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The Cosmological Constant in Theories with Finite Spacetime

Die kosmologische Konstante in Theorien mit endlicher Raumzeit

Master's Thesis

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Abstract

We study the role of the cosmological constant in different theories with finite spacetime. The cosmological constant appears both as an initial condition and as a constant of integration. In the context of the cosmological constant problem a new model will be presented. This modification of general relativity generates a small, non-vanishing cosmological constant, which is radiatively stable. The dynamics of the expansion of the universe in this model will be analyzed. Eventually, we try to solve the emergent problems concerning the generation of accelerated expansion using a quintessence model of dark energy.

Zusammenfassung

Wir studieren die Rolle der kosmologischen Konstante in verschiedenen Theorien mit endlicher Raumzeit. Die kosmologische Konstante tritt sowohl als Anfangsbedingung, als auch als Integrationskonstante auf. Im Kontext des Problems der kosmologischen Konstante wird ein neues Modell präsentiert. Diese Modifizierung der allgemeinen Relativitätstheorie generiert eine kleine, nichtverschwindende kosmologische Konstante, welche stabil gegenüber Strahlungskorrekturen ist. Die Dynamik der Expansion des Universums wird in diesem Modell untersucht. Schließlich versuchen wir das auftretende Problem bezüglich der Erzeugung von beschleunigter Expansion mit Hilfe eines Quintessenz-Modells der Dunklen Energie zu lösen.

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1 Introduction

The cosmological constant problem is a fundamental issue in modern physics [1,2,3]. Basically it means that we are not able to predict the small, but non-vanishing value of the cosmological constant using quantum field theory (QFT).
A lot of reviews are covering this subject [3, 4, 5, 6, 7]. The cosmological constant was originally introduced by Einstein to generate a static solution for our universe. Hubble's discovery of the expansion of the universe made Einstein to abandon this idea. But it is not possible to just drop the cosmological constant because any contribution to the vacuum energy acts like such a term. From Lorentz invariance one gets that the energy momentum tensor of the vacuum energy has the following form $T_{\mu\nu} = -\rho_{\text{vac}} g_{\mu\nu}$. Considering the Einstein field equations shows that one can define an effective cosmological constant as $\Lambda_{\text{eff}} = \Lambda + \frac{1}{M_{\text{Pl}}^2}\rho_{\text{vac}}$. In the following a bare cosmological constant refers to Λ. After the discovery of the accelerated expansion of the universe the concept of non-vanishing cosmological constant is highly attractive, because this term easily implements the observed acceleration. But the identification of the cosmological constant with the vacuum energy, which is well motivated from particle physics, fails. Unfortunately, field theoretical predictions for the vacuum energy do not match with the measured value for the cosmological constant at all. The value of the vacuum energy density from cosmological observation is $\rho_{\text{vac}} \simeq 10^{-48}$ GeV^4 (see [5] for further details). But the field theoretical calculation yields

$$\rho_{\text{vac}} = \int_0^\lambda \frac{4\pi k^2 dk}{(2\pi)^3} \frac{1}{2}\sqrt{k^2 + m^2} \simeq \frac{\lambda^4}{16\pi^2}. \tag{1.1}$$

This gives, if we trust general relativity up to the Planck scale ($\lambda = M_{\text{Pl}}$) $\rho_{\text{vac}} \simeq 10^{72}$ GeV^4. The ratio of these two numbers gives the famous discrepancy of 120 orders of magnitude. If we take λ_{QCD} as the cutoff for the integral we obtain $\rho_{\text{vac}} \simeq 10^{-6}$ GeV^4. Still, 42 orders of magnitude apart from the desired result. To understand the cosmological constant as a UV-divergent quantity allows some more insight. These have to be renormalized in QFT. Their value cannot be predicted, but it must be measured like the quadratically divergent mass of a scalar field. The bare cosmological constant contribution can be interpreted as a counterterm for the vacuum energy. In this picture the procedure is the following: First one regulates ρ_{vac} using a cutoff, then one adds Λ to subtract the UV-sensitive contributions from ρ_{vac} and finally Λ is tuned such that Λ_{eff} agrees with the observed value.

But these considerations are not perturbatively stable. If one fixes ρ_{vac} and Λ up to a certain order in perturbation theory such that Λ_{eff} matches with the observation and then considers the next order, one has to alter the initial conditions dramatically in order to save the agreement. Thus the small value of Λ_{eff} is radiatively unstable.
The problem gets even worse, if one takes contributions from cosmological phase transitions, e.g. the electroweak phase transition, to the cosmological constant into account. In principle one can tune the parameters such that these contributions cancel. But the setup becomes more and more unnatural.
The introduction of supersymmetry has an impact on the cosmological constant problem. Introducing the superpartners to the standard model particles leads to an equal number of bosonic and fermionic states. Thus, the vacuum energy contributions from bosons and fermions cancel exactly, when supersymmety is unbroken. Apparently, supersymmety must be broken in our universe. For a breaking scale M_{SUSY} one expects a vacuum energy of order $\rho_{\text{vac}} \sim M^4_{SUSY}$. That we have not seen any supersymmetric particles in the accelerator experiments implies $M_{\text{SUSY}} \sim$ TeV or even higher. This yields a vacuum energy, which is still off by far.
Different ways to understand the cosmological constant are needed than the common assumption that the energy density induced by quantum fluctuations is equivalent to the cosmological constant. Especially those approaches, which do not consider the determinant of the metric as a dynamical variable, are of main interest for this thesis. In such theories spacetime is finite and the cosmological constant appears in different contexts, as we will see in the following. These theories go under the name of unimodular gravity [8, 9, 10, 11].
A recent modification of general relativity, proposed by N. Kaloper and A. Padilla [12,13], called the KP model in the following, will be studied. It offers new perspectives for the interpretation of the cosmological constant. The most interesting feature of this theory is that it gets rid of the quantum corrections for the cosmological constant and explains why it is small in sufficiently large and old universes. For the model it is necessary that spacetime is finite, i.e. the universe will recollapse in the future. To match the observations the model has to combine accelerated expansion and finite spacetime. If it is not possible to accomplish both tasks the presented model seems to be questionable.
Starting from this setup, our own investigations begin in section 5.5. We find that accelerated expansion generated by a bare cosmological constant is not possible in the KP model. Thus, a dark energy model will be introduced, which generates accelerated expansion using a scalar field. Subsequently, we build a model in Einstein gravity, which is compatible with the observed cosmological parameters. To understand the dynamics of the quintessence model in a closed universe, we solve it numerically. Quintessence is the dark energy model we use. Considering the KP model in terms of the quintessence model

suggests that it is possible to generate accelerated expansion with this approach. The accelerated expansion seems not to stop eventually. This would violate the condition that spacetime has to be finite for the KP model to be consistent with the observations. Thus, our considerations and calculations indicate that there is some tension within the model. But these results are preliminary and have to be confirmed by further considerations.

This work is organized as follows. First some basics of general relativity and cosmology will be reviewed. In chapter 3 it will be shown, that restricted coordinate invariance is sufficient to get the theory of general relativity [11]. The cosmological constant appears in this approach as an initial condition. Unimodular gravity will be introduced in chapter 4 and the corresponding equations of motion are derived with variational methods. The cosmological constant shows up as an constant of integration in this case. The KP model is presented in chapter 5. We derive the Friedmann equations in this model and analyze them. Since accelerated expansion generated by a cosmological constant is not possible, the quintessence model will be introduced in chapter 6. We construct a model for a closed universe using quintessence in Einstein gravity and solve it numerically. Finally, we try to incorporate the dark energy model in the KP model. In the last chapter the main results will be summarized and we give an outlook on possible further investigations.

2 Standard cosmology

Reformulations and modifications of general relativity (GR) and consequently alternative cosmologies are an important subject of this thesis. Therefore it is useful to review some basics of Einstein gravity and Friedmann-Robertson-Walker (FRW) cosmology. The appearing quantities are introduced to set the stage for this work. The review part is mostly based on the textbooks [14, 15, 16].
This work mainly follows the conventions of [15]. In particular we use the mostly plus sign convention for the metric $(-,+,+,+)$. It is convenient to use natural units, where the reduced Planck constant $\hbar$, the speed of light c and the Boltzmann constant k equal

$$\hbar = c = k = 1. \tag{2.1}$$

Lorentz indices are displayed by small Greek letters, e.g. $\mu, \nu = 0, 1, 2, 3$, and spatial indices by Latin letters, e.g. $i, j = 1, 2, 3$. In addition, we use the Einstein summation convention by implicitly summing over repeated indices. The Planck mass is denoted by m_{Pl} and the reduced Planck mass by $M_{\text{Pl}} = \frac{m_{\text{Pl}}}{\sqrt{8\pi}}$.

2.1 Basics of general relativity

The equation of motion of a freely falling particle with coordinates ξ^μ is

$$\frac{d^2\xi^\mu}{d\tau^2} = 0, \tag{2.2}$$

where $d\tau$ is the proper time

$$d\tau^2 = -\eta_{\mu\nu} d\xi^\mu d\xi^\nu \tag{2.3}$$

and $\eta_{\mu\nu}$ the Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.4}$$

Transforming the equation of motion to an arbitrary coordinate system x^μ yields the geodesic equation

$$\frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho{}_{\mu\nu} dx^\mu dx^\nu = 0, \tag{2.5}$$

with the coefficients of the affine connection, called the Christoffel symbols

$$\Gamma^\rho{}_{\mu\nu} = \frac{\partial x^\rho}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}. \tag{2.6}$$

The proper time, which is equal to the length of a line element in natural units, reads in an arbitrary coordinate system with coordinates x^μ

$$ds^2 = -g_{\mu\nu} dx^\mu dx^\nu \tag{2.7}$$

and the metric tensor

$$g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta}. \tag{2.8}$$

The Christoffel symbols can be expressed in terms of the metric

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}). \tag{2.9}$$

They also appear in the covariant derivative

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu{}_{\nu\lambda} V^\lambda, \tag{2.10}$$

$$\nabla_\nu V_\mu = \partial_\nu V_\mu - \Gamma^\lambda{}_{\mu\nu} V_\lambda. \tag{2.11}$$

Note that the metric tensor $g_{\mu\nu}$ is covariantly conserved $\nabla_\lambda g_{\mu\nu} = 0$.
The Riemann curvature tensor is defined as

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\mu\sigma} \tag{2.12}$$

and the torsion tensor (which is zero for the connection defined above, since we are considering general relativity as torsion-free) is

$$T^\lambda{}_{\mu\nu} = 2\Gamma^\lambda{}_{[\mu\nu]}. \tag{2.13}$$

Both quantities appear in the following commutator

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} \nabla_\lambda V^\rho. \tag{2.14}$$

The contraction of the Riemann tensor is called the Ricci tensor

$$R_{\sigma\nu} = R^{\lambda}{}_{\sigma\lambda\nu} \tag{2.15}$$

and the further contraction with the metric tensor is called the Ricci scalar

$$R = g^{\sigma\nu} R_{\sigma\nu} = R^{\nu}{}_{\nu}. \tag{2.16}$$

The Riemann tensor has the following symmetries

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}, \tag{2.17}$$

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} = -R_{\rho\sigma\nu\mu} = R_{\sigma\rho\nu\mu}, \tag{2.18}$$

$$R_{\rho\sigma\mu\nu} + R_{\rho\nu\sigma\mu} + R_{\rho\mu\nu\sigma} = 0 \tag{2.19}$$

and satisfies the Bianchi identities

$$\nabla_\eta R_{\rho\sigma\mu\nu} + \nabla_\nu R_{\rho\sigma\eta\mu} + \nabla_\mu R_{\rho\sigma\nu\eta} = 0, \tag{2.20}$$

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = \nabla_\mu G^{\mu\nu} = 0. \tag{2.21}$$

2.2 Basics of FRW cosmology

The assumption that the universe is isotropic and homogeneous is the foundation of this model. Accordingly, the FRW metric has the form

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right) \tag{2.22}$$

and is invariant under the transformations like $a \to \lambda^{-1} a,\ r \to \lambda r,\ k \to \lambda^{-2} k$. Thus, we can choose a normalization such that k takes the discrete values $k = (-1, 0, +1)$. ($k = -1$ corresponds to an open, $k = 0$ to a flat and and $k = +1$ to a closed universe.) For some applications it is more convenient to use the following normalization

$$ds^2 = -dt^2 + a'(t)^2 \left(\frac{dr'^2}{1 - \kappa r'^2} + r'^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right). \tag{2.23}$$

The parameter $\kappa = \frac{k}{R_0^2}$ describes the geometry of the spacetime, has dimension (length)$^{-2}$ and can take any value. ($\kappa < 0$ corresponds to an open, $\kappa = 0$ to a flat and and $\kappa > 0$ to a closed universe.) In this convention $a'(t) = \frac{a(t)}{R_0}$ is the dimensionless scale factor and $r' = R_0 r$ the radial coordinate with dimension length. In the following the prime will be dropped and the normalizations can be distinguished by use of k or κ.

