Perturbative Algebraic Quantum Field Theory at Finite Temperature

Dissertation
zur Erlangung des Doktorgrades
des Fachbereichs Physik
der Universität Hamburg

vorgelegt von
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Hamburg
2013

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Zusammenfassung

Der algebraische Zugang zur perturbativen Quantenfeldtheorie in der Minkowskiraumzeit wird vorgestellt, wobei ein Schwerpunkt auf die inhärente Zustandsunabhängigkeit des Formalismus gelegt wird. Des Weiteren wird der Zustandsraum der perturbativen QFT eingehend untersucht. Die Dynamik wechselwirkender Theorien wird durch ein neues Verfahren konstruiert, das die Gültigkeit des Zeitschichtaxioms in der kausalen Störungstheorie systematisch ausnutzt. Dies beleuchtet einen bisher unbekannten Zusammenhang zwischen dem statistischen Zugang der Quantenmechanik und der perturbativen Quantenfeldtheorie. Die entwickelten Methoden werden zur expliziten Konstruktion von KMS- und Vakuumzuständen des wechselwirkenden, massiven Klein-Gordon Feldes benutzt und damit mögliche Infrarotdivergenzen der Theorie, also insbesondere der wechselwirkenden Wightman- und zeitgeordneten Funktionen des wechselwirkenden Feldes ausgeschlossen.

Abstract

We present the algebraic approach to perturbative quantum field theory for the real scalar field in Minkowski spacetime. In this work we put a special emphasis on the inherent state-independence of the framework and provide a detailed analysis of the state space. The dynamics of the interacting system is constructed in a novel way by virtue of the time-slice axiom in causal perturbation theory. This method sheds new light in the connection between quantum statistical dynamics and perturbative quantum field theory. In particular it allows the explicit construction of the KMS and vacuum state for the interacting, massive Klein-Gordon field which implies the absence of infrared divergences of the interacting theory at finite temperature, in particular for the interacting Wightman and time-ordered functions.

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Introduction

Historical Remarks

The field of theoretical physics is characterized by the desire to unify models that successfully describe various aspects of nature into a common, superordinate framework. This objective can already be found in what is possibly the pioneering work in theoretical physics, that is Newton's "Philosophiæ Naturalis Principia Mathematica" from 1687. In this seminal work he proposed that the motion of the objects in a physical system is caused by the action of forces on and between the constituents of the system and their dynamical response corresponding to Newton's equations of motion. Therefore the diverse laws of motion in nature – e.g. the elliptical motion of planets around a central star which was discovered by Kepler in 1609, the ballistic motion of projectiles near the surface of the earth or the sinusodial motion induced by springs – could all be derived by classifying the relevant forces that act upon the system and by solving the associated equations of motion. This novel point of view led to a scientific paradigm shift as well as to a simplification of the treatment of mechanical systems.

A breakthrough of similar significance was accomplished by Boltzmann in the second half of the 19th century with his kinetic theory of gases. The theory describes how the phenomenological laws of thermodynamics which govern the macroscopical properties of a system in thermal equilibrium like temperature, energy or entropy can be derived from a statistical description of microscopical systems. This set the cornerstone for the unification of the previously disjoint areas of physics in microscopical and macroscopical systems and is nowadays known as statistical mechanics.

The insight of Boltzmann was without doubt profound and was successfully applied to many models. However shortly after its publication problems were found in the implementation of his ideas into systems that are described by fields. The most prominent and important example in this endeavor is the model of the black body. The system consists of the electromagnetic radiation field whose dynamics is governed by Maxwell's equations and which is in thermal equilibrium with a cavity that incloses the field. If the cavity has a small opening it will emit radiation in such a way that the equilibrium inside it is not disturbed. This is the famous black body radiation. It is expected that the model of black body radiation is a good approximation to the radiation field emitted by macroscopical bodies in thermal equilibrium, such as glowing metals and stars.

The application of Boltzmann's statistical methods to the black body model was done by Rayleigh (1900) and Jeans (1905). The derived Rayleigh-Jeans law states that the energy density $\varrho(\omega)$ per frequency mode ω of the radiation field of the black body with

temperature $\beta^{-1} = k_B T$ is given by

$$\varrho_{\mathrm{RJ}}(\omega) = \frac{\omega^2}{4\pi^3 c^2 \beta}\,, \tag{0.1}$$

where c is the speed of light and k_B is Boltzmann's constant. This result agrees with measured values of the radiation spectrum of hot objects for small frequencies ω, whereas for large values of ω there is a considerable discrepancy. Even worse from a theoretical point of view is that the model is intrinsically inconsistent, since it predicts that the total energy density (that is the integral of ϱ_{RJ} in equation (0.1) over ω) of the radiation field is infinite due to the fact that contributions from the ultra-violet (UV) part of ϱ_{RJ} grow infinitely. This is absolutely not to be expected from the well-established model of electromagnetism. This phenomenon is known as the UV or Rayleigh-Jeans catastrophe. It will be rediscovered in the treatment of classical field theory at finite temperature in section 1.3.3.

It was Planck who predicted that the correct energy density spectrum is given by

$$\varrho_{\mathrm{Pl}}(\omega) = \frac{\hbar\omega^3}{4\pi^3 c^2}\frac{1}{\mathrm{e}^{\beta\hbar\omega} - 1}\,.$$

With this ingenious insight, that took place already in 1900, he saved the model of black body radiation. Despite solving the UV problem of the black body model, this formula could not be derived from electrodynamics with Boltzmann statistics, though. The assertion of Planck was, in retrospective, an anticipation of quantum theory whose breakthrough took place not until the 1920s.

Using the theory of quantized matter as the basis of a statistical analysis *á la* Boltzmann, Bose found in collaboration with Einstein in 1924 that a system of identical and indistinguishable particles can be described by the Bose-Einstein distribution function which, applied to the black body system, predicts the Planck's spectrum ϱ_{Pl} formula for the energy density.

This was the final verification that the thermodynamics of macroscopical systems can actually be derived from the application of statistical methods to microscopical systems, whose dynamics has to be described by the laws of quantum theory. This concept condensed in the notion of quantum statistical mechanics which was used to predict and explain a wide range of interesting phenomena such as

- Bose-Einstein condensation
- superfluidity, superconductivity, stimulated emission
- Fermi's exclusion principle and the degeneracy pressure of bound electrons and neutrons
- semiconductor physics in all its variety

and many more. This closes the historical introduction to the relation between classical and quantum physics on the one hand and the statistical approach on the other. In the light of the preceding discussion the question to ask, from a today's perspective, would be: And what about quantum field theory?

Infrared Problem in Perturbative QFT at Finite Temperature

After the groundbreaking invention of quantum field theory (QFT) by Born, Heisenberg, Jordan (1925) and Dirac (1927), it quickly ascended to become the most powerful tool in describing the interactions of atomic and sub-atomic matter and electromagnetism. With the experimental discovery of more and more structure in the sub-atomic world and the introduction of renormalization theory within the perturbation theory of interacting QFTs, the success of quantum field theoretical models in physics continues until today.

This naturally led to the question whether the statistical methods used in quantum mechanics could also be applied to quantum field theory. From a heuristic point of view many exciting phenomena are expected to be present in quantum field theories in thermal equilibrium. For example it is assumed that quantum chromodynamics exhibits a phase transition at very high temperatures ($\sim 10^{12}$ K), where the quarks and gluons can be described by free, i.e. de-confined, fields. This state is called the quark-gluon plasma and theoretical evidence that supports this claim comes from results in lattice field theory. For a discussion we refer to [ZJ$^+$02].

In addition, the thermal equilibrium state of the quantum fields, that constitute the standard model of high-energy physics, are assumed to provide a good model for the description of the interactions of matter in an early stage of the universe. This model is presumed to be valid as long as the effects of curvature are small – compared to the temperature of the system – but the non-zero density is non-negligible. One of the main features that is expected to be explained by this model is the emergence of an almost homogeneously distributed electromagnetic radiation field that permeates the universe and is called cosmic background radiation. In fact, the cosmic background radiation that is visible today as microwave radiation – thus the name cosmic microwave background (CMB) – seems to be described extremely well by a black body radiation at a temperature around 2.7 K. Furthermore a great amount of information concerning the cosmic evolution of the universe can be extracted from the fluctuations within the CMB (and other data), such as the composition of the energy content of the universe or its age. A detailed introduction to this topic can be found in [Wei08].

From the theoretical side, the pioneering work in the mathematical foundation of statistical methods in free QFTs was done in the works of Araki and Woods [AW63] for the Bose and Araki and Wyss [AW64] for the Fermi gas, where the structure of the thermal equilibrium states was investigated in the algebraic approach to QFT. In the seminal work of Haag, Hugenholtz and Winnink [HHW67] it was shown that a thermal equilibrium state could not be given by a density matrix any more in infinitely extended systems, but it could be characterized by a weaker condition, the so-called KMS condition [Kub57, MS59]. Furthermore, the relation between the mathematical profound theory developed by Tomita and Takesaki [Tak70] in the realm of operator theory and KMS states was explored and established a new connection between mathematics and physics.

The first attempts to include statistical methods in the perturbative description of

interacting QFTs were made in the realm of Euclidean QFT (EQFT) where the formal analogy between the inverse temperature β and the Euclidean time was used to derive a Feynman diagrammatic expansion akin to perturbative expansions for the vacuum state. The Feynman propagator was found to be β-periodic in Euclidean time, hence could be represented as a sum

$$D_F^E(\tau, \boldsymbol{x}) = \frac{1}{(2\pi)^3 \beta} \sum_{n \in \mathbb{Z}} \int \frac{\mathrm{e}^{-i(\omega_n \tau - \boldsymbol{p}\boldsymbol{x})}}{\omega_n^2 + \boldsymbol{p}^2 + m^2} \, \mathrm{d}^3 \boldsymbol{p}, \qquad \omega_n = \frac{2\pi n}{\beta} \, .$$

It was Matsubara who derived the diagrammatic expansion in this formalism [Mat55]. In order to give credit for this, the frequencies ω_n are usually called Matsubara frequencies. This approach was developed further by several authors and standard references for this approach are [FHS65, LB00]. An Osterwalder-Schrader type theorem that can be used to construct the corresponding QFT on Minkowski spacetime from the EQFT, as it is the case in the vacuum state [OS73], is unfortunately not available for the EQFTs that are obtained in this way. Hence only time-independent (static) observables can be directly calculated within this framework.

It was therefore necessary to develop a perturbative theory with real-time arguments, i.e. in Minkowski spacetime. There are three established approaches to formulate the perturbative expansions in the literature. The first one is called Thermo Field Dynamics [ETU57, TU75, Oji81, MOU84]. This approach extrapolates ideas from [HHW67] and modifies the perturbative expansions by introducing unobservable thermal ghost fields in the interaction. The second one is based on a contour integrals in the complexified time plane which was introduced by Schwinger [Sch61] and applied to the finite temperature QFT by Keldysh [Kel65]. Both approaches are equivalent to a large extent which is shown in [MNU84] and in the substantial review of Landsman and van Weert [LvW87].

A third framework was developed by Steinmann in two seminal papers, one dealing with the vacuum state [Ste93] and one with the finite temperature case [Ste95]. The advantage of his formulation was that unlike in the first two approaches also non-time ordered expectation values of observables like the Wightman functions could be derived, which play a major role in the axiomatic classification of the theory.

All three approaches have in common that the (real-time) Feynman propagator of the vacuum theory is replaced by its finite temperature counterpart

$$D_F^\beta(x) = \frac{1}{(2\pi)^4} \int \mathrm{e}^{-i(p_0 x^0 - \boldsymbol{p}\boldsymbol{x})} \underbrace{\left(\frac{i}{p^2 - m^2 + i\varepsilon} + 2\pi \frac{\delta(p^2 - m^2)}{\mathrm{e}^{\beta\hbar|p_0|} - 1} \right)}_{\hat{D}_F^\beta(p)} \mathrm{d}p \, .$$

The use of D_F^β and its related functions introduce severe infra-red (IR) divergences in the formal diagrammatic expansions, in addition to the known UV divergences that are well-understood in the case of the vacuum state and are due to slow decay of the Fourier transform $\hat{D}_F^\beta$. The UV problem can be dealt with by standard methods, in particular the renormalization machinery can be applied independently of the temperature, since the part of $\hat{D}_F^\beta$ that decays slowly is the same as for the vacuum Feynman

propagator. Notice that IR divergences are absent in the EQFT case, which has been shown be Kopper, Müller and Reisz in [KMR01].

The IR problems can be illustrated by the following example: Consider a fish-graph contribution

$$D_F^\beta(x-y)^2 = \frac{1}{(2\pi)^4}\int \mathrm{e}^{-ik(x-y)} \underbrace{\int \hat{D}_F^\beta(p)\hat{D}_F^\beta(k-p)\,\mathrm{d}p}_{=(\hat{D}_F^\beta * \hat{D}_F^\beta)(k)}\,\mathrm{d}k\,.$$

In the following we disregard the UV behavior of the integrals for a moment. The convolution in the integrand of the RHS has a local singularity at $k = 0$ due to the singular thermal contributions in $\hat{D}_F^\beta$ which diverges like $|\boldsymbol{k}|^{-1}$ [Ste95]. Clearly higher order contributions may increase the divergence at the origin, since expressions of the above type are multiplied with each other. This will lead to objects that do not define a distribution anymore.

What makes it even harder to decide whether the mentioned singularities cause the perturbative expansion to diverge is, that in general sums of contributions of the above kind are present in a fixed order of perturbation theory and it is in general hard to say whether the divergences cancel in the sum. Let us consider an example in perturbation theory. The derivation is a straightforward application of the formalism that is developed in section 2.3 which coincides with the respective expansions in the above-mentioned frameworks.

Denote the expectation value of the time-ordered two-point function of the interacting scalar 1fields at points x and y with a cubic interaction $\lambda\phi^3$ in the free thermal equilibrium state by $T(x,y)$. The first non-trivial term in the perturbative expansions of T is in order $O(\lambda^2)$ and is calculated to be

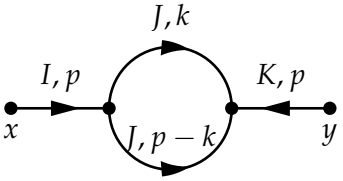
$$\begin{aligned}\hat{T}(x,y) \overset{O(\lambda^2)}{\sim} &(F|F|F)(x,y) - (-|F|+)(x,y) + (-|+|+)(x,y)\\ &+ (-|-|+)(x,y) - (F|-|+)(x,y) - (-|+|F)(x,y)\,,\end{aligned} \tag{0.2}$$

up to constants. The notation $(I|J|K)(x,y)$ denotes the fish graph with functions I, J, K

$$(I|J|K)(x,y) = \int D_I^\beta(x-z_1)D_J^\beta(z_1-z_2)^2 D_K^\beta(z_2-y)\,\mathrm{d}z_1\,\mathrm{d}z_2, \qquad I,J,K \in \{+,-,F\}\,. \tag{0.3}$$

Here $D_+^\beta = (D_-^\beta)^*$ denotes the KMS two-point function. The Fourier transform of $(I|J|K)$ in equation (0.3) is given, up to constants, by

$$\widehat{(I|J|K)}(p) = \hat{D}_I^\beta(p)\cdot\left(\hat{D}_J^\beta * \hat{D}_J^\beta\right)(p)\cdot\hat{D}_K^\beta(p) =$$



The IR singularities of the loop parts can be estimated by the above argument, but the hard part is to show that $\hat{T}(p)$ actually defines a distribution. The problem arises from the fact that the external legs of the diagrams are neither equal nor independent, which would allow a discussion of the loop part alone. They are related by the unitarity relation

$$D_F^\beta + (D_F^\beta)^* = D_+^\beta + D_-^\beta \ .$$

Moreover, the higher order contributions grow rapidly in number and a systematic cancellation is not visible, as Steinmann has remarked in [Ste95]. It is argued in [NS84, LvW87] that the related, so-called pinching singularities are absent for specific time-ordered expectation values. A proof that IR divergences are absent in general is, to our best knowledge, not to be found in the literature.

This presence of IR divergences is a severe problem since it implies that not all interacting expectation values of observables are well-defined. Speaking in more general terms, this means that the interacting theory may not possess an interacting thermal equilibrium state. This would seriously spread doubt on the way perturbation theory is done, because if it only works for the vacuum state then the conceptual depth of the formalism would be limited.

Solution to the Problem, Organization of the Thesis

The divergences arising in the perturbative expansion of the interacting state may be related to the non-vanishing influence of the interaction at asymptotic times in the thermal equilibrium state. This feature is absent in the vacuum sector of the massive interacting theory, where the interacting state is assumed to fulfill the LSZ-asymptotic conditions [LSZ55, LSZ57] – indicating the accessibility of the interacting state by means of scattering theory. This behavior can actually be **derived** from general axioms of quantum field theory (Wightman axioms or Haag-Kastler axioms) in the presence of isolated mass shells in the energy momentum spectrum [Haa92]. It may be interpreted as a consequence of the fact that (stable) particles are far from each other at large times such that their interaction can be neglected.

This comfortable situation is however no longer present in the case that the system is in a thermal equilibrium. A first sign of this has been given in a fundamental work by Narnhofer, Requardt and Thirring who showed in [NRT83] that the existence of stable particles (defined as eigenstates of the mass operator) is incompatible with interaction.

Another crucial observation has been carried out by Bros and Buchholz in [BB02] who analyzed the asymptotic behavior of interacting quantum field theories at finite temperature in an axiomatic and non-perturbative setting. It was found that the expectation values of the interacting theory (with interaction $\lambda\phi^4$ with both $\lambda \lesseqgtr 0$) **differed significantly** from the expectation values of the free field at asymptotic times.

This explains why the methods that are used in scattering theory which include the construction of the expectation values of the interacting theory in terms of the free the-

ory in asymptotic times amongst others, may not be the correct ansatz. This hint is taken seriously in the present approach.

The path to construct the interacting theory in a thermal equilibrium state will be guided by the following principles:

The state-independent construction of the observables of the interacting theory:

In order to accomplish this we use the recently developed approach of perturbative algebraic quantum field theory (pAQFT). The explicit construction of the algebras of free and interacting observables, based on the ideas of causal perturbation theory *á la* Epstein and Glaser [EG73] and developed in a series of papers starting from [BF00, DF00], is presented in the first and second chapter. It leads to a fundamentally different viewpoint of QFT at finite temperature, in which the term "at finite temperature" amounts to **the selection of one particular state** in the state space of the theory of the (free or interacting) scalar field. This is in contrast to setting up a separate theory, as in Thermo Field Dynamics or the Schwinger-Keldysh approach.

A novel contribution in this thesis is a discussion in section 2.3.2 on interacting theories which are induced by the same classical interaction functional $\mathcal{V} \in \mathcal{A}_{\mathrm{cl}}$, but are constructed within two (equivalent) realizations of the free theory $\mathcal{A}_1$ and $\mathcal{A}_2$. Due to the general construction, we know that both interacting theories are equivalent, however not equal, since the isomorphism between the algebras may not be the identity map. We find that the interaction functionals $\mathcal{V}_i \in \mathcal{A}_i$, represented as free fields over the respective algebras, differ by a local interaction which is in general of lower order than $\mathcal{V}$. The relation of this observation with the axiom of perturbative agreement, which is introduced in the work of Hollands and Wald [HW05], is drawn.

This result is illustrated on the example of the quartic interaction in the algebra of Wick polynomials $\mathcal{A}_{\mathrm{vac}}$ and $\mathcal{A}_\beta$, where the Wick-ordering is carried out with respect to the vacuum two-point function and the KMS two-point function with inverse temperature β (with same mass $m \geq 0$) respectively. It is shown that the monomial $\Phi^4_x \in \mathcal{A}_{\mathrm{vac}}$ at the point $x \in M$ is mapped under the isomorphism of $\mathcal{A}_{\mathrm{vac}}$ and $\mathcal{A}_\beta$ as:

$$\frac{\lambda}{4!}\Phi^4_x \in \mathcal{A}_{\mathrm{vac}} \quad \longrightarrow \quad \frac{\lambda}{4!}\Phi^4_x + \underbrace{\frac{\lambda\hbar F(\beta m)}{8\pi^2\beta^2}}_{\frac{1}{2}m^2_{\mathrm{th}}}\Phi^2_x \in \mathcal{A}_\beta\,.$$

The appearance of the Φ^2_x-term is interpreted as the influence of the state, around which the perturbative expansion is done. The term is non-vanishing even if $m = 0$ and corresponds (for $\lambda > 0$) to the so-called **thermal mass**, which is mentioned in many points in the literature, see e.g. [Alt90, BP90, LB00]. In section 2.3.2 the appearance of the thermal mass term $\frac{1}{2}m^2_{\mathrm{th}}$ in the perturbative expansions is commented. It is argued that using the thermal mass term as a part of a modified free action yields an interacting theory, that is not a formal power series in λ in general. This confirms similar observations in the literature, e.g. in [Alt90] for the quartic model.

Discussion of the state space in pAQFT:

The treatment of pAQFT in the literature is concerned with the algebraic constructions to a large extent, however an extensive understanding of the state space will be imperative for the constructions of interacting states in pAQFT. Thus a strong emphasis is put on this topic in this thesis, elaborating important features of the state space of both classical and quantum field theory in sections 1.3.3 and 1.5 for the free theories and in section 2.4 for the interacting theory.

The case of the classical KMS state of the free field is discussed as an interesting example at the end of section 1.3.3. At that point the Rayleigh-Jeans UV catastrophe from equation (0.1) is rediscovered in a more general context. We point out that there is actually a class of observables for which the expectation values in this state exist. The localized fields are, expectably, not contained in this class.

The state space of the free quantum theory is analyzed in section 1.5. There we draw the connection between the traditional notion of states from axiomatic approaches (see e.g. [Haa92]) with the canonical states on quantum algebras, that are obtained by formal deformation quantization [BW98]. Moreover a symmetry analysis of the state space is carried out emphasizing the translation invariant states in relativistic quantum theories.

We present two complementary methods of deriving interacting states for a given, interacting system. The first way is the well-established method of asymptotic expansion, in which the interacting state is assumed to coincide with the free state in

- the past of the interaction $\mathcal{V}_g$, if $g \in \mathcal{D}(M)$ is a spacetime cutoff of the interaction $\mathcal{V}$,
- the **asymptotic past** (with suitable convergence properties) in the case g tends to the constant function $g = 1$ over M.

This method is implicitly used in many approaches, in particular in the founding works of causal perturbation theory in [EG73, BS75, EG76]. By using this ansatz to derive the interacting KMS state we obtain, as an example, exactly the same expansion for the time-ordered expectation values of the interacting fields, as in equation (0.2).

Due to the fact that the status of the possible cancellations of the IR divergences (appearing in individual graphs) in the expansions of general observables is unclear, we abandon this ansatz and introduce a novel approach, which circumvents asymptotic expansions of the interacting state.

Avoiding the asymptotic expansion of the interacting state:

The complicated IR divergences in the perturbative expansions which were indicated in the last section, on the one hand, and the results on the asymptotic influence of the interaction on the interacting state from [BB02] on the other, motivated the search for an alternative perturbative construction of the interacting state. This is accomplished by our novel method that exploits the validity of the **time-slice axiom** (TSA) in perturbative QFTs [CF09].

The TSA, which was first formulated by Haag and Schroer in [HS62], states that the values of the observables in some region $\mathcal{O} \subset M$ can be derived from the knowledge of the algebra of observables, which is restricted to small but finite region Σ_ϵ in Minkowski spacetime M, see figure 1. Here, a time-slice is thought to be a four dimensional submanifold of M that has an tiny extent in time direction and a spacelike extent, that is large enough such that its future (or past) domain of dependence incloses $\mathcal{O}$. Hence in order to compute an observable at an arbitrary point $x \in M$ we need to know the theory only on the full time-slice $(-\epsilon, \epsilon) \times \mathbb{R}^3$ of M.

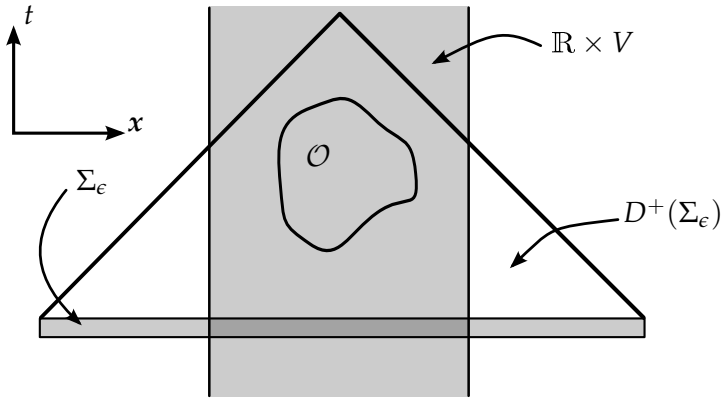


Figure 1.: The time-slice Σ_ϵ which determines the theory in its future domain of dependence $D^+(\Sigma_\epsilon) \supset \mathcal{O}$. A "space-slice" $\mathbb{R} \times V \subset M$ with a finite spatial volume $V \subset \mathbb{R}^3$.

The TSA is a weaker replacement of the uniqueness axiom on the initial value problem that can be formulated in the following way: The values of all observables in a region $\mathcal{O} \subset M$ of the theory are determined by the initial values of the canonical variables **on a Cauchy surface** whose domain of dependence incloses $\mathcal{O}$. This can be seen as a limiting case of the TSA, in which the "thickness" of the time-slice goes to zero. While such a uniqueness property holds in classical field theory, it is expected that the restriction of the interacting quantum fields to a Cauchy surface are, in general, ill-defined. The TSA is a regularization of this uniqueness axiom and can be shown to hold in perturbatively constructed QFTs [CF09].

The validity of the TSA in perturbative QFTs is used in chapter 3 to construct the interacting dynamics $\alpha_t^{\mathcal{V}}$ (seen as a one-parameter group) of observables, which are restricted to a time-slice and are under the influence of the interaction $\mathcal{V}$. Due to the principles of our approach the interacting dynamics $\alpha_t^{\mathcal{V}}$ is constructed in a state-independent manner and depends only on the values of $\mathcal{V}$ in a slightly larger time-slice.

Moreover we show that $\alpha_t^{\mathcal{V}}$ can be implemented by unitaries $W_h(t)$ in the algebra of the free field for any given time, if the spatial extent of the interacting theory is fixed, i.e. for observables in a "space-slice" $\mathbb{R} \times V$. Here, h denotes a test-function in $\mathbb{R}^3$ which cuts the interaction $\mathcal{V}$ off in spacelike directions and is constant on the volume $V \subset \mathbb{R}^3$.

The $W_h(t)$ fulfill the **co-cycle relation**

$$W_h(t+s) = W_h(t)\alpha_t(W_h(s)), \qquad t,s \in \mathbb{R}$$

and are well-known in the literature on quantum statistical mechanics, see e.g. in the textbook [BR02b]. This construction connects the frameworks of causal perturbation theory and quantum statistical mechanics in very transparent and smooth way.

In this way we obtain a regularization of the canonical approach to perturbative QFT, in which the dynamics of the system is derived for the canonical quantum field and its conjugate momentum, restricted to a Cauchy surface Σ. This approach is highly singular, since higher order Wick polynomials, which describe the interactions of relativistic QFTs, have been shown to possess no well-defined restriction to a Cauchy surface and the perturbative expansions of the interacting fields produce additional UV divergences, even after renormalization of the time-ordered products (Stückelberg divergences) [Stü51].

In chapter 4 we present the construction of the interacting thermal equilibrium state in a finite spatial volume V by means of the well-known analytic continuation of the co-cycle $W_h(t)$. It is shown that

$$\omega^{\mathcal{V}}_{\beta,h}(A) = \frac{\omega_\beta(AW_h(i\beta))}{\omega_\beta(W_h(i\beta))}, \qquad \text{supp}\, A \subset \mathcal{O}\,,$$

is a well-defined state on the algebra of observables, where ω_β is the KMS state of the **free theory** and $\mathcal{O} \subset \mathbb{R} \times V$. This state fulfills the

- KMS property in the case $0 < \beta < \infty$,
- ground state property in the case $\beta = \infty$.

by construction, in agreement to the results in quantum statistical mechanics [Ara73, BKR78].

In order to obtain an interacting KMS state on the whole space $\mathbb{R}^3$ we perform the adiabatic (or thermodynamic) limit, in which the cutoff h is set to a constant function on $\mathbb{R}^3$. The corresponding limiting state

$$\omega^{\mathcal{V}}_\beta = \lim_{h\to 1} \omega^{\mathcal{V}}_{\beta,h}$$

is shown exist in the case of a massive theory, both for the vacuum ($\beta = +\infty$) and the thermal equilibrium case ($0 < \beta < \infty$). We are able to show convergence due to the good **spatial cluster properties** of the free vacuum and free thermal equilibrium state, respectively. This is the first proof of existence of an interacting thermal equilibrium state in relativistic QFT, to our best knowledge. Furthermore we show how the thermostatic observables of the interacting theory can be obtained within this limit.

1. Quantization of the Free Scalar Field

The functional approach to QFT in which perturbative algebraic quantum field theory (pAQFT) is formulated has been proposed and developed during the last decade in a series of papers by Brunetti, Dütsch, Fredenhagen, Lauridsen-Ribeiro and Rejzner, starting from [BF00, DF00, DF01]. It aims to unify fundamental aspects of very different approaches to QFT, like

- the Haag-Kastler axiomatic approach to QFT on curved spacetimes
- constructive (non-perturbative) approaches to interacting QFTs
- the path-integral approach to perturbative QFTs

among others. The most important feature of this framework is, in the context of this work, the possibility to

- construct the observables of QFT with either the free or interacting dynamics in an explicit and **state-independent** manner
- separate the short-distance (UV) and the long-range (IR) behavior of the theory; this amounts to separating the purely algebraic aspects and the properties of the state of the theory
- treat the arising objects in a mathematically rigorous way.

The mathematical notions and methods used in this framework differ from the so-called canonical quantization procedure which can be found in standard textbooks, like [Wei96, Ryd96]. One of the goals within this work is to identify the quantities, which have turned out to be useful in the canonical approach and to derive them from first principles.

This chapter is intended to set up all the necessary tools that have been developed within the functional approach to QFT in the case of the free field. It is in particular important to characterize the singularity structure of the arising objects that will be used in perturbation theory in free theory. A special emphasis will be put on the state spaces of either classical (section 1.3.3) and quantum field theory (section 1.5). In the very first section we fix some geometric notions and sign conventions.

1.1. Minkowski Spacetime and Geometric Preliminaries

The description of physics of (quantum) fields in absence of interactions with the gravitational field, i.e. in situations where the influence of gravitational effects on the system

is negligible, is usually done with **Minkowski spacetime** as the background manifold. The application of this approximation to systems with high energies has been extraordinarily successful and an incorporation of curvature effects in accelerator-based experiments has not been of relevance so far.

We begin with the definition of important geometrical features of Minkowski spacetime and Lorentzian geometry, largely to fix the conventions and notations.

Definition 1.1.1 (Minkowski spacetime).
We equip $\mathbb{R}^4$ with a metric tensor η, defined by $\eta_{\mu\nu} = \eta(e_\mu, e_\nu) = \mathrm{diag}(1,-1,-1,-1)$ in the global Cartesian coordinates e_μ of $\mathbb{R}^4$. The tuple $M := (\mathbb{R}^4, \eta)$ is called Minkowski spacetime.

Due to the simple structure of the manifold M we may always identify points $x \in M$ with their image in the global chart given by the standard basis of $\mathbb{R}^4$, i.e. we identify $x = x^\mu e_\mu$ with x^μ. Moreover the tangent TM and the cotangent bundle T^*M over M are given by the trivial bundles $\mathbb{R}^4 \times \mathbb{R}^4$.

The Lorentzian signature of the metric tensor η induces important geometrical relations for points in M, summarized in the next definition.

Definition 1.1.2 (Causal relations in M).
A C^∞-curve $\gamma : \mathbb{R} \supset I \to M$ is called timelike / lightlike / causal / spacelike, if $\eta(\dot\gamma^\mu, \dot\gamma^\nu)$ is positive / null / non-negative / negative along the curve.
Two bounded sets $\mathcal{O}_1, \mathcal{O}_2 \subset M$ are called timelike / lightlike / causal to each other, if all the points in $\mathcal{O}_1$ and $\mathcal{O}_2$ can be connected by timelike / lightlike / timelike or lightlike curves. They are spacelike, if no points from $\mathcal{O}_1$ and $\mathcal{O}_2$ can be connected by a causal curve.
*A three-dimensional smooth submanifold Σ is called a **Cauchy surface** for M, if every inextendible causal curve intersects Σ exactly once.*
*For a bounded region $\mathcal{O} \subset M$ we define the **causal past and future** $J^\pm(\mathcal{O})$ and the **past and future domain of dependence** $D^\pm(\mathcal{O})$ by*

$$J^\pm(\mathcal{O}) = \left\{ x \in M : \exists \text{ causal } \begin{cases} \text{future directed} \\ \text{past directed} \end{cases} \text{ curve } \gamma \text{ from some } y \in \mathcal{O} \text{ to } x \right\}$$

$$D^\pm(\mathcal{O}) = \left\{ x \in M : \text{All causal } \begin{cases} \text{past inextendible} \\ \text{future inextendible} \end{cases} \text{ curves } \gamma \text{ through } x \text{ intersect } \mathcal{O} \right\}.$$

*The **past and future lightcone** $V^\pm(\mathcal{O})$ emerging from $\mathcal{O}$ is the boundary of $J^\pm(\mathcal{O})$. An alternative notation is $\mathcal{O}_1 \gtrsim \mathcal{O}_2$, if $\mathcal{O}_1 \subset J^+(\mathcal{O}_2)$. The symbols V, J, D denote the unions of the respective past and future sets.*

A standard example of a Cauchy surface of M is $\{0\} \times \mathbb{R}^3$ and an associated foliation of M is given by $\cup_{t\in\mathbb{R}} \{t\} \times \Sigma$, making M a globally hyperbolic spacetime.

An important aspect for physical systems, which will be formulated on M, is the following. Since M is a maximally symmetric spacetime (it has 10 smooth and global Killing vector fields) there is no reason for the laws that govern the dynamics of some physical system to depend explicitly on the choice of some local coordinates of M. Physically this amounts to an observer-independent formulation of the theory, as every

set of local coordinates can be interpreted as the reference frame of some observer in M.

In order to formulate this idea in precise mathematical terms, we will use the concept of isometries of M.

Definition 1.1.3 (Poincaré transformations).
The symmetry group of M, i.e. the group of diffeomorphisms of M that leaves the Minkowski length functional $\ell(x,y) = \eta(x-y, x-y)$ invariant is called the group of Poincaré transformations $\mathfrak{P}$. $\mathfrak{P}$ decomposes into four connected components and the identity component $\mathfrak{P}^{\uparrow}_{+}$ will be mostly used in the following. The action of any element $\mathfrak{p} \in \mathfrak{P}^{\uparrow}_{+}$ can be written as

$$(\mathfrak{p}x)^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}, \qquad \det(\Lambda^{\mu}{}_{\nu}) = 1, \quad \Lambda^{0}{}_{0} \geq 1.$$

Finally, for every $\mathfrak{p} \in \mathfrak{P}$ we can find an element $\hat{\mathfrak{p}} \in \mathfrak{P}^{\uparrow}_{+}$, such that $\mathfrak{p} = u \circ \hat{\mathfrak{p}}$ with $u \in \{P, T, P \circ T\}$, where T is the time inversion and P is the spatial reflection operator.

A last convention concerns the Fourier transformation of functions and distributions.

Definition 1.1.4 (Fourier Transformation).
The Fourier transform $\hat{f}$ and inverse Fourier transform of a function $f \in \mathcal{D}$ are defined by

$$\hat{f}(p) = \int f(x) \mathrm{e}^{i\eta_{\mu\nu} p^{\mu} x^{\nu}} \, \mathrm{d}x, \quad f(x) = \frac{1}{(2\pi)^4} \int \hat{f}(p) \mathrm{e}^{-i\eta_{\mu\nu} p^{\mu} x^{\nu}} \, \mathrm{d}p \, .$$

The map $f \mapsto \hat{f}$ extends by $\langle T | f \rangle \mapsto \langle \hat{T} | f \rangle := \left\langle T \middle| \hat{f} \right\rangle$ to the space of distributions $T \in \mathcal{D}'$.

1.2. The Configuration Space and Functionals

The quest of quantizing the theory of the scalar field starts with the discussion of the functional approach to classical field theory in this section. Although the treatment of classical field theory may seem distant to the topic of this thesis, which is the construction of the interacting KMS states in perturbative QFT, it will display many crucial concepts that are exploited in the subsequent chapters.

One of the heavily used concepts is the **off-shell functional approach** to field theory, which is discussed in this section. The main intent is to give a better understanding to this notion and to familiarize the reader with the use of functionals and functional derivatives in field theory.

To this end, we will introduce the **configuration space** of the theory of the real scalar field, which is the theory of interest in this thesis and the following associations are helpful for a physical understanding:

configuration ϕ	:	**pure state** in classical field theory
real-valued functional F	:	**observable** with possible values in $\mathbb{R}$
value $F(\phi) \in \mathbb{R}$	:	outcome of the **measurement** of F in ϕ

It is important that we do not implement any **dynamics** in the configuration space from the beginning. It turns out that, mathematically, it is very convenient to use the

space of configurations, which do not obey any equation of motion. This applies especially to field theory, since the solution space of the theory is infinite dimensional. It is not even a vector space in the case of non-linear equations of motion. Moreover, the idea to pass from a theory described by some equation of motion to another theory described by another equation of motion, as it will be done in perturbative approaches, can be implemented in an easier fashion in such a setting.

1.2.1. Off-Shell Configuration Space

We start off with the case of classical free theory, though the idea of working off-shell will endure throughout the rest of the work. For a detailed introduction and interpretation of the formalism we refer to the books of deWitt [DeW03] (note, he calls the space of configurations the space of histories) and the chapter by Brunetti and Fredenhagen in [BF09].

Definition 1.2.1 (Configuration Space $\mathcal{E}$).
*The space of **off-shell configurations** of a real scalar field over M consists of all real-valued smooth functions, $\mathcal{E} = C^\infty(M, \mathbb{R})$. An important subspace, the space of smooth functions with compact support in M is denoted by $\mathcal{D}$. Both vector spaces are endowed with their standard locally convex topologies.*

The space $\mathcal{E}$ has very nice properties, both from the point of view of differential geometry and functional analysis. In this work, only the latter aspect will play an important role. For a detailed treatment of these structures we refer to [BFR12].

The observables of the theory are seen as functionals over $\mathcal{E}$, i.e. maps $F : \mathcal{E} \to \mathbb{R}$, as indicated in the beginning of the section. This attitude is in a nearly complete analogy to the algebraic approach to Lagrangian **mechanics**, where the observables are functionals on the path space $\mathcal{P} \subset C^\infty(\mathbb{R}, Q)$ of some finite dimensional configuration space Q. The paths in $\mathcal{P}$ are usually taken to be solutions to an Euler-Lagrange equation obtained from an action functional, which distinguishes a specific model in the theory.

The fundamental difference in the mathematical treatment of **fields** on M is that paths in the corresponding path space take values in the function space $Q = C^\infty(\Sigma, \mathbb{R})$, which is infinite dimensional. Many structural results of classical mechanics do unfortunately not generalize in an easy way to the case of infinite dimensional configuration spaces. Nevertheless a large amount of tools have been developed, which will be partly introduced in the next sections.

1.2.2. Observables as Functionals

In order to illustrate the different kinds of functionals which will be important in later parts of this work, we present some special functionals over $\mathcal{E}$ that carry an intuitive physical interpretation. They will in addition be used to elucidate some of the functional-analytic concepts for general functionals on $\mathcal{E}$ in the future discussions.

Example 1.2.2 (Polynomial functionals).
The space of polynomial functionals $\mathcal{F}_{\text{pol}}$ is the vector space, generated by elements

$$F : \mathcal{E} \to \mathbb{R}, \qquad F(\phi) := \left\langle t \middle| \phi^{\otimes n} \right\rangle = \int t(x_1, \dots, x_n)\phi(x_1) \cdots \phi(x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n, \quad (1.1)$$

where t is a distribution of compact support: $t \in \mathcal{E}'(M^n)$, which is, without loss of generality, totally symmetric. $\langle \cdot | \cdot \rangle$ denotes the dual pairing between $\mathcal{E}'(M^n)$ and $\mathcal{E}(M^n)$. The space $\mathcal{F}_{\text{pol}}$ contains the following interesting functionals:

- the smeared **linear field** Φ_g: With $n = 1$ and $t = g \in \mathcal{D}$ we obtain

  $$\Phi_g(\phi) := \int g(x)\phi(x)\, \mathrm{d}x = \int g(x)\Phi_x(\phi)\, \mathrm{d}x \quad (1.2)$$

  where $\Phi_x(\phi) = \phi(x)$ is the **linear field at a point** $x \in M$ (considered as a limiting case, in which $t = \delta_x \in \mathcal{E}'$),

- n-fold products of the linear field: For $t(x_1, \dots, x_n) = g_1(x_1) \cdots g_n(x_n)$:

  $$\Phi_{g_1} \cdots \Phi_{g_n}(\phi) := \int g_1(x_1) \cdots g_n(x_n)\phi(x_1) \cdots \phi(x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

  with $g_k \in \mathcal{D}$, $k = 1, \dots, n$,

- the smeared **field monomials** Φ_g^n: The functionals

  $$\Phi_g^n(\phi) := \int g(x)\phi(x)^n\, \mathrm{d}x = \int g(x)\Phi_x^n(\phi)\, \mathrm{d}x, \qquad \Phi_x^n(\phi) = \phi(x)^n \quad (1.3)$$

  with $t(x_1, \dots, x_n) = g(x_1)\delta(x_1 - x_2) \cdots \delta(x_{n-1} - x_n)$ and $g \in \mathcal{D}$ will be important for interacting field theories, since they serve as interactions,

- the smeared **stress-energy tensor**:

  $$T_f^{\mu\nu}(\phi) := \int f(x) \left(\partial^\mu \phi(x) \partial^\nu \phi(x) - \frac{1}{2}\eta^{\mu\nu} \left(\partial^\mu \phi(x) \partial_\mu \phi(x) - m^2 \phi(x)^2 \right) \right) \mathrm{d}x,$$

- non-locally smeared fields: $t = g \in \mathcal{D}(M^n)$, where g is totally symmetric:

  $$N_g(\phi) = \int g(x_1, \dots, x_n)\phi(x_1) \cdots \phi(x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

From the mathematical point of view it is desirable to find the exponentiated linear fields or **Weyl fields**

$$\mathrm{e}^{i \int g(x)\phi(x)\, \mathrm{d}x} = \mathrm{e}^{i\Phi_g}, \qquad g \in \mathcal{D}$$

in the formalism, too. They can be used to generate the well-known Weyl algebra (with suitably chosen product and equations of motion), for which many structural results are known. The Weyl fields can be seen as a generalization of functionals of the form (1.1) with infinitely many summands (with an appropriate convergence criterion). We discuss the Weyl algebra intensively in the appendix A.2. ♢

One of the nice features of the functional formalism of classical and quantum field theory is the possibility to include non-polynomial observables into the framework. The need for such objects arises in many applications to systems with complex geometries, e.g. in condensed matter systems and also in toy models like the sine- or sinh-Gordon models.

Hence we will characterize functionals directly by suitable differentiability and singularity constraints, rather than using $\mathcal{F}_{\mathrm{pol}}$ in order to obtain all the structures we will need to develop classical and quantum field theory. We begin with the definition of a differential calculus on the space of functionals.

Definition 1.2.3 (Functional derivatives).
*A functional $F : \mathcal{E} \to \mathbb{C}$ is called **smooth**, if all its functional derivatives (or Gâteaux derivatives) at every $\phi \in \mathcal{E}$:*

$$F^{(n)}(\phi)[\psi_1, \dots, \psi_n] := \frac{\partial^n}{\partial\lambda_1 \cdots \partial\lambda_n}\Big|_{\lambda_1=\cdots=\lambda_n=0} F\left(\phi + \sum_{k=1}^{n} \lambda_k \psi_k\right)$$

exist as jointly continuous maps from $\mathcal{E} \times \mathcal{E}^{\otimes n}$ to $\mathbb{C}$. This implies, that for fixed $\phi \in \mathcal{E}$, $F^{(n)}(\phi)$ is a distribution of compact support on M^n for every $n \in \mathbb{N}$. This fact is used to write the n-th functional derivative as

$$F^{(n)}(\phi)[\psi_1, \dots, \psi_n] \equiv \left\langle F^{(n)}(\phi) \middle| \psi_1 \otimes \cdots \otimes \psi_k \right\rangle$$

where $\langle \cdot | \cdot \rangle$ denotes dual pairing between $\mathcal{E}'(M^n)$ and $\mathcal{E}(M^n)$.

This differential calculus possesses important properties which are well-known from calculus on $\mathbb{R}^n$, such as:

- Fundamental theorem of calculus:

$$F(\phi + \psi) - F(\phi) = \int_0^1 \left\langle F^{(1)}(\phi + t\psi) \middle| \psi \right\rangle \mathrm{d}t$$

- Taylor's formula (with remainder):

$$F(\phi + \psi) - F(\phi) = \sum_{k=1}^{n} \frac{1}{k!} \left\langle F^{(k)}(\phi) \middle| \psi^{\otimes k} \right\rangle + \int_0^1 \frac{(1-t)^n}{n!} \left\langle F^{(n+1)}(\phi + t\psi) \middle| \psi^{\otimes (n+1)} \right\rangle \mathrm{d}t$$

- Commutativity, Leibniz formula:

$$(F \cdot G)^{(n)}(\phi)[\psi^{\otimes n}] = \sum_{k=0}^{n} \binom{n}{k} \left\langle F^{(k)}(\phi) \middle| \psi^{\otimes k} \right\rangle \left\langle G^{(n-k)}(\phi) \middle| \psi^{\otimes (n-k)} \right\rangle$$

for smooth functionals F, G. In the example of a polynomial F of the form (1.1), a short calculation shows

$$F^{(k)}(\phi)[x_1, \dots, x_k] = \frac{n!}{(n-k)!} \int t(x_1, \dots, x_k, y_1, \dots, y_{n-k}) \phi(y_1) \cdots \phi(y_{n-k}) \, \mathrm{d}y_1 \cdots \mathrm{d}y_{n-k}, \tag{1.4}$$

where $F^{(k)}(\phi)[x_1, \dots, x_k]$ denotes the distribution kernel of the k-th derivative:

$$F^{(k)}(\phi)[\psi_1, \dots, \psi_k] = \int F^{(k)}(\phi)[x_1, \dots, x_k]\psi(x_1) \cdots \psi(x_k)\, \mathrm{d}x_1 \cdots \mathrm{d}x_k.$$

In particular, every polynomial functional is smooth.

In the next step we want to define the spacetime support of a functional. To this end we extend the notion of the support from the case of distributions, i.e. linear functionals over a function space, to the case of non-linear functionals.

Definition 1.2.4 (Support of a functional).
The support $\operatorname{supp}(F)$ *of a functional F is defined as the closed set of points $x \in M$, such that for all neighborhoods U_x of x there exist configurations $\phi, \psi \in \mathcal{E}$, with the properties* $\operatorname{supp}\psi \subset U_x$ *and*

$$F(\phi + \psi) \neq F(\phi)\ .$$

It has been shown [BFR12] that the support of a smooth functional can be characterized by its first derivatives:

$$\operatorname{supp} F = \overline{\bigcup_{\phi \in \mathcal{E}} \operatorname{supp} F^{(1)}(\phi)}\ .$$

where $\operatorname{supp} F^{(1)}(\phi)$ is understood in the sense of distributional support. Note that this notion has some peculiar features, e.g. the unit functional

$$\mathbb{1}(\phi) = 1$$

has vanishing support. In the following we will only deal with the space of smooth, compactly supported functionals denoted by $\mathcal{F}_0$.

In a last step we identify important subspaces of $\mathcal{F}_0$, which are distinguished by the singularity structure of the functionals. To characterize the singularity structure of a smooth functional, we use the concept of a wavefront set of a distribution with compact support. This concept originated from microlocal analysis [Hör90] and plays a major role in QFT on curved spacetimes since the mid 90's, thanks to a seminal work by Radzikowski [Rad96]. A discussion on this topic can be found after definition 1.4.2.

Definition 1.2.5 (Spaces of smooth functionals).
The following subspaces of smooth and compactly supported functionals $\mathcal{F}_0$ will be used throughout the work:

- ***regular*** *functionals:* $\mathcal{F}_{\mathrm{reg}} = \{F \in \mathcal{F}_0 : F^{(n)}(\phi) \in \mathcal{D}(M^n)\ \forall \phi \in \mathcal{E}\}$
- ***microcausal*** *functionals:* $\mathcal{F}_{\mu c} = \{F \in \mathcal{F}_0 : \mathrm{WF}(F^{(n)}(\phi)) \subset \Xi_n\ \forall \phi \in \mathcal{E}\}$ *with*

$$\Xi_n = \left\{ (x_1, \dots, x_n | k_1, \dots, k_n) \in \dot{T}^* M^n : \sum_{i=1}^{n} k_i = 0 \right\}$$

where $\dot{T}^* M^n$ is the cotangent bundle of M^n with the zero section removed.

- ***local*** *functionals:*

$$\mathcal{F}_{\text{loc}} = \{F \in \mathcal{F}_{\mu c} : \operatorname{supp} F^{(n)}(\phi) \subset \operatorname{diag}(M^n),\ \forall \phi \in \mathcal{E},\ n > 1\},$$

where $\operatorname{diag}(M^n) = \{(x, \dots, x) \in M^n : x \in M\}$ *is the (thin) diagonal of* M^n.

In the literature on QFT on curved spacetimes another definition of microcausal functionals has to be used, since cotangent vectors at different points in a curved manifold cannot be added. A suitable replacement is stated in [BDF09, BFR12] and yields, if applied to the Minkowski spacetime, a weaker condition.

One readily sees that in the case of the polynomial functionals we can translate the different conditions in definition 1.2.5 to the compactly supported distribution $t \in \mathcal{E}'(M^n)$ due to (1.4), which define the generating elements in (1.1). For such a functional $F(\phi) = \langle t|\phi^{\otimes n}\rangle$ with $t \in \mathcal{E}'(M^n)$ we find

- $F \in \mathcal{F}_{\text{reg}}$, if $t \in \mathcal{D}(M^n)$
- $F \in \mathcal{F}_{\mu c}$, if $t \in \mathcal{E}'_{\Xi_n}(M^n) = \{s \in \mathcal{E}'(M^n) : \operatorname{WF}(s) \subset \Xi_n\}$
- $F \in \mathcal{F}_{\text{loc}}$, if there exists a $g \in \mathcal{D}(M)$, such that $t(x_1, \dots, x_n) = g(x_1)\delta(x_1 - x_2) \cdots \delta(x_{n-1} - x_n)$.

Considering the particular examples in example 1.2.2, we find that the linear field Φ_g is regular and local (actually, if a functional is regular and local, then it is linear or constant). The smeared field monomials Φ_g^n and stress-energy tensor $T_f^{\mu\nu}$ provide an example for local functionals, while the non-locally smeared fields N_g, the n-fold products of the linear field and the Weyl fields are regular.

1.2.3. Poincaré Transformations on Functionals

In this section the representations of the Poincaré group on different function and functional spaces will be discussed. The notions of Poincaré covariance and invariance will play an important role in the construction of models in relativistic field theories.

Definition 1.2.6 (Representations of $\mathfrak{P}_+^\uparrow$).
Let $\mathfrak{p} \in \mathfrak{P}_+^\uparrow$, $\mathcal{O}_1 \subset M$ *and* $\mathfrak{p}\mathcal{O}_1 \subset \mathcal{O}_2$, *then*

$$\alpha^{\mathfrak{p}} : \mathcal{E} \supset \mathcal{E}(\mathcal{O}_2) \to \mathcal{E}(\mathcal{O}_1) \subset \mathcal{E}, \qquad \alpha^{\mathfrak{p}}(\phi)(x) = \phi(\mathfrak{p}x)$$

defines a (contravariant) action of $\mathfrak{P}_+^\uparrow$ *on* $\mathcal{E}$. *The dual map*

$$\alpha_{\mathfrak{p}} : \mathcal{F}_0(\mathcal{O}_1) \to \mathcal{F}_0(\mathcal{O}_2), \qquad \alpha_{\mathfrak{p}}(F)(\phi) = F(\alpha^{\mathfrak{p}}\phi)$$

defines a (covariant) action of $\mathfrak{P}_+^\uparrow$ *on the space of functionals.*

The action $\alpha_{\mathfrak{p}}$ is compatible with the canonical action on $\mathcal{D}$ in the following sense. Any compactly supported function $f \in \mathcal{D}$ can be identified as a linear functional $\phi \mapsto \langle f|\phi\rangle$ in $\mathcal{F}_0$. For this functional it holds

$$\alpha_{\mathfrak{p}} \langle f|\phi\rangle = \langle f|\alpha^{\mathfrak{p}}\phi\rangle \overset{\text{lin.}}{=} \langle \alpha_{\mathfrak{p}} f|\phi\rangle\,, \qquad (\alpha_{\mathfrak{p}} f)(x) = \begin{cases} f(\mathfrak{p}^{-1}x) & x \in \mathfrak{p}\mathcal{O}_1 \\ 0 & \text{else} \end{cases}.$$

The words contravariant and covariant are to be understood in a categorical sense, i.e. the arrows in the respective actions are reversed (contravariant) or point in the same direction (covariant) as the arrow $\mathfrak{p} : \mathcal{O}_1 \to \mathcal{O}_2$, as illustrated in the diagram:

$$\begin{array}{ccccccc} \mathcal{E}(\mathcal{O}_1) & \longleftarrow & \mathcal{O}_1 & \longrightarrow & \mathcal{D}(\mathcal{O}_1) & \xrightarrow{\subset} & \mathcal{F}_0(\mathcal{O}_1) \\ \uparrow{\scriptstyle \alpha^{\mathfrak{p}}} & & \downarrow{\scriptstyle \mathfrak{p}} & & \downarrow{\scriptstyle \alpha_{\mathfrak{p}}} & & \downarrow{\scriptstyle \alpha_{\mathfrak{p}}} \\ \mathcal{E}(\mathcal{O}_2) & \longleftarrow & \mathcal{O}_2 & \longrightarrow & \mathcal{D}(\mathcal{O}_2) & \xrightarrow{\subset} & \mathcal{F}_0(\mathcal{O}_2) \end{array}$$

For a polynomial (1.1) we find that $\alpha_{\mathfrak{p}}$ acts as

$$\alpha_{\mathfrak{p}} F(\phi) = \left\langle t \middle| (\alpha^{\mathfrak{p}}\phi)^{\otimes n} \right\rangle = \left\langle \alpha_{\mathfrak{p}} t \middle| \phi^{\otimes n} \right\rangle, \quad \alpha_{\mathfrak{p}} t(x_1, \ldots, x_n) = t(\mathfrak{p}^{-1}x_1, \ldots, \mathfrak{p}^{-1}x_n)$$

which is in agreement with the canonical action of $\mathfrak{P}_+^\uparrow$ on $\mathcal{E}'$. As an application of the notion of Poincaré-covariance we define invariant differential operators, which in turn determine $\mathfrak{P}_+^\uparrow$-invariant dynamics for the system.

Definition 1.2.7 (Wave or D'Alembertian operator, Klein-Gordon operator).
Let $P : \mathcal{E} \to \mathcal{E}$ be a linear, second order partial differential operator on M with smooth coefficients. We say, that P is invariant under $\mathfrak{P}_+^\uparrow$, if P commutes with $\alpha^{\mathfrak{p}}$: $P \circ \alpha^{\mathfrak{p}} = \alpha^{\mathfrak{p}} \circ P$ for all $\mathfrak{p} \in \mathfrak{P}_+^\uparrow$. In this case, P can be written as

$$P\phi = \big(\underbrace{\eta^{\mu\nu}\partial_\mu\partial_\nu}_{=\Box} + m^2\big)\phi, \qquad \eta^{\mu\nu} = (\eta_{\mu\nu})^{-1},\ m^2 \in \mathbb{R}.$$

*The linear operator $P = \Box + m^2$ is called the **Klein-Gordon operator**, which reduces to the **wave operator** or D'Alembertian operator in the case $m^2 = 0$.*

1.3. Classical Field Theory

After the introduction in the calculus of functionals we proceed to the formulation of classical field theory. As a starting point we define the notion of an action functional. The action functional distinguishes a particular model in the theory of the scalar field and induces the equations of motion for the system due to the principle of stationary action.

The most important part in this discussion is the characterization of the full solution space of the free theory by means of advanced and retarded Green's functions and the emergence of the principle of finite speed of information propagation.

1.3.1. Actions and the Solution Space

The traditional approach to obtain an action is to integrate a Lagrangian functional over a fixed time interval. This idea is modified a little bit in this setting. On the one hand we want to implement the relativistic structure of M of the theory, but distinguishing a time interval does not comply with the Poincaré-symmetry. On the other hand it has been noticed by Stückelberg that solving the fixed-time initial value problem that arises from the Euler-Lagrange equations in interacting quantum field theories yields additional singularities, even after renormalization [Stü51]. See section 3.1 for a discussion.

In order to circumvent these problems we will define the action by an integration of the Lagrangian density functional against a test function with compact support. This allows to find a $\mathfrak{P}_+^\uparrow$-covariant action, that is smoothly cut off at large distances by the test function.

Definition 1.3.1 (Actions).
An action $\mathbb{S}$ is a map $\mathbb{S} : \mathcal{D} \to \mathcal{F}_{\text{loc}}$, $f \mapsto \mathbb{S}[f]$ with the properties

- *$f \mapsto \mathbb{S}[f]$ is linear and $\mathbb{S}[f]$ is real: $\mathbb{S}[f]^* = \mathbb{S}[f^*]$.*
- $\operatorname{supp} \mathbb{S}[f] \subset \operatorname{supp} f$
- *Poincaré symmetry: $\alpha_\mathfrak{p}\mathbb{S}[f] = \mathbb{S}[\alpha_\mathfrak{p} f]$ for all $\mathfrak{p} \in \mathfrak{P}_+^\uparrow$.*

We will consider two actions $\mathbb{S}_1$ and $\mathbb{S}_2$ to be equivalent, if

$$\operatorname{supp} (\mathbb{S}_1[f] - \mathbb{S}_2[f]) \subset \operatorname{supp} \mathrm{d}f \qquad \forall f \in \mathcal{D},$$

i.e. if both actions differ by a boundary term.

The case in which $\mathbb{S}$ comes from a Lagrangian density fits in here as follows: Let $\mathcal{L}$ be a functional of the linear fields Φ_x and its derivative $\partial_\mu \Phi_x$ (see example 1.2.2). The action $\mathbb{S}_\mathcal{L}$ induced by $\mathcal{L}$ is defined to be

$$\mathbb{S}_\mathcal{L}[f](\phi) = \int f(x)\mathcal{L}(\Phi_x, \partial_\mu \Phi_x)(\phi)\, \mathrm{d}x = \int f(x)\mathcal{L}(\phi(x), \partial_\mu \phi(x))\, \mathrm{d}x. \tag{1.5}$$

From this picture it is clear that the introduction of a "cut-off" function f is essential to define the action $\mathbb{S}_\mathcal{L}[f]$ as a functional over $\mathcal{E}$. In this way the action $\mathbb{S}_\mathcal{L}$ is $\mathfrak{P}_+^\uparrow$-covariant rather than $\mathfrak{P}_+^\uparrow$-invariant, since f changes under the action of $\alpha_\mathfrak{p}$.

The corresponding Euler-Lagrange equations for a given Lagrangian $\mathcal{L}$

$$\partial_\mu \frac{\delta \mathcal{L}(\phi, \partial_\mu \phi)}{\delta \partial_\mu \phi(x)} - \frac{\delta \mathcal{L}(\phi, \partial_\mu \phi)}{\delta \phi(x)} = 0 \tag{1.6}$$

are clearly independent of any choice of a cutoff function. The question arises, whether one can obtain the Euler-Lagrange equations (1.6) in the present formalism and, in a more general case, the principle of least action for an arbitrary action $\mathbb{S}$. The answer is: locally, yes.

Definition 1.3.2 (Euler-Lagrange operator and local solutions).
*Let $\mathcal{S}$ be an action and $\mathcal{O} \subset M$ be bounded. Fix $f \in \mathcal{D}$ such that $f = 1$ on $\mathcal{O}$. The **Euler-Lagrange operator** for $\mathcal{S}$ is defined by*

$$\delta_{\mathrm{EL}}\mathcal{S} : \mathcal{E} \to \mathcal{D}', \quad \langle \delta_{\mathrm{EL}}\mathcal{S}(\phi_0)|h\rangle := \left\langle \frac{\delta \mathcal{S}[f](\phi_0)}{\delta \phi} \middle| h \right\rangle, \qquad \forall h \in \mathcal{D}(\mathcal{O}) \,.$$

*A local solution or **on-shell configuration** ϕ_0 is a locally stationary point of $\mathcal{S}$, i.e. $\forall h \in \mathcal{D}(\mathcal{O})$: $\langle \delta_{\mathrm{EL}}\mathcal{S}(\phi_0)|h\rangle = 0$.*

The Euler-Lagrange operator $\delta_{\mathrm{EL}}\mathcal{S}$ is independent of the cutoff function f due to the condition that $f = 1$ on the region $\mathcal{O}$, where $\delta_{\mathrm{EL}}\mathcal{S}(\phi)$ is tested. The Euler-Lagrange equation $\delta_{\mathrm{EL}}\mathcal{S}[f](\phi_0) = 0$ is, in general, a non-linear differential equation for ϕ_0.

Note, that in the above definition the choice of $\mathcal{O} \subset M$ was arbitrary, thus the local equation of motion $\delta_{\mathrm{EL}}S(\phi) = 0$ has to hold everywhere in M. Configurations ϕ_0 which satisfy the Euler-Lagrange equation on the whole spacetime M are called global solutions.

If we apply definition 1.3.2 to actions which are induced by Lagrangians $\mathcal{L}$ as in (1.5), a quick calculation shows that

$$\begin{aligned}\delta_{\mathrm{EL}}\mathcal{S}_{\mathcal{L}}(\phi_0) &= \int \underbrace{f(x)}_{=1 \text{ on } \operatorname{supp} h} h(x) \frac{\delta \mathcal{L}(\phi, \partial_\mu \phi)}{\delta \phi}\Big|_{\phi=\phi_0} \mathrm{d}x \\ &= -\int h(x) \left(\partial_\mu \frac{\delta \mathcal{L}(\phi_0, \partial_\mu \phi_0)}{\delta \partial_\mu \phi(x)} - \frac{\delta \mathcal{L}(\phi_0, \partial_\mu \phi_0)}{\delta \phi(x)} \right) \mathrm{d}x = 0 \,,\end{aligned}$$

i.e. local solutions ϕ_0 obey (1.6) in the region $\mathcal{O}$.

