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ϵ-Expansion for non-planar double-boxes

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Abstract: We present calculations for non-planar double-box with four massless/massive external/internal legs/propagators. The results are expressed for arbitrary exponents of propagators and dimension in terms of Lauricella's hypergeometric functions of three variables and hypergeometric-like multiple series.

Keywords: Quantum field theory, negative dimensional integration, double-box integrals, radiative corrections.

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1. Introduction

Quantum field theories have come a long way since its inception and today, all known interactions have been classed within its model description for elementary particles in nature [1]. Within this framework, studies on Feynman loop integrals became even more compelling, challenging us with ever increasing mathematical complexities associated with the perturbative approach. Results for calculations of diagrams with massive internal particles were presented by the authors [2] and others [3]; non-planar double-box scalar and tensorial integrals were studied by Tausk [4] and Smirnov [5] using Mellin-Barnes technique — recently the triple box as well — while Gehrmann and Remiddi [6] using the powerful differential equation method calculated several 2-loop integrals. Bern and collaborators studied dimensionally regularized one-loop pentagon integrals [7] as well as Binoth *et al* [8] (numerically) and the authors (analytically and with arbitrary exponents of propagators) tackle one-loop scalar hexagon integrals [9].

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CIT: gehrmann
CIT: bern-penta
CIT: binoth
CIT: n-point

These three methods are very powerful and interesting: integration-by-parts relates a complicated integral to simpler ones with some exponents of propagators raised to powers greater than one. Differential equation method also relates a complicated graph to simpler ones – that means lesser number of loops or simpler graphs with the same number of loops –, with the exponents of propagators also changing when one uses this method. Mellin-Barnes approach is based on an integral representation in which the resulting integrals can be carried out summing up the residues (either in the right or left complex plane). The latter one therefore yields the two domains of validity available for the hypergeometric functions connected by analytic continuation. These in turn define the two distinct kinematical regions of interest [10].

CIT: boos

An altogether different and elegant approach has been suggested back in the middle of the 1980's by Halliday and Ricotta [11], coined NDIM, where the key point is the introduction of a negative dimensional space to work out the integral. Their seminal ideia was reframed within the context of solving systems of linear algebraic equations and from their original work to our present understanding and experience in it we deem NDIM to be a more compleat tool to handle complex Feynman integrals, covariant and non-covariant alike, and far simpler in its essence to implement, needing only to deal with systems of linear algebraic equations of first degree. All kinematical regions of interest come simultaneously defined and the bonus by-product is that it defines even as yet unknown relationships of analytic continuation among hypergeometric functions of several variables.

CIT: halliday

Our aim in this work is to evaluate some loop integrals which were not considered in the literature up to now using the NDIM technique. They are double-box scalar integrals with six propagators: in the simplest case, i.e., massless internal particles and on-shell external legs the result [4] is well-known for a special case of exponents of propagators (all equal to minus one); we fill the gap presenting the result for arbitrary exponents of propagators and extend our knowledge studying the same graph where four internal particles have arbitrary mass μ and also studying the case where one of the external legs is off-shell. These diagram computations become important as progress in perturbative calculations for fundamental interactions between particles are checked against the background of our

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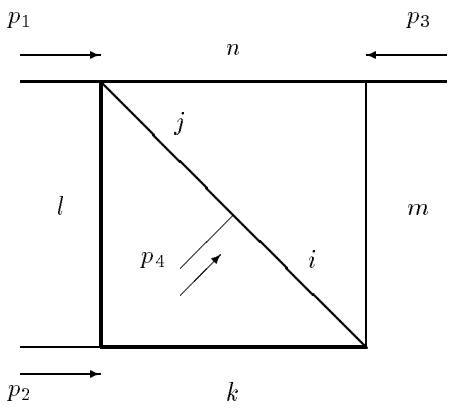

Figure 1: Scalar non-planar double-box with four massive propagators which are represented by thick lines. The labels in the internal lines represent the exponents of propagators. All external momenta are considered to be incoming, so $p_1 + p_2 + p_3 + p_4 = 0$.

present experimental data increases their precision measurements.

