EINSTEIN REVISITED: GRAVITATION IN CURVED SPACETIME
WITHOUT EVENT HORIZONS

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It is well known that Einstein General Relativity can be expressed covariantly in bi-metric spacetime, without the uncertainties which arise from the effects of gravitational energy-momentum pseudotensors. We construct a new bi-metric General Relativity theory of gravitation based on a new physical paradigm which allows the operational procedure of local spacetime measurements in general spacetime frames of reference to be defined in a similar manner as that for local spacetime measurements in inertial Special Relativistic spacetime frames. The new paradigm accomplishes this by using the Principle of Equivalence to define the metric tensor of curved spacetime as an exponential function of a gravitational potential tensor. The resultant gravitational field equations imply that, in addition to the matter energy-momentum tensor, a gravitational field stress-energy tensor appears in the right member of the Einstein field equations. A unique prediction of the new bi-metric General Relativity theory is that massive compact astrophysical objects have no event horizons and can manifest the existence of intrinsic dipole magnetic fields which can affect their accretion disks.

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I. INTRODUCTION

The physical spacetime associated with Special Relativity is the globally flat Inertial Cartesian Minkowski (ICM) spacetime frame where all coordinate and non-inertial effects vanish. The metric $\eta_{\mu\nu}$ associated this flat ICM spacetime is a constant tensor which has the same value for all observers independent of their physical location in the spacetime. Special Relativity also implies that the following two fundamental relativistic rules of observation are valid in the flat ICM spacetime: a) all physical observations involve local measurements by observers, and b) all observers measure the local speed of light to be $v=c$.

Using the rules of General Covariance one can extend Special Relativity from the flat, ICM spacetime into the realm of flat non-Cartesian, non-inertial spacetime. However in this case the metric becomes a variable tensor function of spacetime whose value depends on the physical location of the origin of the observer in the spacetime. In order to obtain a consistent extension of Special Relativity into the realm of flat, non-Cartesian, non-inertial spacetime one can formulate the theory so that the above two fundamental relativistic rules of observation implied by Special Relativity still remain valid in the flat non-Cartesian, non-inertial spacetime

This can be accomplished by taking advantage of the fact that general flat spacetime coordinate transformations contain the inherent freedom to arbitrarily set the origin of the observer coordinates at spacetime locations which do not have to be the same point as that of the origin of symmetry of the coordinate transformation itself. This means that flat spacetime coordinate transformations may be represented in a functional form where the location of the observer is operationally given as part of the physical flat spacetime transformation.
The bi-metric description of spacetime (Rosen [1]) assumes that at every point in spacetime a flat background spacetime metric $\gamma_{\mu\nu}$ and a curved physical spacetime metric $g_{\mu\nu}$ are juxtaposed upon each other in a manner so as to allow an operational definition of physical tensor quantities in terms of "background covariant derivatives". The bi-metric description of spacetime has the distinct advantage of allowing a well-defined tensor description of gravitational energy-momentum to be defined in the Einstein Theory of General Relativity within a class of gravitational gauge transformations [8].

Hence it was shown that the standard Einstein theory of General Relativity could be derived in the context of the bi-metric description of spacetime. However it is important to note that bi-metric spacetime is not necessarily associated with a specific form for the bi-metric gravitational field equations. In this paper we will consider a new form for the bi-metric gravitational field equations which are different from the bi-metric Einstein field equations.

The new bi-metric formulation of General Relativity presented in this paper is based on a physical paradigm [2] which extends the two fundamental rules of observation implied by Special Relativity to include the requirement that the metric of curved spacetime should be defined in an operational manner so that the procedure of local spacetime measurements, as seen by an observer in a general spacetime frame which contains gravitational fields due to spacetime curvature, becomes locally defined in a similar manner as that for the extension of Special Relativity into general flat spacetimes. While it is still true that an observer in a local free fall frame of reference will locally measure the speed of light to be $v = c$, the new theory also requires that an observer in a general frame of reference (which contains both the effects of real gravitation due to spacetime curvature and artificial gravitation due to non-inertial coordinate transformations) will also locally measure the vacuum velocity of light to be $v = c$. 
II BI-METRIC COVARIANT FORMULATION OF GENERAL RELATIVITY IN CURVED SPACETIME WITH AN EXPONENTIAL METRIC

The purpose of this paper is to develop a new bi-metric theory of General Relativity in curved spacetime which uses an exponential metric [2] formulated in a generally covariant manner within the context of the bi-metric description of spacetime [1].

In the new theory the Strong Principle of Equivalence implies that a fundamental symmetric gravitational potential tensor field \( \phi_{\mu\nu} = \phi_{\nu\mu} \) exists in the bi-metric spacetime which is connected to the symmetric metric tensor density field \( g_{\mu\nu} \), by the holonomic exponential metric density constraint relationship

\[
g_{\mu\nu} = \left[ \exp(-2\phi^{\mu}) \gamma^{\alpha\beta} \left[ \exp(-2\phi^{T}) \beta_{\nu} \right] \right] = g_{\nu\mu} \tag{1}
\]

where \( \phi^{\mu} = \phi^{\mu\alpha} g^{\alpha\nu} = \phi^{\nu\mu}, \ (\phi^{T})^{\mu} = \phi^{\nu\mu} \) are mixed tensors and \( \gamma^{\alpha\beta} = (-\gamma)^{-1/2} \gamma^{\alpha\beta} \) is the covariant metric tensor density of the flat background spacetime, associated with the physical spacetime metric tensor density field \( g_{\mu\nu} \) (see Appendix A & B).

It follows from (1) that the metric tensor \( g_{\mu\nu} \) is a nonlinear function of \( \phi^{\mu} \) given by the exponential holonomic relationship

\[
g_{\mu\nu} = \left[ \exp(\Phi^{\mu}) \gamma^{\alpha\beta} \left[ \exp(\Phi^{T}) \beta_{\nu} \right] \right] = g_{\nu\mu} \tag{2}
\]

where \( \Phi^{\mu} = (\text{Tr}(\phi) \delta^{\mu}_{\nu} - 2\phi^{\mu}_{\nu}) = \Phi^{\nu}_{\mu} \)

Now taking the bi-metric covariant derivative "|\( \mu \)" of equations (1) and (2), in a Cartesian bi-metric spacetime where the metrics \( g_{\mu\nu} \) and \( \gamma_{\mu\nu} \) are diagonal, and then using the bi-metric tensor properties of the resulting equations to transform to a general bi-metric spacetime frame of reference, we find in the general frame the following bi-metric tensor equations are valid

\[
\phi^{\mu}_{\nu|\lambda} = -1/4 (g_{\mu\rho} |\lambda g^{\rho\nu}) \quad (3-a)
\]

\[
\Phi^{\mu}_{\nu|\lambda} = 1/2 (g_{\mu\rho} |\lambda g^{\rho\nu}) \quad (3-b)
\]
Since the metric tensor $g_{\mu\nu}$ in the theory is an exponential function of the gravitational potential tensor $\phi_{\mu}^{v}$ (where we assume that $\phi_{\mu\nu}=\phi_{\nu\mu}$ so that $\phi_{\mu}^{v}=\phi_{\mu\alpha}g^{\alpha\nu}=\phi^{v}_{\mu}$ is valid), then gravitational potential tensor $\phi_{\mu}^{v}$ must obey a superposition principle if the matter geodesic equations of motion are to have an N-body interactive form. The physical requirement that the geodesic equations of motion have an N-body interactive form leads directly to field equations for the gravitational potential. This is because the simplest way to satisfy this physical requirement, in a manner which maintains a non-relativistic correspondence limit with the Newtonian theory of gravity, is to postulate that the gravitational field equations have a bi-metric tensor covariant D'Alembertian form (which is gauge-invariant under a class of gravitational gauge transformations)\(^{[*]}\) given by

$$\quad \Box \phi_{\mu}^{v} = g^{\alpha\beta}(\phi_{\mu}^{v} | \alpha | \beta) = \tau_{\mu}^{v} \tag{4}$$

where in (4) above $c = 1 = 4\pi G$ and the symbols " $| \alpha $ " and " $| \beta $ " are bi-metric $\gamma$-covariant derivatives with respect to the flat background metric $\gamma_{\mu\nu}$. (for more information on the N-body interactive form of the geodesic equations of motion associated with this field equation see Appendix H)

It is important to note that the bi-metric d'Alembertian gravitational field equations given by (4) are fundamentally different from those of Rosen's bi-metric gravitational field theory \([6]\), which in our notation can be written in the form

$$\gamma^{\alpha\beta}(\phi_{\mu}^{v} | \alpha | \beta) = \tau_{\mu\alpha}^{\gamma} \tag{5}$$

The key difference involves the fact that the flat spacetime d'Alembertian operator $\gamma^{\alpha\beta}(\phi_{\mu}^{v} | \alpha | \beta)$ acts in Rosen's gravitational field equations instead of the bi-metric spacetime d'Alembertian operator $\Box \phi_{\mu}^{v} = g^{\alpha\beta}(\phi_{\mu}^{v} | \alpha | \beta)$.
Upon inserting equations (1 through 4) into the Einstein tensor $G_{\mu}^{\nu}$ in the bi-metric spacetime we find that, in terms of bi-metric covariant derivatives, equation (4) is equivalent to a modified Einstein equation given by

$$-1/2 G_{\mu}^{\nu} = (\tau_{\mu}^{\nu} + t_{\mu}^{\nu})$$

(5)

In (5) $t_{\mu}^{\nu} = [S_{\mu}^{\nu} - U_{\mu}^{\nu}]$, where $S_{\mu}^{\nu}$ is given by

$$S_{\mu}^{\nu} = \left\{ g^{\alpha\beta} | \beta \phi_{\mu}^{\nu} | \alpha - \left[ (-\kappa)^{1/2} \phi_{\mu}^{\nu} | \alpha \right] | \alpha 
- \left[ (\delta_{\mu}^{\alpha} g^{\nu\lambda} - \delta_{\nu}^{\beta} g^{\mu\alpha}) \phi_{\lambda}^{\beta} | \beta \right] | \alpha \right\} / (-\kappa)^{1/2}$$

(6)

and $U_{\mu}^{\nu}$ is the bi-metric Einstein gravitational stress energy tensor given by

$$U_{\mu}^{\nu} = \left\{ (1/2 \delta_{\mu}^{\nu} W_{\alpha}^{\alpha} - W_{\mu}^{\nu}) \right\}$$

(7)

$$W_{\mu}^{\nu} = \{ Tr(\phi | \mu \phi | \nu) - 2 Tr(\phi | \mu) Tr(\phi | \nu) \} + [g^{\nu\rho} (\phi_{\rho}^{\lambda} | \mu) (\phi_{\lambda}^{\alpha} | \alpha)]$$

(8)

Hence we see that in the bi-metric spacetime context the new General Relativity differs from bi-metric Einstein General Relativity in the fact that the standard Einstein equation $-1/2 G_{\mu}^{\nu} = \tau_{\mu}^{\nu}$ is modified to become $-1/2 G_{\mu}^{\nu} = \tau_{\mu}^{\nu} + t_{\mu}^{\nu}$. This implies that the fundamental field equation (5) is equivalent to modifying the right hand side of the Einstein equation by adding gravitational potential stress energy momentum tensor $t_{\mu}^{\nu} = [S_{\mu}^{\nu} - U_{\mu}^{\nu}]$ to the standard energy momentum tensor of matter $\tau_{\mu}^{\nu}$. It has been shown [3] that gravitational stress-energy tensor contributions of this type cannot be excluded by existing weak-field tests of general relativity. (For a variational principle derivation of this theory see Appendix F).

In the new bi-metric General Relativity the Einstein curvature of spacetime is generated by the action of both the matter energy momentum tensor $\tau_{\mu}^{\nu}$ and the gravitational stress energy momentum tensor given by $t_{\mu}^{\nu}$. Thus gravitation "gravitates" on the right hand side of the modified Einstein equation through the action of the gravitational potential tensor $\phi_{\mu}^{\nu}$ acting inside of the gravitational stress energy momentum tensor given by $t_{\mu}^{\nu}$. 
Now the Bianchi Identity \( G_{\mu}^{\nu} ; \nu = 0 \) and equation (5) implies that

\[
[t_{\mu}^{\nu} + t_{\mu}^{\nu}] ; \nu = 0
\]

where the symbol " ; " denotes the normal g-covariant derivative with respect to curved physical metric in the bi-metric spacetime context. Upon expanding the g-covariant derivatives in (9-a) and using equation (5) we find that the Bianchi Identity becomes the Freud Identity

\[
[(-\kappa)^{-1/2} (t_{\mu}^{\nu} + t_{\mu}^{\nu}) + u_{\mu}^{\nu}] ; \nu = 0
\]

The Bianchi -Freud identities (9-a) and (9-b) are associated with the Einstein tensor \( G_{\mu}^{\nu} \). Consistency of these identities requires that the gravitational energy momentum tensor in the new bi-metric of General Relativity of gravitation be given by \( t_{\mu}^{\nu} = (S_{\mu}^{\nu} - u_{\mu}^{\nu}) \), (For a variational principle approach see Appendix F).

Note that the Bianchi and Freud identities taken together imply that

\[
t_{\mu}^{\nu} ; \nu = [1/(-\kappa)^{-1/2} \{(\sigma + P/c^2) u_{\mu}^{\nu} - P \delta_{\mu}^{\nu}\}
\]

Now suppose we have a classical continuous distribution of matter with proper density \( \sigma \) and proper pressure \( P \), then the matter energy-momentum tensor will be written as

\[
\tau_{\mu}^{\nu} = (\sigma u_{\mu}^{\nu} + P( u_{\mu}^{\nu} / c^2 - \delta_{\mu}^{\nu})) = \{(\sigma + P/c^2) u_{\mu}^{\nu} - P \delta_{\mu}^{\nu}\}
\]

The field equations of the new bi-metric theory of General Relativity of gravitation given by (4) are

\[
\Box \phi_{\mu}^{\nu} = (\phi_{\mu}^{\nu}) \mid_{\alpha} \mid_{\alpha} = \{(\sigma + P/c^2) u_{\mu}^{\nu} - P \delta_{\mu}^{\nu}\}
\]

Solutions to (4) are also solutions to (5)

\[
-1/2 G_{\mu}^{\nu} = \{(\sigma + P/c^2) u_{\mu}^{\nu} - P \delta_{\mu}^{\nu}\} + t_{\mu}^{\nu}
\]
where the gravitational stress energy tensor $t_{\mu}^{\nu}$ is given by

$$t_{\mu}^{\nu} = [S_{\mu}^{\nu} - u_{\mu}^{\nu}]$$  \hspace{1cm} (12-c)

In this context we now substitute (11) into the Bianchi Identity (9-a) and project out the components parallel and perpendicular to the four velocity $u^{\mu}$ to get parallel to $u^{\mu}$

$$[(\sigma + P/c^2)u^{\nu}]_{;\nu} + u^{\mu} t_{\mu}^{\nu} = 0$$  \hspace{1cm} (13-a)

perpendicular to $u^{\mu}$

$$(\sigma + P/c^2) u^{\nu} u^{\mu} = P_{;\mu} - (\delta^{\alpha}_{\mu} - u^{\alpha}u^{\mu}/c^2) t^{\alpha}_{\nu}$$  \hspace{1cm} (13-b)

Since $u^{\mu}u^{\mu} = c^2$ we solve (13-a) for $t_{\mu}^{\nu}$ and using (13-b) we have that

$$t_{\mu}^{\nu} = -u^{\mu} [(\sigma + P/c^2)u^{\nu}]_{;\nu}$$  \hspace{1cm} (14-a)

which implies that $t_{\mu}^{\nu}$ is parallel to the four velocity $u^{\mu}$. From this fact it follows that $(\delta^{\alpha}_{\mu} - u^{\alpha}u^{\mu}/c^2) t^{\alpha}_{\nu} = 0$ and (13-b) becomes

$$(\sigma + P/c^2) u^{\nu} u^{\mu} = P_{;\mu}$$  \hspace{1cm} (14-b)

Using (14-a) we see that (14-b) has the same form as that of the standard Euler equation of motion for matter in the presence of gravitational and non-gravitational pressure forces. Because of this the presence of $t_{\mu}^{\nu}$ does not generate a Nordtvedt effect in the matter equations of motion in the new bi-metric General Relativity.