We want to restrict the attention to the case of the free Klein-Gordon field now. Thus we look at actions $\mathcal{S}$ that are quadratic in the field ϕ and are induced by a very specific Lagrangian density:

$$\begin{aligned}\mathcal{L}_0(\Phi_x, \partial_\mu \Phi_x) &= \frac{1}{2}\eta^{\mu\nu}(\partial_\mu \Phi_x) \cdot (\partial_\nu \Phi_x) - \frac{m^2}{2}\Phi_x^2 \\ \longleftrightarrow \quad \mathcal{S}_{\mathcal{L}_0}[f](\phi) &= \frac{1}{2}\int f(x) \left(\eta^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) - m^2 \phi(x)^2 \right) \mathrm{d}x \\ &= -\frac{1}{2}\int f(x)\phi(x) \left(\Box + m^2 \right) \phi(x) \, \mathrm{d}x \end{aligned} \tag{1.7}$$

It is well-known that in case of the **linear** Euler-Lagrange equations induced by the quadratic action $\mathcal{L}_0$, that is the **Klein-Gordon equation**:

$$\left(\partial_\mu \frac{\delta \mathcal{L}_0}{\delta \partial_\mu \phi(x)} - \frac{\delta \mathcal{L}_0}{\delta \phi(x)} \right)_{\phi=\phi_0} = P\phi_0(x), \qquad P = \Box_x + m^2$$

the complete solution space can be characterized. This is done by solving the initial value problem on a Cauchy surface Σ of M, i.e.

$$\delta_{\mathrm{EL}}\mathcal{S}(\phi) = 0, \qquad \phi_{\restriction\Sigma} = u_0, \quad \partial_{x_0}\phi_{\restriction\Sigma} = u_1 \tag{1.8}$$

for $f \in \mathcal{D}(M)$ and $u_0, u_1 \in \mathcal{D}(\Sigma)$ and is also called the **Cauchy problem** for an action $\mathcal{S}$ (or for its Euler-Lagrange differential operator respectively). For the Klein-Gordon field it holds the following theorem.

Theorem 1 (Fundamental solutions for the Klein-Gordon equation [BF09]).
*Let $P = \Box + m^2$ be the Klein-Gordon operator on M. There exist unique **advanced** G_a and **retarded fundamental solutions** G_r to P with the properties*

$$\begin{aligned} G_\mathrm{a} : \mathcal{D}(M) \to \mathcal{E}(M) : \quad & G_\mathrm{a}(Pf) = f = P(G_\mathrm{a}(f)), \quad \operatorname{supp} G_\mathrm{a}(f) \subset V^-(\operatorname{supp} f) \\ G_\mathrm{r} : \mathcal{D}(M) \to \mathcal{E}(M) : \quad & G_\mathrm{r}(Pf) = f = P(G_\mathrm{r}(f)), \quad \operatorname{supp} G_\mathrm{r}(f) \subset V^+(\operatorname{supp} f) \end{aligned}$$

for all $f \in \mathcal{D}(M)$. Since P is formally self-adjoint, i.e. $\langle Pf|g\rangle = \langle f|Pg\rangle$, it holds $\langle G_\mathrm{r}(f)|g\rangle = \langle f|G_\mathrm{a}(g)\rangle$ in $\mathcal{D}(M)$.

Due to the simple form of the differential operator P, we can explicitly express the fundamental solutions. This is done by representing $G_\mathrm{r}, G_\mathrm{a}$ as bi-distributions $G_{\mathrm{r,a}} : \mathcal{D}' \otimes \mathcal{D}' \to \mathbb{C}$ via the identification $G_\mathrm{r}(f,g) \equiv \langle G_\mathrm{r}(f)|g\rangle$ (using the same symbols). Then

$$G_\mathrm{r}(f,g) = \lim_{\varepsilon \downarrow 0} \frac{-1}{(2\pi)^4} \int \frac{\hat{f}(-p)\hat{g}(p)}{(p_0 + i\varepsilon)^2 - \boldsymbol{p}^2 - m^2} \,\mathrm{d}p, \qquad \boldsymbol{p} = (p_1, p_2, p_3) \tag{1.9}$$

and $G_\mathrm{a}(f,g) = G_\mathrm{r}(g,f)$. The support properties translate into

$$\operatorname{supp} G_\mathrm{r} \subset \{(x,y) \in M \times M : y \in J^+(x)\}, \quad \operatorname{supp} G_\mathrm{a} \subset \{(x,y) \in M \times M : y \in J^-(x)\}$$

The existence of the fundamental solutions is sufficient to completely solve the Cauchy problem (1.8) for the Klein-Gordon operator by means of the causal commutator function.

Definition 1.3.3 (Causal commutator function G_c).
The causal commutator function G_c for the Klein-Gordon operator P on M is defined by $G_\mathrm{c} = G_\mathrm{r} - G_\mathrm{a}$.

It is well-known that G_c characterizes the space of solutions to the Klein-Gordon equations with compactly supported Cauchy data [BF09]. This means:

- Let $\mathcal{E}_P^{\mathrm{sc}}$[1] be the space of solutions to the Cauchy problem (1.8). For every $\phi \in \mathcal{E}_P^{\mathrm{sc}}$ there is a compactly supported function $f \in \mathcal{D}$, such that $\phi = G_\mathrm{c}(f)$. The Cauchy data of ϕ are recovered by $u_0 = G_\mathrm{c}(f)_{|\Sigma}$ and $u_1 = (\partial_{x^0} G_\mathrm{c}(f))_{|\Sigma}$,
- For every $f \in \mathcal{D}$, such that $G_\mathrm{c}(f) = 0$ there is a $g \in \mathcal{D}$ with $f = Pg$, i.e. $\ker(G_\mathrm{c}) = P\mathcal{D}$.

[1] The notation $\mathcal{E}_P^{\mathrm{sc}}$ indicates that the solutions $\phi \in \mathcal{E}_P^{\mathrm{sc}}$ to P have spacelike compact support. A region $\mathcal{O} \subset M$ is called spacelike compact, if $\mathcal{O} \subset \bigcup_{K \subset M} D(K)$, where K runs over all compact subsets in M and D is the domain of dependence, see definition 1.1.2.

From the support and adjointness properties of the advanced and retarded fundamentals solutions it follows directly that

$$\operatorname{supp} G_c(f) \subset J(\operatorname{supp} f), \qquad \langle G_c(f)|g\rangle = -\langle f|G_c(g)\rangle, \qquad f,g \in \mathcal{D}$$

and

$$\operatorname{supp} G_c = \{(x,y) \in M \times M : y \in J(x)\}, \qquad G_c(f,g) = -G_c(g,f)$$

for the associated bi-distribution. The explicit form of G_c follows from (1.9)

$$\begin{aligned} G_c(f,g) &= \frac{1}{(2\pi)^3}\int \hat{f}(-p)\hat{g}(p)\varepsilon(p_0)\delta(p_0^2 - \boldsymbol{p}^2 - m^2)\ \mathrm{d}p, \quad \varepsilon(p_0) = \frac{p_0}{|p_0|} \\ G_c(x,y) &= \frac{1}{(2\pi)^3}\int \mathrm{e}^{-i\eta(p,x-y)}\varepsilon(p_0)\delta(p_0^2 - \boldsymbol{p}^2 - m^2)\ \mathrm{d}p. \end{aligned} \tag{1.10}$$

The choice of G_c as the characterizing function for the solution space has a further advantage, namely we obtain a very important result on the finiteness of the propagation speed of information. Let ϕ be a solution to P with compact Cauchy data $u_0, u_1 \in \mathcal{D}(\Sigma)$. Then there is a $f \in \mathcal{D}$ with $\phi = G_c(f)$ and

$$\operatorname{supp}\phi = \operatorname{supp} G_c(f) \subset J(K), \qquad K = \operatorname{supp} u_0 \cup \operatorname{supp} u_1.$$

This implies, that initial data propagate in M with a **finite speed**, at most with the speed of light. It is a consequence of the hyperbolic nature of $\mathcal{S}_{\mathcal{L}_0}$, meaning that $\delta_{\mathrm{EL}}\mathcal{S}_{\mathcal{L}_0}$ is given by a hyperbolic partial differential operator. This feature is extremely important in classical and quantum field theory. It incorporates the validity of the fundamental principle of special relativity in the respective theories, namely it excludes superluminal motion and information exchange.

For later use we need a generalization to the statements in definition 1.3.3 that involves smooth functions with timelike compact support. The support of f is called timelike compact, if there are two Cauchy surfaces $\Sigma_\pm$ such that Σ_- is in the causal past and Σ_+ is in the causal future of $\operatorname{supp}(f)$. We find that the properties of G_c carry over to a slightly more general situation.

Proposition 1.3.4.
G_c extends to a map $G_c : \mathcal{E}_{tc} \to \mathcal{E}$, where $\mathcal{E}_{tc}$ is the space of smooth functions with timelike compact support and it holds the following:

- *Let $\mathcal{E}_P$ be the space of smooth solutions to P. For every $\phi \in \mathcal{E}_P$ there exists a function $f \in \mathcal{E}_{tc}$, such that $\phi = G_c(f)$ and the smooth Cauchy data $u_0, u_1 \in C^\infty(\Sigma)$.*
- *For every $f \in \mathcal{E}_{tc}$, such that $G_c(f) = 0$ there is a $g \in \mathcal{E}_{tc}$ with $f = Pg$, i.e. $\ker(G_c) = P\mathcal{E}_{tc}$.*

Let us summarize the structure. We begun with the space of smooth functions $\mathcal{E}$ and constructed a well-behaved solution space to the Euler-Lagrange equations for the action $\mathcal{S}_{\mathcal{L}_0}$, with a Lagrangian $\mathcal{L}_0$ that is quadratic. This solutions space in $\mathcal{E}$ is constructed as the space of solutions with compact Cauchy data. It is characterized by the causal commutator function G_c and we have found that information, which is carried by solutions to the theory, can propagate at most with the speed of light in M.

1.3.2. Functional Formalism of Classical Field Theory

In this section we want to lift the results of the previous discussion to the level of observables. The most important feature in this context will be the principle of finite speed of information propagation. The manifestation of this principle at the level of observables can be formulated as follows: Observables whose support is contained in spacelike separated regions of M must be represented within **independent** subsystems of the full theory. See [BF09] for more details on the general formulation of independent subsystems. In classical field theory that is described by Poisson algebras of observables, as it will be done in this work, independent subsystems can be realized by mutually Poisson commuting Poisson subalgebras.

We begin by stating physically motivated axioms which should be fulfilled by any classical field theory complying with the principles of special relativity. The formulation of these axioms is done in the spirit of the Haag-Kastler axioms in QFT, using nets of algebras.

Definition 1.3.5 (Axioms of locally covariant classical field theory).
The theory of a classical scalar field is described by a net of unital Poisson *-algebras $\{\mathfrak{A}_{\mathrm{cl}}(\mathcal{O}) \equiv (\mathfrak{A}_{\mathrm{cl}}(\mathcal{O}), \{\cdot,\cdot\}) : \mathcal{O} \subset M\}$ with the following properties:

CFT1: ***Isotony**: For every isometry $\mathfrak{P}^{\uparrow}_{+} \ni \mathfrak{p}_1 : \mathcal{O}_1 \to \mathcal{O}_2$ there is a unital *-homomorphism $\alpha_{\mathfrak{p}_1} : \mathfrak{A}_{\mathrm{cl}}(\mathcal{O}_1) \to \mathfrak{A}_{\mathrm{cl}}(\mathcal{O}_2)$ and for every further isometry $\mathfrak{P}^{\uparrow}_{+} \ni \mathfrak{p}_2 : \mathcal{O}_2 \supset \mathfrak{p}_1\mathcal{O}_1 \to \mathcal{O}_3$ it holds $\alpha_{\mathfrak{p}_1 \circ \mathfrak{p}_2} = \alpha_{\mathfrak{p}_1} \circ \alpha_{\mathfrak{p}_2}$.*

CFT2: ***Einstein causality**: If $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated, then $\{\mathfrak{A}_{\mathrm{cl}}(\mathcal{O}_1), \mathfrak{A}_{\mathrm{cl}}(\mathcal{O}_2)\} = 0$.*

CFT3: ***Time-slice axiom**: Let $\mathfrak{p} : \mathcal{O}_1 \to \mathcal{O}_2$. If $\mathfrak{p}\mathcal{O}_1$ contains a Cauchy surface for $\mathcal{O}_2$, then $\alpha_{\mathfrak{p}} : \mathfrak{A}_{\mathrm{cl}}(\mathcal{O}_1) \to \mathfrak{A}_{\mathrm{cl}}(\mathcal{O}_2)$ is surjective. It follows that in particular $\mathfrak{A}_{\mathrm{cl}}(\Sigma_\epsilon) = \mathfrak{A}_{\mathrm{cl}}(M)$ for any ϵ-neighborhood Σ_ϵ of a Cauchy surface Σ of M.*

The ϵ-neighborhood Σ_ϵ of Σ in the axiom CFT3 is called a time-slice for M. We see that the main object in classical field theory is a space of observables, that is endowed with a Poisson and an involutive structure.

In the following we construct the classical field theory as Poisson *-algebras of functionals $A \in \mathcal{F}_{\mu\mathrm{c}}$ over $\mathcal{E}$. $\mathcal{F}_{\mu\mathrm{c}}$ contains a lot of physically interesting observables, some of them are given in example 1.2.2, and this space turns out to be stable under deformation quantization. We begin with the definition of the Poisson bracket on $\mathcal{F}_{\mu\mathrm{c}}$.

Definition 1.3.6 (Poisson bracket).
Let G_{c} the the causal commutator function, induced by the Klein-Gordon operator P in definition 1.3.3. The bi-linear map

$$\{\cdot,\cdot\}_{G_{\mathrm{c}}} : \qquad \mathcal{F}_{\mu\mathrm{c}} \times \mathcal{F}_{\mu\mathrm{c}} \ni A \times B \mapsto \{A,B\}_{G_{\mathrm{c}}}(\cdot) = \left\langle iG_{\mathrm{c}} \middle| \frac{\delta A(\cdot)}{\delta\phi} \otimes \frac{\delta B(\cdot)}{\delta\phi} \right\rangle \in \mathcal{F}_{\mu\mathrm{c}} \tag{1.11}$$

*is called the **Poisson bracket** of the classical field theory governed by the action $\mathcal{S}_{\mathcal{L}_0}$ in* (1.7).

The proof, that $\{\cdot,\cdot\}_{G_c}$ fulfills all properties of a Poisson bracket is found in [BFR12] (in the more general context of Peierls brackets, see also section 2.1).

The Poisson bracket can now be used to define a Poisson *-algebra, which will serve as the classical algebra of observables.

Definition 1.3.7 (Poisson *-algebra of off-shell observables, Local net).
The vector space $\mathcal{F}_{\mu c}$, endowed with

- *the pointwise product $\cdot$ of functionals $(A \cdot B)(\phi) = A(\phi)B(\phi)$*
- *the canonical involution $A^*(\phi) = A(\phi)^*$*
- *the Poisson bracket $\{\cdot,\cdot\}_{G_c}$ from definition 1.3.6*

forms a unital Poisson *-algebra. That is, $(\mathcal{F}_{\mu c}, \cdot)$ is a unital, commutative and associative *-algebra and

- *$\{\cdot,\cdot\}_{G_c}$ is bi-linear, anti-symmetric and obeys the Jacobi identity,*
- *$\{\cdot,\cdot\}_{G_c}$ is a derivation in the right argument: $\{A, B\cdot C\}_{G_c} = \{A,B\}_{G_c}\cdot C + B\cdot\{A,C\}_{G_c}$,*
- *$\{\cdot,\cdot\}_{G_c}$ is compatible with the involution: $\{A,B\}^*_{G_c} = \{A^*, B^*\}_{G_c}$.*

*We call the triplet $\mathcal{A}_{\mathrm{cl}} = (\mathcal{F}_{\mu c}, \cdot, \{\cdot,\cdot\}_{G_c})$ the **classical algebra of off-shell observables** of the free scalar field. The local net is obtained by the restricted algebras $\mathcal{A}_{\mathrm{cl}}(\mathcal{O}) = \{A \in \mathcal{A}_{\mathrm{cl}} : \operatorname{supp} A \subset \mathcal{O}\}$.*

Here the use of the factor i in (1.11) is crucial, since iG_c is real-valued. That makes $\{\cdot,\cdot\}_{G_c}$ compatible with the canonical involution on $\mathcal{F}_{\mu c}$.

A nice aspect of this construction is that we have the quasi-local algebra of (off-shell) observables $\mathcal{A}_{\mathrm{cl}} = \mathcal{A}_{\mathrm{cl}}(M)$ directly available. In the general situation in which there is only a net $\mathcal{A}_{\mathrm{cl}}(\mathcal{O})$ at our disposal, the quasi-local algebra is obtained by the inductive limit

$$\mathcal{A}_{\mathrm{cl}}(M) = \varinjlim_{\mathcal{O}\nearrow M} \mathcal{A}_{\mathrm{cl}}(\mathcal{O})$$

with the help of axiom CFT1. With this limiting construction the quasi-local algebra consists of equivalence classes of sequences in the union of all $\mathcal{A}_{\mathrm{cl}}(\mathcal{O})$. This abstract setting can be avoided here.

The resulting net of Poisson *-algebras $\{\mathcal{A}_{\mathrm{cl}}(\mathcal{O}) : \mathcal{O} \subset M\}$ is not endowed with any dynamical structure yet, justifying the term off-shell algebra. It is called kinematical algebra sometimes, pointing out, that $\mathcal{A}_{\mathrm{cl}}$ carries a kinematical (relativistic) structure. This can be used to show, that the first two axioms of classical field theory in definition 1.3.5 are satisfied.

Proposition 1.3.8 ([BFR12]).
The net of Poisson *-algebras $\mathcal{A}_{\mathrm{cl}}(\mathcal{O})$ from definition 1.3.7 fulfills the axioms CFT1 *and* CFT2.

In order to implement the dynamics induced by the quadratic action $\mathcal{S}_{\mathcal{L}_0}$ in (1.7), we use a basic technique known in the algebraic approach: we divide $\mathcal{A}_{\mathrm{cl}}$ by a suitable ideal $\mathcal{I}_P$ that is obtained by the Euler-Lagrange equations of the action. To ensure that the resulting quotient space is a Poisson *-algebra we have to make sure, that $\mathcal{I}_P$ is a Poisson *-ideal in $\mathcal{A}_{\mathrm{cl}}$.

The identification of the correct ideal follows from the following idea. In the canonical approach (or on-shell approach), the observables of the theory are constructed as functionals over the solution space (i.e. functionals over $\mathcal{E}_P$ from proposition 1.3.4). $\mathcal{E}_P$ has a natural embedding $\iota : \mathcal{E}_P \to \mathcal{E}$ into $\mathcal{E}$. The dual map

$$\iota^* : \mathcal{F}_{\mu\mathrm{c}} \to \mathcal{F}_{\mu\mathrm{c}}\Big|_{\mathcal{E}_P}$$

is the restriction of microcausal functional to the solution space. The map ι^* has a non-trivial kernel, given by

$$\ker(\iota^*) = \{F \in \mathcal{F}_{\mu\mathrm{c}} : F(\phi) = 0 \quad \text{if } \phi \in \mathcal{E}_P\}.$$

It turns out that $\ker(\iota^*)$ has exactly the properties of a Poisson *-ideal. This is shown by the following argument: Let $\phi \in \mathcal{E}_P$ and define $\phi_t = \phi + itG_{\mathrm{c}}(A^{(1)}(\phi))$, where we use the notation $G_{\mathrm{c}} : \mathcal{D} \to \mathcal{E}_P$ here. From proposition 1.3.4 it is clear that $G_{\mathrm{c}}(A^{(1)}) \in \mathcal{E}_P$, i.e. we have a family ϕ_t of solutions to P with $\phi_0 = \phi$. This implies that $I(\phi_t) = 0$ for all $t \in \mathbb{R}$ for any $I \in \mathcal{I}_P$ and

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} I(\phi_t)\Big|_{t=0} = i\left\langle G_{\mathrm{c}}\middle| A^{(1)}(\phi) \otimes I^{(1)}(\phi)\right\rangle = \{A, I\}(\phi)$$

which implies the claim. From now on we denote the Poisson *-ideal $\ker(\iota^*)$ by $\mathcal{I}_P$. It motivates the following definition:

Definition 1.3.9 (On-shell functionals).
The quotient

$$\mathcal{F}^P_{\mu\mathrm{c}} = {}^{\mathcal{F}_{\mu\mathrm{c}}}\Big/_{\mathcal{I}_P}, \qquad \mathcal{A}^P_{\mathrm{cl}} = (\mathcal{F}^P_{\mu\mathrm{c}}, \cdot, \{\cdot,\cdot\}_{G_{\mathrm{c}}})$$

*is called the space of on-shell functionals and the corresponding Poisson *-algebra $\mathcal{A}^P_{\mathrm{cl}}$ is termed the **classical algebra of on-shell observables** of the free scalar field.*

We want to discuss this notion within the example of polynomial functionals.

Example 1.3.10 (On-shell polynomial functionals).
Let F be a generating element of the space of polynomial functionals $\mathcal{F}_{\mathrm{pol}}$, i.e. $F(\phi) = \langle t|\phi^{\otimes n}\rangle$ with $t \in \mathcal{E}'_{\Xi_n}(M^n)$, see below definition 1.2.5. The null class of $\mathcal{F}_{\mathrm{pol}}$ with respect to the ideal $\mathcal{I}_P$ is given by

$$[0] = \left\{ F \in \mathcal{F}_{\mathrm{pol}} : F(\phi) = \sum_{n=1}^{N} \langle t_n|\phi^{\otimes n}\rangle = 0 \quad \text{if } P\phi = 0 \right\}.$$

Now we use the fact that for every $\phi \in \mathcal{E}_P$ there is a $f \in \mathcal{E}_{\mathrm{tc}}$, such that $\phi = G_{\mathrm{c}}(f)$. Thus the condition for $[0]$ can be characterized by distributions $t_n \in \mathcal{E}'_{\Xi_n}(M^n)$ for which $G_{\mathrm{c}}^{\otimes n}(t_n) = 0$, i.e. those which lie in the kernel of $G_{\mathrm{c}}^{\otimes n}$ for $n \in \mathbb{N}$. Using the second property in 1.3.3 (which can be generalized to compactly supported distributions with appropriate wavefront sets) we can deduce that distributions $t_n \in \mathcal{E}'_{\Xi_n}(M^n)$, which are contained in the kernel of G_{c} can be written as

$$t_n(x_1, \ldots, x_n) = P_{x_j} u_n(x_1, \ldots, x_n), \qquad P_{x_j} = \Box_{x_j} + m^2, \quad u_n \in \mathcal{E}'_{\Xi_n}(M^n)$$

for some $j \in \{1, \ldots, n\}$. This can be used to determine the equivalence class of any polynomial functional:

$$[F] = F + I, \qquad I(\phi) = \sum_{n=1}^{N} \left\langle P_{x_{j(n)}} u_n \middle| \phi^{\otimes n} \right\rangle, \qquad u_n \in \mathcal{E}'_{\Xi_n}, \quad j(n) \in \{1, \ldots, n\},$$

where $P_{x_{j(n)}} u_n$ can be symmetrized if one restricts the attention to symmetric distributions. This agrees with the equivalence classes defined in other algebraic approaches to free field theory [Haa92]. ♢

Proposition 1.3.11 ([Dim80]).
The net of local Poisson *-algebras $\mathcal{A}^P_{\mathrm{cl}}(\mathcal{O})$ fulfills all of the axioms of locally covariant classical field theory in definition 1.3.5.

We conclude the section with a summary of all the ingredients to the off-shell approach to classical field theory. We started with the definition of a suitable space of observables $\mathcal{F}_{\mu\mathrm{c}}$, which contains functionals over $\mathcal{E}$ with a certain singularity structure for which the Poisson bracket $\{\cdot, \cdot\}_{G_{\mathrm{c}}}$ induced by the free action $\mathcal{S}_{\mathcal{L}_0}$, is well-defined. The Poisson *-algebra $\mathcal{A}_{\mathrm{cl}}$ was shown to comply with the principles of special relativity, since it fulfilled CFT1 and CFT2. In a last step we endowed $\mathcal{A}_{\mathrm{cl}}$ with a dynamics induced by the free action by dividing by the ideal $\mathcal{I}_P$.

This concludes the discussion on the algebraic properties of classical field theory. Many of the presented structures will appear again in the quantization of the classical field theory, which is carried out in the framework of deformation quantization. A key idea behind this procedure is to equip the space $\mathcal{F}_{\mu\mathrm{c}}$ with a non-commutative associative product $\star$, which is compatible with both the principles of special relativity and quantum mechanics. In doing so, we obtain a very nice interpretation of the theory: The space of observables is the same in classical and quantum field theory! Therefore the physical interpretation of, say the energy or angular momentum functional, is stable under quantization. A few functionals of physical relevance have been were denoted in example 1.2.2.

The algebraic characterization of classical field theory by means of nets of Poisson *-algebras puts us in the very comfortable position concerning the space of states of classical field theory, since the construction of $\mathcal{A}_{\mathrm{cl}}$ was carried out completely **state-independent**. The fruits of our labor will be harvested in the following section.

1.3.3. States in Classical Field Theory

In this section we discuss the space of states in classical field theory. Many notions concerning the general structure of the state space of quantum field theory appear already at the classical level akin to the case of the algebra of observables. After an introduction to the basic notions of states on $\mathcal{A}_{\text{cl}}$ we discuss the example of the classical KMS state, whose historical background was highlighted in the introduction of this thesis.

The general concept of a state of a physical system refers to the idea of a **preparation** of a given ensemble of systems. The prescription how to prepare every system in this ensemble of systems is denoted by ω and is called a state of the system. The outcomes of measurements in the ensemble will then follow a probability distribution. This means that the values of the observable $A \in \mathcal{A}_{\text{cl}}$ vary in the individual systems and the probability to find A with value a is $p_a(A) \geq 0$. Thus the relevant quantities which characterize a state are the **expectation values** of the observables, i.e. the average values of A when picking a system from the ensemble at random, as well as higher statistical momenta. If the prepared ensemble (the state) is denoted by ω, this expectation value shall be denoted by $\omega(A) \in \mathbb{R}$.

The concept of states can be realized, under some assumptions on the system, by Borel measures on the phase space of the physical system. For example, in classical mechanics (with configuration space Q and phase space T^*Q) states on the algebra of bounded functions on T^*Q are given by Borel probability measures μ on T^*Q. The expectation value of an observable A in the ensemble of systems ω_μ described by measure μ on T^*Q is

$$\omega_\mu(A) = \int_{T^*Q} A \, \mathrm{d}\mu \, .$$

The case in which the ensemble is prepared in an optimal way, i.e. the uncertainty of the predictions of all possible measurements is minimal, corresponds to a **pure state** of the system. In classical mechanics this minimum is zero and is achieved by states induced from a Dirac measure $\mu_{(q,p)}$ on T^*Q. That is the state defined by

$$\omega_{\mu_{(q,p)}}(A) = \int_{T^*Q} A \, \mathrm{d}\mu_{(q,p)} = A(q,p) \, .$$

We want to generalize the concept of a state to an algebraic notion, i.e. rather than obtaining states by Borel measures on a phase space, we characterize a state by its expectation values on $\mathcal{A}_{\text{cl}}$. Following this line of thought, we see a state as a map $A \mapsto \omega(A)$ which is, by abuse of notation, denoted by $\omega : \mathcal{A}_{\text{cl}} \to \mathbb{C}$. Clearly, this map should be linear. It is crucial that this map describes a probability distribution. This idea can be translated into a condition on the linear functional ω: For this we look at the characteristic functions χ_K for finite regions K in the phase space T^*Q. The probability that the ensemble described by μ is prepared with the values $(q,p) \in K$ is given by

$$\mu(K) = \omega_\mu(\chi_K) = \int_{T^*Q} \chi_K \, \mathrm{d}\mu, \qquad \chi_K(q,p) = \begin{cases} 1 & (q,p) \in K \\ 0 & (q,p) \notin K \end{cases} .$$

The indicator functions χ_K are projection operators on the algebra, i.e. $\chi_K^2 = \chi_K = \chi_K^*$. The property of being a probability distribution can be rephrased into the condition: $0 \leq \omega(\chi_K) \leq 1$ for all $K \subset T^*Q$ and $\omega(\chi_{T^*Q}) = \omega(1) = 1$. Since the first condition is still dependent on the regions $K \subset T^*Q$, we replace it with a sufficient condition, namely positivity: $\omega(A) \geq 0$ if A is a non-negative function. In the case of complex-valued functions, we can write the non-negative ones as $A = B^* \cdot B$.

The discussion above motivates the following definition of states.

Definition 1.3.12 (States in $\mathcal{A}_{\text{cl}}$).
A state over $\mathcal{A}_{\text{cl}}$ is a linear functional ω over $\mathcal{A}_{\text{cl}}$, which is normalized $\omega(\mathbb{1}) = 1$ and positive: For $A^* = A > 0$ the property $\omega(A) \geq 0$ holds or, anticipating the quantum condition of positivity,

$$\omega(A^* \cdot A) \geq 0 \qquad \forall A \in \mathcal{A}_{\text{cl}}.$$

*A state is called **pure**, if the decomposition*

$$\omega(A) = \lambda\omega_1(A) + (1-\lambda)\omega_2(A), \qquad \lambda \in (0,1)$$

*into two other states yields only the trivial solution $\omega = \omega_1 = \omega_2$. Otherwise, ω is called **mixed**. The convex cone of all states over $\mathcal{A}_{\text{cl}}$ is denoted by $\mathfrak{S}(\mathcal{A}_{\text{cl}})$.*

Note that the above states are defined on an algebra, which is "too large", since it contains unphysical off-shell configurations. Though the question arises whether we can obtain states on the algebra of on-shell functionals from elements of $\mathfrak{S}(\mathcal{A}_{\text{cl}})$. This can be answered with the help of the canonical surjection $\iota^* : \mathcal{A}_{\text{cl}} \to \mathcal{A}_{\text{cl}}^P$ that was used to defined the ideal $\mathcal{I}_P$ in definition 1.3.9.

A state $\omega \in \mathfrak{S}(\mathcal{A}_{\text{cl}}^P)$ is said to be induced by a state $\hat{\omega}$ over $\mathcal{A}_{\text{cl}}^P$ if

$$\omega = \hat{\omega} \circ \iota^*$$

Since $\ker \iota^*$ is non-trivial, this relation can not be inverted, i.e. not every state in $\mathfrak{S}(\mathcal{A}_{\text{cl}})$ is induced by an on-shell state. If we restrict the attention to states $\omega \in \mathfrak{S}(\mathcal{A}_{\text{cl}})$ which vanish on $\ker(\iota^*) = \mathcal{I}_P$, then the map can be inverted. This means that we have a bijection between the spaces $\mathfrak{S}(\mathcal{A}_{\text{cl}}^P)$ and $\{\omega \in \mathfrak{S}(\mathcal{A}_{\text{cl}}) : \omega(I) = 0 \quad \forall I \in \mathcal{I}_P\}$. A state ω in the latter set will also be called compatible with the dynamics induced by $\mathcal{I}_P$.

The most important class of states in the functional formalism is given by evaluation functionals.

Definition 1.3.13 (Evaluation functionals).
Linear functionals ev_ψ over $\mathcal{A}_{\text{cl}}$, which are given by

$$\text{ev}_\psi(A) = A(\psi), \qquad \psi \in \mathcal{E}$$

are called evaluation functionals. They are normalized and positive, hence they form a subset of $\mathfrak{S}(\mathcal{A}_{\text{cl}})$. They are pure states.

Due to form of ev_ψ, it is evident that ω_ψ is compatible with the dynamics induced by $\mathcal{I}_P$ if

$$\omega_\psi(I) = I(\psi) \stackrel{!}{=} 0 \qquad \forall I \in \mathcal{I}_P \,.$$

It follows from the definition of $\mathcal{I}_P$ that those functionals have to be evaluations on **solutions** to the differential operator P, i.e. $\psi \in \mathcal{E}_P$.

We end the discussion of the classical state space by two example states, which can be derived from the classical KMS condition. For a detailed introduction and derivation of the KMS condition and the properties of KMS states, which are generalizations of Gibbs states, see [Haa92].

Definition 1.3.14 (Classical KMS States [AGGL76]).
The KMS condition for a Poisson algebra $\mathcal{A}_{\mathrm{cl}}$ with time-evolution α_t is a relation between the functions

$$F_{A,B}(t) = \omega(A \cdot \alpha_t(B)) - \omega(A)\omega(B), \quad \textit{and} \quad G_{A,B}(t) = \omega(\{A, \alpha_t(B)\}).$$

A state ω_β is called a KMS state for α_t on $\mathcal{A}_{\mathrm{cl}}$ with inverse temperature $\beta = (k_B T)^{-1}$, if

$$G_{A,B}(t) = -\beta \frac{\mathrm{d}}{\mathrm{d}t} F_{A,B}(t) \qquad \forall A, B \in \mathcal{A}_{\mathrm{cl}}. \tag{1.12}$$

In the limit $\beta \to \infty$, we obtain the ground state condition $\frac{d}{dt} F_{A,B}(t) = 0$.

Example 1.3.15 (Classical, quasi-free KMS-state).
First, we consider the Poisson *-subalgebra of $\mathcal{A}_{\mathrm{cl}}$ which consists only of polynomial and regular functionals. A quasi-free state is a state, which is determined by the two-point function (with respect to the linear field Φ_f):

$$W_\omega(f, g) = \omega(\Phi_f \cdot \Phi_g) \qquad f, g \in \mathcal{D}.$$

The distribution W_ω determines the expectation value for an arbitrary $A \in \mathcal{A}_{\mathrm{pol}}$ by the formula

$$\omega(A) = \mathrm{ev}_0(\mathrm{e}^{\Gamma_\omega} A), \qquad \Gamma_\omega A = \int W_\omega(x, y) \frac{\delta^2 A}{\delta\phi(x)\delta\phi(y)} \,\mathrm{d}x \,\mathrm{d}y. \tag{1.13}$$

The positivity condition on ω translates into the condition: $W_\omega(f^*, f) \geq 0$ for all $f \in \mathcal{D}$. Combining the KMS condition with the ansatz (1.13) we obtain the following differential equation for the two-point function

$$iG_{\mathrm{c}}(x, y) = -\beta \frac{\partial}{\partial y^0} W_\beta(x, y) \,. \tag{1.14}$$

We assume at this point that ω_β (and thus W_β) is translation invariant (as G_{c} is, too), hence we can assume that

$$W_\beta(x, y) = \frac{1}{(2\pi)^4} \int \hat{W}_\beta(p) \mathrm{e}^{-i\eta(p, x-y)} \,\mathrm{d}p$$

up to a constant coming from the one-point function $\omega(\Phi_x)$[2], which we assume to be zero. The differential equation on W_β transforms into

$$\hat{G_c}(p) = \beta p_0 \hat{W}_\beta(p)$$

which can be solved by inserting G_c from (1.10)

$$\begin{aligned} W_\beta(x,y) &= \frac{1}{(2\pi)^3\beta}\int \mathrm{e}^{-i\eta(p,x-y)}\frac{\delta(p_0^2-\boldsymbol{p}^2-m^2)}{|p_0|}\,\mathrm{d}p \\ &= \frac{1}{2\pi^2\beta}\int_m^\infty \frac{\sin\left(\sqrt{\omega^2-m^2}\,|\boldsymbol{x}-\boldsymbol{y}|\right)\cos\left(\omega(x^0-y^0)\right)}{\sqrt{\omega^2-m^2}\,|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\omega. \end{aligned}$$

The positivity condition for $W_\beta(f^*,f)$ is implied by the positivity of the measure $\hat{W}_\beta(p)$. Moreover W_β is compatible with the dynamics, i.e. it annihilates the ideal $\mathcal{I}_P$, since W_β is a weak bi-solution to P.

In the limit of vanishing mass we can directly compute the integral (using translation invariance):

$$W_\beta(x,0) = \frac{1}{8\pi\beta\,|\boldsymbol{x}|}\left(\varepsilon(x^0+|\boldsymbol{x}|)-\varepsilon(x^0-|\boldsymbol{x}|)\right) = \begin{cases} \frac{1}{4\pi\beta|\boldsymbol{x}|} & |x^0|<|\boldsymbol{x}| \\ 0 & \text{else} \end{cases}.$$

By equation (1.14) it is clear that the wavefront sets of W_β and G_c coincide for $\beta \neq 0$. Unfortunately, we see that $\mathrm{WF}(W_\beta)$ and Ξ_2 can add up to the zero section, thus Hörmanders theorem of multiplication can not be applied to define ω_β by means of (1.13) for an arbitrary $A \in \mathcal{F}_{\mu c}$ (see definition 1.4.2 and below). Moreover, by a direct computation one encounters UV-divergences, e.g. for local observables.

This shows that there is no KMS-state for $\beta \neq 0$, if we choose $\mathcal{F}_{\mu c}$ as the space of observables. This problem is known as the Rayleigh-Jeans (UV-)catastrophe of classical field theory at finite temperature, which has been explained in the introduction. The interpretation of this fact is that classical field theory can only be an effective description, which is valid in a certain range of energies i.e. for observables, which are insensitive to high momenta. As a matter of fact this was one of the reasons to develop quantum mechanics and quantum field theory in the first place. It turns out, that this problem does not occur in QFT as the two-point function there will not obey a Rayleigh-Jeans momentum distribution (the particular form of $\hat{W}_\beta$), but a Planck distribution, as we will show in the section 1.5.

W_ω does, however, define a KMS-state through (1.13) on $\mathcal{F}_{\mathrm{reg}}$, which in turn lacks many interesting observables like the energy density and other characteristic quantities.

The ground state condition in definition 1.3.14 (the limit $\beta \to \infty$) has a better behavior. A quasi-free ground state, which is in addition translation invariant fulfills the following condition

$$\frac{\partial}{\partial y^0}W_\infty(x,y) = 0 = \frac{\partial}{\partial x^0}W_\infty(x,y), \qquad \omega_\infty(\Phi_x) = 0$$

[2]Since the state is assumed to be translation invariant, $\omega(\Phi_x)$ must be a constant.

setting again the one-point function to zero. The only solution to the wave-equation with these Cauchy data is $W_\infty = 0$, resulting in the state $\omega_\infty(A) = A(0)$ for $A \in \mathcal{F}_{\mu c}$. This is the classical, translation invariant ground state with vanishing one-point function defining the **vacuum state** for the classical field theory. However, unlike in quantum theory, where the vacuum state plays a pivotal role, the classical vacuum state has no interesting features, e.g. its expectation value and higher statistical momenta vanish on all polynomials besides the constant. ♢

1.4. Formal Deformation Quantization

This section is concerned with the quantization of the theory of the real scalar field in the framework of the functional approach to quantum field theory. The important guideline for this procedure will be, as it is stressed throughout this thesis, state-independence. A method that works very well with both the functional framework and the algebraic approach to QFT is formal deformation quantization. It exhibits extensive conceptual power and is at the same time a very explicit method to implement a non-commutative structure in the space of functionals $\mathcal{F}_{\mu c}$.

After a short review on the idea of formal deformation quantization we construct the quantum algebra of observables of the real scalar field in the off-shell functional approach. We define the action of the symmetry group $\mathfrak{P}_+^\uparrow$ of M and show that the corresponding net of *-algebras fulfills the Haag-Kastler axioms of QFT, after going "on-shell". A considerable part will be spent on the discussion on the space of quantum states of the free scalar field. There the vacuum and KMS state are presented in an explicit form and crucial properties are highlighted, which will become important in the subsequent construction of the interacting KMS state.

1.4.1. The Idea behind Formal Deformation Quantization

The concept of formal deformation quantization as we use it here has its roots in the question: What is the quantization of a general classical system? There are many proposals to answer this question, such as canonical quantization, the path integral approach or geometric quantization. Whereas the first two are usually considered in the physics community due to a certain simplicity of the framework, these methods unfortunately have little conceptual power and only a limited amount of observables is quantized.

Geometric quantization, on the other hand, has a well-motivated conceptual basis, which takes the Lie-algebraic foundation of classical theory serious and constructs a quantum theory by representing the classical observables as suitable sections on a line bundle over the classical phase space. This procedure has problems in the explicit construction of non-trivial models due to the involved representation-theoretical aspects. We do not go into detail here and refer to standard textbooks, e.g. [Woo97].

This work is concerned with the deformation quantization approach to quantization. One of the foundations to this approach is an observation which dates back to Groe-

newold and van-Hove in the middle of the last century. The authors showed that classical mechanics has no straightforward quantization in the following sense. We view classical mechanics over $\mathbb{R}^n$ as a Poisson-algebra $\mathfrak{A}_{\text{cl}} = (\mathsf{Pol}(\mathbb{R}^{2n}), \cdot, \{\cdot,\cdot\})$, where Pol is the space of polynomial functions and $\{\cdot,\cdot\}$ comes from the standard symplectic form over $\mathbb{R}^n$, making $\mathfrak{A}_{\text{cl}}$ a Lie-algebra, in particular. Quantum mechanics over $\mathbb{R}^n$ can be seen as the Lie-algebra $\mathfrak{A}$, generated by $\{p_i, x_j : i,j = 1,\dots,n\}$ obeying the canonical commutation relations

$$[x_i, p_j] = i\hbar\{x_i, p_j\} = i\hbar\delta_{ij},$$

It can be realized by the canonical quantization:

$$x_i \in \mathcal{S}'(\mathbb{R}^n), \quad x_i(f) = x_i \cdot f, \qquad p_i \in \mathcal{S}'(\mathbb{R}^n), \quad p_i(f) = \frac{\hbar}{i}\frac{\partial f}{\partial x_i}$$

with $\mathcal{S}(\mathbb{R}^n)$[3] as the Hilbert space. A natural definition of the quantization of $\mathfrak{A}_{\text{cl}}$ in this language would be an injective Lie-algebra homomorphism

$$Q : \mathfrak{A}_{\text{cl}} \to \mathfrak{A}\,.$$

The theorem of Groenewold and van-Hove states that there is no such map. Moreover, it can be shown that there is no unital, associative algebra $\mathfrak{A}_{\text{cl}}$, such that there is a Lie-algebra isomorphism $Q : \mathfrak{A}_{\text{cl}} \to \mathfrak{A}$ [Wal07].

At this point, formal deformation quantization sets in. What if the map Q can be realized not as a strict Lie-algebra isomorphism, but up to "quantum corrections"? This means, for the above case of quantum mechanics, that

$$Q(A \cdot B) = A \cdot B + O(\hbar), \qquad Q(\{A,B\}) = \frac{\hbar}{i}[A,B] + O(\hbar^2), \qquad A, B \in \mathfrak{A}_{\text{cl}}. \quad (1.15)$$

This seminal idea was taken serious in [BFF$^+$77], where deformation quantization was first investigated. The concept is introduced as follows. Define the space of formal power series in $\hbar$. That is, for a vector space V, the space $V[[\hbar]]$ which consists of sequences $(v_n \in V : n \in \mathbb{N})$ seen as formal sums

$$v = v_0 + v_1\hbar + v_2\hbar^2 + \dots, \qquad v_n \in V,$$

where we do not impose any convergence conditions on the sum. $V[[\hbar]]$ carries a natural vector space structure.

The quantum algebra $\mathfrak{A}$ is defined as the algebra of formal power series of classical observables, e.g. $\mathfrak{A} = (\mathsf{Pol}(\mathbb{R}^{2n})[[\hbar]], \star)$ with a $\star$-product

$$\star : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}, \qquad A \star B = \sum_{n=0}^{\infty} \hbar^n C_n(A,B) \quad (1.16)$$

[3]The space $\mathcal{S}$ is the space of rapidly decaying functions or Schwartz functions.

where the C_n are bi-$\mathbb{R}[[\hbar]]$-linear. That implies that the n-th order term of the $\star$-product $A \star B$ of $A = (A_n)$, $B = (B_n)$ is given by

$$(A \star B)_n = \sum_{\substack{k,l,m \in \mathbb{N}_0 \\ k+l+m=n}} C_k(A_l, B_m) \,,$$

i.e. only **finitely** many elements of the formal power series A and B contribute to each summand in $A \star B$. This feature lies at the very heart of the the theory of formal deformation quantization.

A further condition on the $\star$-product which sounds quite innocent at first, but has very deep impact on the arising theory, is associativity of the product. It can be expressed in terms of the maps C_n:

$$\sum_{l=0}^{k} C_l(A_1, C_{k-l}(A_2, A_3)) = \sum_{l=0}^{k} C_l(C_{k-l}(A_1, A_2), A_3) \qquad \forall A_1, A_2, A_3 \in \mathfrak{A} \,.$$

It is sufficient to evaluate the maps C_n on elements of order $\hbar^0$ in this context. The enormous impact of this condition is due to its non-linear structure and it turns out that it is very restrictive in the following sense: In the case of quantum mechanics (i.e. $\mathbb{R}^{2n}$) all associative $\star$-products are equivalent (the notion of equivalence is introduced in the next section).

Then, the quantization map $Q : \mathfrak{A}_{\mathrm{cl}} \to \mathfrak{A},\ A \mapsto Q(A) = A\hbar^0$ is well-defined and indeed a Lie-algebra homomorphism, up to higher powers in $\hbar$. For a detailed treatment of this matter we refer to [Wal07].

There are many advantages of deformation quantization, one of which is that the quantization of general systems, which may have a very complicated phase space, is possible. It has been shown by Kontsevich in [Kon03], that every (finite-dimensional) Poisson-manifold admits a formal deformation quantization by means of $\star$-products of the form (1.16).

The success of the $\star$-deformation approach in quantum mechanics posed the question whether these methods could also be applied to field theoretical systems which are described by infinite-dimensional Poisson-algebras, such as the classical algebra of off-shell observables in definition 1.3.7. In a work of Dito [Dit90] this issue was addressed and solved for the free field. He also showed a relation of this approach to the path-integral approach to quantum field theory. In a series of papers starting from [DF00, BF00] the authors Brunetti, Dütsch, Fredenhagen developed a quantum field theoretical framework, based on deformation quantization, that includes special and general covariance (in non-trivial spacetime backgrounds) and can be used for the perturbative construction of interacting theories. For a review we refer the reader to the book [BF09].

1.4.2. Deformation Quantization of Classical Field Theory

In analogy to quantum mechanics we define the space of formal power series over the space of classical observables.

Definition 1.4.1 (Formal power series of functionals).
The space of quantum observables is defined to be the space of formal power series $\mathcal{F}_{\mu c}[[\hbar]]$ of $\mathcal{F}_{\mu c}$. The classical observables are embedded by $\mathcal{F}_{\mu c} \ni A \mapsto A\hbar^0 \in \mathcal{F}_{\mu c}[[\hbar]]$.

A crucial task in the deformation quantization of classical field theory is to specify a $\star$-product or equivalently a set of bi-linear maps C_n in (1.16) such that the arising theory has all the required properties. Moreover, as specified in the introduction, this is required to be done with the help of a state-independent prescription.

A way to bring together the ideas of deformation quantization and the algebraic approach to quantum physics was proposed by Dütsch and Fredenhagen in [DF00]. A crucial ingredient is the definition of a suitable $\star$-product on $\mathcal{F}_{\mu c}[[\hbar]]$. This product is motivated by Wick's theorem for normal ordered operators. We state it here in a form that serves well for the subsequent constructions.

Theorem 2 ([BF00]).
Let $\mathcal{H}$ be the Fock space with respect to a quasi-free state

$$\Delta_+(f,g) = \omega(\varphi_f \varphi_g) = \int f(x)g(y)\omega(\varphi(x)\varphi(y))\,\mathrm{d}x\,\mathrm{d}y$$

of the free Klein-Gordon field φ and $A, B \in \mathcal{F}_{\mathrm{reg}} \cap \mathcal{F}_{\mathrm{pol}}$ be regular and polynomial. Moreover let $:\cdot:$ be the normal ordering of operators on $\mathcal{H}$ with respect to the creation and annihilation operators on $\mathcal{H}$. Then $:A(\varphi):$, $:B(\varphi):$ and the operator product $:A(\varphi)::B(\varphi):$ exist as operator valued functionals on (a dense subspace of) $\mathcal{H}$ and

$$\begin{aligned}&:A(\varphi)::B(\varphi):\\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int \Delta_+(x_1,y_1)\cdots\Delta_+(x_n,y_n) :\frac{\delta^n A(\varphi)}{\delta\phi(x_1)\cdots\delta\phi(x_n)} \frac{\delta^n B(\varphi)}{\delta\phi(y_1)\cdots\delta\phi(y_n)}: \mathrm{d}X\,\mathrm{d}Y,\end{aligned}$$

where the sum extends only over finitely many terms, due to A and B being polynomials. Here $\mathrm{d}X = \mathrm{d}x_1\cdots\mathrm{d}x_n$.

This version of Wick's theorem expresses how the product of two normal ordered operators can be written in terms of a sum of normal ordered operators. In addition the multiplication formula holds on every Hilbert space associated to a quasi-free state over the algebra of the free field. For details on the construction of such a Fock space, see section A.2 in the appendix. The algebraic structure of Wick's theorem is now distilled in the $\star$-product over the space of functionals: For $A, B \in \mathcal{F}_{\mathrm{reg}} \cap \mathcal{F}_{\mathrm{pol}}$ we define

$$\begin{aligned}\mathcal{F}_{\mathrm{pol}} \cap \mathcal{F}_{\mathrm{reg}} \ni A \star B &= \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \Delta_+^{\otimes n} \middle| A^{(n)} \otimes B^{(n)} \right\rangle = m \circ \exp(\Gamma_+)(A \otimes B)\\ \Gamma_+ &= \int \Delta_+(x,y) \frac{\delta}{\delta\phi(x)} \otimes \frac{\delta}{\delta\phi(y)}\,\mathrm{d}x\,\mathrm{d}y, \qquad m(A\otimes B)(\phi) = A(\phi)B(\phi)\end{aligned}$$

The $\star$-product is well-defined since the functional derivatives of A and B are compactly supported functions and the sum extends again only over finitely many terms.

We will now use $\star$ in order to build a non-commutative algebra on the space of all functionals, such that Wick's theorem holds. Since Δ_+ is not symmetric, the resulting

product $\star$ is non-commutative. This is the crucial step in quantizing the free classical field theory. First of all we will need some constraints on Δ_+ in order to get a well-behaved algebra with suitable physical properties.

Definition 1.4.2 (Hadamard two-point functions).
Let H_+ be the set of all distributions $\Delta_+ \in \mathcal{D}' \otimes \mathcal{D}'$, such that for all $f, g \in \mathcal{D}$ the following conditions hold:

H1: ***Commutator**: $\Delta_+(f,g) - \Delta_+(g,f) = \frac{1}{i} G_c(f,g)$ with G_c from definition 1.3.3 for all $f, g \in \mathcal{D}$,*

H2: ***Hermiticity**: $\Delta_+(f,g)^* = \Delta_+(g^*, f^*)$,*

H3: ***Microlocal spectrum condition** or **Hadamard** condition:*

$$\begin{aligned} &\mathrm{WF}(\Delta_+) \subset \Lambda_+ = \{(x,y|k_x,k_y) \in \mathrm{WF}(G_c) : k_x^0 > 0\} \\ &= \{(x,y|k_x,k_y) \in \dot{T}^*M^2 : \eta(x-y,x-y) = 0,\ k_x + k_y = 0,\ k_x \parallel (x-y),\ k_x^0 > 0\}, \end{aligned}$$

H4: ***Equation of motion**: $\Delta_+(Pf,g) = 0 = \Delta_+(f,Pg)$ for the Klein-Gordon operator P with mass parameter $m^2 \geq 0$,*

H5: ***Positivity**: $\Delta_+(f^*,f) \geq 0$.*

The set H_+ is called the set of Hadamard two-point functions for the Klein-Gordon operator P on Minkowski spacetime M.

We will define the $\star$-product on the space $\mathcal{F}_{\mu c}[[\hbar]]$ of formal power series in $\hbar$ with coefficients in $\mathcal{F}_{\mu c}$. Note that in the definition of Hadamard two-point functions H_+ we already "pulled out" a $\hbar$ from Δ_+, meaning $\hbar\Delta_+(f,g) = \omega(\varphi_f \varphi_g)$ in the notation of theorem 2. In this way we obtain a natural definition of $\star$ as a product on formal power series.

Definition 1.4.3 ($\star$-product).
Let Δ_+ be a Hadamard two-point function. The following product

$$\begin{aligned} (A \star B)(\phi) &:= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle \Delta_+^{\otimes n} \middle| A^{(n)}(\phi) \otimes B^{(n)}(\phi) \right\rangle \\ &= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \int \Delta_+(x_1,y_1) \cdots \Delta_+(x_n,y_n) A^{(n)}(\phi)[x_1,\ldots,x_n] B^{(n)}(\phi)[y_1,\ldots,y_n] \, \mathrm{d}X \ \mathrm{d}Y \end{aligned} \tag{1.17}$$

is called the $\star$-product with respect to Δ_+ on $\mathcal{F}_{\mu c}[[\hbar]]$.

The product $\star$ is non-commutative since its antisymmetric part in lowest order (given by G_c) is non-vanishing. The associativity of $\star$ can be easily shown by using the commutativity of the functional derivatives on $\mathcal{F}_{\mu c}$. Here the off-shell viewpoint is extremely helpful, since the latter statement is hard to prove for functionals over the solution

space $\mathcal{E}_P$. Due to the Hermiticity property H2 of $\Delta_+ \in H_+$, $(\mathcal{F}_{\mu c}[[\hbar]], \star)$ becomes a *-algebra with the canonical involution, given by complex conjugation.

The Hadamard property H3 of Δ_+ is important for the well-definedness of $\star$. It implies that the result of the following pointwise multiplication of distributions

$$\begin{aligned}(\mathcal{D}' \otimes \mathcal{D}')^{\otimes n} \times \mathcal{E}'(M^n) \otimes \mathcal{E}'(M^n) &\longrightarrow \mathcal{E}'(M^{2n}) \\ (\Delta_+^{\otimes n}, A^{(n)} \otimes B^{(n)}) &\longmapsto \Delta_+^{\otimes n} \cdot A^{(n)}(\phi) B^{(n)}(\phi)\end{aligned}$$

appearing in the integrand of equation (1.17) defines a distribution of compact support. This is ensured by the multiplication theorem of Hörmander, since the condition

$$\{0\} \notin \mathrm{WF}(\Delta_+^{\otimes n}) \oplus \mathrm{WF}(A^{(n)}(\phi) \otimes B^{(n)}(\phi))$$

is fulfilled for every $\phi \in \mathcal{E}$. See [Hör90] for more details on the multiplication theorem and e.g. [Kel10] for a proof of the above statement. The multiplication theorem gives in addition information on the wavefront set of the resulting product, namely

$$\mathrm{WF}\left\langle \Delta_+^{\otimes n} \middle| A^{(n)}(\phi) \otimes B^{(n)}(\phi) \right\rangle^{(m)} \subset \Xi_m \qquad \forall A, B \in \mathcal{F}_{\mu c},\ \phi \in \mathcal{E}$$

Therefore $\star : \mathcal{F}_{\mu c}[[\hbar]] \times \mathcal{F}_{\mu c}[[\hbar]] \to \mathcal{F}_{\mu c}[[\hbar]]$ and the resulting algebra $(\mathcal{F}_{\mu c}[[\hbar]], \star)$ is closed under $\star$.

With the above choice of the set of admissible two-point functions, we conclude the definition of the non-commutative *-algebra, which is the quantum algebra of observables, induced by the selection of a $\Delta_+ \in H_+$.

Definition 1.4.4 (*-algebra of off-shell observables).
For a fixed $\Delta_+ \in H_+$, the algebra of off-shell observables $\mathcal{A}(\mathcal{O})$, associated to a region $\mathcal{O} \subseteq M$ is defined to be

$$\mathcal{A}(\mathcal{O}) := \left(\left\{A \in \mathcal{F}_{\mu c}[[\hbar]] : \mathrm{supp}(F) \subset \mathcal{O}\right\}, \star\right).$$

In the case $\mathcal{O} = M$ we abbreviate $\mathcal{A}(M) = \mathcal{A}$. The canonical involution for functionals $F^*(\phi) = \overline{F(\phi)}$ turns $\mathcal{A}$ into a *-algebra. We identify the following subspaces:

- ***Regular observables**: $\mathcal{A}_{\mathrm{reg}}(\mathcal{O}) = \{A \in \mathcal{A}(\mathcal{O}) : A \in \mathcal{F}_{\mathrm{reg}}[[\hbar]]\}$, which is a *-subalgebra of $\mathcal{A}(\mathcal{O})$*
- ***Local observables**: $\mathcal{A}_{\mathrm{loc}}(\mathcal{O}) = \{A \in \mathcal{A}(\mathcal{O}) : A \in \mathcal{F}_{\mathrm{loc}}[[\hbar]]\}$, which is only a vector subspace of $\mathcal{A}(\mathcal{O})$*
- ***Polynomial observables**: $\mathcal{A}_{\mathrm{pol}}(\mathcal{O}) = \{A \in \mathcal{F}_{\mathrm{pol}} \cap \mathcal{F}_{\mu c} : \mathrm{supp}\, A \subset \mathcal{O}\}$ is also a *-subalgebra of $\mathcal{A}(\mathcal{O})$.*

Remark 1.4.5.
In the definition of the polynomial observables, the functionals were not taken to be formal power series in $\hbar$. This is due to the fact, that for polynomial functionals the sum in the $\star$-product (1.17) has only finitely many terms. Therefore we obtain a $\hbar$-convergent $\star$-closed subalgebra of $\mathcal{A}$ in which we are allowed to set $\hbar = 1$ in (1.17), which is implicitly assumed in $\mathcal{A}_{\mathrm{pol}}$ from here on. ♢

From a mathematical point of view, the huge algebra $\mathcal{A}$ may seem a bit singular for a nice topology. This is indeed true to some extent. The space of microcausal functionals can be endowed with the initial topology, induced by the mappings

$$A \mapsto A^{(n)}(\phi) \in \mathcal{E}'_{\Xi_n}(M^n) = \{t \in \mathcal{E}'(M^n) : \mathrm{WF}(t) \subset \Xi_n\} \qquad \phi \in \mathcal{E}, n \in \mathbb{N}$$

where the images of the maps carry the Hörmander topology. This resulting topology for $\mathcal{F}_{\mu c}$ has, unfortunately, not many good features, see [BFR12] for more details on the topological structure. The reason why we allow for such singular elements in our algebra anyhow, can be simply be put in the following statement: The interesting interactions in QFT are not regular, they are non-linear local functionals $V \in \mathcal{A}_{\mathrm{loc}}$. To see why regular interactions fail to describe local QFTs see section 2.3.

The deformation quantization for the classical off-shell algebra of observables $\mathcal{A}_{\mathrm{cl}}$ from definition 1.3.7 is complete and the quantization map

$$Q : \mathcal{A}_{\mathrm{cl}} \to \mathcal{A}, \qquad Q(A) = A\hbar^0$$

fulfills the appropriate correspondence properties (1.15) due to the commutator condition H1.

In a next step, we verify that the quantization has actually been done in a **state-independent way**, as announced in the beginning. For this we show, that the algebraic structure of $\mathcal{A}$ is independent of the Hadamard two-point function $\Delta_+ \in H_+$. This statement can be formulated in a precise way by using the notion of equivalent products.

Definition 1.4.6 (Equivalent $\star$-products).
Two $\star$-products $\star_1$ and $\star_2$ on $\mathcal{F}_{\mu c}[[\hbar]]$ are called equivalent, if there exists a formal power series of linear maps $U = \mathbb{1} + \sum_{n=1}^{\infty} \hbar^n U_n$ with $U_n : \mathcal{F}_{\mu c} \to \mathcal{F}_{\mu c}$ such that

$$A \star_1 B = U^{-1}\left(U(A) \star_2 U(B)\right), \quad U(A)^* = U(A^*), \quad U(1) = 1 \quad \forall A, B \in \mathcal{A}\,.$$

The map U provides a *-isomorphism between $(\mathcal{F}_{\mu c}[[\hbar]], \star_1)$ and $(\mathcal{F}_{\mu c}[[\hbar]], \star_2)$.

Thus, two equivalent $\star$-products are merely two different ways to describe the same algebra. We find, that all deformation quantizations of the (off-shell) classical algebra of the free theory are equivalent. That implies that the algebraic structure of $\mathcal{A}$ is independent of the particular choice of $\Delta_+ \in H_+$.

Proposition 1.4.7.
Let $\mathcal{A}_i = (\mathcal{F}_{\mu c}[[\hbar]], \star_i)$ be two *-algebras arising by using two Hadamard two-point functions $\Delta_+^i \in H_+$ for $i = 1, 2$. Then $\mathcal{A}_1$ is isomorphic to $\mathcal{A}_2$.

PROOF. We construct the isomorphism explicitly. Let Δ_+^i, $i = 1, 2$ be two Hadamard two-point functions, then

$$\Delta_+^1(x, y) - \Delta_+^2(x, y) = w^1(x, y) - w^2(x, y) = u(x, y), \qquad u \in C^\infty(M \times M),$$

where the w^i are the state-dependent terms in the Hadamard condition. The w^i and u are smooth by condition H3 and due to the Hermiticity condition real-valued. The map

$$U : \mathcal{F}_{\mu c} \to \mathcal{F}_{\mu c}, \qquad U = \mathrm{e}^{\hbar \int u(x,y) \frac{\delta^2}{\delta\phi(x)\delta\phi(y)} \,\mathrm{d}x\,\mathrm{d}y}$$

intertwines $\star_1$ and $\star_2$, i.e.

$$A \star_2 B = U^{-1}(UA \star_1 UB) \qquad \forall A, B \in \mathcal{F}_{\mu c}.$$

and fulfills the conditions from definition 1.4.6. The proof of the last statement is a consequence of the product rule for functional derivatives, see [Kel10] for a proof in terms of a co-product rule. □

The fact that all algebras of observables $\mathcal{A}$ arising from a Hadamard two-point function $\Delta_+ \in H_+$ are isomorphic can be used further to define an algebra, which is manifestly independent of the Hadamard two-point function Δ_+. The construction, which uses specific sections over the bundle of algebras $(\mathcal{A}_{\Delta_+})_{\Delta_+ \in H_+}$ is found in [FR12].

1.4.3. Axioms of Quantum Field Theory

Concluding the construction of the algebra of off-shell observables of the free scalar field we state the Haag-Kastler axioms for a quantum field theory.

Definition 1.4.8 (Haag-Kastler Axioms of QFT).
] The theory of a scalar field is described by a net of unital *-algebras $\{\mathfrak{A}(\mathcal{O}) : \mathcal{O} \subset M\}$ with the following properties:

QFT1: ***Isotony**: For every isometry $\mathfrak{P}_+^\uparrow \ni \mathfrak{p}_1 : \mathcal{O}_1 \to \mathcal{O}_2$ there is a unital *-homomorphism $\alpha_\mathfrak{p} : \mathfrak{A}(\mathcal{O}_1) \to \mathfrak{A}(\mathcal{O}_2)$ and for every further isometry $\mathfrak{P}_+^\uparrow \ni \mathfrak{p}_2 : \mathcal{O}_2 \supset \mathfrak{p}\mathcal{O}_1 \to \mathcal{O}_3$ it holds $\alpha_{\mathfrak{p}_1 \circ \mathfrak{p}_2} = \alpha_{\mathfrak{p}_1} \circ \alpha_{\mathfrak{p}_2}$.*

QFT2: ***Einstein causality**: If $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated, then $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = 0$.*

QFT3: ***Time-slice axiom**: Let $\mathfrak{p} : \mathcal{O}_1 \to \mathcal{O}_2$. If $\mathfrak{p}\mathcal{O}_1$ contains a Cauchy surface for $\mathcal{O}_2$, then $\alpha_\mathfrak{p} : \mathfrak{A}(\mathcal{O}_1) \to \mathfrak{A}(\mathcal{O}_2)$ is surjective. It follows that in particular $\mathfrak{A}(\Sigma_\epsilon) = \mathfrak{A}(M)$ for any ϵ-neighborhood Σ_ϵ of a Cauchy surface Σ of M.*

The net of off-shell algebras $\{\mathcal{A}(\mathcal{O}) : \mathcal{O} \subset M\}$ satisfies the first two axioms of definition 1.4.8, which are the kinematical ones [BF09]. For the time-slice axiom, which is a statement concerning the dynamics, we will have to include dynamics in $\mathcal{A}$.