The outline for our paper is as follows: in section 2 we study covariant four-point integrals with six propagators – non-planar double-box with four massive propagators – the exact result of which is written in terms of hypergeometric function of three variables. We also perform numerical calculations in order to expand the result in powers of ϵ. Section 3 is concerned with massless non-planar double-box with off-shell external legs and in section 4 we give concluding remarks for our present work.

2. Massive non-planar double-box

To begin with consider the integral for the massive non-planar double-box with six propagators, namely,

$$\mathcal{I}_M = \int d^Dq\, d^Dr\, (q^2-\mu^2)^i[(q-p_3)^2-\mu^2]^j[(q+r)^2-\mu^2]^k[(q+r+p_2)^2-\mu^2]^l(r^2)^m(r-p_4)^{2n}\,. \quad (2.1)$$ LAB: im

This is the scalar integral which arises in the computation of the diagram depicted in figure 1, where the external legs represent on-shell massless particles, i.e., $p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$.

The generating functional for *massless* on-shell non-planar double-box[12] is given by, CIT: pentabox

$$\begin{aligned} G_0 = & \int d^Dq\, d^Dr\ \exp\{-\alpha q^2 - \beta(q-p_3)^2 - \gamma(q+r)^2 - \theta(q+r+p_2)^2 - \phi r^2 \\ & -\omega(r-p_4)^2\} \qquad (2.2) \\ = & \left(\frac{\pi^2}{\lambda}\right)^{D/2} \exp\left[-\frac{1}{\lambda}\left(\beta\gamma\omega s + \alpha\theta\omega t + \beta\theta\phi u\right)\right], \qquad (2.3) \end{aligned}$$ LAB: geradora-on

where we have defined $\lambda' = \alpha + \beta + \gamma + \theta$, $\lambda = \alpha\gamma + \alpha\theta + \beta\gamma + \beta\theta + \lambda'(\phi + \omega)$ and s, t, u are the usual Mandelstam's variables, given by

$$s = 2p_1 \cdot p_2, \qquad t = 2p_1 \cdot p_3, \qquad u = 2p_1 \cdot p_4\,. \tag{2.4}$$

Since $p_i^2 = 0,\ (i = 1, 2, 3, 4)$, observe that $s+t+u = 0$ follows from the above equation.

Then, considering some of the internal particles to be *massive*, that is, diagram of figure 1, one has the generating functional,

$$G_M = \exp\left(\lambda'\mu^2\right) G_0, \tag{2.5}$$

with the *massive* sector factorized, and the following system of algebraic equations,

$$\begin{cases} X_2 + Y_{1234} + W_1 &= i \\ X_{13} + Y_{5678} + W_2 &= j \\ X_1 + Y_{159} + W_3 + Z_1 &= k \\ X_{23} + Y_{26} + W_4 + Z_{23} &= l \\ X_3 + Y_{379} + Z_2 &= m \\ X_{12} + Y_{48} + Z_{13} &= n \\ \Sigma X + \Sigma Y + \Sigma Z &= -D/2 \end{cases}, \tag{2.6}$$

LAB: sistema

where W_j are the indices labelling the pure massive sector. We use the shorthand notation,

$$X_{abc} = X_a + X_b + X_c,$$

and so on. In the last equation $\Sigma X = X_{123}$, $\Sigma Y = Y_{123456789}$, $\Sigma Z = Z_{123}$. Note that the total of sum indices are 19 with 7 constraint equations, so that the result will be a series of 12 indices. This 12-fold sum may be constructed in various ways, in fact, C_7^{12} ways. A huge number of ways. The majority of them – 30,972 – yield vanishing determinant for the algebraic system, so that the solution is trivial in all of these cases. The remaining 29,416 yield non-vanishing determinant and non-trivial solutions expressed as hypergeometric series representations. The several variables that identify these series belong to a subset of the following set

$$\left\{1,\ \frac{t}{s},\ \frac{u}{t},\ \frac{s}{u},\ \frac{s}{t},\ \frac{t}{u},\ \frac{u}{s},\ \frac{t}{4\mu^2},\ \frac{u}{4\mu^2},\ \frac{s}{4\mu^2},\ \frac{4\mu^2}{t},\ \frac{4\mu^2}{u},\ \frac{4\mu^2}{s}\right\}. \tag{2.7}$$