The Einstein field equations yield the dynamics of the universe

$$M_{\rm Pl}^2 G_{\mu\nu} = M_{\rm Pl}^2 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = T_{\mu\nu}, \tag{2.24}$$

with a cosmological constant included

$$M_{\rm Pl}^2 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu}\right) = T_{\mu\nu}. \tag{2.25}$$

. Using the FRW metric one can decompose the Einstein field equations and obtains the Friedmann equations. For this purpose we need to express the Einstein tensor $G_{\mu\nu}$ in terms of the FRW metric. The Ricci tensor now reads [14]

$$R_{\mu\nu} = \begin{pmatrix} -3\frac{\ddot{a}}{a} & 0 & 0 & 0 \\ 0 & \frac{2\kappa+2\dot{a}^2+a\ddot{a}}{1-kr^2} & 0 & 0 \\ 0 & 0 & r^2\left(2(\kappa+\dot{a}^2)+a\ddot{a}\right) & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta\left(2(\kappa+\dot{a}^2)a\ddot{a}\right) \end{pmatrix} \tag{2.26}$$

and the Ricci scalar is

$$R = \frac{6(\kappa + \dot{a}^2 + a\ddot{a})}{a^2}. \tag{2.27}$$

Consequently the Einstein tensor takes the form

$$G_{\mu\nu} = \begin{pmatrix} \frac{3(\kappa+\dot{a}^2)}{a^2} & 0 & 0 & 0 \\ 0 & \frac{-\left(\kappa+\dot{a}^2+2a\ddot{a}\right)}{1-\kappa r^2} & 0 & 0 \\ 0 & 0 & -r^2\left(\kappa+\dot{a}^2+2a\ddot{a}\right) & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta\left(\kappa+\dot{a}^2+2a\ddot{a}\right) \end{pmatrix}. \tag{2.28}$$

In order to extract the first Friedmann equation consider the (0,0)-component of the Einstein field equations. This leads to the following relation

$$M_{\rm Pl}^2 \left(\frac{3\left(\kappa+\dot{a}^2\right)}{a^2}\right) = T_{00}. \tag{2.29}$$

Assume the energy-momentum tensor for a perfect fluid

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu}, \tag{2.30}$$

where u_μ is the 4-velocity (Choosing the time frame yields $u_\mu = (1,0,0,0)$.), $T_{00} = \rho$ the energy density and p the corresponding pressure. Plugging in gives

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_{\rm Pl}^2}\rho - \frac{\kappa}{a^2}, \tag{2.31}$$

where H is the Hubble-parameter. To derive the second Friedmann equation one needs to consider the (i,j)-component of the corresponding Einstein field equations

$$M_{\rm Pl}^2\left(-\frac{(\kappa + \dot{a}^2 + 2a\ddot{a})}{a^2}\right) g_{ij} = p g_{ij} \tag{2.32}$$

$$\Longleftrightarrow \; -\left(\frac{\dot{a}}{a}\right)^2 - 2\frac{\ddot{a}}{a} = \frac{1}{M_{\rm Pl}^2}p + \frac{\kappa}{a^2}. \tag{2.33}$$

Adding the first Friedmann equation (2.31) yields the second Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_{\rm Pl}^2}\left(\rho + 3p\right). \tag{2.34}$$

If one includes a cosmological constant Λ the Friedmann equations become

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_{\rm Pl}^2}\rho + \frac{\Lambda}{3} - \frac{\kappa}{a^2}, \tag{2.35}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_{\rm Pl}^2}\left(\rho + 3p\right) + \frac{\Lambda}{3}. \tag{2.36}$$

Starting from the first Friedmann equation and using the covariant conservation of the energy-momentum ($\nabla_\mu T^{\mu\nu} = 0$ as a consequence of Bianchi identity applied to the Einstein field equations) one gets the second Friedmann equation. In terms of FRW cosmology one has

$$-3a^2 p = \frac{d}{da}\left(a^3\rho\right) \quad \Longleftrightarrow \quad \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p\right) = 0, \tag{2.37}$$

which is sometimes called the continuity equation. Imposing an equation of state for a perfect fluid $p = w\rho$, one finds

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}. \tag{2.38}$$

If w is constant, integration gives

$$\rho \propto a^{-3(1+w)}. \tag{2.39}$$

The range of values for w is bounded due to the Null Energy Condition (see appendix B). This implies $|w| \leq 1$. Now we discuss the different components of the cosmological fluid.

- Matter:

$$\rho_{\mathrm{M}} \propto a^{-3},\ p_{\mathrm{M}} = 0 \tag{2.40}$$

Collisionless, non-relativistic particles, which have zero pressure (dust). The energy density falls off due to the expansion of the universe.

- Radiation:

$$\rho_{\mathrm{R}} \propto a^{-4},\ p_{\mathrm{R}} = \frac{1}{3}\rho_{\mathrm{R}} \tag{2.41}$$

The energy density falls off due to the expansion of the universe and additionally due to the redshift.

- Vacuum:

$$\rho_{\Lambda} \propto a^{0},\ p_{\Lambda} = -\rho_{\Lambda} \tag{2.42}$$

The energy density is constant despite the expansion. It tends to dominate the energy density of an open or flat universe for late times. Anti-de Sitter and de Sitter space are vacuum dominated solutions.

The first Friedmann equation can be rewritten in terms of the dimensionless density parameter $\Omega = \frac{\rho}{\rho_{\mathrm{c}}}$, where $\rho_{\mathrm{c}} = 3H^2 M_{\mathrm{Pl}}^2$ is the critical density. Now, the geometry of the universe is determined by the value of Ω:

$$\Omega < 1 \rightarrow k = -1 \iff \kappa < 0 \tag{2.43}$$
$$\Omega = 1 \rightarrow k = 0 \iff \kappa = 0 \tag{2.44}$$
$$\Omega > 1 \rightarrow k = +1 \iff \kappa > 0 \tag{2.45}$$

One can rewrite the Friedmann equation using the density parameter

$$\Omega - 1 = \frac{\kappa}{H^2 a^2}. \tag{2.46}$$

To solve the Friedmann equation we need to specify the different components ρ_i, which contribute to the total energy density ρ. Knowing the equations of state $p_i = p_i(\rho_i)$ and the spatial curvature κ renders the Friedmann equation solvable. The solution describes the whole evolution of the scale factor $a(t)$. Assuming that the different components of the energy density evolve as power laws

$$\rho_i = \rho_{i,0} a^{-n_i} \tag{2.47}$$

leads to

$$w_i = \frac{1}{3} n_i - 1 \tag{2.48}$$

for the equation-of-state parameters.

source	w_i	n_i
matter	0	3
radiation	$\frac{1}{3}$	4
curvature	$-\frac{1}{3}$	2
vacuum	-1	0

The curvature term can be expressed in a similar way defining $\rho_\kappa = -\frac{3M_{\text{Pl}}^2\kappa}{a^2}$ and $\Omega_\kappa = -\frac{\kappa}{H^2a^2}$. The Friedmann equation then becomes

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_{\text{Pl}}^2}\sum_i \rho_i \tag{2.49}$$

and if one divides by H^2

$$1 = \sum_i \Omega_i \iff \Omega_\kappa = 1 - \Omega. \tag{2.50}$$

The solution of the Friedmann equation, which is dominated by one fluid, is easily obtained for the general case $\rho \propto a^{-n}$ $(\kappa = 0)$

$$\dot{a} \propto a^{1-\frac{n}{2}}, \tag{2.51}$$

thus

$$a \propto t^{\frac{2}{n}}. \tag{2.52}$$

This means for matter domination $a \propto t^{\frac{2}{3}}$ and for radiation domination $a \propto t^{\frac{1}{2}}$.

2.3 Friedmann equations in closed spacetime

To understand the dynamics of the universe, consider the Friedmann equations in Einstein gravity. These are (2.31) and (2.34), but in this case we use the alternative normalization (2.22) of the FRW metric

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_{\text{Pl}}^2}\left(\rho + 3p\right), \tag{2.53}$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_{\text{Pl}}^2}\rho - \frac{k}{a^2}. \tag{2.54}$$

Because we are interested in the cosmological constant in theories with finite spacetime, we will analyze the dynamics of a closed universe $(k = +1)$ in full detail. The main difference to flat and open universes $(k = 0, k = -1)$ is that a recollapse can happen. In this case the Big Bang is followed by a Big Crunch. We start with the first Friedmann equation for a closed universe

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_{\rm Pl}^2}\rho - \frac{1}{a^2}. \tag{2.55}$$

For a Bang/Crunch scenario we need an initially increasing $a(t)$ reaching a maximum, followed by a decrease. This implies that there is a t_0 such that $\dot{a}(t_0) = 0$ and $\ddot{a}(t_0) < 0$. The first condition can be satisfied because the two terms on the right-hand-side of equation (2.55) have opposite signs. This would not be possible in theories with a flat or open universe. For the second condition we consider the second Friedmann equation, which contains the second time derivative of the scale factor. Thus the right-hand side of (2.53) has to be smaller than zero. This is always satisfied, if the fluid component satisfies $p > -\frac{\rho}{3}$. Therefore a Bang/Crunch scenario is not possible, if the theory includes a cosmological constant, with $p = -\rho$. After the start of a phase of accelerated expansion induced by a non-zero positive cosmological constant, it will continue forever even in a closed universe.
In order to understand the dynamics of a closed universe better, an explicit solution for a radiation dominated universe will be presented. For such explicit calculations it is convenient to introduce the conformal time τ, which is defined as

$$dt = d\tau a(\tau). \tag{2.56}$$

The second Friedmann equation in conformal time reads

$$a'^2 = \left(\frac{da}{d\tau}\right)^2 = \frac{1}{3M_{\rm Pl}^2}\rho a^4 - ka^2. \tag{2.57}$$

Applying $\frac{d}{d\tau}$ and using $\dot{\rho} + 3H(\rho + p) = 0$ (2.37) one gets

$$a'' = \frac{1}{6M_{\rm Pl}^2}(\rho - 3p)\, a^3 - ka. \tag{2.58}$$

Alternatively, one can transform the derivatives in the Friedmann equations using conformal time and combine them to get the same result

$$\dot{x} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{x'}{a(\tau)}, \tag{2.59}$$

$$\ddot{x} = \frac{d}{dt}\frac{dx}{dt} = \frac{d}{dt}\left(\frac{dx}{d\tau}\frac{d\tau}{dt}\right) = \frac{d}{dt}\left(\frac{x'}{a(\tau)}\right) \tag{2.60}$$

$$= \frac{d\tau}{dt}\frac{d}{d\tau}\left(\frac{x'}{a(\tau)}\right) = \frac{x''}{a(\tau)^2} - \frac{a'(\tau)x'}{a(\tau)^3}. \tag{2.61}$$

Equation (2.58) simplifies a lot for radiation domination ($p = \frac{\rho}{3}$).

$$a'' + ka = 0 \tag{2.62}$$

For a closed universe ($k = 1$) the solution simply reads

$$a(\tau) = a_0 \sin(\tau), \tag{2.63}$$

where a_0 is a constant of integration and using the condition $a(\tau = 0) = 0$. Integrating the defining relation for the conformal time $dt = d\tau a(\tau)$ yields the physical time

$$t = (1 - \cos(\tau)) \,. \tag{2.64}$$

Combined we get an explicit solution

$$a(t) = a_0 \sin\left(\arccos\left(1 - \frac{t}{a_0}\right)\right), \tag{2.65}$$

which is shown in the figure 2.1 and describes a recollapsing universe.

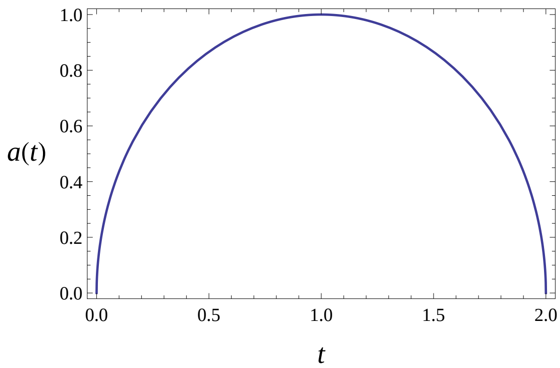

Figure 2.1: Time evolution of the scale factor $a(t)$ in a closed, radiation dominated universe with $a_0 = 1$

3 General relativity from restricted coordinate invariance

This chapter presents basically the idea of W. Buchmüller and N. Dragon that invariance under general coordinate transformations is not necessary to derive the quantities, which appear in general relativity. Invariance under restricted coordinate transformations is sufficient [11].

3.1 Constructing the Lagrangian

Consider an infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu - \epsilon^\mu \iff x^\mu = x'^\mu + \epsilon^\mu. \tag{3.1}$$

This implies

$$\frac{\partial x^\mu}{\partial x'^\nu} = \delta^\mu{}_\nu + \partial_\nu \epsilon^\mu, \; \frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu{}_\nu - \partial_\nu \epsilon^\mu. \tag{3.2}$$

The metric tensor transforms as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x'). \tag{3.3}$$

One can find another transformation such that the metric transforms as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = g'_{\mu\nu}(x' + \epsilon) = \underbrace{g'_{\mu\nu}(x')}_{=\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)} + \epsilon^\lambda \partial_\lambda g_{\mu\nu}(x), \tag{3.4}$$

$$g'_{\mu\nu}(x') = (\delta^\rho{}_\mu + \partial_\mu \epsilon^\rho)(\delta^\sigma{}_\nu + \partial_\nu \epsilon^\sigma) g_{\rho\sigma}(x) \tag{3.5}$$

$$= g_{\mu\nu}(x) + \partial_\mu \epsilon^\rho g_{\rho\nu}(x) + \partial_\nu \epsilon^\sigma g_{\mu\sigma}(x) + \mathcal{O}(\epsilon^2). \tag{3.6}$$

$$\Rightarrow g'_{\mu\nu}(x) = g_{\mu\nu}(x) + \epsilon^\lambda \partial_\lambda g_{\mu\nu}(x) + \partial_\mu \epsilon^\lambda g_{\lambda\nu}(x) + \partial_\nu \epsilon^\lambda g_{\mu\lambda}(x) + \mathcal{O}(\epsilon^2) \tag{3.7}$$

The determinant of the metric denoted by $g = \det(g_{\mu\nu})$ transforms under infinitesimal transformations as

$$\delta g = \delta \det(g_{\mu\nu}) = g g^{\mu\nu} \delta g_{\mu\nu}, \tag{3.8}$$

because the following holds for any square matrix

$$\ln(\det M) = \operatorname{tr} \ln M, \tag{3.9}$$

$$\frac{1}{\det(M)} \delta \det(M) = \operatorname{tr}(M^{-1} \delta M) \Rightarrow \delta g = g(g^{\mu\nu} \delta g_{\mu\nu}) = -g(g_{\mu\nu} \delta g^{\mu\nu}). \tag{3.10}$$

