gr-qc/9912070 16 Dec 1999

Weak-field approximation of a scalar theory of gravitation and some remarks on the propagation effects of gravity

Submitted (October 1998) to Physics Letters A. This is a revised version of February 1999 at the author's initiative, with a few additional modifications (of style only) for the present version.

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A simple example is given of the implementation of the usual method of asymptotic expansions for weak gravitational fields. A scalar, preferred-frame theory of gravitation is considered, but the method is general. The version used allows to account for propagation effects, also in the zone occupied by the system of bodies. As an application, it is shown that the retardation due to propagation at speed c has no dynamical effect below the second order, i.e., below the order $1/c^2$; and that it does have dynamical effects at this order.

1. Introduction

The first attempt to modify Newtonian gravitation (NG) so that the theory become consistent with relativistic dynamics was made by Poincaré [1], who built a Lorentz-invariant 4-vector theory of gravitation. In doing so, he "was led to assume that gravitation propagates with the velocity of light," c. He immediately wondered whether this would be compatible with astronomical observations, "for Laplace has shown", he wrote, "that gravitation should propagate instantaneously, or at least much more rapidly than does light." But Poincaré [1] noted that, in his own work, the assumption that gravitation has speed c was introduced together with a number of other alterations of NG, due to the "Mécanique nouvelle" [i.e., relativistic dynamics], and that, therefore, Laplace?s old result, which referred to the case where the former assumption was the only modification to NG, was not an argument against the new theory. He actually imposed to his relativistic theory of gravitation the *constraint* that no departure from Newton's attraction law should be predicted if one would neglect all v^2/c^2 terms, with v the velocity of a point mass subjected to the gravitational attraction [1b]. Later on, he summarized [2] the results of apparently a number of detailed astronomical calculations that he had done with his new theory of gravitation, and he concluded that "the only sensible effect on astronomical observations would be a motion of Mercury?s perihelion, in the same direction as the observed and unexplained one, though notably smaller" [6 times smaller, indeed].

As is well-known, Einstein?s general relativity (GR) also predicts that gravitation has speed c. The modification of celestial mechanics due to GR is obtained by a perturbative expansion valid for weak *and quasi-stationary* fields, known as the "post-Newtonian approximation" (PNA) [3-7]. But, at the PNA of GR, the gravitational field is characterized by potentials obeying Poisson-like equations, thus

gravitation propagates instantaneously. Would the results of Poincaré [1-2] be the only test we have of the compatibility between astronomical observations in the solar system and a theory in which gravitation really propagates with speed c? There is yet another perturbative approximation for GR, known as the linearized theory, in which the propagation speed c is maintained, thus including retardation effects. But this linear approximation is used only in the "far zone", in order to discuss the question of gravitational radiation. The reaction effects to the emission of gravitational waves on the motion of celestial bodies, the latter being confined to the "near zone", are then evaluated by a very indirect method [8], which seems questionable (see e.g. Ref. [9]). For a long time, a *direct* estimate of the retardation effects due to the propagation speed c in celestial mechanics has been missing. Such direct estimate was first derived for the two-bodies problem [11a-b], by using an iterative "post-Minkowskian" approximation (PMA) leading to wave equations, instead of Poisson equations, for the gravitational potentials - and thus allowing to describe propagation effects [10]. This included a direct (and higher-order) estimate of the mechanical reaction effects to the emission of gravitational radiation, which was found to exactly corroborate [11a-b] the indirect first-order estimate [8]. Moreover, in expanding the retarded quantities as power series in $1/c$, the usual ("Einstein-Infeld-Hoffmann" or EIH) equations of the PNA of GR were recovered at the order $1/c^2$ [10c]. Thus, the retardation effects were found to have no action on the motion of a two-bodies system up to and including the order $1/c^2$. This analysis [10-11] was indeed restricted to two point masses, but it was since generalized to a system made of an arbitrary number of extended bodies (e.g. the solar system), with the same result, i.e., the EIH equations were recovered at the order $1/c^2$ [12]. The latter work constitutes what may be called the standard "post-Minkowskian" approximation (PMA) of GR, in the same way as Refs. [3-7] are (different, but essentially equivalent) standard forms of the PNA of GR.

Common to both the standard PNA and the standard PMA is the fact that only the gravitational field is expanded in a power series, whereas the matter fields are left "unexpanded". If one looks for asymptotic expansions in the usual sense, one is led to expand also the matter fields [13-14]. The expansions obtained thus have a clear status from the viewpoint of the approximation theory. One of the questions that one could thus hope to answer clearly is whether it is completely safe to state that the retardation due to propagation has no effect at the order $1/c$, and even at the order $1/c^2$. This question has a particular relevance since Van Flandern [15] has used a model of gravitation with propagation, based on *a priori* assuming an "aberration formula" for the gravitation force and, therefore, leading to *first-order* ($1/c$) departure from NG - as is also the case for the force law proposed by Beckmann [16]. According to this model with first-order retardation effects, observations in the solar system and observations of binary pulsars would imply that [15] "the speed of gravity is no less than $2.10^{10}\ c$ " !!

