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CLASSICAL OPEN-STRING FIELD THEORY : A_∞-Algebra, Renormalization Group and Boundary States

Toshio NAKATSU

Department of Physics,
Graduate School of Science, Osaka University,
Toyonaka, Osaka 560, JAPAN

Abstract

We investigate classical bosonic open-string field theory from the perspective of the Wilson renormalization group of world-sheet theory. The microscopic action is identified with Witten's covariant cubic action and the short-distance cut-off scale is introduced by length of open-string strip which appears in the Schwinger representation of open-string propagator. *Classical open-string field theory* in the title means open-string field theory governed by a classical part of the low energy action. It is obtained by integrating out suitable tree interactions of open-strings and is of non-polynomial type. We study this theory by using the BV formalism. It turns out to be deeply related with deformation theory of A_∞-algebra. We introduce renormalization group equation of this theory and discuss it from several aspects. It is also discussed that this theory is interpreted as a boundary open-string field theory. Closed-string BRST charge and boundary states of closed-string field theory in the presence of open-string field play important roles.

1 Introduction

In this paper we investigate classical bosonic open-string field theory from the perspective of the Wilson renormalization group of world-sheet theory. The microscopic action is identified with Witten's covariant cubic action [1] of open-string field and the short-distance cut-off scale parameter is introduced by length of open-string strip which appears in the Schwinger representation of open-string propagator.

Low energy action at the cut-off scale ζ is obtained by integrating out all the contributions of open-string interactions at length scales less than ζ. It becomes the cubic action as ζ goes to zero, and provides a macroscopic description of open-string field theory as ζ goes to ∞. The low energy action is of non-polynomial type similarly to the case of closed-string field theory [2, 3] and has an expansion by open-string loops. *Classical open-string field theory* in the title means open-string field theory governed by the classical part of the low energy action. It is understood to be obtained by integrating out the tree contributions. n-valent tree interactions of open-strings beyond the cubic become elementary at the cut-off scale ζ (> 0) and contribute to the action. Geometric structure of these interactions can be handled by homotopy associative algebra (A_∞-algebra) [4]. Study of *classical open-string field theory* turns out to be deeply related with deformation theory of A_∞-algebra which is recently developed by K. Fukaya et al [5, 6].

In Sections 4 and 5 we describe *classical open-string field theory* from the viewpoint of the deformation theory. Differential graded algebra in the microscopic description is open-string gauge algebra [1]. It is non-commutative associative algebra. The gauge algebra is deformed in the macroscopic description or flows in the sense of renormalization group, to A_∞-algebra which is non-commutative and non-associative. Our description is based on the Batalin-Vilkovisky formalism [7]. It was elegantly used in quantization of open-string field theory [8, 9]. It also played an important role in the construction of closed-string field theory [2].

Off-shell open-string fields at different length scales are related with one-another by renormalization group flows. In Section 6 we introduce renormalization group equation of *classical open-string field theory*. This is a rational concept since the action is the classical part of low energy effective action. Our treatment of renormalization group is based on that given in [10, 11].

Nevertheless it may be important to explain the above idea in the context of field theories and to make a comparison between two. For this purpose let us recall briefly the renormalization group treatment in field theories. We consider a real scalar field theory in d-dimensions ($2 < d \leq 4$). We use the momentum representation. Let φ_p be a momentum mode of $\varphi(x) = \frac{1}{(2\pi)^{d/2}} \int dp e^{ipx} \varphi_p$. We put $\varphi \equiv \{\varphi_p\}$. Let Λ be a UV cut-off. We put $\varphi_{\leq \Lambda} \equiv \{\varphi_p\}_{|p| \leq \Lambda}$. Let $S[\varphi_{\leq \Lambda_0} : \Lambda_0]$ be an Euclidean action at Λ_0. For $\Lambda < \Lambda_0$ the Wilsonian action $S[\varphi_{\leq \Lambda} : \Lambda]$ is defined in principle by the following integration on $\varphi_p (\Lambda < |p| \leq \Lambda_0)$ in the original action.

$$\int \prod_{\Lambda < |p| \leq \Lambda_0} d\varphi_p e^{-S[\varphi_{\leq \Lambda_0} : \Lambda_0]} = const. \times e^{-S[\varphi_{\leq \Lambda} : \Lambda]}. \tag{1.1}$$

The integration on the high energy modes changes the values of the coupling constants or generates several interactions at Λ.

It is convenient to describe the cut-off theory by using a regularized propagator $\Delta(p : \Lambda)$. We may put $\Delta(p : \Lambda) = \frac{K(p^2/\Lambda^2)}{p^2 + m^2}$, where K is a cut-off function and m is a bare mass. We put $\mathcal{S}(\Lambda)$ be a set of (running) coupling constants,

$$\mathcal{S}(\Lambda) = \Big\{ S_n(p_1, \cdots, p_n : \Lambda) \Big\}_{n \geq 0}, \tag{1.2}$$

where $S_n(p_1, \cdots, p_n : \Lambda)$ are symmetric functions with respect to p_a. The relevant action has the following form.

$$\begin{aligned} S[\varphi : \mathcal{S}(\Lambda) : \Lambda] &= \frac{1}{2} \int dp \Delta(p : \Lambda)^{-1} \varphi_p \varphi_{-p} + \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{a=1}^{n} dp_a S_n(p_1, \cdots, p_n : \Lambda) \prod_{a=1}^{n} \varphi_{p_a} \\ &\equiv \frac{1}{2} \int dp \Delta(p : \Lambda)^{-1} \varphi_p \varphi_{-p} + S_{int}[\varphi : \Lambda]. \end{aligned} \tag{1.3}$$

Let $Z(\mathcal{S}(\Lambda) : \Lambda)$ be the partition function

$$Z(\mathcal{S}(\Lambda) : \Lambda) = \frac{\int D\varphi e^{-S[\varphi : \mathcal{S}(\Lambda) : \Lambda]}}{\int D\varphi e^{-S[\varphi : \emptyset : \Lambda]}} \qquad (D\varphi \equiv \prod_p d\varphi_p), \tag{1.4}$$

where the denominator is a simple Gaussian integral. The above construction of the Wilsonian action implies

$$\Lambda \frac{d}{d\Lambda} Z(\mathcal{S}(\Lambda) : \Lambda) = 0, \tag{1.5}$$

which states that the coupling constants flow so that the partition function is actually independent of Λ. This equation can be brought to a perturbatively tractable form by using the

expression $Z = e^{-S_{int}[\frac{\delta}{\delta j}]} e^{\frac{1}{2}\int dp \Delta(p;\Lambda) j_p j_{-p}} \Big|_{j=0}$. After some manipulation we can rewrite equation (1.5) in the following form.

$$0 = \int D\varphi \left\{ \Lambda \frac{dS_{int}}{d\Lambda} - \frac{1}{2} \int dq \Lambda \frac{\partial \Delta(q:\Lambda)}{\partial \Lambda} \left(\frac{\partial S_{int}}{\partial \varphi_q} \frac{\partial S_{int}}{\partial \varphi_{-q}} - \frac{\partial^2 S_{int}}{\partial \varphi_q \partial \varphi_{-q}} \right) \right\} e^{-S[\varphi]}. \tag{1.6}$$