However the Bianchi-Freud Identity (9-a) implies the local free fall conservation law $[\tau_{\mu}^{\nu} + t_{\mu}^{\nu}]_{,\nu} = 0$ so that in local free fall the sum of the non-gravitational energy momentum tensor and the gravitational energy momentum tensor is conserved.
In a general coordinate frame of reference the bi-metric covariant D'Alembertian operator acting on the gravitational potential tensor $\phi^\nu_\mu$ field equation in (12-a) is given by

$$\Box \phi^\nu_\mu = (\phi^\nu_\mu | \alpha | \alpha) = g^{\alpha\beta} (\chi^\nu_\mu | \beta)$$

$$= g^{\alpha\beta} \{ [\chi^\nu_\mu | \beta] + \Sigma^\nu_\rho \chi^\rho_\mu \alpha - \Sigma^\rho_\mu \chi^\nu_\rho \alpha - \Sigma^\rho_\alpha \chi^\nu_\rho \}$$

(15-a)

where

$$\chi^\nu_\mu | \alpha = (\phi^\nu_\mu | \alpha) = \{ \phi^\nu_\mu . \alpha + \Sigma^\nu_\rho \phi^\rho_\mu \alpha - \Sigma^\rho_\mu \phi^\nu_\rho \alpha \}$$

(15-b)

In the above expression for $\Box \phi^\nu_\mu$ we assume that the theory is initially formulated in a physical Inertial Cartesian Minkowski (ICM) bi-metric spacetime $x^\alpha_{(ICM)}$, which contains the physical curved spacetime metric written in terms of the ICM coordinates $g_{\mu\nu}^{(ICM)}$ and the inertial flat background spacetime $\eta_{\mu\nu}^{(ICM)}$ where $\Sigma^\lambda_{\mu\nu}^{(ICM)} = 0$. Next we make a transformation to a general spacetime. In the bi-metric context coordinate transformations act on both the $g_{\mu\nu}^{(ICM)}$ and $\eta_{\mu\nu}^{(ICM)}$ hence

$$g_{\mu\nu} = (\partial x^\rho_{(ICM)} / \partial x^\mu) (\partial x^\sigma_{(ICM)} / \partial x^\nu) g_{\rho\sigma}^{(ICM)}$$

(15-d)

$$\gamma_{\mu\nu} = (\partial x^\rho_{(ICM)} / \partial x^\mu) (\partial x^\sigma_{(ICM)} / \partial x^\nu) \eta_{\rho\sigma}^{(ICM)}$$

(15-e)

from which $\Sigma^\lambda_{\mu\nu}$ is given by

$$\Sigma^\lambda_{\mu\nu} = (\partial^2 x^\sigma_{(ICM)} / \partial x^\mu \partial x^\nu) (\partial x^\lambda / \partial x^\sigma_{(ICM)})$$

(15-f)
The fact that the new bi-metric theory of General Relativity can be covariantly formulated in globally flat Inertial Cartesian Minkowski (ICM) background where non-Cartesian and non-inertial effects vanish from the theory is an explicit manifestation of the existence of Mach's Principle in the bi-metric spacetime in which it is formulated. Note that ICM coordinates represent a covariant Cartesian definition of a flat relativistic inertial frame as: \( x^{\mu} = x^{\mu}_{ICM} \), \( R_{\mu\nu\alpha\beta} = 0 \), \( \gamma_{\mu\nu} = \eta_{\mu\nu} \), \( \Sigma^{\lambda}_{\mu\nu} = 0 \). In this context time independent flat coordinate transformations (for which \( \Sigma^{\lambda}_{\mu\nu} \neq 0 \) is time independent) from (ICM) spacetime to Inertial Spherical Minkowski (ISphM) or Inertial Cylindrical Minkowski (ICylM) spacetime coordinates represents a spatially spherical or spatially cylindrical definition of a relativistic inertial spacetime frame.

The field equations (4) and the Euler equations of motion

\[
(\sigma + P/c^2) \ u^\nu u^\mu;_\nu = (\sigma + P/c^2) \ du^\mu/ds = P^\mu = g^{\mu\nu} P_{;\nu}
\]

are to be solved simultaneously for gravitational potential tensor \( \phi_{\mu,\nu} \) solutions. The form of the gravitational stress energy tensor given by \( t_{\mu,\nu} = [S_{\mu,\nu} - u_{\mu,\nu}] \) implies that the modified Einstein equation

\[
-1/2 \ G_{\mu,\nu} = [ (\sigma + P/c^2) \ u_{\mu} u_{\nu} - P \ \delta_{\mu,\nu} ] + t_{\mu,\nu}
\]

satisfies the Bianchi and Freud Identities in the form

\[
[(-\kappa)^{-1/2} \ (\tau_{\mu,\nu} + S_{\mu,\nu})]_{\nu} = 0
\]

Hence in ICM coordinates (18) implies that the total mass of the system is equal to

\[
M = P_0 / c^2
\]
where the total conserved energy-momentum $P_{\mu}$ is given by

$$P_{\mu} = \int dx^3 (-g)^{-1/2} \left\{ \{ \sigma u_{\mu}u^0 + P (u_{\mu}u^0 / c^2 - \delta_{\mu}^0) \} + S_{\mu}^0 \right\}$$  \hspace{1cm} (20-a)

It is interesting to note that to lowest order in the static Newtonian limit (19) can be written as

$$M = \int dx^3 (-g)^{-1/2} \sigma = \int dx^3 (1 + 2\phi^0_0) \sigma = \int dx^3 (\sigma + t^0_0 / c^2) = \int dx^3 (\rho)$$  \hspace{1cm} (20-b)

which implies that the effective mass density $\rho$ contains a gravitational mass energy contribution $t^0_0 / c^2$ in addition to the baryonic mass density $\sigma$ and is given by $\rho = (\sigma + t^0_0 / c^2)$

Let us now assume that we are in the physical frame of reference associated with the ICM spacetime frame of special relativity where $\gamma_{\mu}^\nu = \eta_{\mu}^\nu$. In order to exhibit the fact that the new gravitational field equations have no event horizons in their solutions let us consider the nonlinear, static field, low velocity limit where $v \ll c$ let us consider the case of proper matter density $\sigma$ with no proper pressure $P=0$. Then in this static limit $\tau_{\mu}^\nu \longrightarrow \tau^0_0 = \tau$ and $\phi_{\mu}^\nu \longrightarrow \phi^0_0 = \phi$ and the exponential metric field $g_{\mu\nu}$ defined in equations (1) through (4) is diagonal and is

$$g_{\mu\nu} = \text{diag} \left( \exp[-2\phi], -\exp[2\phi] \right)$$  \hspace{1cm} (21-a)

$$(-g)^{1/2} = \exp[2\text{Tr}(\phi)]$$  \hspace{1cm} (21-b)

$$g^{kj} = \left[ (-g)^{1/2} \right]^{kj} = \exp[2\phi_{\mu}^k \eta_{\mu\nu} \exp[2\phi_{\mu}^l]] = \delta_{\mu}^k \eta_{\mu\nu} \delta_{\mu}^j = \eta^{kj}$$  \hspace{1cm} (21-c)

We also note that in this static case the metric satisfies the Harmonic metric condition

$$g^{\mu\nu, \nu} = 0$$  \hspace{1cm} (21-d)

Then equation (12-1) in the nonlinear, static field, low velocity limit becomes

$$\Box \phi_{\mu}^\nu \left. \biggr|_{\alpha} \right|_{\alpha} = [g^{\alpha\beta}] \left( \partial_\alpha \partial_\beta \phi_{\mu}^\nu \right) = [\eta^{kj} / (-g)^{1/2}] \left( \partial_k \partial_j \phi_{\mu}^\nu \right) = \tau_{\mu}^\nu$$  \hspace{1cm} (22a)
which simplifies to become (now using cgs units)

\[- \nabla^2 \phi = (4\pi G / c^4) (-g)^{1/2} \tau\]  \hspace{1cm} (22-b)

Now in (22-b) if \( \sigma \) contains a superposition of \( N \)-localized neutral mass distributions such that

\[\begin{align*}
[(-g)^{1/2} \tau] &= \left[ (-g)^{1/2} \sigma \right] = \Sigma_{1,2,\ldots,N} [(-g)^{1/2} \sigma(N)] \\
&= \Sigma_{1,2,\ldots,N} [m(N)c^2 \delta^3(x - X(N))] \tag{22-c}
\end{align*}\]

Now assuming that the origin of the observer coordinates is at spatial infinity, so that \( \phi_0^0 = \phi \) approaches zero at spatial infinity, then the exterior \( N \)-body solutions to (22-b) are given by (now using cgs units)

\[\phi(N) = \left[ G \frac{m(N)}{c^2} \right] / |x - X(N)| \tag{22-d}\]

From (2.2-d we have that in the nonlinear, static field, low velocity limit the line element given by \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu \) maintains its exponential form in terms of \( \phi(N) \) and takes on an \( N \)-body form which has no event horizons given by

\[ds^2 = \exp \left[ - \Sigma_{1,2,\ldots,N} \left( 2G \frac{m(N)}{c^2} \right) / |x - X(N)| \right] c^2 dt^2 - \exp \left[ \Sigma_{1,2,\ldots,N} \left( 2G \frac{m(N)}{c^2} \right) / |x - X(N)| \right] (dx^2 + dy^2 + dz^2) \tag{24-a}\]

(Note that this exponential metric embodies a form of Mach's principle since in the vicinity of one particle, the terms due to other more distant masses can be lumped and used to rescale the local measures of distance, time and mass.)

Then for \( N > 1 \) the covariant geodesic equations (14-b) in the nonlinear, static field, low velocity limit, have an \( N \)-body interactive form (see Appendix K) as

\[\begin{align*}
[\Sigma_{1,2,\ldots,N} \sigma(N)] du^\mu / ds &= \left[ \Sigma_{1,2,\ldots,N} \sigma(N) \right] \left[ - \Sigma_{1,2,\ldots,M} \Gamma^{(N)}_{\alpha\beta} u^\alpha u^\beta \right] \\
\Gamma^{(N)}_{\alpha\beta} &= \Phi^{(N)}_{\alpha\beta} + \Phi^{(N)}_{\beta\alpha} - \frac{1}{2} \left( g^{\alpha\beta} \Phi^{(N)}_{\gamma\mu} + g^{\beta\gamma} \Phi^{(N)}_{\mu\gamma} \right) \tag{24-b}
\end{align*}\]
To better illustrate the lack of event horizons let us apply the equations (24) to the N-body case such that the mass of one of the objects is much larger that of the others. This is like the case of the solar system where the mass of the sun dominates the mass of the other planets. The low velocity static limit, described for the gravitational spacetime metric dominated by mass $M$, where $\phi^0_0 = GM/r$, $\exp(-2\phi) = \exp(-2GM/r)$, $\exp(2\phi) = \exp(2GM/r)$, implies that

$$g_{\mu\nu} = \exp[-2rG/r] (c\,dt)^2 - \exp[2rG/r] (dx^2 + dy^2 + dz^2)$$

(25)

where $rG = (GM/c^2)$ is the gravitational radius. Since for $r > r_{ms} = 5.24 \, rG$ the static exponential metric (25) is first order equivalent to the Schwarzschild metric of general relativity in Harmonic spatially isotropic coordinates, the exponential metric predicts the same results to that order as the Schwarzschild metric. However for $r$ on the order of $rG$ the exponential metric in the new bi-metric theory of General Relativity is very different from the Schwarzschild metric since the new metric has no Event Horizon. Instead it has an unstable "Photon Circular Orbit" at $r = 2rG$ and is smooth and continuous all the way to $r=0$ with no singularities. Hence in the context of the new bi-metric theory of General Relativity collapsed objects behave like "Red Holes" (Graber [4]) instead of Black Holes. (see Appendix E). Red holes can possess intrinsic magnetic fields that can interact with accretion disks. Robertson [5] has proposed that such magnetic fields may be responsible for the spectral state switches in the x-ray spectra of stellar mass black hole candidates. The existence of magnetic effects of this type would represent observational support for the new bi-metric theory of General Relativity without event horizons. This is because Einstein theory predicts that these BHC compact objects, should be black holes with event horizons and hence should not exhibit soft/hard spectral switching associated with the presence of a magnetic field intrinsic to these compact objects.
III. CONCLUSIONS

A new bi-metric theory of General Relativity is developed which is based on a new physical paradigm which defines the spacetime metric in an exponential manner so as to allow the operational procedure of local spacetime measurements seen by an observer in a general spacetime frame of reference to be defined in a similar manner as that of Special Relativity. We have shown this theory can be consistently formulated in a generally covariant tensor manner in the context of the Relativistic Bi-metric Spacetime.

The new bi-metric formulation of gravitation presented in this paper was constructed by extending the two fundamental rules of observation implied by Special Relativity to include the requirement that the metric of curved spacetime should be defined in an operational manner so that the procedure of local spacetime measurements, as seen by an observer in a general spacetime frame which contains gravitational fields due to spacetime curvature, becomes locally defined in a similar manner as that for the extension of Special Relativity into general flat spacetimes. Hence while an observer in local free fall frame of reference will still locally measure the speed of light as $v = c$, the new theory also implied that an observer in a general frame of reference, which contains both the effects of real gravitation due to spacetime curvature and artificial gravitation due to non-inertial coordinate transformations, will also locally measure the vacuum velocity of light to be $v = c$.

Since the new bi-metric theory of General Relativity can be derived in a generally covariant tensor context this allows it to be observationally tested in a self consistent manner against the Einstein Theory of Gravitation. Lacking event horizons, compact objects described in the new bi-metric theory of gravitation are not subject to the "Black Holes Have No Hair" theorem that applies to the Einstein Theory.
Hence it follows that a unique observational test of the new bi-metric theory of General Relativity involves the prediction that collapsed objects described in this theoretical context would be expected to possess magnetic dipole moment fields.

Robertson and Leiter [5] have suggested that the spectral state switches displayed by low mass x-ray binaries, including the stellar mass galactic black hole candidates, may be caused by a magnetic propeller effect generated by magnetic dipole fields intrinsic to the compact objects. In addition, the intrinsic magnetic fields of such collapsed objects could have a dominant role in the jet mass ejections which are seen to emerge from the black hole candidates. If confirmed, the existence of intrinsic magnetic dipole fields emerging from such compact objects would represent a strong observational test of the new bi-metric theory of General Relativity versus the Einstein theory of gravitation.

[*] The field equations (4) subject to the constraint relation (2) are form invariant under a class of local (gravitational potential gauge plus coordinate scale) transformations given by

\[ \phi'_{\mu} = \phi_{\mu} + \lambda \delta_{\mu} \]

\[ dx'_{\mu} = \exp(2\lambda) dx_{\mu} \]

where \( \lambda(x) \) is an arbitrary scalar function of space and time obeying \( \Box \lambda = 0 \)

REFERENCES


APPENDIX A: THE RELATIVISTIC BI-METRIC DESCRIPTION OF CURVED SPACETIME

In order to operationally define the dynamic effects of gravitational energy momentum in curved spacetime, independent of the flat geometric and inertial properties of the curved spacetime within which the effects of gravitation act, we assume (Rosen (1963) [1]) that spacetime is fundamentally bi-metric in nature.

This means that there exists at every point of spacetime a flat relativistic background metric tensor $\gamma_{\mu\nu}$ for which the Riemann-Christoffel curvature tensor $\Gamma_{\mu\nu\alpha\beta}$ is assumed to vanish everywhere identically. The relativistic flat background metric tensor $\gamma_{\mu\nu}$ exists in addition to that of the curved metric tensor $g_{\mu\nu}$ for which the Riemann-Christoffel curvature tensor $R_{\mu\nu\alpha\beta}$ is assumed to be non-zero. The physical metric interval $ds^2$ and relativistic background metric interval $d\sigma^2$ in spacetime are given by

$$ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu \, d\sigma^2 = \gamma_{\mu\nu} \, dx^\mu \, dx^\nu$$

(A.1-a)

Since the $\gamma_{\mu\nu}$ describes flat background spacetime, the most general change that $\gamma_{\mu\nu}$ can undergo corresponds to an arbitrary flat spacetime coordinate transformation and which involves four arbitrary tensor functions. Hence in this context it follows that without loss of generality that we are allowed to impose four additional tensor conditions on the metrics $g_{\mu\nu}$ and the $\gamma_{\mu\nu}$ in order to fix the relationship between them. However since these four additional metric connection conditions are tensor conditions they are more general than the usual they do not single out a specific coordinate system.
In the absence of real gravitational forces due to spacetime curvature \( g_{\mu\nu} \) is equal to \( \gamma_{\mu\nu} \). In the presence of real gravitational forces due to spacetime curvature \( g_{\mu\nu} \) is not equal to \( \gamma_{\mu\nu} \). In this context, the difference between the physical metric \( g_{\mu\nu} \) and its first and second derivatives from the flat background spacetime metric \( \gamma_{\mu\nu} \) and its first and second derivatives represents operational data by which the geometry of a given curved spacetime with gravitation may be compared with that of the geometry of the spacetime one would have if the gravitational field were removed.

In this context the initial choice of the flat background spacetime metric \( \gamma_{\mu\nu} \) must be made on the basis of physical considerations. This can be accomplished by using the fact that in the absence of real gravitational forces due to spacetime curvature the Cartesian coordinate frame of Special Relativity represents the global definition of "inertial frame" for which the non-gravitational laws of physics are covariant under the constant velocity Lorentz transformations associated with the Minkowski spacetime metric \( \eta_{\mu\nu} \).