The aim of this paper is to check, in a way as clear-cut as possible, whether or not the effects of the propagation speed do lead to first-order departure from NG in celestial mechanics. However, this will be done, not in the frame of GR, but within a recently proposed *scalar* theory, which is a preferred-frame theory [17-19]. This has a great advantage in simplicity. It has been shown, moreover, that, at the PNA and in the case of a static spherical body (thus the assumptions made in GR for that purpose), this theory predicts no preferred-frame effect for light rays, and in fact recovers just the standard predictions of GR for the gravitational effects on light rays [20]. Since those effects are the best established predictions of GR, this justifies investigating this very simple theory further. The recognition that the cosmic microwave blackbody radiation is isotropic only in one local Lorentz frame, which may be interpreted as an absolute local inertial frame [21], also gives support for investigating a preferred-frame theory. We first present the exact theory. Then we develop its PMA, whose field equation is just the Poisson equation with the d?Alembert operator in the place of Laplace?s. The proposed PMA differs from the

standard PMA used in GR, in that the former is a straightforward application of the usual method of asymptotic expansions and, in particular, all fields are expanded. We finally assess, in the zone occupied by the gravitating system, the magnitude of the retardation effects on the potential and the attraction field: We show that there is no dynamical effect of the retardation below the *second order* in the velocity (the order $1/c^2$).

2. Brief summary of the scalar theory

Space-time is endowed with both a flat metric $\mathbf{g}^0$ and a curved metric $\mathbf{g}$, the latter being interpreted as physical (the "effective metric"). The preferred frame E is assumed to be an inertial frame for the flat metric, i.e., there are Galilean coordinates (x^μ) for $\mathbf{g}^0$, that are bound to the frame E. ["Galilean" coordinates means that, in that coordinates, $(g^0{}_{mn}) = (h_{mn}) \circ \mathrm{diag}(1, -1, -1, -1)$]. The inertial time $t \circ x^0/c$ in the preferred frame, is called the "absolute time". The scalar field of the theory is the contraction factor b affecting the time interval $dt_{\mathbf{x}}$ measured by a clock fixed at point $\mathbf{x}$ bound to the frame E (due to the dilation of the clock period in the gravitational field), or the square $f = b^2$. The latter defines the g_{00} component of the physical space-time metric $\mathbf{g}$ in any coordinates (y^μ), though with $y^0 = ct$, that are bound to the frame E :

$$\mathrm{d}t_{\mathbf{x}}/\mathrm{d}t = \sqrt{(\gamma_{00})_{\mathrm{E}}} \circ \sqrt{f} \circ b \leq 1. \tag{2.1}$$

The field f is also a potential for the assumed gravity acceleration $\mathbf{g}$:

$$\mathbf{g} = -\frac{c^2}{2}\nabla f, \qquad (\nabla f)^i \equiv g^{0ij} f_{,j}\ , \tag{2.2}$$

where $(g^{0\,ij})$ is the inverse matrix of the component matrix $(g^0{}_{ij})$ of $\mathbf{g}^0$, with $\mathbf{g}^0$ the spatial part of the flat metric $\mathbf{g}^0$ in the frame E (thus $\mathbf{g}^0$ is an Euclidean metric). The field b still defines the dilation of an infinitesimal distance $\mathrm{d}l$ measured in the direction of vector $\mathbf{g}$, as compared with the distance $\mathrm{d}l^0$ evaluated using the Euclidean metric: $\mathrm{d}l = \mathrm{d}l^0/b$, while the infinitesimal distances in directions perpendicular to $\mathbf{g}$ are unaffected. (This makes the physical space metric $\mathbf{g}$ in the frame E discontinuous at the isolated points where $\mathbf{g} = 0$. But $\mathbf{g}$ is merely the effective metric and, moreover, it remains bounded in the neighborhood of any such point: no theoretical difficulty occurs [20].) This gives the following expression for metric $\mathbf{g}$ in the frame E (in any coordinates (y^μ) bound to E):

$$g_{ij} = g^0{}_{ij} + \left(\frac{1}{f} - 1\right) h_{ij}\,, h_{ij} \equiv \frac{f_{,i} f_{,j}}{g^{0\,kl} f_{,k} f_{,l}} \tag{2.3}$$

The space-time metric writes, in any coordinates (y^m) bound to the frame E, with $y^0 = ct$:

$$ds^2 = g_{mn}\, dy^m\, dy^n = f\,(dy^0)^2 - dl^2,\ dl^2 = g_{ij}\, dy^i dy^j \ \Rightarrow\ g_{00} = f,\ g_{ij} = -g_{ij},\ g_{0i} = 0. \quad (2.4)$$

That the g_{0i} components are always zero in the ether frame E, is the expression of the assumption that this frame admits a global simultaneity defined with the absolute time t. Except for cosmological problems, the field equation [17] may be rewritten [18] as:

$$\Delta f - \frac{1}{f}\left(\frac{f_{,0}}{f}\right)_{,0} = \frac{8\pi G}{c^2}\,\sigma \qquad (y^0 = ct), \quad (2.5)$$

with D ° div (Ñ) the usual Laplace operator, defined with the Euclidean metric $\mathbf{g}^0$, and where G is Newton's gravitation constant, and s ° $(T^{00})_E$ is the mass-energy density in the ether frame [18]. This equation, which is merely space-covariant, implies that gravitation propagates with the velocity of light c (as measured with physical standards affected by gravitation).

