dQ)\ \mathrm{tr}(dQ^\dagger \cdot Q)}{\mathrm{tr}(Q^\dagger \cdot Q)^2}\ , \tag{3.4.1}$$

where $Q^\dagger \cdot Q = \sum_{a=0}^{n} Q^{a\dagger} Q^a$. Further examples are given by 'quaternionic-Kähler quotients' thereof [123, 131] or symmetric coset spaces of the form G/K with holonomy group $K \supset SU(2) \times Sp(n_H)$. Symmetric QK manifolds (with positive scalar curvature) are classified as the Wolf spaces [133]. They have non-compact analogues with indefinite metric signature. The other homogeneous non-symmetric spaces have been classified by Alekseevskii [134]. Further examples (of dimension $4n$) are given by a certain reduction, the 'superconformal quotient', of hyperkähler cones (of dimension $4n + 4$) [135]. From the type II string theory compactified on Calabi-Yau threefolds, another class of QK manifolds is known as 'special QK' or as QK manifolds in the image of the c-map [105]. These have the special property that there exists a special-Kähler submanifold inside, such that the full QK geometry is governed by the Kähler prepotential of the submanifold. The explicit c-map metric is also known under the name 'Ferrara-Sabharwal metric' [106]. It describes a fibre bundle of Ramond-Ramond scalars over the special-Kähler base space of complex structure deformations.

Our results for the kinetic terms in section 3.3.1 suggest that the scalar target space should also be viewed as certain fibre bundle. All coupling functions only depend on the geometric $K3$ moduli which therefore should span the base manifold. The union of the $K3$ moduli and B-field moduli, encoded in the moduli space $\tilde{\mathcal{M}}_{K3}$ in (2.2.35), is divided by the Yang-Mills scalars which only mix with the B-scalars. However, as mentioned before, the very definition of the matter fields requires the specification of $H^{1,1}(K3, \mathbb{R})$, i.e. a choice of specific complex structure on $K3$. This formally breaks the $SU(2)_R$ symmetry of the hypercomplex structure and hence, the common moduli space $\mathcal{M}_{K3}$ gets divided into complex structure and Kähler moduli.

[6] We refer to the review [130] for the mathematical definition of a quaternionic-Kähler manifold.

However, unlike for Calabi-Yau threefolds this factorization can only be locally defined at each point in the moduli space, and is somewhat arbitrary. It would be helpful to restore manifest $SU(2)_R$ symmetry in our final expressions. Hence, we consider as the base manifold the space of complex structures on $K3$ which is known to be the Grassmannian manifold [108]

$$\mathcal{M}_\Omega = \frac{SO(3,19)}{SO(2)\times SO(1,19)} \ , \tag{3.4.2}$$

of dimension 40.[7] For every fixed complex structure the Kähler form can be chosen as a positive vector in $H^{1,1}(K3,\mathbb{R}) \cong \mathbb{R}^{1,19}$. The volume modulus is counted separately. Hence, over each point in $\mathcal{M}_\Omega$ there is a fibre $\mathcal{M}_J$ of Kähler moduli. Next, over each point in $\mathcal{M}_J$ (and fixed gauge bundle topology) there is the fibre of charged matter fields $\mathcal{M}_C$, an independent fibre of bundle moduli $\mathcal{M}_\xi$ and the fibre of B-scalars $\mathcal{M}_B$. This fibration can be summarized by the projections

$$(\mathcal{M}_C \times \mathcal{M}_\xi)\times \mathcal{M}_B \longrightarrow \mathcal{M}_J \longrightarrow \mathcal{M}_\Omega \ . \tag{3.4.3}$$

The basis for the analysis of the sigma model target $\mathcal{M}$ space is its metric defined by the kinetic terms of section 3.3.1. Since this turns out to be quite involved we will consider in the following certain submanifolds of $\mathcal{M}$. As a first simplification we truncate the bundle moduli out off the theory because their coupling functions are only known in terms of abstract integrals. Second, we neglect the gauging of the matter fields under E_7, i.e. we replace the covariant derivatives by normal differentials. The remaining sigma model metric is then given by

$$\begin{aligned} ds^2 = {} & \tfrac{1}{4}h_{IJ}dt^I_s dt^J_s - \tfrac{1}{8\mathcal{V}^2}d\mathcal{V}d\mathcal{V} - g_{ij}d\bar{C}_{ix}dC^x_j \\ & - \tfrac{1}{6\mathcal{V}}g_{IJ}\big(db^I + M^I_{kl}(C^x_k d\bar{C}_{lx} - \bar{C}_{lx}dC^x_k) + N^I_{ij}\varepsilon_{xy}C^x_i dC^y_j + \bar{N}^I_{ij}\varepsilon^{xy}\bar{C}_{ix}d\bar{C}_{jy})\big)^2 \ . \end{aligned} \tag{3.4.4}$$

This metric should still be quaternionic-Kähler since we truncated a number of full hypermultiplets.

Let us collect some facts about the parametrization of the geometrical moduli space $\tilde{\mathcal{M}}_{K3}$ which we used so far. The 66 fields t^I_s, defined in (2.2.30), are not independent but subject to six normalization conditions (2.2.30) and three equivalence equations (2.4.36). Therefore, we first introduce a reference point $p \in \mathcal{M}_{K3}$ which is defined by the background values

$$\langle t^I_1, t^I_2, t^I_3\rangle = (\delta^I_1, \delta^I_2, \delta^I_3) \ , \quad \langle \mathcal{V}\rangle \in \mathbb{R}^+ \ , \tag{3.4.5}$$

and which satisfy the normalization (2.2.30). That is, p defines a fixed (but arbitrary) background metric on $K3$ where we choose the three-plane $H_+ \subset H^2(K3,\mathbb{R})$ to be

[7] Here we neglect modding out $\mathcal{M}_\Omega$ by the discrete automorphism group of $H^2(K3,\mathbb{Z})$.

spanned by the first three basis elements η_1, η_2, η_3. Infinitesimal metric fluctuations are then orthogonal to H_+ and satisfy the linear equation

$$\rho_{IJ}\langle t_r^I\rangle t_s^J = 0 \ . \tag{3.4.6}$$

Choosing a (non-integral) basis such that $\rho \propto \mathrm{diag}(1,1,1,-1,-1,\ldots)$, the fluctuations can be parametrized by $\mathcal{V}$ and

$$t_1^A, t_2^A, t_3^A \ , \quad A = 4,\ldots,22 \ , \tag{3.4.7}$$

which are 57 independent and unconstrained fields. They define the tangent directions in $T_p\mathcal{M}_{K3}$. Using these fields the only natural hypermultiplet pairings of $\tilde{\mathcal{M}}_{K3}$ are given by[8]

$$\begin{aligned} \Phi^A &= \{b^A, t_1^A, t_2^A, t_3^A\} \ , \quad A = 4,\ldots,22 \\ \Phi^0 &= \{\mathcal{V}, b^1, b^2, b^3\} \ . \end{aligned} \tag{3.4.8}$$

Note however that these scalars do not build complex doublets of $SU(2)_R$ as described in section 2.3.1, but rather (t_1^A, t_2^A, t_3^A) and (b^1, b^2, b^3) are triplets whereas b^A and $\mathcal{V}$ are singlets.

3.4.1 Charged scalar fibre

Before considering the total space let us study the charged matter fibre for fixed geometrical moduli and B-field. First, the charged matter fields occur as half-hypermultiplets, consistent with their pseudoreal gauge representation **56** of E_7. In contrast to the geometric moduli of $\tilde{\mathcal{M}}_{K3}$, the matter fields arise naturally as doublets of the $SU(2)_R$ symmetry, as we already manifestly used in the formulas above. Let us see how the $SU(2)_R$ acts on these fields. Recall from (3.1.8) that the Kaluza-Klein expansion is

$$a_{\bar{1}}^{(\mathbf{56},\mathbf{2})} = \big(C_j^x(\omega_j)^a + \varepsilon^{xy}\varepsilon^a{}_{\bar{b}}\bar{C}_{jy}(\bar{\omega}_j)^{\bar{b}}\big)T_{xa} \ , \tag{3.4.9}$$

where $(\omega_j)^a \in H^{0,1}(K3,\mathcal{T}_{K3})$ and $\varepsilon^a{}_{\bar{b}}(\bar{\omega}_j)^{\bar{b}} \in H^{1,0}(K3,\mathcal{T}_{K3})$. We know from section 2.2 that the tuple $(\mathcal{T}_{K3}, \overline{\mathcal{T}}_{K3})$ and hence also $(\mathcal{T}^*_{K3}, \overline{\mathcal{T}}^*_{K3})$, i.e. the tuple of $(1,0)$- and $(0,1)$-forms, transforms as a doublet under $SU(2)_R$. Therefore the coefficients of the one-forms in (3.4.9) also transform as the (dual) doublets

$$\Phi_j^x = \begin{pmatrix} C_j^x \\ \varepsilon^{xy}\bar{C}_{jy} \end{pmatrix} \ , \quad \bar{\Phi}_{jx} = \begin{pmatrix} \bar{C}_{jx} \\ \varepsilon_{xy}C_j^y \end{pmatrix} \ , \tag{3.4.10}$$

where we recognize the generic half-hypermultiplet structure (2.3.13). This is in fact the reason why we expressed all couplings as quadratic forms involving the doublets in (3.4.10).

[8] There is an analog statement in [98] due to conformal field theory methods.

Let us consider the sigma model metric (3.4.4) restricted to the charged fibre directions

$$ds_C^2 = -g_{ij} d\bar{C}_x^i dC^{jx} - \tfrac{1}{6\mathcal{V}} g_{IJ} \big(M_{kl}^I (C_k^x d\bar{C}_{lx} - \bar{C}_{lx} dC_k^x) + N_{ij}^I \varepsilon_{xy} C_i^x dC_j^y + \bar{N}_{ij}^I \varepsilon^{xy} \bar{C}_{ix} d\bar{C}_{jy} \big)^2 . \tag{3.4.11}$$

This metric resembles a Fubini-Study metric on the quaternionic projective space, written in homogeneous coordinates [131]. For a complete match with the Fubini-Study metric, the coupling functions should combine to a form which can be expressed entirely in terms of g_{ij}. This is however not the case and we rather find

$$\begin{aligned}
g_{IJ} N_{kl}^I \bar{N}_{mn}^J &= 4\rho_{kl}\rho_{mn} \ , \\
g_{IJ} N_{kl}^I N_{mn}^J &= g_{IJ} \bar{N}_{kl}^I \bar{N}_{mn}^J = 0 \ , \\
g_{IJ} M_{kl}^I N_{mn}^J &= i\sqrt{\tfrac{\mathcal{V}}{2}} (\rho_{lI}\rho_{kJ} + \rho_{kI}\rho_{lJ}) t_3^I (t_1^J - i t_2^J) \rho_{mn} \ , \\
g_{IJ} M_{kl}^I M_{mn}^J &= -2\mathcal{V} g_{kl} g_{mn} - 2\mathcal{V} t_{3k} t_{3l} t_{3m} t_{3n} \\
&\quad - \mathcal{V}\big[(t_{1k} + i t_{2k})(t_{1m} - i t_{2m}) t_{3l} t_{3n} + (t_{1l} + i t_{2l})(t_{1n} - i t_{2n}) t_{3k} t_{3m} \\
&\quad + (t_{1k} + i t_{2k})(t_{1n} - i t_{2n}) t_{3l} t_{3m} + (t_{1l} + i t_{2l})(t_{1m} - i t_{2m}) t_{3k} t_{3n} \big] \ .
\end{aligned} \tag{3.4.12}$$

We will see in the next section that these expressions simplify if we consider the limit of frozen complex structure moduli.

3.4.2 Fixed complex structure limit

Since the quaternionic geometry of the full sigma model is hard to handle, we study in this section truncations of the geometry which resemble the more familiar $\mathcal{N} = 1$ case in four dimensions. In particular, we search for Kähler submanifolds of the full target space. We saw in section 3.1 that the matter fields in the standard embedding require the specification of $\mathcal{T}_{K3}$ which requires a specific complex structure on $K3$. Therefore, we formally divided the geometric moduli space $\mathcal{M}_{K3}$ into the moduli space of complex structures and the moduli space of the Kähler class. This division is unnatural from the global point of view but can be done at least locally, in a neighborhood of each point in $\mathcal{M}_{K3}$. Clearly, the hypermultiplet structure is broken by this division. Recall from section 2.2 that a Ricci-flat metric on $K3$ is given by the orthonormalized triple $(J_1, J_2, J_2) \in H^2(K3, \mathbb{R})$ modulo $SO(3)$ rotations. We define the Kähler form J and the holomorphic two-form by

$$J = \sqrt{\mathcal{V}} J_3 \ , \quad \Omega = \tfrac{1}{\sqrt{2}} (J_1 + i J_2) \ . \tag{3.4.13}$$

The corresponding moduli fields t_1^I, t_2^I, t_3^I showed up in the coupling functions (3.3.12) already in a distinguished manner. We take the complex structure moduli space $\mathcal{M}_\Omega$

in (3.4.2) as the base which is parametrized globally by a set of 44 homogeneous coordinates t_1^I, t_2^I subject to the conditions

$$\rho_{IJ} t_a^I t_b^J = \delta_{ab} \ , \qquad t_a^I \sim \tilde{t}_a^I = R_a{}^b t_b^I \ , \quad R \in SO(2) \ . \tag{3.4.14}$$

Locally, it can be parametrized by choosing a reference point $q \in \mathcal{M}_\Omega$ via the background values, say

$$(\langle t_1^I \rangle, \langle t_2^I \rangle) = (\delta_1^I, \delta_2^I) \ . \tag{3.4.15}$$

Here we take the two-plane $\Omega \in H^2(K3, \mathbb{R})$ to be spanned by the first two basis elements η_1, η_2. Infinitesimal deformations are then orthogonal to Ω. Choosing a basis of $H^2(K3, \mathbb{R})$ such that $\rho = \mathrm{diag}(1, 1, 1, -1, -1, \dots)$ the deformations can be parametrized by a set of 40 independent fields

$$t_1^i \ , t_2^i \ , \quad i = 3, \dots, 22 \ . \tag{3.4.16}$$

For every fixed complex structure, the Kähler form can be chosen as a positive vector in $H^{1,1}(K3, \mathbb{R}) \cong \mathbb{R}^{1,19}$, yielding twenty Kähler moduli t^i (with the volume modulus included). They are related to the previous moduli t_3^i by

$$t^i = \sqrt{\mathcal{V}} t_3^i \ , \quad \rho_{ij} t^i t^j = 2\mathcal{V} \ . \tag{3.4.17}$$

Together, two unphysical moduli are included for the time being, which can be eliminated by taking the full hypercomplex symmetry $SO(3)$ into account.