The last equality is true because $g^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}{}_{\nu}$ and the variation of the Kronecker delta vanishes. Therefore one has

$$\delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma}. \tag{3.11}$$

Now all the pieces can be combined to get the variation of the negative squareroot of the determinant of the metric

$$\delta \sqrt{-g} = \frac{1}{2} \frac{\delta g}{\sqrt{-g}} = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu} \tag{3.12}$$

$$= \frac{1}{2} \sqrt{-g} g^{\mu\nu} (\epsilon^{\lambda} \partial_{\lambda} g_{\mu\nu}(x) + \partial_{\mu} \epsilon^{\lambda} g_{\lambda\nu}(x) + \partial_{\nu} \epsilon^{\lambda} g_{\mu\lambda}(x)) \tag{3.13}$$

$$= \frac{1}{2} \sqrt{-g} g^{\mu\nu} \epsilon^{\lambda} \partial_{\lambda} g_{\mu\nu}(x) + \sqrt{-g} \partial_{\lambda} \epsilon^{\lambda} \tag{3.14}$$

$$= \partial_{\lambda} (\sqrt{-g} \epsilon^{\lambda}). \tag{3.15}$$

Under restricted coordinate transformations, which are volume preserving transformations, $\sqrt{-g}$ transforms as a scalar field, namely

$$\delta \sqrt{-g} = \epsilon^{\lambda} \partial_{\lambda} \sqrt{-g}. \tag{3.16}$$

This means that restricted coordinate transformations are infinitesimal transformations with vanishing divergence, i.e. $\partial_{\mu} \epsilon^{\mu}(x) = 0$. Thus, one can factorize the metric tensor into a part with fixed determinant and another part, which satisfies the transformation law

$$g_{\mu\nu} = \mathrm{e}^{2\sigma} \overline{g}_{\mu\nu}, \ \det \overline{g}_{\mu\nu} = -1. \tag{3.17}$$

The tensors, which can be constructed from $\overline{g}_{\mu\nu}$, are identical to the tensors under general coordinate transformations. To show this one first has to consider the Christoffel symbols (2.9), which are defined as

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} (\partial_{\mu} g_{\lambda\nu} + \partial_{\nu} g_{\mu\lambda} - \partial_{\lambda} g_{\mu\nu}), \tag{3.18}$$

and transform under infinitesimal coordinate transformations (in analogy to the metric) like

$$\Gamma^{\rho}{}_{\mu\nu}(x) \to \Gamma'^{\rho}{}_{\mu\nu}(x) = \Gamma'^{\rho}{}_{\mu\nu}(x'+\epsilon) = \underbrace{\Gamma'^{\rho}{}_{\mu\nu}(x')}_{=\frac{\partial x'^{\rho}}{\partial x^{\gamma}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\Gamma^{\gamma}{}_{\alpha\beta}+\frac{\partial x'^{\rho}}{\partial x^{\gamma}}\frac{\partial^2 x^{\gamma}}{\partial x'^{\mu}\partial x'^{\nu}}} + \epsilon^{\lambda}\partial_{\lambda}\Gamma^{\rho}_{\mu\nu}(x), \tag{3.19}$$

$$\begin{aligned}\Gamma'^{\rho}{}_{\mu\nu}(x') &= (\delta^{\rho}{}_{\gamma} - \partial_{\gamma}\epsilon^{\rho})(\delta^{\alpha}{}_{\mu} + \partial_{\mu}\epsilon^{\alpha})(\delta^{\beta}{}_{\nu} + \partial_{\nu}\epsilon^{\beta})\Gamma^{\gamma}{}_{\alpha\beta} \\ &\quad + \underbrace{(\delta^{\rho}{}_{\gamma} - \partial_{\gamma}\epsilon^{\rho})\partial_{\mu}(\delta^{\gamma}{}_{\nu} + \partial_{\nu}\epsilon^{\gamma})}_{=\partial_{\mu}\partial_{\nu}\epsilon^{\rho}+\mathcal{O}(\epsilon^2)} \qquad (3.20) \\ &= \Gamma^{\rho}{}_{\mu\nu} + \partial_{\nu}\epsilon^{\beta}\Gamma^{\rho}{}_{\mu\beta} + \partial_{\mu}\epsilon^{\alpha}\Gamma^{\rho}{}_{\alpha\nu} - \partial_{\gamma}\epsilon^{\rho}\Gamma^{\gamma}{}_{\mu\nu} + \partial_{\mu}\partial_{\nu}\epsilon^{\rho} + \mathcal{O}(\epsilon^2). \qquad (3.21)\end{aligned}$$

$$\begin{aligned}\Rightarrow \Gamma'^{\rho}{}_{\mu\nu}(x) &= \Gamma^{\rho}{}_{\mu\nu}(x) + \epsilon^{\lambda}\partial_{\lambda}\Gamma^{\rho}{}_{\mu\nu}(x) + \partial_{\nu}\epsilon^{\lambda}\Gamma^{\rho}{}_{\mu\lambda} + \partial_{\mu}\epsilon^{\lambda}\Gamma^{\rho}{}_{\lambda\nu} - \partial_{\lambda}\epsilon^{\rho}\Gamma^{\lambda}{}_{\mu\nu} \\ &\quad + \partial_{\mu}\partial_{\nu}\epsilon^{\rho} + \mathcal{O}(\epsilon^2) \qquad (3.22)\end{aligned}$$

The last term of equation (3.22) vanishes for $\overline{\Gamma}^{\mu}{}_{\mu\nu}$ and in case of restricted coordinate transformations ($\partial_{\mu}\epsilon^{\mu} = 0$). But in this case the whole tensor $\overline{\Gamma}^{\mu}{}_{\mu\nu}$ vanishes because

$$\overline{\Gamma}^{\mu}{}_{\mu\nu} = \frac{1}{2}\overline{g}^{\mu\rho}(\partial_{\mu}\overline{g}_{\rho\nu} + \partial_{\nu}\overline{g}_{\mu\rho} - \partial_{\rho}\overline{g}_{\mu\nu}) = \frac{1}{2}\overline{g}^{\mu\rho}\partial_{\nu}\overline{g}_{\mu\rho} = \partial_{\nu}\ln\underbrace{\sqrt{-\overline{g}}}_{=1} = 0. \tag{3.23}$$

To construct the Riemann curvature tensor one needs the antisymmetric combination of the partial derivatives of the Christoffel symbols. The last term of equation (3.22) cancels then $\partial_{\mu}\overline{\Gamma}^{\rho}{}_{\nu\lambda} - \partial_{\nu}\overline{\Gamma}^{\rho}{}_{\mu\lambda}$. Thus it transforms as a tensor and can be completed to the Riemann tensor. It follows that all the important quantities for general relativity can be derived from $\overline{g}_{\mu\nu}$.

The most general Lagrangian is

$$\mathcal{L} = \frac{1}{2}\chi^2(\Phi)\overline{R}(\overline{g}) + \overline{\mathcal{L}}(\overline{g}, \Phi). \tag{3.24}$$

After rescaling the metric using a conformal transformation (see appendix A) $g_{\mu\nu} = \chi^2\overline{g}_{\mu\nu}$ one obtains

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{2}R(g) + 6g^{\mu\nu}\chi^{-2}\partial_{\mu}\chi\partial_{\nu}\chi + \mathcal{L}(g, \Phi)\right), \tag{3.25}$$

with $\overline{\mathcal{L}}(\overline{g}, \Phi) = \chi^4\mathcal{L}(g, \Phi)$ and the constraint $\sqrt{-g} = \chi^4(\Phi)\underbrace{\sqrt{-\overline{g}}}_{=1} = \chi^4$. The Lagrangian contains the standard Einstein-Hilbert term, a matter term and an additional scalar field. The constraint can be implemented via a Lagrange multiplier $\Lambda(x)$. The final Lagrangian reads

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{2}R(g) + \mathcal{L}_{\mathrm{M}}(g,\Phi) - \chi^{-4}(\Phi)\Lambda\right) + \Lambda, \tag{3.26}$$

where the new field is absorbed into $\mathcal{L}_{\mathrm{M}}(g,\Phi)$. Varying with respect to Λ gives the constraint $\sqrt{-g} = \chi^4$ again and by varying with respect to $g^{\mu\nu}$ one obtains the Einstein field equations.

3.2 The cosmological constant as an initial condition

Define the energy-momentum tensor as

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_{\mathrm{M}}}{\delta g^{\mu\nu}}, \tag{3.27}$$

with the matter action

$$S_{\mathrm{M}} = \int d^4x \left(\sqrt{-g}\left(\mathcal{L}_{\mathrm{M}}(g,\Phi) - \chi^{-4}(\Phi)\Lambda\right) + \Lambda\right). \tag{3.28}$$

Hence, general covariance is not ensured due to the last term in the action. The energy-momentum tensor is not automatically covariantly conserved, but just due to the Einstein field equations and Bianchi identities (2.21). The matter action transforms under infinitesimal transformations as a scalar field due to the last term, which breaks the covariance. This implies

$$\delta S_{\mathrm{M}} = \int d^4x \epsilon^\mu \partial_\mu \Lambda \tag{3.29}$$

$$= \int d^4x \left(\frac{\delta S_{\mathrm{M}}}{\delta\Phi}\delta\Phi + \frac{\delta S_{\mathrm{M}}}{\delta\Lambda}\delta\Lambda - \frac{1}{2}\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}\right). \tag{3.30}$$

The first two terms vanish because Φ and Λ satisfy equations of motion.
With the following relation

$$\delta g^{\mu\nu} = -\nabla^\mu\epsilon^\nu - \nabla^\nu\epsilon^\mu \tag{3.31}$$

and integration by parts one obtains

$$\int d^4x\ \epsilon^\mu(\partial_\mu\Lambda + \sqrt{-g}\nabla^\nu T_{\nu\mu}) = 0. \tag{3.32}$$

For an arbitrary ϵ^μ the bracket has to vanish

$$\sqrt{-g}\nabla^\nu T_{\nu\mu} = -\partial_\mu\Lambda. \tag{3.33}$$

Because $T_{\nu\mu}$ is covariantly conserved ($\nabla^{\nu} T_{\nu\mu} = 0$) this means that

$$\Lambda = \text{constant}. \tag{3.34}$$

In the case $\chi^2 = 1$ the introduced Lagrange multiplier Λ gives a covariantly constant contribution to the Lagrangian, which can be interpreted as a cosmological constant. The value of Λ depends on the initial conditions and is not a parameter in the Lagrangian. This is a consequence of considering a theory with finite spacetime induced by invariance under restricted coordinate transformations.

4 Unimodular gravity

The aim of this chapter is to derive the equations of motion for unimodular gravity using the variational calculus. This will be done step by step starting with the variation of the Einstein-Hilbert action. For simplicity we choose Planck units in this section ($M_{Pl} = 1$).

4.1 Einstein-Hilbert action

The Einstein-Hilbert action reads as follows

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} R \right). \tag{4.1}$$

Variation with respect to the metric tensor $g^{\mu\nu}$ yields the following equation

$$\delta S = \int d^4x \left(\delta\sqrt{-g} \left(\frac{1}{2} R \right) + \sqrt{-g} \left(\frac{1}{2} \left(\delta R_{\mu\nu} g^{\mu\nu} + R_{\mu\nu} \delta g^{\mu\nu} \right) \right) \right) = 0. \tag{4.2}$$

The third term has already the desired form, but the first and the second term have to be analyzed further. The variation of $\sqrt{-g}$ is given in equation (3.12) and with (3.11) follows

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \tag{4.3}$$

Before considering the variation of the Ricci tensor it is helpful to find out how the Riemann tensor transforms under infinitesimal transformations of the Christoffel symbol $\Gamma \rightarrow \Gamma + \delta\Gamma$,

$$\begin{aligned} \delta R^{\rho}{}_{\mu\lambda\nu} = {} & \partial_\nu(\delta\Gamma^{\rho}{}_{\mu\lambda}) - \partial_\lambda(\delta\Gamma^{\rho}{}_{\mu\nu}) + \delta\Gamma^{\rho}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\lambda\mu} + \Gamma^{\rho}{}_{\nu\sigma}\delta\Gamma^{\sigma}{}_{\lambda\mu} \\ & - \delta\Gamma^{\rho}{}_{\lambda\sigma}\Gamma^{\sigma}{}_{\nu\mu} - \Gamma^{\rho}{}_{\lambda\sigma}\delta\Gamma^{\sigma}{}_{\nu\mu}. \end{aligned} \tag{4.4}$$

With the covariant derivative of $\delta\Gamma$ one can show that

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\nu \left(\delta\Gamma^{\rho}{}_{\lambda\mu} - \nabla_\lambda \left(\delta\Gamma^{\rho}{}_{\nu\mu} \right) \right), \tag{4.5}$$

$$\delta R_{\mu\nu} = \nabla_\nu \left(\delta\Gamma^{\lambda}{}_{\lambda\mu} - \nabla_\lambda \left(\delta\Gamma^{\lambda}{}_{\nu\mu} \right) \right). \tag{4.6}$$

Now, the second term in (4.2) the variation of the action can be reconsidered

$$\int d^4x\sqrt{-g}\delta R_{\mu\nu}g^{\mu\nu} = \int d^4x\sqrt{-g}\left(\nabla_\nu\left(g^{\mu\nu}\delta\Gamma^\lambda{}_{\lambda\mu}\right) - \nabla_\lambda\left(g^{\mu\nu}\delta\Gamma^\lambda{}_{\nu\mu}\right)\right) \tag{4.7}$$

$$= \int d^4x\sqrt{-g}\nabla_\sigma\left(g^{\mu\sigma}\delta\Gamma^\lambda{}_{\lambda\mu} - g^{\mu\nu}\delta\Gamma^\sigma_{\nu\mu}\right). \tag{4.8}$$

This is a surface term and vanishes, if the variation of $\delta\Gamma$ and δg vanishes at infinity, due to Gauss' theorem.
This means we are left with

$$\delta S = \int d^4x\sqrt{-g}\left(\frac{1}{2}\left(-\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu}\right)\right)\delta g^{\mu\nu} = 0. \tag{4.9}$$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{4.10}$$

These are the Einstein field equations in vacuum.