Motion is governed by a natural *extension of Newton?s second law:*

$$\mathbf{F}_0 + m(v)\mathbf{g} = D\mathbf{P}/Dt_{\mathbf{x}} \quad (2.6)$$

[18], which also may be defined for any theory of gravitation in curved space-time [22]. With the assumed gravity acceleration **g** (Eq.(2.2)), it *implies* Einstein?s motion along geodesics of metric **g**, but only for a *static* gravitational field [22]. In Eq.(2.6), $\mathbf{F}_0$ is the non-gravitational force, v is the modulus of the velocity **v** of the test particle (relative to the frame E), measured with the local time $t_{\mathbf{x}}$ (Eq.(2.1)), and the modulus v is defined with the space metric **g**:

$$v^i \text{ ° } dy^i/dt_{\mathbf{x}}\ ,\ v \text{ ° } [\mathbf{g}(\mathbf{v}, \mathbf{v})]^{1/2} = (g_{ij}\, v^i\, v^j)^{1/2}; \quad (2.7)$$

$m(v)$ ° $m(0).g_v$ ° $m(0).(1 - v^2/c^2)^{-1/2}$ is the relativistic inertial mass, **P** ° $m(v)$ **v** is the momentum, and $D/Dt_{\mathbf{x}}$ is the derivative of a spatial vector appropriate to the case where the Riemannian spatial metric **g** varies with time. In particular, with this derivative, Leibniz? rule for the time derivative of a scalar product **v.w** = **g**(**v**,**w**) is satisfied [18].

Newton?s second law, (2.6) with (2.2), being thus defined for any mass particle, it is thereby defined for a dust, i.e., a continuum made of coherently moving, non-interacting particles, each of which conserves its rest mass. It then mathematically implies, independently of the assumed form for the space-time metric **g** provided it satisfies $g_{0i} = 0$ in the preferred frame, the following dynamical equation for the dust, in terms of its mass tensor

$$T^{mn} = r^*\, U^m\, U^n$$

(with r^* the proper rest-mass density and $U^m = dy^m/ds$ the 4-velocity) [23]:

$$T_\mu{}^\nu{}_{;\nu} = b_\mu . \tag{2.8}$$

Here, b_μ is defined by

$$b_0(\mathbf{T}) \equiv \frac{1}{2} g_{jk,0}\, T^{jk} \quad , \quad b_i(\mathbf{T}) \equiv -\frac{1}{2} g_{ik,0}\, T^{0k} \tag{2.9}$$

(indices are raised and lowered with metric **g**, unless mentioned otherwise; the derivation of Eqs.(2.8)-(2.9) [23] is summarized in Ref. [19]). Equation (2.8), with the definition (2.9), is assumed to hold for any material continuum: accounting for the mass-energy equivalence, this is the expression of the assumed universality of gravitation. In Cartesian coordinates (x^i) for $\mathbf{g}^0$ (i.e. $g^0{}_{ij} = d_{ij}$) and with $x^0 = ct$, Eq.(2.8) becomes [23]:

$$T_0{}^\nu{}_{,\nu} = \frac{1}{2} f_{,0}\, T^{00} \circ \frac{1}{2} f_{,0}\, s , \tag{2.10}$$

$$T_i{}^\nu{}_{,\nu} = \frac{1}{2} \left(f_{,i}\, \sigma - g_{ik,0}\, T^{0k} - g_{jk,i}\, T^{jk} \right) . \tag{2.11}$$

3. Principle of the perturbative expansions

i) Let us state some principles common to any perturbative expansion valid for weak gravitational fields, in this or another theory of gravitation. As for any asymptotic expansion, one must define a small parameter which, at least conceptually, one should be able to make tend towards zero. Hence, one should formally envisage a *family* of gravitating systems depending on a continuous parameter e, with each system S_e being, ideally, *characterized* by the value of e. To be definite, we assume that all matter is in the form of barotropic perfect fluids (one per astronomical body), with mass tensor

$$T^{mn} = (m^* + p/c^2)\, U^m\, U^n - (p/c^2)\, g^{mn} , \tag{3.1}$$

where μ^* is the volume density of the rest-mass plus internal energy in the proper frame, expressed in mass units,

$$m^* \circ r^* (1 + P/c^2), \tag{3.2}$$

with, by hypothese, $m^* = m^*(p)$; then, r^* and P also depend on p only, P being given by [3]

$$P(p) = \Pi(p_0) + \int_{p_0}^{p} \frac{dq}{\rho^*(q)} - \left[\frac{q}{\rho^*(q)}\right]_{p_0}^{p} . \qquad (3.3)$$