We now show that, given a fixed complex structure, the remaining (complexified) Kähler moduli span a Kähler submanifold. Let us choose the reference point q from (3.4.15) and freeze the complex structure deformations $t_{1,2}^I = dt_{1,2}^I = 0$. The sigma model metric of $\mathcal{M}_{K3}$ in this case reduces to

$$ds^2_{\text{fixed c.s.}} = \tfrac{1}{4} h_{ij} dt_3^i dt_3^j - \tfrac{1}{8\mathcal{V}^2} d\mathcal{V} d\mathcal{V} = -\tfrac{1}{4\mathcal{V}} g_{ij} dt^i dt^j \ , \tag{3.4.18}$$

where g_{ij} is the coupling function (3.3.8).[9] This metric can also be written explicitly in a Fubini-Study-like form (with homogeneous coordinates)

$$ds^2_{\text{fixed c.s.}} = \frac{dt \cdot dt}{2t \cdot t} - \frac{(t \cdot dt)(t \cdot dt)}{(t \cdot t)^2} \ , \tag{3.4.19}$$

where we abbreviated $t \cdot t = \rho_{ij} t^i t^j$. From (3.4.18) we see that in this limit, the leading coupling functions of t^i and b^i coincide to be g_{ij}. Hence, if we additionally freeze b^1 and b^2 the remaining fields combine to complexified Kähler moduli whose metric reads

$$ds^2_{\text{compl.K.}} = -\tfrac{1}{4\mathcal{V}} g_{ij} dt^i dt^j - \tfrac{1}{6\mathcal{V}} g_{ij} db^i db^j = K_{ij} dT^i d\bar{T}^j \ . \tag{3.4.20}$$

[9] Note that the background values from (3.4.15) disappear from the metric expression due to the orthogonality condition $\rho_{IJ} \langle t_{1,2}^I \rangle t_3^J = 0 = \rho_{IJ} \langle t_{1,2}^I \rangle dt_3^J$.

The Kähler potential given by

$$K = \log\big(\rho_{ij}(T^i + \bar{T}^j)(T^j + \bar{T}^j)\big)\ , \quad T^i = \tfrac{1}{2}t^i + \tfrac{i}{\sqrt{6}}b^i\ . \tag{3.4.21}$$

This is just the standard Kähler potential of the Kähler moduli, i.e. $K = \log(2\mathcal{V})$. The submanifold of $\tilde{\mathcal{M}}_{K3}$ described by (3.4.21) is the fibre of complexified Kähler moduli over the base of complex structure moduli. This result is invariant under the hypercomplex symmetry $SO(3)$ because for any other choice of the Kähler form, different from (3.4.13), an equivalent submanifold exists.

3.4.3 Inclusion of the matter fields

Starting from the fixed complex structure limit, described in the previous section, we now include the charged matter fields. The truncation of the sigma model metric (3.4.4) takes the form

$$\begin{aligned} d\tilde{s}^2 = & -\tfrac{1}{4\mathcal{V}}g_{ij}dt^i dt^j - g_{ij}d\bar{C}_{ix}dC^x_j \\ & -\tfrac{1}{6\mathcal{V}}g_{ij}\big(db^i + M^i_{kl}(C^x_k d\bar{C}_{lx} - \bar{C}_{lx}dC^x_k)\big)\big(db^j + M^j_{kl}(C^x_k d\bar{C}_{lx} - \bar{C}_{lx}dC^x_k)\big) \end{aligned} \tag{3.4.22}$$

Recall that the terms involving N^i_{kl} were absorbed by the redefinition (3.3.16) of b^1 and b^2 which then are frozen. All couplings now only depend on t^i, hence we can interpret the matter fields as fibred over the (complexified) Kähler moduli space. Moreover, in the fixed complex structure limit we have the following algebraic equation between coupling functions[10]

$$g_{ij} = -\tfrac{i}{2\mathcal{V}}\rho_{kl}M^l_{ij}t^k\ , \tag{3.4.23}$$

which is similar to a relation in the Calabi-Yau three-fold case [91]. The charged scalar fibre metric now takes the form

$$d\tilde{s}^2_C = -g_{ij}d\bar{C}_{ix}dC^x_j + \tfrac{1}{3}(g_{kl}g_{mn} + t_{3k}t_{3l}t_{3m}t_{3n})(C^x_k d\bar{C}_{lx} - \bar{C}_{lx}dC^x_k)(C^y_m d\bar{C}_{ny} - \bar{C}_{ny}dC^y_m)\ . \tag{3.4.24}$$

which resembles the metric on the quaternionic projective space [123]. Note that the metric is diagonal in the **56**-index such that the charged fibre has the structure $\mathcal{C}^{\otimes 56}$ for some manifold $\mathcal{C}$. It is tempting to expect that the full metric (3.4.22) can be described by the standard modification of the Kähler potential

$$K = \log(2\mathcal{V}) = \log\big(\rho_{ij}(T^i + \bar{T}^j + 2iM^i_{kl}\bar{C}_{kx}C^x_l)(T^j + \bar{T}^j + 2iM^i_{kl}\bar{C}_{kx}C^x_l)\big)\ , \tag{3.4.25}$$

where the complexified Kähler moduli are now defined by [92, 93]

$$T^i = \tfrac{1}{2}t^i + \tfrac{i}{\sqrt{6}}b^i - iM^i_{kl}(t)\bar{C}_{kx}C^x_l\ . \tag{3.4.26}$$

[10] We did not find a simple generalization of this equation (including the functions N^I_{kl}) in the non-fixed complex structure case. Regarding only the complex structure moduli, we have the similar relation $\rho_{ij} = \frac{1}{4}\rho_{KL}N^L_{ij}(t^K_1 + it^K_2)$.

However, we find that the metric derived from (3.4.25) not only reproduces the terms in (3.4.22) but also additional terms, most importantly mixing terms between matter and geometric moduli $\sim \bar{C}dCdt + c.c.$. We derive them in appendix B.3. Clearly, they arise due to the moduli dependence of the coupling function $M^i_{kl}(t)$. One may conjecture that the unexpected mixing terms also occur in the dimensional reduction if the Lorenz-Chern-Simons forms are properly taken into account.

3.4.4 Isometries

For the identification of the full quaternionic-Kähler space it might be useful to find the isometries of the metric (3.4.22). For example, it is known that $4n$-dimensional QK manifolds with $n+1$ abelian isometries are in the image of the superconformal quotient and can be classified by a homogeneous function. In this case, the (neutral) hypermultiplets are dual to superconformal tensor multiplets [135–137].

Considering the standard embedding on $K3$ (and neglecting the bundle moduli) we have a spectrum of twenty moduli-hypermultiplets and $20 \cdot 56$ matter half-hypermultiplets, spanning a target space of real dimension 2320. Let us consider the isometries of the hyperscalar sigma model which can be read off from the coordinate expression of the metric (3.4.22). As already mentioned, the charged fibre has the structure $\mathcal{C}^{\otimes 56}$ such that we clearly have $SO(56)$ as part of the isometry group, containing 28 abelian isometries. Additionally, there are circular translations inside each $\mathcal{C}$

$$C^x \mapsto e^{i\phi_x} C^x \ , \tag{3.4.27}$$

constituting 56 further abelian isometries. From the charged fibre metric (3.4.24) we see that the couplings are not diagonal in the generation indices $i, j = 1, \ldots, 20$. Therefore, the isometries (3.4.27) only act simultaneously on all generations. Isometries mixing the generations are only given by the combined action of $O(1,19)$

$$t^i \mapsto O^i{}_j t^j \ , \quad b^i \mapsto O^i{}_j b^j \ , \quad C^{xi} \mapsto O^i{}_j C^{xj} \ . \tag{3.4.28}$$

This corresponds to base transformations in $H^{1,1}(K3,\mathbb{R})$ for a fixed complex structure and contains ten abelian isometries. We also expect the full metric (3.4.4) to be invariant under $SO(3)$ rotations of the hyperkähler structure of $K3$, i.e. under $t^I_s \ \mapsto \ R_s{}^t \ t^I_t$ which is however not manifest in our formulation. Together, the low number of abelian isometries and the fact that we are dealing with charged hypermultiplets seems to make a description in terms of the dual tensor multiplets impossible.

3.4.5 Orbifold limit

In this section we perform a further truncation of the hyperscalar sigma model and compare it to a known orbifold limit of the heterotic string compactified on $T^4/\mathbb{Z}_3$.

An orbifold of $K3$ is a singular deformation limit of its metric such that the non-zero curvature gets asymptotically concentrated at separate discrete points. Apart from these points the manifold is flat such that the string world-sheet can be quantized rigorously on this background. The singular geometry can be described by a flat torus T^4 acted upon by some discrete symmetry operation $\mathbb{Z}_N$. The symmetry action typically has fixed points with non-trivial holonomy, therefore these fixed points can be identified with the singular points of infinite curvature. The massless string excitations on an orbifold are divided into the 'twisted' and 'untwisted' states. The former are states located at the fixed points while the latter are states propagating in the 'bulk', i.e. the flat region in between. The supergravity approximation of string theory assumes that $l_s \ll L$ where L is any characteristic length scale of the compactified space. For orbifold limits of $K3$ it is known that certain two-cycles shrink to zero at the fixed points which clearly violates this assumption. As a consequence, to compare the supergravity approximation to orbifold string calculations it is more reliable to consider only the untwisted state sector.

There exists an orbifold construction which mimics the standard embedding on $K3$, realized by a non-trivial gauge shift vector [117].[11] In this case the unbroken gauge group in six dimensions is $E_7 \times E_8$ and the massless spectrum of the standard embedding is reproduced

$$(\mathbf{56}, \mathbf{1})^{\text{untw}} \oplus 2(\mathbf{1}, \mathbf{1})^{\text{untw}} \oplus 9(\mathbf{56}, \mathbf{1})^{\text{tw}} \oplus 45(\mathbf{1}, \mathbf{1})^{\text{tw}} \oplus 18(\mathbf{1}, \mathbf{1})^{\text{tw}} \,. \tag{3.4.29}$$

In the untwisted sector there are two hypermultiplets of geometrical moduli and one matter hypermultiplet. Furthermore, it is known that the hyperscalar sigma model of the untwisted orbifold spectrum is described by the coset space

$$\mathcal{M}_{\text{orb}} = \frac{SU(2, 2+56)}{U(1) \times SU(2) \times SU(2+56)} \,, \tag{3.4.30}$$

obtained by a compactifying the heterotic string on the torus T^4 and then performing a suitable truncation [94]. The manifold (3.4.30) is known to be quaternionic-Kähler and Kähler, with a metric determined by the Kähler potential

$$K = -\log\det(T + T^\dagger - 2\Phi^x \Phi^\dagger_x) \,. \tag{3.4.31}$$

Here $\Phi^x = (\Phi^x_1, \Phi^x_2)$ is a doublet of complex fields constituting one hypermultiplet as in (2.3.12) and T is a 2×2 complex matrix given by

$$(T_{ij}) = \begin{pmatrix} g_{1\bar{1}} + iB_{1\bar{1}} + \Phi_1\bar{\Phi}_1 & g_{12} + iB_{12} + \Phi_1\bar{\Phi}_2 \\ \bar{g}_{12} + i\bar{B}_{12} + \Phi_2\bar{\Phi}_1 & g_{2\bar{2}} + iB_{2\bar{2}} + \Phi_2\bar{\Phi}_2 \end{pmatrix} \,. \tag{3.4.32}$$

It contains the real $g_{1\bar{1}}$, $g_{2\bar{2}}$ and the complex g_{12} metric elements and the the corresponding components of the B-field. Here the **56**-index was omitted such that

[11] The shift vector corresponds to a vector in the Cartan subalgebra of $E_8 \times E_8$ which partly defines a non-trivial line bundle.

$\Phi_i\bar{\Phi}_j = \Phi_i^x\bar{\Phi}_{jx}$ includes a summation over the 56 components. In the following we compare the orbifold result with the fixed complex structure limit of our smooth compactification. On the orbifold side, the fixed complex structure limit can be realized by setting $g_{12} = 0$. In this limit, the Kähler potential (3.4.31) yields the kinetic terms

$$K_{T_{ij}\bar{T}_{kl}} dT_{ij} d\bar{T}_{kl} = \tfrac{1}{4g_{1\bar{1}}^2} dT_{11} d\bar{T}_{11} + \tfrac{1}{4g_{2\bar{2}}^2} dT_{22} d\bar{T}_{22} + \tfrac{1}{4g_{1\bar{1}}g_{2\bar{2}}} (dT_{12} d\bar{T}_{12} + dT_{21} d\bar{T}_{21}) \ , \quad (3.4.33)$$

$$K_{\Phi_i\bar{\Phi}_j} d\Phi_i d\bar{\Phi}_j = (\tfrac{1}{g_{1\bar{1}}} + \tfrac{\Phi_2\bar{\Phi}_2}{g_{1\bar{1}}g_{2\bar{2}}} + \tfrac{\Phi_1\bar{\Phi}_1}{g_{1\bar{1}}^2}) d\Phi_1 d\bar{\Phi}_1 + (\tfrac{1}{g_{2\bar{2}}} + \tfrac{\Phi_1\bar{\Phi}_1}{g_{1\bar{1}}g_{2\bar{2}}} + \tfrac{\Phi_2\bar{\Phi}_2}{g_{2\bar{2}}^2}) d\Phi_2 d\bar{\Phi}_2 \ , \quad (3.4.34)$$

$$K_{T_{ij}\bar{\Phi}_k} dT_{ij} d\bar{\Phi}_k = -\tfrac{\Phi_1}{2g_{1\bar{1}}^2} dT_{11} d\bar{\Phi}_1 - \tfrac{\Phi_2}{2g_{2\bar{2}}^2} dT_{22} d\bar{\Phi}_2 - \tfrac{\Phi_2}{2g_{1\bar{1}}g_{2\bar{2}}} dT_{12} d\bar{\Phi}_1 - \tfrac{\Phi_1}{2g_{1\bar{1}}g_{2\bar{2}}} dT_{21} d\bar{\Phi}_2 \ . \quad (3.4.35)$$

Inserting (3.4.32) we get the kinetic terms in terms of the Kaluza-Klein modes [93, 94]. The leading term for the charged scalars reads

$$\sum_{i=1,2} \tfrac{1}{g_{i\bar{i}}} d\Phi_i d\bar{\Phi}_i \ . \quad (3.4.36)$$

The terms for the two complexified Kähler moduli read

$$\sum_{i=1,2} \tfrac{1}{4g_{i\bar{i}}^2} |dg_{i\bar{i}} + idB_{i\bar{i}} + \bar{\Phi}_i d\Phi_i - \Phi_i d\bar{\Phi}_i|^2 \ . \quad (3.4.37)$$

The terms for the off-diagonal fields in T read

$$\tfrac{1}{4g_{1\bar{1}}g_{2\bar{2}}} \left(|idB_{12} + \Phi_1 d\bar{\Phi}_2 - \bar{\Phi}_2 d\Phi_1|^2 + |i\bar{dB_{12}} + \Phi_2 d\bar{\Phi}_1 - \bar{\Phi}_1 d\Phi_2|^2 \right) \ . \quad (3.4.38)$$

We now compare the above kinetic couplings with our results (3.3.3) coming from the smooth $K3$. To make contact with the ones just derived, we have to take the orbifold limit and identify the $K3$ moduli related to $g_{i\bar{i}}$. The $T^4/\mathbb{Z}_3$ limit of $K3$ corresponds to taking the three-plane H_+ orthogonal to 18 two-cycles with intersection matrix[12]

$$A_2^{\oplus 9} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}^{\oplus 9} \ , \quad (3.4.39)$$

which corresponds to a vector space of signature $(0, 18)$. The remaining geometrical moduli space is given by the embedding $H_+ \subset \mathbb{R}^{3,1} \subset H_2(K3, \mathbb{Z})$. The basis elements of $\mathbb{R}^{3,1}$ correspond to the two complex two-tori (that we call η_1, η_2) spanned by the coordinates z^i, plus two two-cycles (called η_3, η_4) with positive self-intersection and that are not of type $(1, 1)$. In the following we denote by η_i the two-cycles as well as their Poincaré dual two-forms. They have the following intersection matrix:

$$\begin{pmatrix} 0 & 3 & & \\ 3 & 0 & & \\ & & 2 & -1 \\ & & -1 & 2 \end{pmatrix} \ . \quad (3.4.40)$$

[12]The statements about intersection matrices of singular limits of $K3$ have been computed by Roberto Valandro.