This shows that the flow of $\mathcal{S}(\Lambda)$ needs to satisfy the Wilson-Polchinski's renormalization group equation [10, 11]

$$\Lambda \frac{dS_{int}}{d\Lambda}[\varphi : \mathcal{S}(\Lambda)] = \frac{1}{2} \int dq \Lambda \frac{\partial \Delta(q:\Lambda)}{\partial \Lambda} \left(\frac{\partial S_{int}}{\partial \varphi_q} \frac{\partial S_{int}}{\partial \varphi_{-q}} - \frac{\partial^2 S_{int}}{\partial \varphi_q \partial \varphi_{-q}} \right) [\varphi : \mathcal{S}(\Lambda)]. \tag{1.7}$$

In the text we regulate open-string propagator utilizing the Schwinger representation. The counterpart in the field theory is

$$\Delta(p:\Lambda) = \int_{1/\Lambda^2}^{\infty} ds e^{-s(p^2+m^2)} \qquad \left(= \frac{e^{-\frac{p^2+m^2}{\Lambda^2}}}{p^2+m^2} \right). \tag{1.8}$$

Advantage in taking this regularization is that the action (1.3) becomes equivalent to the following one after the similarity transformation, $\varphi_p \to e^{\frac{p^2+m^2}{2\Lambda^2}} \varphi_p$:

$$\begin{aligned} &\hat{S}[\varphi : \mathcal{S}(\Lambda) : \Lambda] \\ &= \frac{1}{2} \int dp (p^2+m^2) \varphi_p \varphi_{-p} + \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{a=1}^{n} dp_a \left(\prod_{a=1}^{n} e^{-\frac{p_a^2+m^2}{2\Lambda^2}} \right) S_n(p_1, \cdots, p_n : \Lambda) \prod_{a=1}^{n} \varphi_{p_a} \\ &\equiv \frac{1}{2} \int dp (p^2+m^2) \varphi_p \varphi_{-p} + \hat{S}_{int}[\varphi : \mathcal{S}(\Lambda) : \Lambda]. \end{aligned} \tag{1.9}$$

Instead of the propagator the interaction vertices acquire the regularization factors (external legs in the Feynman diagrams). The effective action of open-string field in the text is expressed in a way analogous to this form. The renormalization group equation (1.7) can be written down in terms of $\hat{S}_{int}$ as follows.

$$\begin{aligned} &\Lambda \frac{d\hat{S}_{int}}{d\Lambda}[\varphi : \mathcal{S}(\Lambda) : \Lambda] \\ &= \frac{1}{2} \int dq \frac{1}{\Lambda^2} \left(\frac{\partial \hat{S}_{int}}{\partial \varphi_q} \frac{\partial \hat{S}_{int}}{\partial \varphi_{-q}} - \frac{\partial^2 \hat{S}_{int}}{\partial \varphi_q \partial \varphi_{-q}} \right) [\varphi : \mathcal{S}(\Lambda) : \Lambda] + \Lambda \frac{\partial \hat{S}_{int}}{\partial \Lambda}[\varphi : \mathcal{S}(\Lambda) : \Lambda]. \end{aligned} \tag{1.10}$$

The evolution of the coupling constants $S_n(p_1, \cdots, p_n : \Lambda)$ can be read from this equation.

$$\begin{aligned} &\Lambda \frac{\partial S_n}{\partial \Lambda}(p_1, \cdots, p_n : \Lambda) \\ &= \int dq \frac{e^{-\frac{q^2+m^2}{\Lambda^2}}}{\Lambda^2} \left\{ \sum_{I,J} S_{|I|+1}(q, P_I : \Lambda) S_{|J|+1}(-q, P_J : \Lambda) - S_{n+2}(q, -q, p_1, \cdots, p_n : \Lambda) \right\}, \end{aligned} \tag{1.11}$$

for $n = 0, 1, 2, \cdots$. Here I, J are the multi-indices satisfying $I \cup J = \{1, \cdots, n\}$.

Let us take a closer look on the evolution of the coupling constants. To be definite, we start at Λ_0 with the condition,

$$S_4(p_1, p_2, p_3, p_4 : \Lambda_0) = v_0 \times \delta(\sum_{a=1}^{4} p_a) \qquad \text{and all other } S_k \text{ vanish.} \tag{1.12}$$

[Note : To study the renormalizability of ϕ^4-theory in d-dimensions ($^{i.e.}$ to search an existence of the limit $\Lambda_0 \to \infty$), other coupling constants $S_2(p, q : \Lambda_0) = (\eta_0 p^2 + u_0)\delta^d(p+q)$ are required. We take the above condition just to explain the underlying idea in the text.] Solutions of the evolution equation (1.11) admit to have perturbative expansions. At $\Lambda = \Lambda_0 e^{-\zeta} (\zeta > 0)$, solution in this particular example has the expansions as depicted in Fig. 1. The Feynman graphs are interpreted as world-lines in the figure and the Schwinger parameter s of $e^{-s(p^2+m^2)}$ is regarded as length of the world-line. The higher vertices are generated at Λ by the contribution of graphs all the internal propagators in which have length $(\frac{1}{\Lambda_0})^2 < s \leq (\frac{e^\zeta}{\Lambda_0})^2$. The appearance of loops is originated in the term $\frac{\partial^2 \hat{S}_{int}}{\partial\varphi_q \partial\varphi_{-q}}$ of the renormalization group equation. The proliferation of trees is due to the term $\frac{\partial \hat{S}_{int}}{\partial\varphi_q}\frac{\partial \hat{S}_{int}}{\partial\varphi_{-q}}$.

At this present stage we introduce a concept of *classical* renormalization group flow. It is defined by dropping out the loop effect in the original equation.

$$\Lambda \frac{d\hat{S}_{int}}{d\Lambda}[\varphi : \mathcal{S}(\Lambda) : \Lambda] = \frac{1}{2}\int dq \frac{1}{\Lambda^2}\frac{\partial \hat{S}_{int}}{\partial\varphi_q}\frac{\partial \hat{S}_{int}}{\partial\varphi_{-q}}[\varphi : \mathcal{S}(\Lambda) : \Lambda] + \Lambda \frac{\partial \hat{S}_{int}}{\partial\Lambda}[\varphi : \mathcal{S}(\Lambda) : \Lambda]. \tag{1.13}$$

The evolution of the coupling constants can be read as

$$\Lambda \frac{\partial S_n}{\partial\Lambda}(p_1, \cdots, p_n : \Lambda) = \int dq \frac{e^{-\frac{q^2+m^2}{\Lambda^2}}}{\Lambda^2} \sum_{I,J} S_{|I|+1}(q, P_I : \Lambda) S_{|J|+1}(-q, P_J : \Lambda), \tag{1.14}$$

for $n = 0, 1, 2, \cdots$. With the same initial condition at Λ_0 the evolution of the coupling constants becomes that obtained by neglecting the loops in the previous solution.