The simplest of the hypergeometric series representations for $\mathcal{I}_M$ is given by a *triple series*,

$$\begin{aligned} \mathcal{I}_M = \pi^D (\mu^2)^\sigma\ \Gamma \sum_{X_1, X_2, X_3 = 0}^{\infty} & \frac{X^{X_1} Y^{X_2} Z^{X_3} (-\sigma|X_{123})(-n|X_{12})(-m|X_3)(-l|X_{23})}{X_1! X_2! X_3! (D/2|X_{123})(-i-j|X_{123})} \\ \times & \frac{(-i|X_2)(-j|X_{13})(-k|X_1)(-k-l-m-n-D/2|X_{123})}{(1/2-\sigma/2-m/2-n/2|X_{123})(-\sigma/2-m/2-n/2|X_{123})}, \end{aligned} \tag{2.8}$$

LAB: tripla

where $\sigma = i + j + k + l + m + n + D$, is the sum of exponents and dimension, $(a|b)$ is the Pochhammer symbol,

$$(a|b) \equiv (a)_b = \frac{\Gamma(a+b)}{\Gamma(a)},$$

and

$$\Gamma = (-\sigma - m - n | m+n)(D/2 | m+n)(-i-j | -m-n-D/2)(-k-l | -m-n-D/2), \quad (2.9)$$

where the subset of three variables are

$$X = \frac{s}{4\mu^2}, \quad Y = \frac{t}{4\mu^2}, \quad Z = \frac{u}{4\mu^2}.$$

Observe that the final result, eq. (2.8), has only a three-fold series. However, the expression provided by the solution of system (2.6) was a 12-fold series. It is very easy to understand why this is so. Among the defining variables for the hypergeometric series representations here there are 9 whose variables are just unity and are summable series. In other words, we were able to sum up nine of them using Gauss' summation formula[13]. CIT: luke Our strategy is therefore to choose as many series as possible in which the individual sums can be written as ${}_2F_1(a,\, b;\, c|1)$, then we plug them in a computer program that do the job, i.e., sum up the series using Gauss' summation formula.

Once we have evaluated this result, it is a straightforward exercise to write down several other solutions which are connected by symmetry in the diagram, namely, by exchanging the pairs $(i \leftrightarrow k,\, s \leftrightarrow t)$, and $(j \leftrightarrow l,\, s \leftrightarrow t)$

If we are interested in the primary integral where all the exponents of propagators are equal to minus one, the above result reduces to

$$\mathcal{I}_M = \pi^D (\mu^2)^{D-6} \Gamma(\{-1\}) \sum_{X_1, X_2, X_3 = 0}^{\infty} \frac{X^{X_1} Y^{X_2} Z^{X_3} (6 - D | X_{123})(1|X_{12})(1|X_3)(1|X_{23})}{X_1! X_2! X_3! (D/2|X_{123})(2|X_{123})}$$
$$\times \frac{(1|X_1)(1|X_{13})(1|X_2)}{(9/2 - D/2|X_{123})}, \quad (2.10)$$ LAB: 4mass

where

$$\Gamma(\{-1\}) = \frac{\Gamma(D/2-2)\Gamma(D/2)\Gamma(6-D)\Gamma^2(4-D/2)}{\Gamma(8-D)}\,. \quad (2.11)$$