The complex structure is fixed by restricting Ω to live in the positive definite subspace spanned by $\{\eta_3, \eta_4\}$. Then the remaining Kähler moduli are given by the expansion

$$J = t^1\,\eta_1 + t^2\,\eta_2 \tag{3.4.41}$$

and can be identified as $g_{i\bar{i}} \leftrightarrow t^i$. The volume modulus is given by $\mathcal{V} = 6t^1t^2$. Similarly, we consider the Kalb-Ramond field restricted to η_1, η_2 such that

$$B = b^1\eta_1 + b^2\eta_2 \tag{3.4.42}$$

and we can identify $B_{i\bar{i}} \leftrightarrow b^i$.

The coupling function in front of (3.4.37) matches with our formula (up to a numerical constant) as the truncation of the smooth result reads

$$\tfrac{1}{\mathcal{V}}g_{ij} = \tfrac{1}{\mathcal{V}}\int_{K3}\eta_i\wedge\star\eta_j = \tfrac{1}{6t^1t^2}\begin{pmatrix}\frac{t^2}{t^1} & \\ & \frac{t^1}{t^2}\end{pmatrix} = \begin{pmatrix}\frac{1}{6(t^1)^2} & \\ & \frac{1}{6(t^2)^2}\end{pmatrix} \tag{3.4.43}$$

The truncation of our leading coupling of the charged matter fields reads

$$g_{ij} = \int_{K3}\eta_i\wedge\star\eta_j = \begin{pmatrix}\frac{t^2}{t^1} & \\ & \frac{t^1}{t^2}\end{pmatrix}\ , \tag{3.4.44}$$

which does not match with (3.4.36) but suggests the correspondence

$$\Phi_1 = \sqrt{t^2}C_1\ , \qquad \Phi_2 = \sqrt{t^1}C_2\ . \tag{3.4.45}$$

The same redefinition also works for the match of the coupling function M^I_{kl} which is linear in the Kähler moduli. The truncation of our smooth result reads

$$\begin{aligned} &M^1_{12} = M^2_{12} = M^2_{11} = M^1_{22} = 0\ , \\ &M^1_{11} = 6it^2\ , \qquad M^2_{22} = 6it^1\ . \end{aligned} \tag{3.4.46}$$

However, the redefinition (3.4.45) is problematic since then the Kähler potential (3.4.31) leads to additional kinetic terms as $\sim C_1\bar{C}_1 dt^2 dt^2$ which are unexpected. This problem may be connected to the similar undesired kinetic terms mentioned in the previous section. A possible way out is a refinement of the isomorphism (3.1.10) at the very first steps of the calculation according to

$$(\omega_j)_{\bar{\alpha}}^{\ \beta} = \tfrac{\gamma_j}{\|\Omega\|^2}\bar{\Omega}^{\alpha\beta}(\eta_j)_{\alpha\bar{\alpha}}\ , \tag{3.4.47}$$

where we added a moduli dependent function

$$\gamma_j = \Big(\int_{K3} J\wedge\eta_j\Big)^{-\frac{1}{2}}\ . \tag{3.4.48}$$

If the zero modes ω_j are treated as constant over six-dimensional spacetime, the results from the Kähler potential (3.4.31) are reproduced in the orbifold limit. However, the general sigma model gets complicated dramatically by this change as the simple linear moduli dependence of M^I_{kl} is modified to include square root dependences. Also it should be noted that there is no reason from first principles to include the function γ_j in the isomorphism of zero modes.

Chapter 4

Effective action from line bundles

In this chapter we consider heterotic compactifications on $K3$ different from the standard embedding, i.e. $\mathcal{F} \neq \mathcal{R}$, which satisfy the integrated tadpole condition (2.1.6). Although no Kalb-Ramond flux exists on $K3$ due to $H^3(K3, \mathbb{R}) = 0$, we generically have $\mathcal{H} \neq 0$ such that

$$d\mathcal{H} = \alpha'(\text{tr } \mathcal{R} \wedge \mathcal{R} - \text{tr } \mathcal{F} \wedge \mathcal{F}) \tag{4.0.1}$$

is satisfied locally. The back-reaction onto the internal geometry is well known and supersymmetric solutions satisfy the Strominger equations [76]

$$\begin{aligned} d^* J &= i(\partial - \bar{\partial})\ln\|\Omega\| \ , \\ \mathcal{H} &= \tfrac{i}{2}(\bar{\partial} - \partial) J \ , \\ \|\Omega\|^2 &= e^{-4(\Phi+\Phi_0)} \ , \end{aligned} \tag{4.0.2}$$

where J is the fundamental two-form of a complex structure and Ω is a holomorphic volume form. In particular, $\mathcal{H} \neq 0$ implies that J is not closed, thus Kähler manifolds (and hence Calabi-Yau manifolds) are excluded. In the case of four compact dimensions, it is known that the Strominger equations can be solved by a conformal deformation of the Ricci-flat $K3$ metric

$$g_{mn} = e^{2\Phi} g_{mn}^{K3} \ , \tag{4.0.3}$$

yielding a 'conformal Calabi-Yau' [76]. The hermitean Yang-Mills equations, relating $\mathcal{F}$ and J, are conformally invariant and hence unaffected. Therefore, the analysis of section 2.4 can be applied to the conformal $K3$ without changes. The simplest gauge bundles, apart from the standard embedding, are given by principal bundles with structure group $U(1)$. These backgrounds have been studied intensely [56, 79, 84, 85, 138–143] for being able to yield smaller unbroken gauge groups, including unbroken

$U(1)$ factors, and standard model like particle spectra.[1] All irreducible associated vector bundles are one-dimensional, hence these backgrounds are also denoted as complex line bundles.

In the present chapter we consider $K3$ compactifications with line bundles, where the tadpole condition is solved by assigning $\mathcal{F}$ to be the curvature of one (or several) principal $U(1)$ bundle(s) . Let us, for simplicity, describe a background with one $U(1)$ bundle L inside one E_8 factor. The gauge group is then broken according to

$$E_8 \longrightarrow G \times \langle U(1) \rangle \ , \tag{4.0.4}$$

where G is the commutant of $U(1)$ inside E_8. Since the Lie algebra of $U(1)$ is $i\mathbb{R}$, its field strength is a usual differential two-form. Moreover, $U(1)$ bundles are completely characterized by their first Chern class $c_1(L)$ given by

$$c_1(L) = i\, \mathrm{tr}\mathcal{F} \in H^2(K3, \mathbb{Z}) \ . \tag{4.0.5}$$

Therefore, line bundles are equivalent to integrally quantized, two-form fluxes.[2] A line bundle is specified by a vector X in the Cartan subalgebra $E_8 \times E_8$ and an integral linear combination of the two-cycles of $K3$.[3] X determines the group theoretical embedding and the unbroken gauge group while the two-cycles determine the location of the flux. It can be expanded as

$$i\mathcal{F} = X \otimes m^I \eta_I \ , \qquad I = 1, ..., 22 \ , \tag{4.0.6}$$

with η_I being an integral basis of $H^2(K3, \mathbb{Z})$. Note that in our convention $\mathcal{F}$ is anti-hermitean, i.e. imaginary in the Abelian case, such that $i\mathcal{F}$ is a real valued two-form. The flux satisfies the quantization condition

$$i \int_{\Gamma^I} \mathrm{tr}\mathcal{F} = -\|X\| \; m^I \quad \in \mathbb{Z} \ , \tag{4.0.7}$$

for all integral two-cycles $\Gamma_I \in H_2(K3, \mathbb{Z})$. Here $\|X\|$ is the Euclidean norm in the Cartan subalgebra of E_8. If there are several line bundles we label their field strengths by $\mathcal{F}^n \ , \ n = 1, \ldots, N$

$$i\mathcal{F}^n = X^n \otimes m^{In} \eta_I \ . \tag{4.0.8}$$

[1] In the standard embedding further gauge symmetry breaking of E_7 (or E_6 for Calabi-Yau three-folds) to the standard model gauge group is problematic due to the absence of adjoint scalar multiplets. Non-simply connected Calabi-Yau manifolds and discrete Wilson lines have to be applied in this case.

[2] There exist no abelian local instantons on $K3$ because in four dimensions these are characterized by the winding number of the map $S^3 \mapsto U(1)$ which is trivial, i.e. $\pi_3(U(1)) = 0$.

[3] The specific choice of two-cycles can be motivated by making contact with heterotic orbifold models which arise as singular limits of $K3$ with shrinking two-cycles [56, 57].

Since $E_8 \times E_8$ has rank 16, there are at most 16 independent line bundles available. The integrated tadpole condition then takes the form

$$24 = \tfrac{1}{2} \int_{K3} \mathrm{tr}(\mathcal{F} \wedge \mathcal{F}) \ = -\tfrac{1}{2}(X^n \cdot X^m)\ m^{In} m^{Jm} \rho_{IJ}\ . \tag{4.0.9}$$

Here $\cdot$ is the Euclidean scalar product in the Cartan subalgebra and ρ_{IJ} is the two-cycle intersection matrix of $K3$. The hermitean Yang-Mills equations (2.4.1) for line bundles take a particular simple form. If we denote by $\mathcal{F} \in H_2(K3, \mathbb{Z})$ the two-cycle combination Poincaré dual to the flux and by $H_+ \in H_2(K3, \mathbb{R})$ the positive three-plane defining the $K3$ metric, the bundle is hermitean Yang-Mills if

$$\mathcal{F} \perp H_+\ . \tag{4.0.10}$$

Here orthogonality is defined with respect to the bilinear intersection form ρ_{IJ}. Due to the quantization condition (4.0.7), the gauge flux is rigid and cannot be deformed continuously. According to the general argument in section 3.3.2, this stabilizes some of the geometrical moduli, as we will show in section 4.3.1.

In the following sections we derive the six-dimensional effective action. However, in contrast to the standard embedding, our result involves abstract integrals which we cannot solve explicitly such that the moduli dependence is hidden. Hence, we can only show here that the effective action is consistent with six-dimensional supergravity without knowing the details of the hypermultiplet sigma model. Instead we focus on two aspects which are peculiar for line bundle compactifications: moduli stabilization via Fayet-Iliopoulos terms and the occurrence of Stückelberg masses for (some of) the unbroken $U(1)$ gauge bosons.

4.1 Yang-Mills sector

For simplicity, let us assume that the background is given by a single $U(1)$ bundle over $K3$ as in (4.0.4). The massless matter spectrum is determined by the decomposition of the adjoint representation

$$\mathbf{248} \longrightarrow (\mathfrak{g}, \mathbf{1}_0) \oplus (\mathbf{1}, \mathbf{1}_0) \bigoplus_i \big((\mathbf{R}_i, \mathbf{1}_{q_i}) \oplus (\overline{\mathbf{R}}_i, \mathbf{1}_{-q_i})\big)\ , \tag{4.1.1}$$

where $\mathfrak{g}$ is the adjoint representation of G while the second term includes $\mathbf{1}_0$ as the adjoint representation of $U(1)$. The $\mathbf{R}_i$ are model dependent representations of G and $\mathbf{1}_{q_i}$ are representations of $U(1)$ with charge q_i. The right entries define associated vector bundles $E_{\mathbf{1}_q}$ which are tensor products of the line bundle L with charge q:

$$E_{\mathbf{1}_q} = L^q = L \otimes ... \otimes L\ . \tag{4.1.2}$$

Negative charges correspond to the dual bundle, $L^{-1} = L^*$, and $L^0 = \mathcal{O}$ is the trivial bundle. Since $\mathbf{1}_q$ is a complex representation for $q \neq 0$, the matter fields occur as full hypermultiplets in vector-like representations. Applying the deformation theory of gauge connections to this setup yields the multiplicities of the corresponding massless fields. Specifically, one finds via the chiral index theorem

$$h^{0,1}(L^q) = -2 - q^2 ch_2(L) \ , \tag{4.1.3}$$

where $ch_2(L) = -\frac{1}{2}\int \mathrm{tr}\mathcal{F} \wedge \mathcal{F}$ is the second Chern-character. Moreover, no bundle moduli exist, as $\mathfrak{u}(1) = E_{\mathbf{1}_0}$ is the trivial line bundle. In other words, the endomorphism bundle is End $L = L \otimes L^* = L^0$ such that all transition functions can be chosen to be the identity. According to (2.4.39) the multiplicity of bundle moduli is given by

$$H^{0,1}(\mathrm{End}\ L^q) = H^{0,1}(K3, \mathbb{R}) = 0 \ . \tag{4.1.4}$$

Applying the analysis from section 2.4.2, the Kaluza-Klein expansion of the gauge potential reads

$$\begin{aligned} a_1 &= V^{\mathfrak{g}} + V^{\mathbf{1}} \ , \\ a_{\bar{1}} &= \sum_i \ (C^{\mathbf{R}_i}_{k_i} \otimes \omega^{q_i}_{k_i} + \bar{C}^{\overline{\mathbf{R}}_i}_{k_i} \otimes \bar{\omega}^{-q_i}_{k_i}) + (\bar{D}^{\overline{\mathbf{R}}_i}_{k_i} \otimes \bar{\varpi}^{-q_i}_{k_i} + D^{\mathbf{R}_i}_{k_i} \otimes \varpi^{q_i}_{k_i}) \ . \end{aligned} \tag{4.1.5}$$

Here $V^{\mathfrak{g}}$ is the six-dimensional gauge potential in the adjoint representation of G. Additionally, there is the abelian gauge potential $V^{\mathbf{1}}$, hence, the $U(1)$ defined by the line bundle is (possibly) part of the unbroken gauge group in six dimensions. We will see in section 4.3.2 however that generically the gauge fluxes generate Stückelberg masses for the abelian gauge bosons. For $q_i \neq 0$ the representations in (4.1.1) are complex and always occur pairwise, with corresponding charged scalars C_{k_i} and $\bar{D}_{k_i}$, respectively. Their four real degrees of freedom align in one hypermultiplet in the representation $\mathbf{R}_i \oplus \overline{\mathbf{R}}_i$. The zero modes belong to

$$\begin{aligned} \omega^{q_i}_{k_i} &\in H^{0,1}(L^{q_i}) \ , \qquad \bar{\omega}^{-q_i}_{k_i} \in H^{1,0}(L^{-q_i}) \ , \\ \varpi^{q_i}_{k_i} &\in H^{1,0}(L^{q_i}) \ , \qquad \bar{\varpi}^{-q_i}_{k_i} \in H^{0,1}(L^{-q_i}) \ , \end{aligned} \tag{4.1.6}$$

with multiplicities $k_i = 1, ..., h^{0,1}(L^{q_i})$. Moreover, according to (2.4.24) the zero modes are annihilated by the operators $d_{\mathcal{A}}, \Delta_{\bar{\partial}_{\mathcal{A}}}$ and $\Delta_{\partial_{\mathcal{A}}}$. The scalars are grouped into a hypermultiplet as a doublet of $SU(2)_R$

$$\Phi^{\mathbf{R}_i}_{k_i} = (C^{\mathbf{R}_i}_{k_i}, D^{\mathbf{R}_i}_{k_i}) \ . \tag{4.1.7}$$

The Kaluza-Klein expansion of the field strength is performed analog to (3.1.13) such that we only give the result here

$$f = f^{\mathbf{1}}_2 + f^{\mathfrak{g}}_2 + \sum_i (f^{\mathbf{R}_i}_{1,\bar{1}} + \bar{f}^{\overline{\mathbf{R}}_i}_{1,\bar{1}}) + f_{\bar{2}} \ . \tag{4.1.8}$$