Specifically this means that in the absence of real gravitational forces due to spacetime curvature, physical spacetime must be the inertial Cartesian Minkowski spacetime of Special Relativity. In this case this implies that \( \gamma_{\mu\nu} = \eta_{\mu\nu}, \ g_{\mu\nu} = \eta_{\mu\nu} \) and

\[
\begin{align*}
 ds^2 &= \eta_{\mu\nu} \, dx^\mu \, dx^\nu \\
\sigma^2 &= \eta_{\mu\nu} \, dx^\mu \, dx^\nu
\end{align*}
\tag{A.2-a}
\]

In the presence of real gravitation due to curvature, in the physical inertial Cartesian Minkowski spacetime of Special Relativity, \( g_{\mu\nu} \) is different from \( \eta_{\mu\nu} \) and

\[
\begin{align*}
 ds^2 &= g_{\mu\nu} \, dx^\mu \, dx^\nu \\
\sigma^2 &= \eta_{\mu\nu} \, dx^\mu \, dx^\nu
\end{align*}
\tag{A.2-b}
\]
However when we make general coordinate transformations $dx' \mu = (\partial' x' / \partial x^\nu) dx' ^\nu$ from the physical inertial Cartesian Minkowski spacetime of Special Relativity $dx^\mu$ to a non-inertial and or non-cartesian spacetime $d x' ^\mu$ these coordinate transformations act on both the physical spacetime (g-metric) and the inertial flat background spacetime ($\eta$-metric) together giving

$$g'_{\mu \nu} = (\partial' x' / \partial x^\rho) (\partial x^\sigma / \partial x' ^\nu) g_{\rho \sigma} \quad \gamma_{\mu \nu} = (\partial x^\rho / \partial x' ^\mu) (\partial x^\sigma / \partial x' ^\nu) \eta_{\rho \sigma} \quad (A.3-a)$$

$$ds'^2 = g'_{\mu \nu} dx' ^\mu dx' ^\nu \quad ds'^2 = \gamma_{\mu \nu} dx' ^\mu dx' ^\nu \quad (A.3-b)$$

Note that the inertial background spacetime "$\eta$-metric" transforms in the same relativistic tensor manner as that of the "g-metric" under general coordinate transformations. This choice of Cartesian Minkowski spacetime of Special Relativity as the global definition of inertial frame causes the invariant scalar quantity $(-\kappa)^{1/2} = (-g)^{1/2} / (-\gamma)^{1/2}$ to have the value

$$(-\kappa')^{1/2} = (-g')^{1/2} / (-\gamma')^{1/2} = (-\kappa)^{1/2} = (-g)^{1/2} / (-\eta)^{1/2} = (-g)^{1/2} \quad (A.4)$$

Now in the usual manner associated with the g-metric covariant differentiation (denoted by ";'") there will be a "non-tensor g- Christoffel 3-index symbol" $\Gamma^\lambda_{\mu \nu}$ associated with $g_{\mu \nu} \; ; \; \lambda = 0$ given by

$$\Gamma^\lambda_{\mu \nu} = \frac{1}{2} g^{\lambda \alpha} (\partial_\nu g_{\mu \alpha} + \partial_\mu g_{\nu \alpha} - \partial_\alpha g_{\mu \nu}) \quad (A.5)$$

and in a similar manner associated with the $\gamma$-metric covariant differentiation (denoted by "'|") there will be a "non-tensor $\gamma$- Christoffel 3-index symbol" $Z^\lambda_{\mu \nu}$ associated with $\gamma_{\mu \nu} \; | \; \lambda = 0$ given by

$$Z^\lambda_{\mu \nu} = \frac{1}{2} \gamma^{\lambda \alpha} (\partial_\nu \gamma_{\mu \alpha} + \partial_\mu \gamma_{\nu \alpha} - \partial_\alpha \gamma_{\mu \nu}) \quad (A.6)$$
Now under general spacetime co-ordinate transformations denoted by

\[ dx'{}^\mu = \left( \frac{\partial x'}{\partial x^\mu} \right) dx^v \]  
(A.7-a)

both non-tensor Christoffel-3-index symbols \( \Gamma_{\mu \nu}^{\lambda} \) and \( Z_{\mu \nu}^{\lambda} \) transform in the same non-covariant non-tensor like coordinate frame dependent manner as

\[
\Gamma'_{\mu \nu}^{\lambda} = \left( \frac{\partial x^\rho}{\partial x'\mu} \right) \left( \frac{\partial x^\tau}{\partial x'\nu} \right) \left( \frac{\partial x^\lambda}{\partial x^\rho} \right) \Gamma_{\sigma \tau}^{\rho} + \left( \frac{\partial^2 x^\sigma}{\partial x'\mu \partial x'\nu} \right) \left( \frac{\partial x^\lambda}{\partial x^\sigma} \right) 
\]  
(A.7-b)

\[
Z'_{\mu \nu}^{\lambda} = \left( \frac{\partial x^\sigma}{\partial x'\mu} \right) \left( \frac{\partial x^\tau}{\partial x'\nu} \right) \left( \frac{\partial x^\lambda}{\partial x^\rho} \right) Z_{\sigma \tau}^{\rho} + \left( \frac{\partial^2 x^\sigma}{\partial x'\mu \partial x'\nu} \right) \left( \frac{\partial x^\lambda}{\partial x^\sigma} \right) 
\]  
(A.7-c)

However the difference between the two non-tensor Christoffel symbols \( \Delta_{\mu \nu}^{\lambda} \)

\[
\Delta_{\mu \nu}^{\lambda} = ( \Gamma_{\mu \nu}^{\lambda} - Z_{\mu \nu}^{\lambda} ) 
\]  
(A.7-a)

will transform covariantly like a tensor as

\[
\Delta'_{\mu \nu}^{\lambda} = \left( \frac{\partial x^\rho}{\partial x'\mu} \right) \left( \frac{\partial x^\tau}{\partial x'\nu} \right) \left( \frac{\partial x^\lambda}{\partial x^\rho} \right) \Delta_{\sigma \tau}^{\rho} 
\]  
(A.7-b)

Now let the coordinate transformation

\[
\gamma_{\mu \nu} = (\partial x^\rho / \partial x'\mu) \left( \frac{\partial x^\sigma}{\partial x'\nu} \right) \eta_{\rho \sigma} 
\]  
(A.8-a)

\[
g_{\mu \nu} = (\partial x^\rho / \partial x'\mu) \left( \frac{\partial x^\sigma}{\partial x'\nu} \right) g_{\rho \sigma} 
\]

represent a transformation from a curved spacetime, with \( \Gamma_{\mu \nu}^{\lambda} = \Delta_{\mu \nu}^{\lambda} \neq 0 \) (associated with a flat background Minkowski spacetime where \( Z_{\mu \nu}^{\lambda} = 0 \)) to another curved spacetime with \( \Gamma_{\mu \nu}^{\lambda} \neq 0 \) (associated with a general flat background spacetime where \( Z_{\mu \nu}^{\lambda} \neq 0 \)). Then it follows from the non-tensor transformations (A.7-b and c) that

\[
\Gamma'_{\mu \nu}^{\lambda} = \left( \frac{\partial x^\sigma}{\partial x'\mu} \right) \left( \frac{\partial x^\tau}{\partial x'\nu} \right) \left( \frac{\partial x^\lambda}{\partial x^\rho} \right) \Delta_{\sigma \tau}^{\rho} + Z'_{\mu \nu}^{\lambda} 
\]  
(A.8-b)

where \( Z'_{\mu \nu}^{\lambda} \) is a coordinate frame dependent non-tensor quantity given by

\[
Z'_{\mu \nu}^{\lambda} = 1/2 \gamma_{\mu \alpha} \left( \partial_{\nu \gamma} \gamma'_{\mu \alpha} + \partial_{\mu \gamma} \gamma'_{\nu \alpha} - \partial_{\alpha} \gamma'_{\mu \nu} \right) = \left( \frac{\partial^2 x^\sigma}{\partial x'\mu \partial x'\nu} \right) \left( \frac{\partial x^\lambda}{\partial x^\sigma} \right) 
\]  
(A.8-c)
(This relationship implies that the primed $\Gamma'_\lambda{}^{\mu\nu}$, $\Delta'_\lambda{}^{\mu\nu}$, and $\Gamma'_\lambda{}^{\mu\nu}$ indices can be raised and lowered by the $g'$-metric in the general background case since the unprimed $\Gamma_\lambda{}^{\mu\nu}$ indices are raised and lowered by the $g$-metric in the Minkowski background case).

Hence in general from (A.7) and (A.8) we see that the non-tensor "$g$-Christoffel" 3-index symbol $\Gamma'_\mu{}^{\nu\alpha}$ which represents the total gravitational force can be formally written as

$$\Gamma'_\mu{}^{\nu\alpha} = \Delta'_\mu{}^{\nu\alpha} + \Xi'_\mu{}^{\nu\alpha} \quad (A.9)$$

Hence equation (A.9) represents an explicit breakup of the total gravitational geometry and force carried by the non-tensor Christoffel symbol $\Gamma'_\mu{}^{\nu\alpha}$ into an "actual tensor part" $\Delta'_\mu{}^{\nu\alpha}$ and a "fictitious coordinate frame dependent non-covariant, non-tensor part" $\Xi'_\mu{}^{\nu\alpha}$ which are physically interpreted as follows:

a) the actual gravitational geometry-force carried by the covariant Christoffel tensor $\Delta'_\mu{}^{\nu\alpha}$ due to the presence of actual mass-energy distributions, which generate actual curvature in the spacetime.

b) the fictitious coordinate frame dependent non-tensor gravitational geometry-force carried by the non-covariant Christoffel non-tensor $\Xi'_\mu{}^{\nu\alpha}$ due to general co-ordinate transformations (which do not generate spacetime curvature) from the flat Spacetime of Special Relativity to general flat non-Cartesian, non-inertial spacetimes.

Hence the total non-tensor gravitational geometry and force carried by the Christoffel non-tensor $\Gamma'_\mu{}^{\nu\alpha}$ is the physical sum of the covariant actual tensor gravitational geometry and force $\Delta'_\mu{}^{\nu\alpha}$ and the non-covariant fictitious non-tensor gravitational geometry and force $\Xi'_\mu{}^{\nu\alpha}$.
This result can be thought of as a manifestation of the Weak Principle of Equivalence.
Since the Riemann Christoffel curvature tensor vanishes in general for the arbitrary coordinate transformations (which convert $\eta_{\mu\nu}$ from its inertial Cartesian Minkowski metric form into a non-Cartesian, non-inertial $\gamma_{\mu\nu}$ metric form) it follows that covariant differentiation with respect to the background metric (which we will now call by the name $\gamma$-differentiation and denote by the subscript bar-symbol " | ") can be defined in the same manner as covariant differentiation with respect to the gravitational metric (which we now call $g$-differentiation and denote by the subscript semicolon symbol "; "). However we note that the vanishing curvature in flat $\gamma$-spacetime means that one can interchange the order of "$\gamma$-differentiation" so that it obeys all the rules of ordinary differentiation except that the covariant "$\gamma$-derivative" of the $\gamma_{\mu\nu}$ must vanish.

Dropping the primed notation in what follows and for simplicity now using the shorthand notation $\partial_\mu = ","$ we now have from the vanishing of the $g$-covariant derivative $g_{\mu\nu} ; \lambda$ and equation (A.9) that

\[
g_{\mu\nu} ; \lambda = (g_{\mu\nu} , \lambda - \Gamma^\alpha_{\mu\lambda} g_{\alpha\nu} - \Gamma^\alpha_{\nu\lambda} g_{\alpha\mu} ) = 0 \quad (A.10-a)
\]

\[
[(g_{\mu\nu} , \lambda - Z^\alpha_{\mu\lambda} g_{\alpha\nu} - Z^\alpha_{\nu\lambda} g_{\alpha\mu} ) - \Delta^\alpha_{\mu\lambda} g_{\alpha\nu} - \Delta^\alpha_{\nu\lambda} g_{\alpha\mu} ] = 0 \quad (A.10-b)
\]

Now using the definition of covariant $\gamma$-differentiation of $g_{\mu\nu}$ given by

\[
g_{\mu\nu} | \lambda = g_{\mu\nu} , \lambda - Z^\alpha_{\mu\lambda} g_{\alpha\nu} - Z^\alpha_{\nu\lambda} g_{\alpha\mu} \quad (A.11)
\]

in (A.10) we can write the vanishing of $g_{\mu\nu} ; \lambda$ in the following form

\[
g_{\mu\nu} ; \lambda = [(g_{\mu\nu} | \lambda ) - \Delta^\alpha_{\mu\lambda} g_{\alpha\nu} - \Delta^\alpha_{\nu\lambda} g_{\alpha\mu} ] = 0 \quad (A.12)
\]
It then follows directly from the cyclic properties of the indices in (A.12) that the Christoffel 3-index tensor \( \Delta^\lambda_{\mu \nu} \) defined in (A.9) (which is associated with the geometrical effects of pure gravitation acting in the spacetime independent of the geometrical effects of the background metric) can be written as

\[
\Delta^\lambda_{\mu \nu} = \frac{1}{2} \, g^\lambda_{\alpha} \left( \frac{\partial g_{\mu \alpha}}{\partial \nu} + \frac{\partial g_{\nu \alpha}}{\partial \mu} - \frac{\partial g_{\mu \nu}}{\partial \alpha} \right) \quad (A.13)
\]

Hence from (A.11) it follows that the non-tensor \( g_{\mu \alpha, \nu} \) breaks up into two parts as

\[
g_{\mu \alpha, \nu} = \left( g_{\mu \alpha} | \nu + z_{\mu \alpha \nu} \right) \quad (A.14-a)
\]

where

\[
z_{\mu \alpha \nu} = (Z_{\mu \lambda}^\alpha g_{\alpha \nu} + Z_{\nu \lambda}^\alpha g_{\alpha \mu}) \quad (A.14-b)
\]

Equation (A.14) represents a simple expression of the explicit physical breakup of the total gravitational geometry-force carried by the non-tensor spacetime derivative of the metric \( g_{\mu \alpha, \nu} \) into an actual tensor part \( g_{\mu \alpha} | \nu \) and a fictitious non-tensor coordinate frame dependent part given by \( z_{\mu \alpha \nu} \).

Substituting (A.9) into the Riemann-Christoffel curvature tensor \( R_{\mu \nu} \) and using (A.5) we have (since \( r_{\mu \nu \alpha \beta} = 0 \)) that

\[
R_{\mu \nu} = \left( - \Delta^\alpha_{\mu \nu} | \alpha + \Delta^\alpha_{\alpha \mu} | \nu - \Delta^\alpha_{\alpha \beta} \Delta^\beta_{\mu \nu} + \Delta^\alpha_{\beta \mu} \Delta^\beta_{\alpha \nu} \right) \quad (A.15)
\]

This is the curvature tensor \( R_{\mu \nu} \) associated with the curvature effects of pure gravitation acting in the spacetime independent of the geometrical effects of the background metric. It has no \( \gamma \) - metric coordinate transformation "z-dependence" in it since the \( \gamma \) - metric has vanishing curvature.
In summary we have shown that using the flat background description of spacetime automatically decomposes the ordinary spacetime non-tensors $\Gamma^\mu_{\nu\alpha}$ and $g_{\mu\alpha, \nu}$ respectively, into pure tensor parts $\Lambda^\mu_{\nu\alpha}$ and $g_{\mu\alpha} |_\nu$ and non-covariant, non-tensor coordinate frame dependent terms $Z^\lambda_{\mu\nu}$ and $z_{\mu\alpha\nu}$ as

$$\Gamma^\mu_{\nu\alpha} = (\Lambda^\mu_{\nu\alpha} + Z^\mu_{\nu\alpha})$$  \hspace{1cm} (A.16-a)

where

$$Z^\lambda_{\mu\nu} = \frac{1}{2} \gamma_{\lambda\alpha} (\gamma_{\mu\alpha, \nu} + \gamma_{\nu\alpha, \mu} - \gamma_{\mu\nu, \alpha})$$  \hspace{1cm} (A.16-b)

and

$$g_{\mu\alpha, \nu} = (g_{\mu\alpha} |_\nu + z_{\mu\alpha\nu})$$  \hspace{1cm} (A.17-b)

where

$$z_{\mu\alpha\nu} = (Z^\alpha_{\mu\lambda} g_{\alpha\nu} + Z^\alpha_{\nu\lambda} g_{\alpha\mu})$$  \hspace{1cm} (A.17-b)

and $\gamma^{\mu\nu}$ is the metric of the flat background spacetime upon which the curvature associated with the real gravitational effects has been imposed. Since the covariant derivative $A_{\mu\nu}; \lambda$ of a tensor $A_{\mu\nu}$ remains a tensor in both the ordinary spacetime context and in the bi-metric spacetime context and it does not contain a coordinate frame dependent "z" component.

We see this as follows

$$A_{\mu\nu}; \lambda = (A_{\mu\nu}, \lambda - \Gamma^\alpha_{\mu\lambda} A_{\alpha\nu} - \Gamma^\alpha_{\nu\lambda} A_{\alpha\mu})$$

$$= (A_{\mu\nu} | \lambda - \Lambda^\alpha_{\mu\lambda} A_{\alpha\nu} - \Lambda^\alpha_{\nu\lambda} A_{\alpha\mu})$$  \hspace{1cm} (A.18)

where $\Lambda^\mu_{\nu\alpha} = (\Gamma^\mu_{\nu\alpha} - Z^\mu_{\nu\alpha})$.
Hence we see that, starting from the field equations written in the context of tensor g-covariant derivatives, the spacetime tensor quantities appearing in covariant field equations in curved spacetime can be covariantly written in the Relativistic Background Spacetime context using the following procedure: (a) replace ordinary non-tensor differentiation (denoted by the subscript comma " , " ) by tensor flat background γ-differentiation" (denoted by the subscript bar-symbol " | ") and , (b) replace the non-tensor g- Christoffel 3-index symbol $\Gamma^{\mu}_{\nu\alpha}$ by the Christoffel 3-index tensor $\Delta^{\mu}_{\nu\alpha}$ and, (c) replace $(-g)^{1/2} \mathrm{d}^4x$ by $(-\kappa)^{1/2} \mathrm{d}^4x$ (where $(-\kappa)^{1/2} = (-g)^{1/2} / (-\gamma)^{1/2}$ ) in the invariant volume element.

However in this context the non-tensor quantities $A_{\mu\nu,\lambda}$ that are generated by taking the ordinary spacetime derivative of tensor quantities $A_{\mu\nu}$ in curved spacetime can be formally be broken into a tensor part and a non-tensor part as

$$A_{\mu\alpha,\nu} = (A_{\mu\alpha | \nu} + z_{\mu\alpha\nu})$$

(A.19-a)

where the tensor part $A_{\mu\alpha | \nu}$ associated with the background spacetime derivative is

$$A_{\mu\alpha | \nu} = A_{\mu\alpha,\nu} - (Z^{\alpha}_{\mu\lambda} A_{\alpha\nu} + Z^{\alpha}_{\nu\lambda} A_{\alpha\mu})$$

(A.19-b)

and the non-tensor part $z_{\mu\alpha\nu}$ is given by

$$z_{\mu\alpha\nu} = (Z^{\alpha}_{\mu\lambda} A_{\alpha\nu} + Z^{\alpha}_{\nu\lambda} A_{\alpha\mu})$$

(A.19-c)

Hence we can eliminate the coordinate frame dependent " z-" problem from equations in curved spacetime which contain ordinary spacetime derivatives by formulating those equations entirely in terms of flat background spacetime derivatives.