The original system of linear equations defines a 7×19 rectangular matrix. From it we can draw $50,388$ square submatrices of dimension 7×7, of which $30,972$ yield vanishing determinant, as already said before. Of course, among the 29,416 solvable solutions that remain, NDIM provides other kind of series, such as 5-fold and 7-fold series, i.e., hypergeometric series representations with five and seven variables respectively (meaning seven and five summable series expressed as ${}_2F_1(a, b;\, c|1)$ respectively). And all of them have symmetries among s, t and u, namely,

$$(p_3 \leftrightarrow p_4,\; j \leftrightarrow n,\; i \leftrightarrow m,\; t \leftrightarrow u), \quad (p_2 \leftrightarrow p_3,\; l \leftrightarrow n,\; k \leftrightarrow m,\; t \leftrightarrow s),$$
$$(p_2 \leftrightarrow p_4,\; j \leftrightarrow l,\; i \leftrightarrow k,\; s \leftrightarrow u), \quad (2.12)$$ LAB: simetrias

so, for each hypergeometric series representations provided by NDIM there are other two, also originated from the system of algebraic equations, which represent the same integral and can be transformed in the first using (2.12). This is the case of (2.8).

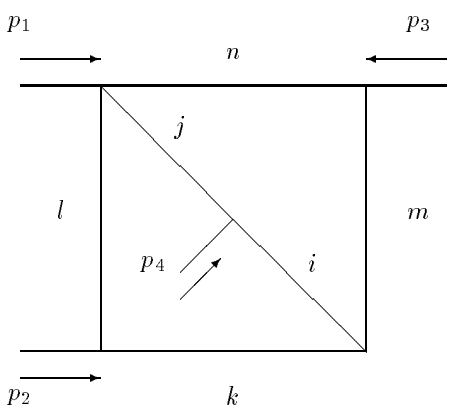

Figure 2: Scalar massless non-planar double-box with six propagators.

2.1 Numerical calculation

Expansion in ϵ for the integral (2.1) can be obtained numerically to all orders since our result (2.10) is exact. Hypergeometric series converge very fast and eq.(2.10) can be truncated after just few terms. We consider two examples, namely, $s = -1, t = -2, s = -3, 4\mu^2 = 25$ and $s = 2, t = -1, u = -1, \mu = 4$.

Table

N	a	b	c
1	-0.15877787089947	0.1144504436314	-0.4375363472289
2	-0.15905593258958	0.1141346540498	-0.4382967223214
3	-0.15904157598309	0.1141585957166	-0.4382477388046
4	-0.15904248795073	0.1141566914709	-0.4382515309963
5	-0.15904248795073	0.1141568521924	-0.4382512066856
6	-0.15904242717753	0.1141568379462	-0.4382512361932
7	-0.15904242673854	0.1141568392587	-0.4382512333925
8	-0.15904242677727	0.1141568391341	-0.4382512336666
9	-0.15904242677373	0.1141568391462	-0.4382512336391
10	-0.15904242677406	0.1141568391450	-0.4382512336419
20	-0.15904242677403	0.1141568391451	-0.4382512336417

N=Number of terms of each series and coefficients of ϵ-expansion for (2.1). Case-I: $s = -1, t = -2, u = -3, 4\mu^2 = 25$. It is outside the physical region and therefore represents a numerical sample. The result is in the form $\frac{a}{\epsilon} + b + c\epsilon$, where a, b, c are given in the table. It is possible to obtain the ϵ-expansion to all orders, since our result is exact.

Table

N	a	b	c
1	-0.16666202713362544092	0.11111642984629631817	-0.4557060127212616592
2	-0.16668058021298683828	0.11109516323349484374	-0.4557569968657018264
3	-0.16668072454737026304	0.11109492182787342273	-0.455757490573366544
4	-0.16668072849754143706	0.1110949135624075636 7	-0.4557575070366868277
5	-0.16668072856344380806	0.1110949134016995572 5	-0.4557575073610663256
6	-0.16668072856499855487	0.1110949133974485926 6	-0.4557575073698739456
7	-0.16668072856503197638	0.1110949133973486074 2	-0.4557575073700873608
8	-0.16668072856503277706	0.1110949133973460293 4	-0.4557575073700930307
9	-0.16668072856503279614	0.1110949133973459640 0	-0.4557575073700931786
10	-0.16668072856503279661	0.1110949133973459622 8	-0.4557575073700931826
20	-0.16668072856503279662	0.1110949133973459622 4	-0.4557575073700931827