We assume that each system S_e is made of separated bodies w_1, ..., w_n. The mass of a body w is defined as $m = \int_w \rho dV$ where r is the density of rest-mass in the preferred frame, given by [15]

$$r = r^* \sqrt{g_v}/b \qquad (3.4)$$

[r is defined with respect to the Euclidean volume measure $dV = \sqrt{g^0}\, dy^1\, dy^2\, dy^3$, where $g^0 = \det(g^0_{ij})$]. We assume that, for the system S_e, the masses are $m_i^e = e^2 m_i$, while the size r_i of the body w_i and the separation distance r_{ij} between w_i and w_j (evaluated, say, with the Euclidean space metric $\mathbf{g}^0$) are like e^0. Therefore, the mass density r^e is like e^2 . [Thus, we may imagine that, as e is varied and tends towards zero, the system S_e is made of bodies whose sizes and separation distances do not depend on e, while their density is decreased, varying like e^2. Of course, in reality, we will have to do with *one* system. However, it is shown in ref. [14] that one may indeed define a family (S_e) of systems, with just those properties, from the mere data of the real system assumed to obey the gravitational equations.] Hence, *in the "near zone",* i.e. at distances of the order of the r_{ij}?s and hence that are like e^0, the Newtonian potential

$$U_N^e = G \int \rho^e / r\, dV ,$$

will also be like e^2. (Obviously, if we go to distances that are like e^{-2}, say, then U_N^e will be like e^4.) Thus, we may physically define the dimensionless small parameter as $e = [(U_N)_{max}/c^2]^{1/2}$ [6]. We assume that, as in NG, the velocity field (of the bodies) in the mass-center frame, **u?**^{e}, is of the same order of magnitude as $\sqrt{U_N}$, thus like e, and the pressure p is of the same order of magnitude as $r_0 U_N$ [7]. (This may be justified in the frame of the theoretical construction of the family (S_e) [14].) Thus, p is like e^4.

Now, we multiply the mass unit by e^2 and the time unit by e^{-1} (note that this does not affect the value of G). It is easy to verify that all quantities: U_N, **u?**, r_0, p, r_i, r_{ij}, are now like e^0, hence the spatial derivatives of those quantities (at distances that are like e^0, e.g. at distances characteristic of the system) will also be like e^0. Moreover $e = c_0/c$ (with c_0 the numerical value of c in the starting units, which are independent of e), thus *the small parameter may really be taken as* $e' = 1/c$. It is then not equivalent to consider that the time derivatives are O(1) with $y^0 = t$ or with $y^0 = ct$, and this is precisely *the departure point between the PNA* ($y^0 = t$) *and the PMA* ($y^0 = ct$), as noted by Fock [3]. However, if $\partial f/\partial t$, say, is

O(1) (thus of order *higher than or equal to* e^0), then $\partial f/\partial(ct)$ is *a fortiori* O(1) [and in fact is even O(e)]. I.e., the domain of validity of the assumption $\partial f/\partial(ct)$ = O(1) ("propagating fields") *includes* that of the assumption $\partial f/\partial t$ = O(1) ("quasi-stationary fields"). In the following, we shall make the weakest hypothese among the two possible ones, or *"PM hypothese":* if f is O(e^k), then $\partial f/\partial(ct)$ is also O(e^k). This hypothese has to be used only when time-differentiating the remainder term in the asymptotic expansions, e.g. f is the O(e^4) term in Eq. (3.5) below. It means that the derivative with respect to ct commutes with the power expansion. The result will be a PMA, with wave operators.

ii) Let us now specialize to the investigated scalar theory. The true *definition* of a weak-gravitational-field limit [14] includes, in particular, the prescription that the effective space-time metric **g** must reduce to the flat metric $\mathbf{g}^0$ in this limit. [In point **i)** above, we have, *as usual*, *preassumed* that NG will be a good approximation in this limit; this must be checked, and will be checked afterwards, even though we know that NG is indeed a good approximation in the *real* solar system!] Due to the assumed form (2.3)-(2.4) for the metric, this means that the scalar field f must have the form $f = 1 + fe^k + O(e^k+1)$ with k a positive integer, and the question is: which integer? Expanding the field equation (2.5), one finds that its left-hand side is then like e^k, whereas its r.h.s. is like e^2 (since $e \mu 1/c$ and since s is like e^0). Hence, one must have $k = 2$. More generally, there is an excellent reason to make the following assumption *(P)* [for "parity"]: *for any relevant field, its asymptotic expansion in powers of e contains only even terms, or only odd terms*. Indeed, we have seen that the small parameter e may be taken to be $1/c$. Now, in the exact field equation (2.5), as well as in the exact dynamical equations for a perfect fluid [Eqs. (4.1) and (4.2) below], it is only $1/c^2$ that enters the formulas. We set therefore

$$f = 1- 2U/c^2 + O(e^4), \tag{3.5}$$

with U = O(1) in units such that $1/c$ is equivalent to const ´ e , see above; however, Eq. (3.5) holds in any units. Note that the absence of the $1/c$ term in (3.5) has been *proved:* it is the absence of the $1/c^3$ term that we postulate as a particular case of assumption (P). The latter is an (only natural) *a priori* restriction on the form of the expansion searched for. Since we will find here (thus up to the order $1/c^2$) that the relevant fields and equations do admit consistent expansions satisfying assumption (P), this assumption will be justified *a posteriori*. The argument used in GR and other metric theories for assuming expansions involving merely even powers of $1/c$, is based on the statement that odd-order terms change sign under time reversal and therefore must represent energy dissipation or absorption by the system [6-7].