In the case of open-string field theory we take a route which is slightly different from the above field theory argument. This is partially because in open-string field theory there are no coupling constants except the string field itself. It indicates that wave function renormalizations suffice for the renormalization group argument. In fact the classical effective action $S[\Phi : \zeta]$, where Φ is open-string field and ζ is the cut-off scale, is given in Section 5 and shown to satisfy the master equation, $\{S, S\} = 0$. This confirms the above observation. Since, as one can

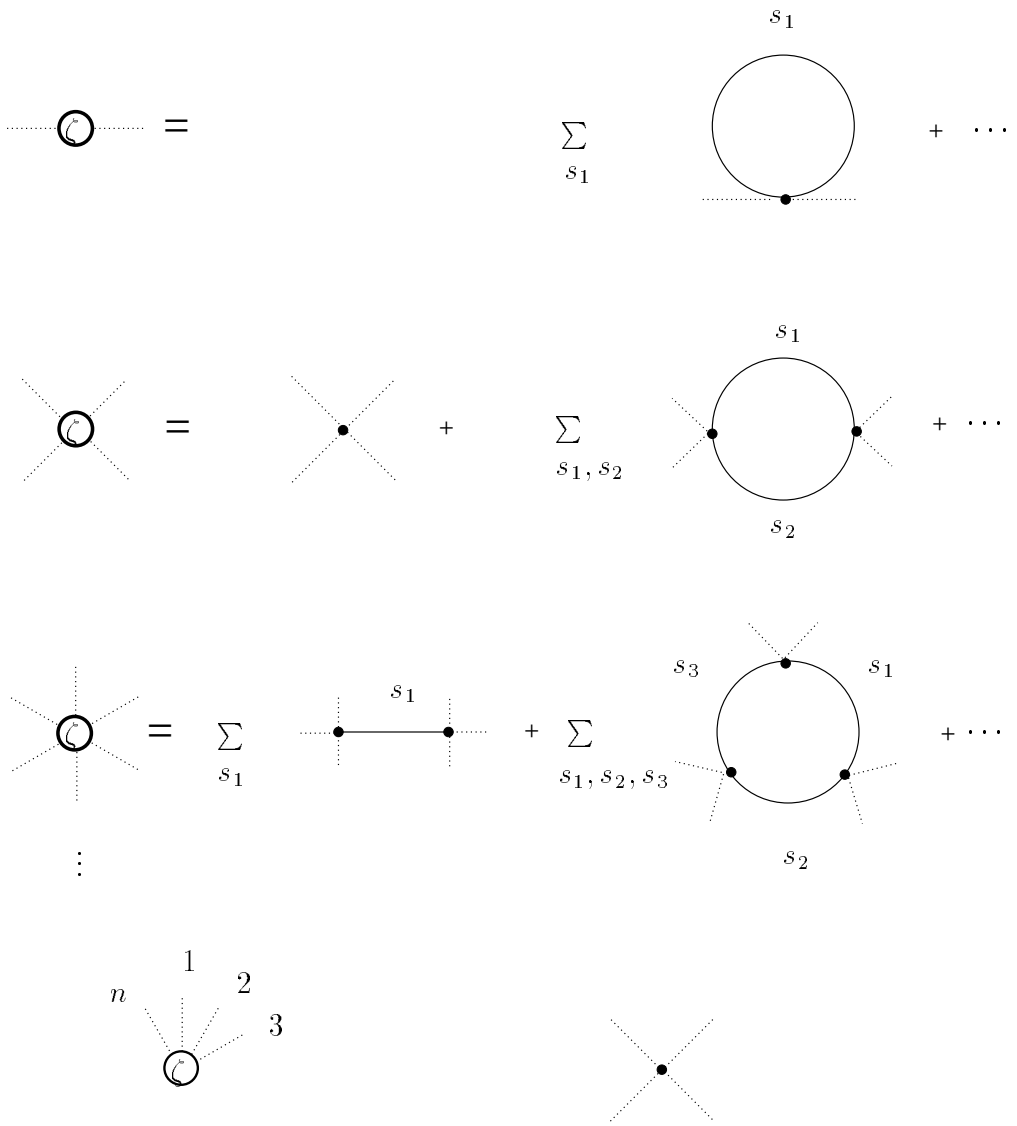

Figure 1: Perturbative expansion of the solution : The Schwinger parameters $s_i (i = 1, 2, 3)$ are integrated over the region $(\frac{1}{\Lambda_0})^2 < s_i \leq (\frac{e^\zeta}{\Lambda_0})^2$. The dotted lines are deleted external legs.

find in the standard argument on renormalizability of gauge field theories, the master equation ensures that the theories maintain gauge symmetries in each step of renormalization group transformation. It is useful to point out that equation (1.5) can be expressed as

$$\Lambda \frac{d\Gamma}{d\Lambda}[\varphi : \mathcal{S}(\Lambda) : \Lambda] = 0, \tag{1.15}$$

where Γ is the effective action. Guided by this expression we then define the renormalization group equation of *classical open-string field theory* by

$$\frac{dS}{d\zeta}[\Phi(\zeta) : \zeta] = 0. \tag{1.16}$$

Description of renormalization group equation of string-field theory along this line was first given for closed-string in [12]. The renormalization group equation turns out to give an infinitesimal variation of the A_∞-algebra but its precise relation with the deformation theory is still left unclear. Open-string field theory analogue of equation (1.13) is derived in equation (6.7) as

$$\omega\left(\frac{d\Phi}{d\zeta}, \left|\frac{\partial S[\Phi:\zeta]}{\partial\Phi}\right\rangle\right) = \omega\left(b_0\left|\frac{\partial S_{int}[\Phi:\zeta]}{\partial\Phi}\right\rangle + L_0\Phi, \left|\frac{\partial S_{int}[\Phi:\zeta]}{\partial\Phi}\right\rangle\right). \tag{1.17}$$

Here ω is the odd symplectic structure of open-string field theory and the ket vectors are the hamiltonian vectors of the classical effective action and its interaction part. They are explained in Section 5 and used to develop the Batalin-Vilkovisky formalism of open-string field theory.

World-lines in field theories are replaced by world-sheets of strings in string field theories. The open-string world-sheets are identified with ribbon graphs while the world-lines are with graphs. The tree ribbon graphs are mapped to 2-disk by the Mandelstam mapping. This cause a great difference with field theory. By this mapping the effective action $S[\Phi : \zeta]$ acquires an interpretation in terms of two-dimensional σ-model. And the classical renormalization group flow can be identified with renormalization group flow of the underlying two-dimensional σ-model. This aspect is studied in the next section.