N=Number of terms of each series and coefficients of ϵ-expansion for (2.1). Case-II: $s = 2, t = -1, u = -1, \mu = 4$. Observe that hypergeometric series converges very fast: twenty terms for each of the three series provides 18 figures precision. The result is in the form $\frac{a}{\epsilon} + b + c\epsilon$, where a, b, c are given in the table.

3. Massless Double-box

Our method can also be used to study Feynman diagrams where external legs are off-shell[14]. Consider for instance the *massless* non-planar double-box of figure 1. Let $p_i^2 \neq 0$, $(i = 1, 2, 3, 4)$, that is, all external legs off-shell. The generating functional is more complicated, namely, CIT: probing

$$G_{4-OFF} = G_0 \exp\left(-\beta\theta\omega\, p_1^2 - a_2\, p_2^2 - a_3\, p_3^2 - a_4\, p_4^2\right), \tag{3.1}$$

where G_0 is the generating functional (2.3) for the on-shell massless diagram and

$$\begin{aligned} a_2 &= \alpha\gamma\theta + \alpha\theta\phi + \beta\gamma\theta + \gamma\theta\phi + \gamma\theta\omega \\ a_3 &= \alpha\beta\gamma + \alpha\beta\theta + \alpha\beta\phi + \alpha\beta\omega + \beta\gamma\phi \\ a_4 &= \alpha\gamma\omega + \alpha\phi\omega + \beta\theta\phi + \gamma\phi\omega + \theta\phi\omega. \end{aligned} \tag{3.2}$$

In the previous case we used the result $s + t + u = 0$, however, for the case at hand Mandelstam variables are,

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ t &= (p_1 + p_3)^2 = (p_2 + p_4)^2 \\ u &= (p_1 + p_4)^2 = (p_2 + p_3)^2, \end{aligned} \tag{3.3}$$

and $s + t + u = p_1^2 + p_2^2 + p_3^2 + p_4^2 \neq 0$.

In the next subsections we will study some particular cases. See Table

Table: Number of systems, solutions and type of results

Diagram	4 Equal Masses	Massless (on)	Massless (I)	Massless (II)
System	7×19	7×15	7×16	7×20
Total number	50,388	5,040	11,440	77,520
Solutions	29,416	2,916	4,632	34,994
Result	Triple Series	Double Series	$F_A^{(3)}$	9-fold series

3.1 One leg off-shell, $p_1^2 \neq 0$

There are two distinct cases to consider when one external leg is off-shell: $p_1^2 \neq 0$ and $p_j^2 \neq 0$, for $j = 2, 3, 4$, since the diagram is symmetric under the change

$$p_2 \leftrightarrow p_3, \qquad p_2 \leftrightarrow p_4, \qquad p_3 \leftrightarrow p_4\,.$$

Observe that the vertex where p_1 is attached is quartic and all the others are triple and can be interchanged leaving the diagram unchanged.

Mandelstam variables must be rewritten as,

$$\begin{aligned} s &= p_1^2 + 2p_1 \cdot p_2 \\ t &= p_1^2 + 2p_1 \cdot p_3 \\ u &= p_1^2 + 2p_1 \cdot p_4, \end{aligned} \tag{3.4}$$

and $s + t + u = p_1^2 = M_1^2$. So the generating functional becomes slightly different,

$$\begin{aligned} G_{1-OFF} &= \left(\frac{\pi^2}{\lambda}\right)^{D/2} \exp\left[-\frac{1}{\lambda}\left(\beta\gamma\omega s + \alpha\theta\omega t + \beta\theta\phi u + \beta\theta\omega M_1^2\right)\right] \\ &= G_0 \exp\left(-\frac{\beta\theta\omega M_1^2}{\lambda}\right), \end{aligned} \tag{3.5}$$

LAB: geradora-off

comparing (3.5) with (2.3) we conclude that the original system gained only one variable, i.e., the former system (double-box with 4 legs on-shell) was 7×15 and the present (double-box with 1 leg off-shell) is 7×16. The total number of solutions is now 11,440 being 6,808 trivial systems and we must deal with 4,632 possible ones.