We can do this by first formulating the theory in the context of a physical spacetime associated with a flat Minkowski background where $A_{\mu\alpha,\nu} = A_{\mu\alpha | \nu}$ and then transforming to a general physical spacetime. In other words we can eliminate this problem by considering all ordinary spacetime derivatives to be flat background Minkowski spacetime derivatives before transforming to a more general set of spacetime coordinates.
We conclude that use of flat background covariant $\gamma$-derivatives of the physical spacetime metric tensor $g_{\mu\nu}$ operationally represents a mathematical language in which the physical geometrical effects of gravitation due to spacetime curvature spontaneously emerge independent of the non-covariant, non-tensor components which are generated by flat spacetime coordinate transformations.
APPENDIX B: extending special relativity to general flat coordinate spacetimes and the new bi-metric general relativity

The physical spacetime associated with Special Relativity is the flat Cartesian inertial Minkowski spacetime frame where all coordinate and non-inertial effects vanish. The metric $\eta_{\mu\nu}$ associated this spacetime is a constant tensor which has the same value for all observers independent of their physical location in the spacetime. However Special Relativity also implies that the following two fundamental relativistic rules of observation are valid in the flat, Cartesian, Minkowski spacetime:

a) all physical observations involve local measurements by observers, and
b) all observers measure the local speed of light to be $v=c$.

Using the rules of General Covariance one can extend Special Relativity in the flat, Minkowski spacetime into the realm of flat non-Cartesian, non-inertial spacetime. However in this case the metric becomes a variable tensor function of spacetime whose value depends on the physical location of observers in the spacetime. Hence in order to obtain a consistent extension of Special Relativity into the realm of flat, non-Cartesian, non-inertial spacetime one must formulate the theory so that the two fundamental relativistic rules of observation implied by Special Relativity:

a) all physical observations involve local measurements by observers, and
b) all observers measure the local speed of light to be $v=c$.

still remain valid in the flat non-Cartesian, non-inertial spacetime.

This can be accomplished by taking advantage of the fact that global flat spacetime coordinate transformations contain the inherent freedom to arbitrarily translate the spacetime origin of the observer to spacetime locations which are not necessarily the same as that of the origin of symmetry of the original coordinate transformation. This means that flat spacetime coordinate transformations must be represented in a
functional form where the location of the observer is operationally given as part of the physical definition of the transformation.

Hence we must define the class of general flat spacetime transformations from a Minkowski spacetime frame to a general flat spacetime frame in an observer-dependent form given by

\[ x'^{\mu} = x^{\mu} (x - x_{\text{obs}}) \]  

(B.1)

where: a) the spacetime origin of symmetry of the flat spacetime coordinate transformation is located at the origin of the unprimed frame at \( x = 0 \), b) the spacetime location of the observer in the primed frame is located at the origin \( x'_{\text{obs}}^{\mu}(0) = 0 \), while in the unprimed frame it is at \( x = x_{\text{obs}} \neq 0 \).

Now assuming that the unprimed spacetime frame is the Minkowski spacetime the flat \( \eta_{\mu\nu} \) metric associated with the spacetime of Special Relativity will transform under the general flat space transformation defined in (1) into an observer-dependent flat background metric tensor \( \gamma'_{\mu\nu} (x' - x'_{\text{obs}}) \) where

\[ \gamma'_{\mu\nu} (x' - x'_{\text{obs}}) = (\partial x^{\rho} / \partial x'^{\mu}) (\partial x^{\sigma} / \partial x'^{\nu}) \eta_{\rho\sigma} \]  

(B.2)

and the invariant spacetime interval \( ds^2 = \eta_{\mu\nu} dx^{\mu} dx^{\nu} \) transforms into the observer-dependent form given by

\[ ds'^2 (x' - x'_{\text{obs}}) = \gamma'_{\mu\nu} (x' - x'_{\text{obs}}) dx^{\mu} dx^{\nu} \]  

(B.3)

where since \( (-\gamma)^{1/2} = (-\eta)^{1/2} = 1 \).
It follows that the extension of Special Relativity into the realm of observer-dependent flat spacetime coordinate transformations defined in (1) satisfies the fundamental rules of relativistic observation that: a)) all physical observations involve local measurements by observers, and b) all observers measure the local speed of light to be $v=c$.

To see this we note that the local coordinate speed of light associated with the observer-dependent line element (B.3) at the location of the observer where $x' = x'_{\text{obs}} = 0$ is given by

$$ ds'^2 (0) = \gamma'_{\mu\nu} (0) \, dx'^\mu \, dx'^\nu = 0 $$

(B.4)

Now even if the primed frame may be a non-inertial frame where $\partial'_{\mu} [\gamma'(0)_{\alpha\beta}] \neq 0$, the metric $\gamma'_{\mu\nu} (0)$ has the following properties in the local vicinity of the observer:

a) $\gamma'_{00} (0) = 1$ and,

b) all off-diagonal elements vanish $K \neq J = 1,2,3$, as $\gamma'_{0K} (0) = \gamma'_{K0} (0) = 0$, $\gamma'_{KJ} (0) = \gamma'_{JK} (0) = 0$

(B.5)

Hence (4) becomes

$$ ds'^2 (0) = dx'^0 \, dx'^0 + \Sigma_K \gamma'_{KK} (0) \, dx'^K \, dx'^K = 0 $$

(B.6)

Dividing (B.6) by $dx'^0 = c \, dt'$ and defining $V'^K = dx'^K / dt'$ we find that the observer at $x' = x'_{\text{obs}}$ will locally measure the speed of light to be

$$ V' = [\Sigma_K (\gamma'_{KK} (0) \, v'^K \, v'^K)]^{1/2} = c $$

(B.7)

even though the observer is located in a general spacetime frame of reference.
The new theory of gravitation is based on a new physical paradigm[2] which extends the observer-dependent nature of the Relativity of flat spacetime coordinates to include the requirement that the metric of the curved spacetime should be defined in a observer-dependent manner so that the operational procedure of local spacetime measurements, as seen by an observer in a general spacetime frame which contains gravitational fields due to spacetime curvature, becomes locally defined in a similar manner as that of Special Relativity. In essence the new paradigm requires that an observer at an arbitrary spacetime point \( x_{\text{obs}} \), in a general frame of reference which contains both the effects of real gravitation due to spacetime curvature and artificial gravitation due to non-inertial coordinate transformations, will locally (i.e. at \( x = x_{\text{obs}} \)) measure the vacuum velocity of light to be \( v = c \).

This can be accomplished on the basis of the following paradigm:

\textbf{a)} In addition to the symmetric spacetime metric field \( g_{\mu \nu} = g_{\nu \mu} \), there exists a fundamental symmetric gravitational potential tensor field \( \Phi_{\mu \nu} = \Phi_{\nu \mu} \), which underlies the metric effects of real gravitation due to spacetime curvature, and

\textbf{b)} In the presence of real gravitational fields an observer-dependent exponential metric relationship exists between the \( g_{\mu \nu} \) and the mixed tensor \( \Phi_{\mu}^{\nu} \), which allows the operational procedure of local spacetime measurements to be defined for an observer located at spacetime coordinates \( x = x_{\text{obs}} \) in a similar manner as that of Special Relativity, and for which space and time dilation effects are put on an equal group theoretical footing, given by

\[
g_{\mu \nu}(x, x_{\text{obs}}) = \exp[(\Phi(x) - \Phi(x_{\text{obs}}))_{\mu}^{\alpha} \gamma_{\alpha\beta}(x-x_{\text{obs}}) \exp[(\Phi^{T}(x) - \Phi^{T}(x_{\text{obs}}))_{\nu}^{B}] (B.8)
\]

since by definition \( \Phi(x_{\text{obs}})_{\mu}^{\nu} = 0 \) at the coordinates of the observer which are not necessarily the same as the origin of the spacetime (i.e. \( x_{\text{obs}} \neq 0 \)).
Note in the above that:

1) \( g_{\mu\nu}(x, x_{obs}) \) depends only on differences in the real gravitational potential tensor field \( \Phi(x)_{\mu}^{\nu} \) and \( \Phi(x_{obs})_{\mu}^{\nu} = 0 \), where \( \Phi(x)_{\mu}^{\alpha} = [\phi(x)_{\alpha}^{\alpha} \delta_{\mu}^{\nu} - 2 \phi(x)_{\mu}^{\nu}] \).

2) \( \gamma_{\alpha\beta}(x-x_{obs}) \) is the observer dependent metric of the flat background inertial spacetime frame of Special Relativity.

3) the asymptotic boundary conditions on \( \Phi(x)_{\mu}^{\alpha} \) are determined by the cosmological boundary conditions on \( g_{\mu\nu}(x, x_{obs}) \) at spatial infinity which are seen by the observer.

On the basis of the above we can distinguish the basic difference between the new bi-metric theory of gravitation and the Einstein theory of Gravitation namely that:

1) In the new theory of gravitation the existence of an exponential connection between the metric tensor \( g_{\mu\nu} \) and a gravitational potential tensor \( \phi_{\mu}^{\nu} \) allows the Strong Principle of Equivalence to propagate the observer-dependent property inherent in the flat spacetime metric \( \gamma'_{\alpha\beta}(x', x'_{obs}) \) into the structure of the curved spacetime metric \( g'_{\mu\nu}(x', x'_{obs}) \).

2) In the Einstein theory of gravitation the connection between the metric tensor \( g'_{\mu\nu}(x') \) and a gravitational potential tensor does not exist, hence the Strong Principle of Equivalence cannot propagate the observer-dependent property inherent in the flat spacetime metric \( \gamma'_{\alpha\beta}(x', x'_{obs}) \) into the structure of the curved spacetime metric \( g'_{\mu\nu}(x') \).
APPENDIX C: LOCAL OBSERVER IN A GENERAL BI-METRIC SPACETIME FRAME OF REFERENCE

In a general physical spacetime frame the associated background spacetime metric $\gamma_{\mu\nu}$ is different from the Minkowski background spacetime metric $\eta_{\mu\nu}$ hence $\Sigma_{\lambda}^{\mu\nu}$ will in general be nonzero. In this case the contravariant equations of motion for particle of mass $m$ being acted on by gravitational and a non-gravitational force $K^\mu$ in this general spacetime frame are

$$K^\mu / m = du^\mu / ds + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = du^\mu / ds + [\Sigma_{\alpha\beta}^\mu + \Delta_{\alpha\beta}^\mu] u^\alpha u^\beta$$  \hspace{1cm} (C.1)

where

$$\Delta_{\lambda}^{\mu\nu} = 1/2 g^{\lambda\alpha} (g_{\mu\alpha} | \nu + g_{\nu\alpha} | \mu - g_{\mu\nu} | \alpha)$$  \hspace{1cm} (C.2-a)

$$\Sigma_{\lambda}^{\mu\nu} = 1/2 \gamma^{\lambda\alpha} (\gamma_{\mu\alpha,\nu} + \gamma_{\nu\alpha,\mu} - \gamma_{\mu\nu,\alpha})$$  \hspace{1cm} (C.2-b)

The associated covariant geodesic equations of motion have a more compact form given by

$$K^\mu / m = du^\mu / ds = 1/2 \left( \gamma^{\alpha\beta,\mu} + g^{\alpha\beta} | \mu \right) u^\alpha u^\beta$$  \hspace{1cm} (C.3)

In the general spacetime frame the observer-dependent exponential metric is given in the bi-metric spacetime context as

$$g_{\mu\nu}(x, x_0) = \exp\left[ (\Phi(x) - \Phi(x_0))_{\mu}^{\alpha} \right] g_{\alpha\beta} (x - x_0) \exp\left[ (\Phi^T(x) - \Phi^T(x_0))_{\beta}^{\nu} \right]$$  \hspace{1cm} (C.4-a)

where

$$\Phi(x)_\mu^{\nu} = (\phi(x)_{\alpha}^{\nu} \delta_{\mu}^{\alpha} - 2 \phi(x)_\mu^{\nu})$$  \hspace{1cm} (C.4-b)

and the observer-dependent line element is given by

$$ds^2 (x, x_0) = g(x, x_0)_{\mu\nu} dx^\mu dx^\nu$$  \hspace{1cm} (C.4-c)

Note that in equations (4) above:

1. $\Phi(x)_\mu^{\nu}$ is the gravitational potential tensor function, associated with real gravitation due to spacetime curvature, generated by the action of the energy momentum tensor $\tau_{\mu}^{\nu}$,
b) the observer-dependent flat space metric $\gamma(x, x_0)_{\mu\nu}$, which is a general solution to the flat space equation $R_{\mu\nu\alpha\beta} = 0$, appears explicitly in the exponential metric and differential line element.

Now using (C.4) we can calculate $g_{\alpha\beta} | \mu$ and $\Gamma_{\alpha\beta}^{\mu} = \Delta_{\alpha\beta}^{\mu}$ in terms of the quantity $\Phi(x)_{\alpha}^{\mu}$. For example $g(x, x_0)_{\alpha\beta} | \mu$ is given (see the end of Appendix D) by

$$g(x, x_0)_{\alpha\beta} | \mu = 2 \Phi_{\alpha}^{\rho}(x) | \mu \ g(x, x_0)_{\rho\beta}$$  \hspace{1cm} (C.5)

Then the local observer's matter equation of motion is given by the (1) where fictitious and real gravitational forces act in addition to non-gravitational forces as

$$K_{\mu} = du_{\mu} / ds(x, x_0) + \Gamma(x, x_0)_{\alpha\beta}^{\mu} u^\alpha u^\beta$$

$$= du_{\mu} / ds(x, x_0) + [\Sigma(x, x_0)_{\alpha\beta}^{\mu} + \Delta(x, x_0)_{\alpha\beta}^{\mu}] u^\alpha u^\beta$$ \hspace{1cm} (C.1)'

while the local observer's line element is given by

$$ds^2 (x, x_0) = g(x, x_0)_{\mu\nu} \ dx^\mu \ dx^\nu$$ \hspace{1cm} (C.4-c)'

Then it follows that for an observer in the general spacetime located at the spacetime point $x_0$, the line element for a light ray measured nonlocally at the spacetime point $x$ is given by

$$ds^2 (x, x_0) = g(x, x_0)_{\mu\nu} \ dx^\mu \ dx^\nu = 0$$ \hspace{1cm} (C.7-a)

where the metric $g_{\mu\nu}$ will now have off-diagonal elements due to spacetime curvature and non-inertial coordinate dependent effects in the general spacetime frame.
However for an observer in the general spacetime located at the spacetime point \( x_0 \), the line element for a light ray measured locally at the spacetime point \( x_0 \) is
\[
\begin{align*}
\text{ds}^2 (x_0, x_0) &= g(x_0, x_0)_{\mu\nu} \, dx^\mu \, dx^\nu \\
&= \gamma(x_0, x_0)_{\mu\nu} \, dx^\mu \, dx^\nu = 0
\end{align*}
\] (C.7-b)

This implies that in general for a local observer who is in a non-inertial frame at \( x = x_0 \), but is not freely falling since \( [\partial_\mu \gamma(x_0, x_0)_{\alpha\beta}] \neq 0 \) we still have that the non-inertial metric \( \gamma(x_0, x_0)_{\mu\nu} \) has the following properties:
\[
\begin{align*}
\gamma(x_0, x_0)_{00} &= 1, \\
\text{all off-diagonal elements vanishing as:} \\
\gamma(x_0, x_0)_{0K} &= \gamma(x_0, x_0)_{K0} = 0, \; K = 1, 2, 3 \\
\gamma(x_0, x_0)_{KJ} &= \gamma(x_0, x_0)_{JK} = 0, \; K \neq J = 1, 2, 3
\end{align*}
\] (C.7-c)

Hence (7-b) implies that a non-freely falling observer at \( x' = x_{\text{obs}}' \) will locally measure the speed of light to be given by
\[
\begin{align*}
v &= \left[ -\gamma(x_0, x_0)_{KK} v^K v^K \right]^{1/2} = c
\end{align*}
\] (C.8-a)

even though as shown in equation (1) 'the local observer is in a non-inertial frame of reference and is not freely falling since
\[
K^{\mu} / m = du^{\mu} / ds(x_0, x_0) + \Gamma(x_0, x_0)^{\mu}_{\alpha\beta} u^\alpha u^\beta \\
= du^{\mu} / ds(x_0, x_0) + [\Sigma(x_0, x_0)^{\mu}_{\alpha\beta} + \Delta(x_0, x_0)^{\mu}_{\alpha\beta}]u^\alpha u^\beta
\] (C.8-b)

here \( \Sigma(x_0, x_0)^{\mu}_{\alpha\beta} \) generates the fictitious gravitational forces and \( \Delta(x_0, x_0)^{\mu} \) generates the real gravitational forces. Note in addition that we also have, but do not necessarily have to use, the option to locally transform the observer-dependent flat space metric \( \gamma(x_0, x_0)_{\mu\nu} \) into a locally Minkowskian form such that \( \gamma(x_0, x_0)_{\rho\sigma} = \eta_{\mu\nu} \) even though the first derivatives \( \gamma(x, x_0)_{\mu\nu, \lambda} \) will not necessarily vanish at \( x = x_0 \) this general flat spacetime frame.
APPENDIX D: LOCAL OBSERVER IN A FREE-FALL BI-METRIC SPACETIME FRAME OF REFERENCE