The preferred method to do this will be the identification of a two-sided *-ideal, similar to the case of classical field theory in definition 1.3.9. Thanks to condition H4 for the Hadamard two-point functions we find that the **classical** ideal $\mathcal{I}_P$ is in fact a two-sided ideal in $\mathcal{A}$. For this let $I \in \mathcal{I}_P$ and $\psi \in \mathcal{E}_P$. The functional derivative $I^{(n)}(\psi)[h^{\otimes n}]$ of I at $\psi \in \mathcal{E}_P$ vanishes if h is a solution to P. This follows from definition 1.2.3. The maps

$$\left\langle \Delta_+^{\otimes n} \middle| A^{(n)}(\psi) \otimes \cdot \right\rangle : \mathcal{E}'_{\Xi_n} \to \mathbb{C}$$

can be seen, by H4, as the limit of a sequences h_n of solutions to P in the Hörmander topology for $\psi \in \mathcal{E}_P$. It follows:

$$(A \star I)(\psi) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle \Delta_+^{\otimes n} \middle| A^{(n)}(\psi) \otimes I^{(n)}(\psi) \right\rangle$$
$$\overset{\text{H4}}{=} A(\psi) \cdot I(\psi) \in \mathcal{I}_P$$

for all $\psi \in \mathcal{E}_P$, i.e. $\mathcal{I}_P$ is a left ideal. The right-ideal property follows by the same argument and so does the *-ideal property. We arrive at:

Definition 1.4.9 (On-shell algebra $\mathcal{A}_{\text{os}}$).
The algebra $\mathcal{A}_{\text{os}}(\mathcal{O})$ of on-shell observables in $\mathcal{O}$ is given by *-algebra of formal power series of on-shell functionals $\mathcal{F}_{\mu c}^P$ in $\hbar$ with the product $\star$: $\mathcal{A}_{\text{os}}(\mathcal{O}) = (\{A \in \mathcal{F}_{\mu c}^P[[\hbar]] : \text{supp}\, A \subset \mathcal{O}\}, \star)$.

The associated net $\{\mathcal{A}_{\text{os}}(\mathcal{O}) : \mathcal{O} \subset M\}$ fulfills all of the Haag-Kastler axioms 1.4.8. A detailed proof of the time-slice axiom is found in [CF09].

This concludes the treatment of the algebra of observables in the free theory of the scalar field. After an introduction to the ideas of formal deformation quantization we defined the notion of Hadamard two-point functions. We showed that all elements $\Delta_+ \in H_+$ can be used to equip $\mathcal{F}_{\mu c}[[\hbar]]$ with non-commutative, associative products which obey certain physically motivated properties. Furthermore all the arising algebras are shown to be equivalent. In the last step we introduced dynamics into the algebra of off-shell observables in a very similar procedure as in classical free theory, namely by going over to the quotient algebra $\mathcal{A}_{\text{os}}$ with respect to the ideal $\mathcal{I}_P$.

1.5. States in Quantum Field Theory

The state space of quantum theories exhibits a distinguishing feature (in comparison to classical theories) besides the non-commutative product on the algebraic side: The pure states of quantum systems are in one-to-one correspondence with normalized vectors in a Hilbert space, which is a linear space. In particular for every normalized vector ψ in the Hilbert space there is a pure state ω_ψ of the quantum system. That implies that e.g. the sum $\hat{\psi} = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$ of **any** two normalized vectors ψ_1, ψ_2 can be identified with a pure state $\omega_{\hat{\psi}}$ again. This feature is called **superposition principle** and implies many interesting effects of quantum theory. Classical theories are shown to lack this attribute.

Due to the state-independence of the algebra of observables $\mathcal{A}$ we are in the same comfortable situation as in classical field theory, as it comes to the discussion of the state space. After the introduction of the concept of states in formal deformation quantization we establish a link between states in axiomatic approaches and the states in formal deformation quantization. We then present a method to obtain states in the our framework, which are close to classical states (coherent states). We close the section with a detailed discussion on the two very important states in relativistic QFT, the quasi-free vacuum and the KMS state.

1.5.1. States in Formal Deformation Quantization

The way of defining states which is discussed here is similar to the one used in classical field theory (section 1.3.3), namely we look at positive linear functionals over the algebra of observables. The above-mentioned distinguishing feature of the superposition principle is not directly visible at this stage, since only the convex sums of such states result again in states: only "classical combinations" of states are present at first sight. We come back to this point later on.

The algebra of observables $\mathcal{A}$ has been constructed as formal power series of functional spaces, i.e. $\mathcal{A} = (\mathcal{F}_{\mu c}[[\hbar]], \star)$. Looking at linear functionals $\omega : \mathcal{A} \to \mathbb{C}$, it was found that such functionals are no suitable candidates for states due to possible convergence issues. In our case it is more natural to look at linear functionals $\omega : \mathcal{A} \to \mathbb{C}[[\hbar]]$. The notion of normalization can be directly extended to those functionals, the condition of positivity for states has to be reconsidered, though.

An appropriate notion for positivity of linear functionals with values in $\mathbb{C}[[\hbar]]$ is motivated and discussed in [BW98, Wal07]. It is shown, that a meaningful definition of positivity can be achieved by demanding that the first non-vanishing coefficient of the formal power series of complex numbers $\omega(A^* \star A)$ shall be real and positive, i.e. there exists a $k \in \mathbb{N}_0$ such that

$$\omega(A^* \star A) = \sum_{n=0}^{\infty} \hbar^n a_n, \qquad a_n = 0 \text{ for } n < k \text{ and } a_k > 0\,. \tag{1.18}$$

For the purposes of this work it is sufficient to work only with states over $\mathcal{A}$ that are induced by states over $\mathcal{A}_{\mathrm{pol}}$.

Definition 1.5.1 (States over $\mathcal{A}_{\mathrm{pol}}$).
A linear functional $\omega : \mathcal{A}_{\mathrm{pol}} \to \mathbb{C}[[\hbar]]$ is called a state, if it is normalized: $\omega(\mathbb{1}) = 1$ and positive $\omega(A^* \star A) \geq 0$. The set of all states over $\mathcal{A}_{\mathrm{pol}}$ is denoted by $\mathfrak{S}(\mathcal{A}_{\mathrm{pol}})$.

A straightforward calculation shows that every state ω over $\mathcal{A}_{\mathrm{pol}}$ induces a normalized linear functional, that fulfills (1.18) by

$$\tilde{\omega} : \mathcal{A} \to \mathbb{C}[[\hbar]], \qquad \tilde{\omega}(A) = \sum_{n=0}^{\infty} \hbar^n \omega(A_n)$$

for $\mathcal{A} \ni A = (A_n)_{n\in\mathbb{N}}$. Hence the set of states $\mathfrak{S}(\mathcal{A})$ over $\mathcal{A}$ is defined as the set of all induced states from $\mathfrak{S}(\mathcal{A}_{\mathrm{pol}})$.

Similar to the issue in classical field theory the state space over $\mathcal{A}$ and over $\mathcal{A}_{\mathrm{os}}$ are not in one-to-one correspondence, see the discussion below definition 1.3.12. Following those ideas, we will restrict the attention to states over $\mathfrak{S}(\mathcal{A})$ which annihilate the ideal $\mathcal{I}_P$. For those states we have a one-to-one correspondence to $\mathfrak{S}(\mathcal{A}_{\mathrm{os}})$.

An important class of states over $\mathcal{A}$ is given by evaluation functionals, which we know from the classical state space. At this point, the positivity condition (H5) of the Hadamard two-point function $\Delta_+ \in H_+$ is crucial. It is actually the only point in the construction, where this property is used.

Proposition 1.5.2.
For a Hadamard two-point function Δ_+ all evaluation functionals ev_ψ with $\psi \in \mathcal{E}$ are states.

PROOF. The linearity and normalization condition is trivial for the evaluation functionals. We look at the formal power series

$$\mathrm{ev}_\psi(A^* \star A) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle \Delta_+^{\otimes n} \middle| A^{(n)}(\psi)^* \otimes A^{(n)}(\psi) \right\rangle .$$

The positivity condition (H5) together with stability of this condition under tensor products yields

$$\left\langle \Delta_+^{\otimes n} \middle| f^* \otimes f \right\rangle \geq 0 \qquad f \in \mathcal{D}(M^n).$$

By a continuity argument one shows, that the non-negativity extends to evaluations over $\mathcal{E}'_{\Xi_n} \otimes \mathcal{E}'_{\Xi_n}$. Hence, every summand is a positive number. This implies both that the evaluation functional is formally positive, as well as positive for convergent subalgebras. □

Unlike in the classical case, evaluation functionals are, in general, not pure states. The question, whether ev_ψ is intimately related to the question, whether the quasi-free state determined by Δ_+ is pure. A discussion on the level of the Weyl algebra is found in section A.2 of the appendix.

The evaluation states can be considered as coherent states of the algebra. The reason is the following: Consider the on-shell algebra $\mathcal{A}_{\mathrm{os}}$ and the evaluation functional ev_ψ with a solution $\psi \in \mathcal{E}$. The state ev_ψ is a quasi-free state for the subalgebra of $\mathcal{A}$, generated by the linear field Φ_f and the expectation value of Φ_x in $\mathcal{A}_{\mathrm{os}}$ at a point $x \in M$, i.e. the one-point function is given by

$$\mathrm{ev}_\psi(\Phi_x) = \psi(x) .$$

Hence, ev_ψ is a quasi-free state with non-vanishing one-point function. This characterizes a coherent state with respect to the quasi-free state Δ_+ defining $\star$ in $\mathcal{A}_{\mathrm{os}}$.

A very important notion related to states is the GNS representation. For this manner let $\mathfrak{A}$ be an abstract *-algebra over $\mathbb{C}$. It is well-known [Haa92] that states can be used to construct a representation of the algebra $\mathfrak{A}$ on a certain Hilbert space.

The GNS construction relies on the fact, that the state $\omega \in \mathfrak{S}(\mathfrak{A})$ induces a sesquilinear form $(A|B)_\omega = \omega(A^*B)$ on $\mathfrak{A}$, seen as a vector space. The degeneracy space of $(\cdot|\cdot)_\omega$ is the Gelfand ideal $\mathfrak{I}_\omega = \{A \in \mathfrak{A} : \omega(A^*A) = 0\}$ of $\mathfrak{A}$. Due to the Cauchy-Schwarz inequality for $(\cdot|\cdot)_\omega$, $\mathfrak{I}_\omega$ is a left-ideal in $\mathfrak{A}$. The quotient

$$\mathcal{H}_\omega = {}^{\mathfrak{A}}\Big/_{\mathfrak{I}_\omega}$$

is a pre-Hilbert space with respect to $(\cdot|\cdot)_\omega$ and its completion yields the Hilbert space $\bar{\mathcal{H}}_\omega$ of the system in the state ω. The GNS representation π_ω is obtained by left multiplication

$$\pi_\omega(A)[B] = [AB], \qquad [B] \in \mathcal{H}_\omega$$

on the dense subspace $\mathcal{H}_\omega$. The cyclic GNS vacuum vector is given by $[\mathbb{1}] = \Omega_\omega$. It holds the following theorem:

Theorem 3 (Uniqueness of the GNS representation [GN43, Seg47]).
Let π be a *-representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$, such that there is a vector $\Omega \in \mathcal{H}$ which is cyclic for $\mathfrak{A}$ and it holds $\omega(A) = (\Omega|\pi(A)\Omega)$. Then π is unitarily equivalent to the GNS representation π_ω.

In physical terms this means that the description of the theory in a fixed Hilbert space can be used to obtain all predictions of the theory (in terms of expectation values of observables) once the state of the system is **fixed**. For the treatment of different states and state independent features of the theory the algebraic approach is superior though. In the case of formal power series of algebras a similar formalism is introduced in [BW98].

At this point we rediscover the linear structure of the space of states. Every algebraic state ω on the algebra of observables $\mathfrak{A}$ induces a Hilbert space $\bar{\mathcal{H}}_\omega$ and every vector $\psi \in \bar{\mathcal{H}}_\omega$ defines an algebraic state over $\mathfrak{A}$ by

$$\omega_\psi(A) = (\psi, \pi_\omega(A)\psi)$$

in return. Hence a superposition of two states, as described in the beginning of this section, can be achieved for the elements of $\bar{\mathcal{H}}_\omega$ on each GNS Hilbert space separately.

1.5.2. Symmetries and States

The symmetries of Minkowski spacetime, namely the elements of the group $\mathfrak{P}^\uparrow_+$, play a decisive role in the determination of the properties of the system. It was shown in section section 1.2.3 that the natural action of $\mathfrak{P}^\uparrow_+$ on the space of functionals induces a *-homomorphism

$$\alpha_\mathfrak{p} : \mathcal{A}(\mathcal{O}) = (\mathcal{F}_{\mu c}(\mathcal{O})[[\hbar]], \star) \longrightarrow (\mathcal{F}_{\mu c}(\mathfrak{p}\mathcal{O})[[\hbar]], \star_\mathfrak{p}) \cong \mathcal{A}(\mathfrak{p}\mathcal{O}) \,.$$

This implies that the algebraic structure of $\mathcal{A}$ is invariant under $\mathfrak{P}^\uparrow_+$. However, the particular realization of $\mathcal{A}$, that specified by the Hadamard two-point function Δ_+ is not invariant. We introduce the notion of an algebra, which is invariant under a symmetry transformation.

Definition 1.5.3 (Invariant algebras).
A $\star$-product given by a Hadamard two-point function Δ_+ is called invariant under $\mathfrak{p} \in \mathfrak{P}^\uparrow_+$, if

$$\alpha_\mathfrak{p}(A \star B) = \alpha_\mathfrak{p}(A) \star \alpha_\mathfrak{p}(B) \qquad \forall A, B \in \mathcal{A} \,.$$

The corresponding algebra $\mathcal{A} = (\mathcal{F}_{\mu c}[[\hbar]], \star)$ is called invariant under $\mathfrak{p} \in \mathfrak{P}^\uparrow_+$, if $\star$ is invariant under $\mathfrak{p}$.

The condition of $\mathfrak{p}$-invariance can be linked to the Hadamard two-point function: The product $\star$ is $\mathfrak{p}$-invariant, iff $\Delta_+(\alpha_\mathfrak{p} f, \alpha_\mathfrak{p} g) = \Delta_+(f, g)$ for all $f, g \in \mathcal{D}(M)$. It follows in particular that $\alpha_\mathfrak{p}$ acts on a $\mathfrak{p}$-invariant quasi-local algebra $\mathcal{A}$ as an automorphism.

These properties trivially extend to algebras, which are invariant under subgroups of $\mathfrak{P}_+^\uparrow$.

On algebras which are invariant under a symmetry $\mathfrak{p}$ we may ask whether there exist states, which are also invariant under $\mathfrak{p}$.

Definition 1.5.4 (Invariant states).
Let $\mathcal{A}$ be invariant under $\mathfrak{p} \in \mathfrak{P}_+^\uparrow$. A state $\omega \in \mathfrak{S}(\mathcal{A})$ is called $\mathfrak{p}$-invariant, if

$$\omega(\alpha_\mathfrak{p}(A)) = \omega(A) \qquad \forall A \in \mathcal{A}.$$

An evaluation functional ev_ψ over $\mathcal{A}$ is invariant under $\mathfrak{p}$, if $\alpha^\mathfrak{p}\psi = \psi$, see definition 1.2.6 for the contravariant action $\alpha^\mathfrak{p}$ of $\mathfrak{P}_+^\uparrow$ on $\mathcal{E}$.

Invariant states on a general *-algebra $\mathfrak{A}$ can be used to construct a representation of the respective symmetry on the GNS Hilbert space. The unitary operator $U_\omega(\mathfrak{p})$, defined by

$$U_\omega(\mathfrak{p})A\Omega_\omega = \pi_\omega(\alpha_\mathfrak{p}(A))\Omega_\omega$$

is easily shown to define a unitary representation of the automorphism $\alpha_\mathfrak{p}$ on $\mathcal{H}_\omega$, which extends to the full Hilbert space.

A very important class of symmetries is the subgroup of spacetime translations. For any translation invariant state we obtain a four parameter group of unitaries $U_\omega(x)$ with $x \in M$. Assuming that the maps

$$x \mapsto F^\omega_{A,B}(x) = \omega(A \star \alpha_x(B))$$

are continuous we can infer that the unitaries $U_\omega(x)$ act strongly continuous on $\bar{\mathcal{H}}_\omega$ and by using the Stone von-Neumann theorem we know, that there is a dense subspace of the GNS Hilbert space, on which the self-adjoint generators

$$P_\omega = (P^0_\omega, \ldots, P^3_\omega), \qquad U_\omega(x) = \mathrm{e}^{i\eta_{\mu\nu}x^\mu P^\mu_\omega}$$

are defined. The P^μ_ω are interpreted as the components of the 4-momentum operator and $\eta_{\mu\nu}P^\mu_\omega P^\nu_\omega$ is the total mass, associated to the systems in the ensemble ω.

In the case of algebras $\mathcal{A}$ obtained by deformation quantization, $\mathcal{A}$ is translation invariant, if Δ_+ is translation invariant. In the search for a translation invariant state we consider the set of evaluation functionals. An evaluation functional ev_ψ is translation invariant, iff

$$\mathrm{ev}_\psi(\alpha_x(A)) = A(\alpha^x\psi) = \mathrm{ev}_\psi(A) = A(\psi), \qquad \forall A \in \mathcal{A}.$$

with the contravariant action α^x from definition 1.2.6. Hence, ψ has to be a constant function. Since ev_ψ must also be compatible with the dynamics, $P\psi = 0$ has to hold at the same time. If $m \neq 0$ the only solution is the evaluation at $\psi = 0$. In the massless case, there is no restriction.

We close the section on the states of the algebra of the scalar field by the two most important states in free relativistic systems. The first state is the famous vacuum state.

Example 1.5.5 (Quasi-free vacuum state).
The existence of a vacuum state in a relativistic quantum field theory is a very important fact and is deeply connected to the stability of the theory. This needs some explanation, which is carried out only to a small extent here. A vacuum state or ground state is defined to be the preparation of the system with lowest possible total energy (the existence of such a state is not guaranteed by the axioms). In addition to that one often requires that the vacuum state is $\mathfrak{P}_+^\uparrow$-invariant to obtain a unique vacuum.

In relativistic QFT "lowest possible energy" is an ambiguous notion, since energy and spatial momentum are mixed by $\mathfrak{P}_+^\uparrow$-transformations, hence are observer dependent. A relativistic criterion is given by the **relativistic spectrum condition**: Let ω be a translation invariant state on $\mathfrak{A}$. Then it is called a vacuum state, if the joint spectrum of P_ω fulfills

$$\operatorname{spec}(P_\omega) \subset J^+(0) \equiv J^+.$$

This is a generalization of the non-relativistic spectrum condition $P_\omega^0 = E_\omega > 0$ and it is implied by the condition

$$\operatorname{supp}(\hat{F}_{A,B}^\omega) \subset J^+ \quad \forall A, B \in \mathfrak{A} \qquad \text{with } F_{A,B}^\omega(x) = \omega(A\alpha_x(B))$$

if the $F_{A,B}^\omega(x)$ exist as tempered distributions over M and where α_x denotes the action of the translation group and $\hat{F}_{A,B}^\omega$ means the Fourier transformation (in the sense of distributions).

The construction of a vacuum state in the case of the algebra $\mathcal{A}$ of the free scalar field is done by cutting off the negative energies in the causal commutator function G_c (1.10) by a step function in momentum space. Let $\Delta_+(x,y) = D_+^{\text{vac}}(x-y)$ on account of translation invariance and

$$D_+^{\text{vac}}(f) = \frac{1}{(2\pi)^3} \int \hat{f}(-p)\theta(p_0)\delta(p^2 - m^2)\,\mathrm{d}p, \qquad f \in \mathcal{D}(M).$$

It is clear, that $D_+^{\text{vac}}(\alpha_\mathfrak{p} f) = D_+^{\text{vac}}(f)$, thus the corresponding $\star$-product is $\mathfrak{P}_+^\uparrow$-invariant. The other properties of definition 1.4.2 are easy to see, e.g. since D_+^{vac} is given by the Fourier transform of a positive measure it follows, that Δ_+ is a distribution of positive type, i.e. $\Delta_+(f^*, f) \geq 0$ for all $f \in \mathcal{D}$.

The vacuum state is defined as the evaluation functional ev_0 in the $\mathfrak{P}_+^\uparrow$-invariant algebra that is defined by D_+^{vac}. To show that this is a vacuum state, consider the expectation value

$$\begin{aligned}\text{ev}_0(A \star \alpha_x(B)) &= \sum_{n=0}^\infty \frac{\hbar^n}{n!} \left\langle ((\mathbb{1} \otimes \alpha_x)\Delta_+)^{\otimes n} \,\middle|\, A^{(n)}(0) \otimes B^{(n)}(0) \right\rangle \\ &= \sum_{n=0}^\infty \frac{\hbar^n}{n!} \int \hat{D}_+^{\text{vac}}(p_1) \cdots \hat{D}_+^{\text{vac}}(p_n) a_n(-p_1, \ldots, -p_n) b_n(p_1, \ldots, p_n) \mathrm{e}^{-i\eta(x, \sum_{k=1}^n p_k)}\,\mathrm{d}P\,,\end{aligned}$$

where the a_n, b_n are the Fourier transforms of the $A^{(n)}(0), B^{(n)}(0)$ respectively. For this particular state, one obtains for the distribution $\hat{F}_{A,B}^{\text{ev}_\psi}(q)$ in n-th order in $\hbar$, up to factors

of 2π

$$\int \hat{D}_+^{\mathrm{vac}}(p_1)\cdots\hat{D}_+^{\mathrm{vac}}(p_n)a_n(-p_1,\ldots-p_n)b_n(p_1,\ldots,p_n)\delta(q-\sum_{k=1}^{n}p_k)\,\mathrm{d}p_1\cdots\mathrm{d}p_n\,.$$

Due to the fact that $\hat{D}_+^{\mathrm{vac}}$ has support in $H_m=\{p\in M:\eta(p,p)=m^2\}$ (which coincides with the forward lightcone in the case $m=0$) one sees, that the support of $\hat{F}_{A,B}^{\mathrm{ev}_\psi}$ is contained in $H_m+H_m+\cdots+H_m\subset J^+$ for $m\geq 0$ at every order in $\hbar$. It follows, that the evaluation functional ev_0 on the algebra $\mathcal{A}$ (with $\star$ given by D_+^{vac}) is a Poincaré invariant vacuum state. The state is induced by ev_0 in the massive case ($m>0$) on the on-shell algebra $\mathcal{A}_{\mathrm{os}}$. In the massless case there are several vacuum states since constant functions are also solutions to $\Box$. The uniqueness of the massive vacuum has been established under very general assumptions in [Seg62]. ♢

A second class of important translation invariant states on $\mathcal{A}$ is given by KMS states with inverse temperatures $\beta=\frac{1}{k_BT}>0$.

Definition 1.5.6 (KMS state with inverse temperature β).
Let α_t be a one-parameter group of automorphisms of $\mathfrak{A}$. A linear functional ω is called a KMS state with inverse temperature β if the function

$$F_{A,B}^{\omega}(t)=\omega(A\alpha_t(B))$$

has an analytic continuation into the strip $S_\beta=\{z\in\mathbb{C}:0<\Im(z)<\beta\}$ and is continuous on the boundary, such that $F_{A,B}^{\omega}(i\beta)=F_{B,A}^{\omega}$ for all $A,B\in\mathfrak{A}$.

KMS states are generalized Gibbs states and exhibit lots of features that justifies to call them thermal equilibrium states. We will use both terms synonymously here. The origin and properties of the KMS states are thoroughly discussed in [Haa92] and will not be repeated here.

Example 1.5.7 (Quasi-free, homogeneous KMS state).
Similar to the vacuum case, we begin the construction of the free KMS state with the choice of a suitable Hadamard two-point function Δ_+. In this case we choose $\Delta_+(x,y)=D_+^\beta(x-y)$, where

$$D_+^\beta(f)=\frac{1}{(2\pi)^3}\int\hat{f}(-p)\frac{\varepsilon(p_0)\delta(p^2-m^2)}{1-\mathrm{e}^{-\beta p^0}}\,\mathrm{d}p,\qquad \beta=\frac{1}{k_BT}>0. \tag{1.19}$$

Clearly D_+^β and thus $\star$ is invariant under translations in M and rotations in $\mathbb{R}^3$ (for a fixed time), but not under Lorentz-boosts. As in the case of the vacuum state, D_+^β induces a Hadamard two-point function.

We consider the evaluation at $\psi=0$, ev_0, and study the function

$$t\mapsto F_{A,B}^{\beta}(t)=\mathrm{ev}_0(A\star\alpha_t(B))$$

for time-translations α_t only. We verify, that $F^{\beta}_{A,B}$ extends analytically to the strip $0 < \Im(t) < \beta$ and

$$\mathrm{ev}_0(A \star \alpha_t(B))\Big|_{t=i\beta} = \mathrm{ev}_0(B \star A) \,,$$

following the calculation in the previous example and replacing the support property of D^{vac}_+ by the KMS property of D^{β}_+: $D^{\beta}_+(f, \alpha_{i\beta} g) = D^{\beta}_+(g, f)$ for $f, g \in \mathcal{D}$. This proves, that ev_0 defines a KMS state over $\mathcal{A}$ with $\star$ given by D^{β}_+. Moreover, the state ev_0 fulfills also the relativistic KMS condition that has been established in [BB94]. The function

$$x \mapsto F^{\beta}_{A,B} = \mathrm{ev}_0(A \star \alpha_x(B))$$

can be shown to have an analytic continuation into the tube-shaped domain

$$\{z \in \mathbb{C}^4 : \Im(z) \in J^+(0) \cap J^-(\beta e_0)\} \,.$$

Again this follows from the fact, that D^{β}_+ has this property.

Furthermore, the quasi-free quantum KMS state, determined by ev_0 on the algebra generated by D^{β}_+, is superior to its classical counterpart, which was studied in example 1.3.15. It exists on all functionals $A \in \mathcal{A}$, thus the quantization has removed the singularities, that appeared in the classical theory. ♢

2. The Theory of the Interacting Scalar Field in the Perturbative Approach

The content of this chapter addresses the mathematical treatment of interactions and the construction of non-linear classical and quantum field theories in a perturbative manner. The use of interacting models is imperative from a physical point of view, since models in free theory are not able to predict well-known observations from high-energy physics like particle creation and annihilation, scattering or thermalization.

This does not imply, that models in free theory are obsolete, quite the contrary! Many aspects of the rich phenomenology of interacting field theories can be very well approximated by using notions from the linearized theory. This led amongst other reasons to the success of interacting quantum field theories in describing the fundamental forces of nature and their interactions with matter and in many other interesting systems.

The interacting scalar field is one of the simplest relativistic field theories in many ways, which contains models that are used e.g. in high-energy physics. In the standard model of particle physics the Higgs field is actually described by a scalar field with quartic self-interaction and couplings to both bosons and fermions of the electroweak sector [Group12]. It induces an additional mass term for the fields it is coupled to, as it remains in a ground state with non-vanishing vacuum expectation value (Higgs mechanism).

The emphasis of the present approach lies on the covariant and state-independent construction of the interacting observables. This is essential for the further discussion concerning the UV-divergences that appear due to very singular nature of pointwise interactions in quantum field theoretical models and the IR divergences that are to be expected from the effects of the finite temperature of the system which was touched on in the introduction.

After introducing the geometric ideas of causal perturbation theory in the setting of classical field theory we proceed to define time-ordered products and the formal S-matrix. These objects are crucial for the construction of the algebra of interacting observables. After showing the state-independence of the construction, we discuss the translation of two particular realizations of the algebra of interacting observables in section 2.3.2. This leads to two important connections: On the one hand the **principle of perturbative agreement**, introduced by Hollands and Wald in [HW05]. On the other hand we draw the connection to the so-called **thermal mass** [LB00] which arises by representing the same interaction term in the algebra of Wick polynomials, induced by the vacuum and the KMS Hadamard two point function.

2.1. Classical Field Theory

To start with a perturbative treatment of pointwise interactions in classical field theory is presented according to the framework of locally covariant field theory [DF03, BFR12]. One of the reasons to present perturbation theory already at the level of classical field theory is that the formalism that is developed here has a very similar form to causal perturbation theory. Thus the geometric nature can be pointed out very well without the usual UV divergences of perturbative QFT.

2.1.1. Configuration Space Picture

Conforming to the ideas in the discussion of the free classical field theory (section 1.3) we start with solving the Euler-Lagrange equations for non-linear actions $\mathcal{S}$. In this work we will only deal with actions which can be written as

$$\mathcal{S}[f] = \mathcal{S}_{\mathcal{L}_0}[f] - V[f], \qquad \mathcal{S}_{\mathcal{L}_0}[f](\phi) = -\frac{1}{2}\int f(x)\phi(x)\left(\Box + m^2\right)\phi(x)\,\mathrm{d}x\,, \tag{2.1}$$

where $\mathcal{S}_{\mathcal{L}_0}$ is the free action, considered in (1.7) and V denotes an interaction term. The possible interactions are restricted as follows:

Definition 2.1.1 (Interactions).
An interaction V a map $V : \mathcal{D} \to \mathcal{F}_{\mathrm{loc}} \cap \mathcal{F}_{\mathrm{pol}}$, satisfying the axioms of definition 1.3.1. In addition we consider only functionals that do not contain derivative couplings.

Any interaction V hence may be written as

$$V[f](\phi) = \int f(x)\mathcal{V}(\Phi_x)\,\mathrm{d}x, \qquad \mathcal{V}(\Phi_x) = \frac{\lambda_{n+1}}{n!}\sum_{n\geq 3}^{N}\Phi_x^n$$

in the notation of (1.3). The $\lambda_n \in \mathbb{R}$ are called coupling constants. Consequently, the full action (2.1) reads $\mathcal{S} = \mathcal{S}_{\mathcal{L}_0} - V = \mathcal{S}_{\mathcal{L}_0 - \mathcal{V}}$.

The construction of local solutions demands solving the non-linear Euler-Lagrange equations

$$0 = \delta_{\mathrm{EL}}(\mathcal{S}_{\mathcal{L}_0-\mathcal{V}})(\phi)(x) = P\phi(x) + \underbrace{\mathcal{V}^{(1)}(\Phi_x(\phi))}_{\equiv \mathcal{V}'(\phi(x))} = P\phi(x) + \sum_{n\geq 2}^{N-1}\frac{\lambda_n}{n!}\phi(x)^n \tag{2.2}$$

in a region $\mathcal{O} \subset M$. We may choose $\mathcal{O}$ to be causally complete: $D(\mathcal{O}) = \mathcal{O}$, where D is the domain of dependence, see definition 1.1.2. This makes $\mathcal{O}$ a (flat) spacetime in its own right and there always exists a Cauchy surface Σ for M, such that $\mathcal{O} \cap \Sigma = \Sigma_{\mathcal{O}}$ is a Cauchy surface for $\mathcal{O}$. The geometrical situation is depicted in figure 2, where a double cone are used as region $\mathcal{O}$. We assume, that (2.2) is well-posed, meaning that the Cauchy problem

$$\delta_{\mathrm{EL}}\mathcal{S}_{\mathcal{L}_0-\mathcal{V}}(\phi) = 0 \quad \text{on } \mathcal{O}, \qquad \phi_{|\Sigma} = u_0, \qquad \partial_{x^0}\phi_{|\Sigma} = u_1 \tag{2.3}$$

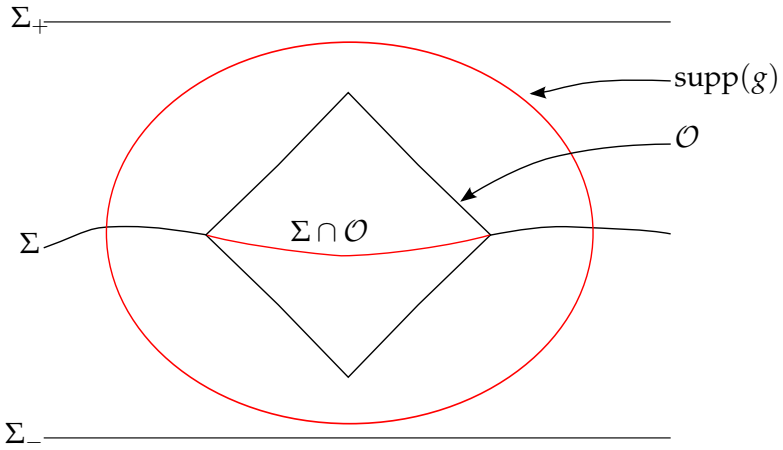


Figure 2.: The geometrical situation in the scattering setting.

has a unique solution ϕ with

$$\operatorname{supp}(\phi) \subset J(K), \qquad K = \operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \Sigma_{\mathcal{O}} \, .$$

Here $u_0, u_1 \in \mathcal{D}(\Sigma_{\mathcal{O}})$, where $\Sigma_{\mathcal{O}}$ a Cauchy surface for $\mathcal{O}$. For sufficiently small Cauchy data (where small is meant with respect to certain Sobolev norms) the Cauchy problem is, in fact, well-posed, see [Hör90].

The quest of constructing solutions to (2.2) by means of methods from scattering theory takes a slightly different approach. To make things precise let us cut off the interaction $\mathcal{V}$ in M with a test function $g \in \mathcal{D}$, such that $g = 1$ on $\mathcal{O}$ and $\operatorname{supp}(g)$ is located between two Cauchy surfaces $\Sigma_\pm$, one in the future and one in the past of $\operatorname{supp}(g)$, see figure 2. Let us denote the modified interaction by

$$\mathcal{V}_g = \int \mathcal{V}_g(\Phi_x)\, \mathrm{d}x, \qquad \mathcal{V}_g(\Phi_x) = g(x)\mathcal{V}(\Phi_x) = \begin{cases} \mathcal{V}(\Phi_x) & x \in \mathcal{O} \\ 0 & x \text{ later than } \Sigma_+ \\ 0 & \Sigma_- \text{ later than } x \end{cases} .$$

Denote the space of smooth solutions to (2.2) with interaction $\mathcal{V}_g$ and compact Cauchy data in $\Sigma_{\mathcal{O}}$ by $\mathcal{E}_{\mathcal{V}_g}$. Then for every $\psi \in \mathcal{E}_{\mathcal{V}_g}$ there exist unique solutions to the **free** EOMs $P\phi_{\text{in/out}} = 0$ such that

$$\phi_{\text{in}},\, \phi_{\text{out}} \in \mathcal{E}_P^{\text{sc}} : \qquad \psi(x) = \begin{cases} \phi_{\text{out}}(x) & x \text{ later than } \Sigma_+ \\ \phi_{\text{in}}(x) & \Sigma_- \text{ later than } x \end{cases}$$

with the same Cauchy data as ψ. Due to the well-posedness of the free and interacting Cauchy problem we know that the maps

$$r_{\mathcal{V}_g} : \mathcal{E}_P^{\text{sc}} \to \mathcal{E}_{\mathcal{V}_g} \, , \quad \phi_{\text{in}} \mapsto \psi, \qquad a_{\mathcal{V}_g} : \mathcal{E}_P^{\text{sc}} \to \mathcal{E}_{\mathcal{V}_g}, \quad \phi_{\text{out}} \mapsto \psi$$

are bijections of the respective solution spaces.

For intuitive reasons we focus the discussion in the following to the retarded Møller map $r_{\mathcal{V}_g}$. Its dual map determines a bijection of functional spaces:

$$\mathcal{R}_{\mathcal{V}_g} : \mathcal{F}(\mathcal{E}_{\mathcal{V}_g}) \to \mathcal{F}(\mathcal{E}_P^{\mathrm{sc}}), \qquad \mathcal{R}_{\mathcal{V}_g} A = A \circ r_{\mathcal{V}_g} \tag{2.4}$$

with the retardation property: $\mathcal{R}_{\mathcal{V}_g} A = A$, if supp g is later than supp A. We calculate $\mathcal{R}_{\mathcal{V}_g}$ in a Taylor expansion around $g = 0$:

$$\mathcal{R}_{\mathcal{V}_g} A = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{R}^n(g^{\otimes n}|A), \qquad \mathcal{R}^n(g^{\otimes n}|A) = \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big|_{\lambda=0} A \circ r_{\lambda \mathcal{V}_g}$$

It was shown in [DF00] that the $\mathcal{R}^n$ are given by

$$\mathcal{R}^n(g^{\otimes n}|A) = n! \int \rho_{\mathrm{r}}(x_1) \cdots \rho_{\mathrm{r}}(x_n) A \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \tag{2.5}$$

where

$$\rho_{\mathrm{r}}(y) = \int \frac{\delta \mathcal{S}_{\mathcal{V}_g}}{\delta \phi(x)} G_{\mathrm{r}}(x,y) \frac{\delta}{\delta \phi(y)} \, \mathrm{d}x = \int g(x) \mathcal{V}'(\phi(x)) G_{\mathrm{r}}(x,y) \frac{\delta}{\delta \phi(y)} \, \mathrm{d}x.$$

Notice, that this formula makes sense only, if the respective derivatives of the involved functionals are interpreted as restrictions from functional derivatives of functionals over $\mathcal{E}$.

In this way we obtain the retarded Møller operator $\mathcal{R}_{\mathcal{V}_g}$ as a **formal power series** in the interaction $\mathcal{V}$, or in g respectively. The question, whether the sum in (2.4) converges to a functional or whether $\mathcal{R}_{\mathcal{V}_g}$ can be constructed by other than perturbative means is discussed in a series of papers, starting with [BFR12] and we will not discuss the question in this work.

2.1.2. Functional Approach

As we pointed out in the last section, the functional derivatives that were used to defined the Møller map $\mathcal{R}_{\mathcal{V}_g}$ must be seen as restrictions of functional derivatives on $\mathcal{E}$. Hence it is quite natural to extend this formalism to the off-shell picture, which has been established in section 1.3.2. It is therefore essential to define a Poisson bracket for the commutative algebra $(\mathcal{F}_{\mu \mathrm{c}}, \cdot)$, which comes from the full action $\mathcal{S}$ and coincides with $\{\cdot,\cdot\}_{G_{\mathrm{c}}}$ from definition 1.3.6 in the case of $\mathcal{V} = 0$.

Definition 2.1.2 (Linearized Euler-Lagrange operator).
The linearized Euler-Lagrange operator δ^2_{EL} for an action $\mathcal{S}$ around a background configuration $\psi \in \mathcal{E}$ is defined by

$$\delta^2_{\mathrm{EL}} \mathcal{S}(\psi) \in \mathcal{D}'(\mathcal{O}) \otimes \mathcal{D}'(\mathcal{O}) : \qquad \big\langle \delta^2_{\mathrm{EL}} \mathcal{S}(\psi) \big| h_1 \otimes h_2 \big\rangle = \frac{\delta^2 \mathcal{S}[f](\psi)}{\delta \phi^2}\Big|_{f=1} [h_1 \otimes h_2]$$

where $f = 1$ on $\mathcal{O}$.

Since $\mathcal{S}[f]$ is local, the distribution $\delta^2_{\mathrm{EL}}\mathcal{S}(\psi)$ has support only on the diagonal and can therefore be identified with a differential operator $P_{\mathrm{lin}}(\psi) : \mathcal{E} \to \mathcal{D}'$.

For the action we are interested in, i.e. $\mathcal{S}_{\mathcal{L}_0-\mathcal{V}_g}$ we obtain

$$\delta^2_{\mathrm{EL}}\mathcal{S}_{\mathcal{L}_0-\mathcal{V}_g}(\psi)[h_1 \otimes h_2] = -\int h_1(x) \underbrace{\left(P + g(x)\mathcal{V}''(\psi(x))\right)}_{P_{\mathrm{lin}}(\psi)} h_2(x)\,\mathrm{d}x \tag{2.6}$$

The linearized Euler-Lagrange operator $P_{\mathrm{lin}}(\psi) : \mathcal{E} \to \mathcal{E}$ is normal hyperbolic for all $\psi \in \mathcal{E}$, hence we can use the following theorem that guarantees the existence and uniqueness of advanced and retarded fundamental solutions to $P_{\mathrm{lin}}(\psi)$.

Theorem 4 ([BF09]).
Let $P_{\mathrm{lin}}(\psi)$ be the normal hyperbolic linear partial differential operator from (2.6)*. Then there exist advanced and retarded fundamental solutions to $P_{\mathrm{lin}}(\psi)$ for ψ and their difference*

$$G_{\mathrm{c},\mathcal{V}_g}(\psi) : \mathcal{D} \to \mathcal{E}$$

satisfies the same properties as G_{c} in definition 1.3.3 and below. $G_{\mathrm{c},\mathcal{V}_g}(\psi)$ is called the causal commutator function for the action $\mathcal{S}$ and the background $\psi \in \mathcal{E}$.

As in the case of free theory, we define the Poisson structure on the algebra of off-shell observables by $G_{\mathrm{c},\mathcal{V}_g}$. The difference in this case is, that the Poisson structure is non-constant due to the fact, that $G_{\mathrm{c},\mathcal{V}_g}$ depends on the background configuration.

Definition 2.1.3 (Peierls bracket).
Let $G_{\mathrm{c},\mathcal{V}_g}$ be the causal commutator function with respect to the action $\mathcal{S}_{\mathcal{L}_0-\mathcal{V}_g}$. The map given by

$$\{\cdot,\cdot\}_{G_{\mathrm{c},\mathcal{V}_g}} : \mathcal{F}_{\mu\mathrm{c}} \times \mathcal{F}_{\mu\mathrm{c}} \to \mathcal{F}_{\mu\mathrm{c}}, \qquad \{A,B\}_{G_{\mathrm{c},\mathcal{V}_g}}(\phi) = \left\langle iG_{\mathrm{c},\mathcal{V}_g}(\phi) \middle| A^{(1)}(\phi) \otimes B^{(1)}(\phi) \right\rangle$$

*is the **Peierls bracket** for the algebra of classical observables $(\mathcal{F}_{\mu\mathrm{c}}, \cdot)$ governed by the action $\mathcal{S}_{\mathcal{L}_0-\mathcal{V}_g}$.*

The bracket $\{\cdot,\cdot\}_{G_{\mathrm{c},\mathcal{V}_g}}$ endows the commutative algebra $(\mathcal{F}_{\mu\mathrm{c}}, \cdot)$ with a Poisson structure, as in the case of free theory. The well-definedness of the bi-linear map $\{\cdot,\cdot\}_{G_{\mathrm{c},\mathcal{V}_g}}$ can be established due to the fact, that $\mathrm{WF}\, G_{\mathrm{c},\mathcal{V}_g}(\phi) = \mathrm{WF}\, G_{\mathrm{c}}$ for all $\phi \in \mathcal{E}$, since both P and P_{lin} have the same principal symbol, see [BF09, Hör90]. Then the arguments for all algebraic properties of the Peierls bracket go along the same lines as for $\{\cdot,\cdot\}_{G_{\mathrm{c}}}$ besides the Jacobi identity, for which a proof can be found in [DF03]. Keeping in mind, that due to the cutoff procedure for $\mathcal{V}$, we have to restrict the support of the observables of the interacting theory to the region $\mathcal{O} \subset M$ on which g was chosen to be constant. We arrive at:

Definition 2.1.4 (Poisson *-algebra of off-shell observables).
*The triplet $\mathcal{A}^{\mathcal{V}_g}_{\mathrm{cl}}(\mathcal{O}) = \left(\mathcal{F}_{\mu\mathrm{c}}(\mathcal{O}), \cdot, \{\cdot,\cdot\}_{G_{\mathrm{c},\mathcal{V}_g}}\right)$ forms a unital Poisson *-algebra for $\mathcal{D} \ni g = 1$ on $\mathcal{O}$. It is called the **local Poisson *-algebra of off-shell observables** in the region $\mathcal{O}$ with respect to the action $\mathcal{S}_{\mathcal{L}_0-\mathcal{V}_g}$.*

Moreover $\{\mathcal{A}_{\mathrm{cl}}^{\mathcal{V}_g}(\mathcal{O}) : \mathcal{O} \subset M\}$ *forms a net of local Poisson* **-algebras, if the* $\mathcal{O}$ *are causally complete. It satisfies the axioms Isotony (CFT1) and Einstein causality (CFT2) from definition 1.3.5.*

The proof of the statements is to a large extent the same as in the case of the quadratic action in section 1.3.2 and can be found in [BF09].

One of the main features of the off-shell formalism of classical field theory and the use of Peierls bracket becomes apparent, if one changes the action $\mathcal{S}$. Though the general formalism can be used to relate any kind of interactions (also non-local ones), we will focus on the relation between $\mathcal{S}_{\mathcal{L}_0}$ and $\mathcal{S}_{\mathcal{L}_0-\mathcal{V}_g}$ here.

Theorem 5 ([DF03]).
Let $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *be two actions and* $\{\cdot,\cdot\}_{1,2}$ *their associated Peierls brackets. Then the retarded Møller map* $r_{12} : \mathcal{E}_{\mathcal{S}_2} \to \mathcal{E}_{\mathcal{S}_1}$*, which is constructed in the same way as above, induces a Poisson* **-isomorphism via* (2.4)*:*

$$\{\mathcal{R}_{12}(A), \mathcal{R}_{12}(B)\}_2 = \mathcal{R}_{12}\left(\{A, B\}_1\right), \qquad \mathcal{R}_{12}(A) = A \circ r_{12}$$

for all A, B *in the algebra* $\mathcal{A}_{\mathrm{cl}}^{\mathcal{S}_1}$*. Speaking in terms of Hamiltonian mechanics,* $\mathcal{R}_{12}$ *is a canonical transformation.*

Notice, that the theorem is proven by purely algebraic properties of the map $\mathcal{R}_{\mathcal{V}_g}$ and does not rely on the expansion as a formal power series.

At this point the off-shell nature of the formulation of the theory is utterly important: Without it such a straightforward construction of the Peierls bracket and the Møller map could not be done. In this sense, changing the dynamics of the system is done best in a formalism, which is constructed independent of the concrete dynamics ($\mathcal{A}_{\mathrm{cl}}^{\mathcal{V}_g}$ depends only on the linearized Euler-Lagrange equations) and allows for implementing the dynamics in a secondary step.

The last theorem is applied for the present situation now: For the quadratic action $\mathcal{S}_{\mathcal{L}_0}$, the linearized Euler-Lagrange operator P_{lin} coincides with the Klein-Gordon operator P, thus the Peierls bracket reduces to the previous Poisson bracket, defined in section 1.3.2. The above theorem allows now to represent the (local) algebra of off-shell observables $\mathcal{A}_{\mathrm{cl}}^{\mathcal{V}_g}$, induced by the full action $\mathcal{S}_{\mathcal{L}_0-\mathcal{V}_g}$ explicitly by objects in the free theory $\mathcal{A}_{\mathrm{cl}}$ (in a formal power series in g!). This approach is very close to the interaction picture of quantum mechanics, in which unitary operators can be used to represent interacting observables by observables of the free theory.

Since $\mathcal{R}_{\mathcal{V}_g} : \mathcal{A}_{\mathrm{cl}}^{\mathcal{V}_g} \to \mathcal{A}_{\mathrm{cl}}$ is a canonical transformation, the physical interpretation of the observable A is stable under the mapping $\mathcal{R}_{\mathcal{V}_g}$. For example, the smeared interacting field Φ_f is represented in $\mathcal{A}_{\mathrm{cl}}$ by

$$\begin{aligned}\mathcal{R}_{\mathcal{V}_g}(\Phi_f)(\phi) &= \sum_{n\geq 0} \int \rho_{\mathrm{r}}(x_1)\cdots\rho_{\mathrm{r}}(x_n)\Phi_f(\phi)\,\mathrm{d}x_1\cdots\mathrm{d}x_n \\ &= \Phi_f(\phi) + \int g(x)\mathcal{V}'(\phi(x))G_{\mathrm{r}}(x,y)f(y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

which is nothing but an off-shell Yang-Feldmann approach to interacting fields, see e.g. [YF50] or [DF04] for an overview.

In addition to the above properties we want to mention that the construction of the algebra of interacting polynomials $\mathcal{A}_{\mathrm{cl}}^{\mathcal{V}_g} \cap \mathcal{F}_{\mathrm{pol}}$ can be done, using the so-called factorization property [DF03]. That is

$$\mathcal{A}_{\mathrm{cl}}^{\mathcal{V}_g}(\mathcal{O}) = \bigvee \left\{ \mathcal{R}_{\mathcal{V}_g}(A) : A \in \mathcal{F}_{\mathrm{pol}},\ \mathrm{supp}\, A \subset \mathcal{O} \right\} \tag{2.7}$$

where $\vee$ means the generated algebra. In this way the properties of the algebra of interacting polynomial observables can be studied by looking at the generators $\mathcal{R}_{\mathcal{V}_g}(A)$ only. This will be of importance in the quantum theory where the analogue of the Møller map $\mathcal{R}_{\mathcal{V}_g}$ fails to be invertible in general. In this case, the RHS of (2.7) defines the interacting theory.

In the end we mention that ideals $\mathcal{I}_P$ and

$$\mathcal{I}_{\mathcal{V}_g} = \{ A \in \mathcal{A}_{\mathrm{cl}}^{\mathcal{V}_g} : A(\phi) = 0 \quad \text{if } \phi \in \mathcal{E}_{\mathcal{V}_g} \}$$

are intertwined by $\mathcal{R}_{\mathcal{V}_g}$ almost by construction. A complete proof of this fact is out of the scope of this work, it is announced to be published in [BFLR]. Using this fact it is obvious that the on-shell algebra of the interacting scalar field

$$\mathcal{A}_{\mathrm{cl}}^{\mathcal{V}_g} \Big/ \mathcal{I}_{\mathcal{V}_g} = \mathcal{R}_{\mathcal{V}_g}^{-1} \left(\mathcal{A}_{\mathrm{cl}}^{P} \right)$$

is isomorphic to $\mathcal{A}_{\mathrm{cl}}^{\mathcal{V}_g}$ and that the representation induced by $\mathcal{R}_{\mathcal{V}_g}$ is compatible with the ideal of the equations of motion. The above quotient algebra fulfills then the time-slice axiom CFT3 from definition 1.3.5.

2.2. Quantum Field Theory and Renormalization

This section will be concerned with the translation of the methods which were used to construct the interacting classical field theory to quantum field theory. The procedure that is being used in this work is causal perturbation theory. The general idea in this approach, as the name suggests, is to define the perturbed theory in such a way, that Einstein causality (definition 1.4.8) of the interacting theory is preserved in the whole process.

The ansatz is to derive a quantized analogue $\mathcal{R}_{\mathcal{V}_g}$ of the Møller map $\mathcal{R}_{\mathcal{V}_g}$ in order to obtain a representation of the interacting quantum theory by objects from the free quantum theory, in the sense of formal power series in the interaction. However it turns out to be quite evolved to identify the domain of this map, we will restrict the attention mostly to algebras, which are generated by a certain set of image points of the map.

We stress at this point, that there are two different notions of formal power series in the forthcoming sections, which are independent of each other. The first one is the

identification of quantum observables as formal power series in $\hbar$, coming from the viewpoint of deformation quantization, see section 1.4. The use of formal power series in $\hbar$ could be avoided by restricting the attention to polynomial observables $\mathcal{A}_{\mathrm{pol}}$ (see definition 1.4.4), where all observables are given by **convergent** power series in $\hbar$. In those cases we are allowed to set $\hbar \approx 6.626 \times 10^{-34}$ Js or to choose a system of units in which $\hbar = 1$.

The **other** notion of formal power series comes into the formalism by the definition of the interacting observables by the Taylor expansion of $\mathcal{R}_{V_g}$ (and its quantum counterpart) around $g = 0$. In the realm of interacting quantum theories with local interactions, the use of formal power series can unfortunately not be avoided in the construction, yet.

The central object in the construction of interacting quantum theory will be the **time-ordered product** for functionals. The time-ordered product eases the perturbative expansion of the retarded Møller map even at the level of classical theory (see (2.5)) and is essential in our approach to perturbative QFT.

2.2.1. Time-Ordered Product on Regular Functionals

In this part the goal is to define a time-ordered product in the free quantum theory, described in the functional off-shell approach of section 1.4. The main idea is to construct this product in similar form to the $\star$-product, i.e. as a functional differential operator on the space of off-shell functionals. In order to omit the difficulties in the definition of such a product – at least in the beginning – we restrict ourselves in this section to regular functionals $A \in \mathcal{A}_{\mathrm{reg}}$ first. The extension to more singular functionals will be dealt with in the next section. The defining property of a time-ordered product $\cdot_{\mathcal{T}}$ is that for any two functionals $A, B \in \mathcal{A}_{\mathrm{reg}}$ it holds

$$A \cdot_{\mathcal{T}} B = \begin{cases} A \star B & \text{if } \mathrm{supp}(A) \text{ later than } \mathrm{supp}(B) \\ B \star B & \text{if } \mathrm{supp}(B) \text{ later than } \mathrm{supp}(A) \end{cases}.$$

It is obvious that the defining property of $\cdot_{\mathcal{T}}$ is intimately tied to the choice of $\Delta_+ \in H_+$, which yields $\star$ in definition 1.4.3. Due to the structure of $\star$, we can derive $\cdot_{\mathcal{T}}$ by evaluating the defining property in lowest orders in $\hbar$, or equivalently by evaluating them at linear functionals:

$$\Phi_f \cdot_{\mathcal{T}} \Phi_g = \Phi_f \cdot \Phi_g + \begin{cases} \Delta_+(f,g) & \mathrm{supp}(f) \text{ later than } \mathrm{supp}(g) \\ \Delta_+(g,f) & \mathrm{supp}(g) \text{ later than } \mathrm{supp}(f) \end{cases}.$$

This relation can be solved by introducing the **Feynman propagator** Δ_F for Δ_+ by

$$\Delta_F(f,g) = \int \left(\theta(x^0 - y^0)\Delta_+(x,y) + \theta(y^0 - x^0)\Delta_+(y,x)\right) f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{2.8}$$

where θ is the Heaviside step function. The pointwise product of the distributions θ and Δ_+ is well-defined for all $\Delta_+ \in H_+$ by Hörmanders theorem [Hör90], since the

wavefront sets

$$\mathrm{WF}(\Delta_+) = \{(x,y|k_x,k_y) \in \dot{T}^*M^2 : x \in V(y),\ k_x + k_y = 0,\ k_x \parallel (x-y),\ k_x^0 > 0\}$$
$$\mathrm{WF}(\theta) = \{(x,y|k_x,k_y) \in \dot{T}^*M^2 : x^0 - y^0 = 0,\ k_x + k_y = 0,\ k_x \parallel e_0\}, \quad e_0 = (1,0,0,0)$$

cannot add up to the zero section: All covectors (in the x-component) of $\mathrm{WF}(\Delta_+)$ are on the boundary of the forward lightcone, where the covectors of $\mathrm{WF}(\theta)$ (where θ is seen as a bi-distribution) are parallel to the e_0-axis, so that they can never add up to zero for non-zero covectors. The wavefront set of the Feynman-propagator is, however, worse than the one of Δ_+ due to the multiplication with θ, namely

$$\begin{aligned}\mathrm{WF}(\Delta_F) = \{(x,y|k_x,k_y) \in \dot{T}^*M^2 :\ & \eta(x-y,x-y) = 0,\ k_x + k_y = 0,\ k_x \parallel (x-y),\\ & k_x \in V^{\pm} \text{ for } (x-y) \in V^{\pm} \text{ and } k_x \in \dot{T}^*_x M \text{ for } x = y\}\end{aligned} \tag{2.9}$$

The fact, that the $\mathrm{WF}(\Delta_F)$ is not one-sided like $\mathrm{WF}(\Delta_+)$ creates the obstruction in defining time-ordered product for more singular functionals, since the pointwise multiplication of Δ_F with itself can not be uniquely defined as a distribution in $\mathcal{D}'(M^2)$. This problem is avoided in the case of regular functionals.

Definition 2.2.1 (Time-ordered product).
The time-ordered product $\cdot_{\mathcal{T}}$ for the algebra of regular functionals $\mathcal{A}_{\mathrm{reg}}$ with $\star$-product defined by $\Delta_+ \in H_+$ is the bi-linear map

$$\cdot_{\mathcal{T}} : \mathcal{A}_{\mathrm{reg}} \times \mathcal{A}_{\mathrm{reg}} \to \mathcal{A}_{\mathrm{reg}}, \qquad (A \cdot_{\mathcal{T}} B) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle \Delta_F^{\otimes n} \middle| A^{(n)}(\phi) \otimes B^{(n)}(\phi) \right\rangle \tag{2.10}$$

Due to the fact that $\Delta_F(f,g) = \Delta_F(g,f)$, $\cdot_{\mathcal{T}}$ is a symmetric product.

In the case of symmetric products like $\cdot_{\mathcal{T}}$ is, there exists a so-called generating functional for the products. A generating functional is a map $\mathcal{S}_0 : \mathcal{A}_{\mathrm{reg}} \to \mathcal{A}_{\mathrm{reg}}$, such that

$$A_1 \cdot_{\mathcal{T}} \cdots \cdot_{\mathcal{T}} A_n = \frac{1}{i^n} \frac{\mathrm{d}^n}{\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_n}\Big|_{\lambda_1 = \ldots = \lambda_n = 0} \mathcal{S}_0\left(i \sum_{k=1}^{n} \lambda_k A_k\right), \qquad \forall n \in \mathbb{N}$$

In the case of $\cdot_{\mathcal{T}}$ it is given by the time-ordered exponential

$$\mathcal{S}_0(A) = \exp_{\cdot_{\mathcal{T}}}(iA) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \underbrace{A \cdot_{\mathcal{T}} \cdots \cdot_{\mathcal{T}} A}_{n \text{ times}}. \tag{2.11}$$

which has to be read in the sense of formal power series. A convenient way to deal with n-fold time-ordered products which will turn up quite often in the following, is to view them as linear maps

$${}^0\mathcal{T}_n : \mathcal{A}_{\mathrm{reg}}^{\otimes n} \to \mathcal{A}_{\mathrm{reg}}, \qquad {}^0\mathcal{T}_n(A_1 \otimes \cdots \otimes A_n) = A_1 \cdot_{\mathcal{T}} \cdots \cdot_{\mathcal{T}} A_n\,. \tag{2.12}$$

In this way, the generating functional for $\cdot_{\mathcal{T}}$ can be written in terms of ${}^0\mathcal{T}$ by

$$\mathcal{S}_0(A) = \sum_{n=0}^{\infty} \frac{i^n}{n!} {}^0\mathcal{T}_n(A^{\otimes n})$$

and due to the fact, that $\mathcal{S}_0$ can only be defined in the sense of formal power series in its argument, the definition of $\mathcal{S}_0$ is equivalent to the specification of the sequence of maps $\{{}^0\mathcal{T}_n : n \in \mathbb{N}\}$ in (2.12).

2.2.2. Time-Ordered Product on Local Functionals, Renormalization

One way to define time-ordered products of physical interactions, i.e. local functionals in $\mathcal{A}_{\text{loc}}$, is to apply a renormalization to $\cdot_{\mathcal{T}}$. To see what goes wrong, if we tried to naively apply the definition 2.2.1 to a non-linear local functionals, consider e.g. the quadratic monomial Φ_f^2 (see (1.3) in example 1.2.2):

$$\Phi_f^2 \cdot_{\mathcal{T}} \Phi_g^2 = \Phi_f^2 \cdot \Phi_g^2 + 4\hbar \left\langle \Phi_f \cdot \Phi_g \cdot \Delta_F \mathbb{1} \middle| f \otimes g \right\rangle + 2\hbar^2 \int \Delta_F(x,y)^2 f(x) g(y) \, \mathrm{d}x \, \mathrm{d}y$$

The last term is ill-defined as a distribution due to the bad wavefront set properties of Δ_F.

In order to formulate the renormalization problem, one defines the time-ordered product for local functionals with **disjoint** support. For example, let $x \neq y$, then we can define

$$\Phi_x^2 \cdot_{\mathcal{T}} \Phi_y^2 = \begin{cases} \Phi_x^2 \cdot \Phi_y^2 + 4\hbar \Phi_x \cdot \Phi_y \Delta_+(x,y) + 2\hbar^2 \Delta_+(x,y)^2 & x \notin J^-(y) \\ \Phi_x^2 \cdot \Phi_y^2 + 4\hbar \Phi_x \cdot \Phi_y \Delta_+(y,x) + 2\hbar^2 \Delta_+(y,x)^2 & y \notin J^-(x) \end{cases}$$

which has a straightforward extension to the case of smeared fields with test-functions with disjoint support. Notice, that for spacelike x, y the upper and lower line agree, since G_{c} vanishes for spacelike separated points, leaving only the symmetric part of Δ_+. It remains to find an extension of the above functional to the four dimensional submanifold $x = y$ of M^2, the total diagonal. It turns out that, that this extension is ambiguous, i.e. there exists a freedom in renormalizing $\cdot_{\mathcal{T}}$.

To start the quest to defining time-ordered products for all local functionals, we summarize the above argument in terms of the maps ${}^0\mathcal{T}_n$ from (2.12): For local functionals $A_1, \ldots, A_n \in \mathcal{A}_{\text{loc}}$ with pairwise disjoint support the map

$${}^0\mathcal{T}_n : \mathcal{A}_{\text{loc}}^{\otimes n} \to \mathcal{A}, \qquad {}^0\mathcal{T}_n(A_1 \otimes \cdots \otimes A_n) = A_1 \cdot_{\mathcal{T}} \ldots \cdot_{\mathcal{T}} A_n$$

is well-defined for every $n \in \mathbb{N}$. This is easily shown, since we can find a permutation π of $\{1, \ldots, n\}$, such that

$$\operatorname{supp}(A_{\pi(1)}) \gtrsim \operatorname{supp}(A_{\pi(2)}) \gtrsim \ldots \gtrsim \operatorname{supp}(A_{\pi(n)})$$

With this permutation we define

$${}^0\mathcal{T}_n(A_1 \otimes \cdots \otimes A_n) = A_{\pi(1)} \star \cdots \star A_{\pi(n)}$$

which has the desired properties, see [BF00] for more details. The most important property is the causal factorization: If the supports of $A_{k+1}, \ldots, A_n$ are later than the supports of $A_1, \ldots, A_k$, then

$${}^0\mathcal{T}_n(A_1 \otimes \cdots \otimes A_n) = {}^0\mathcal{T}_k(A_1 \otimes \cdots \otimes A_k) \star {}^0\mathcal{T}_{n-k}(A_{k+1} \otimes \cdots \otimes A_n). \tag{2.13}$$

This property can be used in many further steps and is the basic ingredient of causal perturbation theory, which has been first exploited by the authors Epstein and Glaser in [EG73].