Finally, recall that the equations of the theory are given in the preferred frame E. The velocity, assumed to be of the order zero (after the change of units explained above, thus of the order ec) in the mass-center frame, will remain so in E, only if the absolute velocity **V** of the mass-center frame is O(ec), which we assume. For the solar system, we have $e^2 \circ (U_N)_{max}/c^2 \approx 10^{-5}$ [7] and if V is assessed by assuming that the preferred frame is at rest with respect to the cosmic microwave background, then V is approximately 300 km/s [7], so that $V/c \approx 10^{-3} < e$.

4. Zero-order post-Minkowskian approximation of the scalar theory

We begin by giving the explicit form of the exact dynamical equations (2.10) and (2.11) for a perfect fluid, with mass tensor (3.1)-(3.2). A straightforward calculation gives respectively

$$\frac{1}{f}\left(\frac{f_{,0}}{f}\right)_{,0} = \frac{-2}{c^2}U_{,0,0} + O(\varepsilon^4), \tag{4.1}$$

$$\frac{\partial}{\partial t}(\rho u_i) + (\rho u_i u^j)_{,j} - \frac{1}{2}\rho u^j\left(\frac{\partial g_{ij}}{\partial t} + g_{jk,i}u^k\right) = \rho g^i - p_{,i}, \tag{4.2}$$

with $u^i = \mathrm{d}x^i/\mathrm{d}t$,

$$\rho \equiv \sigma + \frac{p}{c^2 f} = \left(\mu^* + \frac{p}{c^2}\right)\frac{\gamma_v^2}{f}, \tag{4.3}$$

and

$$u_i \circ g_{ij}\, u^j, \tag{4.4}$$

and where Cartesian space coordinates (x^i) are being used. Let us expand these equations and the field equation (2.5) within the *zero-order PMA*. That is, we search for the equations that replace (2.5), (4.1) and (4.2) when one substitutes for the independent fields f - 1, p, **u** the first term of their powers expansion with respect to $e^2 \mu 1/c^2$ [cf. assumption (P) above], assuming the PM hypothese [see the end of point **i)** in §**3**]. We denote by r_0, p_0, $\mathbf{u}_0$, etc. the zero-order terms in the expansions of r, p, **u**, etc.:

$$r = r_0 + O(e^2), \quad p = p_0 + O(e^2), \mathbf{u} = \mathbf{u}_0 + O(e^2), \ldots \tag{4.5}$$

First, Eq.(3.5) with (2.3) shows that

$$g_{ij} = d_{ij} + O(e^2) \tag{4.6}$$

(in other words, the first-order term in the expansion of the effective spatial metric **g** is the Euclidean metric $\mathbf{g}^0$ with Cartesian components $g^0{}_{ij} = d_{ij}$); hence

$$g_{ij,\,k} = O(e^2), \quad \frac{\partial g_{ij}}{\partial x^0} = O(\varepsilon^2). \tag{4.7}$$

We obtain from (4.6), (3.2), (3.4) and (4.3):

$$u_{0\,i} = u_0{}^i,\ r^*{}_0 = \mu^*{}_0 = r_0 = s_0 = y_0. \tag{4.8}$$

We get from (3.5):

$$\mathrm{D}f = \frac{-2}{c^2}\Delta U + \mathrm{O}(\varepsilon^4),$$

$$\frac{1}{f}\left(\frac{f_{,0}}{f}\right)_{,0} = \frac{-2}{c^2}U_{,0,0} + \mathrm{O}(\varepsilon^4).$$

Substituting this in Eq.(2.5) and using Eq.$(4.8)_2$, we obtain the wave equation

$$\Delta U - U_{,0,0} = -4\pi G\rho_0 . \tag{4.9}$$

The relevant solution of Eq.(4.9) is the retarded potential

$$U(\mathbf{X}, x^0) = \int \frac{G}{R}\rho_0\left(\mathbf{x}, x^0 - R\right)\mathrm{d}V(\mathbf{x}), \quad R \circ |\mathbf{X} - \mathbf{x}| \tag{4.10}$$