We start Section 7 with a further investigation on the renormalization group flow by imposing the Siegel gauge condition on open-string field Φ. We express the renormalization group equation in a form $dT^i/d\zeta = \beta^i(T, \zeta)$, where T^i are the coefficients in a suitable expansion of open-string field Φ (or the space-time variables). The beta functions are shown in Proposition

7.2 to have the form,

$$\beta^i = \sum_j g^{ij} \frac{\partial S}{\partial T^j}, \tag{1.18}$$

where g_{ij} is roughly the renormalized BPZ metric of two-dimensional field theories. Since T^i are open-string field Φ itself this expression implies that zeros of the beta functions are nothing but classical solutions of open-string field theory. Contrary to our naive expectation, the beta functions depend on the cut-off scale parameter ζ *explicitly*. This originates in our regularization scheme of open-string field theory. Open-string field theory is formulated using *two-dimensional* conformal field theory [13]. But the regularization we choose is simply to put a restriction on length of open-string evolution, which is a regularization of *one-dimension*. Actually we have two length scales. The missing scale is length of open-string itself. ζ is the ratio of these two length scales and is a dimensionless parameter. In the end of Section 7 the regularization employed so far is examined from the perspective of world-sheet boundary theories. It corresponds to a point-splitting regularization of short-distance on the boundary when ζ is sufficiently large. The point-splitting is prescribed by the boundary length scale $\sim e^{-\zeta}$. The action $S[\Phi : \zeta]$ is interpreted as an analogue of a generating function of all correlation function of a world-sheet boundary theory regularized by the point-splitting method.

Our discussions in Sections $4 - 7$ are based on a conjecture given in the end of Section 3. The conjecture is related with construction of open-string n-valent vertices at the cut-off scale ζ for $n \geq 3$. Section 3 besides Appendix are devoted to test the conjecture.

One of the motivations of this paper is Sen's conjecture [14] on open-string tachyon condensation. The conjecture has been studied by using open-string field theory in two different formulations. One [15] is based on Witten's covariant cubic open-string field theory and the other [16] [17] is based on the so-called boundary open-string field theory [18]. Relation between these two treatments has not been clarified yet. In Section 8 we investigate their relation. It is discussed that the macroscopic open-string field theory studied in this paper is interpreted as a boundary open-string field theory. Boundary states of closed-string field theory in the presence of open-string field and the closed-string BRST charge play important roles in the discussion. In particular the following equality is found :

$$Q_c \Big| B[\Phi : \zeta] \Big\rangle = \delta_{\mathbf{BRS}} \Big| B[\Phi : \zeta] \Big\rangle. \tag{1.19}$$

The ket vector is a boundary state of closed-string in the presence of open-string field Φ and is given in equation (8.12). Q_c is the (free) closed-string BRST charge. $\delta_{\mathbf{BRS}}$ is the BRST transformation of open-string field Φ which is determined by the action $S[\Phi : \zeta]$ in the framework of the Batalin-Vilkovisky formalism.

We provide a brief review on the cubic open-string field theory in Section 2 to make the paper self-contained as far as possible.

2 Microscopic Open-String Field Theory

In this section we give a brief review on Witten's covariant bosonic open-string field theory [1]. It is described along the work of LeClair, Peskin and Preitschopf [13]. Our goal is Definition 2.1, where the covariant action is given. This section is also devoted to a preparation for later discussions. Basic machinery and concepts which will be used in the subsequent sections are explained here.

Bosonic string field theory is formulated using a two-dimensional conformal field theory ($2D$ CFT). This conformal field theory consists of the matter sector described by X^μ $(0 \leq \mu \leq 25)$ and the ghost sector described by world-sheet reparametrization ghosts (b, c). X^μ and (b, c) are respectively grassmann-even and -odd variables of string coordinates (X^μ, b, c). The matter and ghost sectors are $2D$ CFTs of central charge respectively equal to 26 and -26. The total conformal field theory therefore has central charge equal to zero. Consider the conformal field theory formulated in the z-plane with $z = e^{\tau+i\sigma}$. Mode expansion of the string coordinates on a unit disk $|z| \leq 1$ is

$$\begin{aligned} X^\mu(z) &= x^\mu - ip^\mu \ln z - i \sum_{n\neq 0} \frac{\alpha^\mu_{-n}}{n} z^n, \\ b(z) &= \sum_n b_{-n} z^{n-2}, \qquad\qquad c(z) = \sum_n c_{-n} z^{n+1}, \end{aligned} \tag{2.1}$$

where we set string length scale l_s equal to one. Their first quantization gives the following (anti-)commutation relations.

$$\begin{aligned} &[x^\mu, p^\nu\ ] = i\eta^{\mu\nu}, \qquad [\alpha^\mu_n, \alpha^\nu_m] = n\eta^{\mu\nu}\delta_{n+m,0}, \\ &\{b_n, c_m\} = \delta_{n+m,0}, \end{aligned} \tag{2.2}$$

where $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, \cdots, 1)$ is the Minkowski metric of $\mathbf{R}^{1,25}$. For open-string, the σ of $z = e^{\tau + i\sigma}$ originally runs only over $0 \leq \sigma \leq \pi$. The conformal field theory may be formulated on the upper half-plane Im $z \geq 0$. But we can extend the σ in the equations to run over $0 \leq \sigma \leq 2\pi$ just as for closed-string.

Open-string Hilbert space

Open-string Hilbert space $\mathcal{H}$ is the tensor product $\mathcal{H}_{matter} \otimes \mathcal{H}_{ghost}$, where $\mathcal{H}_{matter}$ and $\mathcal{H}_{ghost}$ are respectively the Fock spaces of the matter and ghost CFTs. It consists of the following vectors.

$$\mathcal{H} = \Big\{ b_{-n_1} \cdots b_{-n_p} c_{-m_1} \cdots c_{-m_q} \alpha^{\mu_1}_{-l_1} \cdots \alpha^{\mu_r}_{-l_r} |k\rangle \Big\}, \tag{2.3}$$

where $|k\rangle \equiv e^{ik^\mu x_\mu} |0\rangle$. We introduce the SL_2-invariant vacuum $|0\rangle$ by the conditions,

$$\begin{aligned} &\alpha^\mu_n |0\rangle = 0 \quad \text{for} \;\; n \geq 0, \qquad (\alpha^\mu_0 \equiv p^\mu), \\ &c_n |0\rangle = 0 \quad \text{for} \;\; n \geq 2, \qquad b_n |0\rangle = 0 \quad \text{for} \;\; n \geq -1. \end{aligned} \tag{2.4}$$

The open-string Hilbert space is $\mathbf{Z}$-graded by ghost number G. The string coordinates have the ghost numbers, $G(X^\mu) = 0, G(b) = -1$ and $G(c) = 1$. The vacuum state $|0\rangle$ is set to have no ghost number and to be a grassmann-even vector. The BRST charge Q acts on this Hilbert space. It is a grassmann-odd operator and has the ghost number equal to one. The vacuum state becomes a BRST-invariant vector. The BRST charge obeys the usual relations,

$$Q^2 = 0, \qquad \Big\{ Q, b(z) \Big\} = T(z), \tag{2.5}$$

where T is the total energy-momentum tensor.