The next step is to formulate the goal of the procedure, namely to give conditions that the time-ordered products should obey. Possible sets of axioms have been discussed in many works [HW01, HW02, HW03, DF00, BDF09] in slightly different settings. We use the following:

Definition 2.2.2 (Axioms of time-ordered products).
Let $\{\mathcal{T}_n : n \in \mathbb{N}\}$ be a sequence of linear maps $\mathcal{T} : \mathcal{A}_{\mathrm{loc}}^{\otimes n} \to \mathcal{A}$. The $\mathcal{T}_n$ are called time-ordered products, if they satisfy

T1: ***Initial conditions**: $\mathcal{T}_0 = 0$ and $\mathcal{T}_1(A) = A$*

T2: ***Symmetry**: $\mathcal{T}_n(A_1 \otimes \ldots \otimes A_n) = \mathcal{T}_n(A_{\pi(1)} \otimes \ldots \otimes A_{\pi(n)})$ for every permutation π of $\{1, \ldots, n\}$*

T3: ***Unitarity**: Let $I = (I_1, \ldots, I_k)$ be a partition of $\{1, \ldots, n\}$ into k pairwise disjoint subsets, then*

$$\mathcal{T}_n(A_1 \otimes \ldots \otimes A_n)^* = \sum_I (-1)^{n+k} \mathcal{T}_{|I_1|}\Big(\bigotimes_{j \in I_1} A_j^*\Big) \star \cdots \star \mathcal{T}_{|I_k|}\Big(\bigotimes_{j \in I_k} A_j^*\Big)$$

where the sum runs over all such partitions I.

T4: ***Causal factorization**: If the supports of $A_1, \ldots, A_k$ are in the causal future of the supports of $A_{k+1}, \ldots, A_n$, then*

$$\mathcal{T}_n(A_1 \otimes \ldots \otimes A_n) = \mathcal{T}_k(A_1 \otimes \ldots \otimes A_k) \star \mathcal{T}_{n-k}(A_{k+1} \otimes \ldots \otimes A_n)$$

T5: ***ϕ-independence** or product rule:*

$$\frac{\delta}{\delta \phi} \mathcal{T}_n\left(A_1 \otimes \cdots \otimes A_n\right) = \sum_{k=1}^{n} \mathcal{T}_n\left(A_1 \otimes \cdots \otimes \frac{\delta A_k}{\delta \phi} \otimes \cdots \otimes A_n\right)$$

T6: ***$\mathfrak{P}_+^{\uparrow}$-invariance**: $\alpha_{\mathfrak{p}} \mathcal{T}_n(A_1 \otimes \cdots \otimes A_n) = \mathcal{T}_n(\alpha_{\mathfrak{p}}(A_1) \otimes \cdots \otimes \alpha_{\mathfrak{p}}(A_n))$*

The renormalized time-ordered product between two local functionals $\mathcal{T}(A \otimes B)$ may be written as $A \cdot_{\mathcal{T}} B$ again for convenience. It has been shown in [FR11] that $\cdot_{\mathcal{T}}$ is actually a binary operation on a suitable subspace of $\mathcal{A}_{\mathrm{loc}}$.

There exist many explicit solutions $\mathcal{T}_n$ in the literature, e.g. using causal splitting of distributions [EG73, Sch89], extensions of distributions [BF00] or dimensional regularization + minimal subtraction [Kel10]. We do not want to go into the details of those constructions here.

We can rephrase the above axioms in a handier way in terms of generating functional $\mathcal{S}$, which define the $\mathcal{T}_n$ in the same way as $\mathcal{S}_0$ in (2.11) for the ${}^0\mathcal{T}$. We will call the generating functional $\mathcal{S}$ formal S-matrix. The origin of the name is discussed below.

Definition 2.2.3 (Axioms of formal S-matrices).
Let $\mathcal{S} : \mathcal{A}_{\mathrm{loc}} \to \mathcal{A}$ be analytic in the neighborhood of the origin, in particular all derivatives

$$\mathcal{T}_n(A^{\otimes n}) := \frac{1}{i^n} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big|_{\lambda=0} \mathcal{S}(\lambda A)$$

exist as functionals in $\mathcal{A}$. Then $\mathcal{S}$ is called a formal S-matrix, if it fulfills

S1: ***Initial condition**: $\mathcal{S}(0) = 1$ and $\frac{\mathrm{d}}{i\,\mathrm{d}\lambda}\big|_{\lambda=0}\mathcal{S}(\lambda A) = A$ concerning the formal power series in the argument. $\mathcal{S} = \mathbb{1} + O(\hbar)$ in $\hbar$ for the series in $\hbar$.*

S2: ***Unitarity**: $\mathcal{S}(A)^* = \mathcal{S}(A^*)^{\star -1}$*

S3: ***Causal factorization:** For A, B, C in $\mathcal{A}_{\mathrm{loc}}$ with* supp A *later than* supp C *it holds* [4]

$$\mathcal{S}(A + B + C) = \mathcal{S}(A + B) \star \mathcal{S}(B)^{\star -1} \star \mathcal{S}(B + C) \tag{2.14}$$

S4: ***ϕ-independence**:*

$$\frac{\delta}{i\delta\phi}\mathcal{S}(A) = \mathcal{T}_1\left(\frac{\delta A}{\delta\phi}\right) \cdot_{\mathcal{T}} \mathcal{S}(A)$$

S5: ***$\mathfrak{P}_+^\uparrow$-invariance**: $\alpha_{\mathfrak{p}}\mathcal{S}(A) = \mathcal{S}(\alpha_{\mathfrak{p}}(A))$*

Any such map $\mathcal{S}$ defines in this way a sequence of time-ordered products

$$\mathcal{T}_n(A_1 \otimes \cdots \otimes A_n) = \frac{\mathrm{d}^n}{\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_n}\Big|_{\lambda_1=\ldots=\lambda_n=0} \mathcal{S}\left(\sum_{k=1}^{n} \lambda_k A_k\right)$$

which obeys the properties from definition 2.2.2.

The name formal S-matrix has its origins in the quantum mechanical scattering theory. There the S-matrix of the interacting system is the **operator** $S(\mathcal{V}_g) : \mathcal{H}_{\mathrm{in}} \to \mathcal{H}_{\mathrm{out}}$ which is obtained by the same analysis as in the classical case 2.1 for interaction $\mathcal{V}$ that has been localized with $g \in \mathcal{D}$. This S-matrix can be expanded around $g = 0$ resulting in Dyson's series, i.e. the time-ordered exponential of the interaction $\mathcal{V}$. For interactions $\mathcal{V}$ that have good decay properties, the physical S-matrix is assumed in the limit $g \to 1$, the **adiabatic limit**. This issue is discussed in section 3.

The adjective formal indicates that in this case the analogy is not complete in this setting: $\mathcal{S}(\mathcal{V}_g)$ is not an operator on the Hilbert space of the free theory. It is rather its algebraic pendant on the level of off-shell functionals. In a representation π of the algebra $\mathcal{A}_{\mathrm{os}}$ (the on-shell algebra of observables) on a Hilbert space, it is expected that the representative of the formal S-matrix $\pi(\mathcal{S}(\mathcal{V}_g))$ tends to the physical S-matrix $S(\mathcal{V})$ in the adiabatic in the case the latter one exists. The convergence has been proven only in the case of the vacuum representation and massive fields (under certain renormalization constraints) so far, see [EG76, BS75].

In addition the formal S-matrix $\mathcal{S}(\mathcal{V}_g)$ is, unlike in many cases in QM, **only** available via a formal power series in the interaction $\mathcal{V}$. The construction of S-matrices of non-linear relativistic quantum field theories in four dimensions has unfortunately not been successful so far. The topic of constructive QFT is despite its very interesting and in many ways complementary approach omitted here, see [BLS11] for an overview.

One of the nice features of the axiomatic approach to time-ordered products and the formal S-matrix is that it can be used to define the interacting theory independent of

[4] This condition is more general than T4, which is obtained by setting $B = 0$.

some particular regularization scheme. Moreover it can be used to show, that every such regularization scheme is equivalent, up to finite renormalizations. The meaning of that statement can be made precise with the help of the renormalization group introduced by Stückelberg and Petermann [SP53, BDF09].

Definition 2.2.4 (Renormalization group in the sense of Stückelberg and Petermann). *Let $\mathcal{Z}$ be the set of maps $Z : \mathcal{A}_{\text{loc}} \to \mathcal{A}_{\text{loc}}$, which are analytic in the neighborhood of the origin and obey*

Z1: ***Starting element**: $Z(0) = 0$ and $\frac{\mathrm{d}}{i\,\mathrm{d}\lambda} Z(\lambda A) = A$ and $Z(A) = A + O(\hbar)$ for all $A \in \mathcal{A}_{\text{loc}}$*

Z2: ***Additivity**: $Z(A+B+C) = Z(A+B) - Z(B) + Z(B+C)$ if $\operatorname{supp}(A) \cap \operatorname{supp}(C) = \varnothing$*

Z3: ***ϕ-independence**: $\frac{\delta}{\delta\phi} Z = 0$ for all $A \in \mathcal{A}_{\text{loc}}$*

Z4: ***$\mathfrak{P}_+^\uparrow$-invariance**: $\alpha_{\mathfrak{p}} Z(A) = Z(\alpha_{\mathfrak{p}}(A))$.*

The set $\mathcal{Z}$ forms a group with respect to composition.

Using the group $\mathcal{Z}$ it is possible to formulate the **main theorem** of renormalization, which has been proven in [BDF09].

Theorem 6 (Main theorem of renormalization).
Let $\mathcal{S}$ be a formal S-matrix. Then $\mathcal{S} \circ Z$ is again a formal S-matrix. Moreover for any two formal S-matrices $\mathcal{S}_1$ and $\mathcal{S}_2$ there exists a $Z \in \mathcal{Z}$, such that $\mathcal{S}_1 = \mathcal{S}_2 \circ Z$.

This shows, that all regularization schemes the can be used to derive a formal S-matrix $\mathcal{S}(\mathcal{V}_g)$ (for the localized interaction $\mathcal{V}_g$) fulfilling the axioms of definition 2.2.3 are equivalent up to a renormalization group transformation $Z \in \mathcal{Z}$ of the interaction.

Unfortunately Z does not respect the **linear** dependence of the interaction $\mathcal{V}_g$ on the cutoff g, i.e. $\mathcal{V}_g = \int g(x)\mathcal{V}(\Phi_x)\,\mathrm{d}x$, see definition 2.1.1 for the notation. This is the reason why a generalized action is introduced as a map $\mathcal{S} : \mathcal{D} \to \mathcal{F}_{\text{loc}}$ which is additive in the sense of R2, rather than linear as in definition 1.3.1. See [BDF09] for the further constructions. We do not engage in this discussion here.

In the subsequent part of this chapter we are going to construct the interacting quantum field theory with localized interaction $\mathcal{V}_g$. The main ingredient is the formal S-matrix and its descendant the relative S-matrix. It is evident from the previous discussion that there is an ambiguity in the definition of the time-ordered products. This freedom must be fixed in order to obtain unique predictions of the resulting theory. At this point the so-called **renormalizable interactions** play a pivotal role in the discussion. In the case of the scalar field, a renormalizable, local interaction consists only of the terms

$$\mathcal{V}_g = \frac{\lambda_3}{3!}\Phi_g^3 + \frac{\lambda_4}{4!}\Phi_g^4\,.$$

in the notation of equation (1.3).

It has been shown that for renormalizable interactions the image of $\mathcal{Z}$ is finite dimensional, i.e. the freedom in extending the time-ordered products can be parametrized by finitely many numbers. For a discussion of this fact within this setting we refer to [DF04].

If one fixes the degree of perturbation theory to say n, i.e. one disregards all contributions to the time-ordered products, which are of order $O(g^{n+1})$, then a renormalizable theory is predictive after the fixing of finitely many numbers. These numbers have to be determined by experimental data for e.g. the masses and the interaction strength of the respective objects in the theory. We refer to [IZ12] for a more detailed discussion to this topic.

In the following it is always assumed, that the time-ordered products $\mathcal{T}_n$ are fixed – or are fixable – to an arbitrary high degree in n, such that we may take the formal S-matrix as a fundamental ingredient to the construction of the theory.

2.3. Algebra of Interacting Observables

The aim of this section is to construct a quantum field theory in terms of a Haag-Kastler net of *-algebras (see 1.4.8) $\{\mathcal{A}^{\mathcal{V}_g}(\mathcal{O}) : \mathcal{O} \subset M\}$ whose elements obey equations of motion, which are induced by the action $\mathcal{S}_{\mathcal{L}_0-\mathcal{V}_g}$.

We will start with defining the observables of the interacting theory with interaction $\mathcal{V}_g$ very similar to the case of classical field theory in section 2.1, namely by an analogue of the classical Møller map $\mathcal{R}_{\mathcal{V}_g}$. In the definition of the interacting observables the formal S-matrix of the interaction will play a pivotal role, however the definition of the time-ordered products and hence the formal S-matrix of the interaction depends to large extent on the chosen Hadamard two-point function $\Delta_+ \in H_+$.

Thence we show in a second step that the interacting theory, i.e. the net of *-algebras $\{\mathcal{A}^{\mathcal{V}_g}(\mathcal{O}) : \mathcal{O} \subset M\}$ generated by the interacting observables is in fact independent of the choice of Δ_+. This is interpreted as the **state independence** of the interacting theory.

The question concerning the thermal equilibrium state will be discussed after the notion of the algebra of interacting observables is fixed. This concludes the disentanglement of the UV and IR regime of the theory that was mentioned in the introduction. The UV divergences that are present in the canonical approaches to perturbative QFT in [Ryd96, Wei96] correspond to using the unrenormalized time-ordered products ${}^0\mathcal{T}_n$ in perturbation theory.

With a fixed choice of renormalized time-ordered products $\mathcal{T}_n$ or equivalently of the renormalized formal S-matrix $\mathcal{S}$ of the theory, there will no UV divergences arise at all in this formalism. Moreover every two different ways to renormalize the time-ordered products can be converted into each other by elements of the Stückelberg-Petermann renormalization group $\mathcal{Z}$.

A novel aspect in this manner is the discussion on the influence of the **background state** on which the perturbative expansion is centered. Although the interacting theories are equivalent in the sense of *-algebras, they are not equal since the isomorphism

between the two algebras is not the identity map in general. This implies in particular that the corresponding quantum interaction functionals $\mathcal{V}$, represented in two different algebras of free fields differ in general by a lower order term. This is interpreted as the influence of the background state on the perturbative expansion of the theory.

A discussion of this fact is done in the example of the (classical) Φ^4_x-interaction term in the case of the vacuum and finite temperature Hadamard two-point function that define the algebra of Wick-polynomials of the free theory. We find that both terms agree up to a mass term (and a constant which is neglected), which does not vanish even if the mass of the free theory is set to $m = 0$.

The IR divergences that are mentioned in the introduction correspond to using an ill-defined state on the algebra of interacting observables in our framework. Consequently the IR problem is reduced to a sole discussion on states of the interacting theory. A detailed analysis of the state space in pAQFT is done in sections 2.4 and chapter 4.

2.3.1. Interacting Observables in a Bounded Region

The method underlying the construction was already pointed out in classical field theory where the factorization property of classical observables yielded

$$\mathcal{A}_{\mathrm{cl}}^{\mathcal{V}_g}(\mathcal{O}) = \bigvee \left\{ \mathcal{R}_{\mathcal{V}_g}(A) : A \in \mathcal{F}_{\mathrm{pol}},\ \mathrm{supp}(A) \subset \mathcal{O} \right\} .$$

While a direct construction of the LHS in quantum theory will not be available for local interactions it is the goal to use the RHS to define the quantum algebra of interacting observables. For this a corresponding linear map to $\mathcal{R}_{\mathcal{V}_g}$ has to be constructed. This is achieved by invoking the so-called relative S-matrices.

The allowed interactions $\mathcal{V}$ of the theory are the same as in classical field theory, namely polynomials in the linear field Φ_x, as described in definition 2.1.1. The polynomials will be cut off with a test-function $g \in \mathcal{D}$ and we identify the functionals

$$\mathcal{S}_{\mathcal{V}_g}[f]\Big|_{f=1} = V[f]\Big|_{f=1} = \int g(x)\mathcal{V}(\Phi_x)\,\mathrm{d}x \equiv \mathcal{V}_g \in \mathcal{F}_{\mathrm{loc}} \cap \mathcal{F}_{\mathrm{pol}}$$

where $\mathcal{S}$ is the action functional and the subscript indicates the Lagrangian density of the action (see definition 1.3.1 and below).

We keep the interaction polynomial $\mathcal{V}$ fixed during this construction, though modifications of the cutoff function $g \in \mathcal{D}$ of $\mathcal{V}$ will play an important role in the discussion.

Definition 2.3.1 (Relative S-matrix).
For two local functionals $A, B \in \mathcal{A}_{\mathrm{loc}} \cap \mathcal{A}_{\mathrm{pol}}$, the relative S-matrix between B and A is defined by

$$\mathcal{S}_B(A) = \mathcal{S}(B)^{\star -1} \star \mathcal{S}(B + A)$$

where $\mathcal{S}$ is the formal S-matrix from 2.2.3. We will only be interested in the case, where B is the interaction $\mathcal{V}_g$ and define the ***relative S-matrix*** *for the interaction $\mathcal{V}_g$ by*

$$\mathcal{S}_g(A) \coloneqq \mathcal{S}_{\mathcal{V}_g}(A) = \mathcal{S}(\mathcal{V}_g)^{\star -1} \star \mathcal{S}(\mathcal{V}_g + A)\,, \tag{2.15}$$

where the shorthand subscript g in $\mathcal{S}_g(A)$ will always refer to the interaction functional $\mathcal{V}_g$, unless stated otherwise.

The following properties of the relative S-matrix are found to be equivalent [EG73] to the causal factorization property (2.14) of the formal S-matrix and play a leading role in the future discussion:

R1 **Causal factorization**:

$$\mathcal{S}_g(A+B) = \mathcal{S}_g(A) \star \mathcal{S}_g(B) \qquad \text{if } \operatorname{supp}(A) \gtrsim \operatorname{supp}(B) \tag{2.16}$$

R2 **Retardation**:

$$\mathcal{S}_{g+g'}(A) = \mathcal{S}_g(A) \qquad \text{if } \operatorname{supp}(g') \gtrsim \operatorname{supp}(A). \tag{2.17}$$

The property (2.17) displays that $\mathcal{S}_g(A)$ indeed only depends on the restriction of g to the past of supp A, i.e. it is, in particular, independent of the behavior of g in the future of its support. Thus it holds $\mathcal{S}_g(A) = \mathcal{S}(A)$ in the case that $\operatorname{supp}(g) \gtrsim \operatorname{supp}(A)$.

It follows directly from the causal factorization property of $\mathcal{S}_g$, that the elements $\mathcal{S}_g(A)$ and $\mathcal{S}_g(B)$ commute, if the supports of A and B are spacelike separated, since in this case $\operatorname{supp}(A) \gtrsim \operatorname{supp}(B)$ and $\operatorname{supp}(B) \gtrsim \operatorname{supp}(A)$ at the same time, thus

$$\mathcal{S}_g(A+B) = \mathcal{S}_g(A) \star \mathcal{S}_g(B) = \mathcal{S}_g(B) \star \mathcal{S}_g(A).$$

Consequently all quantities which are derived from $\mathcal{S}_g(A)$ and $\mathcal{S}_g(B)$ as functional derivatives will mutually commute, if A, B have spacelike separated support. This property will guarantee the Einstein causality (QFT2) property for algebras $\star$-generated by the relative S-matrices and its functional derivatives. The following definition is a result of this argument.

Definition 2.3.2 (Off-shell algebra of retarded interacting observables).
The space of test functions $g \in \mathcal{D}$ with $g \equiv 1$ in $\mathcal{O}$ is denoted by $\mathcal{D}_\mathcal{O}$. For every $g \in \mathcal{D}_\mathcal{O}$ the algebra $\mathcal{A}^{\mathcal{V}_g}(\mathcal{O})$ is defined as the algebra that is $\star$-generated by the functionals $\mathcal{S}_{\mathcal{V}_g}(A)$ with supp $A \subset \mathcal{O}$. *$\mathcal{A}^{\mathcal{V}_g}(\mathcal{O})$ is called the retarded **off-shell algebra of interacting observables** in the region $\mathcal{O}$.*

Proposition 2.3.3.
The *-algebra $\mathcal{A}^{\mathcal{V}_g}(\mathcal{O})$ satisfies Einstein causality (QFT2) from definition 1.4.8 and its algebraic structure is independent of the choice of g.

PROOF. The Einstein causality $[\mathcal{A}^{\mathcal{V}_g}(\mathcal{O}_1), \mathcal{A}^{\mathcal{V}_g}(\mathcal{O}_2)]_\star = 0$ for spacelike separated $\mathcal{O}_1, \mathcal{O}_2$ follows immediately from the causal factorization property R1 of the generators, which has already been discussed.

The independence of $g \in \mathcal{D}_\mathcal{O}$ can be seen by the following argument: Let $g_1, g_2 \in \mathcal{D}_\mathcal{O}$, then there exist test-functions $g_\pm \in \mathcal{D}$ such that $g_1 - g_2 = g_+ + g_-$ and $\operatorname{supp}(g_\pm) \cap J^\mp(\mathcal{O}) = \varnothing$. Using causal factorization we obtain

$$\mathcal{S}_{g_1}(A) = \mathcal{S}_{g_2+g_++g_-}(A) \overset{\text{R2}}{=} \mathcal{S}_{g_2+g_-}(A)\,.$$

Here we need another important property that follows from the causal factorization of the formal S-matrix:

$$\mathcal{S}_{B+C}(A) = \mathcal{S}_B(C)^{-1} \star \mathcal{S}_B(A) \star \mathcal{S}_B(C) \qquad \text{if } \operatorname{supp}(A) \gtrsim \operatorname{supp}(C)\,. \tag{2.18}$$

This has the following consequence for the above formula:

$$\mathcal{S}_{g_1}(A) = \mathcal{S}_{g_2+g_-}(A) = \mathcal{S}_{g_2}(\mathcal{V}_{g_-})^{-1} \star \mathcal{S}_{g_2}(A) \star \mathcal{S}_{g_2}(\mathcal{V}_{g_-})$$

i.e. both generators are unitarily equivalent with the unitary $\mathcal{S}_{g_2}(\mathcal{V}_{g_-})$. This implies that the generated algebras are also unitarily equivalent, hence isomorphic. □

The algebra $\mathcal{A}^{\mathcal{V}_g}(\mathcal{O})$ is given as a formal power series of elements in $\mathcal{A}(J^-(\mathcal{O}))$ in the following sense: Every element $A \in \mathcal{A}^{\lambda\mathcal{V}_g}(\mathcal{O})$ is a formal power series in the parameter λ and every coefficient in this series is a functional in $\mathcal{A}(J^-(\mathcal{O}))$. The support statement follows from the fact, that the generators $\mathcal{S}_g(A)$ depend only the projection of g onto $J^-(\mathcal{O})$, which is expressed in (2.17).

2.3.2. Influence from the Free Theory, Thermal Mass

It is important to notice at this point that the interacting algebra $\mathcal{A}^{\mathcal{V}_g}(\mathcal{O})$ does not only depend on the choice of the interaction $\mathcal{V}_g$, but two different choices of $\star$-products in the free theory will lead to possibly different interacting algebras $\mathcal{A}^{\mathcal{V}_g}(\mathcal{O})$. We relate the generators $\mathcal{S}_g(A)$ that are defined for a $\star$-product $\star_1$ to the ones defined for $\star_2$, where the $\star$-products are defined by Δ_+^1, Δ_+^2, respectively.

Due to Hadamard condition H3 from definition 1.4.2 we know that there exists a real-valued smooth function $u \in \mathcal{E}(M \times M)$, such that $\Delta_+^2 - \Delta_+^1 = u$ and in the proof of proposition 1.4.7 we showed that

$$U : \mathcal{F}_{\mu c} \to \mathcal{F}_{\mu c}, \qquad U = \mathrm{e}^{\hbar \int u(x,y) \frac{\delta^2}{\delta\phi(x)\delta\phi(y)} \,\mathrm{d}x\,\mathrm{d}y}$$

is an isomorphism $U : (\mathcal{F}_{\mu c}[[\hbar]], \star_1) \to (\mathcal{F}_{\mu c}[[\hbar]], \star_2)$. It is clear that any state ω defined on $(\mathcal{F}_{\mu c}[[\hbar]], \star_1)$ is transformed under U to a state $\omega \circ U^{-1}$ to a state on $(\mathcal{F}_{\mu c}[[\hbar]], \star_2)$. The action of U can be seen as **changing** the background state of the free theory.

The time-ordered products behave under U in a very similar way. For the regular functionals U extends to an isomorphism

$$U : (\mathcal{F}_{\mu c}[[\hbar]], \cdot_{\mathcal{T},1}) \to (\mathcal{F}_{\mu c}[[\hbar]], \cdot_{\mathcal{T},2})$$

where $\cdot_{\mathcal{T},i}$ are time-ordered products with Feynman propagators Δ_F^i that differ by the **same** smooth function u as their two-point functions Δ_+^i, i.e. $\Delta_F^2 - \Delta_F^1 = u$. Moreover it is important that $U : \mathcal{F}_{\mathrm{loc}} \to \mathcal{F}_{\mathrm{loc}}$. This holds due to

$$\frac{\delta^k}{\delta\phi^k}(UA)(\phi) = \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \left\langle u^{\otimes n} \middle| A^{(2n+k)}(\phi) \right\rangle ,$$

thus $\operatorname{supp}(UA)^{(k)}(\phi) \subset \operatorname{diag}(M^k)$. It is obvious that U will map polynomials into polynomials.

Consequently the renormalized time-ordered products $\mathcal{T}_n$ change under the action of U as follows

$$U\mathcal{T}_n^1(A_1 \otimes \cdots A_n) = \mathcal{T}_n^2(UA_1 \otimes \cdots UA_n)$$

where $\mathcal{T}_n^1$ and $\mathcal{T}_n^2$ obey the unitarity (T3) and causal factorization rule (T4) of definition 2.2.2 with the respective $\star$-products $\star_1$ and $\star_2$. Equivalently it holds for the formal S-matrices and relative S-matrices

$$U\mathcal{S}^1(A) = \mathcal{S}^2(UA), \qquad U\mathcal{S}^1_{\mathcal{V}_g}(A) = \mathcal{S}^2_{U\mathcal{V}_g}(UA)$$

We deduce that the action of U on the interacting algebra $\mathcal{A}^{\mathcal{V}_g}(\mathcal{O})$ fulfills

$$U\left(\mathcal{S}^1_{\mathcal{V}_g}(A) \star_1 \mathcal{S}^1_{\mathcal{V}_g}(B)\right) = \mathcal{S}^2_{U\mathcal{V}_g}(UA) \star_2 \mathcal{S}^2_{U\mathcal{V}_g}(UB)\,.$$

This implies that U extends to a *-homomorphism $U : \mathcal{A}^{\mathcal{V}_g}(\mathcal{O}) \to \mathcal{A}^{U\mathcal{V}_g}(\mathcal{O})$. Thence U is almost an element of the renormalization group, it fulfills all the axioms in definition 2.2.4 except that it does start with the identity in the $\hbar$-expansions. It modifies the interaction at tree level and thus has physical significance, this why we discuss it separately to the renormalization group.

From a state space point of view the map U alters the asymptotic behavior of the background state, thus influences the perturbative expansion. This can be easily seen by the different asymptotics of the massless and massive vacuum two-point function D_+^{vac}. A similar phenomenon can be observed if considers the D_+^{vac} and D_+^{β} which is discussed in an instant. Thus it is crucial to specify from which realization of the free theory, i.e. from which $\star$-product, the generators $\mathcal{S}_{\mathcal{V}_g}(A) \equiv \mathcal{S}_g(A)$ emerge. A particular example is a system with quartic interaction

$$\mathcal{V}_g = \frac{\lambda}{4!}\Phi_g^4 = \frac{\lambda}{4!}\int g(x)\Phi_x^4\,\mathrm{d}x$$

defined within the algebra with vacuum two-point function D_+^{vac} from example 1.5.5 and KMS two-point function D_+^{β} from example 1.5.7. The field monomials Φ_g^n are defined in example 1.2.2. We denote

$$u_\beta(x-y) = D_+^{\beta}(x-y) - D_+^{\mathrm{vac}}(x-y) = \frac{1}{(2\pi)^3}\int \frac{\delta(p^2-m^2)}{\mathrm{e}^{\beta|p_0|}-1}\mathrm{e}^{-ip(x-y)}\,\mathrm{d}p$$

exploiting the translation invariance of both two-point functions and U_β is the corresponding isomorphism. The full action functional reads

$$\mathcal{S}_{\mathcal{L}_0-\mathcal{V}_g}[f] = \frac{1}{2}\int f(x)\left(\eta^{\mu\nu}(\partial_\mu\Phi_x)(\partial_\nu\Phi_x) - m^2\Phi_x^2\right) - f(x)g(x)\frac{\lambda}{4!}\Phi_x^4\,\mathrm{d}x$$

where $f = 1$ on the region of interest, that is $\mathcal{O} \subset M$, on which also $g = 1$. The free part $\mathcal{S}_{\mathcal{L}_0}$ is invariant under the action of any such transformation U (i.e. for general $u = \Delta_+^2 - \Delta_+^1$)

$$U\mathcal{S}_{\mathcal{L}_0}[f](\phi) = \frac{-1}{2}\int f(x)\delta(x-y)\left(\Box_x + m^2\right)u(x,y)\,\mathrm{d}x = 0$$

since u is a smooth bi-solution to $P = \Box + m^2$. The interaction term changes though:

$$U_\beta \mathcal{V}_g = \mathcal{V}_g + \hbar\left\langle u_\beta \middle| \mathcal{V}_g^{(2)}\right\rangle + \frac{\hbar^2}{2}\left\langle u_\beta^{\otimes 2}\middle|\mathcal{V}_g^{(4)}\right\rangle .$$

The term in order $\hbar^2$ can be neglected for the discussion since it is proportional to the constant functional $\mathbb{1}$, the term in order $\hbar^1$ yields

$$\left\langle u_\beta \middle| \mathcal{V}_g^{(2)}\right\rangle = \frac{\lambda u_\beta(0)}{4}\int g(x)\Phi_x^2\,\mathrm{d}x$$

where the prefactor is given by

$$\begin{aligned} u_\beta(0) &= \frac{1}{(2\pi)^3}\int \frac{1}{\omega_p}\frac{1}{\mathrm{e}^{\beta\omega_p}-1}\,\mathrm{d}^3\boldsymbol{p} = \frac{1}{2\pi^2\beta^2}F(\beta m),\\ F(y) &= \int_y^\infty \frac{\sqrt{x^2-y^2}}{\mathrm{e}^x-1}\,\mathrm{d}x, \quad F(0) = \frac{\pi^2}{6}, \qquad F(y) \sim \sqrt{\frac{\pi y}{2}}\mathrm{e}^{-y}, \quad \text{as } y\to\infty . \end{aligned}$$

The additional term that arises from the action of U on $\mathcal{V}_g$ can be absorbed into the free part $\mathcal{L}_0$

$$\begin{aligned} U_\beta\mathcal{S}_{\mathcal{L}_0-\mathcal{V}_g}[f] &= \mathcal{S}_{\mathcal{L}_0-U_\beta\mathcal{V}_g}[f]\\ &= \frac{1}{2}\int f(x)\left(\eta^{\mu\nu}(\partial_\mu\Phi_x)(\partial_\nu\Phi_x) - \hat{m}^2\Phi_x^2\right) - g(x)f(x)\frac{\lambda}{4!}\Phi_x^4\,\mathrm{d}x \end{aligned}$$

with a modified mass term (if $\beta m > 0$ and $\lambda > 0$):

$$\frac{\hat{m}^2}{2} = \frac{m^2}{2} + \lambda u_\beta(0) = \frac{m^2}{2} + \frac{\lambda}{4}\frac{\hbar}{2\pi^2\beta^2}F(\beta m) .$$

The additional term is often called "thermal mass" in the literature, see e.g. [LB00]. It is used mainly in massless theories, where $m^2 = 0$ and $\lambda > 0$ such that

$$\hat{m}^2 = \frac{\lambda\hbar}{24\beta^2}$$

replaces the absent mass term.

In order to implement this idea, we will have to require an additional constraint on the renormalized time-ordered products: namely the principle of "perturbative agreement" which was introduced by Hollands and Wald in [HW05].

T7: **Principle of perturbative agreement**: If the local interaction is given by a mass term $\mathcal{V}_g = \frac{\lambda^2}{2}\int g(x)\Phi_x^2\,\mathrm{d}x$ then the perturbative expansions agree with the modified free action, i.e. the map

$$\mathcal{A}^\lambda \ni A \mapsto \mathcal{R}_{\mathcal{V}_g}(A) \in \mathcal{A}$$

is an injective *-homomorphism, where $\mathcal{A}^\lambda$ is the *-algebra generated by the same Hadamard two-point function Δ_+ in which m^2 is replaced by $m^2+\lambda^2$

It has not been shown that time-ordered products $\mathcal{T}$ can be defined in such a way that T7 can always be fulfilled. If this is not the case, then both theories do not coincide, but the observables are related by the action of an element of the renormalization group $\mathcal{Z}$.

In addition one has to be careful with modification of the free theory by a term that depends on the interaction strength λ which serves as the formal expansion parameter, since it is not evident that the elements of the arising interacting theory are formal power series in the interaction (i.e. in λ) any more. This is due to the fact that the two-point function and the Feynman propagator of the free theory contain inverse powers of the mass. This has already been remarked in [Alt90]. In order to revise the ill-defined power series expansions in the massless theory, a number of resummation techniques have been proposed, see [LB00, NS84].

2.3.3. Interacting Local Nets (Off-Shell)

A crucial question at this point is, whether one can define a **net of** algebras $\{\mathcal{A}^{\mathcal{V}_g}(\mathcal{O}) : \mathcal{O} \subset M\}$ that satisfies the isotony property. This is not quite straightforward, since $\mathcal{A}^{\mathcal{V}_g}(\mathcal{O})$ depends explicitly on a cutoff $g \in \mathcal{D}_\mathcal{O}$. To avoid this one defines the local net of interacting algebras as a bundle over all $g \in \mathcal{D}_\mathcal{O}$. This removes the explicit dependence on the choice $g \in \mathcal{D}_\mathcal{O}$. This was established in [DF00].

Definition 2.3.4 (Bundle of algebras).
Let $\mathcal{B}^\mathcal{V}(\mathcal{O})$ be the bundle of algebras

$$\mathcal{B}^\mathcal{V}(\mathcal{O}) \coloneqq \bigcup_{g\in\mathcal{D}_\mathcal{O}} \{g\}\times\mathcal{A}\,.$$

A section over $\mathcal{B}^{\mathcal{V}(\mathcal{O})}$ is denoted by $A = g \mapsto (A)_g \in \mathcal{A}(\mathcal{O})$. The space of sections $\Gamma(\mathcal{B}^\mathcal{V}(\mathcal{O}))$ over $\mathcal{B}^\mathcal{V}(\mathcal{O})$ is endowed with fiberwise algebra operations of $\mathcal{A}$, in particular with the product

$$(A\star B)_g = (A)_g \star (B)_g\,.$$

The sections over the bundle exhibit an algebra structure such that it is natural to look for the algebra of interacting observables within the space of sections. The are distinguished by a very important feature: Constructions of the same interacting observable with different cutoffs led to unitarily equivalent results. That is now required for the net of interacting local algebras.

Definition 2.3.5 (Local net of interacting algebras).
*The net local of algebras $\mathcal{A}^{\mathcal{V}}(\mathcal{O})$ consists of **covariant constant sections** over $\mathcal{B}^{\mathcal{V}}(\mathcal{O})$, i.e. elements*

$$g \mapsto (A)_g \in \Gamma(\mathcal{B}^{\mathcal{V}}(\mathcal{O})) : U_{g,g'} \star (A)_g = (A)_{g'} \star U_{g,g'} \quad \forall U_{g,g'} \in \mathcal{U}(g,g')$$

where $\mathcal{U}(g,g')$ denotes the set of all unitaries in $\mathcal{A}$ that obey

$$U_{g,g'} \star \mathcal{S}_g(A) = \mathcal{S}_{g'}(A) \star U_{g,g'} \qquad \forall g,g' \in \mathcal{D}_{\mathcal{O}} \,. \tag{2.19}$$

$\mathcal{A}^{\mathcal{V}}(\mathcal{O})$ is equipped with a covariant action of $\mathfrak{P}_+^{\uparrow}$: For $\mathfrak{p} : \mathcal{O}_1 \to \mathcal{O}_2$ define

$$\alpha_{\mathfrak{p}}^{\mathcal{V}} : \mathcal{A}^{\mathcal{V}}(\mathcal{O}_1) \to \mathcal{A}^{\mathcal{V}}(\mathcal{O}_2) : \quad g \mapsto \left(\alpha_{\mathfrak{p}}^{\mathcal{V}}(A)\right)_g := g \mapsto \alpha_{\mathfrak{p}}(A)_{\alpha^{\mathfrak{p}}(g)} \,.$$

Proposition 2.3.6.
The net $\{\mathcal{A}^{\mathcal{V}}(\mathcal{O}) : \mathcal{O} \subset M\}$ fulfills the isotony axiom, as well as Einstein causality. The generators $\mathcal{S}_g(A)$, seen as maps $g \mapsto \mathcal{S}_g(A)$ are elements of $\mathcal{A}^{\mathcal{V}}(\mathcal{O})$ by construction.

PROOF. The isotony property follows mainly from the fact that every $\mathfrak{p} : \mathcal{O}_1 \to \mathcal{O}_2$ acts in a **contravariant** way the on the spaces $\mathcal{D}_{\mathcal{O}}$:

$$\alpha^{\mathfrak{p}} : \mathcal{D}_{\mathcal{O}_2} \to \mathcal{D}_{\mathcal{O}_1}, \quad g \mapsto \alpha^{\mathfrak{p}} g = g \circ \mathfrak{p} \in \mathcal{D}_{\mathfrak{p}^{-1}\mathcal{O}_2}$$

for $g \in \mathcal{D}_{\mathcal{O}_2}$ with the contravariant action $\alpha^{\mathfrak{p}}$ from definition 1.2.6.

For every two isometric embeddings $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P}_+^{\uparrow}$ that are compatible with each other it holds

$$\left(\alpha_{\mathfrak{p}_1 \circ \mathfrak{p}_2}^{\mathcal{V}}(A)\right)_g = \alpha_{\mathfrak{p}_1} \circ \alpha_{\mathfrak{p}_2}\,(A)_{\alpha^{\mathfrak{p}_1} \circ \alpha^{\mathfrak{p}_2}(g)} = \left(\alpha_{\mathfrak{p}_1}^{\mathcal{V}}\left(\alpha_{\mathfrak{p}_2}^{\mathcal{V}}(A)\right)_{\alpha^{\mathfrak{p}_2}(\cdot)}\right)_g \,.$$

The definition is compatible with the fiberwise $\star$-multiplication and involution on the $\mathcal{B}^{\mathcal{V}}(\mathcal{O})$ and with the notion of covariant constant sections. That is for any $g, g' \in \mathcal{D}_{\mathcal{O}}$ it holds

$$\begin{aligned}
\left(\alpha_{\mathfrak{p}}^{\mathcal{V}} A\right)_{g'} &= U_{g,g'} \star \left(\alpha_{\mathfrak{p}}^{\mathcal{V}} A\right)_g \star U_{g,g'}^{-1} = U_{g,g'} \star (\alpha_{\mathfrak{p}} A)_{\alpha^{\mathfrak{p}}(g)} \star U_{g,g'}^{-1} \\
&= \alpha_{\mathfrak{p}}\left(U_{\alpha^{\mathfrak{p}}(g),\alpha^{\mathfrak{p}}(g')}\right) \star \alpha_{\mathfrak{p}}\,(A)_{\alpha^{\mathfrak{p}}(g)}\, \alpha_{\mathfrak{p}} \star \left(U_{\alpha^{\mathfrak{p}}(g),\alpha^{\mathfrak{p}}(g')}^{-1}\right) \\
&= \alpha_{\mathfrak{p}}\left(U_{\cdot,\alpha^{\mathfrak{p}}(g')} \star A \star U_{\cdot,\alpha^{\mathfrak{p}}(g')}^{-1}\right)_{\alpha^{\mathfrak{p}}(g)} = \left(\alpha_{\mathfrak{p}}^{\mathcal{V}}\left(U_{\cdot,\alpha^{\mathfrak{p}}(g')} \star A \star U_{\cdot,\alpha^{\mathfrak{p}}(g')}^{-1}\right)\right)_g
\end{aligned}$$

where the property

$$\alpha_{\mathfrak{p}} U_{\alpha^{\mathfrak{p}}(g),\alpha^{\mathfrak{p}}(g')} = U_{g,g'} \tag{2.20}$$

together with $\alpha_{\mathfrak{p}} \circ \alpha^{\mathfrak{p}} = \mathbb{1}$ was used. The property (2.20) follows from the $\mathfrak{P}_+^{\uparrow}$-invariance of $\mathcal{S}$ and the defining equation (2.19). Thus $\alpha_{\mathfrak{p}}^{\mathcal{V}} A$ is again a covariant constant section and yields a *-homomorphism $\mathcal{A}^{\mathcal{V}}(\mathcal{O}) \to \mathcal{A}^{\mathcal{V}}(\mathfrak{p}\mathcal{O})$. □

The concept of covariant constant section in $\mathcal{B}^{\mathcal{V}}(\mathcal{O})$ implements the following idea: The generated interacting algebra of observables $\mathcal{A}^{\mathcal{V}_g}(\mathcal{O})$ was shown to be independent of the choice of $g \in \mathcal{D}_{\mathcal{O}}$ due to the fact that the generators $\mathcal{S}_g(A)$ could be converted into each other for two different choices $g, g' \in \mathcal{D}_{\mathcal{O}}$ by using the unitaries $U_{g,g'} = \mathcal{S}_{g'}(\mathcal{V}_{g_-})$ in the notation of the last proof. Thus if an observable was constructed by some $\mathcal{S}_g(A)$ with a special choice of g, all other possible constructions with different test-functions $g' \in \mathcal{D}_{\mathcal{O}}$ could be derived from the original one. The covariant constant section in $\mathcal{B}^{\mathcal{V}}(\mathcal{O})$ behave just in the same way by definition.

Since the local net of algebras $\{\mathcal{A}^{\mathcal{V}}(\mathcal{O}) : \mathcal{O} \subset M\}$ is manifestly independent of the cutoff g that defined the generators $\mathcal{S}_g(A)$ it is possible to obtain the global algebra $\mathcal{A}^{\mathcal{V}}(M)$ by a direct limit (or inductive limit). The covariant action $\alpha^{\mathcal{V}}_{\mathfrak{p}}$ leads to an automorphic action on $\mathcal{A}^{\mathcal{V}}(M)$.

For the generators $\mathcal{S}_g(A)$ from definition 2.3.2 the homomorphisms $\alpha^{\mathcal{V}}_{\mathfrak{p}}$ act by

$$\left(\alpha^{\mathcal{V}}_{\mathfrak{p}}\mathcal{S}_{\cdot}(A)\right)_g = \alpha_{\mathfrak{p}}\mathcal{S}_{\alpha^{\mathfrak{p}}g}(A) = \mathcal{S}_g\left(\alpha_{\mathfrak{p}}(A)\right) \ . \tag{2.21}$$

This formula will play an important role in the discussion of the interacting dynamics in section 3.2 which is induced by the one-parameter group of automorphisms that are the time-translations $\alpha^{\mathcal{V}}_t$ for $t \in \mathbb{R}$.

The definition of the interacting algebra of off-shell observables is inspired by the factorization property (2.7) of the interacting classical field theory. Due to the fact, that a construction of the full interacting theory (i.e. the potential LHS of (2.7) in the quantized case) is unavailable at the moment, the above procedure **defines** full interacting theory, whereas in the classical case both sides of the equation could be independently constructed and compared.

2.3.4. Interacting Local Nets (On-Shell)

It remains to show the validity of the time-slice axiom QFT. However $\mathcal{A}^{\mathcal{V}}(\mathcal{O})$ is modeled over the off-shell algebra $\mathcal{A}$, thus an exploitation of the time-slice axiom within the **free theory** can not be used in $\mathcal{A}^{\mathcal{V}}(\mathcal{O})$. If one constructs the interacting theory over the **on-shell algebra of the free theory** $\mathcal{A}_{\mathrm{os}}$ though, the time-slice axiom for $\mathcal{A}^{\mathcal{V}}(\mathcal{O})$ can be proven.

Proposition 2.3.7 (On-shell algebra [CF09]).
*Let $\mathcal{A}_{\mathrm{os}}$ be the on-shell algebra of the free scalar field from definition 1.4.9. The algebra generated by the sections $g \mapsto \mathcal{S}_g(A)$ with $A \in \mathcal{A}_{\mathrm{loc}} \cap \mathcal{A}_{\mathrm{os}}(\mathcal{O})$ satisfies the axioms QFT1 - QFT3. It is called the **on-shell algebra of the interacting scalar field** with interaction $\mathcal{V}_g$ and is denoted by $\mathcal{A}^{\mathcal{V}}_{\mathrm{os}}(\mathcal{O})$*

The question arises, whether we can **identify** observables in $\mathcal{A}^{\mathcal{V}}(\mathcal{O})$ from the free theory, such as the interacting quantum field or its energy density. This was possible in the classical theory due to the *-**homomorphism** $\mathcal{R}_{\mathcal{V}_g}$ between the theories, the (retarded) Møller map (2.4). The corresponding counterpart in QFT will be the linear part of the relative S-matrix with respect to the observable.

Definition 2.3.8 (Quantum Møller map).
*The linear part of relative S-matrix $\mathcal{S}_g(A)$ with respect to the observable A defines the **quantum Møller map** $\mathcal{R}_{\mathcal{V}_g}$. It is a linear map $\mathcal{R}_{\mathcal{V}_g} : \mathcal{A}_{\mathrm{loc}} \to \mathcal{A}$ given by*

$$\begin{aligned}\mathcal{R}_{\mathcal{V}_g}(A) &= \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0}\mathcal{S}_g(\lambda A) = \mathcal{S}(\mathcal{V}_g)^{\star-1}\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0}\mathcal{S}(\mathcal{V}_g+\lambda A)\\ &= \mathcal{S}(\mathcal{V}_g)^{\star-1}\star\left(\mathcal{S}(\mathcal{V}_g)\cdot_{\mathcal{T}} A\right)\ ,\end{aligned}$$

where the time-ordered product $\mathcal{S}(\mathcal{V}_g)\cdot_{\mathcal{T}} A$ is understood in the following sense (cf. definition 2.2.3):

$$\mathcal{S}(\mathcal{V}_g)\cdot_{\mathcal{T}} A = \sum_{n=0}^{\infty}\frac{i^n}{n!}\mathcal{T}_{n+1}((\mathcal{V}_g)^{\otimes n}\otimes A)\ .$$

*The functional $\mathcal{R}_{\mathcal{V}_g}(A)$ is called the retarded, **interacting observable** A with respect to the interaction $\mathcal{V}_g$ and $g\mapsto\mathcal{R}_{\mathcal{V}_g}$ defines a covariant constant section in $\mathcal{A}^{\mathcal{V}}$.*

The structural properties R1 and R2 of the relative S-matrix are passed down to the quantum Møller map $\mathcal{R}$ as follows:

R′1 **Causal factorization**: If $\mathrm{supp}(A)\gtrsim\mathrm{supp}(B)$, then $A\cdot_{\mathcal{T},\mathcal{V}}B = \mathcal{R}_{\mathcal{V}_g}(A)\star\mathcal{R}_{\mathcal{V}_g}(B)$, where $\cdot_{\mathcal{T},\mathcal{V}}$ denotes the interacting time-ordered product:

$$A\cdot_{\mathcal{T},\mathcal{V}}B = \frac{1}{i^2}\frac{\mathrm{d}^2}{\mathrm{d}\lambda\,\mathrm{d}\kappa}\Big|_{\lambda=\kappa=0}\mathcal{S}_g(\lambda A+\kappa B)$$

between local functionals $A, B\in\mathcal{A}_{\mathrm{loc}}$.

R′2 **Retardation**: If $\mathrm{supp}(g')\gtrsim\mathrm{supp}(A)$ then it holds $\mathcal{R}_{\mathcal{V}_{g+g'}}(A) = \mathcal{R}_{\mathcal{V}_g}(A)$; in particular $\mathcal{R}_{\mathcal{V}_g}(A) = A$ if $\mathrm{supp}(g)\gtrsim\mathrm{supp}(A)$.

From the factorization property (2.7) in the classical theory it seems natural identify interacting observables by the inverse of the quantum Møller map. Unfortunately the map $\mathcal{R}_{\mathcal{V}_g}$ is not surjective, i.e. $\mathcal{R}_{\mathcal{V}_g}$ is only invertible on a suitable subspace $\mathcal{D}_{\mathcal{T}}$ that is found in [Rej11]. This space is not stable under $\star$-multiplication thus $\mathcal{R}_{\mathcal{V}_g}(A)\star\mathcal{R}_{\mathcal{V}_g}(B)$ may have no inverse and naively applying the inverse map

$$\mathcal{R}^{-1}_{\mathcal{V}_g}(A) = \mathcal{S}(-\mathcal{V}_g)\cdot_{\mathcal{T}}\left(\mathcal{S}(\mathcal{V}_g)\star A\right)$$

leads to additional UV-divergences even after renormalization of the time-ordered products $\cdot_{\mathcal{T}}$.

This prevents a complete description of the full interacting theory by the introduction of interacting products

$$A\star_{\mathcal{V}_g}B \coloneqq \mathcal{R}^{-1}_{\mathcal{V}_g}\Big(\mathcal{R}_{\mathcal{V}_g}(A)\star\mathcal{R}_{\mathcal{V}_g}(B)\Big)\ ,\quad A\cdot_{\mathcal{T},\mathcal{V}}B \coloneqq A\cdot_{\mathcal{T}}B \tag{2.22}$$

where the interacting observables are simply the same functionals $A\in\mathcal{F}_{\mu\mathrm{c}}[[\hbar]]$ and only the product changed again. The latter products exist for the regular functionals $\mathcal{F}_{\mathrm{reg}}$ (and a regular interaction), which sadly does not suffice to construct interesting

models in QFT. Notice that a hypothetical algebra with product $\star_{\mathcal{V}_g}$ could always be represented on the algebra of free fields $\mathcal{A}$ with a representation given by $\mathcal{R}_{\mathcal{V}_g}$ by definition of $\star_{\mathcal{V}_g}$.

The properties of the quantum Møller map shows it shares many features with the classical Møller map $\mathcal{R}_{\mathcal{V}_g}$. In fact, by looking at the first order contribution of $\mathcal{R}_{\mathcal{V}_g}$ in $\hbar$ one finds the classical Møller map as the classical limit $\hbar \to 0$ of $\mathcal{R}_{\mathcal{V}_g}$. Using

$$A \cdot_{\mathcal{T}} B = A \cdot B + \hbar \left\langle \Delta_F \middle| A^{(1)} \otimes B^{(1)} \right\rangle + O(\hbar^2)$$
$$A \star B = A \cdot B + \hbar \left\langle \Delta_+ \middle| A^{(1)} \otimes B^{(1)} \right\rangle + O(\hbar^2)$$

and reinserting a $\hbar^{-1}$ into the interaction $\mathcal{V}_g$ one finds [5]

$$\lim_{\hbar \to 0} \mathcal{R}_{\hbar^{-1}\mathcal{V}_g}(A)$$
$$= A + \left\langle \underbrace{i(\Delta_F - \Delta_+)}_{G_{\mathrm{r}}} \middle| \mathcal{V}_g^{(1)} \otimes A^{(1)} \right\rangle + \left\langle G_{\mathrm{r}} \middle| \mathcal{V}_g^{(1)} \otimes \left\langle G_{\mathrm{r}} \middle| \mathcal{V}_g^{(1)} \otimes A^{(1)} \right\rangle \right\rangle + O((\mathcal{V}_g)^3)$$

There are two interesting results appearing in the classical limit analysis: First of all, the classical limit of $\mathcal{R}_{\hbar^{-1}\mathcal{V}_g}(A)$ exists although there is an inverse power of $\hbar$ in the interaction. Such a statement is not true e.g. for the generators of the interacting algebra $\mathcal{S}_{\hbar^{-1}\mathcal{V}_g}(A)$. The well-defined classical limit of the quantum Møller map is assumed due to the fact that no pointwise products appear in the expansion of $\mathcal{R}_{\mathcal{V}_g}$ and the factors $\hbar$ in front of the two-point functions Δ_+ and Feynman propagators Δ_F in (1.17) and (2.10) canceling the negative powers.

Moreover, the cancellation of the higher order contributions (in $\hbar$) to $\mathcal{R}_{\hbar^{-1}\mathcal{V}_g}$ is such that the classical Møller map $\mathcal{R}_{\mathcal{V}_g}$ is assumed in the limit. To prove this one has to calculate the above limit to all orders in $\mathcal{V}_g$. This can conveniently done using a graphical expansion of the S-matrix in terms of Feynman diagrams, see e.g. [Kel10] for a nice derivation of the diagrammatic expansion. In the classical limit, only the "tree-diagrams" remain in the expansion, i.e. the loops that correspond to pointwise powers of Δ_F and Δ_+ disappear.

2.3.5. Comparison to the Path-Integral Approach

A connection to the path-integral formalism can be drawn at this point. The idea behind the path-integral approach is to formulate both the free and interacting quantum field theory in terms of **classical fields** over a measure space. The assertion is, that there exists an oscillating Gaussian measure $\mathcal{D}_F$ with covariance Δ_F on the space of all configurations $\mathcal{E}$

$$\mathcal{D}_F(\varphi) = \frac{1}{\mathcal{N}} \mathrm{e}^{\frac{i}{\hbar} \mathcal{S}_{\mathcal{L}_0}(\varphi)} \, \mathrm{d}\varphi$$

[5] The physical S-matrix of the system is given by the time-ordered exponential of $i\hbar^{-1}\mathcal{S}_{\mathcal{L}}[1]$ in the case of convergent series in $\hbar$. This implies that not every object in perturbation theory has a well defined classical limit, e.g. the formal S-matrix $\mathcal{S}(\hbar^{-1}\mathcal{V}_g)$ itself.

such that the functional Fourier transformation of this measure

$$Z_0(\phi, f) = \int \mathrm{e}^{i\langle\varphi|f\rangle}\mathcal{D}_F(\phi-\varphi), \qquad f \in \mathcal{D}$$

leads to the generating functional Z_0 of the free time-ordered products of the quantum fields in the sense that

$$\frac{\delta^n Z_0(\phi)}{i^n \delta f(x_1)\cdots\delta f(x_n)}\Big|_{f=0} = {}^0\mathcal{T}_n(\Phi_{x_1}\otimes\cdots\otimes\Phi_{x_n})(\phi)$$

where the ${}^0\mathcal{T}_n$ denotes the unrenormalized time-ordered products from definition 2.2.1. For the free scalar field the generating functional can actually be calculated, for on-shell configurations $\phi \in \mathcal{E}_P$:

$$Z_0(\phi, f) = \mathrm{e}^{-\frac{1}{2}\Delta_F(f,f)+i\langle\phi|f\rangle} = \mathcal{S}(\Phi_f)(\phi)\,.$$

where $\mathcal{S}$ is the formal S-matrix that has been defined in the previous section. The path-integral approach has to be modified though, as it comes to non-linear functionals, where normal-ordering has to be taken into account. This is already incorporated in the functional approach to QFT, such that the RHS yields the correct generating functional of time-ordered products.

This becomes important as one wants to define the interacting time-ordered products of linear field with respect to the interaction $\mathcal{V}$ by the famous Feynman-Kac formula

$$Z(\phi, f) = \int \mathrm{e}^{i\langle\varphi|f\rangle}\mathrm{e}^{-\frac{i}{\hbar}\mathcal{S}_\mathcal{V}(\varphi)}\mathcal{D}_F(\phi-\varphi) = \frac{1}{\mathcal{N}}\int \mathrm{e}^{i\langle\varphi|f\rangle}\mathrm{e}^{\frac{i}{\hbar}\mathcal{S}_{\mathcal{L}_0-\mathcal{V}}(\varphi)}\,\mathrm{d}\varphi$$

This is, by the above identification, given by

$$Z(\phi, f) = \mathcal{S}\left(\frac{1}{\hbar}\mathcal{V}+\Phi_f\right)(\phi)\,.$$

This formula coincide with the relative S-matrix (of Φ_f under the interaction $\hbar^{-1}\mathcal{V}$) that is defined in definition 2.3.1 up to the factor $\mathcal{S}(\hbar^{-1}\mathcal{V})$. Moreover the interacting field, that is the first derivative with respect to the linear field

$$\begin{aligned}\frac{\delta}{i\delta f(x)}Z(\phi, f=0) &= \int \frac{\delta}{i\delta f(x)}\Big|_{f=0}\mathrm{e}^{i\langle\varphi|f\rangle}\mathrm{e}^{-\frac{i}{\hbar}\mathcal{S}_\mathcal{V}(\varphi)}\mathcal{D}_F(\phi-\varphi)\\ &= \int \varphi(x)\mathrm{e}^{-\frac{i}{\hbar}\mathcal{S}_\mathcal{V}(\varphi)}\mathcal{D}_F(\phi-\varphi) = \frac{1}{\mathcal{N}}\int \varphi(x)\mathrm{e}^{\frac{i}{\hbar}\mathcal{S}_{\mathcal{L}_0-\mathcal{V}}(\varphi)}\,\mathrm{d}\varphi\\ &= \frac{\delta}{i\delta f(x)}\Big|_{f=0}\mathcal{S}\left(\hbar^{-1}\mathcal{V}+\Phi_f\right)(\phi) = \left(\mathcal{S}\left(\hbar^{-1}\mathcal{V}\right)\cdot_\mathcal{T}\Phi_x\right)(\phi)\end{aligned}$$

coincides with the quantum Møller map up to the same factor.

There are several comments to mention at this point. First of all, the inverse factor $\hbar^{-1}$ is not explicitly written in the relative S-matrix $\mathcal{S}_g(A)$ that was used to define the interacting theory. This is done simply for convenience, and a reinsertion of $\hbar^{-1}$ was shown to result in formal **power** series in $\hbar$ for the interacting observables, that are identified with the quantum Møller map $\mathcal{R}_{\hbar^{-1}\mathcal{V}}$.

A second point concerns the factor $\mathcal{S}(\mathcal{V})$ by that the path integral generating functional differs from the relative S-matrix. This is strongly connected to the adiabatic limit of the whole framework. It is known that for the **vacuum expectation values** of the time-ordered products in the massive theory, it holds the Gell-Mann and Low formula

$$\lim_{g\to 1}\omega_{\mathrm{vac}}(\mathcal{S}_g(\Phi_x)) = \frac{\omega_{\mathrm{vac}}(\mathcal{S}(\mathcal{V})\cdot_{\mathcal{T}}\Phi_x)}{\omega_{\mathrm{vac}}(\mathcal{S}(\mathcal{V}))} = \frac{\delta}{i\delta f(x)}\omega_{\mathrm{vac}}(Z(\phi,f))\Big|_{\phi=0=f}.$$

due to the cluster properties of the massive vacuum state. This is actually not true in general, a discussion on that issue can be found in [Düt97, DF01].

The major difference between both approaches is that the path-integral approach computes the interacting fields under a interaction that is manifestly non-vanishing everywhere, whereas the relative S-matrix (or the quantum Møller map, respectively) computes the interacting fields under a compactly supported interaction $\mathcal{V}_g$, such that the fields coincide with the free fields in the past of the support of $\mathcal{V}_g$.

It is to be expected that both approaches yield the same interacting fields in the adiabatic limit $g \to 1$.

As indicated in the above lines, the path-integral approach intends to define the interacting time-ordered products for a infinitely extended interaction $\mathcal{V}$. The question whether such a path-integral approach can be made rigorous is highly doubtful and lead to a manifestly non-local description of the interacting QFT. This is in particular problematic if one wants to discuss the issue of IR divergences if the propagators are replaced by their finite temperature counterparts, as described in the introduction.

Thus it seems that a more modest approach such as causal perturbation theory is better suited in the quest for a rigorous description of perturbative QFT.

2.4. States in pAQFT

The present section deals with the state space of the interacting theory. In the first part it is shown that every state on the algebra of free field can be lifted to a state on the interacting algebra.

After that we show how one can relate states on both the free and interacting theory with a similar method that was used to related free and interacting observables, namely by using the quantum Møller map $\mathcal{R}_{\mathcal{V}_g}$. This yields a relation between states on $\mathcal{A}^{\mathcal{V}_g}$ and states on $\mathcal{A}$ which coincide in the past of $\mathrm{supp}(g)$. Clearly as we want to remove the spatial cutoff g from the system this description will make sense only for those states, which tend to the free ones in the asymptotic past in an appropriate sense.

This leads to a highly state depended description of the interacting theory. In the case of the vacuum state there exists a well-established method, the Haag-Ruelle scattering theory, see [Haa92], which can be applied to interacting vacuum state of the scalar in the presence of a mass gap.

The case for the KMS state is even worse. Buchholz and Bros [BB02] investigated the asymptotics of an interacting KMS state with quartic interaction in timelike directions and came to the conclusion that it did not converge to the KMS state of the free

theory. The asymptotic state was shown to be quasi-free (in the leading order in time) but its two-point function did not exhibit a mass-shell contribution $\sim \delta(p^2 - m^2)$ in momentum space as in the case of free theory.

This indicates that the scattering setting will not be the appropriate framework to perform a construction of the interacting KMS state. A novel approach to this issue is presented in chapter 4.

2.4.1. States on Formal Power Series of Algebras

A first fact with which has to be dealt with is that the interacting observables are actually formal power series in the interaction $\mathcal{V}_g$ of elements in $\mathcal{A}$. Since we defined the formal S-matrix only for polynomial interactions (see definition 2.1.1) we omit the discussion of the power series in $\hbar$ here, although the formal S-matrices may as well be defined for formal power series of polynomials.

The fact that $\mathcal{A}^{\mathcal{V}_g}(\mathcal{O})$ is a formal power series in $\mathcal{V}_g$ can be made clearer by inserting a number λ in front of the interaction: $\mathcal{A}^{\lambda\mathcal{V}_g}(\mathcal{O})$ is then a formal power series in λ with values in $\mathcal{A}$. This is done purely for the cause of clarity and shall not interpreted as a modification of the interaction in the subsequent discussion.

We will adjust the positivity criterion of states in order to respect that $\mathcal{A}^{\lambda\mathcal{V}_g}$ is a formal power series in the formal parameter λ. There is however a difference in the treatment of the formal parameter λ to $\hbar$: In the case of convergence in λ we may want to consider also coupling constants λ which are negative.

A good positivity criterion is found in [DF99] and will be presented now. By assumption we have $\mathcal{V}_g \in \mathcal{A}_{\mathrm{pol}}$ and we assume that $A \in \mathcal{A}_{\mathrm{pol}}$. Then a linear functional on the algebra of observables is a map

$$\omega : \mathcal{A}^{\lambda\mathcal{V}_g} \to \mathbb{C}[[\lambda]], \qquad \omega(A^* \star A) = \sum_{n=0}^{\infty} a_n \lambda^n \,. \tag{2.23}$$

It is clear that this functional should be normalized and formally positive with respect to the formal expansion as in definition 1.5.1. Since λ might be negative in a convergent case we demand in addition that the first non-vanishing coefficient, say a_k with $k \in \mathbb{N}_0$, in (2.23) is of **even order**: $k \in 2\mathbb{N}_0$. In this case, the first non-vanishing term $a_k\lambda^k$ of the series is positive even though $\lambda < 0$.