(recall that dV is the Euclidean volume element in the frame E). Note that we must first derive a complete expansion of the set of equations in terms of the variable $x^0 = ct$: since

$$x'^0 \circ t \mu e\, x^0,$$

going from x^0 to x'^0 is going from a *"quick"* variable to a *"slow"* one, hence the expansions *will not be the same* in terms of x^0 or x'^0. Therefore, we must rewrite the dynamical equations (4.1) and (4.2) in terms of $x^0 = ct$:

$$\partial_0(\rho f) + \frac{1}{c}\partial_j(\rho f u^j) = \frac{\sigma}{2}\partial_0 f + \frac{1}{c^2}\partial_0 p, \tag{4.11}$$

$$\partial_0(\rho u_i) + \frac{1}{c}\partial_j(\rho u_i u^j) - \frac{1}{2}\rho u^j\left(\partial_0 g_{ij} + \frac{1}{c}g_{jk,i}u^k\right) = \frac{1}{c}\left(\rho g^i - p_{,i}\right). \tag{4.12}$$

Inserting the expansions (3.5) and (4.5) of the exact fields in Eq.(4.11), and using $(4.8)_2$, we get:

$$\partial_0\rho_0 + \frac{1}{c}\partial_j\left(\rho_0 u_0^j\right) + \mathrm{O}(e^2) = \mathrm{O}(e^2). \tag{4.13}$$

If we would identify the coefficients of $1/c^k$ in both sides, we would get the absurd equations $\partial_0 r_0 = 0$ and $\partial_j(r_0 u_0{}^j) = 0$. But we have assumed expansions in *even* powers of $1/c$, hence the l.h.s. must have the form of a polynomial in $1/c^2$ [possibly multiplied by $1/c$, since we may multiply or divide the whole of Eq.(4.13) by $1/c$]. Hence, the two successive powers of $1/c$ in the l.h.s. must actually reduce to a monomial in $1/c^2$ (possibly multiplied by $1/c$). That is, we must have simultaneously, for some integer k,

$$\partial_0 r_0 = \mathrm{ord}(1/c^k) \text{ and } \frac{1}{c}\partial_j\left(r_0 u_0{}^j\right) = \mathrm{ord}(1/c^k), \tag{4.14}$$

where the notation $f = \mathrm{ord}(e^p)$ means that f is of order e^p exactly. Since $\partial_0 r_0 = O(e^0)$ by the PM hypothese and since $\partial_j(r_0 u_0{}^j) = \mathrm{ord}(e^0)$, it follows that one must have $k = 1$ and that in fact

$$\partial_0 r_0 = \mathrm{ord}(e). \tag{4.15}$$

Now by identifying the l.h.s. of (4.13) with the r.h.s., the monomial obtained by summing the two parts of (4.13) must be *zero*, that is

$$\partial_0 r_0 + \frac{1}{c}\partial_j\left(r_0 u_0{}^j\right) = 0. \tag{4.16}$$

In the same way, inserting the expansions (4.5) into Eq.(4.12), using (4.7) and (4.8), and using also (3.5) in (2.2) gives us

$$\partial_0\left(r_0 u_0{}^i\right) + \frac{1}{c}\partial_j\left(r_0 u_0{}^i u_0{}^j\right) + O\left(e^2\right) = \frac{1}{c}\left(r_0 U_{,i} - p_{0,i}\right) + O\left(e^3\right), \tag{4.17}$$

and the line of reasoning already used to go from (4.13) to (4.16) leads then to

$$\partial_0\left(r_0 u_0{}^i\right) + \frac{1}{c}\partial_j\left(r_0 u_0{}^i u_0{}^j\right) = \frac{1}{c}\left(r_0 U_{,i} - p_{0,i}\right). \tag{4.18}$$

We have thereby got the complete system of equations of the zero-order PMA of the scalar theory in terms of the quick variable $x^0 = ct$, Eqs.(4.9), (4.16) and (4.18). Because Eq.(4.9) leads to the retarded potential (4.10), this system is not tractable.

5. Reexpression of the PM expansion in terms of the slow time variable

It turns out to be possible to eliminate the retardation if we reexpress the retarded potentials in terms of the "slow" time coordinate $x'^0 = t$, which makes the small parameter e or $1/c$ appear. Consider any field, even one that is independent of e, such as a coefficient appearing in a PM expansion, e.g. $r_0(\mathbf{x},x^0)$. When this field is reexpressed in terms of the slow variable $t = x^0/c$, it becomes a function of e also. We assume that the new expressions of the relevant coefficients do themselves admit expansions in even powers of e (because, once again, it is e^2 that appears in the exact equations) and we momentarily push the expansions one step further:

$$\tilde{\rho}_0(\mathbf{x}, t) \circ r_0(\mathbf{x}, ct) = r'(\mathbf{x}, t) + r''(\mathbf{x}, t)/c^2 + O(e^4), \qquad (5.1)$$

$$\tilde{p}_0(\mathbf{x}, t) \circ p_0(\mathbf{x}, ct) = p'(\mathbf{x}, t) + p''(\mathbf{x}, t)/c^2 + O(e^4), \qquad (5.2)$$

$$\tilde{\mathbf{u}}_0(\mathbf{x}, t) \circ \mathbf{u}_0(\mathbf{x}, ct) = \mathbf{u}'(\mathbf{x}, t) + \mathbf{u}''(\mathbf{x}, t)/c^2 + O(e^4). \qquad (5.3)$$