Now to define a "local free-fall frame of reference" at the spacetime point \( x = x_0 \) let us consider a local non-linear non-inertial co-ordinate transformation given to lowest order about the general observer spacetime point \( x = x_0 \) defined by

\[
x'{}^\mu = x^\mu - \frac{1}{2} \left( (A^\mu{}_{\alpha\beta}^\prime) (x^\alpha - x_0^\alpha) (x^\beta - x_0^\beta) \right)
\]

\[
dx'{}^\mu = dx^\mu - \frac{1}{2} \left( (A^\mu{}_{\alpha\beta}^\prime) (x^\alpha - x_0^\alpha) \right) dx^\beta
\]

and its inverse transformation given by

\[
x^\mu = x'{}^\mu + \frac{1}{2} \left( (A^\mu{}_{\alpha\beta}) (x^\alpha - x_0^\alpha) \right) (x^\beta - x_0^\beta)
\]

\[
\frac{dx^\mu}{dx'{}^\mu} = \frac{dx'}{dx^\mu} + \frac{1}{2} \left( (A^\mu{}_{\alpha\beta}) (x^\alpha - x_0^\alpha) \right) \frac{dx'}{dx^\beta}
\]

Then at the spacetime point \( x' = x_0 \) of the free fall observer both the gravitational metric \( g_{\mu\nu} \) and the flat background spacetime metric \( \gamma_{\mu\nu} \) will both locally transform as tensors under this local non-inertial transformation to become

\[
g'_{\mu\nu} = (dx'{}^\rho / dx^\mu') (dx'{}^\sigma / dx^\nu') g_{\rho\sigma}
\]

\[
g'_{\mu\nu} = (dx'{}^\rho / dx^\mu') (dx'{}^\sigma / dx^\nu') \gamma_{\rho\sigma}
\]

\[
ds'{}^2 = g'_{\mu\nu} dx'{}^\mu dx'{}^\nu
\]

\[
ds'{}^2 = g'_{\mu\nu} dx'{}^\mu dx'{}^\nu
\]

\[
\Sigma'_{\lambda}{}_{\mu\nu} = \{ (dx'{}^\sigma / dx^\mu') (dx'{}^\tau / dx^\nu') (dx'{}^\lambda' / dx^\rho) \Sigma'_{\rho\sigma\tau} \} + \Delta'_{\lambda}{}_{\mu\nu}
\]

\[
\Delta'_{\lambda}{}_{\mu\nu} = (dx'{}^\sigma / dx^\mu') (dx'{}^\tau / dx^\nu') (dx'{}^\lambda' / dx^\rho) \Delta'_{\rho\sigma\tau}
\]

\[
\Gamma'_{\lambda}{}_{\mu\nu} = \{ (dx'{}^\sigma / dx^\mu') (dx'{}^\tau / dx^\nu') (dx'{}^\lambda' / dx^\rho) \} (\Sigma'_{\rho\sigma\tau} + \Delta'_{\rho\sigma\tau})
\]

\[
= \{ (\Sigma'_{\rho\sigma\tau} + \Delta'_{\rho\sigma\tau}) (dx'{}^\lambda' / dx^\rho) \}
\]

\[
= \{ (\Sigma'_{\rho\sigma\tau} + \Delta'_{\rho\sigma\tau}) (dx'{}^\lambda' / dx^\rho) \}
\]

\[
\Lambda'_{\lambda}{}_{\mu\nu}
\]

\[
= \{ (\Sigma'_{\rho\sigma\tau} + \Delta'_{\rho\sigma\tau}) + \Lambda'_{\lambda}{}_{\mu\nu} \}
\]
The mass of the particle is a scalar so that \( m' = m \), while the non-gravitational force \( K^\mu \) transforms like a tensor as
\[
K'^\mu = (dx'^\mu / dx_\nu) K^\nu
\]  
(D.2-f)
However since (1) are tensor equations then at the spacetime point \( x_0' = x_0 \) they are
\[
K'^\mu / m = du'^\mu / ds(x' 0, x' 0) \Gamma'^\mu_{\nu \alpha}(x' 0, x' 0) u'^\alpha u'^\beta, \]

\[
= \left[ (\Sigma'^\nu_{\nu \alpha}(x' 0, x' 0) + \Delta'^\nu_{\nu \alpha}(x' 0, x' 0)) + A'^\nu_{\nu \alpha} \right] u'^\alpha u'^\beta
\]  
(D.3)
The local free-fall frame of reference is defined by choosing the local non-inertial transformation (9) so that the non-inertial forces, generated by the quantity
\[
\Sigma'^\nu_{\nu \alpha}(x' 0, x' 0) + A'^\nu_{\nu \alpha}
\] at the spacetime point \( x_0' = x_0 \), compensate and cancel out the real gravitational forces \( \Delta'^\nu_{\nu \alpha}(x' 0, x' 0) \) then the total force vanishes since \( \Gamma'^\nu_{\nu \alpha}(x_0, x_0) = 0 \). Then in (D.3)
\[
A'^\mu_{\alpha \beta} = - \left[ \Sigma'^\nu_{\nu \alpha}(x_0, x_0) + \Delta'^\nu_{\nu \alpha}(x_0, x_0) \right]
\]  
(D.4-a)
\[
A'^\mu_{\alpha \beta} = - \left[ \Sigma'^\nu_{\nu \alpha}(x_0, x_0) + \Phi'_{\alpha | \beta}(x_0) + \Phi'_{\beta | \alpha}(x_0)
\right.
\]
\[
- \frac{1}{2}(\Phi'(x_0)_{\lambda | \mu} \gamma'(x_0, x_0)_{\lambda \beta} + \Phi'(x_0)_{\lambda | \mu} \gamma'(x_0, x_0)_{\lambda \alpha})
\]  
(D.4-b)
and for the free fall observer at \( x' = x_0 \)
\[
g'_{\mu \nu}(x_0, x_0) = \gamma(x_0, x_0)_{\mu \nu}
\]
\[
ds'^2(x_0, x_0) = \gamma'(x_0, x_0)_{\mu \nu} dx'^\mu dx'^\nu
\]
\[
\Gamma'^\nu_{\nu \alpha}(x_0, x_0) = 0
\]  
(D.4-c)
Where again we also have, but do not necessarily have to use, the option to locally transform the observer-dependent flat space metric \( \gamma'(x_0, x_0)_{\mu \nu} \) into a locally Minkowskian form such that \( \gamma'(x_0, x_0)_{\rho \sigma} = \eta_{\mu \nu} \) even though the first derivatives \( \gamma'(x_0, x_0)_{\mu \nu}, \lambda \) will not necessarily vanish at \( x = x_0 \) in this general flat spacetime frame.
Under these conditions the local effects of gravitation vanish in equation (D.3) and it becomes locally identical to that of Special Relativity where

\[
K'_{\mu} / m = du'_{\mu} / ds' (x_0, x_0) \tag{D.4-d}
\]

Hence using (D.3) we see that the free fall Special Relativity correspondence limit, in the bi-metric spacetime can be defined at each spacetime point \( x = x_0 \) by a local non-inertial coordinate transformation given in terms of the flat background derivatives of the gravitational potential tensor function \( \Phi_{\mu \nu} = (\delta_{\mu}^{\alpha} \delta_{\nu}^{\beta} - 2 \phi_{\mu}^{\nu}) \) evaluated to first order at the spacetime point \( x'0 = x_0 \) given by

\[
dx'_{\mu} = dx_{\mu} - \left\{ \Sigma '_{\mu \nu} (x_0, x_0) + \Phi_{\alpha \mu} | \beta (x_0) + \Phi_{\beta \mu} | \alpha (x_0) - 1/2 \left( \Phi^{'(x_0)}_{\lambda} | \mu \gamma (x_0, x_0) \lambda_{\mu} + \Phi^{'(x_0)}_{\beta} | \mu \gamma (x_0, x_0) \beta_{\mu} \right) \right\} (x^\nu - x'^\nu) dx^\alpha \tag{D.4-e}
\]

This means that the "\textbf{instantaneous rest Lorentz frame observer}" which Special Relativity uses to define the relativistic particle equations of motion of the form

\[
K'_{\mu} = m (du'_{\mu} / ds' ) \tag{D.5-a}
\]

where

\[
ds'_{2}(x_0, x_0) = \gamma'(x_0, x_0)_{\mu \nu} dx'_{\mu} dx'_{\nu} \tag{D.5-b}
\]

is actually the "\textbf{local freely falling observer}" defined in the context of the bi-metric space-time by the free fall transformation \( (D.4-e) \). Hence this procedure defines the free fall Special Relativity correspondence limit of the new bi-metric General Relativity theory.

Hence we see that the observer-dependent metric measured by the "\textbf{local freely falling instantaneous rest Lorentz frame observer}" given by

\[
ds'_{2}(x_0, x_0) = \gamma'(x_0, x_0)_{\mu \nu} dx'_{\mu} dx'_{\nu} \tag{D.6-a}
\]

is the same as that of the \textbf{local non-inertial rest Frame observer} given by

\[
ds_{2}(x_0, x_0) = \gamma(x_0, x_0)_{\mu \nu} dx_{\mu} dx_{\nu} \tag{D.6-b}
\]
As an elementary example of this consider an elevator in free-fall in the gravitational field of the earth. In the $v << c$ low velocity limit the only non-zero terms which contribute to the gravitational force term $[\Sigma^\mu_{\nu\alpha'} + \Delta^\mu_{\nu\alpha'}]$ in the geodesic equation are the terms associated with a constant acceleration in the $z$-direction given by

$$a^3 = A^3_{00} = \left[ a \big/ c^2 \right] = \Phi^3_{00}(x_0) - \Phi^3_{00}(x_0) - \Phi^3_{00}(x_0)$$

$$cdt' = cdt, \quad dx' = dx, \quad dy' = dy, \quad dz' = \left[ dz - A^3_{00}(ct) cdt \right] = \left[ dz - at \, dt \right]$$

$$\Sigma^3_{00} = \Delta^3_{00} = \frac{a}{c^2} = -\Phi^3_{00}(0) = -g / c^2$$

in the geodesic equations of motion leading to the Special Relativity equation of motion

$$K^\mu = m \frac{du^\mu}{ds'}$$

which again shows how the kinematic force generated by the local non-inertial coordinate transformation to free-fall frame locally compensates the actual gravitational force. Now using a general flat space-time transformation let us transform from the Inertial Cartesian Minkowski coordinates associated with the flat background space-time metric $\eta_{\mu\nu}$, and the curved metric $g_{\mu\nu}$, to a general set of space-time coordinates associated with the flat background space-time metric $g'_{\mu\nu}$ and the curved metric $\gamma'_{\mu\nu}$.

Since $\gamma'_{\mu\nu}$ is different from $\eta_{\mu\nu}$ the flat Christoffel symbols $\Sigma^\lambda_{\mu\nu}$ will in general be nonzero. In this case (now dropping the prime notation) the contravariant equations of motion for proper mass density $\sigma$ being acted on by gravitational and a non-gravitational force $K^\mu$ in this general space-time frame is are

$$K^\mu = \sigma \{ du^\mu / ds + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta \} = \sigma \{ du^\mu / ds + [\Sigma^\mu_{\alpha\beta} + \Delta^\mu_{\alpha\beta}] u^\alpha u^\beta \}$$
which can be also written in the form

\[ \sigma (du^\mu / ds + \Delta^\mu_{\alpha\beta} u^\alpha u^\beta) = (K^\mu - \sigma \Sigma^\mu_{\alpha\beta} u^\alpha u^\beta) \]  \hspace{1cm} (D.8-b)

where

\[ \Sigma^\mu_{\nu \lambda} = 1/2 \gamma^\lambda_{\mu \alpha} (\gamma^\alpha_{\mu \lambda \nu} + \gamma^\alpha_{\nu \lambda \mu} - \gamma^\alpha_{\lambda \alpha \mu \nu}) \]  \hspace{1cm} (D.8-c)

\[ \Delta^\mu_{\nu \lambda} = (\Gamma^\mu_{\nu \lambda} - \Sigma^\mu_{\nu \lambda}) \]

\[ = 1/2 \left[ g^\lambda_{\mu \alpha} (g^\alpha_{\mu \nu} + g^\alpha_{\nu \mu} - g^\alpha_{\mu \nu}) \right] \gamma^\lambda_{\mu \alpha} (\gamma^\alpha_{\mu \lambda \nu} + \gamma^\alpha_{\nu \lambda \mu} - \gamma^\alpha_{\lambda \alpha \mu \nu}) \]

\[ = 1/2 \left[ g^\lambda_{\mu \alpha} (g^\alpha_{\mu \nu} + g^\alpha_{\nu \mu} - g^\alpha_{\mu \nu}) \right] \]

\[ \]  \hspace{1cm} (D.8-d)

and

\[ g^\mu_{\nu \lambda} = g^\mu_{\nu \lambda} + \gamma^\lambda_{\mu \alpha} g^\alpha_{\nu \beta} - \gamma^\lambda_{\nu \alpha} g^\alpha_{\mu \beta} \]  \hspace{1cm} (D.8-e)

In the general bi-metric space-time frame the exponential metric is given by

\[ g^\mu_{\nu \lambda} = \exp(\Phi^\mu_{\nu \lambda}) \exp(\Phi^T_{\lambda \nu}) \]  \hspace{1cm} (D.9-a)

where

\[ \Phi^\mu_{\nu \lambda} = (\phi^\alpha_{\mu \lambda} \delta^\nu_{\alpha \beta} - 2 \phi^\mu_{\nu}) \]  \hspace{1cm} (D.9-b)

and the line element is given by

\[ ds^2 = g^\mu_{\nu \lambda} dx^\mu dx^\nu \]  \hspace{1cm} (D.9-c)

Then inside of the matter equation of motion (1-b) in the general space-time frame

\[ \sigma (du^\mu / ds + \Delta^\mu_{\alpha\beta} u^\alpha u^\beta) = (K^\mu - \sigma \Sigma^\mu_{\alpha\beta} u^\alpha u^\beta) \]  \hspace{1cm} (D.9-b)

where the bi-metric gravitational field equation is given by

\[ \Phi^\mu_{\nu \lambda} = [g^\alpha_{\beta \gamma} \delta_{\alpha \beta}] \mu_{\gamma \nu} = -(8\pi G / c^4) \tau^\mu_{\nu} = -(8\pi G / c^4) \sigma u^\mu u^\nu \]  \hspace{1cm} (D.10-a)

which is the same as

\[ -1/2 G^\mu_{\nu} = (4\pi G / c^4) (\tau^\mu_{\nu} + t^\mu_{\nu}) \]  \hspace{1cm} (D.10-b)

and the Bianchi-Freud Identity is

\[ [\tau^\mu_{\nu} + t^\mu_{\nu}] ; \nu = 0 \]  \hspace{1cm} (D.11-c)
Now equation (D.9-b) will describe a class of local "free-fall" observers if $K^\mu = 0$ and (D.9-b) describes trajectories such that $\Sigma^\mu_{\alpha\beta} = - \Delta^\mu_{\alpha\beta} \neq 0$ at every point on these trajectories. Then the non-inertial gravitational forces associated with $\Sigma^\mu_{\alpha\beta} \neq 0$ cancel out the local real gravitational forces associated with $\Delta^\mu_{\alpha\beta} \neq 0$ so that the local observable force associated with $\Gamma^\mu_{\alpha\beta}$ vanishes locally in these observer's equations of motion as $\Gamma^\mu_{\alpha\beta} = 0$ implying that the class of "free-fall" observer's local equation of motion is given by $\sigma du^\mu / ds = 0$.

However it is also true for these "free-fall" observers that the real gravitational field associated with $\Delta^\mu_{\alpha\beta} \neq 0$ does not locally vanish. This is seen by the fact for these free-fall observers the local bi-metric gravitational field equation is given by

$$\Box \phi^\mu_{\alpha\beta} = \eta_{\alpha\beta} \left( \phi^\mu_{\alpha\beta} \right) |_{\alpha\beta} = - (8\pi G / c^4) \sigma u^\mu u^\nu \quad (D.12-a)$$

where $t^\mu_{\alpha\beta} = (S^\mu_{\alpha\beta} - u^\mu_{\alpha\beta}) \neq 0$ and the local Bianchi-Freud Identity is

$$[\tau^\mu_{\alpha\beta} + t^\mu_{\alpha\beta}],_{\nu} \equiv 0 \quad (D.12-b)$$

If we now assume a weak field approximation in the free fall frame, then in (6-b) we have

$$S^\mu_{\nu} = - 4 \left[ \eta^\nu_{\rho\lambda\beta} (\phi^\rho_{\lambda\alpha}, \phi^\mu_{\alpha\beta}) \right] \quad (D.12-c)$$

$$U^\mu_{\nu} = \left[ W^\mu_{\nu} - 1/2 \delta^\mu_{\nu} \text{Tr}(W) \right] \quad (D.12-d)$$

$$W^\mu_{\nu} = \{ \text{Tr}(\phi^\beta_{\alpha\beta\nu} - 2 (\phi^\beta_{\alpha,\mu} (\phi^\alpha_{\mu\nu}))) + [\eta^\nu_{\rho\lambda\alpha\beta} (\phi^\rho_{\lambda\mu\nu} (\phi^\alpha_{\mu\nu}))] \} \quad (D.12-e)$$
Hence (6-b) implies that if this class of free-fall observers all have trajectories which fall into a common space-time volume $V_4$ then this group of free fall observers all see (6-b) and the free-fall weak field energy balance equation

$$
(1/c) \frac{\partial}{\partial t} \int_{V} dx^3 \left( \tau_{\mu}^0 + t_{\mu}^0 \right) = - \int_{S_V} dS_k \ t_k^k = - \int_{S_V} dS_k \left( S_k^{k} - U_{\mu}^k \right) \quad (D.12-d)
$$