4.2 Einstein-Hilbert action including matter

The next step is to include matter. The corresponding action reads

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}R + \mathcal{L}_\mathrm{M}\right)\ , \text{ with } \mathcal{L}_\mathrm{M}(\varphi, g). \tag{4.11}$$

The variation of the $\sqrt{-g}$ and R has been calculated before. The additional term in this case is

$$\delta\left(\sqrt{-g}\mathcal{L}_\mathrm{M}\right) = -\frac{1}{2}\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}, \tag{4.12}$$

with $T_{\mu\nu}$ the energy-momentum tensor, which is defined as (3.27)

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_\mathrm{M}\right)}{\delta g^{\mu\nu}}. \tag{4.13}$$

$$\Rightarrow \delta S = \int d^4x\sqrt{-g}\left(\frac{1}{2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - T_{\mu\nu}\right)\right)\delta g^{\mu\nu} = 0 \tag{4.14}$$

One obtains the Einstein field equations with matter sources.

$$\Rightarrow R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu} \tag{4.15}$$

4.3 Variation with fixed determinant

In the case of Unimodular Gravity [17] the variation changes because the determinant of the metric is not a dynamical variable anymore. The restricted coordinate invariance in this theory implies that the determinant of the metric is fixed and its variation vanishes

$$\delta g = -g \underbrace{(g_{\mu\nu}\delta g^{\mu\nu})}_{=0} = g \underbrace{(g^{\mu\nu}\delta g_{\mu\nu})}_{=0} = 0. \tag{4.16}$$

This means that just the traceless part of the variation with respect to the metric tensor has to vanish.

4.3.1 Generally covariant formulation

For the same action as before (4.11) this yields

$$\delta S = \int d^4x\sqrt{-g}\left(\frac{1}{2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - T_{\mu\nu}\right)\right)\delta g^{\mu\nu}. \tag{4.17}$$

Now, one can split the terms in the bracket into traceless and tracefull parts ($T = T^{\mu}_{\ \mu}$)

$$\delta S = \int d^4x\sqrt{-g}\left(\frac{1}{2}\Bigg(\underbrace{\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right)}_{\text{traceless}} - \frac{1}{4}g_{\mu\nu}R - \underbrace{\left(T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T\right)}_{\text{traceless}} - \frac{1}{4}g_{\mu\nu}T\Bigg)\right)\delta g^{\mu\nu} = 0. \tag{4.18}$$

$$\Rightarrow \left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) = \left(T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T\right) \tag{4.19}$$

These are the traceless Einstein field equations.

4.3.2 Lagrange multiplier formulation

If one starts with the Lagrangian of section 3.1 one gets the same equations of motion as in 4.3.1. In this case general covariance is broken, due to the last term of the action, which originates from introducing a Lagrange multiplier. The action is (3.26)

$$S = \int d^4x\left(\sqrt{-g}\left(\frac{1}{2}R + \mathcal{L}_{\mathrm{M}} - \left(\chi^{-4}\Lambda\right)\right)\right) + \Lambda\ . \tag{4.20}$$

The variation yields

$$\delta S = \int d^4x\sqrt{-g}\left(\frac{1}{2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - T_{\mu\nu} + g_{\mu\nu}\left(\chi^{-4}\Lambda\right)\right)\right)\delta g^{\mu\nu} = 0. \tag{4.21}$$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - T_{\mu\nu} + g_{\mu\nu}\left(\chi^{-4}\Lambda\right) = 0 \tag{4.22}$$

If one contracts this equation with the inverse metric $g^{\mu\nu}$ implies

$$\left(\chi^{-4}\Lambda\right) = \frac{1}{4}R + \frac{1}{4}T. \tag{4.23}$$

Plugging this back into (4.22) gives the same result as the first approach.

$$\Rightarrow \left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) = \left(T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T\right) \tag{4.24}$$

4.3.3 The cosmological constant as a constant of integration

For unimodular gravity a cosmological constant-like term is automatically included [3]. This can be seen, if one uses the conservation law of the energy-momentum tensor

$$\nabla^{\mu}T_{\mu\nu} = 0 \tag{4.25}$$

and the Bianchi identity (2.21)

$$\nabla^{\mu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 0 \tag{4.26}$$

$$\Longleftrightarrow \nabla^{\mu}\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) = \nabla^{\mu}\left(\frac{1}{4}g_{\mu\nu}R\right). \tag{4.27}$$

If one takes the covariant derivative of the traceless Einstein field equations (4.24) and

$$\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) = \left(T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T\right), \tag{4.28}$$

uses (4.25) and (4.27), one obtains

$$\nabla^{\mu}\left(\frac{1}{4}g_{\mu\nu}R\right) = -\nabla^{\mu}\left(\frac{1}{4}g_{\mu\nu}T\right) \tag{4.29}$$

$$\Longleftrightarrow \frac{1}{4}g_{\mu\nu}\partial^{\mu}\left(R+T\right) = 0. \tag{4.30}$$

$$\Rightarrow R + T = \text{constant} \equiv 4\Lambda \tag{4.31}$$

Plugging this back into equation (4.28) yields

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = T_{\mu\nu}. \tag{4.32}$$

These are the regular Einstein field equations with a cosmological constant-term included. In this case the cosmological constant arises as a constant of integration and has nothing to do with terms in the action or the vacuum energy. The value is not determined by this setup. This term arises from the fact that the determinant of the metric is not a dynamical variable anymore and therefore spacetime is finite. The integral over the spacetime measure, the worldvolume $\int d^4x\sqrt{-g}$, remains finite in this setup.

5 The KP model

Now, the KP model proposed by N. Kaloper and A. Padilla is presented [12, 13]. This proposal contains a mechanism, which is able to predict a small but non-zero cosmological constant for our universe. Furthermore we will analyze this model in terms of the expansion of the universe.
The action of the model reads

$$S = \int d^4x\sqrt{-g}\left(\frac{M_{\rm Pl}^2}{2}R - \Lambda + \lambda^4\mathcal{L}\left(\lambda^{-2}g^{\mu\nu},\Phi\right)\right) + \sigma\left(\frac{\Lambda}{\lambda^4\mu^4}\right). \tag{5.1}$$

Matter couples minimally to the conformally rescaled metric $\tilde{g}_{\mu\nu} = \lambda^2 g_{\mu\nu}$. Therefore the parameter λ sets the mass scale $m_{\rm phys} = \lambda m$. This follows from considering canonical kinetic terms (m = bare mass in the Lagrangian)

$$\sqrt{-\tilde{g}}\mathcal{L} = \frac{1}{2}\sqrt{-\tilde{g}}\left(\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + m^2\varphi^2\right) \tag{5.2}$$

$$= \frac{1}{2}\lambda^4\sqrt{-g}\left(\lambda^{-2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + m^2\varphi^2\right) \tag{5.3}$$

$$= \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\varphi'\partial_\nu\varphi' + m_{\rm phys}^2\varphi'^2\right), \tag{5.4}$$

with $\varphi = \lambda^{-1}\varphi'$ and $m_{\rm phys} \equiv \lambda m$. Note that the function σ appears outside of the integral. The parameter μ is chosen phenomenologically and has mass dimension one. The cosmological constant-like term Λ has mass dimension four. This is different to ordinary GR, where Λ comes with a factor $M_{\rm Pl}^2$ in the Einstein-Hilbert action and therefore has mass dimension two. The whole model is semiclassical. Thus, graviton loops are not considered.

5.1 Equations of motion

In addition to $g_{\mu\nu}$ the parameters Λ and λ are treated as dynamical variables. First, consider the variation with respect to Λ

$$\delta S = \left(-\int d^4x\sqrt{-g} + \sigma'\frac{\delta z}{\delta\Lambda}\right)\delta\Lambda = 0, \tag{5.5}$$

where $\sigma' = \frac{\delta\sigma(z)}{\delta z}$, $z = \frac{\Lambda}{\lambda^4\mu^4}$ and therefore $\frac{\delta z}{\delta\Lambda} = \frac{1}{\lambda^4\mu^4}$.

$$\Rightarrow \frac{\sigma'}{\lambda^4\mu^4} = \int d^4x\sqrt{-g} \tag{5.6}$$

Now, the variation with respect to λ

$$\delta S = \left(\int d^4x\ \frac{\delta\left(\sqrt{-\tilde{g}}\mathcal{L}(\tilde{g}^{\mu\nu},\Phi)\right)}{\delta\lambda} + \sigma'\frac{\delta z}{\delta\lambda}\right)\delta\lambda = 0. \tag{5.7}$$

$$\Rightarrow \frac{\delta\left(\sqrt{-\tilde{g}}\mathcal{L}(\tilde{g}^{\mu\nu},\Phi)\right)}{\delta\tilde{g}^{\mu\nu}}\frac{\delta\tilde{g}^{\mu\nu}}{\delta\lambda} = -\sigma'\frac{\delta z}{\delta\lambda}, \tag{5.8}$$

with $\frac{\delta z}{\delta\lambda} = -\frac{4\Lambda}{\lambda^5\mu^4}$, the energy momentum tensor $\tilde{T}_{\mu\nu} = -\frac{2}{\sqrt{-\tilde{g}}}\frac{\delta S_m}{\delta\tilde{g}^{\mu\nu}}$ and

$$\frac{\delta\tilde{g}^{\mu\nu}}{\delta\lambda} = \frac{\delta\left(\lambda^{-2}g^{\mu\nu}\right)}{\delta\lambda} = -2\lambda^{-3}g^{\mu\nu} = -2\lambda^{-1}\tilde{g}^{\mu\nu}. \tag{5.9}$$

$$\Rightarrow \int d^4x\lambda^4\sqrt{-g}\tilde{T}^{\mu}_{\ \mu} = 4\Lambda\frac{\sigma'}{\lambda^4\mu^4} \tag{5.10}$$

$$\Longleftrightarrow \int d^4x\sqrt{-g}T^{\mu}_{\ \mu} = 4\Lambda\frac{\sigma'}{\lambda^4\mu^4} \tag{5.11}$$

where $\sqrt{-g}\lambda^4\mathcal{L}\left(\lambda^{-2}g^{\mu\nu},\Phi\right) = \sqrt{-\tilde{g}}\mathcal{L}\left(\tilde{g}^{\mu\nu},\Phi\right)$ i.e. $\sqrt{-\tilde{g}} = \lambda^4\sqrt{-g}$ and $T^{\mu}_{\ \mu} = \lambda^4\tilde{T}^{\mu}_{\ \mu}$.
Equation (5.6) and (5.11) can be combined to

$$\Lambda = \frac{1}{4}\langle T^{\alpha}_{\ \alpha}\rangle, \tag{5.12}$$

with the expectation value $\langle Q\rangle = \frac{\int d^4x\sqrt{-g}Q}{\int d^4x\sqrt{-g}}$ is called historic average. Note that this term is non-local. Together with the equations of motion, which are derived by varying with respect to $g^{\mu\nu}$

$$M_{\mathrm{Pl}}^2 G_{\mu\nu} = \lambda^4\tilde{T}_{\mu\nu} - \Lambda g_{\mu\nu}, \tag{5.13}$$

one gets the modified Einstein field equations

$$M_{\mathrm{Pl}}^2 G_{\mu\nu} = T_{\mu\nu} - \frac{1}{4}\langle T^{\alpha}_{\ \alpha}\rangle g_{\mu\nu}. \tag{5.14}$$

This is unlike unimodular gravity where one loses one degree of freedom due to the subtraction of the trace.
In this model constant contributions of the energy-momentum tensor do not contribute to the field equations, which is shown in the following.
If $\mathcal{L} = \Lambda_0 + \rho_{\mathrm{vac}} + \Delta\mathcal{L}$, where Λ is a bare cosmological constant and ρ_{vac} the vacuum

energy, the historic average gives $\langle \Lambda + \rho_{\text{vac}} \rangle = \Lambda + \rho_{\text{vac}}$.
With the definition

$$\tau_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^4x \sqrt{-g} \lambda^4 \Delta \mathcal{L}(\lambda^{-2} g^{\mu\nu}, \Phi), \tag{5.15}$$

one can write $T_{\mu\nu} = -(\Lambda + \rho_{\text{vac}}) g_{\mu\nu} + \tau_{\mu\nu}$ and thus

$$T_{\mu\nu} - \frac{1}{4} \langle T^{\alpha}_{\ \alpha} \rangle g_{\mu\nu} = \tau_{\mu\nu} - \frac{1}{4} \langle \tau^{\alpha}_{\ \alpha} \rangle \ g_{\mu\nu}. \tag{5.16}$$

Then the Einstein field equations then take the following form

$$M_{\text{Pl}}^2 G_{\mu\nu} = \tau_{\mu\nu} - \frac{1}{4} \langle \tau^{\alpha}_{\ \alpha} \rangle g_{\mu\nu}. \tag{5.17}$$

The term on the right-hand side, which includes the historic average of the trace of the local energy-momentum tensor, can be interpreted as a cosmological constant.
There are some models, which show some similarities to the KP model. The first one is motivated from string theory and in the low-energy limit the usual effective gravitational action is divided by the worldvolume [18]. This creates a term in the Einstein field equations of the model, which is similar to the historic average of the KP model. The action in Normalized General Relativity (NGR) is constructed by dividing the standard GR action by the worldvolume [19, 20]. It exhibits a shift symmetry. In contrast to unimodular gravity, where the cosmological constant is an arbitrary integration constant, it is possible to fix its value in the framework of NGR.

5.2 Symmetries

The action of the KP model has two approximate symmetries which ensure the cancellations of the vacuum energy contributions. The first one is a scaling symmetry with

$$\lambda \to \Omega\lambda, \ g_{\mu\nu} \to \Omega^{-2} g_{\mu\nu}, \quad \Lambda \to \Omega^4 \Lambda. \tag{5.18}$$

This transformation leaves the matter Lagrangian invariant, but is broken by the gravitational sector. The action (5.1) changes by

$$\delta S = \frac{M_{\text{Pl}}^2}{2} \Omega^{-2} \int d^4 \sqrt{-g} R = \frac{M_{\text{Pl}}^2}{2} \langle R \rangle \int d^4x \sqrt{-g}. \tag{5.19}$$

The second symmetry is a shift symmetry with

$$\Lambda \to \Lambda - \epsilon \lambda^4 m^4, \ \mathcal{L} \to \mathcal{L} + \epsilon m^4, \tag{5.20}$$

such that the action (5.1) changes by

$$\delta S = \sigma \left(\frac{\Lambda}{\lambda^4 \mu^4} - \epsilon \frac{m^4}{\mu^4} \right) - \sigma \left(\frac{\Lambda}{\lambda^4 \mu^4} \right) \simeq -\epsilon \sigma' \frac{m^4}{\mu^4}. \tag{5.21}$$

These two symmetries provide naturally a small cosmological constant-like term, because they protect from large additions. This mechanism will be discussed in 5.4 in full detail.