Definition 2.4.1 (States in the interacting theory).
A normalized linear functional $\omega : \mathcal{A}^{\lambda\mathcal{V}_g}(O) \to \mathbb{C}[[\lambda]]$ is called a state, if it is formally positive in λ in the following sense:

$$\omega(A^* \star A) = \sum_{n=k}^{\infty} a_n \lambda^n \qquad a_k > 0,\ k \in 2\mathbb{N}_0$$

The space of states on $\mathcal{A}^{\mathcal{V}_g}(\mathcal{O})$ is denoted by $\mathfrak{S}(\mathcal{A}^{\mathcal{V}_g}(\mathcal{O}))$.

In case that one defines the formal S-matrix for formal power series of interactions in $\hbar$, one then assumes that every coefficient in the series in definition 2.4.1 is also a formal power series in $\hbar$ and that a_k with $k \in 2\mathbb{N}_0$ is formally positive in $\hbar$.

This definition implies an important property that is related to the predictability of the interacting theory. The predictions of the theory are, according to the discussion on states in section 1.3.3, expectation values of observables in a state ω that are interpreted as outcomes of measurements of ensembles of systems that are prepared in the state ω.

Clearly physical measurements produce numbers and not formal power series. At this point the concept of perturbation theory comes into play: The interaction is by assumption very small, thus the **truncated theory**, i.e. the computation of expectation values up to a fixed degree λ^N in the interaction is expected to give a good approximation to the physical system. The validity of this statement is to be proven in the experiment which shows stunning agreement even in low orders, e.g. in quantum electrodynamics.

From a conceptual point of view it is important that the notions that have been introduced do not lose their meaning in the case of truncation. The (modified) formal positivity of a state ω on $\mathcal{A}^{\lambda\mathcal{V}_g}$ which is truncated at order N in λ gives the condition

$$\omega(A^* \star A) = a'_{2k}\lambda^{2k} + \sum_{n=2k+1}^{N} a'_n\lambda^n = \lambda^{2k}\left(a_{2k} + \sum_{n=1}^{2k-N} a'_{2k+n}\lambda^n\right) > 0\,, \quad a_{2k} > 0,\ k \in \mathbb{N}.$$

This condition can always be fulfilled, if $|\lambda|$ is chosen small enough, since the condition is equivalent to

$$p(\lambda) = \left(a_{2k} + \sum_{n=1}^{2k-N} a'_{2k+n}\lambda^n\right) > 0\,.$$

The continuity of the polynomial p at $\lambda = 0$ and $p(0) = a_{2k'} > 0$ ensure that $p(\lambda) > 0$ in a neighborhood of $\lambda = 0$. Thus any state in the sense of definition 2.4.1 is positive, if the series is truncated at some order.

Since $\mathcal{A}^{\mathcal{V}_g}(\mathcal{O})$ takes its values in a subalgebra $\mathcal{A}$ in every order in the interaction we obtain lots of states on $\mathcal{A}^{\mathcal{V}_g}(\mathcal{O})$ by simply using states in $\mathfrak{S}(\mathcal{A})$ (definition 1.5.1).

Proposition 2.4.2.
Let ω be a state on $\mathcal{A}$. Then

$$\omega : \mathcal{A}^{\lambda\mathcal{V}_g}(\mathcal{O}) \to \mathbb{C}[[\lambda,\hbar]], \qquad \omega(A) = \sum_{n=0}^{\infty}\omega(a_n)\lambda^n, \quad A = \sum_{n=0}^{\infty} a_n\lambda^n, \quad a_n \in \mathcal{A}$$

is a state on $\mathcal{A}^{\mathcal{V}_g}(\mathcal{O})$.

PROOF. The formal positivity in $\hbar$ is evident, since (formal) positivity is preserved under the restriction from $\mathcal{A}$ to $\mathcal{A}^{\mathcal{V}_g}$. For the formal positivity in the interaction we have to show that

$$\omega(\mathcal{S}_{\lambda\mathcal{V}_g}(A)^* \star \mathcal{S}_{\lambda\mathcal{V}_g}(A)) = \sum_{n=k_0}^{\infty} a_n\lambda^n, \qquad a_{k_0} > 0,\ k_0 \in 2\mathbb{N}_0$$

We start by denoting the formal power series $S_{\lambda\mathcal{V}_g}(A) = \sum_{s=0}^{\infty} s_n\lambda^n$. Then

$$a_n = \sum_{k=0}^{n}\omega(s_k^* \star s_{n-k})$$

The claim follows from the next proposition:

Proposition 2.4.3.
Let ω be a state over $\mathfrak{A}[[\lambda]]$ and $\mathfrak{A}[[\lambda]] \ni A = \sum_{n=0}^{\infty} a_n \lambda^n$. The first non-vanishing term of $\omega(A^*A)$ is of even order $2N$ if and only if $\omega(a_n^* a_n) = 0$ for all $n = 0, \dots, N-1$ and $\omega(a_N^* a_N) > 0$, then

The direction $\Rightarrow$ is trivial. For the other direction we use the Cauchy-Schwarz inequality for states over $\mathfrak{A}$:

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B) \qquad A, B \in \mathfrak{A}\,.$$

Assume $\omega(a_n^* a_n) = 0$ for all $n = 0, \dots, N-1$. Then for $A \in \mathfrak{A}[[\lambda]]$ as in the proposition

$$\omega(A^*A) = \sum_{n=0}^{\infty} \lambda^n \sum_{k=0}^{n} \omega(a_k^* a_{n-k})\,.$$

Every term of odd order in the series up to $2N-1$ vanishes due to

$$\begin{aligned}\left|\sum_{k=0}^{2m-1} \omega(a_k^* a_{2m-1-k})\right|^2 &= \left|\sum_{k=0}^{m-1} \omega(a_k^* a_{m-1-k} + a_{m-1-k}^* a_k)\right|^2 \\ \leq \sum_{k=0}^{m-1} \left|\omega(a_k^* a_{m-1-k} + a_{m-1-k}^* a_k)\right|^2 &\leq \sum_{k=0}^{m-1} 2\omega(a_k^* a_k)\omega(a_{m-1-k}^* a_{m-1-k}) = 0\end{aligned}$$

for $m = 0, \dots, N-1$. Moreover every term of even order up to order $2N-2$ vanishes

$$\sum_{k=0}^{2m} \omega(a_k^* a_{2m-k}) = \omega(a_m^* a_m^*) + \underbrace{\sum_{k=0}^{m-1} \omega(a_k^* a_{m-1-k} + a_{m-1-k}^* a_k)}_{=0} \qquad m = 0, \dots, N-1$$

and by inserting $m = 2N$ in the last equation on sees, that $\sum_{k=0}^{2m} \omega(a_k^* a_{2m-k}) > 0$ by assumption. This proves the intermediary proposition.

Hence, to prove the first proposition one has to check, that

$$\omega(s_n^* \star s_n) \geq 0 \qquad \forall n \in \mathbb{N}$$

which is trivial since $s_n \in \mathcal{A}$ and ω is a state on $\mathcal{A}$. □

We see that the existence of states on the interacting algebra of local observables is shown quite easily. In particular the important class of evaluation functionals ev_ϕ with $\phi \in \mathcal{E}$ can be used to study some explicit examples of states on $\mathcal{A}^{\mathcal{V}_g}(\mathcal{O})$.

2.4.2. States and Symmetries

In this section the notion of spacetime symmetries is studied on states over the algebra of interacting observables. States that are invariant under symmetries play an important role in QFT, as was already mentioned in the section 1.5. For example in translation invariant states we can define energy and momentum operators of the system (via the GNS-representation). The homogeneous KMS states (example 1.5.7) and the vacuum

state (example 1.5.5) were important examples of translation invariant states on the algebra of the free scalar field $\mathcal{A}$.

Since the formal S-matrix is $\mathfrak{P}_+^\uparrow$-invariant (definition 2.2.3) it is clear, that the algebra of interacting observables $\mathcal{A}^{\mathcal{V}}(\mathcal{O})$, seen as sections, inherits the symmetry properties of the $\star$-product:

$$\left(\alpha_{\mathfrak{p}}^{\mathcal{V}}(A \star B)\right)_g = (\alpha_{\mathfrak{p}}(A \star B))_{\alpha^{\mathfrak{p}} g} = (\alpha_{\mathfrak{p}}(A) \star_{\mathfrak{p}} \alpha_{\mathfrak{p}}(B))_{\alpha^{\mathfrak{p}} g} = \left(\alpha_{\mathfrak{p}}^{\mathcal{V}}(A)\right)_g \star_{\mathfrak{p}} \left(\alpha_{\mathfrak{p}}^{\mathcal{V}}(B)\right)_g$$

with the contravariant action $\alpha^{\mathfrak{p}}$ from definition 1.2.6. Choosing a $\mathfrak{p}$-invariant $\star$ in $\mathcal{A}$ yields a $\mathfrak{p}$-covariant algebra $\mathcal{A}^{\mathcal{V}}(\mathcal{O})$ and a $\mathfrak{p}$-invariant algebra $\mathcal{A}^{\mathcal{V}}(M)$.

Hence we can always obtain an explicit $\mathfrak{P}_+^\uparrow$-covariant interacting algebra by using the vacuum two-point function from 1.5.5 as Δ_+ in $\mathcal{A}$. Interacting algebras modeled on different free algebras $\mathcal{A}$ are shown to be equivalent (in the sense of $\star$-products) to the one defined by D_+^{vac}.

The question whether there are invariant states on the interacting algebra is not so easy to answer. We saw that the free states can be restricted to $\mathcal{A}^{\mathcal{V}_g}(\mathcal{O})$ leading to g-depended states. The local net of interacting algebras algebra $\mathcal{A}^{\mathcal{V}}(\mathcal{O})$ and in particular the global algebra have however a different structure, see definition 2.3.5. A state over the global algebra $\mathcal{A}^{\mathcal{V}}(M)$ has to be considered as a section:

$$\omega : \mathcal{A}^{\lambda\mathcal{V}}(M) \to \mathbb{C}[[\lambda, \hbar]], \qquad \omega(A) = g \mapsto \omega((A)_g)$$

which is normalized and positive on the algebra of sections. Every state over $\mathcal{A}(M)$ is a state over $\mathcal{A}^{\mathcal{V}}(M)$ since the normalization and positivity properties are trivially inherited due the fiberwise operations on the bundle $\mathcal{A}^{\mathcal{V}}(M)$. The action of $\mathfrak{P}_+^\uparrow$ on a state ω is given by

$$\left((\alpha_{\mathfrak{p}}^{\mathcal{V}})^* \omega\right)(A) = \omega(\alpha_{\mathfrak{p}}^{\mathcal{V}}(A)) = g \mapsto \omega\left(\alpha_{\mathfrak{p}}(A)_{\alpha^{\mathfrak{p}} g}\right) = g \mapsto (\alpha_{\mathfrak{p}}^* \omega)\left((A)_{\alpha^{\mathfrak{p}}(g)}\right) .$$

We exemplarily discuss the case of translation invariance. As already mentioned we can choose the interacting algebra to be translation invariant by using a translation invariant Hadamard two-point function for $\star$ in $\mathcal{A}$ and in addition we can chose a translation invariant state over $\mathcal{A}$, e.g. the vacuum state ω_{vac} from example 1.5.5.

ω_{vac} yields a state on $\mathcal{A}^{\mathcal{V}}(M)$ as discussed above, however the action $\alpha_{\mathfrak{p}}^{\mathcal{V}}$ with $\mathfrak{p} \in \mathfrak{P}_+^\uparrow$ does not leave ω_{vac} invariant:

$$\begin{aligned}\left((\alpha_{\mathfrak{p}}^{\mathcal{V}})^* \omega_{\text{vac}}\right)(A) &= g \mapsto \omega_{\text{vac}}(A_{\alpha^{\mathfrak{p}}(g)}), \qquad A \in \mathcal{A}^{\mathcal{V}}(M) \\ &\neq g \mapsto \omega_{\text{vac}}(A_g) .\end{aligned}$$

This comes with no surprise since the $\mathfrak{P}_+^\uparrow$-invariance of the theory was explicitly broken by g. For the algebra of observables there was a way around since any observable has support in a finite region $\mathcal{O} \subset M$ and $g \in \mathcal{D}_{\mathcal{O}}$ had only to be chosen such that it is constant on $\mathcal{O}$. This is one of the reasons why the removal the g-dependence on the level of algebras was possible.

For states the notions change. The standard way of defining a state on $\mathcal{A}^{\mathcal{V}}$ would be to define it on a fiber, i.e. by giving a prescription to obtain the (formal power series of) numbers $\omega((A)_g)$ for all $A \in \mathcal{A}^{\mathcal{V}_s}$. Then, in order to obtain an invariant state, one has to send g to a constant function (the **adiabatic limit**). The question whether this sequence will converge can not be answered with the same amount of generality as in the case of the algebras.

In case of the vacuum state a lot of properties concerning the adiabatic limit are already known in the literature which will be summarized in the end of section 3.1. In the case of the KMS state (example 1.5.7) only few properties have been explored up to now. In particular a proof of existence is not worked out so far.

2.4.3. Induced States

The quantum Møller map can also provide an identification of **interacting states** with states on the algebra of free field, similar to the identification of observables which was discussed in section 2.3.4. To illustrate this, let us discuss the issue for a regular interaction $V \in \mathcal{A}_{\text{reg}}$ first. There the interacting algebra can be considered as the algebra of functionals $\mathcal{F}_{\text{reg}}[[\hbar]]$ with the interacting $\star$-product

$$A \star_V B = \mathcal{R}_V^{-1}\left(\mathcal{R}_V(A) \star \mathcal{R}_V(B)\right), \qquad A, B \in \mathcal{F}_{\text{reg}}[[\hbar]] .$$

In this case $\mathcal{R}_V$ provides an isomorphism

$$\mathcal{R}_V : \mathcal{A}_{\text{reg}}^V = (\mathcal{F}_{\text{reg}}[[\hbar]], \star_V) \to \bigvee_{\star}\{\mathcal{R}_V(A) : A \in \mathcal{F}_{\text{reg}}[[\hbar]]\} \subset \mathcal{A}_{\text{reg}} .$$

Thus any state ω on the free algebra of observables $\mathcal{A}_{\text{reg}}$ can be transported to a state on $\mathcal{A}_{\text{reg}}^V$ by $\mathcal{R}_V$.

Definition 2.4.4 (Induced states).
By duality $\mathcal{R}_V$ induces a map

$$\mathcal{R}_V^* : \mathfrak{S}(\mathcal{A}_{\text{reg}}) \to \mathfrak{S}(\mathcal{A}_{\text{reg}}^V), \qquad \mathcal{R}_V^*(\omega) = \omega^V := \omega \circ \mathcal{R}_V.$$

The state $\omega^V = \mathcal{R}_V^*(\omega)$ is called the interacting state induced by ω.

This can be used to define the **Wightman functions** of a local field $A \in \mathcal{A}_{\text{loc}}(\mathcal{O})$ with respect to an interacting state ω^V: Let such a A be of the form

$$A_f(\phi) = \int f(x)\phi(x)^k \,\mathrm{d}x, \qquad f \in \mathcal{D}, \quad \operatorname{supp} f \subset \mathcal{O}, \quad k \in \mathbb{N}$$

The n-point Wightman-function of A_f for an interacting state ω^V is given by

$$W_n^A(f_1, \ldots, f_n) := \omega^V(A_{f_1} \star_V \cdots \star_V A_{f_n}) \equiv \omega(\mathcal{R}_V(A_{f_1}) \star \cdots \star \mathcal{R}_V(A_{f_n})) \tag{2.24}$$

and in addition the interacting time-ordered expectation values are computed using (2.22)

$$\begin{aligned} T_n^A(f_1, \ldots, f_n) &:= \omega^V(A_{f_1} \cdot_{\mathcal{T},\mathcal{V}} \cdots \cdot_{\mathcal{T},\mathcal{V}} A_{f_n}) = \omega \circ \mathcal{R}_V\left(A_{f_1} \cdot_{\mathcal{T}} \cdots \cdot_{\mathcal{T}} A_{f_n}\right) \\ &= \omega\left(\mathcal{S}(V)^{\star -1} \star \left(\mathcal{S}(V) \cdot_{\mathcal{T}} A_{f_1} \cdot_{\mathcal{T}} \cdots \cdot_{\mathcal{T}} A_{f_n}\right)\right) \end{aligned} \tag{2.25}$$

This expressions are in agreement with other approaches to perturbative QFT, see e.g. [Ste93].

In the case of local and polynomial interactions $\mathcal{V}_g$ which induce relativistic interacting algebras $\mathcal{A}^{\mathcal{V}_g}(\mathcal{O})$, the map $\mathcal{R}_{\mathcal{V}_g}$ is not invertible, as we have already mentioned in section 2.3.4. Since the interacting $\star$-product $\star_{\mathcal{V}_g}$ is not available, $\mathcal{R}_{\mathcal{V}_g}$ cannot be an isomorphism of algebras. Nevertheless we can use the equations (2.24) and (2.25) to define the corresponding expectation values W_n^A and T_n^A for local functional interactions $\mathcal{V}_g$.

3. The Adiabatic Limit

In the last chapter it was shown that while removing the cutoff dependence of the algebra of observables was possible in general, the issue of the adiabatic limit for the states is more tricky. Due to the seminal works of Epstein and Glaser [EG73] and Blanchard and Seneor [BS75] an explicit construction of the limit $g \to$ const. for the **vacuum state** is available which yields the existence of an interacting vacuum state on $\mathcal{A}^{\mathcal{V}}(M)$, for more information see the end of section 3.1.

It was also argued that the scattering approach is very likely to be the wrong approach to show the existence of a KMS state due to the fact that long-range correlations of the interacting KMS state spoil an asymptotic description by the KMS state of the free theory. Therefore another approach will developed in section 3.2 which has a close relationship to the so-called canonical approach to perturbative QFT. For this we give a short synopsis of this approach and indicate the main issue within this canonical framework.

We then formulate a new approach to construct the dynamics of the algebra of interacting observables in section 3.2 which brings together both the ideas of the canonical approach and the mathematical rigor that underlies the approach of pAQFT and / or causal perturbation theory. This will ultimately lead to the definition of the interacting KMS state by means of standard arguments from quantum statistical mechanics in chapter 4.

3.1. The Interaction Picture in QFT and the Adiabatic Limit

The construction of the interacting algebra of observables $\mathcal{A}^{\mathcal{V}_g}(\mathcal{O})$ that has been presented in the previous section was derived rather straightforwardly, once one renormalized the time-ordered products. Moreover it is quite concrete: the interacting observables are functionals in $\mathcal{A}$ which can be explicitly computed in terms of Δ_+ and Δ_F.

It arises the question how the present approach is related to the canonical approach to perturbative QFT, i.e. the use of the interacting picture and Dyson's series. This question is rather tricky, since the interaction picture has been shown not to exist in relativistic QFT. This is the famous Haag's theorem [Haa55] which was one of the main inspirations to developed the algebraic approach to QFT in the first place.

In order to show the relation to the still widely used canonical approach we state a formal derivation of Dyson's series in order to highlight similarities in the rigorous approach that follows.

Consider a representation π of the algebra of the free scalar field on a Fock space $\mathfrak{H}$, e.g. the representation induced by the vacuum state, see example 1.5.5. The construction by means of the GNS representation is found in section 1.5. The canonical **fixed time quantum fields** $\varphi_{(t_0,\boldsymbol{x})}$ and $\dot{\varphi}_{(t_0,\boldsymbol{x})}$ at $t_0 \in \mathbb{R}$ are given by the operator valued distributions

$$\mathcal{D}(\Sigma) \ni f \mapsto \varphi(t_0, f) = \int f(\boldsymbol{x}) \pi\left(\Phi_{(t_0,\boldsymbol{x})}\right) \mathrm{d}^3\boldsymbol{x} = \int f(\boldsymbol{x}) \varphi(t_0, \boldsymbol{x}) \, \mathrm{d}^3\boldsymbol{x}$$

on the Cauchy surface $\Sigma = \{t_0\} \times \mathbb{R}^3$, where $\varphi_{(t,\boldsymbol{x})}$ is the linear free field functional (1.2) in the on-shell algebra $\mathcal{A}_{\mathrm{os}}$ and $\dot{\varphi}$ is the time derivative of φ. These fields are subject to the canonical commutation relations (CCR) (at fixed time), i.e.

$$[\varphi_{(t_0,\boldsymbol{x})}, \varphi_{(t_0,\boldsymbol{y})}] = 0 = [\dot{\varphi}_{(t_0,\boldsymbol{x})}, \dot{\varphi}_{(t_0,\boldsymbol{y})}], \qquad [\varphi_{(t_0,\boldsymbol{x})}, \dot{\varphi}_{(t_0,\boldsymbol{y})}] = \delta(\boldsymbol{x} - \boldsymbol{y})$$

which follow directly from the properties of the causal commutator function G_{c} (1.10), when restricted to a Cauchy surface. The translations act by mutually commuting unitary operators $U(t)$ and $T(\boldsymbol{x})$ on $\mathfrak{H}$ and in particular the subgroup of time-translations induces the free dynamics by

$$t \mapsto \varphi(t, \boldsymbol{x}) = U_{t_0}(t) \varphi(t_0, \boldsymbol{x}) U_{t_0}(t)^{-1}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \varphi(t, \boldsymbol{x}) = i[H_0, \varphi(t_0, \boldsymbol{x})] \, .$$

Here H_0 is the Hamiltonian of the free field.

It is assumed, that there exists another one-parameter group of unitary operators $\mathfrak{V}_{t_0}(t)$ on $\mathfrak{H}$ commuting with the spatial translations $T(\boldsymbol{x})$, such the interacting dynamics for the fields is determined by

$$t \mapsto \varphi^V(t, \boldsymbol{x}) = \mathfrak{V}_{t_0}(t) \varphi(t_0, \boldsymbol{x}) \mathfrak{V}_{t_0}(t)^{-1}, \qquad \mathfrak{V}_{t_0}(t_0) = \mathbb{1} \, .$$

Here φ^V stands for the interacting field. Since the equal time CCR are still valid for the fields, the observables of the interacting theory and the free theory can be identified at t_0. The generators of both time-evolutions are assumed to be related by

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \varphi^V(t, \boldsymbol{x}) - \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \varphi(t, \boldsymbol{x}) = i[V, \varphi(t_0, \boldsymbol{x})]$$

where the interaction V is given by a local observable, restricted to Σ:

$$V = \sum_{n=3}^{N} \frac{\lambda_n}{n!} \int \varphi(t_0, \boldsymbol{x})^n \, \mathrm{d}^3\boldsymbol{x} \, . \tag{3.1}$$

It implies that the full dynamics is generated by the commutator of $H = H_0 + V$. The interacting field $\varphi^V(t, \boldsymbol{x})$ can be related to the free field $\varphi(t, \boldsymbol{x})$ at time $t > t_0$ with the help of the unitary

$$W_{t_0}(t) = \mathfrak{V}_{t_0}(t) U_{t_0}(t)^{-1} = \sum_{n=0}^{\infty} i^n \int\limits_{t_0 \le t_1 \le t_2 \le \cdots \le t_n \le t} V_{t_1} \cdots V_{t_n} \, \mathrm{d}t_n \cdots \mathrm{d}t_n \tag{3.2}$$

$$W_{t_0}(t)^{-1} = U_{t_0}(t) \mathfrak{V}_{t_0}(t)^{-1} = \sum_{n=0}^{\infty} (-i)^n \int\limits_{t_0 \le t_1 \le t_2 \le \cdots \le t_n \le t} V_{t_n} \cdots V_{t_1} \, \mathrm{d}t_n \cdots \mathrm{d}t_n$$

$$V_t = U_{t_0}(t) V U_{t_0}(t)^{-1} = \sum_{n=3}^{N} \frac{1}{n!} \int \varphi(t, \boldsymbol{x})^n \, \mathrm{d}^3\boldsymbol{x}$$

where W is a formal power series in the interaction V. The power series expansion is derived from the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}W_{t_0}(t) = iW_{t_0}(t)V_t, \qquad W_{t_0}(t_0) = \mathbb{1}\,.$$

For this operator it holds $W_{t_0}(t_1)W_{t_1}(t) = W_{t_0}(t)$ for $t_0 < t_1 < t$. The series for W is called the Dyson's series.

Thence the interacting field can be written in terms of the free field at time $t > t_0$ by $W_{t_0}(t)$:

$$\varphi^V(t,\boldsymbol{x}) = W_{t_0}(t)\varphi(t,\boldsymbol{x})W_{t_0}(t)^{-1}\,.$$

This is the interaction picture of QM extrapolated to the case of the algebra generated by the CCR for the time-zero quantum fields.

The problem of giving mathematical rigor to the formulas above appear in manifold ways. The first one concerns the definition of the integrand of the interaction Hamiltonian density in (3.1). In order to obtain a well-defined interaction operator, a normal-ordering prescription has to be applied to V, but this has to be done in a state-independent manner. This problem has been solved in [HW02].

The next problem is to restrict the normal-ordered interaction density to a Cauchy surface (here $\{t_0\} \times \mathbb{R}^3$). It has been shown that a restriction of operator-valued distributions over M to operator-valued distributions over Σ is not possible for normal-ordered polynomials of degree larger than one. This problem causes serious trouble in the perturbative expansions of the interacting dynamics: There arise additional UV-divergences in the theory even after renormalization of the time-ordered products. These singularities are called Stückelberg divergences and were discovered in the treatment of QED [Stü51].

Lastly the convergence of the integral in (3.1) is not established. This is intimately related to Haag's theorem [Haa55, HW57], which shows that under the assumptions that $\mathfrak{V}_{t_0}$ commutes with the free spatial translations, then the interaction V must coincide with H_0 up to a constant. Phrased differently: Under the assumption that $\mathfrak{V}_{t_0}$ actually implements a non-trivial interaction, then there is no translation invariant eigenstate to $H_0 + V$ on the Fock space $\mathfrak{H}$. This can be circumvented by either only considering the generators of the time-translations $\delta_V(\varphi(t_0,\boldsymbol{x})) = i[V,\varphi(t_0,\boldsymbol{x})]$. There one cannot proceed to construct the interacting theory by means of $\mathfrak{V}_{t_0}(t)$ though.

Another way to look at the last problem is in the scattering setting of perturbative QFT. There the interacting and the free systems are assumed to coincide at $t_0 \to -\infty$, i.e. we shift the Cauchy surface Σ to $t_0 \to -\infty$ and replace the operators $W_{t_0}(t)$ by $W_{-\infty}(t)$, which yields

$$\varphi^V(t,\boldsymbol{x}) = W_{-\infty}(t)\varphi(t,\boldsymbol{x})W_{-\infty}(t)^{-1}\,.$$

Using a formal time-ordering operator T, this expression can be rewritten in terms of

the (physical) S-matrix of the system $S(V) = \lim_{t\to\infty} W_{-\infty}(t)^{-1}$:

$$\begin{aligned}\varphi^V(t,\boldsymbol{x}) &= \underbrace{W_{-\infty}(+\infty)W_{+\infty}(t)}_{=W_{-\infty}(t)}\varphi(t,\boldsymbol{x})W_{-\infty}(t)^{-1} = W_{-\infty}(\infty)\left(W_{+\infty}(t)\varphi(t,\boldsymbol{x})W_{-\infty}(t)^{-1}\right)\\ &= W_{-\infty}(\infty)T\left(W_{-\infty}(\infty)^{-1}\varphi_{(t,\boldsymbol{x})}\right) = S(V)^{-1}T\left(\varphi(t,\boldsymbol{x})S(V)\right) \qquad (3.3)\end{aligned}$$

where T denotes the time-ordering for operator-valued functions[6] and the physical S-matrix is defined by

$$S(V) = W_{-\infty}(+\infty)^{-1} = T\mathrm{e}^{-i\int_{-\infty}^{\infty} V_t\,\mathrm{d}t}\,.$$

This does not solve the problem in general, but shifts it to the question of how fast the interacting fields converge to the free fields in the asymptotic region. This leads to the famous LSZ asymptotic conditions [LSZ57], which could be proven to hold in the Haag-Ruelle scattering theory [Haa58, Rue62].

It was Bogoliubov in [BS80] who recognized that the formal expansions of (3.3) agree to those using the formal S-matrix $\mathcal{S}(-\mathcal{V}_g)$ (definition 2.2.3), or the respective quantum Møller map $\mathcal{R}_{-\mathcal{V}_g}$ (definition 2.3.8) in the limit:

$$S(V) = \lim_{g\to 1}\pi\left(\mathcal{S}(-\mathcal{V}_g)\right) \qquad (3.4)$$

$$\varphi^V(t,\boldsymbol{x}) = \lim_{g\to 1}\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0}\pi\left(\mathcal{S}_{-g}(\lambda\Phi_{(t,\boldsymbol{x})})\right) = \lim_{g\to 1}\pi\left(\mathcal{R}_{-\mathcal{V}_g}(\Phi_{(t,\boldsymbol{x})})\right)\,. \qquad (3.5)$$

The sign that appears in front of the interaction is due to the change of the Hamiltonian and Lagrangian picture, in which the interaction picks up a minus sign.

The above argumentation in (3.4) and (3.5) is very formal though, since both sides of the equations are ill-defined (in general). However, the RHS gives a starting point for the mathematical discussion. The limit as g tends to a constant function is called the **adiabatic limit** and can be made precise by the following sequence: Let $\{g_n \in \mathcal{D}_{B_n}, n = 2,\ldots\}$ be a sequence of test functions $g_n(x) = g_1(n^{-1}x)$ where g_1 equals unity on the Euclidean ball B_1 in M. Then $\lim_{n\to\infty} g_n = 1$ in the topology of $\mathcal{E}$.

The question whether the limiting procedure in (3.5) can be made rigorous has been discussed in [EG73, EG76, BS75] and more recently in [BF00, Sch89]. Until today there have been three kinds of limiting notions introduced:

- The operator $\mathcal{S}(\mathcal{V}_g)$ tends to a unitary operator on $\mathfrak{H}$ as $g \to 1$: the **strong** adiabatic limit
- The expectation values of (products of) interacting fields (in fixed state) converge to numbers as $g \to 1$: the **weak** adiabatic limit
- The Haag-Kastler net of algebras, generated by the $\mathcal{S}_g(A)$, or by the $\mathcal{R}_{\mathcal{V}_g}$ respectively, tends to a Haag-Kastler net of algebras as $g \to 1$: **algebraic** adiabatic limit

[6] The relation between the formal time-ordering operator T for operators on a Hilbert space and the time-ordered product of off-shell functionals is elucidated in remark 3.2.2.

The following results concerning the adiabatic limit in relativistic QFTs have been established in the literature:

- In the case of the vacuum representation of the scalar field with mass $m^2 > 0$ the strong adiabatic limit exists under further renormalization constraints (i.e. conditions on the formal S-matrix, that guarantee a suitable wave-function and mass renormalization). This is worked out in the famous papers of Epstein and Glaser [EG73, EG76].

- Furthermore it is known, that the strong adiabatic limit of the formal S-matrix of QED does not exist in the adiabatic limit due to infrared divergences. A method to circumvent the IR divergences, in order to obtain physical predictions of the theory, is the introduction of so-called inclusive cross sections [JR54, YFS61].

- Again in the vacuum representation, the weak adiabatic limit of the scalar field with mass $m^2 > 0$ and also for theories with massless particles such as QED or the massless scalar field with polynomial interaction of even type exist for operator and time-ordered products of the interacting fields. [EG73, EG76, BS75].

- Steinmann proved in [Ste95] that **if** the KMS states exists on the algebra $\mathcal{A}^{\mathcal{V}}$ (in the adiabatic limit) and fulfills a certain cluster property in spacelike directions, then the KMS state is unique. An existence proof is not available so far.

- The algebraic adiabatic limit of the interacting theory was shown to exist for all interactions and $m^2 \in \mathbb{R}$ in [BF00] and crucial parts of the construction have already been presented in definition 2.3.5 and below.

There is also a drawback in the algebraic adiabatic limit framework, though. Due to the fact that it is state-independently constructed, it does not guarantee the existence of interesting states over the algebra of interacting observables, namely translation invariant states or even a vacuum state.

This will be compensated in chapter 4 where the translation invariant KMS and the vacuum state will be explicitly constructed on the interacting algebra.

3.2. The Interacting System in a Time-Slice

The non-existence of the interaction picture created a big problem for rigorous approaches to a perturbative treatment of relativistic QFT. A way to circumvent Haag's theorem and to derive the interacting theory in a perturbative manner was presented in section 2.3 and is called causal perturbation theory. Due to its geometric geometric construction, it could even be generalized to systems in a curved background, see [BF00, BF09].

On the other hand, there is a huge amount of mathematical well-established techniques emerging from the perturbation theory of Hamiltonian or Liouvillean dynamics, which is a part of the theory of dynamical systems. This approach gives a very precise

meaning to the interaction picture of quantum mechanics. The most frequently studied systems are dynamical systems of C^*- or W^*-algebras (von-Neumann algebras), see [BR02b] or [DJP03] for more information.

The question whether there is common framework to accommodate both the structures from the Hamiltonian treatment and the geometric constructions in Lagrangian QFT appears to be very natural. A satisfactory answer to this issue has not been given though.

An approach to answering is developed within this section. The first step in this manner is to take the validity of the time-slice axiom (QFT3) for the on-shell algebra $\mathcal{A}_{\mathrm{os}}^{\mathcal{V}}$ of the interacting theory [CF09] more seriously. It suggests that instead of constructing the algebra of interacting observables in an arbitrary region $\mathcal{O} \subset M$ it suffices to construct it in a time-slice of M, i.e. a neighborhood of a Cauchy surface.

More concretely we are going to construct the algebra of observables contained in a region $\mathcal{O} \subset \Sigma_\epsilon$, where Σ_ϵ is a neighborhood of a Cauchy surface. In that algebra a modified Hamiltonian dynamical system approach is possible. To this avail choose the following cover of M:

$$M = M_+ \cup \Sigma_\epsilon \cup M_-, \qquad M_\pm = (\pm\epsilon, \pm\infty) \times \mathbb{R}^3, \quad \Sigma_\epsilon = (-\epsilon, \epsilon) \times \mathbb{R}^3 \,.$$

The "time-slice" Σ_ϵ is an ϵ-neighborhood of the Cauchy surface $\Sigma = \{0\} \times \mathbb{R}^3$. A corresponding partition of unity over M is given by

$$\begin{aligned} \chi + \chi_- + \chi_+ = 1, \qquad & \mathrm{supp}(\chi_\pm) \subset (\pm\epsilon, \pm\infty), \\ \mathrm{supp}(\chi) \subset (-2\epsilon, 2\epsilon), \qquad & \chi(t) = 1 \qquad \text{for } |t| < \epsilon \,. \end{aligned}$$

A two-dimensional representation of this situation is given in figure 3. The interaction Lagrangian $\mathcal{V}_g$ is split using the partition of unity

$$\mathcal{V}_g = \mathcal{V}_{g\chi_+} + \mathcal{V}_{g\chi} + \mathcal{V}_{g\chi_-}$$

and the causal factorization of the relative S-matrices $\mathcal{S}_g(A)$ for $A \subset \mathcal{A}_{\mathrm{os}}(\mathcal{O})$ yields

$$\begin{aligned} \mathcal{S}_g(A) = \mathcal{S}_{g(\chi_+ + \chi + \chi_-)}(A) &\overset{(2.17)}{=} \mathcal{S}_{g(\chi+\chi_-)}(A) \\ &\overset{(2.18)}{=} \mathcal{S}_{g\chi}(\mathcal{V}_{g\chi_-})^{-1} \star \mathcal{S}_{g\chi}(A) \star \mathcal{S}_{g\chi_-}(\mathcal{V}_{g\chi_-}) \,. \end{aligned}$$

Since the relative S-matrix is a unitary in $\mathcal{A}_{\mathrm{os}}$ it is evident that the algebras generated by the relative S-matrices $\mathcal{S}_{g\chi}(A)$ with $\mathrm{supp}(A) \subset \mathcal{O}$ and $\mathcal{A}_{\mathrm{os}}^{\mathcal{V}}(\mathcal{O})$ coincide. This is great news concerning the adiabatic limit. Whereas the generator $\mathcal{S}_g(A)$ does not assume a limit in $\mathcal{A}_{\mathrm{os}}$ as $g \to 1$, the limit of the new generator can be given explicitly:

$$\lim_{g \to 1} \mathcal{S}_{(g\chi)}(A) = \mathcal{S}_\chi(A) \in \mathcal{A}_{\mathrm{os}} \,,$$

since it dependeds only on the projection of $g\chi$ onto $J^-(\mathcal{O})$ in the first place. But $C(\mathcal{O}) = \mathrm{supp}(\chi) \cap J^-(\mathcal{O})$ is compact, see figure 3. This implies that for a sequence

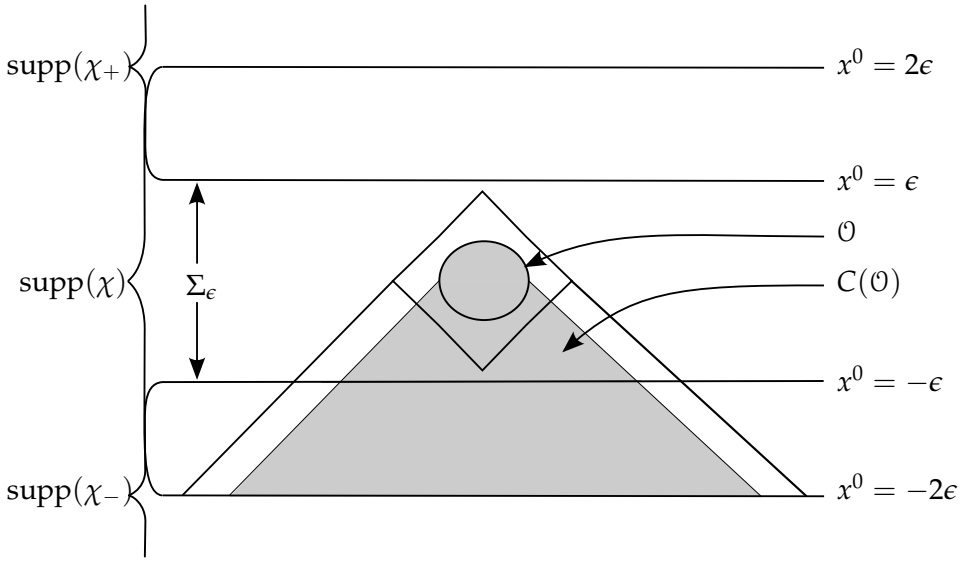


Figure 3.: The cover of M (projected onto the $x^0 - x^1$ plane) and the associated partition of unity. In addition the dependence region $C(\mathcal{O})$ of $\mathcal{S}_{g\chi}(A)$ is drawn as the shaded region.

of test functions g_n which equal unity on a (Euclidean) ball of radius n the above limit is assumed after finitely many elements of the sequence.

With this insight we see that the algebra of interacting observables in $\mathcal{O} \subset \Sigma_\epsilon$ can be constructed by formal power series of elements of the free theory in a compact region located in a slightly larger time-slice $C(\mathcal{O}) \subset \Sigma_{2\epsilon}$.

This will now be exploited to perform a perturbative Hamiltonian description of the interacting system similar to the ideas of the previous section. We will, by abuse of notation, denote the sub-group of translations in $\mathfrak{P}_+^\uparrow$ on $\mathcal{A}$ by α_t (and the interacting time-translations by $\alpha_t^{\mathcal{V}}$ likewise) in this section. By the translation invariance of the S-matrix (definition 2.2.3) we find that the free time-translations act on the generators $\mathcal{S}_\chi(A)$ as

$$\alpha_t \mathcal{S}_\chi(A) = \mathcal{S}_{\alpha_t\chi}(\alpha_t(A)), \qquad (\alpha_t\chi)(x^0) = \chi(x^0 - t)\,.$$

The covariant action of the time-translations for the algebra of interacting observables has been defined by

$$\alpha_t^{\mathcal{V}} \mathcal{S}_\chi(A) = \mathcal{S}_\chi(\alpha_t(A))$$

in (2.21). Following the ideas from the previous section we construct a unitary $W(t)$ which intertwines both time-translations, see (3.2). This amounts to

$$\alpha_t^{\mathcal{V}}(\mathcal{S}_\chi(A)) = W(t) \star \alpha_t(\mathcal{S}_\chi(A)) \star W(t)^{\star -1}\,. \tag{3.6}$$

Solving for $W(t)$ in this equation requires the introduction of a spatial cutoff which is denoted by $h \in \mathcal{D}(\mathbb{R}^3)$ with $h = 1$ in a neighborhood of the biggest spatial extent of $\mathcal{O} \subset \Sigma_\epsilon$. The solution of (3.6) where χ is replaced by $h\chi$ will be denoted by $W_h(t)$.

Moreover one uses that the difference of the time-shifted interaction with the original expression can be written as

$$(\alpha_t\chi) - \chi = \rho_t^+ - \rho_t^-, \qquad \mathrm{supp}(\rho_t^\pm) \cap J^\mp(\mathcal{O}) = \varnothing \,.$$

Let $\delta > 0$ be such that $\mathcal{O} + t \subset \Sigma_\epsilon$ for all $|t| < \delta$. Such a δ can always be found by choosing an appropriate $\mathcal{O}$. We find

$$\begin{aligned}\alpha_t\mathcal{S}_{h\chi}(A) = \mathcal{S}_{h\alpha_t(\chi)}(\alpha_t(A)) = \mathcal{S}_{h(\chi+\rho_t^+ + \rho_t^-)}(\alpha_t(A)) &\overset{(2.17)}{=} \mathcal{S}_{h(\chi+\rho_t^-)}(\alpha_t(A)) \\ &\overset{(2.18)}{=} \underbrace{\mathcal{S}_{h\chi}(\mathcal{V}_{h\rho_t^-})^{\star-1}}_{=W_h(t)^{\star-1}} \star \underbrace{\mathcal{S}_{h\chi}(\alpha_t(A))}_{\alpha_t^{\mathcal{V}}(\mathcal{S}_{h\chi}(A))} \star \mathcal{S}_{h\chi}(\mathcal{V}_{h\rho_t^-}) \\ W_h(t) = \mathcal{S}_{h\chi}(\mathcal{V}_{h\rho_t^-}) &\end{aligned} \tag{3.7}$$

for all t with $|t| < \delta$. Since $\mathcal{V}_{h\rho_t}$ is real, $W_h(t)$ a unitary in $\mathcal{A}_{\mathrm{os}}$. By (2.17) one sees that $W_h(t)$ depends on the projection of h onto $J^-\,\mathrm{supp}(\rho_t^-)$. Therefore the limit $h \to 1$ of $W_h(t)$ is not assumed in $\mathcal{A}_{\mathrm{os}}$. Its adjoint action is still defined through the LHS of (3.6).

Due to the defining equation (3.6) $W_h(t)$ fulfills a co-cycle relation:

Proposition 3.2.1.
The map $t \mapsto W_h(t)$ satisfies

$$W_h(t+s) = W_h(t) \star \alpha_t(W_h(s)) \tag{3.8}$$

for t, s with $|t|, |s| < \delta$. It follows that $t \mapsto W_h(t)$ can be uniquely extended to any $t \in \mathbb{R}$.

PROOF. Let $\Theta^- = \theta(-t)$, where θ is the Heaviside step function on $\mathbb{R}$. Then $\rho_t^- = \Theta^-(\alpha_t(\chi) - \chi)$ and for t, s sufficiently small we have

$$\rho_{t+s}^- = \Theta^-(\alpha_{t+s}(\chi) - \chi) = \Theta^-(\alpha_t(\chi) - \chi) + \Theta^-(\alpha_{t+s}(\chi) - \alpha_t(\chi)) = \rho_t^- + \alpha_t(\rho_s^-) \,.$$

We then find

$$W_h(t)^{\star-1} \star W_h(t+s) = \mathcal{S}_{h\chi}(\mathcal{V}_{h\rho_t^-})^{\star-1} \star \mathcal{S}_{h\chi}(\mathcal{V}_{h\rho_t^- + h\alpha_t(\rho_s^-)}) = \mathcal{S}_{h\chi + h\rho_t^-}(\mathcal{V}_{h\alpha_t(\rho_s^-)}).$$

Using $\chi + \rho_t^- = \alpha_t(\chi) - \rho_t^+$ and the fact that the support of ρ_t^+ lies in the causal future of $\alpha_t(\rho_s^-)$, the factorization rule (2.17) yields for the RHS:

$$\mathcal{S}_{h\alpha_t(\chi) + h\rho_t^+}(\mathcal{V}_{h\alpha_t(\rho_s^-)}) = \mathcal{S}_{h\alpha_t(\chi)}(\mathcal{V}_{h\alpha_t(\rho_s^-)}) = \alpha_t\left(\mathcal{S}_{h\chi}(\mathcal{V}_{h\rho_s^-})\right) = \alpha_t(W_h(s)) \,.$$

This shows the first claim. For the second claim we write

$$\mathbb{R} \ni t = \sum_{k=1}^n t_k, \qquad |t_k| < \delta \tag{3.9}$$

Then $W_h(t)$ with $t \in \mathbb{R}$ is defined by recursively using (3.8):

$$\begin{aligned} W_h(t) &= W_h(t_1) \star \alpha_{t_1}(W_h(t_2)) \star \alpha_{t_1+t_2}(W_h(t_3)) \star \cdots \star \alpha_{t-t_n}(W_h(t_n)) \\ &= W_h(t_1) \star \prod_{k=1}^{n-1}{}^{\star} \alpha_{t_1+\cdots+t_k}(W_h(t_{k+1})) \,. \end{aligned}$$

This shows the existence of $W_h(t)$. For the uniqueness consider another sum $t = \sum_{k=1}^{m} s_m$ with $|s_m| < \delta$ leading to a possibly different co-cycle

$$W_h^1(t) = W_h(s_1) \star \alpha_{s_1}(W_h(s_2)) \star \alpha_{s_1+s_2}(W_h(s_3)) \star \cdots \star \alpha_{t-s_m}(W_h(s_n)) \,.$$

From the above two expansions we construct a third one

$$t = \sum_{i=1}^{n}\sum_{j=1}^{m} u_{ij}, \qquad 2u_{ij} = t_i + s_j, \qquad |u_{ij}| < \delta \,,$$

which is strictly finer than the above two, defining a third unitary $W_h^3(t)$. The strategy now is to prove that any refinement of a fixed partition of t results in the same co-cycle $W_h(t)$, thereby showing that $W_h(t)$ and $W_h^1(t)$ both coincide with $W_h^2(t)$, thus with each other.
For this consider a refinement of the initial partition $t = \sum_{k=1}^{n} t_k$, given by

$$t = \sum_{k=1}^{n} t_k + \sum_{k=1}^{m} v_k, \qquad |v_k| < \delta, \qquad \sum_{k=1}^{m} v_k = 0 \,.$$

Without loss of generality we choose an ordering of the v_k such that the partial sums

$$|V_\ell| < \delta, \qquad \text{where} \quad V_\ell = \sum_{k=1}^{\ell} v_k \,. \tag{3.10}$$

This can be done recursively: Let $\{v_1', \ldots, v_n' \in (-\delta, \delta), \sum_{k=1}^{m} v_k' = 0\}$ be the unordered partition. Take a positive element out of it and call it $v_1 > 0$. Then there must exist an element $v_{n_0}' < 0$, since all terms add up to zero. Define $v_2 = v_{n_0}'$. Clearly $|v_1 + v_2| < \delta$. If $v_1 + v_2 \gtrless 0$ then there must exist an element $v_{n_1}' \lesseqgtr 0$ in the remaining partition. Define this one as $v_3 = v_{n_1}'$. Again $|v_1 + v_2 + v_3| < \delta$. Proceed in the same way until (3.10) is accomplished.

For the co-cycle $W_h'(t)$ defined by the refined partition of t it holds

$$\begin{aligned} W_h'(t) =& W_h(t_1)\alpha_{t_1}(W_h(t_2))\alpha_{t_1+t_2}(W_h(t_3)) \cdots \alpha_{\sum_{k=1}^{n-1} t_k}(W_h(t_n)) \times \\ & \times \alpha_t \left(W_h(v_1)\alpha_{v_1}(W_h(v_2)) \cdots \alpha_{\sum_{k=1}^{m-1} v_k}(W_h(v_m)) \right) \\ =& W_h(t)\alpha_t \left(W_h(v_1)\alpha_{v_1}(W_h(v_2)) \cdots \alpha_{\sum_{k=1}^{m-1} v_k}(W_h(v_m)) \right) \end{aligned}$$

where in the last line we inserted the unitary $W_h(t)$ defined by the initial partition. Due to (3.10) we find

$$W_h(v_1)\alpha_{v_1}(W_h(v_2)) \cdots \alpha_{\sum_{k=1}^{m-1} v_k}(W_h(v_m)) = W_h(\underbrace{v_1 + v_2}_{V_2})\alpha_{v_1+v_2+v_3} W_h(v_3) \cdots$$

$$= W_h(\underbrace{v_1 + v_2 + v_3}_{V_3})\alpha_{v_1+v_2+v_3} W_h(v_4) \cdots = W_h\left(\sum_{k=1}^{n} v_n\right) = W_h(0) = \mathbb{1} \,.$$

This proves the claim. □

The co-cycle can computed explicitly in terms of a formal power series in its generator K_h, by applying standard methods from [BR02b]. Using (3.8) one can derive a differential equation for W_h:

$$\frac{\mathrm{d}}{i\,\mathrm{d}t}W_h(t) = \frac{\mathrm{d}}{i\,\mathrm{d}s}\Big|_{s=0} W_h(t+s) = W_h(t) \star \alpha_t\left(\frac{\mathrm{d}}{i\,\mathrm{d}s}\Big|_{s=0} W_h(s)\right), \quad \frac{\mathrm{d}}{i\,\mathrm{d}s}\Big|_{s=0} W_h(s) =: K_h \tag{3.11}$$

Since $W_h(t)$ is of the form (3.7) we find for K_h:

$$K_h = \mathcal{S}(\mathcal{V}_{h\chi})^{\star -1} \star \frac{\mathrm{d}}{i\,\mathrm{d}\lambda}\Big|_{\lambda=0} \mathcal{S}\left(\mathcal{V}_{h\chi} - \lambda \mathcal{V}_{h\Theta^-\dot{\chi}}\right) = -\mathcal{R}_{\mathcal{V}_{h\chi}}(\mathcal{V}_{h\Theta^-\dot{\chi}}) \tag{3.12}$$

in the notation of definition 2.3.8. For this we used the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \rho_t^-(x^0) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \theta(-x^0)(\chi(x^0-t)-\chi(x^0)) = -\theta(-x^0)\dot{\chi}(x^0) = -\Theta^-(x^0)\dot{\chi}(x^0)$$

where $\dot{\chi}$ means the first derivative of χ. The differential equation (3.11) can be solved (in the sense of formal power series in the generator) by

$$\begin{aligned} W_h(t) - \mathbb{1} &= i\int_0^t W_h(s) \star \alpha_s(K_h)\,\mathrm{d}s \\ &= i\int_0^t \alpha_s(K_h)\,\mathrm{d}s + i^2 \int\limits_{0\le t_1\le t_2\le t} W_h(t_1)\star\alpha_{t_1}(K_h)\star\alpha_{t_2}(K_h)\,\mathrm{d}t_1\,\mathrm{d}t_2 = \cdots \\ &= \sum_{n=1}^{\infty} i^n \int\limits_{0\le t_1\le\cdots\le t_n\le t} \alpha_{t_1}(K_h)\star\cdots\star\alpha_{t_n}(K_h)\,\mathrm{d}t_1\cdots\mathrm{d}t_n \end{aligned} \tag{3.13}$$

for $t > 0$. By a change of coordinates one finds

$$W_h(t) = \sum_{n=0}^{\infty} (it)^n \int\limits_{0\le s_1\le\cdots\le s_n\le 1} \alpha_{ts_1}(K_h)\star\cdots\star\alpha_{ts_n}(K_h)\,\mathrm{d}s_1\cdots\mathrm{d}s_n$$

which is valid for all $t \in \mathbb{R}$.

The above proof of the extension of $W_h(t)$ from infinitesimal t to arbitrary values could have been done in a simpler way by defining $W_h(t)$ for any $t \in \mathbb{R}$ by the RHS of equation (3.13). The method in the proof does not use any formal power series expansion or differentiability condition on $W_h(t)$ though.

The inverse $W_h(t)^{\star -1}$ is computed along the same lines as above:

$$W_h(t)^{\star -1} = \sum_{n=0}^{\infty} (-it)^n \int\limits_{0\le s_1\le\cdots\le s_n\le 1} \alpha_{ts_n}(K_h)\star\cdots\star\alpha_{ts_1}(K_h)\,\mathrm{d}s_1\cdots\mathrm{d}s_n\,.$$

Note that the ordering in the integrand is reversed to the ordering in $W_h(t)$.

Remark 3.2.2.
A very important comment at this point concerns the "time-ordered" products in the above perturbative expansion of the co-cycle $W_h(t)$. They are very different compared

to the time-ordered products that are introduced in section 2.2. The time-ordered product of pAQFT which is denoted by $\cdot_{\mathcal{T}}$ is defined as a binary operation on local functional in the off-shell setting. It is shown that a binary product for smeared Wick-polynomials

$$A_f \cdot_{\mathcal{T}} A_g = \int f(x)g(y) \begin{cases} A_x \star A_y & x^0 > y^0 \\ A_y \star A_x & y^0 > x^0 \end{cases} \mathrm{d}x\,\mathrm{d}y$$

becomes meaningless for operator-valued distributions on the Hilbert space of the free theory. This is intimately related to the fact, that higher-order Wick polynomials cannot be restricted to spacelike hypersurfaces. In the book of Scharf [Sch89] this is called the incorrect splitting of distributions.

Yet another notion of time-ordering is sometimes used in order to simplify the notation the above co-cycle expansion. There one introduces a time-ordering operator for operator valued (or algebra-valued functions). Let $t \mapsto A_t$ be function with values in an algebra $\mathfrak{A}$. Moreover let π be a permutation of $\{1, \dots, n\}$ such that $t_{\pi(1)} \leq \cdots \leq t_{\pi(n)}$. Then the time-ordering operator is a map

$$T_n : \mathbb{R}^n \to \mathfrak{A}, \quad T_n(t_1, \dots, t_n) = A_{t_{\pi(n)}} \cdots A_{t_{\pi(1)}} .$$

An inverse time-ordering operator $\hat{T}$ can be defined in a similar way. In this notation we find

$$W_h(t) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \hat{T}_n \int_0^t \cdots \int_0^t \alpha_{t_1}(K_h) \star \cdots \star \alpha_{t_n}(K_h)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n = \hat{T} \exp_\star^{i \int_0^t \alpha_s(K_h)\, \mathrm{d}s}$$

where $\hat{T}$ is the direct sum of the $\hat{T}_n$. An equivalent formula can be found for the inverse $W_h(t)^{\star -1}$. This notation is used in the corresponding publication in [FL]. We do not intend to use the time-ordering symbols in this work to avoid confusion. ♢

A closer look on the generator

$$-K_h = \mathcal{R}_{\mathcal{V}_{h\chi}} \left(\mathcal{V}_{h\Theta^- \dot{\chi}} \right) = \int \mathcal{R}_{\mathcal{V}_{h\chi}}(\mathcal{V}(\Phi_x)) \Theta^-(x^0) \dot{\chi}(x^0) h(\boldsymbol{x})\, \mathrm{d}x^0\, \mathrm{d}^3\boldsymbol{x}$$

reveals that it is the image of the interaction Lagrangian $\mathcal{V}$ under the quantum Møller map $\mathcal{R}_{\mathcal{V}_g}$. Loosely speaking it can be considered as the interaction Lagrangian $\mathcal{V}_{h\Theta^- \dot{\chi}}$ in the interaction picture (with interaction $\mathcal{V}_{h\chi}$). In addition $\mathcal{R}_{\mathcal{V}_{h\chi}}(\mathcal{V}(\Phi_x))$ is not arbitrarily smeared in the time-component, but with the test-function that integrates to one:

$$\Theta^- \dot{\chi} \in \mathcal{D}(\mathbb{R}), \qquad \int_{-\infty}^{\infty} \Theta^-(t) \dot{\chi}(t)\, \mathrm{d}t = 1 .$$

Thus the generator K_h is a **time-average** over the interval in the negative real axis where χ is non-constant, see figure 4.

The full interacting dynamics on $\mathcal{A}_{\mathrm{os}}^{\mathcal{V}}$

$$\alpha_t^{\mathcal{V}}(A) = W_h(t) \star \alpha_t(A) \star W_h(t)^{\star -1}$$

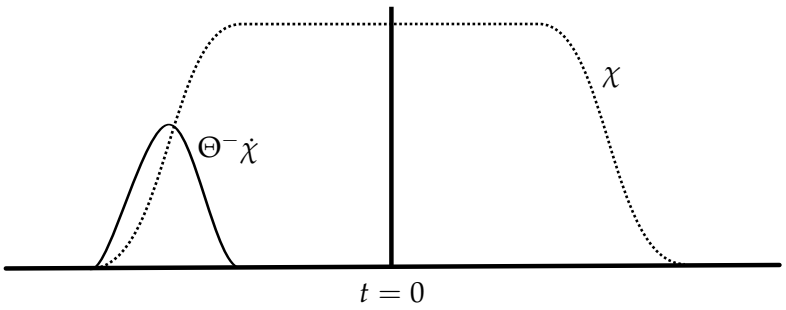


Figure 4.: The cutoff function χ as a dashed line. The first derivative of χ for $t < 0$, i.e. $\Theta^- \dot{\chi}$ as solid line.

can be explicitly calculated by either multiplying the formal power series of $W_h(t)$ with its inverse or by using

$$\alpha_t^{\mathcal{V}}(A) - \alpha_t(A) = \left(\alpha_t^{\mathcal{V}} \circ \alpha_{-t} - \mathbb{1}\right) \alpha_t(A) = \left(\mathrm{Ad}_\star\left(W_h(t)\right) - \mathbb{1}\right) \alpha_t(A)$$

where $\mathrm{Ad}_\star$ is the adjoint action with respect to the $\star$-product and expanding the bracket in the following fashion

$$\begin{aligned}
&\left(\mathrm{Ad}_\star\left(W_h(t)\right) - \mathbb{1}\right)(B) = i \int_0^t \frac{\mathrm{d}}{i\,\mathrm{d}s} \mathrm{Ad}_\star\left(W_h(s)\right)(B)\,\mathrm{d}s \\
=& i \int_0^t W_h(s)\alpha_s(K_h)BW_h(s)^{\star-1} - W_h(s)B\alpha_s(K_h)W_h(s)^{\star-1}\,\mathrm{d}s \\
=& i \int_0^t \mathrm{Ad}_\star\left(W_h(s)\right)\left[\alpha_s(K_h), B\right]_\star \mathrm{d}s \\
=& i \int_0^t \left[\alpha_s(K_h), B\right]_\star \mathrm{d}s + i^2 \int\limits_{0\le t_1\le t_2\le t} \mathrm{Ad}_\star\left(W_h(t_1)\right)\left[\alpha_{t_1}(K_h), \left[\alpha_{t_2}(K_h), B\right]_\star\right]_\star \mathrm{d}t_1\,\mathrm{d}t_2 \\
=& \cdots = \sum_{n=1}^{\infty} i^n \int\limits_{0\le t_1\le\cdots\le t_n\le t} \left[\alpha_{t_1}(K_h), \left[\alpha_{t_2}(K_h), \ldots, \left[\alpha_{t_n}(K_h), B\right]_\star \ldots\right]_\star\right]_\star \mathrm{d}t_1 \cdots \mathrm{d}t_n\,.
\end{aligned}$$

Inserting the above ansatz in the result yields

$$\begin{aligned}
\alpha_t^{\mathcal{V}}(A) =& \alpha_t(A) + \\
&+ \sum_{n=1}^{\infty} i^n \int\limits_{0\le t_1\le\cdots\le t_n\le t} \left[\alpha_{t_1}(K_h), \left[\alpha_{t_2}(K_h), \ldots, \left[\alpha_{t_n}(K_h), \alpha_t(A)\right]_\star \ldots\right]_\star\right]_\star \mathrm{d}t_1 \cdots \mathrm{d}t_n\,.
\end{aligned}$$

which is a well-known formula in quantum statistical mechanics and dynamical systems. It is noteworthy that such a formula has not been shown to hold in QFT due to the severe UV-divergences that appear, if one restricts the attention to observables on a Cauchy surface Σ, as indicated in 3.1.

A formal comparison of the present situation with the canonical approach from section 3.1 is obtained in the limit, where χ tends to the characteristic function of $[-\epsilon, \epsilon]$

and ϵ is very small. In this limit the quantum Møller map tends to the identity mapping:

$$K_h = -\mathcal{R}_{\mathcal{V}_{h\chi}}\left(\mathcal{V}_{h\Theta^-\dot{\chi}}\right) \longrightarrow -\mathcal{V}_{h\Theta^-\dot{\chi}} \longrightarrow -\int \mathcal{V}(\Phi_x)h(\boldsymbol{x})\delta(x^0+\epsilon)\,\mathrm{d}x$$

since the region in which K_h is different from the RHS of the last equation, that is $\operatorname{supp}(h\chi) \cap J^-(\operatorname{supp}\Theta^-\dot{\chi})$, tends to zero. Finally as $\epsilon \to 0$ we find that the generator of $W_h(t)$ tends to the spatially smeared fixed-time interaction term (with $t_0 = 0$ here).

$$K_h \longrightarrow -\int \mathcal{V}(\varphi_{(0,\boldsymbol{x})})h(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x} \tag{3.14}$$

which should generate the interacting dynamics in the latter section, see equation (3.1). Notice that the shape of the interaction, apart from the spatial cutoff h, is exactly the same, only the sign has changed due to the change of Hamiltonian and Lagrangian picture.

As already mentioned in the last section, such a limiting procedure leads to very singular objects whose mathematical description is status is not clear. In particular the RHS of (3.14) is neither an operator on the Hilbert space of the free theory nor an element of the off-shell algebra of free theory $\mathcal{A}$.

Thus the present approach can be interpreted as a regularization of the treatment of interacting dynamics that has a clear-cut connection to the covariant approach which is described in section 2.3. It yields a well-defined unitary co-cycle $W_h(t) \in \mathcal{A}_{\mathrm{os}}$ with a generator $K_h \in \mathcal{A}_{\mathrm{os}}$ which is the time-averaged, spatially smeared interaction (3.12).

The price that is to pay is that the description of the interacting system is only available in terms of formal power series in the interaction. In case that an interacting theory was defined in a non-perturbative fashion using either relative S-matrices or a unitary co-cycle $W_h(t)$ that solves (3.8), then the present framework can still be used to relate both approaches. This is true thanks to the time-slice axiom.

Let us summarize the previous accomplishments: We presented a novel way to construct the interacting theory – by means of the interacting dynamics $\alpha_t^{\mathcal{V}}$ – in a time-slice Σ_ϵ. The validity of the time-slice axiom QFT3 in pAQFT (see proposition 2.3.7) guaranteed that the on-shell algebra of the interacting scalar field $\mathcal{A}_{\mathrm{os}}^{\mathcal{V}}(\mathcal{O})$ can be constructed by elements in $\mathcal{A}_{\mathrm{os}}^{\mathcal{V}}(\mathcal{O})$ such that $\mathcal{O} \subset \Sigma_\epsilon$ and $\mathcal{O} \subset D(\mathcal{O})$, where D denotes the domain of dependence (see definition 1.1.2).