Here $f_2^{\mathbf{1}} = dV^{\mathbf{1}}$ is the abelian field strength and $f_2^{\mathfrak{g}} = dV^{\mathfrak{g}} + \frac{1}{2}[V^{\mathfrak{g}}, V^{\mathfrak{g}}]$ is the non-abelian field strength in six dimensions. The terms with one external and one internal tangent index give rise to gauge covariant derivatives of the charged scalars,

$$f_{1,\bar{1}}^{\mathbf{R}_i} = \mathcal{D}\Phi_{k_i}^{\mathbf{R}_i} \wedge \omega_{k_i}^{q_i}, \qquad \mathcal{D}\Phi^{\mathbf{R}_i} = d\Phi^{\mathbf{R}_i} - q_i V^{\mathbf{1}} \Phi^{\mathbf{R}_i} - V^a(\tau_a \Phi)^{\mathbf{R}_i} . \tag{4.1.9}$$

Finally, the term $f_{\bar{2}}$ contains possible contributions to the scalar potential. By the same reasoning as in (3.3.26) all terms with values in non-trivial internal bundles have vanishing selfdual components such that the only relevant terms for the scalar potential read

$$f_{\bar{2}}^{(\mathfrak{g},\mathbf{1}_0)} = \sum_i \begin{pmatrix} \bar{C}^{\bar{\mathbf{R}}_i} \\ \bar{D}^{\bar{\mathbf{R}}_i} \end{pmatrix}^T \begin{pmatrix} \bar{\omega}^{-q_i} \wedge \omega^{q_i} & \bar{\omega}^{-q_i} \wedge \varpi^{q_i} \\ \bar{\varpi}^{-q_i} \wedge \omega^{q_i} & \bar{\varpi}^{-q_i} \wedge \varpi^{q_i} \end{pmatrix} (\tau^a) \begin{pmatrix} C^{\mathbf{R}_i} \\ D^{\mathbf{R}_i} \end{pmatrix} , \tag{4.1.10}$$

$$f_{\bar{2}}^{(\mathbf{1},\mathbf{1}_0)} = \sum_i q_i \begin{pmatrix} \bar{C}^{\bar{\mathbf{R}}_i} \\ \bar{D}^{\bar{\mathbf{R}}_i} \end{pmatrix}^T \begin{pmatrix} \bar{\omega}^{-q_i} \wedge \omega^{q_i} & \bar{\omega}^{-q_i} \wedge \varpi^{q_i} \\ \bar{\varpi}^{-q_i} \wedge \omega^{q_i} & \bar{\varpi}^{-q_i} \wedge \varpi^{q_i} \end{pmatrix} (\mathbb{I}) \begin{pmatrix} C^{\mathbf{R}_i} \\ D^{\mathbf{R}_i} \end{pmatrix} , \tag{4.1.11}$$

where we suppressed the multiplicity indices. Here τ^a are the $\mathfrak{g}$-generators in the appropriate representation $\mathbf{R}_i$. The products of zero modes belong to $H^2(L^{q_i} \otimes L^{-q_i}) = H^2(K3, \mathbb{R})$.

4.2 Kalb-Ramond sector

The Kaluza-Klein reduction of the Kalb-Ramond field is essentially the same as in section 3.2, so we only present the new features. The coupling between the b-scalars and the charged scalars again arises from the $\omega_{1,\bar{2}}^{YM}$ component of the Chern-Simons three-form. But due to the abelian character of the flux the non-vanishing terms are

$$\omega_{1,\bar{2}}^{YM} = \sum_i \mathrm{tr}\Big(f_{1,\bar{1}}^{\bar{\mathbf{R}}_i} \wedge a_{\bar{1}}^{(\mathbf{R}_i,\mathbf{1}_{q_i})} + c.c. \Big) + \mathrm{tr}\Big(\mathcal{F}^{\mathbf{1}} \wedge a_1^{\mathbf{1}} \Big) . \tag{4.2.1}$$

Compared to the standard embedding result (3.2.8) we see that the second term in (4.2.1) is new here since in the present case there exists a six-dimensional gauge boson in the same representation as the background field strength $\mathcal{F}$. The first term in (4.2.1) generates the skew-symmetric $\bar{\Phi} \overleftrightarrow{\mathcal{D}} \Phi$ couplings and the second term affinely gauges the b-scalars under the unbroken $U(1)$. Using the expansion $\mathcal{F} = -iX \otimes m^I \eta_I$ we get

$$dB_{1,\bar{2}} + \alpha' \omega_{1,\bar{2}}^{YM} = \big(db^I - \alpha' V^{\mathbf{1}} \|X\|^2 m^I \big) \eta_I + \alpha' \sum_i \mathrm{tr}\Big(\bar{\Phi}^{\bar{\mathbf{R}}_i} \overleftrightarrow{\mathcal{D}} \Phi^{\mathbf{R}_i} \Big) , \tag{4.2.2}$$

where the skew-symmetric derivatives (including the zero modes) are given by

$$\begin{aligned}\bar{\Phi}^{\bar{\mathbf{R}}_i}\overleftrightarrow{\mathcal{D}}\Phi^{\mathbf{R}_i} &= \tfrac{1}{2}\begin{pmatrix}\bar{C}^{\bar{\mathbf{R}}_i}_{k_i}\\ \bar{D}^{\bar{\mathbf{R}}_i}_{k_i}\end{pmatrix}^T\begin{pmatrix}\bar{\omega}^{-q_i}_{k_i}\wedge\omega^{q_i}_{l_i} & \bar{\omega}^{-q_i}_{k_i}\wedge\varpi^{q_i}_{l_i}\\ \bar{\varpi}^{-q_i}_{k_i}\wedge\omega^{q_i}_{l_i} & \bar{\varpi}^{-q_i}_{k_i}\wedge\varpi^{q_i}_{l_i}\end{pmatrix}\begin{pmatrix}\mathcal{D}C^{\mathbf{R}_i}_{l_i}\\ \mathcal{D}D^{\mathbf{R}_i}_{l_i}\end{pmatrix}\\ &\quad-\tfrac{1}{2}\begin{pmatrix}\mathcal{D}\bar{C}^{\bar{\mathbf{R}}_i}_{k_i}\\ \mathcal{D}\bar{D}^{\bar{\mathbf{R}}_i}_{k_i}\end{pmatrix}^T\begin{pmatrix}\bar{\omega}^{-q_i}_{k_i}\wedge\omega^{q_i}_{l_i} & \bar{\omega}^{-q_i}_{k_i}\wedge\varpi^{q_i}_{l_i}\\ \bar{\varpi}^{-q_i}_{k_i}\wedge\omega^{q_i}_{l_i} & \bar{\varpi}^{-q_i}_{k_i}\wedge\varpi^{q_i}_{l_i}\end{pmatrix}\begin{pmatrix}C^{\mathbf{R}_i}_{l_i}\\ D^{\mathbf{R}_i}_{l_i}\end{pmatrix}\ .\end{aligned}\tag{4.2.3}$$

4.3 6D Effective action

Combining the previous results, the effective action in six dimensions takes the form

$$\begin{aligned}\mathcal{L}_6 = {}&\tfrac{1}{2}R*1 \;-\; \tfrac{1}{6}e^{-2\phi}H\wedge *H \;+\; \tfrac{\alpha'}{2}e^{-\phi}\mathrm{tr}F^{\mathfrak{g}}\wedge *F^{\mathfrak{g}} \;-\; \tfrac{\alpha'}{2}e^{-\phi}\|X\|^2F^{\mathbf{1}}\wedge *F^{\mathbf{1}}\\ &+\tfrac{9}{2}d\phi\wedge *d\phi \;+\; \tfrac{1}{4}h_{IJ}dt^I_s\wedge *dt^J_s \;-\; \tfrac{1}{8\mathcal{V}^2}d\mathcal{V}\wedge *d\mathcal{V}\\ &-\;\alpha'\sum_i G^{\mathbf{R}_i}_{k_il_i}\mathrm{tr}\big(\mathcal{D}\bar{\Phi}^{\bar{\mathbf{R}}_i}_{k_i}\wedge *\mathcal{D}\Phi^{\mathbf{R}_i}_{l_i}\big) \;-\; \tfrac{1}{6\mathcal{V}}\,g_{IJ}\mathcal{D}b^I\wedge *\mathcal{D}b^J - V*1\ .\end{aligned}\tag{4.3.1}$$

$F^{\mathfrak{g}}$ is the Yang-Mills field strength of the semi-simple part of the unbroken gauge group and $F^{\mathbf{1}}$ is the field strength of the unbroken $U(1)$ corresponding to the line bundle. The derivatives of the scalars read

$$\begin{aligned}\mathcal{D}\Phi^{\mathbf{R}_i} &= d\Phi^{\mathbf{R}_i} - q_iV^{\mathbf{1}}\Phi^{\mathbf{R}_i} - V^a(\tau_a\Phi)^{\mathbf{R}_i}\ ,\\ \mathcal{D}b^I &= db^I - \alpha'V^{\mathbf{1}}\|X\|^2m^I + \alpha'\rho^{IJ}\mathrm{tr}\Big(\bar{\Phi}^{\bar{\mathbf{R}}_i}_{k_i}(N^{q_i}_{Jk_il_i})\overleftrightarrow{\mathcal{D}}\Phi^{\mathbf{R}_i}_{l_i}\Big)\ .\end{aligned}\tag{4.3.2}$$

We see that the scalars $\Phi^{\mathbf{R}_i}_{k_i} = (C^{\mathbf{R}_i}_{k_i}, D^{\mathbf{R}_i}_{k_i})$ are linearly gauged under the entire unbroken gauge group. The b-scalars are affinely gauged under the unbroken $U(1)$ due to the flux of the line bundle, with charges given by the flux vector m^I. The Killing vectors are easily identified as

$$\begin{aligned}K^a_{k_i} &= (\tau^a\Phi_{k_i})^{\mathbf{R}_i}\ ,\\ K^{\mathbf{1}}_{k_i} &= q_i\Phi^{\mathbf{R}_i}_{k_i}\ ,\\ K^{I\mathbf{1}} &= \|X\|^2m^I\ ,\end{aligned}\tag{4.3.3}$$

The coupling matrix $(N^{q_i}_{Jk_il_i})$ is given by

$$(N^{q_i}_{Jk_il_i}) = \int_{K3}\eta_J\wedge\begin{pmatrix}\bar{\omega}^{-q_i}_{k_i}\wedge\omega^{q_i}_{l_i} & \bar{\omega}^{-q_i}_{k_i}\wedge\varpi^{q_i}_{l_i}\\ \bar{\varpi}^{-q_i}_{k_i}\wedge\omega^{q_i}_{l_i} & \bar{\varpi}^{-q_i}_{k_i}\wedge\varpi^{q_i}_{l_i}\end{pmatrix}\ .\tag{4.3.4}$$

This integral is expected to depend on the geometrical moduli, however we cannot find an explicit expression. The leading kinetic metrics of the hyperscalars were already given for h_{IJ} and g_{IJ} in (3.3.4) and (B.1.5), respectively. The leading kinetic metric for the matter fields is given by

$$(G^{\mathbf{R}_i}_{k_i l_i}) = \mathcal{V}^{-1} \int_{K3} \begin{pmatrix} \omega^{q_i}_{k_i} \wedge \star\bar{\omega}^{-q_i}_{l_i} & 0 \\ 0 & \varpi^{q_i}_{k_i} \wedge \star\bar{\varpi}^{-q_i}_{l_i} \end{pmatrix} , \tag{4.3.5}$$

which is diagonal in the $C^{\mathbf{R}_i}$ and $D^{\mathbf{R}_i}$ fields. Unfortunately, we cannot compute the moduli dependence of this coupling function either. Therefore it is not possible to analyze the hyperscalar sigma model in more detail for the line bundle background.

We now turn to the scalar potential. As shown in (3.3.18), only the selfdual components of the field strength fluctuations $f_{\bar{2}}$ contribute to the scalar potential. Applying the vanishing argument (3.3.26), the only non-trivial contributions arising from (4.1.10) and (4.1.11) read

$$\begin{aligned} f^{\mathfrak{g}}_{\bar{2}+} &= \sum_i \bar{\Phi}^{\overline{\mathbf{R}}_i}_{k_i} (U_{k_i l_i}) (\tau^a \Phi)^{\mathbf{R}_i}_{l_i} , \\ f^{\mathbf{1}}_{\bar{2}+} &= \sum_i \bar{\Phi}^{\overline{\mathbf{R}}_i}_{k_i} (U_{k_i l_i}) q_i \Phi^{\mathbf{R}_i}_{l_i} , \end{aligned} \tag{4.3.6}$$

where the coupling matrix is given by

$$(U_{k_i l_i}) = \begin{pmatrix} \frac{i}{2} G^C_{k_i l_i} J & \frac{1}{2} c_{k_i l_i} \Omega \\ \frac{1}{2} \bar{c}_{k_i l_i} \overline{\Omega} & \frac{i}{2} G^D_{k_i l_i} J \end{pmatrix} . \tag{4.3.7}$$

Note that a is used for the adjoint $\mathfrak{g}$ index and that the matrix $(U_{k_i l_i}$ depends on the representation $\mathbf{R}_i$. As in the standard embedding we find on the diagonal the scalar metrics $G^C_{k_i l_i}$ and $G^D_{k_i l_i}$ which are the two matrix elements of (4.3.5). In the off-diagonal entries we find a generalized 'intersection matrix'

$$c_{k_i l_i} = \int_{K3} \bar{\omega}^{-q_i}_{k_i} \wedge \varpi^{q_i}_{l_i} \wedge \bar{\Omega} , \tag{4.3.8}$$

which we cannot compute explicitly. The terms (4.3.6), (4.3.7) are proportional to the first two Killing vectors in (4.3.3), hence they are consistent with the generic D-term scalar potential in six dimensions. Together, the scalar potential reads

$$V = -\tfrac{\alpha'}{\mathcal{V}} e^{\phi} \int_{K3} \mathrm{tr}\left((f^{\mathbf{1}}_{\bar{2}+} + \mathcal{F}_+) \wedge \star (f^{\mathbf{1}}_{\bar{2}+} + \mathcal{F}_+) \right) - \tfrac{\alpha'}{\mathcal{V}} e^{\phi} \int_{K3} \mathrm{tr}\left(f^{\mathfrak{g}}_{\bar{2}+} \wedge \star f^{\mathfrak{g}}_{\bar{2}+} \right) . \tag{4.3.9}$$

Here the selfdual component of the gauge flux appears as a Fayet-Iliopoulos term. Equivalently to (3.3.29) the single D-terms can be extracted from (4.3.9) by the integrals

$$\begin{aligned} D^{a(s)} &= \tfrac{1}{\sqrt{\mathcal{V}}} \int_{K3} f^{a}_{\bar{2}+} \wedge J_s , \\ D^{\mathbf{1}(s)} &= \tfrac{1}{\sqrt{\mathcal{V}}} \int_{K3} (f^{\mathbf{1}}_{\bar{2}+} + \mathcal{F}_+) \wedge J_s . \end{aligned} \tag{4.3.10}$$

The generalization of the above results to several line bundles is straightforward. The b-scalars are then gauged under all abelian factors $U(1)_m$ with charges proportional to the flux vectors m^{In}. For line bundles which are not orthogonal in the Cartan subalgebra, $X^n \cdot X^m \neq 0$, kinetic mixing of the different abelian field strengths occurs

$$\mathcal{L}_6^{mix} = -\tfrac{\alpha'}{2} e^{-\phi} \sum_{m,n} (X^m \cdot X^n) F^{\mathbf{1}_m} \wedge *F^{\mathbf{1}_n} \ . \tag{4.3.11}$$

In this case also the six-dimensional Kalb-Ramond field contains mixed abelian Chern-Simons couplings of the form (2.3.6). The scalar potential contains similar mixing terms

$$V = \tfrac{\alpha'}{\mathcal{V}} e^{\phi} (X^n \cdot X^m) \int_{K3} (f^{\mathbf{1}_n}_{\bar{2}+} + \mathcal{F}^n_+) \wedge \star (f^{\mathbf{1}_m}_{\bar{2}+} + \mathcal{F}^m_+) - \tfrac{\alpha'}{\mathcal{V}} e^{\phi} \int_{K3} \mathrm{tr} \left(f^{\mathfrak{g}}_{\bar{2}+} \wedge \star f^{\mathfrak{g}}_{\bar{2}+} \right) \ . \tag{4.3.12}$$

Here $f^{\mathbf{1}_n}_{\bar{2}+}$ is the direct generalization of (4.1.11) containing all charged matter fields charged under $U(1)_n$. The explicit form of the scalar potential reads

$$\begin{aligned} V = \tfrac{\alpha' e^{\phi}}{\mathcal{V}} (X^n \cdot X^m) \int_{K3} \Big(\sum_i \bar{\Phi}^{\overline{\mathbf{R}}_i}_{k_i} (U_{k_i l_i}) q_i^n \Phi^{\mathbf{R}_i}_{l_i} + \mathcal{F}^n_+ \Big) \wedge \star \Big(\sum_i \bar{\Phi}^{\overline{\mathbf{R}}_i}_{k_i} (U_{k_i l_i}) q_i^m \Phi^{\mathbf{R}_i}_{l_i} + \mathcal{F}^m_+ \Big) \\ + \tfrac{\alpha' e^{\phi}}{\mathcal{V}} \sum_a \int_{K3} \Big(\sum_i \bar{\Phi}^{\overline{\mathbf{R}}_i}_{k_i} (U_{k_i l_i}) (\tau^a \Phi)^{\mathbf{R}_i}_{l_i} \Big) \wedge \star \Big(\sum_i \bar{\Phi}^{\overline{\mathbf{R}}_i}_{k_i} (U_{k_i l_i}) (\tau^a \Phi)^{\mathbf{R}_i}_{l_i} \Big) \ , \end{aligned} \tag{4.3.13}$$

where q_i^n is the charge of the field $\Phi^{\mathbf{R}_i}$ under the group $U(1)_n$.