5.3 Approximation of the historic average

The historic average, which is defined as $\langle \tau^{\alpha}_{\ \alpha} \rangle = \frac{\int d^4x \sqrt{-g} \tau^{\alpha}_{\ \alpha}}{\int d^4x \sqrt{-g}}$, has to be analyzed to find the numerical value for the cosmological constant for our universe. In particular it has to be finite. This implies that the universe needs to be spatially compact, starting in a Bang and ending in a Crunch (i.e. has a finite lifetime). These two integrals have to be evaluated

$$\int d^4x \sqrt{-g} \sim \mathrm{Vol}_3 \int_{t_{\mathrm{bang}}}^{t_{\mathrm{crunch}}} dt a^3, \tag{5.22}$$

$$\int d^4x \sqrt{-g} \tau^{\alpha}_{\ \alpha} \sim \mathrm{Vol}_3 \int_{t_{\mathrm{bang}}}^{t_{\mathrm{crunch}}} dt(-\rho + 3p), \tag{5.23}$$

with the assumption that the universe is homogeneous and with the comoving spatial volume Vol_3. For sources with standard energy conditions $|p/\rho| \leq 1$ (see appendix B) these integrals are regulated by the spacetime singularities. This means we integrate over a finite time interval. In particular demanding that the universe starts with a Bang and ends in a Crunch gives a non-zero mass gap. If $\int d^4x \sqrt{-g}$ is finite (5.6) implies that λ is non-zero. Thus $m_{\mathrm{phys}} = \lambda m$ is non-zero, too. The largest contribution of those integrals will come from the phase close to the turning point, where the universe has its maximal size. During this phase near the turning point the universe is approximately static with a constant scale factor $a = a_{\mathrm{max}}$ over a time interval Δt. The turnaround happens at the time, when the universe has its maximal size and. All timescales are considered as approximately of the same order of magnitude $T \simeq \Delta t \simeq H_{\mathrm{age}}^{-1}$. The curvature of the universe at that time is $|R| \sim \frac{1}{a_{\mathrm{max}}^2} \sim H_{\mathrm{age}}^2$. Using these approximations one finds

$$\int_{T-\Delta t/2}^{T+\Delta t/2} dt a^3 \simeq a_{\mathrm{max}}^3 \Delta t \sim \frac{a_{\mathrm{max}}^3}{H_{\mathrm{age}}} \sim \frac{1}{H_{\mathrm{age}}^4}, \tag{5.24}$$

$$\int_{T-\Delta t/2}^{T+\Delta t/2} dt(-\rho + 3p) \simeq \mathcal{O}(1) a_{\mathrm{max}}^3 \rho_{\mathrm{age}} \Delta t \sim \frac{\rho_{\mathrm{age}}}{H_{\mathrm{age}}^4}, \tag{5.25}$$

with the characteristic energy density of the universe $\rho_{\rm age} \sim M_{\rm Pl}^2 H_{\rm age}^2$ (by virtue of the Friedmann equation). Thus one gets for (5.22) and (5.23)

$$\int d^4x\sqrt{-g} \sim \mathcal{O}(1)\frac{\mathrm{Vol}_3}{H_{\rm age}^4}, \tag{5.26}$$

$$\int d^4x\sqrt{-g}\tau^{\alpha}_{\ \alpha} \sim \mathcal{O}(1)\frac{\mathrm{Vol}_3\rho_{\rm age}}{H_{\rm age}^4}. \tag{5.27}$$

Consequently one has for the cosmological constant in this model

$$\Lambda = \frac{1}{4}\langle\tau^{\alpha}_{\ \alpha}\rangle \simeq \mathcal{O}(1)\rho_{\rm age} \simeq \mathcal{O}(1)M_{\rm Pl}^2 H_{\rm age}^2 < M_{\rm Pl}^2 H_0^2, \tag{5.28}$$

where H_0 is the Hubble parameter today. This gives a small, but non-zero cosmological constant for our universe today. Further it implies naturally that the value never exceeds the critical density of the universe today, because the universe will have to grow at least older than it is now ($H_0^{-1} \simeq 10^{10}$ years) in the Bang/Crunch scenario. The sign is controlled by the dominant contribution to the cosmological fluid close to the turning point. For $w > \frac{1}{3}$ the cosmological constant term is positive and for $w < \frac{1}{3}$ it is negative. Since the universe must collapse in the future and the universe has to be spatially closed, our currently observed accelerated expansion must be transient.

5.4 The cosmological constant problem in the KP model

The cosmological constant problem is addressed by use of the two approximate symmetries described in chapter 5.2. They appear because of the modifications of the action in the gravitational sector. In particular, introducing the two global variables and the corresponding constraints is the key to the creation of these symmetries. The scaling symmetry is broken by the Einstein-Hilbert term and is therefore approximate just like the shift symmetry. As stated in the introduction, if one interprets the cosmological constant as a UV-divergent quantity the radiative instability is the main issue. Actually, the UV-divergence causes the non-locality of the historic average [13]. In supersymmetic theories or theories featuring conformal symmetry the corresponding symmetry protects the cosmological constant from higher order loop corrections. In the KP model the shift symmetry is used to cancel the matter vacuum energy and its quantum corrections. Additionally, the scaling symmetry ensures that the vacuum energy in every loop order couples to gravity the same way as the classical contribution. This is necessary in order to make sure that the cancellation by the shift symmetry works at every loop order. Therefore the cosmological constant is radiatively stable. The reason that one has a residual small cosmological constant in the KP model is that the shift symmetry is approximate (5.21),

with (5.6) one gets $\delta S \simeq -\epsilon m^4 \lambda^4 \int d^4x\sqrt{-g}$.
In order to understand better how the cancellation works, consider the mechanism on the level of the field equations. First note that the action (5.1) is in fact invariant under the scaling symmetry (5.18). Since the historic average of the right-hand side of (5.14) is zero, $\langle R \rangle$ vanishes. The change of the matter Lagrangian under the shift symmetry (5.20) can be expressed in terms of a variation of the energy-momentum tensor $\tilde{T}_{\mu\nu} \to \tilde{T}_{\mu\nu} - \epsilon m^4 g_{\mu\nu}$. The constraints (5.6) and (5.11) have to be modified in order to match with the shifted energy-momentum tensor

$$\hat{g}_{\mu\nu} = g_{\mu\nu}, \;\; \hat{\Lambda} = \Lambda \frac{\hat{z}\sigma'(\hat{z})}{z\sigma'(z)}, \;\; \hat{\lambda}^4 = \lambda^4 \frac{\sigma'(\hat{z})}{\sigma'(z)}. \tag{5.29}$$

Most important is the fact that the metric stays unchanged. Hence a shift of the vacuum energy, like taking higher loop orders into account, does not change the geometry. Those shifts will be absorbed by the auxiliary fields λ and Λ, which can be integrated out. This means that the worldvolume $\int d^4x\sqrt{-g}$ stays fixed and Λ is forced to adjust in this model. The setup is not spoiled by phase transitions in the early universe and is compatible with inflation models, in particular Starobinski-like and monomial inflation models [12, 13].

5.5 Dynamics of the KP model

After introducing the model and its features in the first part of this chapter, we now analyze the dynamics of the model. For this purpose we will derive the Friedmann equations in the KP model and analyze the result.

5.5.1 Friedmann equations

Starting from the action (5.1) one obtains (5.17) by varying with respect to the metric tensor and using the constraints (see section 5.1)

$$M_{\mathrm{Pl}}^2 G_{\mu\nu} = \tau_{\mu\nu} - \frac{1}{4}\langle \tau^{\alpha}{}_{\alpha} \rangle g_{\mu\nu}. \tag{5.30}$$

With the assumption of a FRW-metric (2.23), which reads

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \right), \tag{5.31}$$

we want to get the Friedmann equations in the KP model. We will use the result for the Einstein tensor in FRW cosmology (2.28). The procedure is similar to the derivation presented in chapter 2.2, but we get additional terms including the historic average. First we consider the (0,0)-component of the Einstein field equations. This yields the relation

$$M_{\rm Pl}^2 = \left(\frac{3(\kappa + \dot{a}^2)}{a^2}\right) = \tau_{00} + \frac{1}{4}\langle \tau^\alpha_{\ \alpha} \rangle. \tag{5.32}$$

Assume the energy-momentum tensor for a perfect fluid for (5.15)

$$\tau_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu}, \tag{5.33}$$

where u_μ is the time-like 4-velocity $u_\mu = (1,0,0,0)$, $\rho = \rho_\Lambda + \rho_{M,0}\left(\frac{a_0}{a}\right)^3 + \rho_{R,0}\left(\frac{a_0}{a}\right)^4$ and p the corresponding pressures (see section 2.2). The trace of this energy-momentum tensor is $\tau^\alpha_{\ \alpha} = -\rho + 3p$ and $\tau_{00} = \rho$. We find

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_{\rm Pl}^2}\left(\rho + \frac{1}{4}\langle -\rho + 3p \rangle\right) - \frac{\kappa}{a^2}. \tag{5.34}$$

In order to derive the second Friedmann equation in this case, we need to consider the (i,j)-component of the corresponding Einstein field equations

$$M_{\rm Pl}^2 \left(-\frac{(\kappa + \dot{a}^2 + 2a\ddot{a})}{a^2}\right) g_{ij} = p g_{ij} - \frac{1}{4} g_{ij} \langle \tau^\alpha_{\ \alpha} \rangle \tag{5.35}$$

$$\Longleftrightarrow \ -\left(\frac{\dot{a}}{a}\right)^2 - 2\frac{\ddot{a}}{a} = \frac{1}{M_{\rm Pl}^2}\left(p - \frac{1}{4}\langle \tau^\alpha_{\ \alpha} \rangle\right) + \frac{\kappa}{a^2}. \tag{5.36}$$

Adding the first Friedmann equation (5.34) yields the second Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_{\rm Pl}^2}\left(\rho + 3p\right) + \frac{1}{3M_{\rm Pl}^2}\left(\frac{1}{4}\langle -\rho + 3p \rangle\right). \tag{5.37}$$

5.5.2 Consistency of the Friedmann equations

In the case of Einstein gravity one can derive the second Friedmann equation from the first Friedmann equation using conservation of energy-momentum. Covariant conservation of energy-momentum follows from the Bianchi identity applied to the Einstein field equation. Since the Einstein equations are modified in the KP case, the consistency of the Friedmann equation has to be checked. The Einstein equation (5.17) with both indices upstairs reads

$$M_{\rm Pl}^2 G^{\mu\nu} = \tau^{\mu\nu} - \frac{1}{4}\langle \tau^\alpha_{\ \alpha} \rangle g^{\mu\nu}. \tag{5.38}$$

The Bianchi identity $\nabla_\nu G^{\mu\nu} = 0$ (2.21) implies

$$\nabla_\nu \tau^{\mu\nu} = \nabla_\nu \left(\frac{1}{4}\langle \tau^\alpha_{\ \alpha} \rangle g^{\mu\nu}\right) = \frac{1}{4}\partial^\mu \langle \tau^\alpha_{\ \alpha} \rangle. \tag{5.39}$$

Evaluating this modified conservation of energy-momentum gives [15]

$$\nabla_\nu \tau^{\mu\nu} = \frac{1}{4}\partial^\mu \langle \tau^\alpha_{\ \alpha}\rangle \tag{5.40}$$

$$\iff \frac{\partial p}{\partial x^\nu} g^{\mu\nu} + \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\left(\sqrt{-g}\,(\rho+p)\,u^\mu u^\nu\right) + \Gamma^\mu_{\ \nu\lambda}\,(p+\rho)\,u^\nu u^\lambda = \frac{1}{4}\partial^\mu\langle -\rho+3p\rangle. \tag{5.41}$$

Using $\Gamma^\mu_{\ 00} = 0$ and $u^\mu = (1,0,0,0)$ one finds for $\mu = 0$ that

$$\iff -\frac{dp}{dt}a^3 + \frac{d}{dt}\left(a^3(\rho+p)\right) = -\frac{1}{4}\frac{d}{dt}\langle -\rho+3p\rangle a^3 \tag{5.42}$$

$$\iff \frac{d}{dt}\left(a^3\rho\right) + 3a^2\frac{d}{dt}p = -\frac{1}{4}\frac{d}{dt}\langle -\rho+3p\rangle a^3 \tag{5.43}$$

$$\iff \frac{d}{da}\left(a^3\rho\right) + 3a^2 p = -\frac{1}{4}\frac{d}{da}\langle -\rho+3p\rangle a^3. \tag{5.44}$$

This implies that the continuity equation (2.37) changes in the following way

$$\dot\rho + 3\frac{\dot a}{a}(\rho+p) + \frac{1}{4}\frac{d}{dt}\langle -\rho+3p\rangle = 0. \tag{5.45}$$

For $\mu = i$

$$\partial^i p = \frac{1}{4}\partial^i\langle -\rho+3p\rangle. \tag{5.46}$$

Now, consider the first Friedmann equation and differentiate with respect to time after multiplying with a^2

$$2\dot a\ddot a = \frac{1}{3M_{\rm Pl}^2}\left(2a\dot a\left(\rho + \frac{1}{4}\langle -\rho+3p\rangle\right) + \frac{d\rho}{dt}a^2 + \frac{1}{4}\frac{d}{dt}\langle -\rho+3p\rangle a^2\right) \tag{5.47}$$

$$= \frac{1}{3M_{\rm Pl}^2}\left(\frac{\dot a}{a}\left(2a^2\rho + a^3\frac{d\rho}{da}\right) + 2a\dot a\frac{1}{4}\frac{d}{dt}\langle -\rho+3p\rangle + \frac{1}{4}\langle -\rho+3p\rangle a^2\right) \tag{5.48}$$

$$= \frac{1}{3M_{\rm Pl}^2}\left(\frac{\dot a}{a}\left(-\rho a^2 + \underbrace{\frac{d}{da}\left(a^3\rho\right)}_{=-3a^2p-\frac{1}{4}\frac{d}{da}\langle -\rho+3p\rangle a^3}\right) + 2a\dot a\frac{1}{4}\langle -\rho+3p\rangle + \frac{1}{4}\frac{d}{dt}\langle -\rho+3p\rangle a^2\right) \tag{5.49}$$

$$= \frac{1}{3M_{\rm Pl}^2}\left(\dot a a\left(-\rho-3p\right) + 2a\dot a\frac{1}{4}\langle -\rho+3p\rangle \underbrace{- \frac{1}{4}\frac{d}{da}\langle -\rho+3p\rangle \dot a a^2}_{\frac{1}{4}\frac{d}{dt}\langle -\rho+3p\rangle a^2} + \frac{1}{4}\frac{d}{dt}\langle -\rho+3p\rangle a^2\right). \tag{5.50}$$

The additional term, appearing because of the time derivative, cancels exactly with the term originating from the modified covariant conservation of energy-momentum. Thus we end up with the second Friedmann equation in the KP model

$$\iff \quad \frac{\ddot{a}}{a} = -\frac{1}{6M_{\mathrm{Pl}}^2}\left(\rho + 3p\right) + \frac{1}{3M_{\mathrm{Pl}}^2}\left(\frac{1}{4}\langle -\rho + 3p\rangle\right). \tag{5.51}$$

The Friedmann equations are consistent under the assumption of the modified covariant conservation of energy-momentum (5.40).