The dual Hilbert space $\mathcal{H}^* \equiv \mathcal{H}^*_{mattter} \otimes \mathcal{H}^*_{ghost}$ consists of the following vectors.

$$\mathcal{H}^* = \Big\{ \langle k| \alpha^{\mu_1}_{l_1} \cdots \alpha^{\mu_r}_{l_r} c_{m_1} \cdots c_{m_q} b_{n_1} \cdots b_{n_p} \Big\}, \tag{2.6}$$

where $\langle k| \equiv \langle 0| e^{ik^\mu x_\mu}$. We introduce the state $\langle 0|$ as the SL_2-invariant vacuum of $\mathcal{H}^*$ by imposing the conditions,

$$\begin{aligned} &\langle 0| \alpha^\mu_n = 0 \quad \text{for} \;\; n \leq 0, \\ &\langle 0| c_n = 0 \quad \text{for} \;\; n \leq -2, \qquad \langle 0| b_n = 0 \quad \text{for} \;\; n \leq 1. \end{aligned} \tag{2.7}$$

These conditions also make $\langle 0|$ a BRST-invariant state. Dual pairing between $\mathcal{H}$ and $\mathcal{H}^*$ is prescribed based on $\langle k'|c_{-1}c_0c_1|k\rangle = \delta_{k'+k,0}$, where $c_{\pm 1,0}$ is the ghost zero modes on $\mathbf{CP}_1$. The pairing between any two vectors can be computed by using the (anti-)commutation relations (2.2) and taking account of the conditions (2.4) and (2.7). For the consistency the SL_2-invariant vacuum $\langle 0|$ must be grassmann-odd. We set $\langle 0|$ to have the ghost number equal to -1.

BPZ conjugation

The Belavin-Polyakov-Zamolodchikov (BPZ) conjugation is a general operation of two-dimensional conformal field theories. First we explain this operation in a generic CFT and then apply it to the case of open-string field theory.

Let $U_{0,\infty}$ be two-disks on $\mathbf{CP}_1$ respectively given by $|z| \leq 1$ and $|z| \geq 1$. For the convenience of later application to open-string field theory, local coordinates around 0 and ∞ are chosen to be $z_0 \equiv z$ and $z_\infty \equiv -1/z$. The coordinate transform is given by a map $I : z_\infty \mapsto z_0 = -1/z_\infty$. We have a set of local field operators on each coordinate patch. Let $\mathcal{O}\Big[(U_0, z_0)\Big]$ and $\mathcal{O}\Big[(U_\infty, z_\infty)\Big]$ be sets of local field operators associated with the patches (U_0, z_0) and (U_∞, z_∞). Quantum field φ of the theory on $\mathbf{CP}_1$ is considered as a collection of $\varphi^{(0)} \in \mathcal{O}\Big[(U_0, z_0)\Big]$ and $\varphi^{(\infty)} \in \mathcal{O}\Big[(U_\infty, z_\infty)\Big]$. These two field operators are not independent. In particular, when φ is primary, their relation becomes simple. It is given by

$$\varphi^{(0)}(z_0)\ (dz_0)^\Delta = \varphi^{(\infty)}(z_\infty)\ (dz_\infty)^\Delta, \tag{2.8}$$

where Δ is the conformal dimension of φ.

We attach the Hilbert space $\mathcal{H}$ (representation of the Virasoro algebra) to each coordinate patch in the operator formalism. Let $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(\infty)}$ be the Hilbert spaces attached to (U_0, z_0) and (U_∞, z_∞). Consider expansions of the primary field φ at 0 and ∞. They are given by $\varphi^{(0)}(z_0) = \sum_n \varphi_n^{(0)} z_0^{-n-\Delta}$ and $\varphi^{(\infty)}(z_\infty) = \sum_n \varphi_n^{(\infty)} z_\infty^{-n-\Delta}$. The coefficients $\varphi_n^{(0)}$ and $\varphi_n^{(\infty)}$ are operators which generate the building blocks of $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(\infty)}$. The transformation (2.8) determines a map between $\varphi_n^{(0)}$ and $\varphi_n^{(\infty)}$,

$$\varphi_n^{(\infty)} \longmapsto (\varphi_n^{(\infty)})^T \equiv (-)^{n+\Delta} \varphi_{-n}^{(0)}. \tag{2.9}$$

It can be generalized to any product of operators by letting $(\varphi_{n_1}^{(\infty)} \cdots \varphi_{n_l}^{(\infty)})^T = (\varphi_{n_1}^{(\infty)})^T \cdots (\varphi_{n_l}^{(\infty)})^T$.

This generalization induces a linear map $T : \mathcal{H}^{(\infty)} \to (\mathcal{H}^{(0)})^*$ by

$$|A\rangle_\infty = O_A^{(\infty)}|0\rangle_\infty \longmapsto {}_0\langle A^T| \equiv {}_0\langle 0|(O_A^{(\infty)})^T, \tag{2.10}$$

where $|A\rangle$ is arbitrary state of $\mathcal{H}$, and O_A is an operator which gives $|A\rangle$ when it acts on the vacuum. The map (2.10) is the BPZ conjugation in the operator formalism. In the particular case of open-string field theory the BPZ conjugation becomes as follows.

$$\begin{aligned} \mathcal{H} \ \ni \ & |A\rangle = b_{-n_1}\cdots b_{-n_p}c_{-m_1}\cdots c_{-m_q}\alpha_{-l_1}^{\mu_1}\cdots\alpha_{-l_r}^{\mu_r}|k\rangle \\ & \longmapsto \langle A^T| = \langle k|b_{-n_1}^T\cdots b_{-n_p}^T c_{-m_1}^T\cdots c_{-m_q}^T(\alpha_{-l_1}^{\mu_1})^T\cdots(\alpha_{-l_r}^{\mu_r})^T \in \mathcal{H}^*, \end{aligned} \tag{2.11}$$

where the conjugations of the oscillator modes are given by

$$(\alpha_n^\mu)^T = (-)^{n+1}\alpha_{-n}^\mu, \quad b_n{}^T = (-)^n b_{-n}, \quad c_n{}^T = (-)^{n-1}c_{-n}. \tag{2.12}$$

Since the BRST current and total energy-momentum tensor are primary fields of $\Delta = 1$ and 2, the BPZ conjugates of the BRST charge and Virasoro generators become

$$Q^T = -Q, \qquad L_n{}^T = (-)^n L_{-n}. \tag{2.13}$$

The linear map T gives a map at the level of local field operators. One-to-one correspondence between states and local field operators is known in two-dimensional conformal field theories. We have a local field operator $\varphi_A^{(\infty)}(z_\infty)$ for arbitrary state $|A\rangle_\infty \in \mathcal{H}^{(\infty)}$. The correspondence may be seen by $\lim_{z_\infty\to 0}\varphi_A^{(\infty)}(z_\infty)|0\rangle_\infty = |A\rangle_\infty$. For the BPZ conjugate ${}_0\langle A^T| \in (\mathcal{H}^{(0)})^*$, we have another local field operator, which we call $I[\varphi_A]^{(0)}(z_0)$. The correspondence may be seen by $\lim_{z_0\to\infty}{}_0\langle 0|I[\varphi_A]^{(0)}(z_0) = {}_0\langle A^T|$. Thus we obtain [13] a linear map $I : \mathcal{O}\Big[(U_\infty, z_\infty)\Big] \to \mathcal{O}\Big[(U_0, z_0)\Big]$. (We use the same name as the coordinate transform.) We also obtain the following commutative diagram.

$$\begin{array}{ccc} \mathcal{H}^{(\infty)} & \xrightarrow{T} & (\mathcal{H}^{(0)})^* \\ \downarrow & & \downarrow \\ \mathcal{O}\Big[(U_\infty, z_\infty)\Big] & \xrightarrow{I} & \mathcal{O}\Big[(U_0, z_0)\Big]. \end{array} \tag{2.14}$$