4.3.1 Moduli stabilization

Recalling the general argument in section 3.3.2, the rigid gauge flux of a line bundle background stabilizes some of the $K3$ moduli. The hermitean Yang-Mills equations, or equivalently the anti-selfduality condition, takes the simple form

$$\mathcal{F} \perp H_+ \ , \tag{4.3.14}$$

in terms of two-cycles and the bilinear intersection form ρ. Since $\mathcal{F}$ is discretely quantized and H_+ is determined by the $K3$ metric there exist metric deformations which violate (4.3.14). If $\mathcal{F}$ does not satisfy (4.3.14) it contains a selfdual component which appears as the Fayet-Iliopoulos term in the scalar potential (4.3.9). This is equivalent to a quadratic mass term for geometrical moduli as we saw in (3.3.19). Hence, truly massless deformations of the $K3$ metric are given by all motions of H_+, preserving (4.3.14). Obviously, the more gauge fluxes are present, independent in $H_2(K3, \mathbb{Z})$, the more constrained is this motion and the remaining moduli space is reduced. Let us consider N line bundles characterized by the flux vectors $\{m^1, ..., m^N\}$. If all N

flux vector are linearly independent, the remaining moduli space is described by the Grassmannian manifold

$$\tilde{\mathcal{M}}_{K3} = \frac{O(3,19-N)}{O(3)\times O(19-N)} \times \mathbb{R}^+, \qquad (4.3.15)$$

so there are $3N$ moduli stabilized and $\dim(\tilde{\mathcal{M}}_{K3}) = 58 - 3N$. For $E_8 \times E_8$ we have $N_{max} = 16$, which stabilizes all but ten moduli and leaves the gauge group $U(1)^{16}$ unbroken. For a GUT group to survive at the compactification scale a larger number of moduli necessarily stays unfixed. Finally, let us mention that the moduli space of vacua also contains all D-flat directions of non-zero charged matter fields, satisfying $D^a = D^{\mathbf{1}} = 0$. These non-trivial vacua define the Higgs branch of the six-dimensional theory where the gauge group is broken further and the spectrum of massless hypermultiplets is reduced [144].

4.3.2 Stückelberg mechanism and massive U(1)'s

We close this thesis by analyzing the effect of the affinely gauged scalars b^I (cf. (4.3.2)). Let us first focus on one line bundle for simplicity. In this case the $U(1)$ gauge symmetry acts according to

$$V^{\mathbf{1}} \longrightarrow V^{\mathbf{1}} + d\chi \ , \qquad b^I \longrightarrow b^I + \alpha' m^I \chi \ . \qquad (4.3.16)$$

This implies that one combination of b^I can be gauged to zero with $V^{\mathbf{1}}$ becoming massive which is known as the Stückelberg mechanism.[4] The mass term (in the Einstein frame) is found from (4.3.2) to be

$$\tfrac{\alpha'^2}{6\mathcal{V}} \|X\|^2 V^{\mathbf{1}} \wedge *V^{\mathbf{1}} \int_{K3} \mathrm{tr}(\mathcal{F} \wedge \star\mathcal{F}) = -\tfrac{\alpha'^2}{6\mathcal{V}} \|X\|^4 V^{\mathbf{1}} \wedge *V^{\mathbf{1}} \rho_{IJ} m^I m^J \ , \qquad (4.3.17)$$

where we used the ASD condition $\star\mathcal{F} = -\mathcal{F}$. To identify the physical mass we need to absorb a factor $\sqrt{\alpha'}\|X\|$ into $V^{\mathbf{1}}$ in order to get a canonical kinetic term as can be seen from (4.3.1). Using the tadpole condition (4.0.9) the physical mass reads

$$m = 4\sqrt{\tfrac{\alpha'}{\mathcal{V}}} \ . \qquad (4.3.18)$$

Note that the physical mass only depends on the $K3$ volume.

If there are N line bundles with flux parameters $m^{In} = (m^{I1}, ..., m^{IN})$, the b^I are coupled to all of them and generically all $U(1)$'s become massive. However, if some flux vectors are linearly dependent, dim span$\{m^1, ..., m^N\} = K < N$, the rank of the mass matrix is reduced and there remain $N - K$ massless $U(1)$'s in the spectrum.

[4] In six dimensions this effect is independent of possible abelian anomalies [85].

Let us show which combination of b_I-scalars is eaten by which combination of $U(1)$'s. In an integral basis of $H^2(K3,\mathbb{Z})$ we define $q^{In} = \|X^n\| m^{In} \in \mathbb{Z}$ and look for the orthogonalization

$$\mathcal{L}_6 \sim g_{IJ}(db^I - q^{In}V_n^{\mathbf{1}})^2 = \tilde{g}_{IJ}(d\tilde{b}^I - \lambda^{In}\tilde{V}_n^{\mathbf{1}})^2 \ . \tag{4.3.19}$$

For K linear independent flux vectors the $22 \times N$ matrix q^{In} has rank K and hence can be be brought to the following form (e.g. $N = 3, K = 2$)

$$q^{In} \mapsto O^I{}_J q^{Jm} U^n{}_m = \lambda^{In} = \begin{pmatrix} \lambda^1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & \dots \\ 0 & 0 & 0 & \dots \end{pmatrix} \ , \tag{4.3.20}$$

where $O \in O(22)$ and $U \in O(N)$. This determines the preferred basis

$$\tilde{V}_n^{\mathbf{1}} = U_n{}^p V_p^{\mathbf{1}} \ , \qquad \tilde{b}^I = O^I{}_J b^J \ , \tag{4.3.21}$$

in which the first K $\tilde{b}$ scalars are the Goldstone bosons of the first K gauge potentials. More precisely, one goes to a basis of $H^2(K3,\mathbb{Z})$ where the flux hyperplane $\mathrm{span}(m^1,\dots,m^n)$ is spanned by the first K harmonic 2-forms $\tilde{\eta}_1,\dots\tilde{\eta}_K$. The special form of λ^{In} however does not tell us if this basis is orthogonal with respect to the intersection matrix ρ_{IJ}. Since we have $\star\mathcal{F}^n = -\mathcal{F}^n$ for each gauge flux, the mass terms read

$$\tfrac{\alpha'}{6\mathcal{V}} V_n^{\mathbf{1}} \wedge * V_m^{\mathbf{1}} \int \mathcal{F}^n \wedge \star \mathcal{F}^m = -\tfrac{\alpha'}{6\mathcal{V}} \tilde{V}_n^{\mathbf{1}} \wedge * \tilde{V}_m^{\mathbf{1}} \tilde{\rho}_{IJ} \lambda^{In} \lambda^{Jm} \ , \tag{4.3.22}$$

where $\tilde{\rho}_{IJ} = O_I{}^K O_J{}^L \rho_{KL}$. In general $\tilde{\rho}_{IJ}$ will not be diagonal and hence the mass term will not be diagonal in n, m. Therefore, the mass eigenbasis is generically different from the 'Goldstone eigenbasis'. Note that again the mass matrix only depends on the volume modulus and the topological flux configuration. The trace of the (squared) mass matrix is fixed by the tadpole condition

$$\mathrm{tr}(M^2) = \sum_n (-\tfrac{1}{3}\tfrac{\alpha'}{\mathcal{V}} \tilde{\rho}_{IJ} \lambda^{In} \lambda^{Jn}) = 16 \, \tfrac{\alpha'}{\mathcal{V}} \ . \tag{4.3.23}$$

Chapter 5

Conclusions

In this thesis we derive the six dimensional low energy effective action of the heterotic string compactified on $K3$ up to terms of order α'^2. Consistent backgrounds require the existence of a non-trivial gauge bundle on $K3$ with instanton number 24. We choose to study two classes of backgrounds, the well-known standard embedding of the tangent bundle into the gauge bundle and $U(1)$ line bundles. On $K3$ the hermitean Yang-Mills equations preserving supersymmetry are equivalent to the anti-selfduality of the gauge curvature. For the Kaluza-Klein reduction of the gauge sector we study the local deformation theory of gauge connections. The effective mass operators are studied in detail and the relevant properties of the zero modes, corresponding to massless matter fields and bundle moduli in six dimensions, are derived. For $K3$ this has, to our knowledge, not been discussed in the literature before. We focus on the hypermultiplet sector such that our results, due to invariance under further compactification on $K3 \times T^2$, can be interpreted as α' corrections to the (charged) hypermultiplet moduli space of an $\mathcal{N} = 2$ locally supersymmetric theory in four dimensions.

For the standard embedding we compute the couplings of the charged matter fields with explicit dependence on the geometrical moduli. The non-trivial couplings arise from the Kaluza-Klein reduction of the Chern-Simons three-forms in ten dimensions. While we can give explicit results for the Yang-Mills Chern-Simons form, the reduction of the Lorentz Chern-Simons form is obscure and we have to leave it for future studies. Using a topological vanishing argument we show that the scalar potential only consists of D-terms, consistent with six-dimensional supergravity. We focus on the non-linear sigma model of the hyperscalars defined by their kinetic terms. By general arguments of supersymmetry it is governed by quaternionic-Kähler geometry. Our result implies that the target space is a fibre bundle of the matter fields and bundle moduli over the base of geometrical and B-field moduli. Hence, we expect that the target space is not contained in the set of symmetric quaternionic-Kähler manifolds but rather has some similarities to the Ferrara-Sabharwal metric. The metric on the charged scalar fibre has resemblance to the quaternionic projective space, however a full match seems to be not possible. Subsequently, we consider submanifolds which correspond to the more familiar case of $\mathcal{N} = 1$ Kähler manifolds. The very definition of the matter fields suggest to consider a fibration of the Kähler moduli over the base of complex structure moduli. For a fixed point in the base the remaining Kähler moduli and B-field moduli are shown to span a Kähler manifold. The canonical inclusion of the matter fields, known from Calabi-Yau threefold compactifications, however does not reproduce our results due to the moduli dependence of some coupling functions. Finally, the Kähler submanifold known from an orbifold limit of $K3$ can only be discovered via a truncation and unnatural field redefinition of our results.

For the line bundle background we compute the effective action and show its consistency with six-dimensional supergravity. However, most coupling functions can only be given as abstract integrals whose moduli dependence is implicit. The line bundle curvature is equivalent to two-form gauge fluxes which, due to their rigidity, stabilize those geometrical moduli which violate the hermitean Yang-Mills equations. In the six-dimensional theory these are stabilized via Fayet-Iliopoulos terms containing the gauge flux parameters. The $U(1)$ gauge bosons, first appearing in the unbroken gauge group, gauge the axionic B-field moduli in an affine way such that they generically acquire masses via the Stückelberg mechanism. For several gauge fluxes some of which are linearly dependent in cohomology, some abelian gauge bosons may stay massless. These two mechanisms occur simultaneously and combine hypermultiplets and vector-multiplets to massive vector multiplets.

Appendix A

Mathematical supplementary

A.1 Spinor reduction

In this section we derive the six-dimensional supersymmetry parameter spinor arising from the heterotic compactification on $K3$. We first consider the decomposition of the rotation group yielding a factorization of the spinor bundle. In the ten-dimensional heterotic string theory the supersymmetry parameter is a Majorana-Weyl spinor $\hat{\epsilon} \in \mathbf{16}$ of the local rotation group $SO(9,1)$. For a four-dimensional compactification the rotation group factorizes as $SO(9,1) \to SO(5,1) \times SO(4)$. Since there is a subtlety in spinor reality conditions in six dimensions, let us first consider a Weyl-spinor $\hat{\epsilon} \in \mathbf{16}_{\mathbb{C}}$ which decomposes as [26]

$$\begin{aligned} \mathbf{16}_{\mathbb{C}} &\to (\mathbf{4}_{\mathbb{C}}, \mathbf{2}_{\mathbb{C}}) \oplus (\mathbf{4}'_{\mathbb{C}}, \mathbf{2}'_{\mathbb{C}}) \ , \\ \hat{\epsilon} &\to \epsilon_+ \otimes \eta_+ + \epsilon_- \otimes \eta_- \ . \end{aligned} \tag{A.1.1}$$

Now, imposing the Majorana reality condition $\hat{\epsilon} = B\bar{\hat{\epsilon}}$ in ten dimensions, it descends with $B = B_6 \otimes B_4$ to the lower dimensional spinors as

$$\epsilon_+ \otimes \eta_+ + \epsilon_- \otimes \eta_- = B_6\bar{\epsilon}_+ \otimes B_4\bar{\eta}_+ + B_6\bar{\epsilon}_- \otimes B_4\bar{\eta}_- \ . \tag{A.1.2}$$

Here no reality conditions like $\epsilon_+ = B_6\bar{\epsilon}_+$ and $\eta_+ = B_4\bar{\eta}_+$ can be set, because the single spinor factors transform in pseudoreal representations and $\bar{B}_6 B_6 = \bar{B}_4 B_4 = -1$. In other words, in six-dimensional Minkowski space as well as in four-dimensional Riemannian space the Weyl-condition is incompatible with a further Majorana-condition. However, the product of two pseudoreal representations is real, so there exists a reality condition only for the whole products

$$\begin{aligned} \epsilon_+ \otimes \eta_+ &= B_6\bar{\epsilon}_+ \otimes B_4\bar{\eta}_+ \ , \\ \epsilon_- \otimes \eta_- &= B_6\bar{\epsilon}_- \otimes B_4\bar{\eta}_- \ . \end{aligned} \tag{A.1.3}$$

This reduces the complex degrees of freedom of each product to four, so we have for $SO(9,1) \to SO(5,1)$

$$\mathbf{16}_{\mathbb{R}} \longrightarrow \mathbf{4}_{\mathbb{C}} \oplus \mathbf{4}'_{\mathbb{C}} \ . \tag{A.1.4}$$

Hence, the ten-dimensional Majorana-Weyl spinor reduces to $\mathcal{N} = (1,1)$ supersymmetry in six dimensions, considering just the rotation groups. For a compactification on the flat torus T^4 (A.1.4) would be the complete answer.