5.5.3 Problem concerning accelerated expansion

To check if accelerated expansion ($\ddot{a} > 0$) driven by a non-zero cosmological constant is possible, one has to consider the second Friedmann equation (5.37). For a Λ-dominated universe one has $\rho = \rho_\Lambda = \mathrm{const}$ and $p = -\rho_\Lambda$. We find

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_{\mathrm{Pl}}^2}\left(\rho + 3p\right) + \frac{1}{3M_{\mathrm{Pl}}^2}\left(\frac{1}{4}\langle -\rho + 3p\rangle\right) \tag{5.52}$$

$$= -\frac{1}{6M_{\mathrm{Pl}}^2}\left(-2\rho_\Lambda\right) + \frac{1}{3M_{\mathrm{Pl}}^2}\left(\frac{1}{4}\langle -4\rho_\Lambda\rangle\right) \tag{5.53}$$

$$= \frac{1}{3M_{\mathrm{Pl}}^2}\rho_\Lambda - \frac{1}{3M_{\mathrm{Pl}}^2}\langle \rho_\Lambda\rangle, \tag{5.54}$$

which exactly cancels because ρ_Λ is a constant and hence can be pulled out of the historic average. For matter domination ($p_{\mathrm{M}} = 0$) one gets

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_{\mathrm{Pl}}^2}\rho_{\mathrm{M}} + \frac{1}{3M_{\mathrm{Pl}}^2}\left(\frac{1}{4}\langle -\rho_{\mathrm{M}}\rangle\right) \tag{5.55}$$

and for radiation domination ($p_{\mathrm{R}} = \frac{1}{3}\rho_{\mathrm{R}}$)

$$\frac{\ddot{a}}{a} = -\frac{1}{3M_{\mathrm{Pl}}^2}\rho_{\mathrm{R}}. \tag{5.56}$$

This shows that a positive $\ddot{a}$ is not possible and therefore accelerated expansion cannot be generated this way. This violates the observation of the accelerated expansion of the universe and would rule out the model without further modifications. Nevertheless we need to check other possibilities to generate accelerated expansion of the universe.

6 Quintessence

Quintessence, a certain dark energy model, will be introduced in this chapter. By means of it we will build a model in Einstein gravity to understand the dynamics in a closed universe.

6.1 Introducing quintessence

This introduction follows in principle the review [21]. Dark Energy models like quintessence are another possibility to generate accelerated expansion of the universe. These models were developed for the case that the cosmological constant vanishes. Especially with the finding of the previous chapter, where the cosmological constant drops out of the Friedmann equation, it is worth to study these models in more detail. Inspired by the theory of inflation [22, 23], where a scalar field, the inflaton, drives an epoch of accelerated expansion in the early universe, one introduces a scalar field. These arise naturally in particle physics and act as a candidate for dark energy. Using this idea gives the possibility to create a late time inflation-like period. Dark energy means in particular that the equation of state of the cosmological constant

$$w = \frac{p}{\rho} = -1 \tag{6.1}$$

should be satisfied today, but is time dependent in general and thus can change with the time evolution.

Quintessence is a scalar field minimally coupled to gravity. In this introduction $\kappa = 0$. The action reads (in FRW background)

$$S = \int \sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right). \tag{6.2}$$

The scalar field satisfies the following equation of motion

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0, \tag{6.3}$$

which can be obtained by varying the action with respect to φ. The energy-momentum tensor $T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}$ reads

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - \left(\frac{1}{2} \partial^\sigma \varphi \partial_\sigma \varphi + V(\varphi) \right) g_{\mu\nu}. \tag{6.4}$$

This yields the following energy density and pressure in analogy to an ideal fluid

$$\rho = T_{00} = \frac{1}{2} \dot{\varphi}^2 + V(\varphi), \tag{6.5}$$

$$p = T_{ii} = \frac{1}{2} \dot{\varphi}^2 - V(\varphi). \tag{6.6}$$

The Friedmann equations in this case are given by

$$H^2 = \left(\frac{\dot{a}}{a} \right)^2 = \frac{1}{3M_{\text{Pl}}^2} \rho \tag{6.7}$$

$$= \frac{1}{3M_{\text{Pl}}^2} \left(\frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right) \tag{6.8}$$

$$\tag{6.9}$$

and

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_{\text{Pl}}^2} (\rho + 3p) \tag{6.10}$$

$$= -\frac{1}{3M_{\text{Pl}}^2} \left(\dot{\varphi}^2 - V(\varphi) \right). \tag{6.11}$$

This implies for the equation of state

$$w_\varphi = \frac{p}{\rho} = \frac{\dot{\varphi}^2 - 2V(\varphi)}{\dot{\varphi}^2 + 2V(\varphi)}. \tag{6.12}$$

The integrated form of the continuity equation $\dot{\rho} + 3H(\rho + p) = 0$, which follows from the conservation of the energy-momentum tensor, reads

$$\rho = \rho_0 \exp \left(- \int 3(1 + w_\varphi) \frac{da}{a} \right), \tag{6.13}$$

with the integration constant ρ_0. Now, one can read off the behavior of the energy density in terms of the scale factor for different limits. First for the slow roll limit $\dot{\varphi}^2 \ll V(\varphi)$ one gets $w_\varphi = -1$. This implies $\rho =$ constant. In the other limit $\dot{\varphi}^2 \gg V(\varphi)$ one gets $w_\varphi = 1$ and for the energy density $\rho \propto a^{-6}$. The range of w_φ is from 1 to -1 and therefore $\rho \propto a^{-m},\ 0 \leq m \leq 6$. The threshold for accelerated expansion is $w_\varphi < -\frac{1}{3}$.
Now we will discuss a potential, which gives rise to a power law expansion for the scale

factor [24, 25, 26]

$$a(t) \propto t^p. \tag{6.14}$$

For $p > 1$ accelerated expansion is generated. This is implemented by the following potential

$$V(\varphi) = v\left(-\sqrt{\frac{16\pi}{p}}\frac{\varphi}{m_{\mathrm{Pl}}}\right), \tag{6.15}$$

where v is a constant. Note that $p > 1$ is necessary for accelerated expansion.
Another possibility to generate accelerated expansion is an inverse power-law potential. The original quintessence models [27, 28] used this kind of potential

$$V(\varphi) = \frac{M^{4+\alpha}}{\varphi^\alpha}, \tag{6.16}$$

where α is a positive number and M a constant.

6.2 Application to a closed universe

In order to generate accelerated expansion in the KP model, we develop a model using a scalar field with a quintessence potential. We start in Einstein gravity. The equation of motion of a scalar field in a expanding background reads

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0. \tag{6.17}$$

The equation, which determines the dynamics of the scale factor, is the first Friedmann equation (2.31)

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\frac{1}{2}\dot{\varphi}^2 + V(\varphi)}{3M_{\mathrm{Pl}}^2} - \frac{\kappa}{a^2}. \tag{6.18}$$

Note that we are considering a closed universe, which means that $\kappa > 0$. The limit for small a for this system is a curvature dominated closed universe

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = -\frac{\kappa}{a^2}. \tag{6.19}$$

The solution of this system is an imaginary scale factor, which is unphysical. Thus, we introduce a matter-term in the Friedmann equation, which goes like $\rho_{\mathrm{M},0}\left(\frac{a_0}{a}\right)^3$ and therefore dominates for small a. This regulates the unphysical behavior. Set the value of the scale factor today $a_0 = 1$. Then one has for small values of the scale factor $a \propto t^{\frac{2}{3}}$ (see section 2.2).

The full system now reads

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0, \tag{6.20}$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\frac{1}{2}\dot{\varphi}^2 + V(\varphi) + \frac{\rho_{\mathrm{M},0}}{a^3}}{3M_{\mathrm{Pl}}^2} - \frac{\kappa}{a^2}. \tag{6.21}$$

For a Bang/Crunch scenario one gets a period of curvature domination for late times. The curvature term is the only term on the right-hand side of the Friedmann equation , which is negative and thus can generate the contraction of the universe. But in this case one runs into the same problem as before. Namely that the scale factor becomes imaginary. The resolution of this issue is to solve the second Friedmann equation

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{1}{6M_{\mathrm{Pl}}^2}\left(\rho + 3p\right). \tag{6.22}$$

6.2.1 Numerical results

The system of equations, which will be solved numerically using *Mathematica* is

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0, \tag{6.23}$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\frac{1}{2}\dot{\varphi}^2 + V(\varphi) + \frac{\rho_{\mathrm{M},0}}{a^3}}{3} - \frac{\kappa}{a^2}, \tag{6.24}$$

$$V(\varphi) = v \exp\left(-\sqrt{\frac{2}{p}}\varphi\right). \tag{6.25}$$

Now, we construct a model, which describes our universe in the framework of Einstein gravity. This means the values of the cosmological parameters today should be approximately compatible with the Planck-data [29]. A convenient choice is

$$\Omega_{\mathrm{DE},0} = 0.7,\ \Omega_{\mathrm{M},0} = 0.301,\ \Omega_{\kappa,0} = 0.001 \tag{6.26}$$

at the time $t = 0$ (today). To realize this one needs to specify the initial conditions for solving this system of equations. We choose

$$\varphi(t=0) = 1,\ \dot{\varphi}(t=0) = 0,\ a(t=0) = 1. \tag{6.27}$$

Additionally we work in Planck units ($M_{\mathrm{Pl}} = 1$) and set the Hubble constant today $H_0 = 1$, which means we are computing in units of Hubble time. In order to obtain the mentioned values for the density parameters, consider (6.24) at $t = 0$

$$1 = \underbrace{\frac{V(1)}{3}}_{=\Omega_{\mathrm{DE},0}} + \underbrace{\frac{\rho_{\mathrm{M},0}}{3}}_{=\Omega_{\mathrm{M},0}} - \underbrace{\kappa}_{=-\Omega_{\kappa,0}} . \tag{6.28}$$

This implies that we have to choose the parameters in the equations as $\kappa = 0.001$, $\rho_{\mathrm{M},0} = 0.903$. For the parameters appearing in the potential v and p we have a certain freedom of choice. The condition reads $V(1) = v \exp\left(-\sqrt{\frac{2}{p}}\right) = 2.1$ and the parameter space is given by the line in figure 6.1.

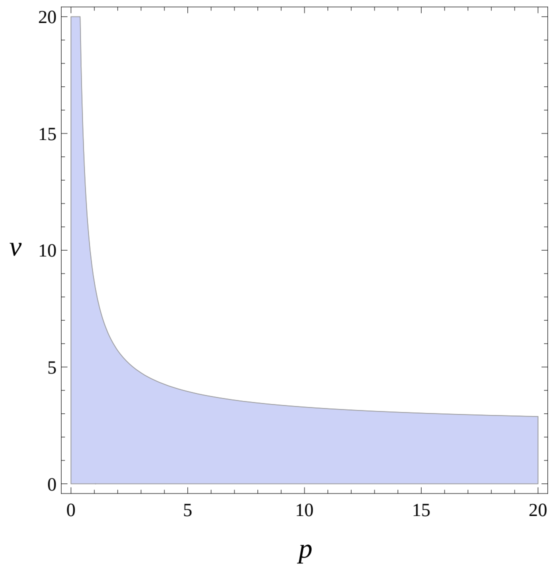

Figure 6.1: Parameter space of the parameters p and v of the quintessence potential

The solutions (figures 6.2 and 6.3 on the next page) of the numerical calculations show the expected properties. There is a transition of the dynamics at $p = 1$. For $p < 1$ we get a recollapsing universe and for $p > 1$ we get accelerated expansion for large $a(t)$. Note that the lifetime of the recollapsing universe seems unnaturally large. This could be due to the fact that the universe is closed, but the curvature parameter κ is very small. Thus, it takes a large amount of the cosmological time evolution to reach the curvature dominated epoch and to generate a decreasing scale factor.