The BPZ conjugation gives a non-degenerate pairing between $\mathcal{H}^{(\infty)}$ and $\mathcal{H}^{(0)}$, which we call the BPZ pairing. It is defined by ${}_0\langle A^T|B\rangle_0$ for any two states $|A\rangle_\infty$ and $|B\rangle_0$. The BPZ pairing is equal to the following two-point function on the z-plane.

$$\langle A^T|B\rangle = \Big\langle I[\varphi_A](\infty)\varphi_B(0)\Big\rangle. \tag{2.15}$$

In the particular case of open-string field theory, existence of the ghost zero-modes $c_{\pm 1,0}$ indicates the following selection rule.

$$\langle A^T|B\rangle \neq 0 \Rightarrow G(A)+G(B)=3. \tag{2.16}$$

Reflector $\left\langle\omega_{12}\right|$

Reflector $\left\langle\omega_{12}\right|$ is a vector of $(\mathcal{H}^{\otimes 2})^*$ and is determined by the condition,

$$\left\langle\omega_{12}\middle|A\right\rangle_1|B\rangle_2 = \langle A^T|B\rangle. \tag{2.17}$$

The subscripts in the LHS of the equation label the open-string Hilbert spaces appearing in the tensor product, $^{i.e.}$ $\left\langle\omega_{12}\right| \in (\mathcal{H}^{(1)}\otimes\mathcal{H}^{(2)})^*$ and $|A\rangle_1|B\rangle_2 \in \mathcal{H}^{(1)}\otimes\mathcal{H}^{(2)}$. The Hilbert spaces $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ are understood to be attached respectively to the coordinate patches (U_∞, z_∞) and (U_0, z_0). By the condition (2.17) the reflector enjoys the following properties.

$$\left\langle\omega_{12}\middle|O^{(1)} = \left\langle\omega_{12}\right|(O^T)^{(2)} \quad (O \text{ is arbitrary operator}), \tag{2.18}$$

$$\left\langle\omega_{12}\middle|k\right\rangle_1 = {}_2\langle k|. \tag{2.19}$$

The superscripts of O in eq.(2.18) indicate the Hilbert spaces on which the operator acts. If we take the BRST charge Q as O in eq.(2.18), we obtain the BRST invariance of the reflector.

$$\left\langle\omega_{12}\right|\left(Q^{(1)}+Q^{(2)}\right) = 0. \tag{2.20}$$

If we regard the reflector as an element of $Hom(\mathcal{H}^{(1)}, (\mathcal{H}^*)^{(2)})$, it gives the BPZ conjugation.

$$\left\langle\omega_{12}\middle|A\right\rangle_1 = {}_2\langle A^T|. \tag{2.21}$$

This is shown by using the properties (2.18) and (2.19) as follows.

$$\begin{aligned}
\left\langle\omega_{12}\middle|A\right\rangle_1 &= \left\langle\omega_{12}\middle|O_A^{(1)}|0\right\rangle_1 \\
&= \left\langle\omega_{12}\right|\left(O_A^T\right)^{(2)}|0\rangle_1 \\
&= \left\langle\omega_{12}\middle|0\right\rangle_1\left(O_A^T\right)^{(2)} \\
&= {}_2\langle 0|\left(O_A^T\right)^{(2)} \\
&= {}_2\langle A^T|.
\end{aligned} \tag{2.22}$$

The reflector has the following oscillator representation [8].

$$\begin{aligned}\Big\langle\omega_{12}\Big| &= \sum_k {}_1\langle k|c_{-1}^{(1)}\otimes {}_2\langle -k|c_{-1}^{(2)}\left(c_0^{(1)}+c_0^{(2)}\right)\\ &\quad\times\prod_{n=1}^{\infty}\exp\left\{\frac{(-)^{n+1}}{n}\sum_\mu \alpha_n^{\mu(1)}\alpha_{n\mu}^{(2)}+(-)^{n+1}\left(c_n^{(1)}b_n^{(2)}+c_n^{(2)}b_n^{(1)}\right)\right\},\end{aligned}\tag{2.23}$$

where the oscillators α_n^μ, b_n and c_n act on the bras according to their superscripts. The bra $\langle k|$ is a grassmann-odd vector with the ghost number -1. Thus the reflector is a grassmann-odd vector and has the ghost number equal to one.

As usual in the description of many body system, we identify $\mathcal{H}^{(1)}\otimes\mathcal{H}^{(2)}$ with $\mathcal{H}^{(2)}\otimes\mathcal{H}^{(1)}$ by imposing the relation,

$$|A\rangle_1|B\rangle_2 = (-)^{G(A)G(B)}|B\rangle_2|A\rangle_1.\tag{2.24}$$

This identification is used implicitly throughout this paper. We can see that the reflector is symmetric under the exchange of open-string indices.

$$\Big\langle\omega_{12}\Big| = \Big\langle\omega_{21}\Big|.\tag{2.25}$$

This actually means the identity, $\Big\langle\omega_{12}\Big|A\rangle_1 B\rangle_2 = \Big\langle\omega_{21}\Big|A\rangle_1|B\rangle_2$ for any two states. By the equivalence relation (2.24) we can identify $\langle\omega_{21}|A\rangle_1|B\rangle_2$ with $(-)^{G(A)G(B)}\langle\omega_{21}|B\rangle_2|A\rangle_1$. This turns out to be $(-)^{G(A)G(B)}\langle B^T|A\rangle$. Thus eq.(2.25) means $\langle A^T|B\rangle = (-)^{G(A)G(B)}\langle B^T|A\rangle$.

Inverse reflector $\Big|S_{12}\Big\rangle$

The inverse reflector $\Big|S_{12}\Big\rangle\in\mathcal{H}^{\otimes 2}$ is introduced as the BPZ-conjugate of the reflector. It enjoys the same properties as (2.18) and (2.19),

$$O^{(1)}\Big|S_{12}\Big\rangle = (O^T)^{(2)}\Big|S_{12}\Big\rangle \quad (O \text{ is arbitrary operator}),\tag{2.26}$$

$${}_1\langle k\Big|S_{12}\Big\rangle = |k\rangle_2.\tag{2.27}$$

The first equation implies the BRST invariance of the inverse reflector,

$$\left(Q^{(1)}+Q^{(2)}\right)\Big|S_{12}\Big\rangle = 0.\tag{2.28}$$

If we regard the inverse reflector as an element of $Hom((\mathcal{H}^*)^{(1)},\mathcal{H}^{(2)})$, it gives

$${}_1\langle A^T\Big|S_{12}\Big\rangle = (-)^{G(A)}|A\rangle_2.\tag{2.29}$$

This can be shown by using (2.26) and (2.27) similarly to the case of eq.(2.21).