Let us rewrite the dynamical equations of the zero-order PMA, Eqs.(4.16) and (4.18), in terms of the slow variable t :

$$\partial_t \tilde{\rho}_0 + \partial_j\left(\tilde{\rho}_0 \tilde{u}_0{}^j\right) = 0, \qquad (5.4)$$

$$\partial_t\left(\tilde{\rho}_0 \tilde{u}_0{}^i\right) + \partial_j\left(\tilde{\rho}_0 \tilde{u}_0{}^i \tilde{u}_0{}^j\right) = \tilde{\rho}_0 \tilde{U}_{,i} - \tilde{p}_{0,i}. \qquad (5.5)$$

These are not the *expanded* equations of the "PMA-with-slow-variable", for they involve the "tilded" fields, which *depend* on the small parameter $1/c$ as is apparent in Eqs.(5.1)-(5.3). To obtain the expanded equations of the "PMA-with-slow-variable", we must precisely substitute the expansions (5.1)-(5.3) for the tilded fields. But we also need the expansion of the retarded potential (4.10); in terms of $x'^0 = t$, the potential is

$$\tilde{U}(\mathbf{X}, t) \equiv U(\mathbf{X}, ct) = \int \frac{G}{R} \tilde{\rho}_0\left(\mathbf{x}, t - \frac{R}{c}\right) dV(\mathbf{x}). \qquad (5.6)$$

We should not *a priori* assume an expansion in even powers of $1/c$ for the latter, because the expansion of the retarded density with respect to the retardation *a priori* introduces also odd powers:

$$\tilde{\rho}_0\left(\mathbf{x}, t - \frac{R}{c}\right) = \tilde{\rho}_0(\mathbf{x}, t) - \frac{R}{c}\frac{\partial \tilde{\rho}_0}{\partial t}(\mathbf{x}, t) + \frac{R^2}{2c^2}\frac{\partial^2 \tilde{\rho}_0}{\partial t^2}(\mathbf{x}, t) + O\left(\varepsilon^3\right), \qquad (5.7)$$

which we have to introduce in (5.6). [Differentiating (5.4) twice, one shows first that $\partial^2 \tilde{\rho}_0/\partial t^2 = O(e^0)$, then that $\partial^3 \tilde{\rho}_0/\partial t^3 = O(e^0)$.] However, due to the continuity equation (5.4), we have

$$\int -\frac{G}{R}\frac{R}{c}\partial_t \tilde{\rho}_0(\mathbf{x}, t) dV(\mathbf{x}) = \int \frac{G}{c} \operatorname{div}\left(\tilde{\rho}_0 \tilde{\mathbf{u}}_0\right) dV = 0, \qquad (5.8)$$

i.e., the contribution of the $1/c$ term in the r.h.s. of Eq.(5.7) cancels. We get

$$\tilde{U}(\mathbf{X}, t) = \int \frac{G}{R} \tilde{\rho}_0(\mathbf{x}, t)\, dV(\mathbf{x}) + \frac{1}{2c^2}\int G R \frac{\partial^2 \tilde{\rho}_0}{\partial t^2}(\mathbf{x}, t)\, dV(\mathbf{x}) + O\left(\varepsilon^3\right). \qquad (5.9)$$

Inserting the expansion (5.1) into (5.9), we obtain an even asymptotic expansion for $\tilde{U}$:

$$\tilde{U}(\mathbf{X}, t) = U'(\mathbf{X}, t) + U''(\mathbf{X}, t)/c^2 + \mathrm{O}(e^3), \tag{5.10}$$

with

$$U'(\mathbf{X},t) = \int \frac{G}{R}\rho'(\mathbf{x},t)\, \mathrm{d}V(\mathbf{x}) \tag{5.11}$$

and

$$U''(\mathbf{X},t) = \int \frac{G}{R}\rho''(\mathbf{x},t)\, \mathrm{d}V(\mathbf{x}) + \frac{1}{2}\int G R \frac{\partial^2 \rho'}{\partial t^2}(\mathbf{x},t)\, \mathrm{d}V(\mathbf{x}). \tag{5.12}$$

Now we may substitute into (5.4) and (5.5) the expansions (5.1)-(5.3) and (5.10) for the tilded fields, and thus obtain the expanded equations of the "PMA-with-slow-variable". Evidently, if we consider only the zero-order expansion, we get simply

$$\partial_t \rho' + \left(\rho' u'^j\right)_{,j} = 0, \tag{5.13}$$

$$\partial_t\left(\rho' u'^i\right) + \partial_j\left(\rho' u'^i u'^j\right) = \rho' U'_{,i} - p'_{,i} \tag{5.14}$$

in which the potential U' is given by (5.11). Thus, at the order zero, the "PMA-with-slow-variable" is just Newtonian gravity. Moreover, the next order is definitely the order $1/c^2$, not the order $1/c$.