A co-cycle W_h, relating both the free and the interacting dynamics, has been constructed as a formal power series in the interaction $\mathcal{V}_{h\chi}$ by using Dyson's series. Here the interaction $\mathcal{V}$ has been cut off in spatial directions by a test function $h \in \mathcal{D}$ that is set to constant value one in a neighborhood of the biggest spatial extent of $\mathcal{O} \subset \Sigma_\epsilon$. If we fix this region to the ball $B_r = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| \leq r\}$, i.e. $h = 1$ in a neighborhood of B_r, then the interacting dynamics of the real scalar field with interaction $\mathcal{V}$ is determined by

$$\alpha_t^{\mathcal{V}}(A) = W_h(t) \star \alpha_t(A) \star W_h(t)^{\star -1} \qquad \forall A \in \mathcal{A}_{\mathrm{os}}, \quad \operatorname{supp} A \subset \mathbb{R} \times B_r\,.$$

Thus the perturbative description of the interacting system is achieved with in the "space-slice" $\mathbb{R} \times B_r$ in a perturbative way, as we announced in the introduction.

In the adiabatic limit, where h is replaced by the constant function on $\mathbb{R}^3$, we know that the adjoint action of the W_h on $\alpha_t(A)$ tends to $\alpha_t^{\mathcal{V}}(A)$, however W_h alone does not define a unitary element in $\mathcal{A}_{\text{os}}$ anymore.

This approach provides a new tool for the study of perturbatively constructed QFTs in Minkowski spacetime. A first example that demonstrates the strength of it will be shown in the next chapter, where the existence of both the interacting vacuum and interacting KMS states can be shown for the interacting algebras of observables. Furthermore a solid connection between the mathematically thoroughly studied field of C^*-dynamical systems and the treatment of interacting relativistic QFTs, which cannot described by normable algebras, is established.

4. Existence of a KMS-State in Perturbative QFT

In this chapter we finally construct the promised KMS state for the interacting dynamics. This will be done by exploiting the existence of a co-cycle W_h that intertwines the interacting dynamics $\alpha_t^{\mathcal{V}}$ with the free one.

The course of action is as follows: In the first section of this chapter we present an outline of the (convergent) perturbation theory of C^*-dynamical systems, which highlights the construction of an interacting KMS state by means of a co-cycle. This area of mathematical physics has been thoroughly studied in the past and standard references are [BR02a, BR02b, AJP06].

The second section is concerned with the immediate application of this method to the interacting system, consisting of a relativistic scalar quantum field in the space-slice $\mathbb{R} \times B_r$. We establish that, for a given KMS state ω_β with respect to the free dynamics α_t, the linear functional

$$A \mapsto \omega_{\beta,h}^{\mathcal{V}}(A) = \frac{\omega_\beta(A \star W_h(i\beta))}{\omega_\beta(W_h(i\beta))} \tag{4.1}$$

defines an interacting KMS state with respect to the interaction $\mathcal{V}$. This fact follows almost immediately from the Araki's analysis for C^*-dynamical systems [Ara73].

The third section is concerned with the question, whether the KMS states (4.1) still exist in the limit as the spatial cutoff h tends to the constant function. A first important result in this direction is a **sufficient condition** for the existence of the the state $\omega_{\beta,h}^{\mathcal{V}}$ in terms of the connected correlation functions of the respective free state ω_β in section 4.3.2.

An extensive analysis of the algebra of observables as well as the massive vacuum and massive KMS state reveals that the interacting KMS state with inverse temperature $0 < \beta < \infty$ and $\beta = +\infty$ (the vacuum state) fulfill the condition from section 4.3.2. The proofs of the statements are found in section 4.3.3 for the vacuum case and in section 4.3.4 for the thermal equilibrium case, in which it is also shown that the limiting functionals maintain the symmetries of their respective free counterparts.

In the last section we define the thermostatic observables of the interacting theory at finite temperature in a finite spatial volume and show their existence in the adiabatic limit.

4.1. Motivation in C^*-Dynamical Systems

The basic object that we will work with is an abstract C^*-algebra $\mathfrak{A}$, as the name C^*-dynamical systems suggests. This is interpreted as the algebra of observables of some physical system, e.g. the Weyl algebra of the scalar field (see section A.2 of the appendix).

Definition 4.1.1 (C^*-dynamical system).
A C^*-dynamical system is a C^*-algebra $\mathfrak{A}$ with a strongly continuous one-parameter group of automorphisms α_t on $\mathfrak{A}$. α_t is called implemented, if there exists a unital *-representation $\pi : \mathfrak{A} \to \mathcal{H}$ and strongly continuous one-parameter group of unitaries $U(t)$ on $\mathcal{H}$ such that

$$\pi(\alpha_t(A)) = U(t)\pi(A)U(t)^{-1} \qquad \forall A \in \mathfrak{A}\,.$$

The conditions on the dynamics are very restrictive, in particular that strong continuity of α_t. The conditions can be significantly weakened (see [DJP03]), but we don't want to go into detail here.

Now we want to look at a system whose dynamics is perturbed in the following sense. To each dynamics α_t we can associate a **generator** δ

$$\alpha_t(A) = \mathrm{e}^{t\delta}A\,.$$

The most prominent case is, if α_t is obtained by a Hamiltonian $H \in \mathfrak{A}$, then $\delta(A) = i[H, A]$. Consider a perturbed dynamics

$$\alpha_t^P(A) = \mathrm{e}^{\delta_P}(A), \qquad \delta_P(A) = \delta(A) + i[P, A]\,.$$

In this case both dynamics can be intertwined by a co-cycle $W(t)$. The following theorem is taken from [BR02b].

Theorem 7 (Perturbed dynamical systems).
Let $P \in \mathfrak{A}$ be a self-adjoint. The map

$$t \mapsto \alpha_t^P(A) = \sum_{n=0}^{\infty} i^n \int\limits_{0\le t_1\le\cdots\le t_n<t} [\alpha_{t_1}(P), [\alpha_{t_2}(P), \cdots [\alpha_{t_n}(P), \alpha_t(A)]]\cdots]\,\mathrm{d}t_1\cdots\mathrm{d}t_n$$

defines a strongly continuous one-parameter group of automorphisms on $\mathfrak{A}$ with generator $\delta_P = \delta + i[P, \cdot]$. The unitary co-cycles defined by

$$\mathfrak{A} \ni W(t) = \sum_{n=0}^{\infty} (it)^n \int\limits_{0\le s_1\le\cdots\le s_n\le 1} \alpha_{ts_1}(P)\cdots\alpha_{ts_n}(P)\,\mathrm{d}s_1\cdots\mathrm{d}s_n$$

intertwine the one-parameter groups α_t and α_t^P, i.e. $\alpha_t^P(A) = W(t)\alpha_t(A)W(t)^{-1}$.

We find that in particular if the dynamics α_t is induced by a Hamiltonian H, then the new dynamics α_t^P is induced by $H + P$. The co-cycle $W(t)$ will now be used to construct a KMS state for α_t^P out of a KMS state for α_t.

Theorem 8 (Perturbed KMS states).
Let ω_β be a KMS state on $\mathfrak{A}$ with inverse temperature β with respect to α_t. Then the function

$$t \mapsto \omega_\beta(AW(t))$$

has an analytic continuation into the strip $\{z \in \mathbb{C} : 0 < \Im(z) < \beta\}$ for every $A \in \mathfrak{A}$ and the linear functional

$$A \mapsto \omega_\beta^P(A) = \frac{\omega_\beta(AW(i\beta))}{\omega_\beta(W(i\beta))}$$

defines a KMS state on $\mathfrak{A}$ with respect to α_t^P.

The last theorem goes back to the work of Araki [Ara73] though it could be generalized to the case in which α_t acts only σ-weakly continuous on a von-Neumann algebra and the perturbation is an essentially self-adjoint operator, affiliated with the von-Neumann algebra [DJP03].

This result will give a guideline to the construction of the interacting KMS state. We have already completed half of the work, since we found the co-cycle $W_h(t)$ analogous to the one in theorem 7. Now we set forth to prove an analogue to theorem 8.

4.2. The Case of Local Interactions with Spatial Cutoff

The outcome of section 3.2 can be summarized as follows: Let $h \in \mathcal{D}(\mathbb{R}^3)$ such that $h = 1$ in a neighborhood of the Euclidean ball $B_r \subset \mathbb{R}^3$ with radius r. Furthermore let $\mathcal{O} \subset \Sigma_\epsilon$ be an open subset of the time-slice Σ_ϵ such that $\mathcal{O} \subset \mathbb{R} \times B_r$. An illustration of this setting is given in figure 5. Then the interacting dynamics $\alpha_t^{\mathcal{V}}$, defined by

$$\alpha_t^{\mathcal{V}}(\mathcal{S}_{\mathcal{V}_{h\chi}}(A)) = \mathcal{S}_{\mathcal{V}_{h\chi}}(\alpha_t(A)), \qquad \operatorname{supp}(A) \subset \mathcal{O}$$

is intertwined with the free dynamics α_t by the co-cycle

$$W_h(t) = \sum_{n=0}^{\infty} (it)^n \int_{\mathfrak{s}_n} \alpha_{ts_1}(K_h) \star \cdots \star \alpha_{ts_n}(K_h)\, \mathrm{d}s_1 \cdots \mathrm{d}s_n$$
$$\mathfrak{s}_n = \{(s_1, \ldots, s_n) \in \mathbb{R}^n : 0 \leq s_1 \leq \cdots \leq s_n \leq 1\}, \qquad K_h = -\mathcal{R}_{\mathcal{V}_{h\chi}}(\mathcal{V}_{h\Theta^- \dot{\chi}})\,.$$

The intertwining relation

$$\alpha_t^{\mathcal{V}}(A) = W_h(t) \star \alpha_t(A) \star W_h(t)^{\star -1}$$

was shown to hold in a neighborhood of $t = 0$ at first, but due to the group law it holds for all on-shell functionals $A \in \mathcal{A}_{\mathrm{os}}$ with $\operatorname{supp}(A) \subset \mathbb{R} \times B_r$. The method to determine the interacting dynamics $\alpha_t^{\mathcal{V}}$ and the co-cycle W_h (for a given interaction $\mathcal{V}$) that is shown above depends only on the cutoff functions h and χ, but is, in particular, state-independent.

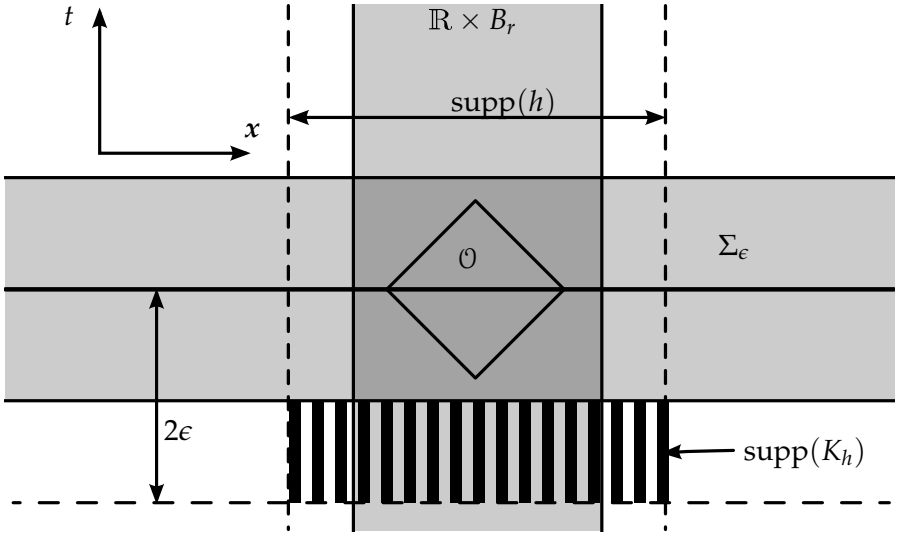


Figure 5.: The time-slice Σ_ϵ and the space-slice $\mathbb{R} \times B_r$ as shaded regions. The support of the generator K_h of W_h is marked as the streaked area.

Using these concepts we will approach the task to prove a version of theorem 8 within this framework, which is achieved in the end of this section. However, this will only be an intermediary result, since the perturbed KMS state that is obtained in this way, is restricted to a system with a finite spatial extent B_r. From the spacetime point of view this means that the interacting state is defined on the algebra of observables which are restricted to the "space-slice" $\mathbb{R} \times B_r$, see figure 5. The construction of the adiabatic limit of the state $\omega^{\mathcal{V}}_{\beta,h}$ – the state that is obtained as h tends to the constant function on $\mathbb{R}^3$ – needs more evolved methods due to possible convergence issues. It is postponed to the next section.

We start from the homogeneous KMS state ω_β with inverse temperature β for the free dynamics α_t. This state has been explicitly discussed in example 1.5.7. The $\star$-product and the (unrenormalized) time-ordered product $\cdot_{\mathcal{T}}$ on $\mathcal{A}_{\text{os}}$ are obtained by the translation invariant Hadamard two-point function Δ_+ and Feynman propagator Δ_F

$$\Delta_+(x,y) = D^\beta_+(x-y), \qquad \Delta_F(x,y) = D^\beta_F(x-y)$$

where D^β_+ is given in (1.19) and $D^\beta_F = D^\beta_+ + iG_{\text{a}}$ with the advanced fundamental solution from theorem 1. This leads to the well-known form of the Feynman propagator D^β_F at finite temperature:

$$D^\beta_F(x) = \lim_{\varepsilon\to 0} \frac{i}{(2\pi)^4} \int \left(\frac{1}{p^2 - m^2 + i\varepsilon} + 2\pi i \frac{\delta(p^2-m^2)}{\mathrm{e}^{\beta|p_0|} - 1} \right) \mathrm{e}^{-i\eta_{\mu\nu}p^\mu x^\nu} \,\mathrm{d}p \tag{4.2}$$

The homogeneous KMS state ω_β on $\mathcal{A}_{\text{os}}$ is simply the evaluation at $\phi = 0$. [7]

[7] Actually, any Hadamard two-point function Δ_+, that fulfills the KMS condition (definition 1.5.6), will result in an interacting KMS state using the following procedure.

We prove an important proposition which concerns the analytic extension of the correlation functions of the observables $A_1, \dots, A_n$ in ω_β. In C^*-algebraic setting a similar theorem is known to hold as a pure consequence of KMS condition [BR02b]. An extension to algebras of unbounded operators has not been worked out to such an amount of generality, to our best knowledge.

Proposition 4.2.1.
Let ω_β and $\star$ be as above. Then for every $A_1, \dots, A_n \in \mathcal{A}_{\mathrm{os}}$ the functions

$$(t_1, \dots, t_n) \mapsto \omega_\beta(\alpha_{t_1}(A_1) \star \cdots \star \alpha_{t_n}(A_n)) \tag{4.3}$$

have an analytic continuation into

$$\mathfrak{T}_\beta^n = \{(z_1, \dots z_n) \in \mathbb{C}^n : -\beta < \Im(z_i - z_j) < 0 \quad \forall 1 \le i < j \le n\} .$$

and

$$\begin{aligned} &\omega_\beta(\alpha_{t_1}(A_1) \cdots \alpha_{t_k}(A_k)\alpha_{t_{k+1}+i\beta}(A_{k+1}) \cdots \alpha_{t_n+i\beta}(A_n)) \\ =&\omega_\beta(\alpha_{t_{k+1}}(A_{k+1}) \cdots \alpha_{t_n}(A_n)\alpha_{t_1}(A_1) \cdots \alpha_{t_k}(A_k)) \end{aligned}$$

for all $k \in \{1, \dots, n\}$.

PROOF. A multiple product of observables can be written as

$$\begin{aligned} (A_1 \star \cdots \star A_n)(\phi) &= \prod_{1 \le i < j \le n} \mathrm{e}^{\Gamma_+^{ij}} (A_1 \otimes \cdots \otimes A_n)\Big|_{\phi_1 \otimes \cdots \otimes \phi_n = \phi^{\otimes n}} \\ \Gamma_+^{ij} &= \int D_+^\beta(x - y) \frac{\delta}{\delta\phi_i(x)} \otimes \frac{\delta}{\delta\phi_j(y)} \,\mathrm{d}x \,\mathrm{d}y \end{aligned}$$

using the Leibniz rule of differential calculus on functionals (see section 1.2.2 and 1.4).

The time-translations $\alpha_x(A)$ on $\mathcal{A}_{\mathrm{os}}$ are simply given by $\alpha_t(A)(\phi) = A(\alpha^t\phi)$ where $\alpha^t(\phi)(x) = \phi(x^0 + t, \boldsymbol{x})$ conforming with the notation in definition 1.2.6. Thus, the expectation value in (4.3) can be written as

$$\begin{aligned} \omega_\beta(\alpha_{t_1}(A_1) \star \cdots \star \alpha_{t_n}(A_n)) &= (\alpha_{t_1}(A_1) \star \cdots \star \alpha_{t_n}(A_n))\,(\phi = 0) \\ &= \prod_{1 \le i < j \le n} \mathrm{e}^{\Gamma_+^{ij}} (\alpha_{t_1}(A_1) \otimes \cdots \otimes \alpha_{t_n}(A_n))\Big|_{\phi_1 \otimes \cdots \otimes \phi_n = 0} \\ &= \prod_{1 \le i < j \le n} \mathrm{e}^{\Gamma_+^{ij}(t_i, t_j)} (A_1 \otimes \cdots \otimes A_n)\Big|_{\phi_1 \otimes \cdots \otimes \phi_n = 0} \\ \Gamma_+^{ij}(t_i, t_j) &= \int D_+^\beta(x^0 - y^0 + (t_i - t_j), \boldsymbol{x} - \boldsymbol{y}) \frac{\delta}{\delta\phi_i(x)} \otimes \frac{\delta}{\delta\phi_j(y)} \,\mathrm{d}x \,\mathrm{d}y \,. \end{aligned}$$

As $t \mapsto D_+^\beta(t, \boldsymbol{x})$ has an analytic continuation into $-S_\beta = \{z \in \mathbb{C} : -\beta < \Im(z) < 0\}$, the function

$$(t, t') \mapsto \mathrm{e}^{\Gamma_+^{ij}(t,t')} (A_1 \otimes \cdots \otimes A_n)\Big|_{\phi_1 \otimes \cdots \otimes \phi_n = 0}$$

has an analytic continuation to $\{(z_1, z_2) \in \mathbb{C}^2 : -\beta < \Im(z_1 - z_2) < 0\}$ for every $A_1, \ldots, A_n \in \mathcal{A}_{\text{os}}$. Thus for the full expectation value we obtain that

$$(t_1, \ldots, t_n) \mapsto \prod_{1 \leq i < j \leq n} \mathrm{e}^{\Gamma_2^{ij}(t_i, t_j)} (A_1 \otimes \cdots \otimes A_n)\Big|_{\phi_1 \otimes \cdots \otimes \phi_n = 0}$$

has an extension into

$$\mathfrak{T}_\beta^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : -\beta < \Im(z_i - z_j) < 0 \quad \forall 1 \leq i < j \leq n\} .$$

The KMS conditions for the expectation values

$$\omega_\beta(\alpha_{t_1}(A_1) \cdots \alpha_{t_n}(A_n)) = \prod_{1 \leq i < j \leq n} \mathrm{e}^{\Gamma_2^{ij}(t_i, t_j)} (A_1 \otimes \cdots \otimes A_n)\Big|_{\phi_1 \otimes \cdots \otimes \phi_n = 0}$$

follow directly from the fact that $\Gamma_2^{ij}(t_i, t_j + i\beta) = \Gamma_2^{ji}(t_j, t_i)$ for $t_i, t_j \in \mathbb{R}$. □

The interacting KMS state $\omega_{\beta,h}^{\mathcal{V}}$ for the interacting dynamics $\alpha_t^{\mathcal{V}}$ with respect to the cut off interaction $\mathcal{V}_{h\chi}$ is be obtained by applying the proof of theorem 8 from the last section to this situation.

Proposition 4.2.2.
Given a KMS state ω_β on $\mathcal{A}_{\text{os}}$ with respect to the free time-evolution α_t, then following statements hold:

- *The linear functional $A \mapsto \omega_\beta(A \star W_h(t))$ has an analytic continuation into the strip S_β and is bounded on the boundary for all functionals A with* $\operatorname{supp}(A) \subset \mathcal{O}$.
- *A KMS state for the interacting dynamics $\alpha_t^{\mathcal{V}}$ for observables contained in the space-slice $\mathbb{R} \times B_r$ is given by the formula*

$$\omega_{\beta,h}^{\mathcal{V}}(A) = \frac{\omega_\beta(A \star W_h(i\beta))}{\omega_\beta(W_h(i\beta))}, \tag{4.4}$$

  where $h \in \mathcal{D}(\mathbb{R}^3)$ with $h(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in B_r$.

PROOF. The proof uses the ideas which are already present in the original work by Araki [Ara73]. To show that the analytic continuation of $\omega_\beta(A \star W_h(t))$ is well-defined, it is useful to construct a unitary operator intertwining the dynamics at different times. For this, consider

$$U_h(t, s) = W_h(t)^{\star -1} \star W_h(s) .$$

Due to equation (3.11) the U_h fulfill, for fixed $t \in \mathbb{R}$, the following differential equation

$$\frac{1}{i} \frac{\mathrm{d}}{\mathrm{d}s} U_h(t, s) = W_h(t)^{\star -1} \star W_h(s) \star \alpha_s(K_h) = U_h(t, s) \star \alpha_s(K_h) .$$

With the initial condition $U_h(t, t) = \mathbb{1}$ one obtains the power series expansion

$$U_h(t, s) = \sum_{n=0}^{\infty} (-i)^n \int\limits_{s \leq t_1 \leq \cdots \leq t_n \leq t} \alpha_{t_1}(K_h) \star \cdots \star \alpha_{t_n}(K_h) \, \mathrm{d}t_1 \cdots \mathrm{d}t_n$$

for $t > s$ similar to $W_h(t)$ in (3.13). This can be rewritten as

$$U_h(t,s) = \sum_{n=0}^{\infty} (-i(t-s))^n \int_{\mathfrak{s}_n} \alpha_{s+u_1(t-s)}(K_h) \star \cdots \star \alpha_{s+u_n(t-s)}(K_h)\, \mathrm{d}u_1 \cdots \mathrm{d}u_n \,,$$

where $\mathfrak{s}_n$ denotes the unit simplex

$$\mathfrak{s}_n = \{(u_1, \ldots, u_n) \in \mathbb{R}^n : 0 \leq u_1 \leq \ldots \leq u_n \leq 1\} \,.$$

We will discuss the analytic continuation of the following expression, which constitutes the numerator of $\omega^{\mathcal{V}}_{\beta,h}(A \star \alpha_t(B))$:

$$\begin{aligned} G_{A,B}(t,s) &= \omega_\beta(A \star \alpha^{\mathcal{V}}_t(B) \star W_h(s)) = \omega_\beta(A \star W_h(t) \star \alpha_t(B) \star W_h(t)^{\star -1} \star W_h(s)) \\ &= \omega_\beta\left(A \star W_h(t) \star \alpha_t(B) \star U_h(t,s)\right) \,. \end{aligned}$$

The power series expansion of $G_{A,B}(t,s)$ in the generator K_h yields

$$\begin{aligned} &G_{A,B}(t,s) = \omega_\beta\left(A \star W_h(t) \star \alpha_t(B) \star U_h(t,s)\right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (it)^{n-k} (is - it)^k \int_{\mathfrak{s}_{n-k}} \mathrm{d}u_1 \cdots \mathrm{d}u_{n-k} \int_{\mathfrak{s}_k} \mathrm{d}v_1 \cdots \mathrm{d}v_k \times \\ &\times \omega_\beta\left(A \star \alpha_{u_1 t}(K_h) \star \cdots \star \alpha_{u_{n-k} t}(K_h) \star \alpha_t(B) \star \alpha_{s+v_1(t-s)}(K_h) \star \cdots \star \alpha_{s+v_k(t-s)}(K_h)\right) . \end{aligned}$$

Using proposition 4.2.1 we infer that the domain of analyticity of the integrand is given by $\mathfrak{T}^{n+2}_\beta$ which can be recast in the form

$$\mathfrak{T}^{n+2}_\beta = \{(z_1, \ldots, z_{n+2}) \in \mathbb{C}^{n+2} : 0 < \Im(z_2) < \cdots < \Im(z_{n+2}) < \beta,\ \Im(z_1) = 0\}$$

in our case, that is the first variable having vanishing imaginary part. This is due to the factor A in front the expectation value $G_{A,B}$. We conclude that

$$\omega_\beta\left(A \star \alpha_{u_1 t}(K_h) \star \cdots \star \alpha_{u_{n-k} t}(K_h) \star \alpha_t(B) \star \alpha_{s+v_1(t-s)}(K_h) \star \cdots \star \alpha_{s+v_k(t-s)}(K_h)\right)$$

has an analytic continuation into

$$\begin{aligned} &0 < u_1 \Im(t) < \ldots < u_{n-k} \Im(t) < \Im(t) < \\ &< \Im(s) + v_1 \Im(t-s) < \cdots < \Im(s) + v_n \Im(t-s) < \beta \,. \end{aligned}$$

Hence $G_{A,B}(t,s)$ can be analytically continued into the simplex $\{(z_1, z_2) \in \mathbb{C}^2 : 0 < \Im(z_1) < \Im(z_2) < \beta\}$ due to the conditions on the integration variables. Moreover $G_{A,B}(t,s)$ remains bounded on the boundaries of the simplex and fulfills the KMS property: The function $t \mapsto G_{A,B}(t, i\beta)$

$$G_{A,B}(t, i\beta) = \omega_\beta(A \star \alpha^{\mathcal{V}}_t(B) \star W_h(i\beta))$$

admits an extension to values $0 < \Im(t) < \beta$ for all $A, B \in \mathcal{A}_{\mathrm{os}}(\mathbb{R} \times B_r)$ and

$$\begin{aligned} G_{A,B}(i\beta, i\beta) &= \omega_\beta(A \star \alpha^{\mathcal{V}}_{i\beta}(B) \star W_h(i\beta)) = \omega_\beta(A \star W_h(i\beta) \star \alpha_{i\beta}(B)) \\ &= \omega_\beta(B \star A \star W_h(i\beta)) \end{aligned}$$

holds using the KMS condition for ω_β.

It remains to show that the functional is formally positive in $\hbar$ and $\mathcal{V}$ (see definition 2.4.1). For this let us denote $2\gamma = \beta$:

$$\begin{aligned} A \mapsto \omega_\beta(A^* \star A \star W_h(i\beta)) &= \omega_\beta(A^* \star A \star W_h(i\gamma) \star \alpha_{i\gamma}(W_h(i\gamma))) \\ &= \omega_\beta\left(\alpha_{-i\gamma}(W_h(i\gamma)) \star A^* \star A \star W_h(i\gamma)\right) = \omega_\beta\left(W_h(i\gamma)^* \star A^* \star A \star W_h(i\gamma)\right) \\ &= \omega_\beta(B^* \star B) \end{aligned}$$

where $B = A \star W_h(i\gamma)$. In this computation the analytically continued co-cycle $W_h(i\gamma)$ and the KMS condition for ω_β was used. The formal positivity in $\hbar$ is directly inherited from the formal positivity of ω_β and the formal positivity in $\mathcal{V}$ comes from the fact, that $B^* \star B$ is a square of formal power series in $\mathcal{V}$.

In order to obtain a normalized functional, hence a state, one has to divide by the factor $\omega_\beta(W_h(i\beta))$ and arrives at the claim. □

This is a first, non-trivial step towards the final goal. The interacting KMS state $\omega^{\mathcal{V}}_{\beta,h}$ on $\mathbb{R} \times B_r$ was constructed in a very simple fashion by using a well-known construction from quantum statistical mechanics and – what was in particular important – without invoking observables at asymptotic times or their correlations, respectively. What is maybe more astonishing is the fact that the construction makes heavy use of the correlation functions of the interaction functional $\mathcal{V}$ (or more precisely its image under the quantum Møller map) at **Euclidean times**. This aspect has not appeared in such a prominent role in the hitherto publications on constructions of interacting KMS states, except in the realm of Euclidean QFTs, of course.

A subtlety in the definition of $\omega^{\mathcal{V}}_{\beta,h}$

$$\omega^{\mathcal{V}}_{\beta,h}(\mathcal{S}_{h\chi}(A)) = \frac{\omega_\beta(\mathcal{S}_{h\chi}(A) \star W_h(i\beta))}{\omega_\beta(W_h(i\beta))}, \qquad \mathrm{supp}(A) \subset \mathcal{O} \tag{4.5}$$

is the possible dependence of the state on the choice of χ. In section 3.2, equation (3.7), the co-cycle $W_h(t)$ was defined by

$$W_h(t) \equiv W_{h\chi}(t) = \mathcal{S}_{h\chi}(\mathcal{V}_{h\rho^-_t}), \qquad \rho_t = \Theta^-(\alpha_t(\chi) - \chi)\,, \tag{4.6}$$

thus the co-cycle exhibits a manifest dependence on the values of χ. Nevertheless it holds the following:

Proposition 4.2.3.
Let $\chi_i \in \mathcal{D}(\mathbb{R})$, such that $\chi_i(t) = 1$ for $|t| < \epsilon$ and $\mathrm{supp}(\chi_i) \subset (-2\epsilon, 2\epsilon)$ for $i = 1, 2$. Then, for $i = 1, 2$, the states $\omega^{\mathcal{V}}_{\beta,h\chi_i}$, defined by equation (4.5) with $W_{h\chi_i}$ defined by (4.6) coincide.

PROOF. We begin with calculating a relation between the two co-cycles $W_{h\chi_1}, W_{h\chi_2}$ first. To this avail, denote $\chi_1 - \chi_2 = \sigma_+ + \sigma_-$ with $\mathrm{supp}(\sigma_\pm) \subset J^\pm(\Sigma_\epsilon)$. Due to equation (2.18) we know that

$$\mathcal{S}_{h\chi_2}(A) = V \star \mathcal{S}_{h\chi_1}(A) \star V^{\star -1}, \qquad V = \mathcal{S}_{h\chi_2}(\mathcal{V}_{h\sigma_-}), \qquad \mathrm{supp}(A) \subset \Sigma_\epsilon \times B_r$$

where $h = 1$ on B_r. Here, $B_r \subset \mathbb{R}^3$ is the open Euclidean ball with radius r. This allows us to relate the co-cycles, induced by equation (4.6) with χ_2, with the one induced by χ_1:

$$\begin{aligned}\alpha_t^{\mathcal{V}}(\mathcal{S}_{h\chi_2}(A)) &= \mathcal{S}_{h\chi_2}(\alpha_t(A)) = V \star \mathcal{S}_{h\chi_1}(\alpha_t(A)) \star V^{\star -1} \\ &= V \star W_{h\chi_1}(t)\alpha_t(\mathcal{S}_{h\chi_1}(A)) \star W_{h\chi_1}(t)^{\star -1} \star V^{\star -1} \\ &= \underbrace{V \star W_{h\chi_1}(t) \star \alpha_t(V^{\star -1})}_{:= W_{h\chi_2}(t)} \star \alpha_t(\mathcal{S}_{h\chi_2}(A)) \star \alpha_t(V) \star W_{h\chi_1}(t)^{\star -1} \star V^{\star -1}\end{aligned}$$

The definition of $W_{h\chi_2}(t)$ is unique, up to elements that commute with all elements in $\mathcal{A}_{\mathrm{os}}((-\epsilon, \epsilon) \times B_r)$. This is removed in the adiabatic limit $h \to 1$, where $W_{h\chi_2}$ is, however, not unitary in $\mathcal{A}$ anymore.

The state $\omega^{\mathcal{V}}_{\beta,h\chi_2}$ is defined by equation (4.5) with $W_{h\chi_2}$ inserted. Looking at the numerator we find that

$$\begin{aligned}\omega_\beta(\mathcal{S}_{h\chi_2}(A) \star W_{h\chi_2}(i\beta)) &= \omega_\beta\left(V \star \mathcal{S}_{h\chi_1}(A) \star V^{-1} \star V \star W_{h\chi_1}(i\beta) \star \alpha_{i\beta}(V^{\star -1})\right) \\ &= \omega_\beta\left(\mathcal{S}_{h\chi_1}(A) \star W_{h\chi_1}(i\beta)\right)\end{aligned}$$

due to the KMS condition. This implies the statement. □

4.3. Removing the Spatial Cutoff

This section is devoted to porving the existence of the adiabatic limit of the interacting KMS state, i.e. the limit $h \to 1$ for the state $\omega^{\mathcal{V}}_{\beta,h}$. Once such a state is shown to exist on all observables in the time-slice Σ_ϵ, the time-slice axiom implies that we can derive the expectation values of all observables $A \in \mathcal{A}^{\mathcal{V}}_{\mathrm{os}}(\mathcal{O})$ for all $\mathcal{O} \subset M$. An algorithm to achieve this derivation is found in [CF09].

One of the major advantages of the time-slice approach will be exploited also for this manner: The behavior of the theory in timelike directions is irrelevant in order to construct the interacting KMS state, for it is only the **spacelike decay** of the correlation functions (with Euclidean time arguments) on which a full control is needed.

This is, at least in the case of KMS states, superior to the canonical way to proving existence of an interacting state, which is to estimate the perturbative expansions of the induced state (see section 2.4.3) as the cutoff g of the interaction functional $\mathcal{V}_g$ tends to the constant function on M.

It is known that in massive theories the vacuum state exhibits a decay in all spacetime directions is sufficient to guarantee the existence of the interacting state, which is exploited in the proof by Epstein and Glaser [EG73].

We will show that interacting, massive theories admit KMS states with inverse temperature $0 < \beta < \infty$ and vacuum states (KMS states with $\beta = +\infty$) due to the exponential decay of the correlation functions of the respective free states in spatial directions, thereby obtaining an alternative proof for the existence of the vacuum state.

4.3.1. Connected Correlation Functions

In order to obtain more control on the perturbative expansions of the interacting state

$$\omega^{\mathcal{V}}_{\beta,h}(A) = \frac{\omega_\beta(A \star W_h(i\beta)}{\omega_\beta(W_h(i\beta))} \tag{4.7}$$

we show a convenient reformulation of (4.7) in terms of the connected[8] correlation functions of the free theory that will play an important role in the discussion on the spatial cluster properties.

Before proving the proposition we introduce the notion of the connected part of a state ω on some algebra $\mathfrak{A}$. Consider the tensor algebra $\mathfrak{TA}$ over $\mathfrak{A}$

$$\mathfrak{TA} = \bigoplus_{n=0}^{\infty} \mathfrak{A}^{\otimes n}$$

in which the tensor-powers of $\mathfrak{A}^{\otimes n}$ are understood in tensor powers of the vector space $\mathfrak{A}$. The space of linear functionals on $\mathfrak{TA}$ carries an associative product:

$$(\nu\mu)(A_1 \otimes \cdots \otimes A_n) = \sum_{I \subset \{1,\ldots,n\}} \nu\left(\bigotimes_{i\in I} A_i\right) \mu\left(\bigotimes_{j\in I^c} A_j\right) \tag{4.8}$$

where I^c is the complement of I in $\{1,\ldots,n\}$. The unit element with respect to this product is $1(A_1 \otimes \cdots \otimes A_n) = \delta_{n0}$. Let m denote the canonical linear map $m : \mathfrak{TA} \to \mathfrak{A}$, $m(A_1 \otimes \ldots \otimes A_n) = A_1 \cdots A_n$. Then the connected part $\omega^{\mathrm{c}} : \mathfrak{TA} \to \mathbb{C}$ of $\omega : \mathfrak{A} \to \mathbb{C}$ is defined by

$$\omega \circ m = \mathrm{e}^{\omega^{\mathrm{c}}} = \sum_{n=0}^{\infty} \frac{1}{n!} (\omega^c)^n$$

in the sense of the products of linear functionals. The formula for the inverse is given:

$$\omega^{\mathrm{c}} = \log(\omega \circ m) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (\omega \circ m - 1)^n$$

with the condition $\omega^{\mathrm{c}}(1) = 0$. Taking the k-th power of ω^{c} one obtains

$$(\omega^{\mathrm{c}})^k(A_1 \otimes \cdots \otimes A_n) = \sum_{I_1,\ldots,I_k \subset \{1,\ldots,n\}} \prod_{j=1}^{k} \omega^{\mathrm{c}}\left(\bigotimes_{i\in I_j} A_i\right)$$

with pairwise disjoint sets $\{I_j \subset \{1,\ldots,n\} : j = 1,\ldots,k\}$ such that $\bigcup_{j=1}^{k} = \{1,\ldots,k\}$ by iteration of (4.8). The contributions in which one of the I_j is empty vanishes due to $\omega^{\mathrm{c}}(1) = 0$. This allows to rewrite the sum as a sum over partitions of $\{1,\ldots,n\}$ into k subsets (where each partition occurs $k!$ times):

$$(\omega^{\mathrm{c}})^k(A_1 \otimes \cdots \otimes A_n) = k! \sum_{\substack{\{I_1,\ldots,I_k\} \in \\ \mathrm{Part}_k\{1,\ldots,n\}}} \prod_{k=1}^{n} \omega^{\mathrm{c}}\left(\bigotimes_{i\in I_k} A_i\right)$$

[8] Many authors use the term truncated correlation functions in this context.

Thus one obtains the (recursive) formula for ω^c

$$\begin{aligned}\omega(A_1\cdots A_n) =&(\omega\circ m)(A_1\otimes\cdots\otimes A_n) = \mathrm{e}^{\omega^{\mathrm{c}}}(A_1\otimes\cdots\otimes A_n)\\ &=\sum_{k=1}^{n}\sum_{\substack{\{I_1,\dots,I_k\}\in\\ \mathrm{Part}_k\{1,\dots,n\}}}\prod_{k=1}^{n}\omega^{\mathrm{c}}\left(\bigotimes_{i\in I}A_i\right)=\sum_{P\in\mathrm{Part}\{1,\dots,n\}}\prod_{I\in P}\omega^{\mathrm{c}}\left(\bigotimes_{i\in I}A_i\right)\end{aligned}$$

which is found in the literature. A simple consequence of this definition is the formula

$$\omega^{\mathrm{c}}(\exp_{\otimes}(A)) = \ln\left(\omega(\mathrm{e}^{A})\right), \qquad A\in\mathfrak{A}$$

which will be of use later on.

The following proposition was first proven by [BKR78] in the C^*-algebraic setting.

Proposition 4.3.1.
The expectation values $\omega^{\mathcal{V}}_{\beta,h}(A)$ can be written in terms of the connected correlation functions:

$$\begin{aligned}\omega^{\mathcal{V}}_{\beta,h}(A) = \sum_{n=0}^{\infty}\int_{\beta\mathfrak{s}_n}\int_{\mathbb{R}^{3n}} h(\boldsymbol{x}_1)\cdots h(\boldsymbol{x}_n)\times\\ \times\,\omega^{\mathrm{c}}_{\beta}\left(A\otimes\mathfrak{V}_h(u_1,\boldsymbol{x}_1)\otimes\cdots\otimes\mathfrak{V}_h(u_n,\boldsymbol{x}_n)\right)\mathrm{d}\boldsymbol{x}_1\cdots\mathrm{d}\boldsymbol{x}_n\,\mathrm{d}u_1\cdots\mathrm{d}u_n\end{aligned}\tag{4.9}$$

where $\beta\mathfrak{s}_n=\{(u_1,\dots,u_n)\in\mathbb{R}^n : 0\le u_1\le\cdots\le u_n\le\beta\}$ and

$$\mathfrak{V}_h(u,\boldsymbol{x}) = \int\Theta^{-}(t)\dot{\chi}(t)\alpha_{iu}\left(\mathcal{R}_{\mathcal{V}_{h_\chi}}(\mathcal{V}(\Phi_{t,\boldsymbol{x}}))\right)\mathrm{d}t\,.$$

*Here, $\omega^{\mathrm{c}}_{\beta}$ denotes the **connected** part of the state ω_β.*

PROOF. The proof of formula (4.9) goes along the same lines as the original proof in [BKR78] and is only sketched here. For this we introduce the following expansions in the interaction h:

$$\omega^{\mathcal{V}}_{\beta,\lambda h}(A) = \sum_{n=0}^{\infty}\lambda^n\Omega_n(A)\,, \qquad \omega_\beta(A\star W_{\lambda h}(i\beta)) = \sum_{n=0}^{\infty}\lambda^n\nu_n(A)\,.$$

The coefficients ν_n are indeed all known from the above construction of $W_h(t)$:

$$\nu_n(A) = (-1)^n\int_{\beta\mathfrak{s}_n}\omega_\beta(A\star\alpha_{iu_1}(K_h)\star\cdots\star\alpha_{iu_n}(K_h))\,\mathrm{d}u_1\cdots\mathrm{d}u_n, \quad \nu_0(A)=\omega_\beta(A)\,.$$

By definition of the interacting states $\omega^{\mathcal{V}}_{\beta,h}$ it holds

$$\omega_\beta(A\star W_{\lambda h}(i\beta)) = \omega_\beta(W_{\lambda h}(i\beta))\omega^{\mathcal{V}}_{\beta,\lambda h}(A)\,,$$

thus by comparing the coefficients of the expansions on both sides on gets

$$\nu_n(A) = \sum_{k=0}^{n}\nu_k(\mathbb{1})\Omega_{n-k}(A), \qquad \Omega_n(A) = \nu_n(A) - \sum_{k=1}^{n}\nu_k(\mathbb{1})\Omega_{n-k}(A)\,.$$

By induction it is then shown that

$$\Omega_n(A) = (-1)^n\int_{\beta\mathfrak{s}_n}\omega^{\mathrm{c}}_{\beta}\left(A\otimes\alpha_{iu_1}(K_h)\otimes\cdots\otimes\alpha_{iu_n}(K_h)\right)\mathrm{d}u_1\cdots\mathrm{d}u_n\,.$$

The claim follows by inserting (3.12) in the formal power series expansion of $\omega^{\mathcal{V}}_{\beta,h}$. □

4.3.2. Condition on the Correlation Functions

The last proposition allows us to formulate the problem in a feasible manner: We want to show that expectation values $\omega^{\mathcal{V}}_{\beta,h}(A)$ with $A \in \mathcal{A}_{\mathrm{os}}(\mathbb{R} \times B_r)$ which are determined by (4.9) converge as $h \to 1$ in the sense of formal power series in $\mathcal{V}$.

A first step to formulate a condition that ensures the existence of such a limit is done by the following observation: The generators of the co-cycle W_h

$$-K_h = \mathcal{R}_{\mathcal{V}_{h\chi}}(\mathcal{V}_{h\Theta^-\dot{\chi}}) = \mathcal{S}(\mathcal{V}_{h\chi})^{\star -1} \star \left(\mathcal{S}(\mathcal{V}_{h\chi}) \cdot_{\mathcal{T}} \mathcal{V}_{h\Theta^-\dot{\chi}}\right)$$

are (formal power series of) elements of the free theory $\mathcal{A}(B)$ where B is a region containing

$$\big(\operatorname{supp}(\chi) \times \operatorname{supp}(h)\big) \cap J^-\Big(\operatorname{supp}(\Theta^-\dot{\chi}) \times \operatorname{supp}(h)\Big)\,,$$

see figure 3. Hence we can replace

$$\mathcal{R}_{\mathcal{V}_{h\chi}}(\mathcal{V}_{h\Theta^-\dot{\chi}}) \quad \longrightarrow \quad \mathcal{R}_{\mathcal{V}_{\chi}}(\mathcal{V}_{h\Theta^-\dot{\chi}}) = \mathcal{S}(\mathcal{V}_{\chi})^{\star -1} \star \left(\mathcal{S}(\mathcal{V}_{\chi}) \cdot_{\mathcal{T}} \mathcal{V}_{h\Theta^-\dot{\chi}}\right) =: -K'_h \tag{4.10}$$

and still obtain a well-defined generator K'_h which has support in the compact set

$$\big(\operatorname{supp}(\chi) \times \mathbb{R}^3\big) \cap J^-\Big(\operatorname{supp}(\Theta^-\dot{\chi}) \times \operatorname{supp}(h)\Big)\,.$$

The latter generator has the advantage that it is a **linear functional** of h. In particular we only have to control the decay behavior of the connected correlation functions in order to prove the existence of the adiabatic limit. We rewrite the expectation value with the new generator (interpreted as a partial adiabatic limit of $\omega^{V}_{\beta,h}$):

$$\begin{aligned}
\omega^{\mathcal{V}}_{\beta,h}(A) &= \sum_{n=0}^{\infty} \int_{\beta\mathfrak{s}_n} \int_{\mathbb{R}^{3n}} h(\boldsymbol{x}_1) \cdots h(\boldsymbol{x}_n)\; \omega^{\mathrm{c}}_{\beta}\left(A \otimes \mathfrak{V}(u_1, \boldsymbol{x}_1) \otimes \cdots \otimes \mathfrak{V}(u_n, \boldsymbol{x}_n)\right) \mathrm{d}\boldsymbol{X}\, \mathrm{d}U \\
\mathfrak{V}(u, \boldsymbol{x}) &= \int \Theta^-(t)\dot{\chi}(t)\alpha_{iu}\left(\mathcal{R}_{\mathcal{V}_{\chi}}(\mathcal{V}(\Phi_{t,\boldsymbol{x}}))\right) \mathrm{d}t\,.
\end{aligned} \tag{4.11}$$

The notation for $\mathfrak{V}$ has to be taken with care, since $\alpha_{iu}(\mathcal{R}_{\mathcal{V}_{\chi}}(\mathcal{V}_{h\Theta^-\dot{\chi}}))$ is – as an algebraic object – not well-defined since there are no analytic elements in the algebra of observables (analytic elements cannot have compact support, see [Haa92]). However, the expectation values of time-translations in imaginary directions of the respective elements in the KMS state ω_β are well-defined in their domain of analyticity $\beta\mathfrak{s}_n$, as proven in proposition 4.2.2.

The latter result will now be used to formulate a condition on the connected expectation values that guarantees the existence of the adiabatic limit.

Proposition 4.3.2.
Let $\omega^{\mathcal{V}}_{\beta,h}$ be the interacting KMS state for $\alpha^{\mathcal{V}}_t$ for the interaction $\mathcal{V}_\chi$, defined by

$$\omega^{\mathcal{V}}_{\beta,h}(A) = \sum_{n=0}^{\infty} \int_{\beta\mathfrak{s}_n} \int_{\mathbb{R}^{3n}} h(\boldsymbol{x}_1) \cdots h(\boldsymbol{x}_n) \omega^{\mathrm{c}}_{\beta}\left(A \otimes \mathfrak{V}(u_1, \boldsymbol{x}_1) \otimes \cdots \otimes \mathfrak{V}(u_n, \boldsymbol{x}_n)\right) \mathrm{d}\boldsymbol{X}\, \mathrm{d}U\,.$$

Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of test functions with the following properties

$$h_n(\boldsymbol{x}) = 1 \quad \textit{for } |\boldsymbol{x}| < n, \qquad |h_n(\boldsymbol{x})| \leq 1 \qquad \forall x \in \mathbb{R}^3,\ n \in \mathbb{N}\,.$$

If the functions

$$F_n(u_1, \boldsymbol{x}_1; \cdots; u_n, \boldsymbol{x}_n) = \omega_\beta^{\mathrm{c}}\left(A_0 \otimes \alpha_{iu_1,\boldsymbol{x}_1}(A_1) \otimes \cdots \otimes \alpha_{iu_n,\boldsymbol{x}_n}(A_n)\right)$$

are contained in the space $L^1(\beta\mathfrak{s}_n \times \mathbb{R}^{3n})$ for all $A_i \in \mathcal{A}_{\mathrm{os}}$ with $i = 0, \ldots, n$, $n \in \mathbb{N}$ and $0 < \beta \leq +\infty$, then the limit

$$\lim_{n\to\infty} \omega^{\mathcal{V}}_{\beta,h_n}(A) = \omega^{\mathcal{V}}_\beta(A) \qquad A \in \mathcal{A}_{\mathrm{os}}$$

exists and defines a state over $\mathcal{A}_{\mathrm{os}}$.

PROOF. Due to proposition 4.3.1 we know that the expectation values of the interacting state can be written in terms of the connected correlation functions

$$\omega^{\mathcal{V}}_{\beta,h}(A) = \sum_{n=0}^{\infty}(-1)^n \int_{\beta\mathfrak{s}_n} \omega_\beta^{\mathrm{c}}\left(A \otimes \alpha_{iu_1}(K'_h) \otimes \cdots \otimes \alpha_{iu_n}(K'_h)\right) \mathrm{d}U\,.$$

in the sense of formal power series in $\mathcal{V}$ by proposition 4.3.1. The n-th term of this series can be obtained by the composition law of formal power series. To this end let $K_h^{\prime(j)}$ be the j-th term in the formal power series of K'_h in $\mathcal{V}$. Then

$$\left[\omega^{\mathcal{V}}_{\beta,h}(A)\right]_n = \sum_{k\in\mathbb{N}} \sum_{\substack{\ell\in\mathbb{N}^{k+1}\\ |\ell|=n}} (-1)^k \int_{\beta\mathfrak{s}_k} \omega_\beta^{\mathrm{c}}\left(A^{(\ell_0)} \otimes \alpha_{iu_1}(K_h^{\prime(\ell_1)}) \otimes \cdots \otimes \alpha_{iu_k}(K_h^{\prime(\ell_k)})\right) \mathrm{d}U\,.$$

By inserting the K'_h from (4.10), using the form (4.11) and abbreviating the ℓ-th term of $\mathfrak{V}$ by $\mathfrak{V}^{(\ell)}$, we obtain

$$\begin{aligned}\left[\omega^{\mathcal{V}}_{\beta,h}(A)\right]_n = \sum_{k\in\mathbb{N}} \sum_{\substack{\ell\in\mathbb{N}^{k+1}\\ |\ell|=n}} \int_{\beta\mathfrak{s}_k}\int_{\mathbb{R}^{3k}} h(\boldsymbol{x}_1)\cdots h(\boldsymbol{x}_k)\times \\ \times\, \omega_\beta^{\mathrm{c}}\left(A^{(\ell_0)} \otimes \mathfrak{V}(u_1,\boldsymbol{x}_1)^{(\ell_1)} \otimes \cdots \otimes \mathfrak{V}(u_k,\boldsymbol{x}_k)^{(\ell_k)}\right) \mathrm{d}\boldsymbol{x}_1\cdots\mathrm{d}\boldsymbol{x}_k\,\mathrm{d}u_1\cdots\mathrm{d}u_k\,.\end{aligned}$$

But again due to (4.11) the $\mathfrak{V}(u,\boldsymbol{x})^{(\ell_j)}$ can be written as

$$\mathfrak{V}(u,\boldsymbol{x})^{(\ell_j)} = \alpha_{iu,\boldsymbol{x}}\left(\int \Theta^-(t)\dot{\chi}(t)\mathcal{R}^{(\ell_j)}_{\mathcal{V}_\chi}(\mathcal{V}(\Phi_{t,0}))\,\mathrm{d}t\right) =: \alpha_{iu,\boldsymbol{x}}(R^{(\ell_j)})$$

where $R^{(\ell_j)} \in \mathcal{A}_{\mathrm{os}}(B)$ where B is an open neighborhood of $[-\epsilon, 0] \times J^-$. The notation $\alpha_{iu,\boldsymbol{x}}(R)$ is also formal in the same sense as explained above. Using these results in the initial formula for the interacting state, we find

$$\begin{aligned}\left[\omega^{\mathcal{V}}_{\beta,h}(A)\right]_n &= \sum_{k\in\mathbb{N}} \sum_{\substack{\ell\in\mathbb{N}^{k+1}\\ |\ell|=n}} \Omega^h_\ell \\ \Omega^h_\ell &= \int_{\beta\mathfrak{s}_k}\int_{\mathbb{R}^{3k}} h(\boldsymbol{x}_1)\cdots h(\boldsymbol{x}_k)\omega_\beta^{\mathrm{c}}\left(A^{(\ell_0)} \otimes \alpha_{iu_1,\boldsymbol{x}_1}(R^{(\ell_1)}) \otimes \cdots \otimes \alpha_{iu_k,\boldsymbol{x}_k}(R^{(\ell_k)})\right) \mathrm{d}\boldsymbol{X}\,\mathrm{d}U.\end{aligned}$$

Hence the integrand is, for a fixed partition ℓ of $\{1,\dots,n\}$ and up to the factors $h(\boldsymbol{x})$, a function of the form

$$F_k(u_1,\boldsymbol{x}_1;\cdots;u_k,\boldsymbol{x}_k)=\omega_\beta^{\mathrm{c}}\left(A_0\otimes\alpha_{iu_1,\boldsymbol{x}_1}(A_1)\otimes\cdots\otimes\alpha_{iu_k,\boldsymbol{x}_k}(A_k)\right)$$

with $A_i\in\mathcal{A}_{\mathrm{os}}$ for $i=0,\dots,n$. Thus it suffices to show the assertion for this function. The claim will now be proven by the theorem on dominated convergence. The integrand can, by assumption, be majorized by

$$|h_n(x_1)\cdots h_n(x_k)F_k(u_1,\boldsymbol{x}_1;\dots;u_k,\boldsymbol{x}_k)|\leq\|F_k\|_{L^1}\ ,$$

hence we can switch the limit and the integration and obtain

$$\begin{aligned}\omega_\beta^{\mathcal{V}}(A)&=\lim_{k\to\infty}\sum_{n=0}^{\infty}\int_{\beta\mathfrak{s}_n}\int_{\mathbb{R}^{3n}}h_k(\boldsymbol{x}_1)\cdots h_k(\boldsymbol{x}_n)\times\omega_\beta^{\mathrm{c}}\left(A\otimes\mathfrak{V}(u_1,\boldsymbol{x}_1)\otimes\cdots\otimes\mathfrak{V}(u_n,\boldsymbol{x}_n)\right)\mathrm{d}\boldsymbol{X}\,\mathrm{d}U\\&=\sum_{n=0}^{\infty}\int_{\beta\mathfrak{s}_n}\int_{\mathbb{R}^{3n}}\omega_\beta^{\mathrm{c}}\left(A\otimes\mathfrak{V}(u_1,\boldsymbol{x}_1)\otimes\cdots\otimes\mathfrak{V}(u_n,\boldsymbol{x}_n)\right)\mathrm{d}\boldsymbol{X}\,\mathrm{d}U\\&=\sum_{n=0}^{\infty}\sum_{k\in\mathbb{N}}\sum_{\substack{\ell\in\mathbb{N}^{k+1}\\|\ell|=n}}\int_{\beta\mathfrak{s}_k}\int_{\mathbb{R}^{3k}}\omega_\beta^{\mathrm{c}}\left(A^{(\ell_0)}\otimes\alpha_{iu_1,\boldsymbol{x}_1}(R^{(\ell_1)})\otimes\cdots\otimes\alpha_{iu_k,\boldsymbol{x}_k}(R^{(\ell_k)})\right)\mathrm{d}\boldsymbol{X}\,\mathrm{d}U.\end{aligned}$$

with

$$R=\int\Theta^-(t)\dot\chi(t)\mathcal{R}_{\mathcal{V}_\chi}(\mathcal{V}(\Phi_{t,0}))\,\mathrm{d}t\ . \tag{4.12}$$

□

We remark that the functional R is slightly more singular than a microcausal functional in the sense of 1.2.5. Its first term in the formal power series in $\mathcal{V}$

$$R^{(1)}=\int\Theta^-(t)\dot\chi(t)\mathcal{V}(\Phi_{t,0})$$

exhibits already a non-trivial wavefront set, which is not possible for a microcausal functional. In general, the wavefront set of these functionals are given by

$$\mathrm{WF}\,\frac{\delta^nR^{(\ell)}}{\delta\phi^n}(\phi)\subset\left\{(x_1,\dots,x_n|p_1,\dots,p_n)\in\dot T^*M^n:(p_1,\dots p_n)\notin(J^+)^n\cup(J^-)^n\right\}\ .$$

This is, in fact, very close to the standard definition of microcausal functionals in the more general pAQFT-approach on curved spacetimes [BF09]. In particular, all structural results on the algebra of observables extend to this more singular class of functionals.

4.3.3. Proof for the Vacuum State

Using the result of proposition 4.3.2 it remains to show that the connected correlation functions (with imaginary time arguments) of the KMS state of the free theory are integrable. The strategy to prove this statement takes a detour to the vacuum state. We will

show that the free massive vacuum state will have an the mentioned property, thereby giving an alternative proof of the existence of a vacuum state for interacting massive theory to [EG73].

The proof that will be shown in the end of this section will make use of these two intermediary results: The first one concerns the singular directions of functional derivatives that appear in the expansion of the truncated vacuum expectation values.

Proposition 4.3.3.
Define for $A_0, \dots, A_n \in \mathcal{A}$ the compactly supported distribution

$$\Psi(x_1, \dots, x_k, y_1, \dots, y_k) = \prod_{l=1}^{k} \frac{\delta^2}{\delta\phi_{i(l)}(x_l)\delta\phi_{j(l)}(y_l)} (A_0 \otimes \cdots \otimes A_n)\Big|_{\phi_0 \otimes \cdots \otimes \phi_n = 0}$$

where $i, j : \{1, \dots, k\} \to \{0, \dots, n\}$ such that $i(l) < j(l)$ for all $l \in \{1, \dots, k\}$. Then

$$(p_1, \dots, p_k) \mapsto \hat{\Psi}(-p_1, \dots, -p_k, p_1, \dots p_k)$$

is rapidly decreasing inside the k-fold product of either the forward lightcone $(J^+)^k$ or the backward lightcone $(J^-)^k$.

PROOF. Using the tensor product rule for wavefront sets (see [Hör90]) and the fact that the functionals A_i are microcausal (see definition 1.2.5) one finds that $\hat{\Psi}(-P, P)$ is rapidly decaying in every direction, except the cone defined by

$$\left\{ (p_1, \dots, p_k) \in \dot{T}^* M^k : \sum_{\substack{l=1,\dots,k \\ i(l)=m}} p_l - \sum_{\substack{l=1,\dots,k \\ r(l)=m}} p_l = 0, \quad m = 0, \dots, n \right\}$$

Assume that all of the momenta lie either inside the forward or backward lightcone. Taking the first condition ($m = 0$) we see that that $\{l \in \{1, \dots, k\} : r(l) = 0\} = \varnothing$. Since all the momenta are contained in J^+ or J^- this implies

$$\sum_{\substack{l=1,\dots,k \\ s(l)=0}} p_l = \sum_{\substack{l=1,\dots,k \\ r(l)=0}} p_l = 0 \quad \Longrightarrow \quad p_l = 0 \quad \forall l \in \{1, \dots, k\} : s(l) = 0\,.$$

But the set $\{l \in \{1, \dots, k\} : s(l) = 0\}$ contains in particular indices $\{l \in \{1, \dots, k\} : r(l) = 1\}$. This information can be put into the next wavefront set condition $m = 1$ which yields

$$\sum_{\substack{l=1,\dots,k \\ s(l)=1}} p_l - \underbrace{\sum_{\substack{l=1,\dots,k \\ r(l)=1}} p_l}_{=0} = 0 \quad \Longrightarrow \quad p_l = 0 \quad \forall l \in \{1, \dots, k\} : s(l) = 1$$

again since all the directions are contained in one of the lightcones J^+, J^-. This can be iterated until $m = n$ with the result that all momenta $\{p_l : l = 1, \dots, k\}$ vanish. Note that the same statement holds for functionals which are restricted to a point in space, but are smeared in time, such as the generator R of W in the adiabatic limit, see (4.12).□

The second property that we will need in the following shows that the vacuum two-point function with imaginary time-arguments decays exponentially, if there is a non-vanishing mass.

Proposition 4.3.4.
Let $f \in \mathcal{D}(\mathbb{R}^4)$ with $\operatorname{supp}(f) \subset B_R$. Then the function

$$I_f(x_0, \boldsymbol{x}) = \int \mathrm{e}^{-i(p_0 x_0 - \boldsymbol{p}\boldsymbol{x})} \theta(p_0) \delta(p^2 - m^2) \hat{f}(p_0, \boldsymbol{p}) \, \mathrm{d}p$$

has an analytic continuation into the lower half plane $\mathbb{C}_- \times \mathbb{R}^3$ and for $m > 0$ it holds

$$\left| I_f(-ix_0, \boldsymbol{x}) \right| \leq c \, \mathrm{e}^{-mr}, \qquad r = \sqrt{x_0^2 + \boldsymbol{x}^2} \, .$$

uniform for $r > 2R + \delta$ with $\delta > 0$.