The simplest hypergeometric series representations for $\mathcal{I}_{1-off}$ are triple series,

$$\mathcal{I}_{1-off} = \pi^D f_1 \sum_{X_{1,2,3}=0}^{\infty} \frac{P_1^{X_1} P_2^{X_2} P_3^{X_3} (-\sigma|X_{123})(-k|X_1)(-i|X_2)(-m|X_3)}{X_1! X_2! X_3! (1+l-\sigma|X_1)(1+j-\sigma|X_2)(1+n-\sigma|X_3)}, \tag{3.6}$$

LAB: tripla-1off-p1

where $P_1 = s/M_1^2$, $P_2 = t/M_1^2$, $P_3 = u/M_1^2$, and following the usual approach for massless diagrams in the NDIM context we have summed up 6 series. The above triple series is a Lauricella's function of three variables[13], namely,

CIT: luke

$$\mathcal{I}_{1-off} = \pi^D f_1 F_A^{(3)}\left[\begin{matrix} -\sigma; -k, -i, -m \\ 1+l-\sigma, 1+j-\sigma, 1+n-\sigma \end{matrix}\,\middle|\, P_1, P_2, P_3\right], \tag{3.7}$$

where for convergence $|P_1|+|P_2|+|P_3|<1$. We also define,

$$\begin{aligned} f_1 = & (M_1^2)^{\sigma}(\sigma+D/2|-2\sigma-D/2)(i+j+m+n+D|-m-n-D/2) \\ & \times(k+l+m+n+D|-k-l-D/2)(i+j+k+l+D|-i-j-D/2), \\ & \times(-j|\sigma)(-l|\sigma)(-n|\sigma), \end{aligned} \tag{3.8}$$

the symmetries of the diagram

$$i\leftrightarrow k,\ \ j\leftrightarrow l,\ \ s\leftrightarrow t; \qquad i\leftrightarrow m,\ \ j\leftrightarrow n,\ \ s\leftrightarrow u; \qquad k\leftrightarrow m,\ \ l\leftrightarrow n,\ \ t\leftrightarrow u, \tag{3.9}$$

are expressed in our final result.

3.1.1 Special Case

In the special case where $i=j=k=l=m=n=-1$ we have,

$$\mathcal{I}_{1-off}=\pi^D f_1(\{-1\})F_A^{(3)}\left[\begin{matrix}6-D;1,1,1\\6-D,6-D,6-D\end{matrix}\,\middle|\,P_1,P_2,P_3\right], \tag{3.10}$$

with

$$f_1(\{-1\})=(p_1^2)^{6-D}\frac{\Gamma^3(D-5)\Gamma^3(D/2-2)\Gamma(6-D)}{\Gamma^3(D-4)\Gamma(3D/2-6)}, \tag{3.11}$$

which have a double pole in the $D=4-2\epsilon$ limit.