Let us now impose the restrictions coming from the nontrivial topology of $K3$. From (A.1.1) only those spinors which have a global non-vanishing section on $K3$ give rise to supersymmetry spinors in six dimensions. On $K3$ the rotation group is reducible, $SO(4) \cong SU(2) \times SU(2)_R$, and the spinor representations in (A.1.1) actually are

$$(\mathbf{4}_{\mathbb{C}}, \mathbf{1}, \mathbf{2}_{\mathbb{C}}) \oplus (\mathbf{4}'_{\mathbb{C}}, \mathbf{2}'_{\mathbb{C}}, \mathbf{1}) \ . \tag{A.1.5}$$

A non-vanishing spinor section can only exist in the (associated) trivial $SU(2)_R$ bundle, i.e. is a singlet of the holonomy group. Thus, only the first term in (A.1.5) globally exists on $K3$, yielding the dimensional reduction

$$\hat{\epsilon} \xrightarrow{K3} \epsilon_+ \otimes \eta_+ \ . \tag{A.1.6}$$

We now again impose the Majorana condition such that

$$\epsilon_+ \otimes \eta_+ = B_6 \bar{\epsilon}_+ \otimes B_4 \bar{\eta}_+ \ . \tag{A.1.7}$$

Therefore, the number of real supercharges is reduced to eight and we arrive at the chiral $\mathcal{N} = (1,0)$ supergravity in six dimensions. (A.1.7) is explicitly solved by the spinor superposition

$$\epsilon_+ \otimes \eta_+ + \epsilon_+^c \otimes \eta_+^c \ , \tag{A.1.8}$$

where $\epsilon_+^c = B_6 \bar{\epsilon}_+$ and $\eta_+^c = B_4 \bar{\epsilon}_+$. It defines a real eight-dimensional subspace inside the vector space defined by $(\mathbf{4}_{\mathbb{C}}, \mathbf{2}_{\mathbb{C}})$. Alternatively, $(\mathbf{4}_{\mathbb{C}}, \mathbf{2}_{\mathbb{C}})$ can be considered as two six-dimensional Weyl-spinors (of the same chirality) transforming as a doublet under $SU(2)_R$ due to (A.1.5). The Majorana condition can then be solved by setting a symplectic Majorana condition

$$\begin{pmatrix} \varepsilon^1 \\ \varepsilon^2 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \bar{\varepsilon}^1 \\ \bar{\varepsilon}^2 \end{pmatrix} = \begin{pmatrix} \varepsilon^2 \\ -\varepsilon^1 \end{pmatrix} \ . \tag{A.1.9}$$

The components are related to the previous solution by

$$\varepsilon^1 = \tfrac{1}{2}(\epsilon_+ + \epsilon_+^c) \ , \quad \varepsilon^2 = \tfrac{1}{2}(\epsilon_+ - \epsilon_+^c) \ . \tag{A.1.10}$$

A.2 Lie algebra valued forms

Throughout this thesis we deal with Lie algebra valued differential forms which are the zero modes of the (matter) fields in the Yang-Mills sector. We want to collect here some symmetry properties of wedge products of these forms which differ from wedge products of usual differential forms. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra and M a manifold of Euclidean signature. A $\mathfrak{g}$-valued p-form $\alpha \in \mathfrak{g} \otimes \Lambda^p(M)$ then can be expanded in the generators

$$\alpha = \alpha^a T_a \ , \quad T_a \in \mathfrak{g} \ , \tag{A.2.1}$$

where $\alpha^a, \ a = 1, \ldots, \dim(G)$ is a set of usual p-forms

$$\alpha^a = \tfrac{1}{p!} \alpha^a_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \ . \tag{A.2.2}$$

The natural product of Lie algebra valued p-forms is given by

$$[\alpha, \beta] = \alpha^a \wedge \beta^b [T_a, T_b] = \alpha^a \wedge \beta^b f^c{}_{ab} T_c \ , \tag{A.2.3}$$

where $f^c{}_{ab}$ are the structure constants of the Lie algebra. It follows that, for $\alpha \in \mathfrak{g} \otimes \Lambda^p(M)$ and $\beta \in \mathfrak{g} \otimes \Lambda^q(M)$ we have the symmetry property

$$[\alpha, \beta] = (-1)^{pq+1} [\beta, \alpha] \ . \tag{A.2.4}$$

If G is defined as a matrix Lie group also its generators have a matrix representation and a second product can be defined

$$\alpha \wedge \beta = \alpha^a \wedge \beta^b T_a T_b \ . \tag{A.2.5}$$

This product is generically not $\mathfrak{g}$-valued but is used in the construction of characteristic classes such as

$$\mathrm{tr}(F \wedge F) = F^a \wedge F^b \mathrm{tr}(T_a T_b) \ , \tag{A.2.6}$$

where $F \in \mathfrak{g} \otimes \Lambda^2(M)$ is the Yang-Mills field strength. The first product (A.2.3) can be written in terms of the second product (A.2.5), and for two odd forms we have the unusual symmetry relation

$$[\alpha, \beta] = \alpha \wedge \beta + \beta \wedge \alpha \ . \tag{A.2.7}$$

In particular, for a one-form $A \in \mathfrak{g} \otimes \Lambda^1(M)$ we have

$$[A, A] = 2 A \wedge A \ , \tag{A.2.8}$$

which is the reason for the two equivalent notations for the Yang-Mills field strength

$$F = dA + \tfrac{1}{2}[A, A] = dA + A \wedge A \ . \tag{A.2.9}$$

Moreover, in the Kaluza-Klein reduction of the Chern-Simons three-form we encountered in section 3.2.1 double and triple products. For α_p, β_q, etc. Lie algebra valued p- and q-forms we have the following symmetry properties

$$\begin{aligned} \mathrm{tr}(\alpha_2 \wedge \beta_1) &= \mathrm{tr}(\beta_1 \wedge \alpha_2) \ , \\ \mathrm{tr}(\alpha_1 \wedge \beta_1 \wedge \gamma_1) &= \mathrm{tr}(\beta_1 \wedge \gamma_1 \wedge \alpha_1) = \mathrm{tr}(\gamma_1 \wedge \alpha_1 \wedge \beta_1) \ , \\ \mathrm{tr}([\alpha_1, \beta_1] \wedge \gamma_1) &= \mathrm{tr}(\alpha_1 \wedge [\beta_1, \gamma_1]) = \mathrm{tr}(\beta_1 \wedge [\gamma_1, \alpha_1]) \ . \end{aligned} \tag{A.2.10}$$

A.3 Identities on a HYM-bundle

The Kaluza-Klein reduction of the Yang-Mills sector is based on the mass operators (2.4.13) and (2.4.14). Massless fields in six dimensions arise from their corresponding zero modes. Since the mass operators are not usual Laplacians the properties of the zero modes are only recognizable using some operator identities (2.4.16), (A.3.5) and (A.3.11) which we prove in this appendix. For sections ψ of a HYM-bundle we have

$$d^*_{\mathcal{A}} d_{\mathcal{A}} = 2\bar{\partial}^*_{\mathcal{A}} \bar{\partial}_{\mathcal{A}} \ . \tag{A.3.1}$$

PROOF:

$$\begin{aligned} d^*_{\mathcal{A}} d_{\mathcal{A}} \psi &= (\partial^*_{\mathcal{A}} + \bar{\partial}^*_{\mathcal{A}})(\partial_{\mathcal{A}} + \bar{\partial}_{\mathcal{A}})\psi \\ &= \big(-i[\bar{\partial}_{\mathcal{A}}, J\cdot] + i[\partial_{\mathcal{A}}, J\cdot]\big)(\partial_{\mathcal{A}} + \bar{\partial}_{\mathcal{A}})\psi \\ &= (J \cdot \bar{\partial}_{\mathcal{A}} - iJ \cdot \partial_{\mathcal{A}})(\partial_{\mathcal{A}} + \bar{\partial}_{\mathcal{A}})\psi \\ &= iJ \cdot (\bar{\partial}_{\mathcal{A}} \partial_{\mathcal{A}} - \partial_{\mathcal{A}} \bar{\partial}_{\mathcal{A}}) \\ &= iJ \cdot (\mathcal{F} - 2\partial_{\mathcal{A}} \bar{\partial}_{\mathcal{A}}) \\ &= iJ \cdot \mathcal{F} - 2i[J\cdot, \partial_{\mathcal{A}}]\bar{\partial}_{\mathcal{A}} \\ &= iJ \cdot \mathcal{F} + 2\bar{\partial}^*_{\mathcal{A}} \bar{\partial}_{\mathcal{A}} \ . \end{aligned} \tag{A.3.2}$$

Here $J\cdot$ is the contraction with the Kähler form and (A.3.1) follows after using the HYM property $J \cdot \mathcal{F} = 0$. In the second and the last line of (A.3.2) we made use of the well-known Kähler identities

$$\partial^*_{\mathcal{A}} = -i[\bar{\partial}_{\mathcal{A}}, J\cdot] \ , \quad \bar{\partial}^*_{\mathcal{A}} = i[\partial_{\mathcal{A}}, J\cdot] \ , \tag{A.3.3}$$

In the fifth line we identified the field strength as

$$\mathcal{F} = d^2_{\mathcal{A}} = \bar{\partial}_{\mathcal{A}} \partial_{\mathcal{A}} + \partial_{\mathcal{A}} \bar{\partial}_{\mathcal{A}} \ . \tag{A.3.4}$$

For one-forms with values in the HYM-bundle we find that the mass operator Δ_{YM} in (2.4.17) can be simplified using the second identity

$$d^*_{\mathcal{A}} d_{\mathcal{A}} a_{\bar{1}} = 2\Delta_{\bar{\partial}_{\mathcal{A}}} a_{\bar{1}} - d_{\mathcal{A}} d^*_{\mathcal{A}} a_{\bar{1}} + iJ \cdot [\mathcal{F}, a_{\bar{1}}] \ . \tag{A.3.5}$$

PROOF: We expand the gauge covariant derivatives for $a \in \Lambda^1(K3, E)$

$$d^*_{\mathcal{A}} d_{\mathcal{A}} a = (\bar\partial^*_{\mathcal{A}} \bar\partial_{\mathcal{A}} + \partial^*_{\mathcal{A}} \partial_{\mathcal{A}}) a + (\bar\partial^*_{\mathcal{A}} \partial_{\mathcal{A}} + \partial^*_{\mathcal{A}} \bar\partial_{\mathcal{A}}) a \tag{A.3.6}$$

The first term can be written as

$$\begin{aligned}(\bar\partial^*_{\mathcal{A}} \bar\partial_{\mathcal{A}} + \partial^*_{\mathcal{A}} \partial_{\mathcal{A}}) a &= i([\partial_{\mathcal{A}}, J\cdot]\bar\partial_{\mathcal{A}} - [\bar\partial_{\mathcal{A}}, J\cdot]\partial_{\mathcal{A}}) a \\ &= i(\partial_{\mathcal{A}} J \cdot \bar\partial_{\mathcal{A}} - \bar\partial_{\mathcal{A}} J \cdot \partial_{\mathcal{A}}) a - iJ \cdot (\partial_{\mathcal{A}} \bar\partial_{\mathcal{A}} a - \bar\partial_{\mathcal{A}} \partial_{\mathcal{A}} a) \\ &= (\partial_{\mathcal{A}} \partial^*_{\mathcal{A}} + \bar\partial_{\mathcal{A}} \bar\partial^*_{\mathcal{A}}) a + iJ \cdot [\mathcal{F}, a] - 2iJ \cdot (\partial_{\mathcal{A}} \bar\partial_{\mathcal{A}} a) \ ,\end{aligned} \tag{A.3.7}$$

Here we used the Kähler identities (A.3.3) as well as $J \cdot \partial_{\mathcal{A}} a = [J\cdot, \partial_{\mathcal{A}}] a$ since $J \cdot a = 0$. We now write the last term in (A.3.7) as

$$\begin{aligned} 2iJ \cdot (\partial_{\mathcal{A}} \bar\partial_{\mathcal{A}} a) &= 2i([J\cdot, \partial_{\mathcal{A}}] + \partial_{\mathcal{A}} J\cdot) \bar\partial_{\mathcal{A}} a \\ &= -2\bar\partial^*_{\mathcal{A}} \bar\partial_{\mathcal{A}} a + 2i \partial_{\mathcal{A}} [J\cdot, \bar\partial_{\mathcal{A}}] a \\ &= -2\bar\partial^*_{\mathcal{A}} \bar\partial_{\mathcal{A}} a + 2\partial_{\mathcal{A}} \partial^*_{\mathcal{A}} a \ .\end{aligned} \tag{A.3.8}$$

With this we arrive at

$$(\bar\partial^*_{\mathcal{A}} \bar\partial_{\mathcal{A}} + \partial^*_{\mathcal{A}} \partial_{\mathcal{A}}) a = (\bar\partial_{\mathcal{A}} \bar\partial^*_{\mathcal{A}} - \partial_{\mathcal{A}} \partial^*_{\mathcal{A}}) a + iJ \cdot [\mathcal{F}, a] + 2\bar\partial^*_{\mathcal{A}} \bar\partial_{\mathcal{A}} a \ . \tag{A.3.9}$$

Now we consider the second term in (A.3.6)

$$(\bar\partial^*_{\mathcal{A}} \partial_{\mathcal{A}} + \partial^*_{\mathcal{A}} \bar\partial_{\mathcal{A}}) a = (\partial_{\mathcal{A}} \bar\partial^*_{\mathcal{A}} + \bar\partial_{\mathcal{A}} \partial^*_{\mathcal{A}}) a = d_{\mathcal{A}} d^*_{\mathcal{A}} a - (\partial_{\mathcal{A}} \partial^*_{\mathcal{A}} + \bar\partial_{\mathcal{A}} \bar\partial^*_{\mathcal{A}}) a \ , \tag{A.3.10}$$

where we used $\{\partial_{\mathcal{A}}, \bar\partial^*_{\mathcal{A}}\} = 0$ (which follows from the Kähler identities). Together we end up with the claimed result (A.3.5)

Finally, on a complex Kähler surface with HYM-bundle we have the third identity

$$\star\, [\mathcal{F}, a_{\bar 1}] = -iJ \cdot [\mathcal{F}, a_{\bar 1}] \ , \tag{A.3.11}$$

PROOF: The left hand side reads in complex components

$$\begin{aligned} \star[\mathcal{F}, a_{\bar 1}] &= \left[\mathcal{F}_{\alpha\bar\beta}, a_\gamma\right] \star (dz^\alpha \wedge d\bar z^{\bar\beta} \wedge dz^\gamma) + [\mathcal{F}_{\alpha\bar\beta}, \bar a_{\bar\gamma}] \star (dz^\alpha \wedge d\bar z^{\bar\beta} \wedge d\bar z^{\bar\gamma}) \\ &= g^{\gamma\bar\gamma} [\varepsilon^\alpha{}_{\bar\gamma} \varepsilon^{\bar\beta}{}_\delta \mathcal{F}_{\alpha\bar\beta}, a_\gamma] dz^\delta + g^{\bar\gamma\gamma} [\varepsilon^{\bar\beta}{}_\gamma \varepsilon^\alpha{}_{\bar\delta} \mathcal{F}_{\alpha\bar\beta}, \bar a_{\bar\gamma}] d\bar z^{\bar\delta} \\ &= -g^{\gamma\bar\gamma} [\mathcal{F}_{\bar\gamma\delta}, a_\gamma] dz^\delta - g^{\gamma\bar\gamma} [\mathcal{F}_{\bar\delta\gamma}, \bar a_{\bar\gamma}] d\bar z^{\bar\delta} \\ &= -iJ \cdot [\mathcal{F}, a_{\bar 1}] \end{aligned} \tag{A.3.12}$$