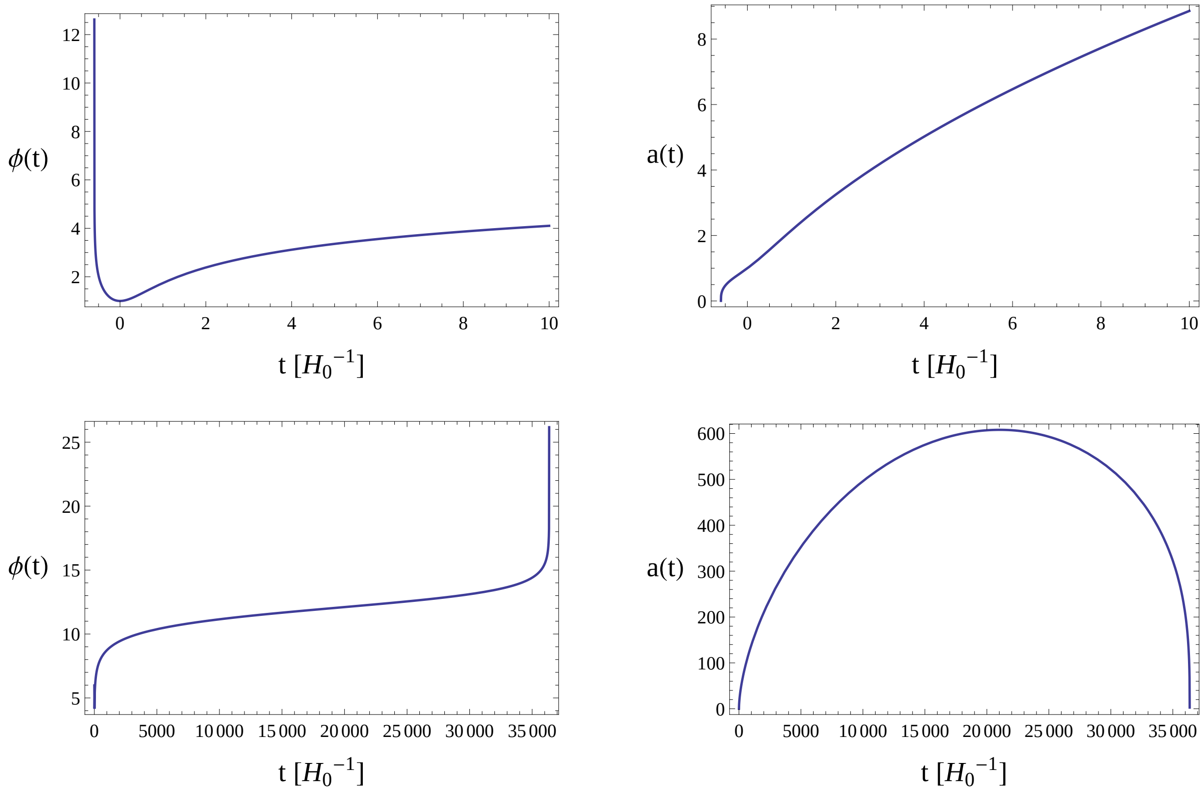

Figure 6.2: Time evolution of the scale factor $a(t)$ and the scalar field $\varphi(t)$ in Einstein gravity for the parameter values $p = 0.5$ and $v = 15.52$

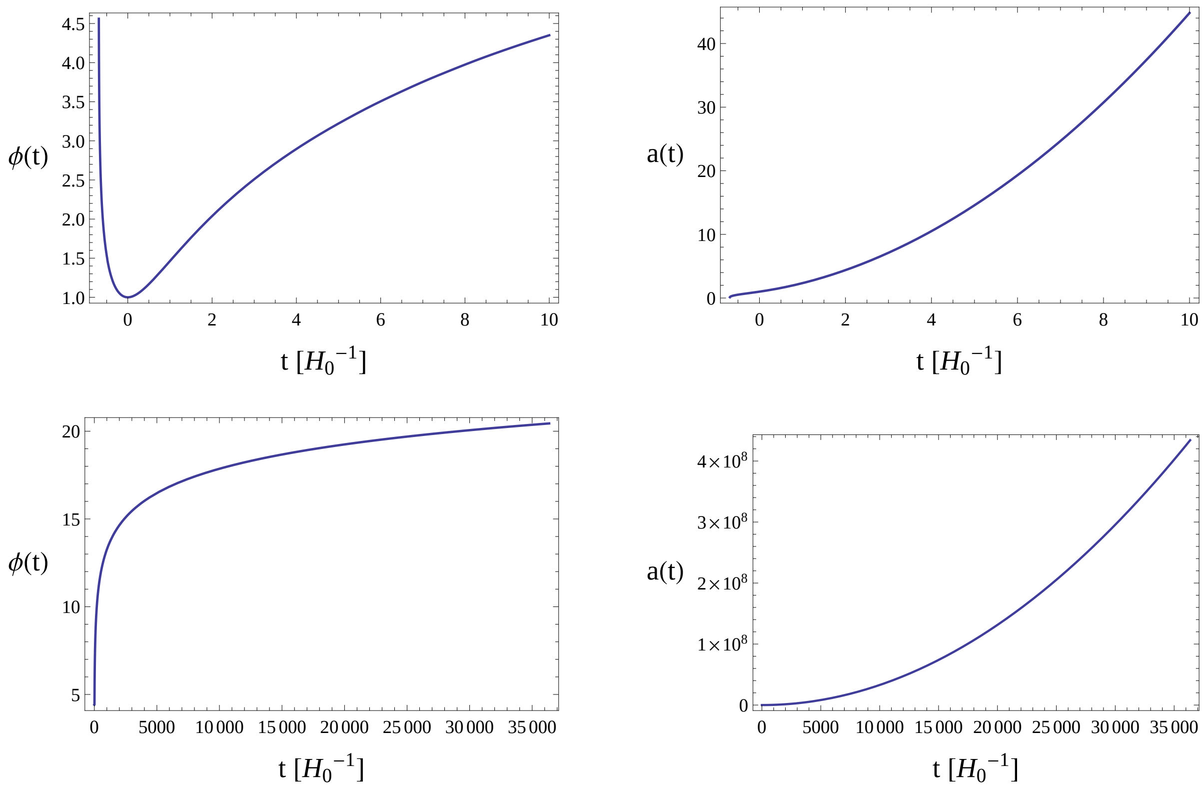

Figure 6.3: Time evolution of the scale factor $a(t)$ and the scalar field $\varphi(t)$ in Einstein gravity for the parameter values $p = 2$ and $v = 5.71$

6.3 Using quintessence to test the KP model

After we studied the dynamics of the quintessence model in Einstein gravity, we will use this model to address the problem concerning accelerated expansion in the KP model.

6.3.1 Friedmann equations for a scalar field

To get the Friedmann equations for a scalar field in the KP model, one has to calculate the energy-momentum tensor via $\tau_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta \Delta S}{\delta g^{\mu\nu}}$ with $\Delta S = \int d^4x\sqrt{-g}\lambda^4\Delta\mathcal{L}\left(\lambda^{-2}g^{\mu\nu},\varphi\right)$. The local Lagrangian of a scalar field reads $\Delta\mathcal{L} = -\frac{1}{2}g^{\mu\nu}\lambda^{-2}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)$. The variation leads to

$$\tau_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\Delta S}{\delta g^{\mu\nu}} \tag{6.29}$$

$$= -\frac{2}{\sqrt{-g}}\left(\left(-\frac{1}{2}g_{\mu\nu}\sqrt{-g}\right)\lambda^4\left(-\frac{1}{2}\lambda^{-2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right)\right. \tag{6.30}$$

$$\left. + \sqrt{-g}\lambda^4\left(-\frac{1}{2}\lambda^{-2}\partial_\mu\varphi\partial_\nu\varphi\right)\right) \tag{6.31}$$

$$= \lambda^2\partial_\mu\varphi\partial_\nu\varphi + \left(-\frac{1}{2}\lambda^2\partial^\rho\varphi\partial_\rho\varphi - \lambda^4 V(\varphi)\right)g_{\mu\nu} \tag{6.32}$$

$$= \partial_\mu\varphi'\partial_\nu\varphi' + \left(-\frac{1}{2}\partial^\rho\varphi'\partial_\rho\varphi' - \lambda^4 V(\lambda^{-1}\varphi')\right)g_{\mu\nu}, \tag{6.33}$$

where $\varphi = \lambda^{-1}\varphi'$. In the following we will drop the prime. To relate the result to the energy-momentum tensor of a perfect fluid $\tau_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$, one needs to do the following identifications (assuming homogeneity): for the pressure

$$p = -\frac{1}{2}\partial^\rho\varphi\partial_\rho\varphi - \lambda^4 V(\lambda^{-1}\varphi) \tag{6.34}$$

$$= \frac{1}{2}\dot{\varphi}^2 - \lambda^4 V(\lambda^{-1}\varphi), \tag{6.35}$$

$$\tag{6.36}$$

for the energy density

$$\rho = -\frac{1}{2}\partial^\rho\varphi\partial_\rho\varphi + \lambda^4 V(\lambda^{-1}\varphi) \tag{6.37}$$

$$= \frac{1}{2}\dot{\varphi}^2 + \lambda^4 V(\lambda^{-1}\varphi) \tag{6.38}$$

$$\tag{6.39}$$

and for the 4-velocity

$$u_\mu = \partial_\mu \varphi \frac{1}{\sqrt{-\partial^\rho \varphi \partial_\rho \varphi}}. \tag{6.40}$$

This implies for the first Friedmann equation (5.34)

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{1}{3M_{\rm Pl}^2}\left(\rho + \frac{1}{4}\langle -\rho + 3p\rangle\right) - \frac{\kappa}{a^2} \tag{6.41}$$

$$= \frac{1}{3M_{\rm Pl}^2}\left(\frac{1}{2}\dot\varphi^2 + \lambda^4 V(\lambda^{-1}\varphi) + \frac{1}{4}\langle \dot\varphi^2 - 4\lambda^4 V(\lambda^{-1}\varphi)\rangle\right) - \frac{\kappa}{a^2} \tag{6.42}$$

and for the second (5.37)

$$\frac{\ddot a}{a} = -\frac{1}{6M_{\rm Pl}^2}(\rho + 3p) + \frac{1}{3M_{\rm Pl}^2}\left(\frac{1}{4}\langle -\rho + 3p\rangle\right) \tag{6.43}$$

$$= -\frac{1}{6M_{\rm Pl}^2}\left(2\dot\varphi^2 - 2\lambda^4 V(\lambda^{-1}\varphi)\right) + \frac{1}{3M_{\rm Pl}^2}\left(\frac{1}{4}\langle \dot\varphi^2 - 4\lambda^4 V(\lambda^{-1}\varphi)\rangle\right). \tag{6.44}$$

For example the first Friedmann equation simplifies a lot in the slow roll limit: $\dot\varphi^2 \ll V(\varphi)$ and $|\ddot\varphi| \ll |3H\dot\varphi|, |\frac{\partial V}{\partial \varphi}|$ [22, 30]

$$\left(\frac{\dot a}{a}\right)^2 = \frac{1}{3M_{\rm Pl}^2}\left(\lambda^4 V(\lambda^{-1}\varphi) - \langle \lambda^4 V(\lambda^{-1}\varphi)\rangle\right) - \frac{\kappa}{a^2}, \tag{6.45}$$

where $\langle \lambda^4 V(\lambda^{-1}\varphi)\rangle = \frac{\int d^4x\sqrt{-g}\left(\lambda^4 V(\lambda^{-1}\varphi)\right)}{\int d^4x\sqrt{-g}}$, $p = -\lambda^4 V(\lambda^{-1}\varphi)$ and $\rho = \lambda^4 V(\lambda^{-1}\varphi)$.

6.3.2 Application to the KP model

Like emphasized before we want to test, if it is possible to generate accelerated expansion in the KP model using the quintessence model. We derived the first Friedmann equation for a scalar field (6.41) and with $\rho = \frac{1}{2}\dot\varphi^2 + \lambda^4 V(\lambda^{-1}\varphi) + \frac{\rho_{\rm M,0}}{a^3}$ and $p = \frac{1}{2}\dot\varphi^2 - \lambda^4 V(\lambda^{-1}\varphi)$ we have

$$\left(\frac{\dot a}{a}\right)^2 = \frac{1}{3M_{\rm Pl}^2}\left(\frac{1}{2}\dot\varphi^2 + \lambda^4 V(\lambda^{-1}\varphi) + \frac{\rho_{\rm M,0}}{a^3} + \frac{1}{4}\langle \dot\varphi^2 - 4\lambda^4 V(\lambda^{-1}\varphi) - \frac{\rho_{\rm M,0}}{a^3}\rangle\right) - \frac{\kappa}{a^2}. \tag{6.46}$$

In order to do computations in this model we first need to discuss the parameter λ phenomenologically [13]. From the variation with respect to Λ one gets the following relation $\lambda = \sigma'/\left(\mu^4 \int d^4x\sqrt{-g}\right)^{1/4}$ (see (5.6)). Specifying the cutoff for the vacuum energy contributions, gives control over λ. It is given by $M_{\rm UV}^{\rm phys} = \lambda M_{\rm UV}$ (cf. $m_{\rm phys} = \lambda m$) and can be as high as $M_{\rm Pl}$. To get $M_{\rm UV}^{\rm phys} \simeq M_{\rm UV} \simeq M_{\rm Pl}$ the model should be able to provide $\lambda \sim \mathcal{O}(1)$. This is satisfied by choosing $\sigma(z) \simeq \exp z$, for $z > 1$. Since $\sigma(z)$ needs to be an

odd function, one can take $\sigma(z) \simeq \sinh z$. In the simplest case $\sigma(z) = z$, i.e. linear, one gets $\lambda = 1/\left(\mu\left(\int d^4x\sqrt{-g}\right)^{1/4}\right)$ and this implies $\lambda \simeq H_{\text{age}}/\mu$ with (5.26). If one takes the cutoff to be $M_{\text{UV}} \sim M_{\text{Pl}}$,then λ has to be very small $\lambda \sim 10^{-15}$ to get $m_{\text{phys}} \sim$ TeV. For $H_{\text{age}} \sim H_0 \sim 10^{-33}$eV this requires an unnatural small value for $\mu \sim 10^{-18}$eV. In addition, this setup gets less attractive because it is very sensitive to changes of the cosmological conditions.
Thus, it is well motivated that $\lambda \sim \mathcal{O}(1)$ and we choose $\lambda = 1$ for simplicity.

6.3.3 Numerical results

Now, in the KP model the system of equations to solve is ($M_{\text{Pl}} = 1$)

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0, \tag{6.47}$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi) + \frac{\rho_{\text{M},0}}{a^3} + \frac{1}{4}\langle\dot{\varphi}^2 - 4V(\varphi) - \frac{\rho_{\text{M},0}}{a^3}\rangle\right) - \frac{\kappa}{a^2}, \tag{6.48}$$

$$V(\varphi) = v\exp\left(-\sqrt{\frac{2}{p}}\varphi\right). \tag{6.49}$$

Using the same initial conditions and the same choice for the values of the parameters $\rho_{\text{M},0}$ and κ as before in the case of Einstein gravity, we get indeed that it is possible to generate accelerated expansion with the quintessence model for a certain range of the parameters p and v.