Hence these free fall observers have a total conserved energy-momentum which includes the effects of the gravitational potential energy tensor $t_{\mu}^\nu \neq 0$. In addition when $t_{\mu}^\nu$ contains the flux of gravitational waves of $\Phi_{\mu}^\nu = 0$ within it then, since gravitation waves $t_{\mu}^\nu = -(c^4/8\pi G) G_{\mu}^\nu \neq 0$ are space-time curvature waves (6-d) implies

$$
(1/c) \frac{\partial}{\partial t} \int_{V} dx^3 \left( \tau_{\mu}^0 + t_{\mu}^0 \right) = \int_{S_V} dS_k \left( c^4/8\pi G \right) G_{\mu}^k \quad (D.12-e)
$$

the emission or absorption of these gravitational waves in the context of the induced geodesic deviation $s^\mu$ between elements of this class of free fall observers given by

$$
D^2 s^\mu / D\tau^2 = - R_{\alpha\beta}^\mu \ s^\sigma u^\alpha u^\beta \quad (D.13)
$$

which these gravitational curvature waves would cause the class of free-fall observers to undergo. Hence a class of free-fall observers within a common $V_4$ represent an operational scenario for which the emission or absorption of gravitational radiation can be described (see Appendix I).
We end this appendix by noting that in terms of the matrix notation given by

\[ g = g_{\mu\nu}, \quad \phi = \phi^\nu = (\phi_{\mu\alpha} g^{\alpha\nu}) = \phi^\nu_{\mu}, \quad \phi^T = \phi_{\nu}^{\mu}, \quad g = \exp(-2\phi) \gamma \exp(-2\phi^T) \]  (D.14)

we see that in the local free fall frame (where locally \( g = \eta \) ) the following relationships between \( \gamma, \gamma, \mu \) and \( \phi, \phi_{\mu} \) hold:

\[ \gamma = \exp(2\phi) \eta \exp(2\phi^T), \quad \gamma, \mu = -g_{\mu} \neq 0 \quad \text{(non-diagonal symmetric)}, \]  (D.15)

\[ \phi = \phi_0 \neq 0, \quad \phi_{\mu} \neq 0, \quad (\phi_{\lambda})^T = \gamma^{-1} (\phi_{\lambda}) \gamma \neq 0 \quad \text{(non-diagonal symmetric)} \]
APPENDIX E. COLLAPSE TO A RED HOLE WITHOUT EVENT HORIZONS IN THE BI-METRIC THEORY OF GRAVITATION

From the main text we have seen that the bi-metric formalism for the new theory of gravitation in ICM coordinates is given by:

**BI-METRIC GRAVITATIONAL FIELD EQUATIONS:**

\[
\Box \Phi^\mu_\nu = [\mathcal{G}^{\alpha\beta} \partial_\alpha \partial_\beta] \Phi^\nu_\mu = - (8\pi G / c^4) \left\{ (\sigma + P / c^2) u_\mu u^\nu - 1/2 \delta^\nu_\mu [\sigma c^2 - P] \right\} \quad (E.1)
\]

**COVARIANT EULER EQUATION FOR MATTER WITH PRESSURE:**

\[
(\sigma + P / c^2) du_\mu / d\tau = 1/2 (\sigma + P / c^2) g_{\alpha\beta} \left( u^\alpha u^\beta \right) + P, \mu \quad (E.2-a)
\]

where

\[
g_{\alpha\beta} \mid_\mu = (\Phi)_{\alpha}^\rho \mid_\mu g_{\rho\beta} + \exp[\left( (\Phi)_{\alpha}^\mu \right) \gamma_{\rho\alpha} (\Phi^T)_{\beta}^\rho \mid_\mu \exp[\left( \Phi^T \right)] = 2 (\Phi)_{\alpha}^\rho \mid_\mu g_{\rho\beta} \quad (E.2-b)
\]

Within the above equations of motion the bi-metric exponential metric is given by

\[
g_{\mu\nu} = \exp(\Phi)_{\mu}^{\alpha} \gamma_{\alpha\beta} \exp(\Phi^T) \bar{\gamma}_{\nu} \quad (E.3-a)
\]

where

\[
\Phi^\mu_\nu \equiv (\text{Tr}(\phi) \delta^\nu_\mu - 2\phi^\nu_\mu) \quad (E.3-b)
\]

Since the physical line element is \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu \) and \( d\tau = ds / c \), this implies that the proper time interval \( d\tau \) of a clock in motion has a relationship to the coordinate time interval \( dt \) by the

\[
d\tau = dt / \gamma = dt [g_{\mu\nu} (v^\mu / c) (v^\nu / c)]^{1/2} \quad (E.4)
\]

Then this implies that if \( u^\mu = dx^\mu / d\tau = \gamma v^\mu \), \( v^\mu = dx^\mu / dt \), \( u^\mu u_\mu = c^2 \) then \( \gamma \) is given by

\[
\gamma = (u^0 / c) = [g_{\mu\nu} (v^\mu / c) (v^\nu / c)]^{-1/2} \quad (E.5-a)
\]

where \( u^\mu \) and \( u_\mu \) are given respectively by

\[
u^\mu = \gamma v^\mu = [g_{\mu\nu} (v^\mu / c) (v^\nu / c)]^{-1/2} v^\mu \quad (E.5-b)
\]

\[
u_\mu = g_{\mu\nu} v^\nu \quad (E.5-c)
\]
Hence in the Euler equations of motion the gravitational redshift of coordinate time seen by an observer at infinity is controlled by

\[ dt = \gamma \, d\tau = d\tau / \left[ g_{\mu\nu} \left( v^\mu/c \right) \left( v^\nu/c \right) \right]^{1/2} \quad (E.6) \]

Now at the surface \( R \) of a radially collapsing fluid of mass \( M \) (for which gravitational radiation will be absent) the exterior and interior values of \( g_{\mu\nu}, ds^2 \) and \( \Phi_{\mu\nu} \) match on the boundary. Hence this means that on the surface \( r = R(t) \) the exterior line element will be given in isotropic cartesian Minkowski ICM coordinates by

\[ ds^2(R(t)) = \exp\left[-2r_{\text{grav}}/R(t)\right] c^2 dt^2 - \exp\left[2r_{\text{grav}}/R(t)\right] (dx^2 + dy^2 + dz^2) \quad (E.7-a) \]

or in isotropic spherical Minkowski ISphM coordinates by

\[ ds^2(R(t)) = \exp\left[-2r_{\text{grav}}/R(t)\right] c^2 dt^2 - \exp\left[2r_{\text{grav}}/R(t)\right] (dr^2 + r^2 d\Omega^2) \quad (E.7-b) \]

where \( r_{\text{grav}} = GM/c^2 \).

Note that since \( R(t) \geq 0 \) then the signature of \( ds^2(R(t)) \) never changes. Hence a surface of infinite redshift can be approached as \( R(t) \to 0 \) without generating an event horizon. Then it follows that since

\[ dt = d\tau / \left\{ g_{00} \left( v^0/c \right) \left( v^0/c \right) + g_{kk} \left( v^k/c \right)^2 \right\}^{1/2} \quad (E.8-a) \]

then this implies that neglecting the effects of rotation

\[ dt = \gamma \, d\tau = d\tau / \left\{ \exp\left[-2r_{\text{grav}}/R(t)\right] - \exp\left[2r_{\text{grav}}/R(t)\right] (v/c)^2 \right\}^{1/2} \quad (E.8-b) \]

which implies that the coordinate velocity \( v \) seen by an observer at infinity is

\[ v = u/\gamma = u \left\{ \exp\left[-2r_{\text{grav}}/R(t)\right] - \exp\left[2r_{\text{grav}}/R(t)\right] (v/c)^2 \right\}^{1/2} \]

\[ = u \exp\left[-r_{\text{grav}}/R(t)\right] \left\{ 1 - \exp\left[4r_{\text{grav}}/R(t)\right] (v/c)^2 \right\}^{1/2} \quad (E.9) \]
Now suppose that $R(t)$ is changing with a finite value of proper radial velocity $(u/c)$ seen by a local observer. Then we can solve for the coordinate radial velocity $(v/c)$ seen by a distant observer at spatial infinity as

$$(v/c)^2 = (u/c)^2 \{ \exp[-2r_{\text{grav}} / R(t)] - \exp[2r_{\text{grav}} / R(t)] (v/c)^2 \} \quad (E.10-a)$$

Then solving this equation for $(v/c)$ we have

$$(v/c) = \pm \frac{(u/c) \exp[-r_{\text{grav}} / R(t)]}{\{1 + (u/c)^2 \exp[2r_{\text{grav}} / R(t)]\}^{1/2}} \quad (E.10-b)$$

Hence for finite values of proper radial velocity $(u/c)$ we see that as $R(t) \to 0$ then the coordinate radial velocity $(v/c) \to 0$. Hence the radial collapse of $R(t)$ between $0 < R < R_{\text{initial}}$ will appear to take an infinite time as seen by a distant observer.

While local proper physical processes are associated with the proper speed $0 < u/c < 1$ the asymptotically external observer sees these physical processes as being associated with the time-dilated coordinate speed $v/c$. Hence to the asymptotically external observer these physical processes appear to begin to slow down significantly for $GM/c^2 < R(t) < 2GM/c^2$ and to asymptotically "freeze up" for $0.5GM/c^2 < R(t) < GM/c^2$. Since physical processes appear to an external observer to be "freezing up" within the radius $R(t) < GM/c^2$, with no event horizon occurring, then this object appears to be a "Red Hole" instead of a “Black Hole”. [Graber] [4].

From the above analysis we see that spherical collapse between $0 < R < r_{\text{ph}}$ (as seen by an observer at infinity) will be constrained by the fact that process is strongly gravitationally red-shifted via the effects of $d\tau$ in the Euler equations of motion. This will also be true in the rotating case as well (see Appendix G), except that gravitational radiation will damp out all quadrupole moments.
Note that RED HOLES are distinguishable from BLACK HOLES in that RED HOLES can exhibit intrinsic multipole magnetic fields (including intrinsic magnetic dipole fields) which can move out into regions beyond $2GM/c^2$ and interact with the accretion disks which may be surrounding them.

If the effects of rotation on the exterior metric are significant then for special observers who look into the rotation of the RED HOLE relativistic doppler effects could apparently "un-freeze" some local physical processes.

Since curvature singularities do not occur for the exterior solution, then matching of the exterior $\Phi^\nu_\mu$ at $r = R$ will prevent curvature singularities from forming in the interior as $R \rightarrow 0$.

A more realistic scenario which includes angular momentum will turn this process into a highly red-shifted rotating torus or Red Hole for which gravitational radiation and non-gravitational forces will also damp it into an equilibrium orbit inside of the exterior photon orbit.
APPENDIX F. VARIATIONAL PRINCIPLE FORMULATION FOR THE NEW BI-MERIC THEORY
GENERAL RELATIVITY

The bi-metric space-time inertial space-time frame of Special Relativity (ICM) is
associated with the (g-metric γ-metric) for which all non-cartesian coordinate and
non-inertial effects vanish. In this bi-metric space-time we derive the new bi-metric
gravitational theory from an action principle with constraints. Then the tensor
properties of the formalism allow this result to be generalized to arbitrary spacetimes.
In the bi-metric space-time contained within the variational principle, the action
principle treats both the metric tensor $g_{\mu\nu}$ and the gravitational potential energy
tensor $\phi_{\mu}^{\nu}$ as independent dynamical variables connected by the bi-metric exponential
metric connection

$$
\begin{align*}
g_{\mu\nu} &= \exp \left[ \left( \text{Tr}(\phi) \delta_{\mu}^{\alpha} - 2\phi_{\mu}^{\alpha} \right) \right] \eta_{\alpha\beta} \exp \left[ \left( \text{Tr}(\phi^T) \delta_{\beta}^{\nu} - 2(\phi)^T_{\beta\nu} \right) \right] \tag{F.1}
\end{align*}
$$

which plays the role of a holonomic constraint between the curved metric $g_{\mu\nu}$ and the
flat metric $\eta_{\mu\nu}$. For simplicity we will assume that holonomic constraint (F.1) obeys the
harmonic metric gauge condition

$$
\partial_{\alpha} g^{\nu\alpha} = 0 ,
$$

which is equivalent to the assuming that the gravitational potential tensor $\phi_{\mu}^{\nu}$ within (F.1) obeys the potential gauge condition

$$
\partial_{\nu} \phi_{\mu}^{\nu} = 0. 
$$

Since the form of the field equations we obtain will be invariant to this choice
of gauge, this result will be completely general.

In this context the new action is a modified version of the Einstein action given by

$$
I = \int dx^4 (-g)^{1/2} \left[ K \{ (LMATTER) + LGRAV. P. E. ( g_{\mu\nu}, \partial_{\alpha} \phi_{\beta}^{\gamma}) \} - R/4 \right] \tag{F.2-a}
$$

where $R$ is the Riemann curvature scalar, $K$ is a constant to be determined, and the
variation of $I$ is performed with respect to $\delta g_{\mu\nu}$ and $\delta \phi_{\mu}^{\nu}$ subject to the holonomic
constraint (F.1).
In the action of the new bi-metric theory of gravitation given by (F.2-a) the gravitational potential tensor lagrangian \( L_{\text{GRAV. P.E.}} \) is

\[
(-g)^{1/2} K (L_{\text{GRAV. P.E.}}) = -1/2 (-g)^{1/2} t_{\mu}^\mu = 1/2 g_{\alpha\beta} t^{\alpha\beta}
\]  

(F.2-b)

Since \( t_{\mu}^\nu = (S_{\mu}^\nu - u_{\mu}^\nu) \), then using \( \partial_{\nu} g^{\mu\nu} = 0 \) and \( \partial_{\nu} \phi_{\mu} = 0 \) implies that

\[
(-g)^{1/2} t_{\mu}^\nu = (-g)^{1/2} \left[ S_{\mu}^\nu - \text{Tr}(\partial_{\mu} \phi \partial^\nu \phi) - 2 \text{Tr}(\partial_{\mu} \phi) \text{Tr}(\partial^\nu \phi) \right]
\]  

(F.2-c)

where in terms of the Freud tensor \( U_{\mu}^\nu \) we find that \( S_{\mu}^\nu \) is given by

\[
(-g)^{1/2} S_{\mu}^\nu = \left[ U_{\mu}^\nu - (g^{\rho\sigma} \partial_{\rho} \partial_{\sigma}) \phi_{\mu} \right]
\]

\[
= - \partial_{\nu} \phi_{\mu} \partial_{\alpha} \phi \partial^{\alpha} g_{\nu\beta}
\]

\[
= 1/4 (-g)^{1/2} (\partial_{\nu} \phi_{\mu} \partial_{\alpha} \phi \partial^{\alpha} g_{\nu\beta})
\]  

(F.2-d)

Note that using the harmonic gauge condition \( \partial_{\nu} \phi_{\mu} = 0 \) and ignoring the 4-divergence term \(-1/2 \partial_{\alpha} (\phi^{\mu\beta} \partial_{\mu} \phi^{\alpha})\) we find that the \((-g)^{1/2} S_{\mu}^\nu\) term does not contribute to the final form of the gravitational potential tensor lagrangian (F.2-b).

Hence we find that

\[
(-g)^{1/2} K (L_{\text{GRAV. P.E.}}) = -1/2 (-g)^{1/2} t_{\mu}^\mu
\]

\[
= -1/2 (-g)^{1/2} \left[ S_{\mu}^\nu - u_{\mu}^\nu \right]
\]

\[
= 1/2 (-g)^{1/2} \left[ \{ \text{Tr}(\partial_{\mu} \phi \partial^\nu \phi) - 2 \text{Tr}(\partial_{\mu} \phi) \text{Tr}(\partial^\nu \phi) \} \right]
\]  

(F.2-e)

In the context of this bi-metric variational principle with constraints, we will show that the Lagrange multiplier tensor density \((-g)^{1/2} \lambda_{\mu\nu}\) will be determined from the consistency of the resulting field equations. We begin by setting the variation \( \delta l \) of (F.2), with respect to independent variations of the metric tensor \( g_{\mu\nu} \) and the
gravitational potential energy tensor $\phi^\mu_\nu$, equal to zero subject to the differential variation of the holonomic constraint equation (F.1) multiplied by a Lagrange tensor density multiplier $(-g)^{1/2}\lambda^\mu_\nu$.