PROOF. The domain of analyticity of I_f is obvious by the fact that

$$I_f(-ix_0, \boldsymbol{x}) = \int \frac{\mathrm{e}^{i\boldsymbol{p}\boldsymbol{x} - x^0 \sqrt{\boldsymbol{p}^2 + m^2}}}{2\sqrt{\boldsymbol{p}^2 + m^2}} \hat{f}\left(\sqrt{\boldsymbol{p}^2 + m^2}, \boldsymbol{p} \right) \mathrm{d}^3 \boldsymbol{p}$$

decays exponentially when $m^2 \geq 0$. Using the identity

$$\frac{1}{2\pi} \int \frac{\mathrm{e}^{ikx}}{k^2 + \omega^2} \, \mathrm{d}k = \frac{\mathrm{e}^{-\omega x}}{2\omega}, \qquad x, \omega > 0$$

we can rewrite the following integral

$$\begin{aligned} I_f(-ix_0, \boldsymbol{x}) &= \int \frac{\mathrm{e}^{i\boldsymbol{p}\boldsymbol{x} - x^0 \sqrt{\boldsymbol{p}^2 + m^2}}}{2\sqrt{\boldsymbol{p}^2 + m^2}} \hat{f}\left(\sqrt{\boldsymbol{p}^2 + m^2}, \boldsymbol{p} \right) \mathrm{d}^3 \boldsymbol{p} \\ &= \frac{1}{2\pi} \int \frac{\mathrm{e}^{i \sum_i p_i x_i}}{p_0^2 + \boldsymbol{p}^2 + m^2} \hat{f}\left(\sqrt{\boldsymbol{p}^2 + m^2}, \boldsymbol{p} \right) \mathrm{d}p_0 \, \mathrm{d}^3 \boldsymbol{p} \end{aligned}$$

With loss of generality we choose the coordinates $\boldsymbol{x} = \boldsymbol{n} r \cos(\alpha)$ and $x^0 = r \sin(\alpha)$ with $\boldsymbol{n} = (1, 0, 0)$ and $0 < 2\alpha < \pi$. The following change in the momentum variables is helpful:

$$k_0 = p_0 \sin(\alpha) + p_1 \cos(\alpha), \quad k_1 = p_0 \cos(\alpha) + p_1 \sin(\alpha), \quad k_{2/3} = p_{2/3} \, .$$

The integral is of the form

$$\begin{aligned} I_f(-ix_0, \boldsymbol{x}) = \frac{1}{2\pi} \int \frac{\mathrm{e}^{ik_0 r} \hat{f}\left(\omega(k_i), \boldsymbol{p}(k_i)\right)}{k_0^2 + k_1^2 + k_2^2 + k_3^2 + m^2} \, \mathrm{d}k = \frac{1}{2\pi} \int \frac{\mathrm{e}^{ik_0 r} \hat{f}\left(\omega(k_i), \boldsymbol{p}(k_i)\right)}{k_0^2 + k_1^2 + \Omega^2} \, \mathrm{d}k, \\ \omega(k_i)^2 = (k_0 \cos(\alpha) - k_1 \sin(\alpha))^2 + \Omega^2, \quad p_1(k_i) = k_0 \cos(\alpha) - k_1 \sin(\alpha) \end{aligned}$$

with the abbreviation $\Omega^2 = k_2^2 + k_3^2 + m^2$. We replace the integration in the k_0-variable by replacing contour integration in upper half plane. By Paley-Wiener theorem [Str03] we know that for $(\omega, \boldsymbol{p})$ in the upper half plane $\mathbb{C}_+^4$:

$$\left| \hat{f}(\omega, \boldsymbol{p}) \right| \leq c \, \mathrm{e}^{R\sqrt{|\omega|^2 + |\boldsymbol{p}|^2}}$$

thus the integrand will exponentially decay for $r > R$. The contour integral yields

$$\frac{1}{2\pi}\int_C \frac{\mathrm{e}^{izr}}{z^2+k_1^2+\Omega^2}\hat{f}(\omega,\boldsymbol{p})\,\mathrm{d}z\,,$$

where the dependence of ω and $\boldsymbol{p}$ on the variables $k_0 = z$ and k_i is suppressed in the notation of $\hat{f}$. We see that the integrand has a pole at

$$z = i\sqrt{k_1^2+\Omega^2} = i\sqrt{\boldsymbol{k}^2+m^2}\,.$$

The principal square root in $\omega(z,k_i)$ has a branch cut on the negative real axis, we get a branch cut along the vertical axis, starting from

$$z = k_1\tan(\alpha) + i\frac{\Omega}{\cos(\alpha)}\,.$$

Thus we choose a contour that avoids both the pole and the branch cut, see figure 6

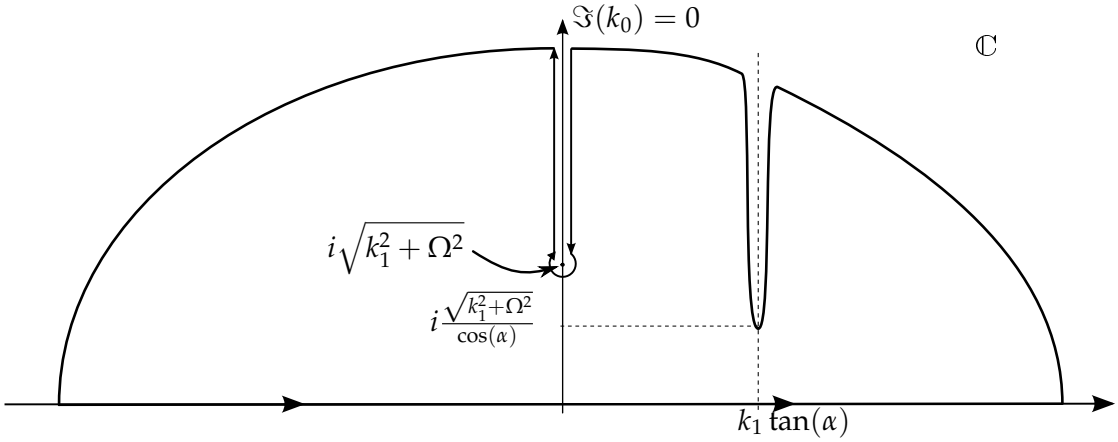


Figure 6.: The integration contour C. The semicircle has to be extended to infinite size and the orientation is positive.

such that the contour integral vanishes due to the exponential decay of the integrand:

$$\begin{aligned}
0 &= \frac{1}{2\pi}\int_C \frac{\mathrm{e}^{izr}}{z^2+k_1^2+\Omega^2}\hat{f}(\omega,\boldsymbol{p})\,\mathrm{d}z \\
&= I_f(x_0,\boldsymbol{x}) + \frac{1}{2\pi}\oint\limits_{\text{pole}} \frac{\mathrm{e}^{izr}}{z^2+k_1^2+\Omega^2}\hat{f}(\omega,\boldsymbol{p})\,\mathrm{d}z \\
&\qquad + \frac{1}{2\pi}\int\limits_{\text{branch}} \frac{\mathrm{e}^{izr}}{z^2+k_1^2+\Omega^2}\hat{f}(\omega,\boldsymbol{p})\,\mathrm{d}z\,.
\end{aligned}$$

The pole contour can be calculated using the residue theorem

$$\begin{aligned}
&\frac{1}{2\pi} \oint\limits_{z=i\sqrt{k_1^2+\Omega^2}} \frac{\mathrm{e}^{izr}}{z^2+k_1^2+\Omega^2} \hat{f}(\omega,\boldsymbol{p})\,\mathrm{d}z \\
&= i\,\mathrm{Res}_{z=i\sqrt{k_1^2+\Omega^2}} \frac{\mathrm{e}^{izr}\hat{f}(\omega,\boldsymbol{p})}{\left(z-i\sqrt{k_1^2+\Omega^2}\right)\left(z+i\sqrt{k_1^2+\Omega^2}\right)} \\
&= \frac{\mathrm{e}^{-r\sqrt{\boldsymbol{k}^2+m^2}}\hat{f}(\omega,\boldsymbol{p})}{2\sqrt{\boldsymbol{k}^2+m^2}}\Big|_{k_0=i\sqrt{\boldsymbol{k}^2+m^2}},
\end{aligned}$$

thus the full pole contribution to I is

$$I_{\mathrm{pole}}(-ix_0,\boldsymbol{x}) = \int \frac{\mathrm{e}^{-r\sqrt{\boldsymbol{k}^2+m^2}}\hat{f}(\omega,\boldsymbol{p})}{2\sqrt{\boldsymbol{k}^2+m^2}}\Big|_{k_0=i\sqrt{\boldsymbol{k}^2+m^2}} \mathrm{d}^3\boldsymbol{k}\,.$$

The branch cut contributes with

$$\int_\Omega^\infty \frac{\mathrm{e}^{irk_1\tan(\alpha)}\mathrm{e}^{-\frac{r\tau}{\cos(\alpha)}}}{z(\tau)^2+k_1^2+\Omega^2}\left(\hat{f}(\omega,\boldsymbol{p})\big|_{k_0=z(\tau)+i\epsilon} - \hat{f}(\omega,\boldsymbol{p})\big|_{k_0=z(\tau)-i\epsilon}\right)\mathrm{d}\tau$$

where $z(\tau) = k_1\tan(\alpha) + i\frac{\tau}{\cos(\alpha)}$. The arguments of $\hat{f}$ in this parametrization of the branch cut are given as

$$\begin{aligned}
\omega(k_0 = z(\tau), k_i) &= \pm i\sqrt{\tau^2-\Omega^2} \\
p_1(k_0 = z(\tau), k_i) &= i\tau
\end{aligned}$$

i.e. $\hat{f}$ does not depend on k_1 on the branch cut. We invoke the k_1-integration to find

$$\int\int_\Omega^\infty \frac{\mathrm{e}^{irk_1\tan(\alpha)}\mathrm{e}^{-\frac{r\tau}{\cos(\alpha)}}}{z(\tau)^2+k_1^2+\Omega^2}(\hat{f}(\omega,\boldsymbol{p})_+ - \hat{f}(\omega,\boldsymbol{p})_-)\,\mathrm{d}\tau\,\mathrm{d}k_1\,.$$

We replace the k_1-integration by a contour-integration along a semi-circle in the upper half plane, where the integrand falls off exponentially for $r > R$:

$$\int_C\int_\Omega^\infty \frac{\mathrm{e}^{irw\tan(\alpha)}\mathrm{e}^{-\frac{r\tau}{\cos(\alpha)}}}{z(\tau)^2+w^2+\Omega^2}(\hat{f}(\omega,\boldsymbol{p})_+ - \hat{f}(\omega,\boldsymbol{p})_-)\,\mathrm{d}\tau\,\mathrm{d}w\,.$$

The fact that $\hat{f}$ does not depend on k_1 on the branch cut implies that the only contribution to the integral comes from the poles of the integrand, which are located at

$$w_{\mathrm{pole}} = -i\tau\sin(\alpha) \pm \cos(\alpha)\sqrt{\tau^2-\Omega^2}\,.$$

Since by assumption $0 < 2\alpha < \pi$, the poles lie in the lower half plane, thus the full contour integral vanishes. Moreover the integrand falls off exponentially in the upper half place, such that the branch cut does not contribute to I_f at all.

Thus we have $I_f(-ix_0, \boldsymbol{x}) = I_{\text{pole}}(-ix_0, \boldsymbol{x})$ and this contribution can be estimated by

$$\begin{aligned}\left|I_f(-ix_0, \boldsymbol{x})\right| &= \left| \int \frac{\mathrm{e}^{-r\sqrt{\boldsymbol{k}^2+m^2}} \hat{f}(\omega(\boldsymbol{k}), \boldsymbol{p}(\boldsymbol{k}))}{2\sqrt{\boldsymbol{k}^2+m^2}} \Big|_{k_0 = i\sqrt{\boldsymbol{k}^2+m^2}} \mathrm{d}^3\boldsymbol{k} \right| \\ &\leq c \int \frac{\mathrm{e}^{-r\sqrt{\boldsymbol{k}^2+m^2}} \mathrm{e}^{R\sqrt{|\omega(\boldsymbol{k})|^2+|p_1(\boldsymbol{k})|^2+k_2^2+k_3^2}}}{2\sqrt{\boldsymbol{k}^2+m^2}} \mathrm{d}^3\boldsymbol{k} \,.\end{aligned}$$

The square root can be estimated

$$\begin{aligned}&\left(|\omega(k_i)|^2 + |p_1(k_i)|^2\right) \Big|_{k_0 = i\sqrt{\boldsymbol{k}^2+m^2}} + k_2^2 + k_3^2 \\ &= \left|\sin(\alpha)\sqrt{\boldsymbol{k}^2+m^2} - ik_1\cos(\alpha)\right|^2 + \left|i\cos(\alpha)\sqrt{\boldsymbol{k}^2+m^2} - k_1\sin(\alpha)\right|^2 + k_2^2 + k_3^2 \\ &\leq 4(\boldsymbol{k}^2+m^2) \,.\end{aligned}$$

This implies that the integral decays exponentially in $|\boldsymbol{x}| = r$ uniformly in r as $r > 2R + \delta$ with $\delta > 0$:

$$\begin{aligned}\left|I_f(-ix_0, \boldsymbol{x})\right| &\leq c \int \frac{\mathrm{e}^{-r\sqrt{\boldsymbol{k}^2+m^2}} \mathrm{e}^{2R\sqrt{\boldsymbol{k}^2+m^2}}}{2\sqrt{\boldsymbol{k}^2+m^2}} \mathrm{d}^3\boldsymbol{k} \leq c\, \mathrm{e}^{-(r-2R)m} \int \frac{\mathrm{e}^{-(r-2R)(\sqrt{\boldsymbol{k}^2+m^2}-m)}}{2\sqrt{\boldsymbol{k}^2+m^2}} \mathrm{d}^3\boldsymbol{k} \\ &\leq c\, \mathrm{e}^{-(r-2R)m} \underbrace{\int \frac{\mathrm{e}^{-\delta(\sqrt{\boldsymbol{k}^2+m^2}-m)}}{2\sqrt{\boldsymbol{k}^2+m^2}} \mathrm{d}^3\boldsymbol{k}}_{<\infty} \,.\end{aligned}$$

This proves the claim. □

Finally we can present the proof of the following theorem about the integrability of the connected correlation functions of the vacuum state. This in turn imply the existence of the adiabatic limit of the interacting vacuum state due to proposition 4.3.2. To this end, we fix the algebra $\mathcal{A}_{\text{os}}$ to be given by a $\star$-product induced by the vacuum Hadamard two-point function (4.13) and the state to be the evaluation at $\phi = 0$.

Theorem 9.
Let ω_{vac} be the vacuum state of the free Klein-Gordon field with mass $m > 0$ induced by the translation invariant two-point function

$$\Delta_+(x,y) = D_+^{\text{vac}}(x-y), \quad D_+^{\text{vac}}(x) = \frac{1}{(2\pi)^3} \int \theta(p_0)\delta(p^2-m^2)\mathrm{e}^{-i\eta_{\mu\nu}p^\mu x^\nu} \,\mathrm{d}p \,. \tag{4.13}$$

Then the connected correlation functions

$$F_n^{\text{vac}}(u_1, \boldsymbol{x}_1; \cdots; u_n, \boldsymbol{x}_n) = \omega_{\text{vac}}^{\text{c}}\left(A_0 \otimes \alpha_{iu_1, \boldsymbol{x}_1}(A_1) \otimes \cdots \otimes \alpha_{iu_n, \boldsymbol{x}_n}(A_n)\right)$$

for $A_0, \ldots, A_n \in \mathcal{A}$ are contained in $L^1(\mathfrak{s}_n^\infty \times \mathbb{R}^{3n})$, where

$$\mathfrak{s}_n^\infty = \lim_{\beta\to\infty} \beta\mathfrak{s}_n = \{(u_1, \ldots, u_n) \in \mathbb{R}^n : 0 < u_1 \leq \cdots \leq u_n\} \,.$$

PROOF. The connected correlation functions ω^c_{vac} can be written in terms of the functional differential operator Γ_2^{ij} from the proof of Proposition 4.2.1, where the KMS two-point function D^β_+ has to be replaced by D^{vac}_+. The correlation function ω_{vac} itself ca be written as

$$\omega_{\mathrm{vac}}(A_0 \otimes A_1 \otimes \cdots \otimes A_n) = \prod_{0\leq i<j\leq n} \mathrm{e}^{\Gamma_2^{ij}} (A_0 \otimes \cdots \otimes A_n)\Big|_{\phi_0\otimes\cdots\otimes\phi_n=0} .$$

Here the product of exponentials can be rewritten as

$$\prod_{0\leq i<j\leq n} \mathrm{e}^{\Gamma_2^{ij}} = \prod_{0\leq i<j\leq n} \sum_{m=0}^{\infty} \frac{(\Gamma_2^{ij})^m}{m!} = \sum_{\substack{l=(l_{ij})_{i<j} \\ l_{ij}\in\mathbb{N}_0}} \prod_{i<j} \frac{(\Gamma_2^{ij})^{l_{ij}}}{l_{ij}!}$$

which reads in terms of a graphical expansion

$$\sum_{\substack{l=(l_{ij})_{i<j} \\ l_{ij}\in\mathbb{N}_0}} \prod_{i<j} \frac{(\Gamma_2^{ij})^{l_{ij}}}{l_{ij}!} = \sum_{G\in\mathcal{G}_{n+1}} \Gamma_G, \qquad \Gamma_G = \prod_{i<j} \frac{(\Gamma_2^{ij})^{l_{ij}}}{l_{ij}!}$$

where $\mathcal{G}_n$ denotes the set of all graphs G with n vertices and l_{ij} are the number of lines joining the vertices i and j. Rewriting the products of exponentials in another way and using a similar argument as above one finds

$$\prod_{0\leq i<j\leq n} \mathrm{e}^{\Gamma_2^{ij}} = \prod_{0\leq i<j\leq n} \left(\mathrm{e}^{\Gamma_2^{ij}} - 1 + 1\right) = \sum_{G\in\mathcal{G}_{n+1}} \prod_{G'\in[G]} \prod_{i<j} \left(\frac{(\Gamma_2^{ij})^{l_{ij}}}{l_{ij}!}\right) \tag{4.14}$$

where $[G]$ denotes the set of connected components of G. The connected correlation functions can be consequently written as

$$\omega^{\mathrm{c}}_{\mathrm{vac}}(A_0 \otimes \cdots \otimes A_n) = \sum_{G\in\mathcal{G}^{\mathrm{c}}_{n+1}} \prod_{i<j} \left(\frac{(\Gamma_2^{ij})^{l_{ij}}}{l_{ij}!}\right) (A_0 \otimes \cdots \otimes A_n)\Big|_{\phi_0\otimes\cdots\otimes\phi_n=0}$$

where $\mathcal{G}^{\mathrm{c}}_n$ denotes only the connected graphs with n vertices. The last equation can be verified by showing that the recursion formula for the connected correlation functions pick out exactly the connected components in the graphical expansion on the RHS of (4.14).

Then the functions F^{vac}_n can be written as

$$\begin{aligned}
F^{\mathrm{vac}}_n(u_1,\boldsymbol{x}_1;\cdots;u_n,\boldsymbol{x}_n) &= \omega^{\mathrm{c}}_{\mathrm{vac}}\left(A_0 \otimes \alpha_{iu_1,\boldsymbol{x}_1}(A_1) \otimes \cdots \otimes \alpha_{iu_n,\boldsymbol{x}_n}(A_n)\right) \\
&= \sum_{G\in\mathcal{G}^{\mathrm{c}}_{n+1}} \prod_{i<j} \left(\frac{(\Gamma_2^{ij})^{l_{ij}}}{l_{ij}!}\right) (A_0 \otimes \cdots \otimes \alpha_{iu_n,\boldsymbol{x}_n}(A_n))\Big|_{\phi_0\otimes\cdots\otimes\phi_n=0} \\
&=: \sum_{G\in\mathcal{G}^{\mathrm{c}}_{n+1}} \frac{1}{\mathrm{Symm}(G)} F^{\mathrm{vac}}_{n,G}(u_1,\boldsymbol{x}_1;\cdots;u_n,\boldsymbol{x}_n) \\
F^{\mathrm{vac}}_{n,G}(u_1,\boldsymbol{x}_1;\cdots;u_n,\boldsymbol{x}_n) &= \prod_{i<j} (\Gamma_2^{ij})^{l_{ij}} (A_0 \otimes \alpha_{iu_1,\boldsymbol{x}_1}(A_n) \otimes \cdots \otimes \alpha_{iu_n,\boldsymbol{x}_n}(A_n))\Big|_{\phi_0\otimes\cdots\otimes\phi_n=0}
\end{aligned}$$

similar to the terminology of the proof of Lemma 4.2.1. The source and range of the line l is denoted with $s(l)$ and $r(l)$ respectively here and X, Y contain all points which are connected by the lines $l \in E(G)$. The last line defined the contribution of G to F_n^{vac} and $\text{Symm}(G)$ is the symmetry factor of G. Instead of indexing all vertices we can also index the graph G by all its edges $l \in E(G)$. The contribution of a fixed, connected graph G can be calculated to:

$$\begin{aligned}
&F_{n,G}^{\text{vac}}(u_1, \boldsymbol{z}_1; \cdots; u_n, \boldsymbol{z}_n) \\
&= \int \prod_{l \in E(G)} D_+^{\text{vac}}(x_l - y_l) \frac{\delta^2}{\delta\phi_{s(l)}(x_l)\delta\phi_{r(l)}(y_l)} (A_0 \otimes \cdots \otimes \alpha_{iu_n, \boldsymbol{z}_n}(A_n)) \Big|_{\phi_0 \otimes \cdots \otimes \phi_n = 0} \mathrm{d}X \, \mathrm{d}Y \\
&= \int \prod_{l \in E(G)} D_+^{\text{vac}}(\bar{x}_l - \bar{y}_l) \, \Psi(X, Y) \, \mathrm{d}X \, \mathrm{d}Y
\end{aligned}$$

with the abbreviations $\bar{x}_l = (x_l^0 + iu_{s(l)}, \boldsymbol{x}_l + \boldsymbol{z}_{s(l)})$ and $\bar{y}_l = (y_l^0 + iu_{r(l)}, \boldsymbol{y}_l + \boldsymbol{z}_{r(l)})$ and the functional derivatives

$$\Psi(X, Y) = \prod_{l \in E(G)} \frac{\delta^2}{\delta\phi_{s(l)}(x_l)\delta\phi_{r(l)}(y_l)} \, (A_0 \otimes \cdots \otimes A_n) \Big|_{\phi_0 \otimes \cdots \otimes \phi_n = 0} .$$

The $F_{n,G}^{\text{vac}}$ can be written as integrals in momentum space

$$\begin{aligned}
F_{n,G}^{\text{vac}}(U, \boldsymbol{Z}) &= \int \prod_{l \in E(G)} \mathrm{e}^{-p_l^0(u_{r(l)} - u_{s(l)}) + i\boldsymbol{p}_l(\boldsymbol{z}_{s(l)} - \boldsymbol{z}_{r(l)})} \hat{D}_+^{\text{vac}}(p_l) \hat{\Psi}(-P, P) \, \mathrm{d}P \\
&= \int \prod_{l \in E(G)} \left(\mathrm{e}^{-\omega_{\boldsymbol{p}_l}(u_{r(l)} - u_{s(l)}) + i\boldsymbol{p}_l(\boldsymbol{z}_{s(l)} - \boldsymbol{z}_{r(l)})} \frac{1}{2\omega_{\boldsymbol{p}_l}} \right) \hat{\Psi}(-P, P) \Big|_{p_l^0 = \omega_{\boldsymbol{p}_l}} \mathrm{d}\boldsymbol{P}
\end{aligned}$$

where $\omega_{\boldsymbol{p}_l} = \sqrt{\boldsymbol{p}_l^2 + m^2}$. By proposition 4.3.3 we know that $\hat{\Psi}(-P, P)$ is rapidly decreasing in the forward lightcone and since $\operatorname{supp} \hat{D}_+^{\text{vac}} \subset H_m$ with

$$H_m = \{p \in M : p_0^2 - \boldsymbol{p}^2 = m^2\} \subset J^+$$

the above integral converges absolutely since by assumption $u_{r(l)} - u_{s(l)} > 0$. Therefore we can make use of proposition 4.3.4 and obtain the estimate

$$\left| F_{n,G}^{\text{vac}}(u_1, \ldots, u_n, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \right| \leq c \, \mathrm{e}^{-mr}, \qquad r^2 = \sum_{l \in G} \left(u_{r(l)} - u_{s(l)} \right)^2 + \left| \boldsymbol{x}_{r(l)} - \boldsymbol{x}_{s(l)} \right|^2 .$$

Since the graph G is connected, i.e. every vertex can be reached from $(u_0, \boldsymbol{x}_0) = 0$, we can use

$$\begin{aligned}
r &= \sqrt{\sum_{l \in G} \left(u_{r(l)} - u_{s(l)} \right)^2 + \left| \boldsymbol{x}_{r(l)} - \boldsymbol{x}_{s(l)} \right|^2} \geq \sqrt{\sum_{i<j} (u_j - u_i)^2 + \left| \boldsymbol{x}_j - \boldsymbol{x}_i \right|^2} \\
&\geq \max_{i \in \{1, \ldots, n\}} u_i + |\boldsymbol{x}_i| \geq \sqrt{\frac{1}{n} \sum_{i=1}^n u_i^2 + |\boldsymbol{x}_i|^2} = \frac{1}{\sqrt{n}} \sqrt{\sum_{i=1}^n u_i^2 + |\boldsymbol{x}_i|^2}
\end{aligned}$$

which yields

$$\left|F^{\mathrm{vac}}_{n,G}(u_1,\dots,u_n,\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)\right| \leq c' \mathrm{e}^{-\frac{m}{\sqrt{n}} r_e}, \qquad r_e^2 = \sum_{i=1}^{n} u_i^2 + |\boldsymbol{x}_i|^2 \ .$$

This shows that $F^{\mathrm{vac}}_{n,G}$ actually decays exponentially in every variable $(u_i, \boldsymbol{x}_i)$, instead of only in the difference variables. Consequently also the summed expression

$$F^{\mathrm{vac}}_{n}(u_1,\dots,u_n,\boldsymbol{x}_1,\dots,\boldsymbol{x}_n) = \sum_{G\in\mathcal{G}^{\mathrm{c}}_{n+1}} \frac{1}{\mathrm{Symm}(G)} F^{\mathrm{vac}}_{n,G}(u_1,\dots,u_n,\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)$$

is exponentially decaying in its variables and is thus integrable over $\mathfrak{s}^{\infty}_n \times \mathbb{R}^{3n}$. □

Using the analytic properties of the free vacuum state ω_{vac} we know that, for all $A \in \mathcal{A}$ with $\mathrm{supp}(A) \subset \mathbb{R} \times B_r$ the function $t \mapsto \omega_{\mathrm{vac}}(A \star W_h(t))$ admits an analytic extension to the full upper half plane, using the first point proposition 4.2.1 for the limiting case $\beta \to +\infty$. In particular the linear functional

$$\begin{aligned}\omega^{\mathcal{V}}_{\mathrm{vac},h}(A) &= \frac{\omega_{\mathrm{vac}}(A \star W_h(i\infty))}{\omega_{\mathrm{vac}}(W_h(i\infty))} \\ &= \sum_{n=0}^{\infty} (-1)^n \int_{\mathfrak{s}^{\infty}_n} \omega^{\mathrm{c}}_{\mathrm{vac}}\left(A \otimes \alpha_{iu_1}(K_h) \otimes \cdots \otimes \alpha_{iu_n}(K_h)\right) \mathrm{d}u_1 \cdots \mathrm{d}u_n\end{aligned}$$

exists and is positive. The last theorem implies that the adiabatic limit

$$\begin{aligned}\omega^{\mathcal{V}}_{\mathrm{vac}}(A) = \lim_{h\to 1} \omega^{\mathcal{V}}_{\mathrm{vac},h}(A) &= \sum_{n=0}^{\infty} \int_{\mathfrak{s}^{\infty}_n} \int_{\mathbb{R}^{3n}} \omega^{\mathrm{c}}_{\mathrm{vac}}\left(A \otimes \alpha_{iu_1,\boldsymbol{x}_1}(R) \otimes \cdots \otimes \alpha_{iu_n,\boldsymbol{x}_n}(R)\right) \mathrm{d}U\, \mathrm{d}\boldsymbol{X} \\ R &= \int \mathcal{R}_{\mathcal{V}_\chi}\left(\mathcal{V}(\Phi_{(t,0)})\right) \Theta^-(t)\dot{\chi}(t)\, \mathrm{d}t\end{aligned}$$

exists. Moreover, we find that the function

$$t \mapsto \omega^{\mathcal{V}}_{\mathrm{vac}}(A \star \alpha^{\mathcal{V}}_t(B)) = \frac{\omega_{\mathrm{vac}}(A \star W_1(t) \star \alpha_t(B) \star U_1(t, i\infty))}{\omega_{\mathrm{vac}}(W_1(i\infty))}$$

has an analytic continuation into the whole upper half plane. Moreover the state $\omega^{\mathcal{V}}$ is $\mathfrak{P}^{\uparrow}_{+}$-invariant as a consequence of the $\mathfrak{P}^{\uparrow}_{+}$-invariance of the chosen $\star$-product and free vacuum state $\omega_{\mathrm{vac}} = \mathrm{ev}_0$. Thus $\omega^{\mathcal{V}}_{\beta}$ is a $\mathfrak{P}^{\uparrow}_{+}$-invariant ground state.

4.3.4. Proof for the KMS State

In order to show that the quasi-free KMS state for $0 < \beta < \infty$ and $m > 0$ fulfills the integrability condition from Proposition 4.3.2 we first have to state a slightly modified version of Proposition 4.3.4.

Proposition 4.3.5.
Let $f \in \mathcal{D}'(\mathbb{R}^4)$ with $\mathrm{supp}(f) \subset B_R \subset M$. Then the function

$$I^b_f(x_0, \boldsymbol{x}) = \int \mathrm{e}^{i\eta_{\mu\nu} p^\mu x^\nu} \mathrm{e}^{-b\omega_p} \hat{f}(-\omega_p, \boldsymbol{p})\, \mathrm{d}p, \qquad \omega_p = \sqrt{\boldsymbol{p}^2 + m^2},\ b > 0$$

has an analytic continuation into the lower half plane $\mathbb{C}_- \times \mathbb{R}^3$ *and for* $m > 0$ *it holds*

$$\left| I_f^b(-iu, \boldsymbol{x}) \right| \leq c\, \mathrm{e}^{-mr}, \qquad r = \sqrt{u^2 + \boldsymbol{x}^2}, \quad u \geq 0\,.$$

uniform for $r \geq 2R$.

PROOF. The proof of Proposition 4.3.5 is very similar to the proof of Proposition 4.3.4. A difference in this version is that here $\hat{f}$ is not of rapid decrease since $f \in \mathcal{D}'$. $\hat{f}$ is a polynomially bounded function, which increases exponentially in imaginary directions by another version of the Paley-Wiener theorem. The convergence of the integral is assured by the exponential $\mathrm{e}^{-b\omega_p}$. We begin by rewriting

$$I_f^b(-iu, \boldsymbol{x}) = \frac{1}{2\pi} \int \frac{\mathrm{e}^{ik_0 r} \mathrm{e}^{-b\omega} \hat{f}(-\omega, \boldsymbol{p})}{k_0^2 + \boldsymbol{k}^2 + m^2}\, \mathrm{d}k_0\, \mathrm{d}\boldsymbol{k}, \quad r = \sqrt{u^2 + |\boldsymbol{x}|^2}$$

where

$$\begin{aligned} \omega(k_i) &= \sqrt{(k_0 \cos(\alpha) - k_1 \sin(\alpha))^2 + k_2^2 + k_3^2 + m^2} \\ \boldsymbol{p}(k_i) &= (k_0 \cos(\alpha) - k_1 \sin(\alpha), k_2, k_3)\,. \end{aligned}$$

and $0 < 2\alpha < \pi$. By choosing an integration contour C as in figure 6 for k_0 we see that I_f^b can be calculated using the residue theorem (the contribution from the branch cut vanishes again):

$$\begin{aligned} I_f^b(-iu, \boldsymbol{x}) &= \int \frac{\mathrm{e}^{-r\sqrt{\boldsymbol{k}^2+m^2}} \mathrm{e}^{-b\omega(\boldsymbol{k})} \hat{f}(-\omega(\boldsymbol{k}), \boldsymbol{p}(\boldsymbol{k}))}{2\sqrt{\boldsymbol{k}^2 + m^2}} \Big|_{k_0 = i\sqrt{\boldsymbol{k}^2+m^2}} \mathrm{d}\boldsymbol{k} \\ \omega(\boldsymbol{k}) &= \sqrt{\boldsymbol{k}^2 + m^2} \sin(\alpha) - ik_1 \cos(\alpha), \quad p_1(\boldsymbol{k}) = i\sqrt{\boldsymbol{k}^2 + m^2} \cos(\alpha) - k_1 \sin(\alpha)\,. \end{aligned}$$

We apply the exponential estimate from the Paley-Wiener theorem (with $\mathrm{supp}(f) \subset B_R$)

$$\left| \hat{f}(\omega, \boldsymbol{p}) \right| \leq c\, \mathrm{e}^{R\sqrt{|\omega|^2 + |\boldsymbol{p}|^2}}$$

to establish

$$\begin{aligned} \left| I_f^b(-iu, \boldsymbol{x}) \right| &\leq c \int \frac{\mathrm{e}^{(2R-b-r)\sqrt{\boldsymbol{k}^2+m^2}}}{2\sqrt{\boldsymbol{k}^2 + m^2}}\, \mathrm{d}\boldsymbol{k} \\ &\leq c\, \mathrm{e}^{-m(r-2R+b)} \underbrace{\int \frac{\mathrm{e}^{-(r-2R+b)(\sqrt{\boldsymbol{k}^2+m^2}-m)}}{2\sqrt{\boldsymbol{k}^2 + m^2}}\, \mathrm{d}\boldsymbol{k}}_{\leq \infty}\,. \end{aligned}$$

Thus

$$\left| I_f^b(-iu, \boldsymbol{x}) \right| \leq c\, \mathrm{e}^{-m(r-2R+b)}, \qquad r = \sqrt{u^2 + |\boldsymbol{x}|^2}$$

uniformly for $r \geq 2R$. □

Now the main theorem of this work can be proven which shows, that the KMS state of perturbatively constructed massive scalar field theories exists. The proof of this fact is very similar to the case of the vacuum, as far as the perturbative expansions are concerned. A main difference arises in the investigation of the decay behavior due to the fact that the KMS state has contributions from positive and negative energies. The KMS condition and the exponential decay of the negative energy part turn out to be crucial to show the convergence of the state in the adiabatic limit.

Theorem 10.
Let ω_β be the KMS state of the free Klein-Gordon field with mass $m > 0$ and inverse temperature $0 < \beta < \infty$, induced by the translation invariant Hadamard two-point function

$$\Delta_+(x,y) = D_+^\beta(x-y), \quad D_+^\beta(x) = \frac{1}{(2\pi)^3}\int \frac{\varepsilon(p_0)\delta(p^2-m^2)}{1-\mathrm{e}^{-\beta p_0}}\mathrm{e}^{-i\eta_{\mu\nu}p^\mu x^\nu}\,\mathrm{d}p\,. \tag{4.15}$$

Then the connected correlation functions

$$F_n^\beta(u_1,\boldsymbol{x}_1;\ldots;u_n,\boldsymbol{x}_n) = \omega_\beta^{\mathrm{c}}\left(A_0\otimes\alpha_{iu_1,\boldsymbol{x}_1}(A_1)\otimes\cdots\otimes\alpha_{iu_n,\boldsymbol{x}_n}(A_n)\right)$$

for $A_0,\ldots,A_n \in \mathcal{A}$ are contained in $L^1(\beta\mathfrak{s}_n\times\mathbb{R}^{3n})$ with $\beta\mathfrak{s}_n$ from proposition 4.3.1.

PROOF. We proceed in the same manner as for the vacuum state. To this end we write

$$\begin{aligned}
F_n^\beta(u_1,\boldsymbol{x}_1;\ldots;u_n,\boldsymbol{x}_n) &= \omega_\beta^{\mathrm{c}}\left(A_0\otimes\alpha_{iu_1,\boldsymbol{x}_1}(A_1)\otimes\cdots\otimes\alpha_{iu_n,\boldsymbol{x}_n}(A_n)\right)\\
&= \sum_{G\in\mathcal{G}_{n+1}^{\mathrm{c}}}\frac{1}{\mathrm{Symm}(G)}F_{n,G}^\beta(u_1,\boldsymbol{x}_1;\ldots;u_n,\boldsymbol{x}_n)\\
F_{n,G}^\beta(u_1,\boldsymbol{x}_1;\ldots;u_n,\boldsymbol{x}_n) &= \prod_{i<j}(\Gamma_2^{ij})^{l_{ij}}\left(A_0\otimes\alpha_{iu_1,\boldsymbol{x}_1}(A_1)\otimes\cdots\otimes\alpha_{iu_n,\boldsymbol{x}_n}(A_n)\right)\Big|_{\phi_0\otimes\cdots\otimes\phi_n=0}
\end{aligned}$$

where the Γ_2^{ij} are the functional differential operators from the proof of proposition 4.2.1. The differential operator is now re-written in terms of the vacuum two-point function

$$\Gamma_+^{ij} = \int D_+^\beta(x^0-y^0,\boldsymbol{x}-\boldsymbol{y})\frac{\delta^2}{\delta\phi_i(x)\delta\phi_j(y)}\,\mathrm{d}x\,\mathrm{d}y$$

By switching to a product over the lines $l\in E(G)$ of the graph G we find

$$\begin{aligned}
F_{n,G}^\beta(U,\boldsymbol{Z}) &= \int\prod_{l\in E(G)}\mathrm{e}^{p_l^0(u_{s(l)}-u_{r(l)})+i\boldsymbol{p}_l(\boldsymbol{z}_{s(l)}-\boldsymbol{z}_{r(l)})}\hat{D}_+^\beta(p_l)\hat{\Psi}(-P,P)\,\mathrm{d}P\\
&= \int\prod_{l\in E(G)}\frac{\mathrm{e}^{i\boldsymbol{p}_l(\boldsymbol{z}_{s(l)}-\boldsymbol{z}_{r(l)})}\left(\lambda_+(p_l)+\lambda_-(p_l)\right)}{2\omega_l\left(1-\mathrm{e}^{-\beta\omega_l}\right)}\hat{\Psi}(-P,P)\,\mathrm{d}P\\
\lambda_+(p_l) &= \mathrm{e}^{\omega_l(u_{s(l)}-u_{r(l)})}\delta(p_l^0-\omega_l), \qquad \lambda_-(p_l) = \mathrm{e}^{-\beta\omega_l}\mathrm{e}^{\omega_l(u_{r(l)}-u_{s(l)})}\delta(p_l^0+\omega_l)
\end{aligned}$$

with $\omega_l \equiv \omega_{p_l}$ in analogy to the calculation for the vacuum state. The difference here is that $\hat{D}_+^\beta$ is not supported purely in the forward lightcone, it has positive and negative

mass-shell part $\lambda_\pm$. The functional derivatives are, again, given by

$$\Psi(X,Y) = \prod_{l\in E(G)} \frac{\delta^2}{\delta\phi_{s(l)}(x_l)\delta\phi_{r(l)}(y_l)} \; (A_0\otimes\cdots\otimes A_n)\Big|_{\phi_0\otimes\cdots\otimes\phi_n=0}. \tag{4.16}$$

Due to the fact, that the integration momenta p_l can lie in both the forward and backward lightcone, we cannot use the same argumentation as in the case of the vacuum state. In order to prove the convergence of the integral we will show that all negative energy parts λ_- are actually exponentially decreasing.

To this end, we use the KMS condition in the original function

$$F_n^\beta(u_1,\boldsymbol{x}_1;\ldots;u_n,\boldsymbol{x}_n) = \omega_\beta^c\left(A_0\otimes\alpha_{iu_1,\boldsymbol{x}_1}(A_1)\otimes\cdots\otimes\alpha_{iu_n,\boldsymbol{x}_n}(A_n)\right)$$

together with the identification $(u_0,\boldsymbol{x}_0)=0$ to rearrange the time-translations in imaginary directions:

$$\begin{aligned}&\omega_\beta^c\left(A_0\otimes\alpha_{iu_1,\boldsymbol{x}_1}(A_1)\otimes\cdots\otimes\alpha_{iu_n,\boldsymbol{x}_n}(A_n)\right)\\ =&\omega_\beta^c\Big(\alpha_{iu_m,\boldsymbol{x}_m}(A_m)\otimes\cdots\otimes\alpha_{iu_n,\boldsymbol{x}_n}(A_n)\otimes\alpha_{i\beta}(A_0)\otimes\alpha_{i(u_1+\beta),\boldsymbol{x}_1}(A_1)\otimes\cdots\\ &\cdots\otimes\alpha_{i(u_{m-1}+\beta),\boldsymbol{x}_{m-1}}(A_{m-1})\Big).\end{aligned}$$

The equality holds irrespective of the choice $m\in\{1,\ldots,n\}$. The non-trivial point is made now: There exists an $m\in\{1,\ldots,n\}$ such that $u_m-u_{m-1}\geq\frac{\beta}{n+1}$. We simply rename all of the variables to

$$F_{n,G}^\beta(U,\boldsymbol{X}) = F'^\beta_{n,G}(V,\boldsymbol{Y}) = \omega_\beta^c\left(\alpha_{iv_0,\boldsymbol{y}_0}(B_0)\otimes\alpha_{iv_1,\boldsymbol{y}_1}(B_1)\otimes\cdots\otimes\alpha_{iv_n,\boldsymbol{y}_n}(B_n)\right)$$

where $B_0=A_m, B_1=A_{m+1},\ldots,B_n=A_{m-1}$,

$$\begin{pmatrix}v_0\\ \vdots\\ v_{n-m}\end{pmatrix} = \begin{pmatrix}u_m\\ \vdots\\ u_n\end{pmatrix}, \qquad \begin{pmatrix}v_{n-m+1}\\ \vdots\\ v_n\end{pmatrix} = \begin{pmatrix}u_0+\beta\\ \vdots\\ u_{m-1}+\beta\end{pmatrix} \tag{4.17}$$

and the similar relabeling is done for the spatial variables $\boldsymbol{y}_i$ with respect to $\boldsymbol{x}_i$. Now the analogous derivation for F' yields

$$\begin{aligned}&F'^\beta_{n,G}(v_0,\boldsymbol{y}_0;\ldots;v_n,\boldsymbol{y}_n) = \int\prod_{l\in E(G)}\frac{\mathrm{e}^{i\boldsymbol{p}_l(\boldsymbol{y}_{s(l)}-\boldsymbol{y}_{r(l)})}\left(\lambda_+(p_l)+\lambda_-(p_l)\right)}{2\omega_l\left(1-\mathrm{e}^{-\beta\omega_l}\right)}\hat{\Psi}_B(-P,P)\,\mathrm{d}P\\ &\lambda_+(p_l)=\mathrm{e}^{\omega_l(v_{s(l)}-v_{r(l)})}\delta(p_l^0-\omega_l), \qquad \lambda_-(p_l)=\mathrm{e}^{\omega_l(v_{r(l)}-v_{s(l)}-\beta)}\delta(p_l^0+\omega_l)\end{aligned}$$

where now $v_0\leq v_1\leq\ldots\leq v_n$ and Ψ_B is the functional derivative from equation (4.16) in which the (A_i) are replaced by (B_i).

We expand the products of the sum of $\lambda_\pm$ by replacing every line $l\in E(G)$ by either a line l_+ or l_-, to which we associate the factors $\lambda_\pm$, and summing over all possibilities to distribute pluses and minuses on all lines in G. This is done by introducing a function

$$\varepsilon : E(G)\to\{+,-\}$$

that associates signs to all the lines in the graph. Denoting $E_\pm(G) = \{l \in G : \varepsilon(l) = \pm\}$ we find

$$\begin{aligned}
&F'^{\beta}_{n,G}(v_0, \boldsymbol{y}_0; \ldots; v_n, \boldsymbol{y}_n) \\
&= \sum_\varepsilon \int \prod_{l_+ \in E_+(G)} \left[\frac{\mathrm{e}^{i\boldsymbol{p}_{l_+}(\boldsymbol{y}_{s(l_+)} - \boldsymbol{y}_{r(l_+)})}}{2\omega_{l_+}\left(1 - \mathrm{e}^{-\beta\omega_{l_+}}\right)} \lambda_+(p_{l_+}) \right] \times \\
&\quad \times \prod_{l_- \in E_-(G)} \left[\frac{\mathrm{e}^{i\boldsymbol{p}_{l_-}(\boldsymbol{y}_{s(l_-)} - \boldsymbol{y}_{r(l_-)})}}{2\omega_{l_-}\left(1 - \mathrm{e}^{-\beta\omega_{l_-}}\right)} \lambda_-(p_{l_-}) \right] \hat{\Psi}_B(-P, P)\, \mathrm{d}P.
\end{aligned}$$

Now we estimate the largest difference between the v_i

$$\max_{i<j}(v_j - v_i) = v_{n-1} - v_0 = \beta + u_{m-1} - u_m = \beta - (u_m - u_{m-1}) \leq \underbrace{\frac{\beta}{n+1}}_{=:c_\beta} < \beta\,.$$

Thus we rewrite

$$\mathrm{e}^{-\omega_{l_-}\beta}\mathrm{e}^{\omega_{l_-}(v_{r(l_-)} - v_{s(l_-)})} = \mathrm{e}^{-\omega_{l_-}(\beta - c_\beta)}\mathrm{e}^{\omega_{l_-}(v_{r(l_-)} - v_{s(l_-)} - c_\beta)} = \mathrm{e}^{-\omega_{l_-}\frac{n\beta}{n+1}} \underbrace{\mathrm{e}^{\omega_{l_-}(v_{r(l_-)} - v_{s(l_-)} - c_\beta)}}_{\leq 1}$$

which shows the claim, that the integrand of F'^{β}_n decays fast in the momentum variables associated to lines l_-. The remaining integration variables (those associated to l_+) are located in the forward lightcone, due to the form of λ_+. Thus the argument, shown in the discussion of the vacuum state, can be applied to those variables. This implies that

$$\prod_{l_- \in E_-(G)} \mathrm{e}^{\omega_{l_-}(v_{r(l_-)} - v_{s(l_-)} - \beta)} \hat{\Psi}_B(-P, P)\Big|_{p^0_{l_+} = \omega_{l_+},\, p^0_{l_-} = -\omega_{l_-}}$$

is rapidly decreasing in all spatial momenta $\boldsymbol{P} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{E(G)})$. We use the geometric series

$$\frac{1}{1 - \mathrm{e}^{-\beta\omega}} = \sum_{n=0}^{\infty} \mathrm{e}^{-\beta n\omega}$$

to rewrite the integrand

$$\begin{aligned}
&\sum_\varepsilon \prod_{l_+ \in E_+(G)} \frac{1}{2\omega_{l_+}} \frac{\mathrm{e}^{i\boldsymbol{p}_{l_+}(\boldsymbol{y}_{s(l_+)} - \boldsymbol{y}_{r(l_+)}) - \omega_{l_+}(v_{r(l_+)} - v_{s(l_+)})}}{1 - \mathrm{e}^{-\beta\omega_{l_+}}} \times \\
&\qquad \times \prod_{l_- \in E_-(G)} \frac{1}{2\omega_{l_-}} \frac{\mathrm{e}^{i\boldsymbol{p}_{l_-}(\boldsymbol{y}_{s(l_-)} - \boldsymbol{y}_{r(l_-)}) - \omega_{l_-}(c_\beta - (v_{r(l)} - v_{s(l)}))} \mathrm{e}^{-\omega_{l_-}\frac{n\beta}{n+1}}}{1 - \mathrm{e}^{-\beta\omega_{l_-}}} \\
&= \sum_\varepsilon \sum_{\boldsymbol{n} \in \mathbb{N}_0^{|E(G)|}} \prod_{l_+ \in E_+(G)} \frac{1}{2\omega_{l_+}} \mathrm{e}^{i\boldsymbol{p}_{l_+}(\boldsymbol{y}_{s(l_+)} - \boldsymbol{y}_{r(l_+)}) - \omega_{l_+}(v_{r(l_+)} - v_{s(l_+)} + \beta n_{l_+})} \times \\
&\qquad \times \prod_{l_- \in E_-(G)} \frac{1}{2\omega_{l_-}} \mathrm{e}^{i\boldsymbol{p}_{l_-}(\boldsymbol{y}_{s(l_-)} - \boldsymbol{y}_{r(l_-)}) - \omega_{l_-}(c_\beta - (v_{r(l)} - v_{s(l)}) + \beta n_{l_-})} \mathrm{e}^{-\omega_{l_-}\frac{n\beta}{n+1}}\,.
\end{aligned}$$

Hence the function F' is of the form

$$\begin{aligned}
&F'^{\beta}_{n,G}(v_0,\boldsymbol{y}_0;\ldots;v_n,\boldsymbol{y}_n)\\
&=\sum_{\varepsilon}\sum_{\boldsymbol{n}\in\mathbb{N}_0^{|E(G)|}}\int\prod_{l_+\in E_+(G)}\frac{1}{2\omega_{l_+}}\mathrm{e}^{i\boldsymbol{p}_{l_+}(\boldsymbol{y}_{s(l_+)}-\boldsymbol{y}_{r(l_+)})}\mathrm{e}^{-\omega_{l_+}(v_{r(l_+)}-v_{s(l_+)}+\beta n_{l_+})}\times\\
&\qquad\times\prod_{l_-\in E_-(G)}\frac{1}{2\omega_{l_-}}\mathrm{e}^{i\boldsymbol{p}_{l_-}(\boldsymbol{y}_{s(l_-)}-\boldsymbol{y}_{r(l_-)})}\mathrm{e}^{-\omega_{l_-}(c_\beta-(v_{r(l)}-v_{s(l)})+\beta n_{l_-})}\;\Xi(\boldsymbol{P})\,\mathrm{d}\boldsymbol{P}\\
&\Xi(\boldsymbol{P})=\prod_{l_-\in E_-(G)}\mathrm{e}^{-\omega_{l_-}\frac{n\beta}{n+1}}\hat{\Psi}_B(-P,P)\Big|_{p^0_{l_+}=\omega_{l_+},\,p^0_{l_-}=-\omega_{l_-}}\,.
\end{aligned}$$

By the above argumentation, $\Xi(\boldsymbol{P})$ is rapidly decreasing in all its variables. Fixing the sign-function ε and a multi-index $\boldsymbol{n}$, we can use Proposition 4.3.4 for the positive energies and Proposition 4.3.5 for the negative energies to find the estimate

$$F'^{\beta}_{n,G,\varepsilon,\boldsymbol{n}}(v_0,\boldsymbol{y}_0;\ldots,v_n\boldsymbol{y}_n)\leq c\prod_{l\in E(G)}\mathrm{e}^{-m\sqrt{|\boldsymbol{x}_{\partial l}|^2+(\beta n_l)^2}}$$

where $\boldsymbol{x}_{\partial l}=\boldsymbol{x}_{r(l)}-\boldsymbol{x}_{s(l)}$. In this estimate we used the fact, that the v_i range only over a finite intervall and the differences

$$c_\beta-(v_{r(l)}-v_{s(l)})\geq 0$$

are bounded from below by zero. The sum over n_l yields

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathrm{e}^{-m\sqrt{q^2+(\beta n)^2}}&=\sum_{\beta n<q}\mathrm{e}^{-m\sqrt{q^2+(\beta n)^2}}+\sum_{\beta n\geq q}\mathrm{e}^{-m\sqrt{q^2+(\beta n)^2}}\\
&\leq\frac{q}{\beta}\mathrm{e}^{-mq}+\frac{\mathrm{e}^{-mq}}{1-\mathrm{e}^{-m\beta}}\leq c'\,\mathrm{e}^{-mq}
\end{aligned}$$

for $q>0$. This implies that

$$F'^{\beta}_{n,G,\epsilon}(V,\boldsymbol{Y})=\sum_{\boldsymbol{n}\in\mathbb{N}^{E(G)}}F'^{\beta}_{n,G,\varepsilon,\boldsymbol{n}}(V,\boldsymbol{Y})\leq cc'\prod_{i<j}\mathrm{e}^{-m\sqrt{|\boldsymbol{x}_j-\boldsymbol{x}_i|^2}}\leq c''\mathrm{e}^{-\frac{m}{\sqrt{n}}r_e},\quad r_e=\sqrt{\sum_{k=0}^{n}|\boldsymbol{x}_k|^2}$$

by the same means as in the case of the vacuum state. The exponential decay for $F^{\beta}_{n,G'}$ i.e.

$$\left|F^{\beta}_{n,G}(u_1,\ldots,u_n,\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\right|\leq c'\,\mathrm{e}^{-\frac{m}{\sqrt{n}}r_e},\qquad r^2=\sum_{k=1}^{n}\boldsymbol{x}_k^2\,,$$

follows by the simple coordinate change in equation (4.17), thus $F^{\beta}_{n,G}$ decays exponentially in all its variables. The same decay properties hold for F^{β}_{n}, which is the sum over all connected graphs of $F^{\beta}_{n,G}$ divided by the symmetry factor of G. This proves the assertion. □

As in the vacuum case we can exploit the analytic properties of the KMS state ω_β to show that the limiting state obeys the KMS condition by using proposition 4.2.1. We find an explicit formula for the adiabatic limit of the state $\omega^{\mathcal{V}}_{\beta,h}$

$$\omega^{\mathcal{V}}_\beta(A) = \lim_{h\to 1} \omega^{\mathcal{V}}_{\beta,h}(A) = \sum_{n=0}^{\infty} \int_{\beta\mathfrak{s}_n} \int_{\mathbb{R}^{3n}} \omega^{\mathrm{c}}_\beta \left(A \otimes \alpha_{iu_1,\boldsymbol{x}_1}(R) \otimes \cdots \otimes \alpha_{iu_n,\boldsymbol{x}_n}(R)\right) \mathrm{d}U \, \mathrm{d}\boldsymbol{X}$$
$$R = \int \mathcal{R}_{\mathcal{V}_\chi}\left(\mathcal{V}(\Phi_{(t,0)})\right) \Theta^-(t)\dot{\chi}(t)\,\mathrm{d}t$$

exists, defines a state on $\mathcal{A}$, and the function

$$t \mapsto \omega^{\mathcal{V}}_\beta(A \star \alpha^{\mathcal{V}}_t(B)) = \frac{\omega_\beta(A \star W_1(t) \star \alpha_t(B) \star U_1(t, i\beta))}{\omega_\beta(W_1(i\beta))}$$

has an analytic continuation into the strip S_β and is continuous on the boundary with the value

$$\omega^{\mathcal{V}}_\beta(A \star \alpha^{\mathcal{V}}_{t+i\beta}(B)) = \omega^{\mathcal{V}}_\beta(\alpha^{\mathcal{V}}_t(B) \star A)\,.$$

In order to prove these statements one has simply to replace the limiting KMS state $\omega^{\mathcal{V}}_\beta$ with the ones on the finite volume $\omega^{\mathcal{V}}_{\beta,h}$ in the proof of proposition 4.2.2. Since the arising integrands are absolutely integrable, we can exchange the limits in the integrations and obtain the desired statements. In particular we find that, due to the fact that the free KMS state is invariant under all spacetime translations and spatial rotations, so is the interacting state $\omega^{\mathcal{V}}_\beta$ in the adiabatic limit.

4.4. Thermostatics

In the last section we provided a construction of the interacting KMS state, in particular we gave prescription to calculate the expectation values of interacting observables in this state as a formal power series in the interaction. In this section how thermostatic quantities can be derived within this framework using the **partition function**.

In order to find the connection of the present approach to the well-known thermostatic notions in Hamiltonian dynamical systems, we will make a quick detour into this framework.

4.4.1. Analogy in Hamiltonian Dynamical Systems

Assume that the algebra of observables $\mathfrak{A}$ is given by a *-subalgebra of $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is the Hilbert space of the theory. Moreover, we have a free α_t and an interacting dynamics α^V_t on $\mathfrak{A}$, which are generated by the self-adjoint Hamiltonians H_0 and $H_0 + V$.

Assume further that there exist two Gibbs states associated to both dynamics, i.e. the self-adjoint operators

$$\mathrm{e}^{-\beta H_0} \qquad \text{and} \qquad \mathrm{e}^{-\beta(H_0+V)}$$

are trace class. Then the co-cycle

$$W(t) = \mathrm{e}^{-itH_0}\mathrm{e}^{it(H_0+V)}$$

intertwines the free and interacting dynamics and

$$\omega_\beta^V(A) = \frac{\mathrm{Tr}\left(\mathrm{e}^{-\beta(H_0+V)}A\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta(H_0+V)}\right)} = \frac{\mathrm{Tr}\left(\mathrm{e}^{-\beta H_0}AW(i\beta)\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta H_0}W(i\beta)\right)} = \frac{\omega_\beta(AW(i\beta))}{\omega_\beta(W(i\beta))}\,.$$

This brief overview should remind us on the background of the constructions in the last section. The free and interacting **partition function** of the thermal equilibrium are given by

$$Z(\beta) = \mathrm{Tr}\,\mathrm{e}^{-\beta H_0}, \qquad Z^V(\beta) = \mathrm{Tr}\,\mathrm{e}^{-\beta(H_0+V)}\,.$$

From the partition function $Z(\beta)$ we can derive the **mean energy** $\langle E\rangle_\beta$ and the **mean free energy** $\langle F\rangle_\beta$ of the free system

$$\begin{aligned}\langle E\rangle_\beta &= -\frac{\partial}{\partial\beta}\ln Z(\beta) = \frac{\mathrm{Tr}\,H_0\mathrm{e}^{-\beta H_0}}{\mathrm{Tr}\,\mathrm{e}^{\beta H_0}} = \omega_\beta(H_0)\\ \langle F\rangle_\beta &= -\frac{1}{\beta}\ln Z(\beta)\end{aligned}$$

and for the interacting system correspondingly.

Since we do not have access to the partition function in our formalism, we will use the co-cycle $W(i\beta)$ for our calculations. This allows the calculation of the differences

$$\begin{aligned}\delta E &= \langle E\rangle_\beta^V - \langle E\rangle_\beta = -\frac{\partial}{\partial\beta}\ln\frac{Z^V(\beta)}{Z(\beta)}\\ \delta F &= \langle F\rangle_\beta^V - \langle F\rangle_\beta = -\frac{1}{\beta}\ln\frac{Z^V(\beta)}{Z(\beta)}\,.\end{aligned}$$

only. In both cases the **relative partition function**

$$\mathcal{Z}^V(\beta) = \frac{Z^V(\beta)}{Z(\beta)} = \frac{\mathrm{Tr}\,\mathrm{e}^{-\beta(H_0+V)}}{\mathrm{Tr}\,\mathrm{e}^{-\beta H_0}} = \frac{\mathrm{Tr}\,\mathrm{e}^{-\beta H_0}\mathrm{e}^{-\beta(H_0+V)}\mathrm{e}^{\beta H_0}}{\mathrm{Tr}\,\mathrm{e}^{-\beta H_0}} = \omega_\beta(W(i\beta))$$

can be expressed by means of the co-cycle and the free Gibbs state ω_β. The mean energy and the mean free energy are related by the entropy

$$S = \beta k_B(\langle E\rangle_\beta - \langle F\rangle_\beta) = -k_B\,\mathrm{Tr}(\varrho\ln\varrho), \qquad \varrho = \frac{\mathrm{e}^{-\beta H_0}}{\mathrm{Tr}\,\mathrm{e}^{-\beta H_0}}$$

The same relation holds for the entropy S^V of the interacting system and thus for the entropy difference δS, too. This is the starting point for our discussion on the thermostatic observables in the pAQFT setting.

4.4.2. Thermostatic Observables in pAQFT

The relative partition function between the free and interacting system (with interaction $\mathcal{V}$) in a **finite volume** V is defined by

$$\mathcal{Z}_h^{\mathcal{V}}(\beta) = \omega_\beta(W_h(i\beta)) \,.$$

Here the test-function $h \in \mathcal{D}(\mathbb{R}^3)$ is chosen to be equal to one one the volume $V \subset \mathbb{R}^3$. Since we want to deal with the infinitely extended system in the end, we will only be interested in the intensive quantities, i.e. in the (free) energy density or entropy density.

Definition 4.4.1 (Thermostatic Observables).
The difference of the mean energy densities $\delta\mathcal{E}$ and mean free energy densities $\delta\mathcal{F}$ of the free and interacting thermal equilibrium state (with interaction $\mathcal{V}$) is defined to be

$$\delta\mathcal{F} = -\lim_{h\to 1} \frac{1}{\beta} \frac{\ln \mathcal{Z}_h^{\mathcal{V}}(\beta)}{\int h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}} \,, \qquad \delta\mathcal{E} = -\lim_{h\to 1} \frac{\partial}{\partial\beta} \frac{\ln \mathcal{Z}_h^{\mathcal{V}}(\beta)}{\int h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}} \,.$$

The difference of the entropy densities is given by $\delta S = \beta k_B(\delta\mathcal{E} - \delta\mathcal{F})$.

A first comment concerns well-definedness of the logarithm of the relative partition function $\mathcal{Z}_h^{\mathcal{V}}(\beta)$: The composition of ln and $\mathcal{Z}_h^{\mathcal{V}}$ is defines again a formal power series in the interaction $\mathcal{V}$, since $\mathcal{Z}_h^{\lambda\mathcal{V}}(\beta) = 1 + \mathcal{O}(\lambda)$, due to fact that $W_h(i\beta)$ also starts with the identity.

The respective quantities at finite volume, e.g. the difference of the free energy-densities

$$-\frac{1}{\beta} \frac{\ln \mathcal{Z}_h^{\mathcal{V}}(\beta)}{\int h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}$$

approximates the correct quantity, if h approaches the characteristic function of the volume V. The focus here is, however, put on the adiabatic limit $h \to 1$ of the quantities.

Proposition 4.4.2.
Let h_n be a sequence of test-functions of the following form:

$$0 \leq h_n(\boldsymbol{x}) \leq 1 \quad \forall x \in \mathbb{R}^3, \qquad h_n(\boldsymbol{x}) = \begin{cases} 1 & |\boldsymbol{x}| < n \\ 0 & |\boldsymbol{x}| > n+1 \end{cases} \,.$$

The the adiabatic limit

$$\lim_{h\to 1} \frac{\ln \mathcal{Z}_h^{\mathcal{V}}(\beta)}{\int h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}} := \lim_{n\to\infty} \frac{\ln \mathcal{Z}_{h_n}^{\mathcal{V}}(\beta)}{\int h_n(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}$$

exists and is smooth in β.