In the third line we used the ASD condition $\varepsilon^\alpha{}_{\bar\gamma} \varepsilon^{\bar\beta}{}_\delta \mathcal{F}_{\alpha\bar\beta} = -\mathcal{F}_{\bar\gamma\delta}$ and in the last line we used the vanishing of the contraction $J^{\alpha\bar\beta} \mathcal{F}_{\alpha\bar\beta}$. Together with the Lorenz-gauge $d^*_{\mathcal{A}} a_{\bar 1} = 0$ this proves the crucial relation

$$\Delta_{YM} = 2\Delta_{\bar\partial_{\mathcal{A}}} = 2\Delta_{\partial_{\mathcal{A}}} \ . \tag{A.3.13}$$

Appendix B

Coupling functions in the standard embedding

In this appendix we derive the coupling functions of the effective action. First we consider the kinetic terms in (3.3.3) and in particular the couplings of the charged scalars. Due to the correspondence of their zero-modes to harmonic $(1,1)$-forms (3.1.10) these functions exhibit a characteristic dependence on the $K3$ moduli.[1] The very definition of $\mathcal{T}_{K3}$ in the standard embedding defines the charged scalar zero modes only with respect to a chosen complex structure. Hence, the discussion of their couplings implicitly requires a formal breaking of the Hyperkähler structure of $K3$. In the end, we argue that all results are invariant under $SU(2)_R$ which rotates the hypercomplex structure. Defining the complex structure via the (periods of the) holomorphic two-form $\Omega = \frac{1}{\sqrt{2}}(J_1 + iJ_2)$, the harmonic $(1,1)$-forms are defined by the projection

$$P^{1,1} : H^2(K3,\mathbb{R}) \to H^{1,1}(K3,\mathbb{R}) \ , \quad {(P^{1,1})_I}^J = \delta_I^J - \sum_{s=1,2} \rho_{IK} t_s^K t_s^J \ , \tag{B.0.1}$$

where ρ_{IJ} is the intersection form on $K3$. They depend on the complex structure moduli t_1^I, t_2^I. Hence, it is natural to (arbitrarily) fix the complex structure of $K3$ and discuss the dependence of the charged scalar couplings on the remaining Kähler moduli t_3^I. As in (3.1.10) we denote with a lower case index, η_i , $i = 3, \ldots 22$, a basis of $H^{1,1}(K3,\mathbb{R})$ with respect to the fixed complex structure. The remaining Kähler moduli are denoted by t_3^i , $i = 3, \ldots, 22$.

[1]Recall that on $K3$ the embedding $H^{1,1}(K3,\mathbb{R}) \subset H^2(K3,\mathbb{R})$ is a moduli dependent subspace.

B.1 Kinetic terms

The metric on the geometrical moduli space of $K3$ was already presented in section 2.2.29, which we repeat here for completeness

$$ds^2_{K3} = \tfrac{1}{2}\big(\rho_{IJ} - \tfrac{1}{2}\rho_{IK}\rho_{JL}t_r^K t_r^L\big)dt_s^I dt_s^J - \tfrac{1}{4\mathcal{V}^2}d\mathcal{V}d\mathcal{V}\ . \tag{B.1.1}$$

Here the variations δ in (2.2.33) are converted to spacetime derivatives d in $\mathbb{M}^{5,1}$. Turning to the Kalb-Ramond field, the leading kinetic term for the B-field scalars (in the six-dimensional Einstein frame) is obtained by the integrals

$$-\tfrac{1}{6\mathcal{V}}\int_{K3} H_{1,\bar{2}} \wedge H_{1,\bar{2}} = -\tfrac{1}{6\mathcal{V}}g_{IJ}db^I \wedge *db^J + \dots\ , \tag{B.1.2}$$

where we first neglected the Chern-Simons couplings. The Hodge star operator acts on $H^2(K3,\mathbb{R})$ as $\star|_{H_+} = \mathrm{id}_{H_+}$ and $\star|_{H_-} = -\mathrm{id}_{H_-}$. Using the projector

$$P_+ : H^2(K3,\mathbb{R}) \to H_+\ , \quad {(P_+)_I}^J = \tfrac{1}{2}\rho_{IK}t_s^K t_s^J \tag{B.1.3}$$

the Hodge star operator can be explicitly written as

$${(\star)_I}^J = {(P_+)_I}^J - \big(\delta_I^J - {(P_+)_I}^J\big) = -\delta_I^J + \rho_{IK}t_s^K t_s^J\ . \tag{B.1.4}$$

Hence, the leading coupling function takes the form

$$g_{IJ} = \int_{K3} \eta_I \wedge \star\eta_J = {(\star)_J}^K \rho_{IK} = -\rho_{IJ} + \rho_{IK}\rho_{JL}t_s^K t_s^L\ . \tag{B.1.5}$$

It depends on the geometrical moduli of $K3$ but is clearly independent of $SU(2)_R$ rotations of the hypercomplex structure.

Let us turn to the charged matter scalars whose leading kinetic term (in the six-dimensional Einstein frame) reads

$$\tfrac{1}{2\mathcal{V}}\int_{K3} \mathrm{tr}\big(f_{1,\bar{1}}^{(\mathbf{56},\mathbf{2})} \wedge *f_{1,\bar{1}}^{(\mathbf{56},\mathbf{2})}\big) = -g_{ij}\ \mathcal{D}C_i^x \wedge *\mathcal{D}\bar{C}_{jx}\ , \tag{B.1.6}$$

where the coupling functions is given by the integral

$$\begin{aligned} g_{ij} &= \tfrac{1}{\mathcal{V}}\int_{K3} {(\omega_i)_{\bar{\alpha}}}^{\alpha} {(\bar{\omega}_j)_{\beta}}^{\bar{\beta}} g^{\bar{\alpha}\beta} g_{\alpha\bar{\beta}}\sqrt{g}\ d^4x \\ &= \tfrac{1}{\mathcal{V}\|\Omega\|^4}\int_{K3} \bar{\Omega}^{\gamma\alpha}(\eta_i)_{\gamma\bar{\alpha}}\Omega^{\bar{\gamma}\bar{\beta}}(\eta_j)_{\beta\bar{\gamma}} g^{\bar{\alpha}\beta} g_{\alpha\bar{\beta}}\sqrt{g}\ d^4x \\ &= \tfrac{1}{\mathcal{V}\|\Omega\|^2}\int_{K3} (\eta_i)_{\gamma\bar{\alpha}}(\eta_j)_{\beta\bar{\gamma}} g^{\gamma\bar{\gamma}} g^{\beta\bar{\alpha}}\sqrt{g}\ d^4x \\ &= \int_{K3} \eta_i \wedge \star\eta_j\ . \end{aligned} \tag{B.1.7}$$

Here we used in the second line the isomorphism (3.1.10) and in the third line the Calabi-Yau relations

$$\Omega_{\alpha\beta} = f(z)\varepsilon_{\alpha\beta}\ , \quad |f|^2 = \|\Omega\|^2\sqrt{g}\ . \tag{B.1.8}$$

In the last line we used the normalization $\|\Omega\|^2 = \mathcal{V}^{-1}$ from (2.2.30). Interestingly, the coupling function (B.1.7) is the restriction of (B.1.5) to the subspace $H^{1,1}(K3,\mathbb{R}) \subset H^2(K3,\mathbb{R})$. Since the charged matter fields are only defined with respect to a fixed complex structure, (B.1.7) actually only depends on the remaining Kähler moduli t_3^i , $i = 3, \ldots, 22$.

We turn to the Chern-Simons couplings between the B-field scalars and the scalars from the gauge bundle. Recall from (3.2.28) that the Kaluza-Klein expansion of $H_{1,\bar{2}}$ reads

$$\begin{aligned} H_{1,\bar{2}} = db^I \wedge \eta_I + \alpha' \begin{pmatrix} d\bar{\xi}_k \\ d\xi_k \end{pmatrix}^T \begin{pmatrix} \bar{\alpha}_k^s \wedge \alpha_l^s & \bar{\alpha}_k^s \wedge \bar{\alpha}_l^s \\ \alpha_k^s \wedge \alpha_l^s & \alpha_k^s \wedge \bar{\alpha}_l^s \end{pmatrix} \begin{pmatrix} \xi_l \\ \bar{\xi}_l \end{pmatrix} \\ + \alpha' \begin{pmatrix} \mathcal{D}\bar{C}_{ix} \\ \varepsilon_{xy}\mathcal{D}C_i^y \end{pmatrix}^T \begin{pmatrix} \bar{\omega}_i^{\bar{\alpha}} \wedge \omega_j^\beta\ g_{\bar{\alpha}\beta} & \bar{\omega}_i^{\bar{\alpha}} \wedge \bar{\omega}_j^{\bar{\beta}}\ \bar{\Omega}_{\bar{\alpha}\bar{\beta}} \\ \omega_i^\alpha \wedge \omega_j^\beta\ \Omega_{\alpha\beta} & \omega_i^\alpha \wedge \bar{\omega}_j^{\bar{\beta}}\ g_{\alpha\bar{\beta}} \end{pmatrix} \begin{pmatrix} C_j^x \\ \varepsilon^{xz}\bar{C}_{jz} \end{pmatrix} \ . \end{aligned} \tag{B.1.9}$$

Considering first the matter fields, we compute the products of zero modes. The off-diagonal terms read

$$\begin{aligned} N_{ij} &:= \Omega_{\alpha\beta}\omega_i^\alpha \wedge \omega_j^\beta \\ &= \tfrac{1}{2\|\Omega\|^4}\Omega_{\alpha\beta}\bar{\Omega}^{\alpha\gamma}\bar{\Omega}^{\beta\delta}(\eta_i)_{\gamma\bar{\gamma}}(\eta_j)_{\delta\bar{\delta}}d\bar{z}^{\bar{\gamma}} \wedge d\bar{z}^{\bar{\delta}} \\ &= \tfrac{1}{\|\Omega\|^2}\bar{\Omega}\cdot(\eta_i \wedge \eta_j) \\ &= \rho_{ij}\bar{\Omega} \\ &= \tfrac{1}{\sqrt{2}}\rho_{ij}(J_1 - iJ_2)\ . \end{aligned} \tag{B.1.10}$$

In the fourth line we used that $\eta_i \wedge \eta_j = \rho_{ij}\mathrm{vol}_1$ is a constant four-form and that the contraction $\bar{\Omega}\cdot\mathrm{vol}_1 = \mathcal{V}^{-1}\bar{\Omega}$.[2]. On the other off-diagonal entry in (B.1.9) we have the complex conjugate

$$\bar{N}_{ij} = \bar{\Omega}_{\bar{\alpha}\bar{\beta}}\bar{\omega}_i^{\bar{\alpha}} \wedge \bar{\omega}_j^{\bar{\beta}} = \tfrac{1}{\sqrt{2}}\rho_{ij}(J_1 + iJ_2)\ . \tag{B.1.11}$$

The second diagonal term reads

$$\begin{aligned} M_{ij} &:= g_{\alpha\bar{\beta}}\omega_i^\alpha \wedge \bar{\omega}_j^{\bar{\beta}} \\ &= \tfrac{1}{\|\Omega\|^4}g_{\alpha\bar{\beta}}\Omega^{\alpha\beta}\bar{\Omega}^{\bar{\beta}\bar{\delta}}(\eta_i)_{\beta\bar{\gamma}}(\eta_j)_{\gamma\bar{\delta}}d\bar{z}^{\bar{\gamma}} \wedge dz^\gamma \\ &= \tfrac{1}{\|\Omega\|^2}g^{\beta\bar{\delta}}(\eta_i)_{\beta\bar{\gamma}}(\eta_j)_{\gamma\bar{\delta}}d\bar{z}^{\bar{\gamma}} \wedge dz^\gamma \end{aligned} \tag{B.1.12}$$

[2]Note that in (B.1.19) products like $\eta_i \wedge \eta_j$ are independent of the $K3$ metric whereas every contraction $\cdot$ involves an inverse metric and hence, volume dependence.

Identifying the components of the Kähler form as $g_{\alpha\bar{\beta}} = -iJ_{\alpha\bar{\beta}}$ and $g_{\bar{\alpha}\beta} = iJ_{\bar{\alpha}\beta}$ we can express M_{ij} as the special contraction

$$\begin{aligned} M_{ij} &= -\, i\mathcal{V}\big(J\cdot(\eta_i\wedge\eta_j) - (J\cdot\eta_i)\eta_j - (J\cdot\eta_j)\eta_i\big) \\ &= -\, i\mathcal{V}\big(\mathcal{V}^{-1}\rho_{ij}J - \mathcal{V}^{-1}\langle J,\eta_i\rangle\eta_j - \mathcal{V}^{-1}\langle J,\eta_j\rangle\eta_i\big) \\ &= -\, i\mathcal{V}^{\frac{1}{2}}\big(\rho_{ij}J_3 - \langle J_3,\eta_i\rangle\eta_j - \langle J_3,\eta_j\rangle\eta_i\big)\ . \end{aligned} \tag{B.1.13}$$

Here we used that $J\cdot \mathrm{vol}_1 = \mathcal{V}^{-1}J$ and that the contraction $J\cdot\eta_i$ is a constant scalar function along $K3$ satisfying

$$(J\cdot\eta_i)\sqrt{g}\ d^4y = J\wedge\eta_i\ . \tag{B.1.14}$$

From the last line in (B.1.13) we see that that M_{ij} is proportional to the square root of the volume modulus $\mathcal{V}$. The second diagonal element in (B.1.9) reads

$$g_{\bar{\alpha}\beta}\bar{\omega}_i^{\bar{\alpha}}\wedge\omega_j^{\beta} = -M_{ij}\ . \tag{B.1.15}$$

From the results (B.1.19) and (B.1.12) it is clear that $N_{ij}, M_{ij}\in H^2(K3,\mathbb{R})$ i.e. the traced products of matter zero modes exhibit the cohomology ring

$$H^1(K3,\mathcal{T}_{K3})\otimes H^1(K3,\overline{\mathcal{T}}_{K3}) = H^2(K3,\mathbb{R})\ . \tag{B.1.16}$$

The products are d-closed

$$d\mathrm{tr}(\omega_i^{\mathbf{2}}\wedge\bar{\omega}_j^{\bar{\mathbf{2}}}) = \mathrm{tr}(d_{\mathcal{A}}\omega_i^{\mathbf{2}}\wedge\bar{\omega}_j^{\bar{\mathbf{2}}}) - \mathrm{tr}(\omega_i^{\mathbf{2}}\wedge d_{\mathcal{A}}\bar{\omega}_j^{\bar{\mathbf{2}}}) = 0\ , \tag{B.1.17}$$

due to the zero mode property $d_{\mathcal{A}}\omega_i^{\mathbf{2}} = 0$, and also d-co-closed by the analog argument. Hence, M_{ij} and N_{ij} are harmonic two-forms which can be expanded into η_I

$$N_{ij} = N_{ij}^I\eta_I\ ,\quad M_{ij} = M_{ij}^I\eta_I\ . \tag{B.1.18}$$

The coefficients are given by the integrals

$$\begin{aligned} N_{ij}^I &= \rho^{IJ}\int_{K3} N_{ij}\wedge\eta_J \\ &= \tfrac{1}{\sqrt{2}}\rho_{ij}\rho^{IJ}(\langle J_1,\eta_J\rangle - i\langle J_2,\eta_J\rangle) \\ &= \tfrac{1}{\sqrt{2}}\rho_{ij}(t_1^I - it_2^I)\ , \end{aligned} \tag{B.1.19}$$

$$\begin{aligned} M_{ij}^I &= \rho^{IJ}\int_{K3} M_{ij}\wedge\eta_J \\ &= -i\mathcal{V}^{\frac{1}{2}}\rho^{IJ}\big(\rho_{ij}\langle J_3,\eta_J\rangle - \langle J_3,\eta_i\rangle\langle\eta_j,\eta_J\rangle - \langle J_3,\eta_j\rangle\langle\eta_i,\eta_J\rangle\big) \\ &= -i\mathcal{V}^{\frac{1}{2}}\big(\rho_{ij}t_3^I - \delta_j^I\rho_{Ki}t_3^K - \delta_i^I\rho_{Kj}t_3^K\big)\ . \end{aligned} \tag{B.1.20}$$

We see here explicitly that these integrals are not topological but depend on the $K3$ moduli.