By taking the bi-metric covariant derivative "|\mu" of constraint equation (F.1), in the context of an arbitrary local free fall frame (see the end of Appendix D), and then transforming back from the local free fall frame to the general frame of reference, it can be shown that an equivalent tensor differential constraint relationship holds for all spacetime points in the general frame given by

$$\partial^\alpha (\phi^\beta_\mu g^\mu_\nu - 2\phi^\mu_\nu) - 1/2 \partial^\alpha g^\mu_\beta g^\nu_\alpha = 0$$  (F.3-a)

from which the differential constraint relationship between $d\phi^\mu_\nu$ and $dg^\alpha_\beta$

$$d\phi^\mu_\nu + 1/4 [\delta^\mu_\beta g^\alpha_\nu - 1/2 \delta^\mu_\nu g^\alpha_\beta] dg^\alpha_\beta = 0$$  (F.3-b)

associated with the holonomic constraint (F.1) can be found. Equation (F.3-b), when multiplied by the Lagrange tensor density multiplier $- (-g)^{1/2}\lambda^\mu_\nu$ and integrated over $dx^4$, gives the variational constraint action

$$\int dx^4 \{-(-g)^{1/2}\lambda^\mu_\nu [d\phi^\mu_\nu + 1/4 [\delta^\mu_\beta g^\alpha_\nu - 1/2 \delta^\mu_\nu g^\alpha_\beta] dg^\alpha_\beta] \} = 0$$  (F.4)

The variational constraint action (F.4) is added to the variational action obtained from (F.2) and following the usual variational calculus and grouping terms together the resultant Hamiltonian derivatives (i.e. the coefficients of the of the $\delta g^\mu_\nu$ and the $\delta \phi^\mu_\nu$) are obtained in this manner. In this context setting the Hamiltonian derivative coefficients of independent variations of the $\delta g^\mu_\nu$ equal to zero gives for the equation of motion for $g^\mu_\nu$ as

$$1/2 G^\mu_\nu = \{K (\tau^\mu_\nu + t^\mu_\nu) + 1/2 [\lambda^\mu_\nu - 1/2 g^\mu_\nu \lambda^\alpha_\nu] \}$$  (F.5-a)
where the matter energy-momentum tensor $\tau_{\mu\nu}$ and the gravitational potential energy-momentum tensor $t_{\mu\nu}$ are given by

\[
\tau_{\mu\nu} = 2 / (-g)^{1/2} \left[ \partial \left\{ (-g)^{1/2} L_{\text{GRAV. P. E.}} \right\} / \partial g^{\mu\nu} \right] \quad (F.6-a)
\]

\[
t_{\mu\nu} = 2 / (-g)^{1/2} \left[ \partial \left\{ (-g)^{1/2} L_{\text{MATTER}} \right\} / \partial g^{\mu\nu} \right] \quad (F.6-b)
\]

In the same context setting the Hamiltonian derivative coefficients of independent variations of the $\delta \phi_{\mu\nu}$ equal to zero, and taking $\partial_{\nu} g^{\mu\nu} = 0$ and $\partial_{\nu} \phi_{\mu} = 0$ into account, gives for the second equation of motion

\[
\left[ \delta \left\{ K (-g)^{1/2} L_{\text{GRAV. P. E.}} \right\} / \delta \phi_{\mu\nu} \right] = (-g)^{1/2} \left[ 2 \phi_{\mu\nu} - \delta_{\mu\nu} \text{Tr}(\phi) \right]
\]

which implies that

\[
\Box \phi_{\mu\nu} = -1/2(\lambda_{\mu\nu} - 1/2 \delta_{\mu\nu} \lambda_{\alpha\alpha}) \quad (F.7-b)
\]

Now (F.6-a) can be written as

\[
1/2 G_{\mu}^{\nu} = \Box \phi_{\mu}^{\nu} + (K S_{\mu}^{\nu} + H_{\mu}^{\nu})
\]

where

\[
H_{\mu}^{\nu} = \left\{ -2 \left[ \text{Tr}(\partial_{\mu} \phi \partial_{\nu} \phi) - 1/2 \delta_{\mu\nu} \text{Tr}(\partial_{\lambda} \phi \partial_{\lambda} \phi) \right] 
\right.
\]

\[
+ \left. \left[ \text{Tr}(\partial_{\mu} \phi) \text{Tr}(\partial_{\nu} \phi) - 1/2 \delta_{\mu\nu} \text{Tr}(\partial_{\lambda} \phi) \text{Tr}(\partial_{\lambda} \phi) \right] \right \} \quad (F.8-b)
\]

where we recall that

\[
(-g)^{1/2} K S_{\mu}^{\nu} = U_{\mu}^{\nu} - (g^{ps} \partial_{p} \partial_{s}) \phi_{\mu}^{\nu}
\]

\[
= - \partial_{\beta} \phi_{\mu}^{\alpha} \partial_{\alpha} g^{\nu\beta}
\]

\[
= 1/4 (-g)^{1/2} (\partial_{\beta} \phi_{\mu}^{\alpha} \partial_{\alpha} \phi^{\nu\beta}) \quad (F.8-c)
\]
Since \( t_\mu^v = (S_\mu^v - u_\mu^v) \) and \( H_\mu^v = - K u_\mu^v \) then it follows that in (F.8-a) we have

\[
K t_\mu^v = -K (S_\mu^v - u_\mu^v) = (KS_\mu^v + H_\mu^v)
\]

and (F.8-a) becomes

\[
\Box \phi_\mu^v = [K \tau_\mu^v + 1/2(\lambda_\mu^v - 1/2 \delta_\mu^v \lambda_\alpha^\alpha)]
\]  \(\text{(F.8-d)}\)

Now requiring that the field equations (F.7-b) and (F.8-d) are consistent with each other implies that

\[
[K \tau_\mu^v + 1/2(\lambda_\mu^v - 1/2 \delta_\mu^v \lambda_\alpha^\alpha)] = [-1/2(\lambda_\mu^v - 1/2 \delta_\mu^v \lambda_\alpha^\alpha)]
\]  \(\text{(F.8-e)}\)

which implies that

\[
(\lambda_\mu^v - 1/2 \delta_\mu^v \lambda_\alpha^\alpha) = (-K \tau_\mu^v)
\]  \(\text{(F.8-f)}\)

so that the field equations (F.7-b) and (F.8-b) become

\[
\Box \phi_\mu^v = (g_\alpha^\beta \partial_\alpha \partial_\beta) \phi_\mu^v = K/2 \tau_\mu^v
\]  \(\text{(F.9-a)}\)

Solving (F.9-b) for \( \lambda_\mu^v \) we see that the Lagrange multiplier tensor, which is associated with the exponential bi-metric constraint (F.1), is a linear function of the energy-momentum tensor of matter \( \tau_\mu^v \) given by

\[
\lambda_\mu^v = K/2 [\delta_\mu^v \tau_\alpha^\alpha - 2 \tau_\mu^v]
\]  \(\text{(F.9-b)}\)

where \( K \) is a constant determined from the Newtonian correspondence limit of the theory. Then equations (F.9-a,b) represent the field equations for the theory.

Now these equations will have a consistent Newtonian correspondence limit if we choose the coupling constant as \( K = 8\pi G/ c^4 \) which gives for the final form of the gravitational field equations as

\[
\Box \phi_\mu^v = (g_\alpha^\beta \partial_\alpha \partial_\beta) \phi_\mu^v = (4\pi G/ c^4) \tau_\mu^v
\]  \(\text{(F.10-a)}\)
where

$$g_{\mu\nu} = \exp \left( (\text{Tr}(\phi) \delta_{\mu}^{\alpha} - 2\phi_{\mu}^{\alpha}) \eta_{\alpha\beta} \exp \left( (\text{Tr}(\phi) \delta_{\mu}^{\beta} - 2\phi_{\mu}^{\beta}) \right) \right)$$

(F.10-b)

implies that

$$\frac{1}{2} G_{\mu\nu} = \left( \frac{4\pi G}{c^4} \right) (\tau_{\mu\nu} + t_{\mu\nu})$$

(F.10-c)

with $t_{\mu\nu} = (S_{\mu\nu} - U_{\mu\nu})$ and

$$S_{\mu\nu} = \left[ \frac{\mathbf{U}_{\mu\nu} - (g^{\rho\sigma} \partial_{\rho} \partial_{\sigma}) \phi_{\mu\nu}}{(-g)^{1/2}(4\pi G/c^4)} \right]$$

(F.10-c)

This also implies that in the ICM coordinates which were chosen that

$$(\tau_{\mu\nu} + t_{\mu\nu}) ; \nu = 0 \quad \text{BIANCHI IDENTITY} \quad (F.10-e)$$

and

$$[-g^{1/2}(\tau_{\mu\nu} + t_{\mu\nu} + U_{\mu\nu})] , \nu = 0 \quad \text{FREUD IDENTITY} \quad (F.10-f)$$
APPENDIX G .  STATIONARY STATE SOLUTIONS WITH CONSTANT ROTATING MATTER CURRENTS

Consider the case of the stationary state of a stellar object, involving neutral matter density and pressure which is rotating with constant azimuthal angular velocity (i.e. rigid body rotation). Then if we transform into the rest frame of the stellar object this implies that \( x^\mu = x^\mu [x(\text{ICM})] \) can be taken to be the flat space-time transformation for constant azimuthal rotation in spherical polar coordinates. Now when written in terms of \( \Phi_\mu^\nu = (\text{Tr}(\phi) \delta_\mu^\nu - 2\phi_\mu^\nu) \) the field equations are

\[
\Box \Phi_\mu^\nu = (4\pi G/c^4) [\exp[\text{Tr}(\Phi)] (\text{Tr}(\tau) \delta_\mu^\nu - 2\tau_\mu^\nu)] \tag{G.1-a}
\]

Now from the rotating frame transformation \( x^\mu = x^\mu [x(\text{ICM})] \) the flat background metric tensor is

\[
\gamma_{\mu\nu} = \frac{\partial x^\rho (\text{ICM})}{\partial x^\mu} \frac{\partial x^\sigma (\text{ICM})}{\partial x^\nu} \eta_{\rho\sigma} \tag{G.1-b}
\]

Hence in this case the stationary rotating d'Alembertian operator can be written as

\[
\Box \Phi_\mu^\nu = g^{kj} \{\chi_\mu^\nu k|j\} + g^{ko} \{\chi_\mu^\nu k|o\} + g^{oj} \{\chi_\mu^\nu o|j\} + g^{oo} \{\chi_\mu^\nu o|o\} \tag{G.1-c}
\]

where

\[
g^{kj} \{\chi_\mu^\nu k|j\} = g^{kj} \{\chi_\mu^\nu k,j\} + \Sigma^\nu_p j \chi_\mu^p k - \Sigma^p_j \chi_\rho^v k - \Sigma^p_k j \chi_\mu^v \rho \tag{G.1-d}
\]

\[
g^{ko} \{\chi_\mu^\nu k|o\} = g^{ko} \{\chi_\mu^\nu k,o\} + \Sigma^\nu_p o \chi_\mu^p k - \Sigma^p_o \chi_\rho^v k - \Sigma^p_k o \chi_\mu^v \rho \tag{G.1-e}
\]

\[
g^{oj} \{\chi_\mu^\nu o|j\} = g^{oj} \{\chi_\mu^\nu o,j\} + \Sigma^\nu_p j \chi_\mu^p o - \Sigma^p_j \chi_\rho^v o - \Sigma^p_o j \chi_\mu^v \rho \tag{G.1-f}
\]

\[
g^{oo} \{\chi_\mu^\nu o|o\} = g^{oo} \{\chi_\mu^\nu o,o\} + \Sigma^\nu_p o \chi_\mu^p o - \Sigma^p_o \chi_\rho^v o - \Sigma^p_oo \chi_\mu^v \rho \tag{G.1-g}
\]

and

\[
\chi_\mu^v k = \{(\phi_\mu^v) k\} = [\phi_\mu^v , k + \Sigma^v_p k \phi_\mu^p - \Sigma^p_k \phi_\rho^v] \tag{G.1-h}
\]

\[
\chi_\mu^v o = \{(\phi_\mu^v) o\} = [\Sigma^o_p o \phi_\mu^p - \Sigma^p_oo \phi_\rho^v] \tag{G.1-i}
\]

\[
\Sigma^\alpha_{\mu\nu} = 1/2 \gamma^{\rho\sigma}(\gamma_{\mu\rho} \gamma_{\nu\sigma} + \gamma_{\nu\rho} \gamma_{\mu\sigma} - \gamma_{\mu\nu} \gamma_{\rho\sigma}) = (\partial^2 x^\sigma (\text{ICM})/ \partial x^\mu \partial x^\nu) (\partial x^\alpha/ \partial x^\sigma (\text{ICM})) \tag{G.1-j}
\]
where in the above equations we used \( g^{k0} \{ \chi_{\mu}^v \} \), \( \phi_{\mu}^v , o = 0 \) and \( \phi_{\mu}^o, o = 0 \) since in terms of spherical coordinates \((r, \theta, \phi)\), constant rotation about the z-axis is given by the holonomic condition \( \phi = \omega t \) where \( \omega \) is a constant. In the context of the above field equations (G.1)

In the rotating spherical bi-metric space-time frame the exponential metric is

\[
g_{\mu\nu} = \exp \left[ (\Phi)^{\alpha}_{\mu} \right] \eta_{\alpha\beta} \exp \left[ (\Phi)^{\beta}_{\nu} \right]
\]

\[
= (\partial x^\mu_{(ICM)} / \partial x^\nu_{(ICM)}) (\partial x^\sigma_{(ICM)} / \partial x^\nu_{(ICM)}) g_{\rho\sigma(ICM)} \quad (G.1-g)
\]

where in the rotating bi-metric spherical space-time frame the gravitational potential tensor \( \Phi_{\mu\nu} \equiv (\text{Tr}(\phi) \delta_{\mu\nu} - 2\phi_{\mu\nu}) \) is given by

\[
\Phi_{\mu\nu} = \left\{ \left( \partial x^\nu_{(ICM)} / \partial x^{\mu(ICM)} \right) \left( \partial x^\sigma_{(ICM)} / \partial x^{\mu(ICM)} \right) \right\} \Phi^\rho_{\sigma(ICM)} \quad (G.1-h)
\]

The above field equations (G.1) are to be solved simultaneously with the stationary state Euler equations of motion in the rest frame of the rotating stellar object (for which \( u_{\mu} = (c, 0, 0, 0) \) and \( du_{\mu} / ds = 0 \)) using the transformation \( \text{ICM} \rightarrow \text{ISM} \rightarrow \text{RSM} \) are given by

\[
P, k = -1/2 (\sigma c^2 + P) g_{00,k}^v c^2 = -1/2 (\sigma c^2 + P) [\gamma_{00,k} + g_{00,k}] \quad (G.2-a)
\]

This can also be written terms of \( \Phi_{\mu\nu} \) as

\[
P, k = -(\sigma c^2 + P) g_{v0} \Phi_{0\nu}^v, k = -(\sigma c^2 + P) [1/2 \gamma_{00,k} + g_{v0} \Phi_{0\nu}^v, k] \quad (G.2-b)
\]

where using

\[
\Phi_{\mu\nu} | \alpha = [\Phi_{\mu\nu}, \alpha + \Sigma_{\nu} \sigma_{\alpha} \Phi_{\mu}^\sigma - \Sigma_{\mu} \alpha \Phi_{\sigma}^\nu] \quad (G.2-c)
\]

implies that in (G.2-b)

\[
\Phi_{0\nu}^v, k = [\Phi_{0\nu}^v, k + \Sigma_{\nu} \sigma_{k} \Phi_{0}^\sigma - \Sigma_{\sigma} \nu_{k} \Phi_{0}^\sigma] \quad (G.2-d)
\]

and \( \phi_{\mu\nu} \) are solutions of the field equations in the rotating rest frame in spherical coordinates where \( x^\mu = (ct, r, \theta, \phi) \), \( u^\mu = (c, 0, 0, 0) \), given by (G.1-c) as

\[
\Box \Phi_{\mu\nu} = (4\pi G/c^4) \left[ \sigma c^2 (\delta_{\mu\nu} - 2u_{\mu} u_{\nu} / c^2) - P(\delta_{\mu\nu} + 2u_{\mu} u_{\nu} / c^2) \right] = (4\pi G/c^4) \chi_{\mu\nu} \quad (G.3)
\]
where now (G.3) is equivalent to the four equations

\[ \Box \Phi^0_\theta = -(4\pi G/c^4) [\sigma c^2 + 3P] \]  \hspace{1cm} (G.4-a)
\[ \Box \Phi^r_r = (4\pi G/c^4) [\sigma c^2 - P] \]  \hspace{1cm} (G.4-b)
\[ \Box \Phi^\theta_\theta = (4\pi G/c^4) [\sigma c^2 - P] \]  \hspace{1cm} (G.4-c)
\[ \Box \Phi^\phi_\phi = (4\pi G/c^4) [\sigma c^2 - P] \]  \hspace{1cm} (G.4-d)

then in the rotating frame \( \Phi^\mu_\nu \) is diagonal and \( \Phi^r_r = \Phi^\theta_\theta = \Phi^\phi_\phi \) which implies that in this frame the metric tensor \( g^{\mu\nu} = [\exp(\Phi)]^{\mu\alpha} [\exp(\Phi)]^{\nu\beta} g^{\alpha\beta} \) is also diagonal.

Note however that for the observer located in the rotating rest frame of the fluid \( \sigma = \sigma(r, \theta) \) and \( P = P(r, \theta) \) because of the centrifugal force which appears in that frame. This is because the transformation from the ISM frame to \( \phi = \omega t \) where \( \omega \) is a constant puts the observer into a non-inertial space at rest with the fluid.