6. Conclusion

One may consistently implement the usual method of asymptotic expansions to obtain tractable equations in relativistic celestial mechanics. This is done by first associating a one-parameter family of gravitating systems with the given system, the parameter e measuring the strength of the gravitational field (this is more detailed in Ref. [14]; after this work was done, the author learned that a similar method was initiated by Futamase & Schutz [13b]). Then a suitable change of units, depending on e, makes $1/c$ equal (or proportional) to e and all relevant fields are of the order of unity. In these varying units, the derivation of asymptotic expansions is thus straightforward (automatical), although there is a bifurcation as to which of the two time variables $x^0 = ct$ and $x'^0 = t$ is chosen in the initial derivation of the expansions.

Using this procedure, it has been shown (focusing, in this paper, on the *zero-order* expansion of a *scalar* theory for simplicity) that: **(i)** The effect of the propagation of gravity may be easily and consistently retained in the process of the asymptotic expansion, not only in the wave zone but also in the zone occupied by the gravitating system. Indeed, the retainment of the propagation is obtained simply by expanding the whole set of equations when the time variable is $x^0 = ct$. One thus obtains equations with the d?Alembert operator, instead of the Laplacian. Hence the dynamics is governed by retarded potentials and is thus not tractable. **(ii)** In a second step, one may re-express the set of the expanded

equations in terms of the "slow" time variable $x'^0 = t$. As a result, one has to re-expand the equations. In the case of the retarded potential, this re-expansion automatically involves an expansion with respect to the retardation. The latter expansion *a priori* contains odd powers of $1/c$ as well as even powers, whereas in all other expansions there is simply no reason to consider odd powers of $1/c$ (because the actual small parameter is $1/c^2$). But, due to the continuity equation, the first odd power, thus the first order in $1/c$, is cancelled out in the expansion of the retarded potential. Therefore, up to the *second-order* [$O(1/c^2)$] terms, the whole set of expanded equations is that of Newtonian gravity. (In particular, there is no preferred-frame effect up to the order $1/c^2$.) **(iii)** However, Eqs.(5.10) and (5.12) show that the retardation effects *do influence* the attraction field (and hence *a priori* also the whole dynamics) already at the second order, which is also the order of the "relativistic" corrections to celestial mechanics. This is confirmed by the full expansion at the order $1/c^2$, which is a straightforward continuation of the present work [14].

Although these three conclusions have been obtained within an alternative, scalar theory of gravitation, it seems to the author that the same conclusions are valid for GR. First, it is well-known that equations involving the d?Alembert operator (with sources), instead of the Laplacian, govern the weak gravitational fields in GR under the harmonic gauge [3, §68; 24, p. 317], and after all this also applies in the near zone, as is already known from the standard PMA [10-12]. Second, if one uses in GR the same method of asymptotic expansion that has been used here, the dynamical equation for a perfect fluid in the zero-order approximation will *not* be the "flat space-time" equation $T_m{}^n{}_{,n} = 0$: instead, the Newtonian equations (4.16) and (4.18) should be obtained, just as here. The main reason is that the exact dynamical equation in the scalar theory, Eq.(2.8), differs from the equation $T_m{}^n{}_{;n} = 0$ of GR only by the r.h.s., which for a perfect fluid does not leave contributions in the first approximation.

In the solar system, the propagation at speed c is thus likely to imply very small effects (in the range of a 10^{-6} fraction of the Newtonian term) and should lead to an as good agreement with astrodynamical observations as does *Newtonian* celestial mechanics. On the other hand, the propagation effects should *a priori* have the same order of magnitude as the usual *relativistic* corrections such as the relativistic contribution to Mercury?s perihelion shift. But the relativistic corrections are calculated using the Einstein-Infeld-Hoffmann Lagrangian, obtained in assuming quasi-stationary fields with instantaneous propagation (although the EIH Lagrangian is also obtained when one uses the standard PMA [10c, 12]). One thus may ask if these usual corrections are altered by the propagation effects when one takes the latter into account within the PMA outlined here, which is based on the usual method of asymptotic expansion and differs from the standard PMA.

It would also be worth investigating the quantitative consequences of the following point: a number of astrodynamical parameters such as the masses of the planets are obtained by a fitting of observational data. The fitting model, hence the masses, are theory-dependent. Today it is admitted that NG is merely the zero-order approximation of a more correct theory (be it GR or some alternative theory). But then, one must expect that the masses $M_{i\mathrm{N}}$ obtained by fitting observations within NG alone should differ from the " true " masses of the first approximation, M_{i0}, by quantities of the same order as the second-approximation corrections [19]. Hence, if one assumes $M_{i\mathrm{N}} = M_{i0}$, one makes an error which is *a priori* of the same magnitude as the second-approximation corrections, i.e., as the *first corrections to NG*. (This is true in celestial mechanics, thus excluding light rays: for those, the first pN corrections are unaffected.)

Acknowledgement

The author is indebted to Professor Jean-Pierre Vigier who pointed out Ref. [15] and suggested that the effect of the propagation of gravity at speed *c* could be important.

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