For the bundle moduli term in (B.1.9) we similarly define the functions

$$\begin{aligned}\mathcal{N}_{kl} &:= \mathrm{tr}(\alpha_k^{\mathbf{3}} \wedge \alpha_l^{\mathbf{3}}) \in H^{0,2}(K3,\mathbb{R}) \ , \\ \mathcal{M}_{kl} &:= \mathrm{tr}(\alpha_k^{\mathbf{3}} \wedge \bar{\alpha}_l^{\mathbf{3}}) \in H^{1,1}(K3,\mathbb{R}) \ ,\end{aligned} \tag{B.1.21}$$

where summation over $s = 1, 2, 3$ is understood. They are two-forms which are d-closed

$$d\mathrm{tr}(\alpha_k^{\mathbf{3}} \wedge \bar{\alpha}_l^{\mathbf{3}}) = \mathrm{tr}(d_{\mathcal{A}}\alpha_k^{\mathbf{3}} \wedge \bar{\alpha}_l^{\mathbf{3}}) - \mathrm{tr}(\alpha_k^{\mathbf{3}} \wedge d_{\mathcal{A}}\bar{\alpha}_l^{\mathbf{3}}) = 0 \ , \tag{B.1.22}$$

due to the (linearized) zero mode property $d_{\mathcal{A}}\alpha_k^{\mathbf{3}} = 0$, and also d-co-closed by the analog argument. Hence, $\mathcal{M}_{kl}$ and $\mathcal{N}_{kl}$ are harmonic two-forms which can be expanded as

$$\mathcal{N}_{kl} = \mathcal{N}_{kl}^I \eta_I \ , \quad \mathcal{M}_{kl} = \mathcal{M}_{kl}^I \eta_I \ , \tag{B.1.23}$$

The coefficients are given by the integrals

$$\mathcal{N}_{kl}^I = \rho^{IJ} \int_{K3} \mathcal{N}_{kl} \wedge \eta_J \ , \quad \mathcal{M}_{kl}^I = \rho^{IJ} \int_{K3} \mathcal{M}_{kl} \wedge \eta_J \ . \tag{B.1.24}$$

These integrals are expected to depend on the $K3$ moduli as well. However due to the lack of an analog isomorphism (3.1.9) for the bundle moduli zero modes, we cannot find this dependence explicitly. Using the above results we can express the Chern-Simons couplings (B.1.9) as

$$\begin{aligned}H_{1,\bar{2}} =& \Big(db^I + \alpha' M_{ij}^I (C_j^x \mathcal{D}\bar{C}_{ix} - \bar{C}_{ix}\mathcal{D}C_j^x) + \alpha'(N_{ij}^I \varepsilon_{xy} C_i^x \mathcal{D}C_j^y + \bar{N}_{ij}^I \varepsilon^{xy}\bar{C}_{ix}\mathcal{D}\bar{C}_{jy}) \\ &+ \alpha' \mathcal{M}_{kl}^I (\bar{\xi}_k d\xi_l - \xi_l d\bar{\xi}_k) + \alpha'(\mathcal{N}_{kl}^I \xi_k d\xi_l + \bar{\mathcal{N}}_{kl}^I \bar{\xi}_k d\bar{\xi}_l)\Big) \wedge \eta_I\end{aligned} \tag{B.1.25}$$

B.2 Scalar potential

In this appendix we compute the selfdual component of $f_{\bar{2}}^{(\mathbf{133},\mathbf{1})}$ which is the only term in (3.1.14) surviving the vanishing theorem of section 3.3.2. Recall from (3.1.28) that we have

$$f_{\bar{2}}^{(\mathbf{133},\mathbf{1})} = \frac{1}{2} \begin{pmatrix} \bar{C}_{ix} \\ \varepsilon_{xz} C_i^z \end{pmatrix}^T \begin{pmatrix} \bar{\omega}_i^{\bar{\alpha}} \wedge \omega_j^{\beta}\, g_{\bar{\alpha}\beta} & \bar{\omega}_i^{\bar{\alpha}} \wedge \bar{\omega}_j^{\bar{\beta}}\, \bar{\Omega}_{\bar{\alpha}\bar{\beta}} \\ \omega_i^{\alpha} \wedge \omega_j^{\beta}\, \Omega_{\alpha\beta} & \omega_i^{\alpha} \wedge \bar{\omega}_j^{\bar{\beta}}\, g_{\alpha\bar{\beta}} \end{pmatrix} (\tau^n)^x{}_y \begin{pmatrix} C_j^y \\ \varepsilon^{yz}\bar{C}_{jz} \end{pmatrix} T_n \ , \tag{B.2.1}$$

First note that the matrix elements are the same two-forms $M_{ij}, N_{ij}, \bar{N}_{ij}$ as we encountered in the Chern-Simons couplings (B.1.9)

$$f_{\bar{2}}^{(\mathbf{133},\mathbf{1})} = \frac{1}{2} \begin{pmatrix} \bar{C}_{ix} \\ \varepsilon_{xz} C_i^z \end{pmatrix}^T \begin{pmatrix} -M_{ij} & \bar{N}_{ij} \\ N_{ij} & M_{ij} \end{pmatrix} (\tau^n)^x{}_y \begin{pmatrix} C_j^y \\ \varepsilon^{yz}\bar{C}_{jz} \end{pmatrix} T_n \ . \tag{B.2.2}$$

The elements on the off-diagonal are already selfdual because $N_{ij} \propto \bar{\Omega}$ and $\bar{N}_{ij} \propto \Omega$. The diagonal elements are harmonic $(1,1)$-forms so their selfdual projection is given by

$$\begin{aligned} M_{ij+} &= \tfrac{1}{2}\left(\int M_{ij} \wedge J_3\right) J_3 \\ &= -i\mathcal{V}^{\frac{1}{2}}\big(\rho_{ij} - \langle J_3, \eta_i\rangle\langle J_3, \eta_j\rangle\big) J_3 \\ &= -i\mathcal{V}^{\frac{1}{2}}\big(\rho_{ij} - \rho_{ik}\rho_{jl}t_3^k t_3^l\big) J_3 \\ &= i g_{ij} J \ . \end{aligned} \tag{B.2.3}$$

Here we identified the leading kinetic coupling function g_{ij} known from (3.3.8) for a fixed complex structure. Finally, we arrive at

$$f^{(\mathbf{133},\mathbf{1})}_{\bar{2}+} = \begin{pmatrix} \bar{C}_{ix} \\ \varepsilon_{xz} C^z_i \end{pmatrix}^T \begin{pmatrix} -ig_{ij}J & \frac{1}{2}\rho_{ij}\Omega \\ \frac{1}{2}\rho_{ij}\bar{\Omega} & ig_{ij}J \end{pmatrix} (\tau^n)^x{}_y \begin{pmatrix} C^y_j \\ \varepsilon^{yz}\bar{C}_{jz} \end{pmatrix} T_n \ . \tag{B.2.4}$$

B.3 Details of the fixed complex structure limit

When including the charged matter fields into the sigma model of the fixed complex structure limit, the resulting target space metric (3.4.22) strongly resembles known expressions from $\mathcal{N} = 1$ compactifications in four dimensions. It therefore seems likely that its geometry can be described by a Kähler potential of standard type

$$K = \log(2\mathcal{V}) = \log\big(\rho_{ij}(T^i + \bar{T}^i - iM^i_{kl}\bar{C}^k C^l)(T^j + \bar{T}^j - iM^i_{mn}\bar{C}^m C^n)\big) \ , \tag{B.3.1}$$

where the Kähler coordinates are defined by

$$T^i = \tfrac{1}{2}t^i + \tfrac{i}{\sqrt{6}}b^i + \tfrac{i}{2}M^i_{kl}(t)\bar{C}^k C^l \tag{B.3.2}$$

Here we omitted the **56** index of the matter fields for brevity. Note that the C-dependent shift in (B.3.2) belongs to the real part of T^i. We know from (3.3.12) that the coupling function $M^i_{kl}(t)$ is linear in the Kähler moduli t^i, hence we introduce the notation

$$M^i_{kl} = -iX^i_{jkl}t^j \ , \qquad X^i_{jkl} = \rho_{kl}\delta^i_j - \rho_{jk}\delta^i_l - \rho_{jl}\delta^i_k \ . \tag{B.3.3}$$

Furthermore, we write the Kähler coordinates as

$$T^i = \tfrac{1}{2}Y^i{}_j t^j + \tfrac{i}{\sqrt{6}}b^i \ , \qquad Y^i{}_j = \delta^i_j + X^i_{jkl}\bar{C}^k C^l \ . \tag{B.3.4}$$

We can easily solve this equation for t^i, yielding

$$t^i = (Y^{-1})^i_j(T^j + \bar{T}^j) \ . \tag{B.3.5}$$

The second derivatives of the Kähler potential then read

$$
\begin{aligned}
K_{T^i\bar{T}^j} =& \tfrac{1}{\mathcal{V}} g_{kl} (Y^{-1})^k{}_i (Y^{-1})^l{}_j \ , \\
K_{C^i\bar{C}^j} =& -\tfrac{1}{\mathcal{V}} g_{kl} (Y^{-1}\cdot \tfrac{\partial Y}{\partial C^i})^k{}_p (Y^{-1}\cdot \tfrac{\partial Y}{\partial \bar{C}^j})^l{}_q t^p t^q - \tfrac{1}{\mathcal{V}} \rho_{kl} t^l \tfrac{\partial}{\partial \bar{C}^j} (Y^{-1}\cdot \tfrac{\partial Y}{\partial C^i}\cdot Y^{-1})^k{}_m t^m \ , \\
K_{T^i\bar{C}^j} =& -\tfrac{1}{\mathcal{V}^2} \rho_{kl} t^k (Y^{-1})^l{}_i \rho_{mn} t^m (Y^{-1}\cdot \tfrac{\partial Y}{\partial \bar{C}^j})^n{}_p t^p \\
& + \tfrac{1}{\mathcal{V}} \rho_{kl} \big(t^k (Y^{-1}\cdot \tfrac{\partial Y}{\partial \bar{C}^j}\cdot Y^{-1})^l{}_i + (Y^{-1})^k{}_i (Y^{-1}\cdot \tfrac{\partial Y}{\partial \bar{C}^j})^l{}_m t^m \big) \ , \\
K_{C^i\bar{T}^j} =& -\tfrac{1}{\mathcal{V}^2} \rho_{kl} t^k (Y^{-1})^l{}_j \rho_{mn} t^m (Y^{-1}\cdot \tfrac{\partial Y}{\partial C^i})^n{}_p t^p \\
& + \tfrac{1}{\mathcal{V}} \rho_{kl} \big(t^k (Y^{-1}\cdot \tfrac{\partial Y}{\partial C^i}\cdot Y^{-1})^l{}_j + (Y^{-1})^k{}_j (Y^{-1}\cdot \tfrac{\partial Y}{\partial C^i})^l{}_m t^m \big) \ .
\end{aligned}
\tag{B.3.6}
$$

To make contact with our results from the dimensional reduction the Kähler metric has to be restricted to terms up to order C^2. We have to take into account that the differential dT, written in the original variables, already contains terms of order C

$$
dT^i = \tfrac{1}{2} Y^i{}_j dt^j + \tfrac{1}{2} t^j X^i_{jkl} (\bar{C}^k dC^l + C^l d\bar{C}^k) + \tfrac{i}{\sqrt{6}} db^i \ . \tag{B.3.7}
$$

We also have the approximation for small matter fields

$$
(Y^{-1})^i{}_j \approx \delta^i_j - X^i_{jkl} \bar{C}^k C^l \ . \tag{B.3.8}
$$

Using the above formulas we arrive at the following metric (up to C^2-terms)

$$
\begin{aligned}
K_{T^i\bar{T}^j} dT^i d\bar{T}^j =& \tfrac{1}{4\mathcal{V}} g_{kl} dt^i dt^j + \tfrac{1}{6\mathcal{V}} g_{ij} db^i db^j - \tfrac{1}{6\mathcal{V}} (g_{ik} X^k_{jmn} + g_{jk} X^k_{imn}) \bar{C}^m C^n db^i db^j \\
& - \tfrac{1}{4\mathcal{V}} g_{ij} M^i_{kl} (\bar{C}^k dC^l + C^l d\bar{C}^k) M^j_{mn} (\bar{C}^m dC^n + C^n d\bar{C}^m) \\
& + \tfrac{i}{2\mathcal{V}} g_{ij} M^j_{kl} dt^i (\bar{C}^m dC^n + C^n d\bar{C}^m) \ ,
\end{aligned}
\tag{B.3.9}
$$

$$
\begin{aligned}
K_{C^i\bar{C}^j} dC^i d\bar{C}^j =& 2 g_{ij} dC^i d\bar{C}^j + \tfrac{1}{\mathcal{V}} g_{kl} M^k_{im} \bar{C}^m dC^i M^l_{jn} C^n d\bar{C}^j \\
& - \tfrac{1}{\mathcal{V}} \rho_{kl} (M^k_{pq} M^l_{ij} + M^k_{ij} M^l_{pq} + M^k_{jq} M^l_{ip} + M^k_{ip} M^l_{jp}) \bar{C}^p C^q dC^i d\bar{C}^j \ ,
\end{aligned}
\tag{B.3.10}
$$

$$
\begin{aligned}
K_{T^i\bar{C}^j} dT^i d\bar{C}^j =& + \tfrac{1}{2\mathcal{V}} g_{ik} M^k_{lj} (C^l d\bar{C}^j) M^i_{mn} (\bar{C}^m dC^n + C^n d\bar{C}^m) \\
& + \tfrac{1}{\mathcal{V}} \rho_{ki} t^k dt^i g_{lj} C^l d\bar{C}^j + \tfrac{1}{2\mathcal{V}} \rho_{kl} X^l_{imj} (t^k dt^i + dt^k t^i) C^m d\bar{C}^j \\
& + \tfrac{1}{2\mathcal{V}} \rho_{kl} t^k X^l_{iqj} X^i_{pmn} t^p (\bar{C}^m dC^n + C^n d\bar{C}^m) C^q d\bar{C}^j \\
& + \tfrac{1}{\mathcal{V}\sqrt{6}} g_{ik} M^k_{qj} db^i C^q d\bar{C}^j + \tfrac{i}{\mathcal{V}\sqrt{6}} \rho_{kl} t^k X^l_{ipj} db^i C^p d\bar{C}^j \ .
\end{aligned}
\tag{B.3.11}
$$

Together, the Kähler potential reproduces our derived kinetic terms (3.4.22), however also additional terms $\sim t\bar{C}dCdt + c.c.$ appear which we did not find in the dimensional reduction.

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