Oscillator representation of the inverse reflector becomes

$$\begin{aligned}\Big|S_{12}\Big\rangle &= \prod_{n=1}^{\infty} \exp\left\{\frac{(-)^{n+1}}{n}\sum_{\mu}\alpha_{-n}^{\mu(1)}\alpha_{-n\mu}^{(2)} + (-)^n\left(c_{-n}^{(1)}b_{-n}^{(2)} + c_{-n}^{(2)}b_{-n}^{(1)}\right)\right\} \\ &\quad \times \sum_{k}\left(c_0^{(1)} + c_0^{(2)}\right)c_1^{(1)}|k\rangle_1 \otimes c_1^{(2)}|-k\rangle_2. \end{aligned} \tag{2.30}$$

If one takes the conjugation of eq.(2.30), one finds that it becomes the representation (2.23). The ket $|k\rangle$ is a grassmann-even vector with no ghost number. Hence the inverse reflector is a grassmann-odd vector and has the ghost number equal to three. We also see that it is anti-symmetric under the exchange of open-string indices.

$$\Big|S_{12}\Big\rangle = -\Big|S_{21}\Big\rangle. \tag{2.31}$$

Let us explain why the BPZ-conjugate of the reflector is called *inverse* reflector. Take arbitrary state $A \in \mathcal{H}^{(1)}$. We examine the state $\Big\langle\omega_{12}\Big|S_{23}\Big\rangle|A\rangle_1 \in \mathcal{H}^{(3)}$. It turns out to be $A \in \mathcal{H}^{(3)}$ by eqs.(2.21) and (2.29) as follows.

$$\begin{aligned}\Big\langle\omega_{12}\Big|S_{23}\Big\rangle|A\rangle_1 &= (-)^{G(A)}\Big\langle\omega_{12}\Big|A\rangle_1\Big|S_{23}\Big\rangle \\ &= (-)^{G(A)}\ {}_2\langle A^T\Big|S_{23}\Big\rangle \\ &= |A\rangle_3. \end{aligned} \tag{2.32}$$

If one regards $\Big\langle\omega_{12}\Big|S_{23}\Big\rangle$ as an element of $Hom(\mathcal{H}^{(1)}, \mathcal{H}^{(3)})$, it can be expressed in a convenient form,

$$\Big\langle\omega_{12}\Big|S_{23}\Big\rangle = {}_3P_1, \tag{2.33}$$

where ${}_3P_1$ is the identity which maps $|A\rangle_1$ to $|A\rangle_3$.

Symmetric 3-vertex $\Big\langle 1\ 2\ 3\Big|$

Let V_i for $i = 1, 2, 3$ be unit disks $|v_i| \leq 1$ on the v_i-plane. Let f_i for $i = 1, 2, 3$ be holomorphic maps from V_i to the z-plane of the following forms.

$$f_1 = F, \quad f_2 = S \circ F, \quad f_3 = S \circ S \circ F, \tag{2.34}$$

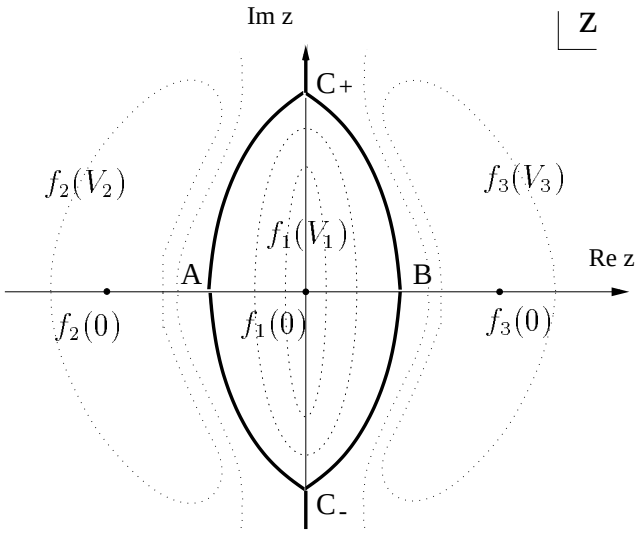

Figure 2: Images of $f_i(V_i)$ on the z-plane. They exactly cover the z-plane. Their boundaries are the bold solid lines. $A, B = \pm 1/\sqrt{3}$. $C_\pm = \pm i$. $f_1(0) = 0, f_2(0) = -\sqrt{3}, f_3(0) = \sqrt{3}$.

where F and S are meromorphic functions given by

$$F(v) = i\frac{1 - \left(\frac{1+iv}{1-iv}\right)^{2/3}}{1 + \left(\frac{1+iv}{1-iv}\right)^{2/3}}, \qquad S(z) = \frac{\frac{1}{2}z - \frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}z + \frac{1}{2}}. \tag{2.35}$$

$F(v)$ has branch points at $v = i, \infty$. For the description we need three Riemann sheets. It is understood in (2.35) that the cut of F is taken so that unit disk $|v| \leq 1$ is on a single sheet. $S(z)$ is the projective action of $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$. $\left\{1, S, S^2\right\}$ is a $\mathbf{Z}_3$ subgroup of $SL_2(\mathbf{R})$. The images $f_i(V_i)$ are depicted in Figure 2. In open-string field theory it may be convenient to use the w- and u-planes instead of the z- and v-planes by

$$z \mapsto w = \frac{1+iz}{1-iz}, \qquad v \mapsto u = \frac{1+iv}{1-iv}. \tag{2.36}$$

The unit disks and upper half-planes are mapped respectively to the right half-planes and unit disks by these maps. The unit disk V_i on the v_i-plane is mapped to the right half-plane on the u_i-plane. If one regards f_i as holomorphic maps from the right half-plane (on the u_i-plane) to

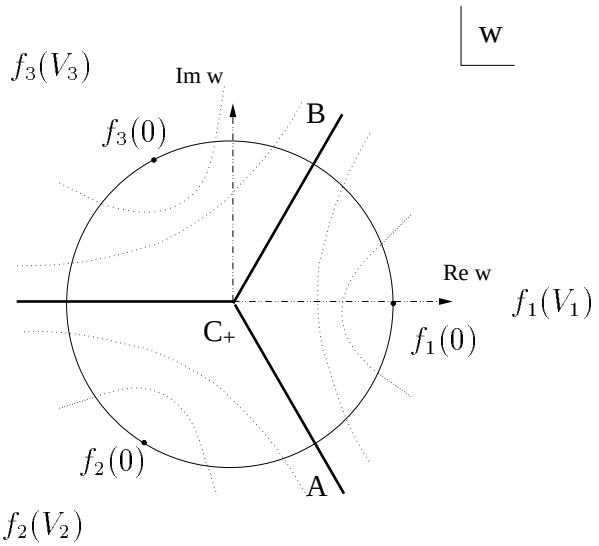

Figure 3: Images of $f_i(V_i)$ on the w-plane. They exactly cover the w-plane. Their boundaries are the bold solid lines. $A, B = e^{\pm\frac{\pi i}{3}}$. $f_1(0) = 1, f_2(0) = e^{\frac{2\pi i}{3}}, f_3(0) = e^{\frac{4\pi i}{3}}$.

the w-plane, F and S in (2.35) are given by [1]

$$F(u) = u^{2/3}, \qquad S(w) = e^{-\frac{2\pi i}{3}} w. \tag{2.37}$$

The images $f_i(V_i)$ on the w-plane are depicted in Figure 3.