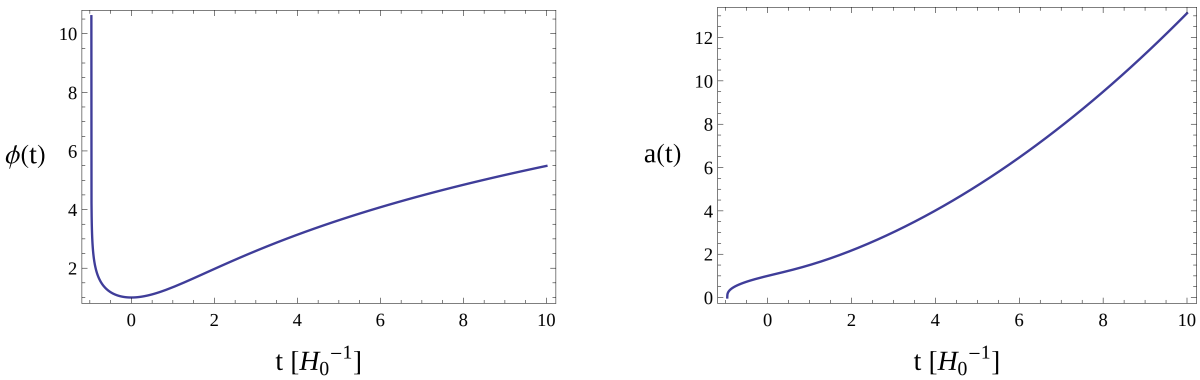

Figure 6.4: Time evolution of the scale factor $a(t)$ and the scalar field $\varphi(t)$ in the KP model for the parameter values $p = 8$ and $v = 3.46$

This result is preliminary because it cannot be reproduced by solving the second Friedmann equation, what is necessary for consistency reasons. It indicates that there is some tension within the model. But if we take this result seriously there is another problem appearing. For the KP model it is required to have a closed universe such that the world-volume $\int d^4x\sqrt{-g}$ is finite. Otherwise it would fail to describe our universe and especially the matter sector. We observe that accelerated expansion continues forever, once it starts. If we assume that this holds, it violates the consistency of the model because the world-volume diverges and therefore spacetime is not finite anymore. The parameter λ goes to zero in this case. This implies that the theory has no mass gap and no mass scales at all. Additionally the cosmological constant vanishes.

7 Conclusion

As elaborated in this work theories with finite spacetime show some interesting features regarding the cosmological constant. Especially the cosmological constant problem is a huge motivation to think about alternatives to the identification of the cosmological constant with the vacuum energy. Starting from a theory with restricted coordinate invariance (i.e. invariance under volume preserving transformations) one finds that the cosmological constant appears as an initial condition. Spacetime becomes finite because the determinant of the metric is not a dynamical variable anymore. The basic idea of unimodular gravity is that the variation of the determinant of the metric vanishes. The Einstein field equations become traceless in this theory and the cosmological constant appears as a constant of integration. Its value is not determined.
The KP model is able to generate a small, but non-vanishing cosmological constant-like term for sufficiently old and big universes like ours. The critical density of the universe today is the upper limit for the value of the cosmological constant. This is accomplished by introducing two new global variables and two constraints in the gravitational sector. The particle sector i.e., local QFT, is unaffected by these modifications. The term, which acts as a cosmological constant, is protected by two approximate symmetries and hence radiatively stable. A shift symmetry gives the possibility to cancel the renormalized vacuum energy and a scaling symmetry ensures that the shift symmetry works at all scales below the cutoff. A small cosmological constant remains because these symmetries are just approximate. It is identified with the historic average of the trace of the energy-momentum tensor $\Lambda = \frac{1}{4}\langle \tau^{\alpha}_{\ \alpha} \rangle$, which is a non-local term. This is due to fact that the cosmological constant is regarded as a UV-divergent quantity.
A bare cosmological constant never enters the equations of motion in the model. Therefore, it is not possible to generate accelerated expansion this way. This was our motivation to introduce a scalar field with a quintessence potential to generate accelerated expansion in this model. This procedure is similar to the mechanism of generating accelerated expansion in an inflationary epoch of the early universe. If we are able to generate expansion, which does not stop, this implies that spacetime is not finite anymore. But this is necessary for the KP model to describe our universe. By means of this approach we are able to generate accelerated expansion in the KP model for a certain range of the parameters . Thus, our considerations indicate that there is a problem in the model

because spacetime does not remain finite. Since the time for working on a master thesis is limited, it was not possible to bring all calculations to a satisfying end. It remains to verify the solution of the first Friedmann equation in the KP case using the second Friedmann equation. The fact that we were not able to do so, could suggest also that there is some additional issue within model. Considering the KP model gives an attractive perspective on the cosmological constant, but more time and further thoughts are needed to verify or exclude it. Even the authors postpone resolving the issues concerning accelerated expansion to a future publication [13]. This is a hint that our considerations and preliminary results are pointing in the right direction.
It seems difficult to create a scenario, in which accelerated expansion stops eventually and a recollapse is possible. There are some models with bouncing cosmologies, which are able to generate an epoch of accelerated expansion followed by a recollapse. Those oscillating universes, called ekpyrotic universes [31,32,33], are not finite in lifetime. They feature a series of Bangs and Crunches. This violates the condition of the KP model that the worldvolume $\int d^4x\sqrt{-g}$ has to be finite. In future investigations one could try to implement different potentials for the scalar field, e.g. those arising from inflation. These could be interesting because inflation has to stop eventually like the epoch of accelerated expansion in the KP model.
The bottom line of our investigations is that the KP model is questionable. The issues about the observed accelerated expansion of the universe remain. First, it is not possible to generate accelerated expansion simply using a bare cosmological constant. Moreover, if one is able to generate accelerated expansion, which does not stop, like in our model, the model is ruled out. In summary, the KP model is most interesting in the context of the cosmological constant problem, but our results do not support its validity.

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A Conformal transformation

An important tool used in chapter 2 is the conformal transformation. This is why it is explained in detail here (for reference see [34] and [14]).
Consider the following transformation of the metric tensor

$$g_{\mu\nu} = \chi^2 \overline{g}_{\mu\nu} \iff \overline{g}_{\mu\nu} = \chi^{-2} g_{\mu\nu}. \tag{A.1}$$

This implies that the inverse metric transforms as

$$g^{\mu\nu} = \chi^{-2} \overline{g}^{\mu\nu} \iff \overline{g}^{\mu\nu} = \chi^2 g^{\mu\nu}. \tag{A.2}$$

The task is to determine how the Ricci scalar transforms under such transformations. This is important because it appears in the canonical Einstein-Hilbert action. To achieve this one has to find out how the Christoffel symbols transform. They are defined as given in equation (2.9). The Christoffel symbols are the coefficients of the connection, which appears in the covariant derivative of general relativity. To find out how the connection transforms under conformal transformations, one considers the difference of the covariant derivatives with respect to the original and to the transformed metric

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu{}_{\nu\lambda} V^\lambda, \tag{A.3}$$

$$\overline{\nabla}_\nu V^\mu = \partial_\nu V^\mu + \overline{\Gamma}^\mu{}_{\nu\lambda} V^\lambda, \tag{A.4}$$

$$\nabla_\nu V^\mu - \overline{\nabla}_\nu V^\mu = \underbrace{\left(\Gamma^\mu{}_{\nu\lambda} - \overline{\Gamma}^\mu{}_{\nu\lambda}\right)}_{:=\delta\Gamma^\mu_{\nu\lambda}} V^\lambda. \tag{A.5}$$

The covariant derivative of a vector transforms as a tensor. This is why the left-hand side of equation (A.5) transforms as a tensor and hence the right-hand side has to transform as a tensor, too. Thus $\delta\Gamma^\mu_{\nu\lambda}$ has to transform as a (1,2)-tensor, if its contraction with a contravariant vector transforms as a (1,1)-tensor. To get the right transformation law one has to replace the partial derivatives with covariant derivatives and obtains

$$\delta\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}\overline{g}^{\lambda\rho}(\nabla_\mu \overline{g}_{\rho\nu} + \nabla_\nu \overline{g}_{\mu\rho} - \nabla_\rho \overline{g}_{\mu\nu}). \tag{A.6}$$

If one plugs in the transformation for the metric one gets the following result

$$\delta\Gamma^{\lambda}{}_{\mu\nu} = -\chi^{-1}(\nabla_\mu\chi\delta^{\lambda}{}_{\nu} + \nabla_\nu\chi\delta^{\lambda}{}_{\mu} - \nabla_\rho\chi g_{\mu\nu}g^{\lambda\rho}). \tag{A.7}$$

The next step is to consider the Riemann tensor, which is defined as

$$R^{\lambda}{}_{\mu\kappa\nu} = \partial_\nu\Gamma^{\lambda}{}_{\mu\kappa} - \partial_\kappa\Gamma^{\lambda}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\mu\kappa}\Gamma^{\lambda}{}_{\nu\alpha} - \Gamma^{\alpha}{}_{\mu\nu}\Gamma^{\lambda}{}_{\kappa\alpha}. \tag{A.8}$$

Considering the transformation of the Christoffel symbols, one has to replace the partial derivatives of $\delta\Gamma$ by covariant derivatives to get the right transformation law. Otherwise the transformed Riemann tensor would not transform as a tensor anymore. This yields the following transformation, under a transformation of the Chrisoffel symbol like $\overline{\Gamma} = \Gamma + \delta\Gamma$,

$$\begin{aligned} \overline{R}^{\lambda}{}_{\mu\kappa\nu} &= R^{\lambda}{}_{\mu\kappa\nu} + \nabla_\nu(\delta\Gamma^{\lambda}{}_{\mu\kappa}) - \nabla_\kappa(\Gamma^{\lambda}{}_{\mu\nu}) \\ &\quad + (\delta\Gamma^{\alpha}{}_{\mu\kappa})(\delta\Gamma^{\lambda}{}_{\nu\alpha}) - (\delta\Gamma^{\alpha}{}_{\mu\nu})(\delta\Gamma^{\lambda}{}_{\kappa\alpha}), \end{aligned} \tag{A.9}$$

$$\begin{aligned} \overline{R}_{\mu\nu} = \overline{R}^{\lambda}{}_{\mu\lambda\nu} &= R^{\lambda}{}_{\mu\lambda\nu} + \nabla_\nu(\delta\Gamma^{\lambda}{}_{\mu\lambda}) - \nabla_\lambda(\Gamma^{\lambda}{}_{\mu\nu}) \\ &\quad + (\delta\Gamma^{\alpha}{}_{\mu\lambda})(\delta\Gamma^{\lambda}{}_{\nu\alpha}) - (\delta\Gamma^{\alpha}{}_{\mu\nu})(\delta\Gamma^{\lambda}{}_{\lambda\alpha}), \end{aligned} \tag{A.10}$$

with

$$\delta\Gamma^{\lambda}{}_{\mu\lambda} = -\chi^{-1}(\nabla_\mu\chi \underbrace{\delta^{\lambda}{}_{\lambda}}_{=d} + \nabla_\lambda\chi\delta^{\lambda}{}_{\mu} - \nabla_\rho\chi\delta^{\rho}{}_{\mu}) \tag{A.11}$$

$$= -d\chi^{-1}\nabla_\mu\chi. \tag{A.12}$$

One finds for the Ricci scalar

$$\overline{R} = \overline{g}^{\mu\nu}\overline{R}_{\mu\nu} = \chi^2 R + d(d-1)\nabla_\mu\chi\nabla_\nu\chi g^{\mu\nu} - \underbrace{2(d-1)\chi\nabla_\mu\nabla_\nu\chi g^{\mu\nu}}_{\text{surface term}}, \tag{A.13}$$

for d=4

$$\overline{R} = \chi^2 R + 12\nabla_\mu\chi\nabla_\nu\chi g^{\mu\nu} - \underbrace{6\chi\nabla_\mu\nabla_\nu\chi g^{\mu\nu}}_{\text{surface term}}. \tag{A.14}$$

Due to Gauss' theorem in curved space-time the surface term can be neglected

$$\int_\Omega d^4x\sqrt{-g}\nabla_\mu(\nabla^\mu\chi) = \int_\Omega d^4x\partial_\mu(\sqrt{-g}\nabla^\mu\chi) \tag{A.15}$$

$$= \int_{\delta\Omega} dS_\mu\nabla^\mu\chi. \tag{A.16}$$

This term is zero, if $\partial_\mu \chi$ vanishes at infinity, which is the case here. This means we are left with

$$\Rightarrow \overline{R} = \chi^2 R + 12\nabla_\mu \chi \nabla_\nu \chi g^{\mu\nu} \tag{A.17}$$

$$= \chi^2 R + 12\partial_\mu \chi \partial_\nu \chi g^{\mu\nu}. \tag{A.18}$$

B Energy conditions

If one does not specify the energy momentum tensor $T_{\mu\nu}$ any metric will obey the Einstein equation. But if we are interested in realistic solutions we have to specify the sources of energy and momentum. That is why energy conditions are imposed [14]. These additional constraints prevent properties, which are regarded as unphysical. The conditions will be translated in terms of a energy-momentum tensor for a perfect fluid

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu}. \tag{B.1}$$

There are several possibilities as

- Weak Energy Condition (WEC):
  $T_{\mu\nu} t^\mu t^\nu \geq 0$ for all timelike vectors t^μ or $\rho \geq 0$ and $\rho + p \geq 0$
- Null Energy Condition (NEC):
  $T_{\mu\nu} l^\mu l^\nu \geq 0$ for all null vectors l^μ or $\rho + p \geq 0$
- Dominant Energy Condition (DEC):
  $T_{\mu\nu} t^\mu t^\nu \geq 0$ for all timelike vectors t^μ and $T^{\mu\nu} t_\mu$ a non-spacelike vector or $\rho \geq |p|$
- Null Dominant Energy Condition (NDEC): (the DEC for null vectors)
  $T_{\mu\nu} l^\mu l^\nu \geq 0$ for all null vectors l^μ and $T^{\mu\nu} l_\mu$ a non-spacelike vector or $p = -\rho$
- Strong Energy Condition (SEC):
  $T_{\mu\nu} t^\mu t^\nu \geq \frac{1}{2} t^\lambda{}_\lambda t^\sigma t_\sigma$ for all timelike vectors t^μ or $\rho + p \geq 0$ and $\rho + 3p \geq 0$

Most classical forms of matter obey the DEC and therefore the less restrictive ones (WEC, NEC, NDEC).

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Erklärung

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Hamburg, 6. August 2014, ________________

Janis Kummer