Since acceleration of observers in defined with respect to the inertial ICM frame in the new bi-metric theory of gravitation, the situation represented by equations (G.4) is physically distinguishable from the situation which starts with the observer at rest in an inertial spherical Minkowski space-time (ISM) frame with respect to the spherically symmetric static fluid (i.e. \( u^\mu = (c,0,0,0) \) with \( \sigma = \sigma(r) \) and \( P = P(r) \) are

\[ -\nabla^2 \Phi^0_\theta = -(4\pi G/c^4) [\sigma c^2 + 3P] \]  \hspace{1cm} (G.5-a)
\[ -\nabla^2 \Phi^r_r = (4\pi G/c^4) [\sigma c^2 - P] \]  \hspace{1cm} (G.5-b)
\[ -\nabla^2 \Phi^\theta_\theta = (4\pi G/c^4) [\sigma c^2 - P] \]  \hspace{1cm} (G.5-c)
\[ -\nabla^2 \Phi^\phi_\phi = (4\pi G/c^4) [\sigma c^2 - P] \]  \hspace{1cm} (G.5-d)

(which also implies that \( \Phi^0_\theta \) and \( \Phi^r_r = \Phi^\theta_\theta = \Phi^\phi_\phi \) are spherically symmetric)

and then transforms the observer into a non-inertial rotating (RS) spherical frame for which \( \phi = -\omega t \) and \( u^\mu = (c,0,0,-\omega t) \) and \( \Phi^\nu_\mu(RS) \) has non-diagonal components as

\[ \Phi^\nu_\mu(RS) = \{(\partial x^\nu/\partial x(RS)^\rho) (\partial x^\sigma(RS)/\partial x^\mu)\} \Phi^\rho_\sigma(ISM) \]  \hspace{1cm} (G.6-a)
\[ g_{\mu\nu}(RS) = \gamma_{\mu\sigma} \exp[2\Phi]^\sigma_\nu = (\partial x^\rho(RS)/\partial x^\mu)(\partial x^\sigma(RS)/\partial x^\nu) g_{\rho\sigma}(ISM) \]  \hspace{1cm} (G.6-b)
Hence in the new bi-metric theory of gravitation we see that rotating the fluid with the observer fixed within it (i.e., equation (G.4)) is physically distinguishable from holding the static fluid fixed and rotating the observer since then the fluid appears to flow in the opposite direction and the observer feels fictitious rotational gravitational forces (e.g., equation (G.5)).
In the absence of pressure $P = 0$ the contravariant geodesic equations of motion for matter are given by (14-b) as

$$\sigma \frac{du^\mu}{ds} = - \Gamma^\mu_{\alpha\beta} (\sigma u^\alpha u^\beta) = - (Z^\mu_{\alpha\beta} + \Delta^\mu_{\alpha\beta}) (\sigma u^\alpha u^\beta) \quad (H.1-a)$$

where in the inertial physical space-time associated with the Minkowski background space-time $Z^\mu_{\alpha\beta} = 0$ and the Christoffel symbol $\Gamma^\mu_{\alpha\beta}$ is equal to the Christoffel tensor $\Delta^\mu_{\alpha\beta}$

$$\Gamma^\mu_{\alpha\beta} = 1/2 g^{\mu\rho} (g_{\alpha\rho,\beta} + g_{\beta\rho,\alpha} - g_{\alpha\beta, \rho}) = \Delta^\mu_{\alpha\beta} \quad (H.1-b)$$

We recall that in the new bi-metric gravitational the field equations the metric $g_{\mu\nu}$ is defined in terms of the symmetric gravitational potential tensor matrix $\Phi^\mu_{\nu} = \Phi^\mu_{\nu}$ (using space-time 4x4 matrix notation) as

$$g_{\mu\nu} = \exp [((\Phi)_\mu^\alpha] \eta_{\alpha\beta} \exp [((\Phi)^T)_\beta^\nu)] \quad (H.2-a)$$

where $\Phi^\mu_{\nu}$ is a function of the gravitational potential tensor $\phi^\mu_{\nu}$ given by

$$\Phi = \Phi^\mu_{\nu} = (\text{Tr}(\phi) \delta^\mu_{\nu} - 2\phi^\mu_{\nu}) \quad (H.2-b)$$

and in (J.2-a) the matrix exponential is defined in terms of the power series

$$[\exp (\Phi)]_\mu^\nu = \delta^\nu_{\mu} + \Sigma_{N=1,2,...} [(\Phi)^N]_\mu^\nu / N! \quad (H.2-c)$$

Now from section I equation (3-b) we have that

$$1/2 g^{\nu\rho} g_{\mu\rho, \alpha} = \Phi^T_{\mu}^\nu \cdot \alpha \quad (H.3-a)$$

and

$$1/2 g^{\mu\rho} g_{\alpha\beta, \rho} = 1/2 g^{\mu\rho} (g_{\sigma\alpha} \Phi^T_{\beta}^\sigma \cdot \rho + g_{\sigma\beta} \Phi^T_{\alpha}^\sigma \cdot \rho) \quad (H.3-b)$$

Substituting (H.3-a,b) into (H.1-b) we have that the Christoffel symbol $\Gamma^\mu_{\alpha\beta}$ can be written as

$$\Gamma^\mu_{\alpha\beta} = \Phi^T_{\alpha}^\mu \cdot \beta + \Phi^T_{\beta}^\mu \cdot \alpha - 1/2 (g_{\sigma\alpha} \Phi^T_{\beta}^\sigma \cdot \mu + g_{\sigma\beta} \Phi^T_{\alpha}^\sigma \cdot \mu) \quad (H.4)$$
Hence the contravariant geodesic equation of motion can be written in terms of the derivatives of \( \Phi^\mu_{\alpha} \) as

\[
\sigma \frac{du^\mu}{ds} = - \Gamma^\mu_{\alpha\beta} (\sigma u^\alpha u^\beta)
\]

\[
= - \left[ \Phi^T_{\alpha \beta}, \delta^\mu_{\alpha} + \Phi^T_{\beta \mu}, \delta^\mu_{\beta} - 1/2 (g^\sigma_{\alpha} \Phi^T_{\beta \sigma}, \delta^\mu_{\mu} + g^\sigma_{\beta} \Phi^T_{\alpha \sigma}, \delta^\mu_{\mu}) \right] (\sigma u^\alpha u^\beta)
\]  

(H.5)

Now in the new bi-metric General Relativity the field equations (H.5) can be written in terms of the gravitational potential tensor \( \Phi^\mu_{\mu} \equiv (\text{Tr}(\phi), \delta^\mu_{\mu} - 2\phi^\mu_{\mu}) \) as

\[
\Box \Phi^\mu_{\mu} = [g^{\alpha\beta} \partial_\alpha \partial_\beta] \Phi^\mu_{\mu} = -(8\pi G / c^4) [\tau^\mu_{\mu} - 1/2 \delta^\mu_{\mu} \text{Tr}(\tau)]
\]  

(H.6-a)

where \( \Box \equiv (g^{\alpha\beta} \partial_\alpha \partial_\beta) \) is the bi-metric covariant d’Alembertian operator.

Now to see the gravitational N-body problem in the new bi-metric General Relativity we assume for the case neutral matter that the matter energy momentum tensor can be broken up into N distinct parts given by \( \tau^\mu_{\nu} = \sum_{1,2,\ldots,N} \tau(N)_{\mu}^\nu \), where the total energy momentum tensor is conserved as \( \left(-g\right)^{1/2} \sum_{1,2,\ldots,N} \tau(N)_{\mu}^\nu \right)_{\nu} = 0 \). Then it follows from the form of the gravitational field equation (H.6) that the gravitational potential tensor \( \Phi^\mu_{\mu} \) can be represented by a sum of N distinct potentials given by \( \Phi_{\mu}^\nu = \sum_{1,2,\ldots,N} \Phi_{\mu}^\nu(N) + K_{\mu\nu} \) where the zero baseline value of the gravitational potential \( \Phi_{\mu\nu} \), associated with the location of the origin of the observer and \( x = x_{\text{obs}} \), is determined by the choice of the constant tensor \( K_{\mu\nu} \). This tensor constant \( K_{\mu\nu} \) is chosen so that at the origin of the observer \( \Phi(x_{\text{obs}})_{\mu\nu} = 0 \) and the metric becomes locally Minkowskian as \( g(x_{\text{obs}})_{\mu\nu} = \eta_{\mu\nu} \). This occurs even though the observer is in a general space-time frame of reference and is not necessarily in free fall. The gravitational potential tensors \( \Phi_{\mu}^\nu(N) \) satisfy the equations \( N = 1,2,\ldots \)

\[
[g^{\alpha\beta} \partial_\alpha \partial_\beta] \Phi_{\mu}^\nu(N) = -(8\pi G / c^4) [\tau(N)_{\mu}^\nu - 1/2 \delta_{\mu\nu} \text{Tr}(\tau(N))] 
\]  

(H.6-b)
and using (H.6-c) in (H.2) we see that the metric tensor is given by

\[ g_{\alpha\beta} = \exp(\sum_{1,2,\ldots,M} \Phi(M)_{\alpha}^{\rho}) \eta^{\rho\sigma} \exp(\sum_{1,2,\ldots,M} \Phi^{T}(M)_{\sigma}^{\beta}) \]  

(H.6-c)

What is remarkable is the fact that in the context of this exact solution to (H.6) the Christoffel symbol breaks up into a sum of \(N\) terms as

\[ \Gamma_{\alpha\beta}^{\mu} = \sum_{1,2,\ldots,N} \Gamma^{(N)}_{\alpha\beta}^{\mu} \]  

(H.7-a)

where

\[ \Gamma^{(N)}_{\alpha\beta}^{\mu} = \Phi^{T}(N)_{\alpha}^{\mu},_{\beta} + \Phi^{T}(N)_{\beta}^{\mu},_{\alpha} - 1/2 (g_{\nu\alpha} \Phi^{T}(N)_{\beta}^{\nu},_{\mu} + g_{\nu\beta} \Phi^{T}(N)_{\alpha}^{\nu},_{\mu}) \]  

(H.7-b)

and the matter equations of motion (H.5) take on the \(N\)-body interactive form

\[ (\sum_{1,2,\ldots,N} \sigma(N)) \frac{d\mu^{\mu}}{ds} = (\sum_{1,2,\ldots,N} \sigma(N)) \left[ - \sum_{1,2,\ldots,M} \Gamma^{(N)}_{\alpha\beta}^{\mu} u^{\alpha}_{\mu} u^{\beta} \right] \]  

(H.8)

Note that with respect to the new gravitational field equations (H.6-a) this is an exact result involving only the assumption that the energy-momentum tensor can be broken up into to \(N\) distinct parts and no other approximations. The associated breakup of the Christoffel symbol into \(N\) distinct parts occurs in the new bi-metric theory of gravitation because of the exponential metric connection to the gravitational potential tensor inherent within the formalism. The lack of such an exponential metric connection in the Einstein theory is the reason why this breakup of the Christoffel symbol does not occur as an exact result and is what prevents such an exact \(N\)-body solution from appearing in Einstein general relativity.
APPENDIX I: GRAVITATIONAL WAVES IN THE NEW BI-METRIC GENERAL RELATIVITY

The bi-metric gravitational field equations can be written in terms of the gravitational potential tensor \( \phi^\mu_\nu \) as

\[
\Box \phi^\mu_\nu = g^{\alpha\beta}(\phi^\mu_\nu | \alpha | \beta) = (4\pi G/c^4) \tau^\mu_\nu \tag{I.1-a}
\]

where

\[
g_{\mu\nu} = \exp [(\Phi^\mu_\nu | \alpha | \beta) \eta_{\alpha\beta} \exp [(\Phi^\tau_\nu | \beta))] \tag{I.1-b}
\]

with \( \phi^\mu_\nu = \text{Tr}(\phi) \delta^\mu_\nu - 2\phi^\mu_\nu \).

The Bianchi-Freud Identity is

\[
(\tau^\mu_\nu + t^\mu_\nu) : \nu = [(-\kappa)^{1/2}(\tau^\mu_\nu + t^\mu_\nu + U^\mu_\nu)] | \nu = 0 \tag{I.2-a}
\]

where \((-\kappa)^{1/2}t^\nu_\mu = (-\kappa)^{1/2}(S^\nu_\mu - U^\nu_\mu)\) and

\[
(-\kappa)^{1/2}S^\nu_\mu = 1/4 \{ g^\mu_\alpha (g^{\nu\alpha} \ g^{\omega\beta} - g^{\omega\alpha} g^{\nu\beta}) | \beta) | \alpha - g^{\alpha\beta}(\phi^\mu_\nu | \alpha | \beta) \\
= - [(-\kappa)^{1/2}\phi^\mu_\nu | \alpha | \beta + g^{\alpha\beta} | \beta \phi^\mu_\nu | \alpha - [(\delta^\mu_\alpha g^{\nu\lambda} - \delta^\nu_\mu g^{\alpha\lambda}) \phi^\lambda_\beta | \beta] | \alpha} \tag{1.2-b}
\]

\[
U^\nu_\mu = [W^\nu_\mu - 1/2 \delta^\nu_\mu \text{Tr}(W)] \tag{I.2-c}
\]

\[
W^\nu_\mu = \{ \text{Tr}(\phi^{\beta_\mu_\alpha \lambda_\nu}) - 2 (\phi^{\alpha_\mu_\nu}) (\phi^{\alpha_\mu_\nu}) + [g^{\nu\rho} (\phi^{\lambda_\mu_\nu}) (\phi^{\lambda_\mu_\nu})] \} \tag{1.2-d}
\]
Equations (F.1-a) is equivalent to the modified bi-metric Einstein equations

\[-1/2 \, G^\nu_{\mu} = (4\pi G/c^4) (\tau^\nu_{\mu} + t^\nu_{\mu}) \]  

(I.3)

For gravitational wave emission we transform to local free fall observers who are located near the center of mass-energy of the time dependent mass distribution. For such observers the non-inertial forces which are generated by \( \Sigma^\mu_{\alpha\beta} \) (due to the observers acceleration with respect to the global Inertial Cartesian Minkowski (ICM) space-time) compensates and cancels out the real gravitational forces generated by Christoffel tensor \( \Delta^\mu_{\alpha\beta} \) so that \( \Gamma^\mu_{\alpha\beta} = \Sigma^\mu_{\alpha\beta} + \Delta^\mu_{\alpha\beta} = 0 \) (i.e. in local free fall \( \Sigma^\mu_{\alpha\beta} = -\Delta^\mu_{\alpha\beta} \) is valid).

For observers in local free fall the field equation (F.1-a) is

\[ \Box \phi^\nu_{\mu} = \eta_{\alpha\beta}(\phi^\nu_{\mu} | \alpha | \beta) = \tau^\nu_{\mu} \]

(I.4-a)

where inside of the background covariant derivative symbols denoted by “ \(| \alpha | \beta \)” the free fall condition \( \Sigma^\mu_{\alpha\beta} = -\Delta^\mu_{\alpha\beta} \) is assumed to hold. Also since \( \Gamma^\mu_{\alpha\beta} = 0 \) the Bianchi-Freud identity in this local free fall frame is given by

\[ (\tau^\nu_{\mu} + t^\nu_{\mu})_{, \nu} = 0 \]

(I.4-b)

Then for observers in local free fall, and in a weak field approximation to lowest order in \( \Sigma^\mu_{\alpha\beta} = -\Delta^\mu_{\alpha\beta} \), equation (I.4-a) can be written as

\[ \Box \phi^\nu_{\mu} = (\eta_{\alpha\beta} \partial_{\alpha} \partial_{\beta}) \phi^\nu_{\mu} = (4\pi G / c^4) \tau^\nu_{\mu} \]

(I.4-a)

For observers in local free fall the weak field Bianchi-Freud identity is still given by

\[ (\tau^\nu_{\mu} + t^\nu_{\mu})_{, \nu} = 0 \]

(I.2-b’)
where $t^v_\mu = (S^v_\mu - u^v_\mu)$ except now in the weak field approximation
\begin{align*}
S^v_\mu &= -4 \left[ \eta^{\nu\rho} \phi^\beta_\rho \alpha \phi^\alpha_\mu, \beta \right] \\
u^v_\mu &= \left[ W^v_\mu - \frac{1}{2} \delta^v_\mu \text{Tr}(W) \right] \\
W^v_\mu &= \left\{ \text{Tr}(\phi^\beta_\alpha, \mu \phi^\alpha_\beta, \nu) - 2 (\phi^\alpha_\alpha, \mu) (\phi^\alpha_\alpha, \nu) \right\} + \left[ \eta^{\nu\rho} (\phi^\lambda_\rho, \mu) (\phi^\alpha_\lambda, \alpha) \right]
\end{align*}

Hence (I.2-b') implies for observers in local free fall in $V_4$ we have the weak field energy balance equation
\begin{equation}
(1/c) \partial_t \int_{V} dx_3 \left( \tau^0_\mu + t^0_\mu \right) = - \int dS_k t^k_\mu = - \int dS_k (S^k_\mu - u^k_\mu) \tag{I.3}
\end{equation}

Now in the weak field approximation let us consider the emission of gravitational radiation from a time varying mass-energy source in $V_4$, as seen by a class of freely falling observers in $V_4$ who are located at the center of mass-energy of the source. Then equations (F.1 thru F.5) for the gravitational potential tensor in the weak field limit obey
\begin{align*}
\Box \phi_\mu^v &= \left[ \eta^{\alpha\beta} \partial^\alpha \partial^\beta \right] \phi_\mu^v = \left( 4\pi G / c^4 \right) \tau_\mu^v \tag{L.4-a} \\
\tau_\mu^v, v &= 0 \tag{I.4-c}
\end{align*}

We solve these weak field equations in the wave zone for retarded transverse-traceless gravitational radiation solutions (i.e. a $\phi_\mu^v \text{(ret)}$ solution whose wave zone properties are compatible with the transverse-traceless conditions $\phi_\mu^v, \nu = 0$, $\phi^\alpha_\alpha, \nu = 0$ associated with gravitational waves). In the weak-field context we find that these gravitational wave solutions have the form\footnote{\cite{7}}
\begin{equation}
\phi^{kj \text{(ret)}} = \left( G / 2c^4 \right) \left[ Q^{kj}(t - r/c) / 3r \right] \tag{1.4-c}
\end{equation}

where
\begin{equation}
Q^{kj} = \int dx'^3 \left( 3 x'^k x'^j - r'^2 \delta^k_j \right) \sigma(x') \tag{1.4-d}
\end{equation}
is the mass quadrupole moment of the system and the symbol " $\bullet$ " denotes the time derivative with respect to the retarded time $(t - r/c)$. 

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The extracted text is a continuation of a discussion on the weak field approximation in gravitational theory, focusing on the energy balance equation and the emission of gravitational radiation from a time-varying mass-energy source. The text delves into the mathematical expressions for the weak field potential tensor and solutions to the gravitational field equations in the weak-field limit. It concludes with a representation of gravitational wave solutions compatible with the transverse-traceless conditions.
Now from (I.3) for the case of weak field gravitational fields we have that

\[-\frac{1}{c} \partial_t E_V = \int_{S_V} dS_k t_0^k = \int_{S_V} dS_k (S_0^k - u_0^k) \tag{I.5-a}\]

where $E_V$ is the total energy contained in $V$ given by

\[E_V = \int_V dx^3 (\tau^0_\mu + t^0_\mu) \tag{I.5-b}\]

Now substituting (I.4-c,d) into (I.2) and (I.5-a,b) we find the energy loss due to gravitational wave emission observed in this context, obeys the standard quadrupole formula as

\[-\partial_t E_V = (\pi G^2 / 45c^7) Q^{k j}(t - r/c) Q^{k j}(t - r/c) \tag{I.6}\]