PROOF. To start with we use equation (3.12) for the perturbative expansion of $W_h(i\beta)$ and a relation to the connected part of ω_β from proposition 4.3.1:

$$\begin{aligned}\mathcal{Z}_h^{\mathcal{V}}(\beta) &= \sum_{n=0}^{\infty}(-1)^n \int_{\beta\mathfrak{s}_n} \omega_\beta\left(\alpha_{iu_1}(K_h') \star \cdots \star \alpha_{iu_n}(K_h')\right) \mathrm{d}u_1 \cdots \mathrm{d}u_n \\ &= \sum_{n=0}^{\infty}(-1)^n \int_{\beta\mathfrak{s}_n} \exp(\omega_\beta^{\mathrm{c}})\left(\alpha_{iu_1}(K_h') \otimes \cdots \otimes \alpha_{iu_n}(K_h')\right) \mathrm{d}u_1 \cdots \mathrm{d}u_n \\ &= \exp\left(\sum_{n=0}^{\infty}(-1)^n \int_{\beta\mathfrak{s}_n} \omega_\beta^{\mathrm{c}}\left(\alpha_{iu_1}(K_h') \otimes \cdots \otimes \alpha_{iu_n}(K_h')\right) \mathrm{d}u_1 \cdots \mathrm{d}u_n\right) .\end{aligned}$$

Thus the logarithm of the expression is given by

$$\begin{aligned}&\ln\left(\mathcal{Z}_h^{\mathcal{V}}(\beta)\right) = \sum_{n=0}^{\infty}(-1)^n \int_{\beta\mathfrak{s}_n} \omega_\beta^{\mathrm{c}}\left(\alpha_{iu_1}(K_h') \otimes \cdots \otimes \alpha_{iu_n}(K_h')\right) \mathrm{d}u_1 \cdots \mathrm{d}u_n \\ &= \sum_{n=0}^{\infty} \int_{\beta\mathfrak{s}_n} \int_{\mathbb{R}^{3n}} h(\boldsymbol{x}_1) \cdots h(\boldsymbol{x}_n) \times \\ &\times \omega_\beta^{\mathrm{c}}\left(\mathfrak{V}_h(u_1, \boldsymbol{x}_1) \otimes \cdots \otimes \mathfrak{V}_h(u_n, \boldsymbol{x}_n)\right) \mathrm{d}\boldsymbol{x}_1 \cdots \mathrm{d}\boldsymbol{x}_n \, \mathrm{d}u_1 \cdots \mathrm{d}u_n .\end{aligned}$$

using the notation of proposition 4.3.1. As in the case of the adiabatic limit of the interacting state we replace the $\mathfrak{V}_h$ by a partial adiabatic limit

$$\mathfrak{V}(u, \boldsymbol{x}) = -\int \dot{\chi}(t)\Theta^-(t)\alpha_{iu,\boldsymbol{x}}\left(\mathcal{R}_{\mathcal{V}_\chi}(\mathcal{V}(\Phi_{t,0}))\right) \mathrm{d}t = -\alpha_{iu,\boldsymbol{x}}(R) .$$

See section 4.3.2 for more information. Thus the remaining expression that is to be analyzed is

$$I_m = \int_{\beta\mathfrak{s}_n} \int_{\mathbb{R}^{3n}} \frac{h_m(\boldsymbol{x}_1) \cdots h_m(\boldsymbol{x}_n) \omega_\beta^{\mathrm{c}}\left(\mathfrak{V}(u_1, \boldsymbol{x}_1) \otimes \cdots \otimes \mathfrak{V}(u_n, \boldsymbol{x}_n)\right)}{\int h_m(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}} \mathrm{d}\boldsymbol{x}_1 \cdots \mathrm{d}\boldsymbol{x}_n \, \mathrm{d}u_1 \cdots \mathrm{d}u_n .$$

Using the translation invariance of the connected correlation function, we can write

$$\omega_\beta^{\mathrm{c}}\left(\mathfrak{V}(u_1, \boldsymbol{x}_1) \otimes \cdots \otimes \mathfrak{V}(u_n, \boldsymbol{x}_n)\right) = \omega_\beta^{\mathrm{c}}\left(\mathfrak{V}(u_1, 0) \otimes \mathfrak{V}(u_2, \boldsymbol{y}_2) \otimes \cdots \otimes \mathfrak{V}(u_n, \boldsymbol{y}_n)\right)$$

where $\boldsymbol{y}_j = \boldsymbol{x}_j - \boldsymbol{x}_1$. Relabeling integration coordinates we find

$$\begin{aligned}I_m = \int_{\beta\mathfrak{s}_n} \int_{\mathbb{R}^{3n}} &\frac{h_m(\boldsymbol{x}_1) \prod_{i=2}^n h_m(\boldsymbol{x}_1 + \boldsymbol{x}_i)}{\int h_m(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}} \times \\ &\times \omega_\beta^{\mathrm{c}}\left(\mathfrak{V}(u_1, 0) \otimes \mathfrak{V}(u_2, \boldsymbol{x}_2) \otimes \cdots \otimes \mathfrak{V}(u_n, \boldsymbol{x}_n)\right) \mathrm{d}\boldsymbol{x}_1 \cdots \mathrm{d}\boldsymbol{x}_n \, \mathrm{d}u_1 \cdots \mathrm{d}u_n . \qquad (4.18)\end{aligned}$$

We estimate the contribution in the $\boldsymbol{x}_1$-variable for large m:

$$\frac{\mathrm{Vol}(B_{m-|\hat{\boldsymbol{x}}|})}{\mathrm{Vol}(B_{m+1})} \leq \frac{\int h_m(\boldsymbol{x}_1) \prod_{i=2}^n h_m(\boldsymbol{x}_i + \boldsymbol{x}_1) \, \mathrm{d}\boldsymbol{x}_1}{\int h_m(\boldsymbol{x}) \, \mathrm{d}x} \leq 1$$

where $\hat{\boldsymbol{x}}$ is the longest vector of all $\{\boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\}$ The upper bound is quickly established by taking the suprema of the shifted h_m, while for the lower bound we used

$$\frac{\int h_m(\boldsymbol{x}_1) \prod_{i=2}^n h_m(\boldsymbol{x}_i + \boldsymbol{x}_1)\, \mathrm{d}\boldsymbol{x}_1}{\int h_m(\boldsymbol{x})\, \mathrm{d}x} \geq \frac{\int h_m(\boldsymbol{x}_1) \prod_{i=2}^n h_m(\boldsymbol{x}_i + \boldsymbol{x}_1)\, \mathrm{d}\boldsymbol{x}_1}{\mathrm{Vol}(B_{m+1})}$$
$$\geq \frac{\int_{B_m} \prod_{i=2}^n h_m(\boldsymbol{x}_i + \boldsymbol{x}_1)\, \mathrm{d}\boldsymbol{x}_1}{\mathrm{Vol}(B_{m+1})} \geq \frac{\mathrm{Vol}(B_{m-|\hat{\boldsymbol{x}}|})}{\mathrm{Vol}(B_{m+1})} \overset{m\to\infty}{\longrightarrow} 1$$

where in the last line we assumed that m was larger than $2\,|\hat{\boldsymbol{x}}|$. We thus obtain

$$\lim_{m\to\infty} \frac{\int h_m(\boldsymbol{x}_1) \prod_{i=2}^n h_m(\boldsymbol{x}_i + \boldsymbol{x}_1)\, \mathrm{d}\boldsymbol{x}_1}{\int h_m(\boldsymbol{x})\, \mathrm{d}x} = 1$$

by the squeeze theorem. Since the integrand is an element of $L^1(\beta\mathfrak{s}_n \times \mathbb{R}^{3(n-1)}$ due to theorem 10 we can exchange integration and limits and use the pointwise convergence of the integrand to show

$$\lim_{m\to\infty} I_m = \int_{\beta\mathfrak{s}_n} \int_{\mathbb{R}^{3(n-1)}} \omega_\beta^{\mathrm{c}} \left(\mathfrak{V}(u_1, 0) \otimes \mathfrak{V}(u_2, \boldsymbol{x}_2) \otimes \cdots \otimes \mathfrak{V}(u_n, \boldsymbol{x}_n)\right) \mathrm{d}\boldsymbol{x}_2 \cdots \mathrm{d}\boldsymbol{x}_n\, \mathrm{d}u_1 \cdots \mathrm{d}u_n\,.$$

The point, in which the argument of the connected correlation function is put to zero (or some other, fixed value) is irrelevant due to translation invariance of $\omega_\beta^{\mathrm{c}}$. The differentiability of the quotient is due to the fact, that derivative in β give, at most, polynomials in the arguments of the integrand, where the connected correlation function decay exponentially. □

The thermostatic observables of the interacting theory are usually derived in the Euclidean version of the respective QFT, see e.g. [LB00]. The present method allows a more direct approach to thermostatic observables in the spirit of quantum statistical mechanics. In order to show that both approaches result in the same results, we present an explicit example in perturbation theory in the next section.

We remark that the form of the adiabatic limit, i.e. the choice of the sequence of test-functions h_n is related to the van-Hove thermodynamic limit in quantum statistical mechanics. The concept which was introduced by van-Hove and extensively used in the book of Ruelle [Rue69] and at its core is the idea, that the adiabatic (or thermodynamic) limit can only exist, if the "boundary terms" are negligible in the limit of large volumes.

4.4.3. Free Energy in Lowest Nontrivial Order

In this section we want to give an example calculation within our framework. To this avail, we choose the difference of the free energies $\delta\mathcal{F}$ as the observable. We assume that a renormalization of the time-ordered product has been done to a sufficiently high order, such that distributions of the form

$$\Delta_F(x - y)^4$$

are well defined and exhibit the appropriate properties (see definition 2.2.2). The interaction is chosen to be a quartic interaction

$$\mathcal{V}(\Phi_x) = \frac{\lambda}{4!}\Phi_x^4 \,.$$

The expansion of $\delta\mathcal{F}$ in the interaction is given by

$$\begin{aligned}
\delta\mathcal{F} &= -\frac{1}{\beta}\lim_{h\to 1}\frac{\ln \mathcal{Z}_h(\beta)}{\int h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}} = -\frac{1}{\beta}\sum_{n=1}^{\infty}\lambda^n \sum_{k\in\mathbb{N}}\sum_{\substack{\ell\in\mathbb{N}^k\\ |l|=n}}\Omega^\ell \\
\Omega^\ell &= \int_{\beta\mathfrak{s}_k}\int_{\mathbb{R}^{3(k-1)}} \omega_\beta^{\mathrm{c}}(\mathfrak{V}^{(\ell_1)}(u_1,0)\otimes\cdots\otimes\mathfrak{V}^{(\ell_k)}(u_k,\boldsymbol{x}_k))\,\mathrm{d}\boldsymbol{x}_1\cdots\mathrm{d}\boldsymbol{x}_{k-1}\,\mathrm{d}u_1\cdots\mathrm{d}u_{k-1} \\
\mathfrak{V}(u,\boldsymbol{x}) &= \int \alpha_{iu}\left(\mathcal{R}_{\mathcal{V}_\chi}\left(\mathcal{V}(\Phi_{t,\boldsymbol{x}})\right)\ \dot{\chi}^-(t)\,\mathrm{d}t\right.
\end{aligned}$$

The term in first order vanishes, since the interaction has vanishing expectation value:

$$\omega_\beta^{\mathrm{c}}(\mathfrak{V}^{(1)}(u,\boldsymbol{x})) = \int \omega_\beta(\alpha_{iu}\left(\mathcal{V}(\Phi_{t,\boldsymbol{x}})\right))\ \dot{\chi}(t)\Theta^-(t)\,\mathrm{d}t = 0\,.$$

The second order term contributes with

$$\begin{aligned}
-\beta\delta\mathcal{F}^{(2)} &= \int_0^\beta \omega_\beta^{\mathrm{c}}(\mathfrak{V}^{(2)}(u,0))\,\mathrm{d}u + \int_{\beta\mathfrak{s}_2}\int_{\mathbb{R}^3}\omega_\beta^{\mathrm{c}}(\mathfrak{V}^{(1)}(u_1,0)\otimes\mathfrak{V}^{(1)}(u_2,\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}u_1\,\mathrm{d}u_2 \\
&= \beta\omega_\beta\left(\mathcal{R}^{(2)}_{\mathcal{V}_\chi}\left(\mathcal{V}_{\delta_0\dot{\chi}^-}\right)\right) + \frac{\beta}{2}\int_0^\beta\int_{\mathbb{R}^3}\omega_\beta\left(\mathcal{R}^{(1)}_{\mathcal{V}_\chi}(\mathcal{V}_{\delta_0\dot{\chi}^-})\star\alpha_{iu}\left(\mathcal{R}^{(1)}_{\mathcal{V}_\chi}(\mathcal{V}_{\delta_x\dot{\chi}^-})\right)\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}u
\end{aligned}$$

where $\dot{\chi}^- = \dot{\chi}\Theta^-$ and $\delta_{\boldsymbol{x}}(\boldsymbol{y}) = \delta(\boldsymbol{y}-\boldsymbol{x})$ and where we used translation invariance of ω_β^c. The quantum Møller map of $\mathcal{V}$ in first second order reads

$$\mathcal{R}^{(1)}_{\mathcal{V}_\chi}(\mathcal{V}_{\delta_0\dot{\chi}^-}) = \mathcal{V}_{\delta_0\dot{\chi}^-} \qquad \mathcal{R}^{(2)}_{\mathcal{V}_\chi}(\mathcal{V}_{\delta_0\dot{\chi}^-}) = i\left(\mathcal{V}_\chi\cdot_{\mathcal{T}}\mathcal{V}_{\delta_0\dot{\chi}^-} - \mathcal{V}_\chi\star\mathcal{V}_{\delta_0\dot{\chi}^-}\right)$$

which gives

$$-\beta\delta\mathcal{F}^{(2)} = i\beta\,\omega_\beta\left(\mathcal{V}_\chi\cdot_{\mathcal{T}}\mathcal{V}_{\delta_0\dot{\chi}^-} - \mathcal{V}_\chi\star\mathcal{V}_{\delta_0\dot{\chi}^-}\right) + \frac{\beta}{2}\int_0^\beta\int_{\mathbb{R}^3}\omega_\beta\left(\mathcal{V}_{\delta_0\dot{\chi}^-}\star\alpha_{iu}\mathcal{V}_{\delta_x\dot{\chi}^-}\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}u\,. \tag{4.19}$$

Since the time-ordered product has been renormalized, n-fold products pointwise products of the Feynman propagator have been defined, thus we can write the first term as

$$\begin{aligned}
&i\beta\lambda^2\hbar^4 \int \mathrm{d}t_1 \int \mathrm{d}t_2 \int \mathrm{d}\boldsymbol{x} \left(\Delta_F(t_1-t_2,\boldsymbol{x})^4 - \Delta_+(t_1-t_2,\boldsymbol{x})^4\right) \chi(t_1)\dot{\chi}^-(t_2) \\
&= i\beta\lambda^2\hbar^4 \int \mathrm{d}t \int \mathrm{d}\boldsymbol{x} \left(\Delta_-(t,\boldsymbol{x})^4 - \Delta_+(t,\boldsymbol{x})^4\right) \theta(-t) \int \mathrm{d}\tau\, \chi(\tau+t)\dot{\chi}^-(\tau) \\
&= i\beta\lambda^2\hbar^4 \int_{-\infty}^{0} \mathrm{d}t \int \mathrm{d}\boldsymbol{x} \left(\int_0^\beta \mathrm{d}u \frac{\mathrm{d}}{\mathrm{d}u}\Delta_+(t-iu,\boldsymbol{x})^4\right) \int \mathrm{d}\tau\, \chi(\tau+t)\dot{\chi}^-(\tau) \\
&= \beta\lambda^2\hbar^4 \int_{-\infty}^{0} \mathrm{d}t \int \mathrm{d}\boldsymbol{x} \left(\int_0^\beta \mathrm{d}u \frac{\mathrm{d}}{\mathrm{d}t}\Delta_+(t-iu,\boldsymbol{x})^4\right) \int \mathrm{d}\tau\, \chi(\tau+t)\dot{\chi}^-(\tau) \\
&\overset{\text{p.i.}}{=} \beta\lambda^2\hbar^4 \int \mathrm{d}\boldsymbol{x} \int_0^\beta \mathrm{d}u\; \Delta_+(-iu,\boldsymbol{x})^4 \underbrace{\int_{-\infty}^{0} \mathrm{d}\tau \chi(\tau)\dot{\chi}^-(\tau)}_{\int_{-\infty}^0 \mathrm{d}\tau \frac{\mathrm{d}}{\mathrm{d}\tau}\chi(\tau)^2 = 1} \\
&- \beta\lambda^2\hbar^4 \int_{-\infty}^{0} \mathrm{d}t \int \mathrm{d}\boldsymbol{x} \int_0^\beta \mathrm{d}u\; \Delta_+(t-iu,\boldsymbol{x})^4 \underbrace{\int \mathrm{d}\tau \dot{\chi}(\tau+t)\dot{\chi}^-(\tau)}_{X(t)}
\end{aligned}$$

The first term after partial integration is the Euclidean contribution to $\delta\mathcal{F}^{(2)}$ and the second term can be compared with the second contribution in equation (4.19):

$$\begin{aligned}
&\frac{\beta}{2}\int_0^\beta \int_{\mathbb{R}^3} \omega_\beta\left(\mathcal{V}_{\delta_0\dot{\chi}^-} \star \alpha_{iu}\mathcal{V}_{\delta_{\boldsymbol{x}}\dot{\chi}^-}\right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}u \\
&= \frac{\beta\lambda^2\hbar^4}{2}\int_0^\beta \mathrm{d}u \int \mathrm{d}t_1 \int \mathrm{d}t_2 \int \mathrm{d}\boldsymbol{x}\; \Delta_+(t_1-t_2-iu,-\boldsymbol{x})^4\; \dot{\chi}^-(t_1)\dot{\chi}^-(t_2) \\
&= \frac{\beta\lambda^2\hbar^4}{2}\int_0^\beta \mathrm{d}u \int \mathrm{d}t \int \mathrm{d}\boldsymbol{x}\; \Delta_+(t-iu,\boldsymbol{x})^4 \int \mathrm{d}\tau\; \dot{\chi}^-(\tau+t)\dot{\chi}^-(\tau)
\end{aligned}$$

Combining both (non-Euclidean) contributions we get

$$\beta\lambda^2\hbar^4 \int_{\mathbb{R}} \mathrm{d}t \int_{\mathbb{R}^3} \mathrm{d}\boldsymbol{x}\; f(t,\boldsymbol{x})X(t), \qquad f(t) = \frac{1}{2}\int_0^\beta \mathrm{d}u\; \varepsilon(t)\Delta_+(t-iu,\boldsymbol{x})^4$$

with the sign function ε. However X is an even function since

$$X(-t) = \int \mathrm{d}\tau\; \dot{\chi}(\tau-t)\dot{\chi}^-(\tau) = \int \mathrm{d}\tau\; \dot{\chi}(\tau)\dot{\chi}^-(\tau+t) = X(t)$$

and $f(t,\boldsymbol{x})$ is odd in t due to

$$\begin{aligned}
f(-t,\boldsymbol{x}) &= -\frac{1}{2}\int_0^\beta \mathrm{d}u\; \varepsilon(t)\Delta_+(-t-iu,\boldsymbol{x})^4 \overset{\text{KMS}}{=} -\frac{1}{2}\int_0^\beta \mathrm{d}u\; \varepsilon(t)\Delta_+(t-i(u+\beta),\boldsymbol{x})^4 \\
&= -\frac{1}{2}\int_0^\beta \mathrm{d}v\; \varepsilon(t)\Delta_+(t-iv,\boldsymbol{x})^4 = -f(t,\boldsymbol{x})\,.
\end{aligned}$$

Thus the integration

$$\int \mathrm{d}t\; f(t,\boldsymbol{x})X(t) = 0$$

vanishes, hence there is no non-Euclidean contribution. The result

$$\delta\mathcal{F}^{(2)} = -\lambda^2\hbar^4 \int_{\mathbb{R}^3} \mathrm{d}\boldsymbol{x} \int_0^\beta \mathrm{d}u\, \Delta_+(-iu,\boldsymbol{x})^4 = -\lambda^2 \int_{\mathbb{R}^3} \mathrm{d}\boldsymbol{x} \int_0^\beta \mathrm{d}u\, \omega_\beta \left(\frac{\Phi^4_{-iu,\boldsymbol{x}}}{4!} \star \frac{\Phi^4_{0,0}}{4!} \right)$$

is independent of the choice of the time-slice cut-off function χ, as we have expected from proposition 4.2.3. It coincides with results that are found in the literature, see e.g. [Lai13].

We remark that one often finds interactions, which are not normal-ordered in the calculations in the literature. Then, so-called tadpole graphs appear in the expansions, which are graphs that contain lines that end at their starting point. The use of these "classical interactions" is, however, unsatisfactory from a conceptual point of view, since they are not elements of the algebra of free fields $\mathcal{A}$. The contributions can be seen from the more general point of view that is proposed in this thesis, as the influence of the free theory, which is described in more detail in section 2.3.2.

Conclusion

Summary

In this thesis the algebraic approach to perturbative QFT (pAQFT) was used to investigate the long known IR (or pinching) divergences appearing in the perturbative expansions in thermal equilibrium states. The state-independent perturbative analysis of the interacting QFT within the framework of pAQFT displayed the absence of any singularities in the theory of the scalar field, which is described by the algebra of observables.

The source of the IR problem of QFT at finite temperature was traced back to the perturbative expansions of the KMS state of the interacting theory. In the hitherto constructions in Thermo Field Dynamics and the Schwinger-Keldysh contour approach [LvW87, MNU84] and also in the framework of the causal perturbation theory [Ste95] the interacting state was perturbatively expanded around the free state in the asymptotic past, complying with the principles of scattering theory. Certainly this method is meaningful only in the case that the interacting state converges to the free one in the respective asymptotic region, as it is shown e.g. for the interacting (massive) vacuum state [Haa58, Rue62].

In a remarkable work the authors Bros and Buchholz [BB02] casted serious doubts on this ansatz by showing that the interacting state does not behave as its free counterpart at timelike infinity. The asymptotic dynamics exhibits a strong imprint of the interaction, indicating that the scattering approach may be problematic.

Consequently, another path was chosen in this thesis using ideas from quantum statistical mechanics, where interacting KMS states can be constructed at a finite time in a Hamiltonian approach. In order to implement these ideas, the gap between the causal perturbation theory, whose emphasis lies in the relativistic covariance of the theory, and quantum statistical mechanics, in which a time-direction is distinguished through a one-parameter group of automorphisms α_t, had to be filled.

This goal is achieved in section 3.2 with the help of the time-slice axiom, which was shown to be valid in perturbatively constructed quantum field theories in causal perturbation theory [CF09]. This allowed the description of the interacting theory at any given region $\mathcal{O} \subset M$ in terms of observables which are restricted to a time-slice Σ_ϵ, see figure 1. We then showed how the interacting dynamics can be expressed in terms of the free one by a co-cycle $W_h(t)$ in the algebra of the free field, as long as the interaction had a finite spatial extent. This illuminates how a treatment of the interacting relativistic system can be realized in terms of quantum dynamical systems. Moreover the Stückelberg divergences [Stü51] that arise in the canonical description of QFTs (in terms of fixed time quantum fields) are avoided through the use of small, but finitely

extended time-slices.

The construction of the KMS state for the interacting system with finite spatial extent was done in section 4.2 by using a well-known method established by Araki in [Ara73], that used an analytic continuation of the co-cycle W_h. The adiabatic limit in which the cutoff of the interaction is removed, reduced to the limit of infinite **spatial** extent of the system in our case. The existence of this limit is established in the case of massive scalar field in section 4.3 using the exponential decay of the connected part of the free KMS state in spatial directions.

In the proof of existence the non-zero mass of the theory is indispensable, since the correlation functions of massless theories exhibit a too slow decay at spatial infinity. The consequences of this have been observed in many applications of the massless theories at finite temperature and is sometimes referred to as **the IR problem** of perturbative QFT at finite temperature, see e.g. [Alt90].

Outlook

It is evident from the last part of the summary that more information on the behavior of the massless theory is desirable, especially since many models in high-energy physics use massless fields (such as QED and QCD).

One way do deal with the bad IR behavior of the theory is to introduce a non-zero auxiliary mass term into the free theory and which is removed in the interaction term. It is clear that an additional mass term in the free theory will drastically alter the asymptotic behavior of the theory in a way that yields convergent expectation values. In order to obtain the the correct interacting theory back in the end, conditions have to be put on the time-ordered products. We expect that this procedure exhibits many similarities to the method of Lowenstein and Zimmermann [LZ75, Low76]. They show the absence of IR singularities in the (time-ordered) vacuum expectation values of theories that contain massless fields by the introduction of auxiliary masses for theses fields, together with a specific renormalization procedure, which is called BPHZL due to its close connection to the BPHZ framework.

An extension of their results to the case of QFT at finite temperature would be a worthwhile result that would shed new light on the methods that used in the literature, which involve the shift of the thermal mass term into the free theory [LB00, BP90].

Another interesting question is whether the asymptotic behavior of the KMS state ω_β^ν in timelike directions coincides with the asymptotics of the interacting KMS state which was found in [BB02].

In a more general context the timelike asymptotics of ω_β^ν is important to characterize its stability properties under further perturbations. See [HKTP74, Haa92, BR02b] for an extensive discussion. As it was shown in [BKR78], an important condition in this respect is the cluster property in time:

$$\lim_{t\to\pm\infty} \omega_\beta(A \star \alpha_t(B)) = \omega_\beta(A)\omega_\beta(B) \qquad \forall A, B \in \mathcal{A}\,.$$

If this condition holds true, then the asymptotic (retarded) expansion of $\omega_\beta^{\mathcal{V}}$ and the time-slice expansion coincide in the C^*-algebraic setting. A thorough investigation in the present setting, containing algebras of unbounded operators, would be highly interesting. If a similar result holds true also in the pAQFT framework, then the equivalence of the expansion at asymptotic times and the time-slice construction would be established.

In addition, the present formalism provides the tools to further study spontaneous symmetry breaking in perturbatively constructed QFTs, e.g. the breaking of the $\mathbb{Z}_2$ symmetry $\Phi \to -\Phi$ in models with $m^2 < 0$ and quartic self-interaction. A discussion on this topic has taken place only at the level of effective, non-local actions, see e.g. [DJ74], or in axiomatic settings, e.g. in [Haa92].

In the pAQFT approach, the symmetry is called spontaneously broken, if the algebra of observables $\mathfrak{A}$ is represented on a Hilbert space, in which the automorphism $\Phi \to -\Phi$ is not unitarily implemented. This holds in representations induced by non-invariant states.

In our case such non-invariant states arise as evaluation functionals ev_ϕ on non-trivial background configurations ϕ, e.g. at the one of the minima of the potential

$$\mathcal{V} = \frac{\lambda}{4!}\phi^4 + \frac{m^2}{2}\phi^2, \qquad \lambda > 0, m^2 < 0\,.$$

The results of this work can serve as a starting point of an detailed analysis of the spontaneous breaking of the $\mathbb{Z}_2$ symmetry for the interacting ground and KMS states.

A. Weyl Algebra at Finite Temperature and Purification

This chapter is intended to give an overview on the intimate relationship between the theory of C^*- and von-Neumann-algebras with KMS states in quantum physics. It is one of the examples in the history of science, where two areas of research come into contact which were disjointly investigated for a long time and whose collaboration resulted into deep insights and lots of fascinating structures for both fields. For further historical remarks and detailed introductions in both subjects it referred to the textbooks of Bratteli/Robinson [BR02a, BR02b] and Haag [Haa92].

In the beginning we introduce some elements of the theory of von-Neumann algebras and Tomita-Takesaki modular theory. After that we discuss the algebras of bounded operators that appear in relativistic quantum theory: the Weyl-algebras. The Weyl-algebra of the free field is constructed in the deformation quantization setting and the relation between Weyl-algebras arising from different $\star$-products is discussed.

In a subsequent section the special case of the Weyl algebra at finite temperature is considered and its modular structure is highlighted. Using the modular conjugation of the theory we construct the **purification** of the system, that is we enlarge the algebra of observables in such a way, that the KMS state extends to a pure state on the larger algebra. This is in close connection to the purification map, introduced by Woronowicz [Wor72, Wor73]. Both the enlarged algebra and the state are given in a very explicit way and the purity of the state is proven by the Kay-Wald criterion [KW91].

The enlarged algebra, together with the purified KMS state marks the starting point of the Thermo Field Dynamics treatment of QFT at finite temperature, see e.g. [Oji81, KMMS09]. In the language of the off-shell functional approach to QFT, Thermo Field Dynamics can thus be seen as an off-shell extension to the algebra of observables $\mathcal{A}$. A full treatment of this extension in the present formalism seemed, however, to go to far for this thesis.

A.1. Prelude in Mathematics: Tomita-Takesaki Modular Theory

A crucial question after the first developments of quantum physics was, how the objects that were used in the description of quantum systems, namely linear operators on Hilbert spaces, could be systematically investigated. Due to complications that arise in the general treatment of unbounded operators on Hilbert spaces the main focus was to classify the structures in spaces of bounded operators at first.

The operations that are of main importance in the use of operators in quantum theory were identified to be: An involution *, mapping an operator to its adjoint, which can be used to single out the observables (self-adjoint operators). Moreover the algebra structure of the operators must not be neglected, since it is the non-commutative product structure that lies at the very heart of quantum theory.

Thus a natural candidate to study is the *-algebra of all bounded operators $\mathfrak{B}(\mathcal{H})$ (or suitable subsets of $\mathfrak{B}(\mathcal{H})$) on a Hilbert space $\mathcal{H}$ with scalar product $(\cdot|\cdot)$. An important ingredient for the mathematical study of this space is the operator norm $\|A\|$ of a bounded operator. It is defined by

$$\|A\| = \sup_{\substack{\psi\in\mathcal{H}\\(\psi|\psi)=1}} |(\psi|A\psi)| \ .$$

The algebra $\mathfrak{B}(\mathcal{H})$ has many different topologies, there are three of them that turned out to be important in quantum theory. These are

- the **norm** or uniform topology, which is induced by the operator norm $\|A\|$
- the **strong operator** topology, which is induced by the family of seminorms $p^s_\psi(A) = \sqrt{(A\psi|A\psi)}$ with $\psi \in \mathcal{H}$
- the **weak operator** topology, which is induced by the family of seminorms $p^w_{\psi,\phi} = |(\phi|A\psi)|$ with $\psi, \phi \in \mathcal{H}$.

Using the first topology leads to the notion of C^*-algebras and the continuous functional calculus for self-adjoint operators.

The latter (weak and strong operator) topologies provide the starting point in the construction and discussion of von-Neumann-algebras and the weak operator topology yields an extension of the functional calculus, that allows the definition of Borel functions of self-adjoint operators.

From a physical point of view the weak operator topology could also be seen as the most natural one: The matrix-elements $(\phi|A\psi)$ of self-adjoint operators A are interpreted as predictions of measurements in quantum theory.

Definition A.1.1 (Von-Neumann-algebras).
Let $\mathcal{H}$ be a Hilbert space and $\mathfrak{B}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. A self-adjoint, unital subset $\mathfrak{M} \subset \mathfrak{B}(\mathcal{H})$ is called a von-Neumann-algebra, if $\mathfrak{M}$ is closed in the weak operator topology.

Thus every von-Neumann-algebra is, in particular, a C^*-algebra.

Definition A.1.2 (Commutant).
Let $\mathfrak{M}$ be a subset of $\mathfrak{B}(\mathcal{H})$. The commutant $\mathfrak{M}'$ of $\mathfrak{M}$ is defined by

$$\mathfrak{M}' = \{A \in \mathfrak{B}(\mathcal{H}) : [A, B] = 0, \quad \forall B \in \mathfrak{M}\}.$$

The double commutant is given by iteration: $\mathfrak{M}'' = (\mathfrak{M}')'$.

Theorem 11 (Von-Neumann double commutant theorem [vN29, MvN43]).
Let $\mathfrak{M}$ be a self-adjoint, unital subset of $\mathfrak{B}(\mathcal{H})$. Then

- *$\mathfrak{M}$ is a von-Neumann-algebra, iff $\mathfrak{M}'' = \mathfrak{M}$*
- *$\mathfrak{M}'$ is a von-Neumann-algebra*
- *$\mathfrak{M}''$ is the smallest von-Neumann-algebra, which contains $\mathfrak{M}$.*

Theorem 12 (GNS theorem [GN43, Seg47]).
Let $\mathfrak{A}$ be a C^*-algebra and ω a state on $\mathfrak{A}$. There exists a Hilbert space $\mathcal{H}_\omega$, a vector $\Omega_\omega \in \mathcal{H}_\omega$ and a *-representation $\pi_\omega : \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_\omega)$, such that

$$\omega(A) = (\Omega_\omega | \pi_\omega(A)\Omega_\omega)\,, \qquad \pi_\omega(\mathfrak{A})\Omega_\omega \overset{\text{dense}}{\subset} \mathcal{H}_\omega. \tag{A.1}$$

The triplet $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ associated to $\mathfrak{A}$ and ω is unique, up to unitary equivalence. Furthermore the following statements are equivalent:

- *ω is a **pure** state, i.e. for states ω_1, ω_2 over $\mathfrak{A}$ the equation $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ with $0 < \lambda < 1$ implies $\omega_1 = \omega_2 = \omega$*
- *π_ω is an **irreducible** representation*
- *$\pi_\omega(\mathfrak{A})' = \{\lambda \mathbb{1} : \lambda \in \mathbb{C}\} \implies \pi_\omega(\mathfrak{A})'' = \mathfrak{B}(\mathcal{H}_\omega)$.*

The vector Ω_ω is called the implementing vector of ω or the GNS vacuum and the latter property of Ω_ω in (A.1) is called cyclicity.

There is an explicit way of constructing a representation $(H_\omega, \pi_\omega, \Omega_\omega)$ from a given state ω over the algebra $\mathfrak{A}$, the GNS construction. It can be found in section 1.5. The above theorem is a refinement of theorem 3 that is valid in this formulation only in the case of C^*-algebras. It allows the classification of quantum states into pure and mixed states. That will be important in the next section, where a purification of a mixed state is discussed.

We turn the focus now on the discussion of cyclic and separating vectors in a given von-Neumann-algebra $\mathfrak{M}$. To define this, let $\mathfrak{M} \subset \mathfrak{B}(\mathcal{H})$ be a von-Neumann-algebra acting on $\mathcal{H}$. A vector $\psi \in \mathcal{H}$ is called separating for $\mathfrak{M}$, if

$$\forall A \in \mathfrak{M} : \; A\Omega = 0 \;\Rightarrow\; A = 0\,.$$

Then there exists a conjugate linear (unbounded) operator S on $\mathcal{H}$, the Tomita operator, which is the closure of the densely defined operator

$$SA\Omega = A^*\Omega \qquad \forall A \in \mathfrak{M}\,.$$

We identify S with its closure and make use of the unique polar decomposition of S, i.e. a positive linear (unbounded) operator $\Delta = S^*S$ and a conjugate linear operator J and on $\mathcal{H}$, such that

$$S = J\Delta^{1/2} = \Delta^{-1/2}J\,.$$

Here J is called the **modular conjugation** and Δ is the **modular operator** for $\mathfrak{M}$ and Ω. The following relations for J and Δ hold [BR02a]:

$$J = J^{-1} = J^*, \quad J\Omega = \Omega,$$
$$\Delta = \Delta^*, \quad \Delta > 0, \quad \Delta\Omega = \Omega\,.$$

It was a deep insight of Tomita that the above constructed operators J and Δ can be used to characterize the commutant of $\mathfrak{M}$ and to show a relation between the separating vector Ω and KMS states. This finding remained unpublished though and it was due to Takesaki [Tak70] to publish the theorem and find the interesting applications, for which it is named Tomita-Takesaki theorem.

Theorem 13 (Tomita-Takesaki).
Let $\mathfrak{M}$ be a von-Neumann-algebra with cyclic and separating vector Ω and J, Δ as above. Then

$$J\mathfrak{M}J = \mathfrak{M}' \quad \textit{and} \quad \alpha_t(A) := \Delta^{-it/\beta} A \Delta^{it/\beta}$$

is a one-parameter group of automorphisms of $\mathfrak{M}$ and for $\mathfrak{M}'$ for every $\beta \in \mathbb{R}$.

Actually the introduction of the factor β^{-1} in α_t was not necessary for the result, nevertheless it turns out to be useful for the formulation of an important consequence of the theorem.

Proposition A.1.3.
For every $\beta \in \mathbb{R}$ the algebraic state ω_Ω over $\mathfrak{M}$ given by $\omega_\Omega(A) = (\Omega|A\Omega)$ is a β-KMS state with respect to the modular group α_t with inverse temperature $\beta \in \mathbb{R}$.

PROOF. Let $A, B \in \mathfrak{M}$, then

$$\begin{aligned}\big(\Omega\big|A\alpha_{i\beta}(B)\Omega\big) &= (A^*\Omega|\Delta B\Omega) = \underbrace{(\Delta^{1/2}A^*\Omega|\Delta^{1/2}B\Omega)}_{(JSA^*\Omega|JSB\Omega)} = (JA\Omega|JB^*\Omega) = (A\Omega|B^*\Omega)^* \\ &= (\Omega|BA\Omega)\ .\end{aligned}$$

Thus ω_Ω fulfills the static KMS condition with respect to to α_t. This implies the dynamic KMS condition given in definition 1.5.6. □

The Tomita-Takesaki theorem shows that every cyclic and separating vector Ω for a von-Neumann-algebra $\mathfrak{M}$ yields a one-parameter group of automorphisms on $\mathfrak{M}$ and an associated KMS state. The converse statement hold true as well, since every KMS state on a C^*-algebra is implemented by a separating vector. Thus results and methods that are developed for von-Neumann-algebras and for quantum systems with a KMS state can be compared with the help of this theorem and many new structures have been found by using this duality. The theorem had also a great impact on other aspects of quantum field theory, for a review on this issue see, e.g. [Bor00].

A.2. Weyl Algebra at Finite Temperature and Purification

In this section we want to study the Weyl-algebra of QFT. Unfortunately the term Weyl-algebra is used to denote very different objects in the literature. In this work we mean by Weyl algebra the $*$-algebra that is generated by exponentiated linear quantum fields.

To explain this in more detail we consider the free scalar field φ_f that fulfills the canonical commutation relations (CCR)

$$[\varphi_f, \varphi_g] = iG_c(f,g), \qquad f,g \in \mathcal{D}$$

on M. φ_f can be seen as representative of the linear field $\Phi_f \in \mathcal{A}$ on a Hilbert space. A quick calculation shows that its exponential $W_f = \exp(i\varphi_f)$ will fulfill the relation

$$W_f W_g = \mathrm{e}^{-\frac{i}{2}G_c(f,g)} W_{f+g}\,,$$

which is known as the CCR in exponentiated form. This observation can be used to establish a connection between the mathematically rigorous results from the last section and relativistic quantum theory. While the free fields φ_f can only be described by unbounded operators, their exponentiated counterparts W_f are bounded unitary operators (if f is real-valued). Thus the generated algebra will have many nice mathematical features that more general algebras including unbounded operators lack.

Moreover there is no need to choose a positive part Δ_+ of G_c in order to construct the quantum algebra (which is generated by the W_f), unlike in the approach of section 1.4. This is a consequence of the regularity of the regarded objects: All elements of this algebra can be described by **regular functionals**.

We begin with some general aspects of the Weyl algebra, associated to the linear field. Afterwards it will be shown how to realize the Weyl algebra by $\star$-exponentials within the algebra of the free scalar field $\mathcal{A}$, the vacuum and KMS state will be discussed and a purification of the KMS state will exhibit some interesting features that will be important in Thermo Field Dynamics.

A.2.1. General Properties of the Weyl Algebra

We now discuss some very general aspects of the Weyl algebra associated to the linear field and the classification of its state space. For that we will construct a Hilbert space representation of the Weyl algebra and make use a key result of Kay and Wald [KW91] to find a characterizing property for a state to be pure or mixed.

Definition A.2.1 (Weyl algebra associated to a symplectic vector space).
Let $(\mathfrak{L}, \sigma)$ be a real symplectic vector space, i.e. σ is a real bi-linear form, which is anti-symmetric and non-degenerate. The Weyl algebra $\mathfrak{W}(\mathfrak{L}, \sigma)$ associated to $(\mathfrak{L}, \sigma)$ is the *-algebra generated by elements $\{W_f : f \in \mathfrak{L}\}$ obeying the following relations

$$W_f^* = W_{-f}, \qquad W_0 = \mathbb{1}, \qquad W_f W_g = \mathrm{e}^{i\sigma(f,g)} W_{f+g}. \tag{A.2}$$

There exists a (unique) C^*-norm $\|\cdot\|$ for every $\mathfrak{W}(\mathfrak{L}, \sigma)$, hence $\overline{\mathfrak{W}}(\mathfrak{L}, \sigma) = \overline{\mathfrak{W}(\mathfrak{L}, \sigma)}^{\|\cdot\|}$ is a C^*-algebra.

Proposition A.2.2.
Let μ be a positive, symmetric bi-linear form on $\mathfrak{L}$. If μ dominates σ in the sense, that

$$\sigma(f,g)^2 \leq \mu(f,f)\mu(g,g), \qquad \forall f,g \in \mathfrak{L}, \tag{A.3}$$

then the linear functional ω_μ on $\mathfrak{W}(\mathfrak{L},\sigma)$ defined by

$$\omega_\mu(W_f) = \mathrm{e}^{-\frac{1}{2}\mu(f,f)}$$

is a state. Moreover it uniquely extends to a state over $\overline{\mathfrak{W}}(\mathfrak{L},\sigma)$. States of this form are called ***quasi-free states****.*

The proof of the statement can be found in [Pet90]. It relies on the fact, that the inequality (A.3) implies positivity of the bi-linear form $\mu - i\sigma$ over $\mathbb{C}\cdot\mathfrak{L}$. Then

$$\Big[\mu(f_i,f_j) - i\sigma(f_i,f_j)\Big]_{i,j=1,\dots,N}$$

defines a positive semi-definite matrix. Moreover semi-positivity is preserved under the element-wise multiplication of matrices (or the Hadamard product).

Due to the positivity condition on μ and the non-degeneracy of σ it follows, that every such μ defines an inner product on $\mathfrak{L}$, which we denote by $(\cdot|\cdot)_\mu$. The completion $\overline{\mathfrak{L}}^\mu$ of $\mathfrak{L}$ with respect to the norm associated to $(\cdot|\cdot)_\mu$ is a real Hilbert space.

In order to construct a (complex) Hilbert space on which a representation of $\mathfrak{W}(\mathfrak{L},\sigma)$ can be defined and which contains a vector, that implements the state ω_μ, we are going to complexify the real vector space $\overline{\mathfrak{L}}^\mu$ by using the real bi-linear forms μ and σ.

On $\overline{\mathfrak{L}}^\mu$ we can always construct the bounded, linear operator $\kappa : \overline{\mathfrak{L}}^\mu \to \overline{\mathfrak{L}}^\mu$ with the properties

$$(f|\kappa g)_\mu = \sigma(f,g), \qquad \kappa^* = -\kappa, \quad \|\kappa\|_\mu \leq 1.$$

The operator κ can be used to define a complex structure over $\overline{\mathfrak{L}}^\mu$. This is done as follows: We obtain the operator I by the polar decomposition $\kappa = I\,|\kappa|$. The properties of κ imply the relations $I\,|\kappa| = |\kappa|\,I$ and $I^2 = -\mathbb{1}$. A complex structure over $\overline{\mathfrak{L}}^\mu$ is then introduced by

$$(a+ib)f \coloneqq af - bIf, \qquad \forall f \in \overline{\mathfrak{L}}^\mu,\ a,b \in \mathbb{R}$$

The resulting complex vector space is denoted by $\overline{\mathfrak{L}}^\mu_{\mathbb{C}}$. It is equipped with a complex scalar product

$$(f|g)_{\mathbb{C}} = \mu(f,g) - i\sigma(f,g), \quad f,g \in \overline{\mathfrak{L}}^\mu_{\mathbb{C}}.$$

It is clear, that this construction does not lead to a complex Hilbert space yet, since $(\cdot|\cdot)_{\mathbb{C}}$ may be degenerate on $\overline{\mathfrak{L}}^\mu_{\mathbb{C}}$. Consequently the quotient

$$\mathcal{H}_1 \coloneqq \overline{\mathfrak{L}}^\mu_{\mathbb{C}} \Big/ \ker\,(\cdot|\cdot)_{\mathbb{C}}$$

defines a complex Hilbert space. It is well-known [BR02a] that the GNS representation of $\mathfrak{W}(\mathfrak{L},\sigma)$ induced by a quasi-free state ω_μ is isomorphic to the Weyl algebra, generated by the **Weyl operators** associate to the linear field φ_f in the bosonic Fock space

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \bigotimes_{\text{symm.}}^{n} \mathcal{H}_1,$$

on which the Weyl operators are obtained using the canonical ladder operators on $\mathcal{H}$. The space $\mathcal{H}_1$ is called the one-particle Hilbert space[9], associated to the state ω_μ. The degeneracy space $\ker\left(\cdot|\cdot\right)_\mu$ is also known as the Gelfand ideal of the state ω_μ, when extended to the algebra of the free field, see section 1.5.

The operator κ is of further use for a characterization of pure states:

Proposition A.2.3 ([KW91]).
Let ω_μ be a state on $\overline{\mathfrak{W}}(\mathfrak{L},\sigma)$, given as in definition A.2.1. Then ω_μ is a pure state, if and only if $\kappa^2 = -1$.

A.2.2. Weyl Algebra in the Functional Formalism

In the following we want to apply the theory, which has been developed in the above section to concrete examples in the functional formalism of QFT. At first we are going to introduce the Weyl algebra as a *-subalgebra of the algebra of observables $\mathcal{A}$ in the deformation quantization setting, that was introduced in section 1.4.

Therefore let $\star$ be given by an admissible two-point function $\Delta_+ \in H_+$, see definition 1.4.2 for details. For convenience we introduce symbols for the (anti-) symmetric part of Δ_+

$$\begin{aligned}\Delta_+(f,g) &= \Delta_1(f,g) + iG_c(f,g)\\ \Delta_1(f,g) &= \frac{1}{2}\Big(\Delta_+(f,g) + \underbrace{\Delta_+(g,f)}_{=\overline{\Delta_+(f,g)}}\Big), \quad G_c(f,g) = \frac{1}{2i}\Big(\Delta_+(f,g) - \Delta_+(g,f)\Big)\end{aligned}$$

In the case of real-valued test functions f,g, both $\Delta_1(f,g)$ and $G_c(f,g)$ are real. We define the **Weyl field functional**, associate to the linear field Φ_f as the $\star$-exponential

$$W_f(\phi) = \exp_\star(i\Phi_f(\phi)), \qquad \Phi_f(\phi) = \int f(x)\phi(x)\,\mathrm{d}x, \qquad f \in \mathcal{D}_{\mathbb{R}}.$$

It can be shown, that

$$\begin{aligned}W_f(\phi) &= \exp\left[i\left(\Phi_f(\phi) + \frac{i\hbar}{2}\Delta_+(f,f)\right)\right] = \mathrm{e}^{-\frac{\hbar}{2}\Delta_1(f,f)}\mathrm{e}^{i\Phi_f(\phi)}\\ (W_f \star W_g)(\phi) &= \mathrm{e}^{-i\hbar G_c(f,g)}W_{f+g}(\phi)\end{aligned}$$

[9] The one-particle Hilbert space is said to carry a fundamental degree of freedom, if there exists an irreducible representation of the Poincaré group on $\mathcal{H}_1$. This is true in the case of the vacuum state of the (Weyl algebra associated to the) free field.

by comparing the power series in λ arising on both sides, if we replace f by λf. We obtain

$$\bigvee_{\star}\{W_f = \exp_\star(i\Phi_f) : f \in \mathcal{D}_\mathbb{R}\}$$

as the Weyl algebra, associated to linear scalar field Φ_f in $\mathcal{A}$, see example 1.2.2. It is a *-subalgebra of $\mathcal{A}$, which is convergent in $\hbar$, i.e. we may associate Planck's constant to $\hbar$ in the algebra. An easy calculation shows, that the generators W_f satisfy the defining Weyl relations (A.2) and

$$\bigvee_{\star}\{W_f = \exp_\star(i\Phi_f) : f \in \mathfrak{L}\} = \mathfrak{W}(\mathfrak{L}, \sigma)$$

with the symplectic space

$$\mathfrak{L} = \{[f]_\sim : f \in \mathcal{D}_\mathbb{R},\ f \sim g \text{ if } f - g = Ph \text{ for some } h \in \mathcal{D}_\mathbb{R}\}, \quad \sigma(f,g) = -\hbar G_\mathrm{c}(f,g).$$

After fixing the notion of the Weyl algebra, we show the relation between the (canonical) evaluation functionals ev_0 (see section 1.5) in the functional formalism and the quasi-free states from proposition A.2.2. Namely we have

$$\mathrm{ev}_0(W_f) = \mathrm{e}^{-\frac{\hbar}{2}\Delta_1(f,f)},$$

i.e. ev_0 is a quasi-free state given by the real symmetric bi-linear form μ

$$\mu(f,g) = \hbar\Delta_1(f,g).$$

The positivity of Δ_+ as a bi-distribution over $\mathcal{D}$ (i.e. the property H5 in 1.4.2) implies the positivity condition (A.3). We see, that the degeneracy space of $(\cdot|\cdot)_\mathbb{C}$ on the complexified space $\overline{\mathfrak{L}}^\mu_\mathbb{C}$

$$\ker(\cdot|\cdot)_\mathbb{C} = \{f \in \overline{\mathfrak{L}}^\mu_\mathbb{C} : (f^*|f)_\mathbb{C} = 0\}$$

is exactly given by the Gelfand ideal of Δ_+, because

$$(f^*|f)_\mathbb{C} = \mu(f^*,f) - i\sigma(f^*,f) = \Delta_1(f^*,f) + iG_\mathrm{c}(f^*,f) = \Delta_+(f^*,f).$$

Moreover for every solution ψ to P the state ev_ψ, defined by

$$\mathrm{ev}_\psi(W_f) = \mathrm{e}^{-\frac{1}{2}\mu(f,f)}\mathrm{e}^{i\int\psi(x)f(x)\,\mathrm{d}x}$$

coincides with a **coherent** state on the Weyl algebra.

Let Δ_+ now be given by the two-point function of the homogeneous KMS state from example 1.5.7 with inverse temperature $\beta > 0$, i.e.

$$\Delta_+(f,g) = \int f(x)D^\beta_+(x-y)g(y)\,\mathrm{d}x\,\mathrm{d}y, \qquad f,g \in \mathcal{D}_\mathbb{R}$$

We find, that the symmetric and anti-symmetric part of Δ_+ are given by

$$\sigma(f,g) = \frac{1}{(2\pi)^3}\frac{i\hbar}{2}\int \hat{f}(-p)\hat{g}(p)\varepsilon(p_0)\delta(p^2-m^2)\,\mathrm{d}p$$
$$\mu(f,g) = \frac{1}{(2\pi)^3}\hbar\int \hat{f}(-p)\hat{g}(p)\frac{\varepsilon(p_0)\delta(p^2-m^2)}{2\coth\left(\frac{\beta p_0}{2}\right)}\,\mathrm{d}p.$$

On the completion $\overline{\mathfrak{L}}^{\mu}$ the operator κ, which is associated to the quasi-free state

$$\mathrm{ev}_0(W_f) = \mathrm{e}^{-\frac{\hbar}{2}\Delta_1(f,f)}$$

is given by a multiplication operator in momentum space:

$$(\kappa f)(x) = \frac{1}{i}\int \mathrm{e}^{-ipx}\tanh\left(\frac{\beta p_0}{2}\right)\tilde{f}(p)\,\mathrm{d}p.$$

This is done by a direct comparison of both sides of the defining relation $(f|\kappa g)_{\mu} = \sigma(f,g)$. We see immediately, that

$$(\kappa^2 f)(x) = -\int \mathrm{e}^{-ipx}\tanh^2\left(\frac{\beta p_0}{2}\right)\tilde{f}(p)\,\mathrm{d}p = -f(x) + \int \mathrm{e}^{-ipx}\frac{\hat{f}(p)}{\cosh^2\left(\frac{\beta p_0}{2}\right)}\,\mathrm{d}p$$

i.e. $\kappa^2 \neq -\mathbb{1}$, unless $\beta = +\infty$.

We obtain a well-known result, namely that

- the homogeneous KMS state for $0 < \beta < \infty$ is a mixed state
- the vacuum state ($\beta = +\infty$) is a pure state

over the Weyl algebra of the free scalar field. This result coincides with the physical intuition of the situation: A mixed state is interpreted as an imperfectly prepared ensemble of systems and a pure state as a perfectly prepared ensemble. But the thermal equilibrium states and its generalizations, the KMS states, prescribe a probabilistic distribution of the systems in the ensemble, whereas the vacuum is by definition a preparation of the system in the lowest energy state.

A.2.3. Enlarged Weyl Algebra

We want to reconsider the case of finite temperature, i.e. the case $0 < \beta < \infty$. We will identify $\mathfrak{W}(\mathfrak{L},\sigma)$ with its GNS representation with respect to the KMS state on the respective Hilbert space. In particular we omit the $\star$, since the products are now products of operators on a Hilbert space. It is easy to show that the vector Ω_{μ} that implements the quasi-free state ω_{μ}, is separating for the Weyl-algebra $\mathfrak{W}(\mathfrak{L},\sigma)$ and its C^*-completion $\overline{\mathfrak{W}}(\mathfrak{L},\sigma)$. This is a simple consequence of the fact, that the Gelfand ideal is trivial for any quasi-free KMS state, with the consequence, that the GNS representation is faithful.

Thus the conditions of the theorem 13 of Tomita-Takesaki are satisfied for the von-Neumann-algebra $\mathfrak{M} = \overline{\mathfrak{W}}(\mathfrak{L}, \sigma)'' \supset \mathfrak{W}(\mathfrak{L}, \sigma)$ that is associated to $\mathfrak{W}(\mathfrak{L}, \sigma)$. In particular there exists a conjugation operator J on the GNS Hilbert space, such that

$$j : \mathfrak{W}(\mathfrak{L}, \sigma) \subset \mathfrak{M} \to \mathfrak{W}(\mathfrak{L}, \sigma)' = \mathfrak{M}', \qquad j(A) \coloneqq JAJ. \tag{A.4}$$

We call the image of the generator W_f under j the **tilde Weyl field functional** $\tilde{W}_f$. Since J is conjugate linear and satisfies $J^2 = \mathbb{1}$, it holds

$$\tilde{W}_f \tilde{W}_g = J W_f J^2 W_g J = \mathrm{e}^{-i\sigma(f,g)} \tilde{W}_{f+g} \tag{A.5}$$

and $\tilde{W}_f$ commutes with W_f. The following generated algebra

$$\bigvee \{W_f : f \in \mathfrak{L}\} \cup \{\tilde{W}_f : f \in \mathfrak{L}\} = \mathfrak{W}(\mathfrak{L}, \sigma) \bigvee \mathfrak{W}(\mathfrak{L}, \sigma)' \subset \mathfrak{M} \bigvee \mathfrak{M}',$$

will be called **enlarged Weyl algebra**. Taking the similar form of Weyl relations of both generators W_f and $\tilde{W}_f$ into account, it is convenient to consider the functional

$$W_{(f,g)} = W_f \tilde{W}_g = \tilde{W}_g W_f.$$

The modular conjugation j acts as on the generators by

$$j(W_{(f,g)}) = W_{(g,f)}$$

and the $W_{(f,g)}$ satisfy the following relations:

$$\begin{aligned} W_{(f_1,f_2)} W_{(g_1,g_2)} &= \mathrm{e}^{i\sigma(f_1,g_1) - i\sigma(f_2,g_2)} W_{(f_1+g_1, f_2+g_2)} = \mathrm{e}^{i\hat{\sigma}[(f_1,f_2),(g_1,g_2)]} W_{(f_1+f_2, g_1+g_2)} \\ \hat{\sigma}[(f_1,f_2),(g_1,g_2)] &= \sigma(f_1,g_1) - \sigma(f_2,g_2). \end{aligned} \tag{A.6}$$

Here $\hat{\sigma}$ defines a symplectic form on $\mathfrak{L} \oplus \mathfrak{L}$. Therefore the enlarged Weyl algebra can be seen as

$$\bigvee \{W_{(f,g)} : f, g \in \mathfrak{L}\} = \mathfrak{W}(\hat{\mathfrak{L}}, \hat{\sigma}), \qquad \hat{\mathfrak{L}} = \mathfrak{L} \oplus \mathfrak{L}.$$

The algebraic state, which is given by the expectation value with respect to Ω_μ extends to a state over the enlarged Weyl algebra as well and has a representation by a symmetric bi-linear form $\hat{\mu}$ on $\mathfrak{L} \oplus \mathfrak{L}$:

$$\begin{aligned} \omega_{\hat{\mu}}(W_{(f,g)}) &= \mathrm{e}^{-\frac{1}{2}\hat{\mu}[(f,g),(f,g)]}, \\ \hat{\mu}[(f_1,f_2),(g_1,g_2)] &= \mu(f_1,g_1) + \mu_m(f_1,g_2) + \mu_m(f_2,g_1) + \mu(f_2,g_2) \end{aligned} \tag{A.7}$$

where the mixed term μ_m is given by

$$\begin{aligned} \mu_m(f,g) &= \hbar \int f(x) D_+^\beta \left(x - y - \frac{i\beta}{2} e^0 \right) g(y) \, \mathrm{d}x \, \mathrm{d}y \\ &= \frac{\hbar}{(2\pi)^3} \int \hat{f}(-p) \hat{g}(p) \frac{\varepsilon(p_0) \delta(p^2 - m^2)}{2 \sinh\left(\frac{\beta p_0}{2}\right)} \, \mathrm{d}p \end{aligned} \tag{A.8}$$

which is symmetric due to the KMS condition. The mixed term μ_m represents the correlations between the generators of W_f and $\tilde{W}_g$ in the state Ω_μ

$$\left(\Omega_\mu \middle| \Phi_f J \Phi_g J \Omega_\mu\right) = \hbar \Delta_m(f,g) = \int f(x) \underbrace{\Delta_+\left(x, y + \frac{i\beta e_0}{2}\right)}_{D_+^\beta\left(x-y-\frac{i\beta}{2}e_0\right)} g(y)\,\mathrm{d}x\,\mathrm{d}y, \qquad f,g \in \mathcal{D}_{\mathbb{R}}$$

The positivity condition (A.3) is guaranteed by the fact, that $\mu - i\sigma$ is a positive semi-definite over the complexified space $\overline{\mathfrak{L}}_{\mathbb{C}}$:

$$\begin{aligned}
&(\mu - i\sigma)[(f_1^*, f_2^*), (f_1, f_2)] = \hbar\Big(\Delta_+(f_1^*, f_1) + \Delta_m(f_1^*, f_2) + \Delta_m(f_2^*, f_1) + \Delta_+(f_2^*, f_2)^*\Big) \\
&= \hbar \int \left(\left|\hat{f}_1(p)\right|^2 + \hat{f}_1(p)^* \hat{f}_2(p) \mathrm{e}^{-\frac{\beta p_0}{2}} + \hat{f}_1(p) \hat{f}_2(p)^* \mathrm{e}^{-\frac{\beta p_0}{2}} + \left|\hat{f}_2(p)\right|^2 \mathrm{e}^{-\beta p_0} \right) \times \\
&\qquad \times \frac{\varepsilon(p_0)\delta(p^2 - m^2)}{1 - \mathrm{e}^{-\beta p_0}}\,\mathrm{d}p \\
&= \hbar \int \left(\hat{f}_1(p) + f_2(p)\mathrm{e}^{-\frac{\beta p_0}{2}}\right)^* \left(\hat{f}_1(p) + f_2(p)\mathrm{e}^{-\frac{\beta p_0}{2}}\right) \frac{\varepsilon(p_0)\delta(p^2 - m^2)}{1 - \mathrm{e}^{-\beta p_0}}\,\mathrm{d}p \geq 0.
\end{aligned}$$

We are able to find an explicit form of the operator κ, associated to the quasi-free state $\omega_{\hat{\mu}}$. It acts on the completion of the **doubled space** $\hat{\mathfrak{L}}_{\mathbb{C}}$. Its action can be written by a matrix valued multiplication operator in momentum space:

$$\begin{aligned}
\kappa(g_1, g_2)^T &= \begin{pmatrix} \kappa_{11} g_1 + \kappa_{12} g_2 \\ \kappa_{21} g_1 + \kappa_{22} g_2 \end{pmatrix} \\
\begin{pmatrix} \hat{\kappa}_{11}(p) & \hat{\kappa}_{12}(p) \\ \hat{\kappa}_{21}(p) & \hat{\kappa}_{22}(p) \end{pmatrix} &= \frac{1}{i \sinh\left(\frac{\beta p_0}{2}\right)} \begin{pmatrix} -\cosh\left(\frac{\beta p_0}{2}\right) & -1 \\ 1 & \cosh\left(\frac{\beta p_0}{2}\right) \end{pmatrix}.
\end{aligned}$$

One sees that $\kappa^2 = -\mathbb{1}$, i.e. $\omega_{\hat{\mu}}$ is a **pure state** over the enlarged algebra. The limit $\beta \to +\infty$ is also possible in this construction. Then the mixed correlations μ_m vanish and the resulting state over the enlarged algebra is just the product state of the vacuum state with itself, which is also pure. This can also be directly computed via

$$\lim_{\beta \to +\infty} \begin{pmatrix} \hat{\kappa}_{11}(p) & \hat{\kappa}_{12}(p) \\ \hat{\kappa}_{21}(p) & \hat{\kappa}_{22}(p) \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

An interpretation of this construction is available with the following argument: The mixed nature of the KMS state is interpreted as the impossibility to obtain the full information of the state via measurements of the observables of the systems, which are represented by the algebra $\mathfrak{W}(\mathfrak{L}, \sigma)$.

This picture changes, if we include the algebra of "tilde-fields", i.e. if we allow hypothetical measurements of the enlarged algebra. Then the same state appears to be pure. However, this procedure is just a mathematical procedure to obtain a pure state, the tilde-fields are not observables of the system!

One might think of these tilde fields as observables of the thermal bath, which surrounds the physical system and keeps it in the thermal equilibrium. This idea is flawed though, since the physical system, which is discussed above, fills the whole Minkowski spacetime and there is no surrounding thermal bath.

The mathematical construction can be seen as a special case of the purification map, which has been studied in [Wor72] for case of general C^*-algebras. Let $\mathfrak{A}$ be a C^*-algebra and ω a state on $\mathfrak{A}$, such that the generated von-Neumann-algebra $\pi_\omega(\mathfrak{A})''$ is a factor. Then, under suitable conditions, there exists a unique extension of ω to a state over

$$\mathfrak{A} \otimes \mathfrak{A}^{\mathrm{o}} \tag{A.9}$$

where $\mathfrak{A}^{\mathrm{o}}$ is the opposite algebra. We have encountered exactly the same structure here, since

$$\mathfrak{W}(\hat{\mathfrak{L}}, \hat{\sigma}) = \mathfrak{W}(\mathfrak{L}, \sigma) \otimes \mathfrak{W}(\mathfrak{L}, \sigma)^{\mathrm{o}} = \mathfrak{W}(\mathfrak{L}, \sigma) \otimes \mathfrak{W}(\mathfrak{L}, -\sigma)$$

as *-algebras.

We close this section with a summary of the algebraic structure of the enlarged Weyl algebra with respect to the modular conjugation $j : A \mapsto JAJ$ which extends to $\mathfrak{W}(\hat{\mathfrak{L}}, \hat{\sigma})$. For convenience we identify the weak closure $\mathfrak{M} = \mathfrak{W}(\mathfrak{L}, \sigma)''$ as the algebra of observables and $\hat{\mathfrak{M}} = \mathfrak{M} \bigvee \mathfrak{M}'$ as the enlarged algebra. Then

M1 **Embedding**: The algebra of observables $\mathfrak{M}$ is a *-subalgebra of $\hat{\mathfrak{M}}$.

M2 **Commutant**: The commutant $\mathfrak{M}'$ is equal to $j(\mathfrak{M})$, i.e. j is a conjugate linear isomorphism between the algebra of observables and its commutant.

M3 **State**: The KMS state ω_μ over $\mathfrak{M}$ extends to a pure state $\omega_{\hat{\mu}}$ over $\hat{\mathfrak{M}}$, which obeys $\omega_{\hat{\mu}}(j(\hat{A})) = \omega_{\hat{\mu}}(\hat{A})^*$ for every $\hat{A} \in \hat{\mathfrak{M}}$ and $\omega_{\hat{\mu}}(j(A)A) \geq 0$ for every $A \in \mathfrak{M}$.

The two conditions in *M*3 are called *j*-positivity and *j*-invariance in [Wor72]. With these conditions, the purification map $\omega_\mu \mapsto \omega_{\hat{\mu}}$ is actually unique [Wor73].

Acknowledgements

First of all I want to express my deepest gratitude to my supervisor, Prof. Klaus Fredenhagen, for proposing this very interesting and challenging topic to me. In the innumerable discussions with him I have gained many insights not only in the field of quantum field theory and mathematical physics, but also in the scientific method in general. For showing me how to properly formulate intuitions into solid mathematics I want to thank you very warmly.

I am also thankful to Prof. Bahns for agreeing to referee my thesis and to Prof. Louis to be the second referee to my disputation.

Moreover, I want to pronounce my tremendous gratitude to the whole AQFT group of Hamburg for helping me on various problems and for giving me the most pleasant time of my scientific career! This includes especially Claudio Dappiaggi, Benjamin Lang and Christian Pfeifer, who made science not only fun, but awesome.

Furthermore, I received invaluable help along the twisting road of my thesis from Andreas Degner, Thomas-Paul Hack, Kai Keller, Daniel Paulino, Nicola Pinamonti, Katarzyna Rejzner, Jan Schlemmer and Jochen Zahn. In addition, I want to thank the former students Marcos Brum, Paniz Imani, Mathias Makedonski and Sebastian Schubert.

I am very grateful to the theory groups of Hamburg University and DESY, especially to Sarah Andreas, Sergei Bobrovskyi, Valerie Domcke, Elina Fuchs, Cecelie Hector, Reinke Sven Isermann, Patrick Rieth, Markus Rummel, Martin Schasny, Kai Schmitz and Václav Tlapák for inspiring discussions on physics and philosophy as well as numerous activities outside the DESY campus.

Many thanks to the secretaries, Elizabeth Monteiro-Duarte and Wiebke Kircheisen, of the II. Institute for Theoretical Physics for giving a helping hand and providing me with sweets in the times in which they were needed.

Am Ende möchte ich den wichtigsten Begleitern auf meinem Weg danken: Meinen Eltern Heike und Jens, meinem Bruder Paul und meinen Großeltern Renate und Jörg: Ohne Eure immerwährende Unterstützung und Euren Rückhalt wäre diese Arbeit nicht möglich gewesen. Für Deine Liebe, Deinen Beistand und für so vieles mehr: Danke, Nicole.

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