If one were interested in writing the above result in terms of more complicated functions, namely, logarithms, polylogarithms and $S_{a,b}$ integrals, then the following integral representation for $F_A^{(3)}$, is in order

$$\begin{aligned} F_A^{(3)}\left[\begin{matrix}\alpha;\beta,\beta',\beta''\\ \gamma,\gamma',\gamma''\end{matrix}\,\middle|\,P_1,P_2,P_3\right] = & \frac{1}{\Gamma_A}\int_0^1 dx_1dx_2dx_3\frac{x_1^{\beta-1}x_2^{\beta'-1}x_3^{\beta''-1}(1-x_1)^{\gamma-\beta-1}}{(1-x_1P_1-x_2P_2-x_3P_3)^{\alpha}} \\ & \times(1-x_2)^{\gamma'-\beta'-1}(1-x_3)^{\gamma''-\beta''-1}, \end{aligned} \tag{3.12}$$

where the integration is constrained to $x_1+x_2+x_3=1$ and

$$\Gamma_A=\frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\beta'')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')\Gamma(\gamma''-\beta'')}{\Gamma(\gamma)\Gamma(\gamma')\Gamma(\gamma'')},$$

in the present case one get,

$$\begin{aligned} F_A^{(3)}\left[\begin{matrix}6-D;1,1,1\\6-D,6-D,6-D\end{matrix}\,\middle|\,P_1,P_2,P_3\right] = & \frac{1}{\Gamma_A}\int_0^1 dx_1dx_2dx_3\frac{(1-x_1)^{4-D}}{(1-x_1P_1-x_2P_2-x_3P_3)^{6-D}} \\ & \times(1-x_2)^{4-D}(1-x_3)^{4-D}, \end{aligned} \tag{3.13}$$

then one take $D=4-2\epsilon$ and uses Taylor expansion. However, to carry out the integral of second derivatives of such integral representation can not be an easy task. For this reason we claim hypergeometric series representations are simpler than the ones in terms of polylogarithms: fast convergence, compact expressions and analytic continuation relations among (i.e. kinematical regions) them.

3.2 One leg off-shell, $p_2^2 \neq 0$

Now we turn to the last case, the one where the external leg attached to a quartic vertex is off-shell. The generating functional is,

$$G_{1-OFF} = G_0 \exp\left[-\frac{\alpha\gamma\theta + \alpha\theta\phi + \beta\gamma\theta + \gamma\theta\phi + \gamma\theta\omega}{\lambda} p_2^2\right], \tag{3.14}$$

LAB: geradora-p2

we see immediately that there will be four extra sums. Simply compare (3.14) and (3.5), the former has four arguments (which properly expanded in Taylor series will produce the referred extra sums) more than the latter.

The system of algebraic equations will be slightly different than (2.6), i.e., for the present case one have,

$$\left\{\begin{array}{rcl} X_2 + Y_{1234} + U_{12} & = & i \\ X_{13} + Y_{5678} + U_3 & = & j \\ X_1 + Y_{159} + Z_1 + U_{1345} & = & k \\ X_{23} + Y_{26} + Z_{23} + U_{12345} & = & l \\ X_3 + Y_{379} + Z_2 + U_{24} & = & m \\ X_{12} + Y_{48} + Z_{13} + U_5 & = & n \\ \Sigma X + \Sigma Y + \Sigma Z + \Sigma U & = & -D/2 \end{array}\right. , \tag{3.15}$$

LAB: sistema-p2

note that as the diagram does not have massive internal lines, the system have not "variables" (sum indices) W_j. Indices U_j are concerned with Taylor expansion of,

$$\exp\left[-\frac{\alpha\gamma\theta + \alpha\theta\phi + \beta\gamma\theta + \gamma\theta\phi + \gamma\theta\omega}{\lambda} p_2^2\right].$$

Such system has in principle 77,520 (C_7^{20}) solutions. Determinant vanish in 42,526 of them, i.e., we must search hypergeometric type functions representing the original two-loop integral in a total of 34,994, less than half of possible solutions for the 7×20 system. However, in this case such series in far more complicated, since they are 13-fold ones. The results of such analysis will be presented elsewhere.

4. Conclusion

We studied in this work several integrals pertaining to non-planar double-box diagrams. Firstly we considered the case where four internal particles have mass μ, then write down the result in terms of a triple hypergeometric series. The second part deals with massless internal particles but off-shell external legs. We calculated the generating functional for our negative-dimensional integrals and then presented some particular cases of interest.

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