Let $\mathcal{H}^{(i)}$ for $i = 1, 2, 3$ be the open-string Hilbert spaces attached to the coordinate patches (V_i, v_i). Open-string trivalent vertex $\Big\langle 1\ 2\ 3\Big|$ is a vector of $(\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \otimes \mathcal{H}^{(3)})^*$. We formulate this vector along the line given in [13]. Let U and V be respectively unit disks $|z| \leq 1$ and $|v| \leq 1$. Let $\mathcal{O}\Big[(U,z)\Big]$ and $\mathcal{O}\Big[(V,v)\Big]$ be the sets of local field operators associated with the coordinates patches (U,z) and (V,v). Let $\mathcal{H}^{(U,z)}$ and $\mathcal{H}^{(V,v)}$ be the Hilbert spaces attached to these coordinates patches. We regard F in (2.35) as a holomorphic map from V to the z-plane. This holomorphic map induces a linear map from $\mathcal{H}^{(V,v)}$ to $\mathcal{H}^{(U,z)}$. Since F is biholomorphic at $F(0) = 0$, the vacuum state of $\mathcal{H}^{(V,v)}$ should be mapped to the vacuum state of $\mathcal{H}^{(U,z)}$. For a primary field φ, the relation between $\varphi^U(z) \in \mathcal{O}\Big[(U,z)\Big]$ and $\varphi^V(v) \in \mathcal{O}\Big[(V,v)\Big]$ is given by

$$\varphi^U(z)\ (dz)^\Delta = \varphi^V(v)\ (dv)^\Delta. \tag{2.38}$$

[1] F in (2.37) shows that the cut can be taken along the negative real line in the u-plane. In the v-plane it is the line on the imaginary axis starting at i.

Analogously to the map T in the BPZ conjugation, this relation is used to introduce a linear map between the operators acting on $\mathcal{H}^{(U,z)}$ and $\mathcal{H}^{(V,v)}$. We write it as $O^{(V,v)} \mapsto F[O]^{(U,z)}$. Thus we obtain the induced map, which we also call F, as follows.

$$\begin{array}{ccc} \mathcal{H}^{(V,v)} & \xrightarrow{F} & \mathcal{H}^{(U,z)} \\ |A\rangle_{(V,v)} = O_A^{(V,v)}|0\rangle_{(V,v)} & \longmapsto & |F[A]\rangle_{(U,z)} \equiv F[O_A]^{(U,z)}|0\rangle_{(U,z)}. \end{array} \tag{2.39}$$

Using the correspondence between states and local field operators, the induced map (2.39) is equivalently described as a linear map between $\mathcal{O}\Big[(V,v)\Big]$ and $\mathcal{O}\Big[(U,z)\Big]$,

$$\begin{array}{ccc} \mathcal{O}\Big[(V,v)\Big] & \xrightarrow{F} & \mathcal{O}\Big[(U,z)\Big] \\ \varphi_A^V & \longmapsto & F[\varphi_A]^U \equiv \varphi^U_{F[A]}. \end{array} \tag{2.40}$$

Let U_i for $i = 1, 2, 3$ be unit disks on the z-plane centered respectively at $z = S^{i-1}(0)$. We put $z_i \equiv z - S^{i-1}(0)$. Let $\mathcal{O}\Big[(U_i, z_i)\Big]$ be the sets of local field operators associated with the coordinate patches (U_i, z_i). We regard S^{1-i} as a holomorphic map from U_i to U. Following the same argument as above, we obtain linear maps S^{1-i} between $\mathcal{O}\Big[(U_i, z_i)\Big]$ and $\mathcal{O}\Big[(U,z)\Big]$,

$$\begin{array}{ccc} \mathcal{O}\Big[(U_i, z_i)\Big] & \xrightarrow{S^{1-i}} & \mathcal{O}\left[(U,z)\right] \\ \varphi_A^{U_i} & \longmapsto & S^{1-i}[\varphi_A]^U \equiv \varphi^U_{S^{1-i}[A]}. \end{array} \tag{2.41}$$

We can now introduce open-string trivalent vertex. For each holomorphic map f_i in (2.34), we consider $S^{1-i}F$, which is a combination of the maps (2.40) and (2.41). It is understood as a map from $\mathcal{O}\Big[(V_i, v_i)\Big]$ to $\mathcal{O}\Big[(U,z)\Big]$ as follows.

$$\mathcal{O}\Big[(V_i, v_i)\Big] \xrightarrow{F} \mathcal{O}\Big[(U_i, z_i)\Big] \xrightarrow{S^{1-i}} \mathcal{O}\Big[(U,z)\Big]. \tag{2.42}$$

The trivalent vertex $\Big\langle 1\ 2\ 3\Big|$ is given [13] by

$$\Big\langle 1\ 2\ 3\Big|A\rangle_1|B\rangle_2|C\rangle_3 \;=\; \Big\langle F[\varphi_A](0)\ S^2F[\varphi_B](-\sqrt{3})\ SF[\varphi_C](\sqrt{3})\Big\rangle, \tag{2.43}$$

where the RHS is the three point function on the z-plane. One often depicts the trivalent vertex as Figure 4.

The most significant property of the trivalent vertex is the BRST invariance,

$$\Big\langle 1\ 2\ 3\Big|\left(Q^{(1)} + Q^{(2)} + Q^{(3)}\right) = 0. \tag{2.44}$$

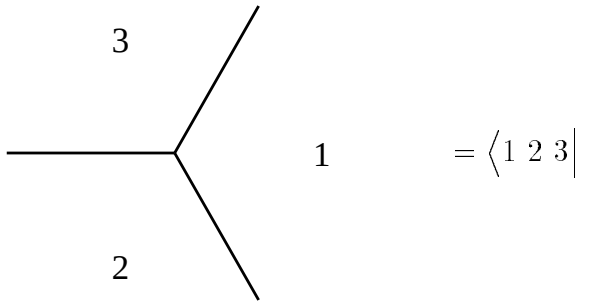

Figure 4: Witten's trivalent vertex. The indices $1, 2$ and 3 label three open-strings.

Let us show eq.(2.44). We first compute $\langle 1\ 2\ 3| \left(Q^{(1)}|A\rangle_1\right) |B\rangle_2 |C\rangle_3$ as follows.

$$\begin{aligned}
\langle 1\ 2\ 3| \left(Q^{(1)}|A\rangle_1\right) |B\rangle_2 |C\rangle_3 &= \langle F[\varphi_{QA}]\ S^2F[\varphi_B]\ SF[\varphi_C]\rangle \\
&= \langle F[\delta_{BRS}\varphi_A]\ S^2F[\varphi_B]\ SF[\varphi_C]\rangle \\
&= \langle \delta_{BRS}\Big(F[\varphi_A]\Big)\ S^2F[\varphi_B]\ SF[\varphi_C]\rangle, \qquad (2.45)
\end{aligned}$$

where $\delta_{BRS}\varphi_A$ is the BRST transform of φ_A and is equal to φ_{QA}. We also use the relation $F[\delta_{BRS}\varphi] = \delta_{BRS}F[\varphi]$. It is ensured by the scalar property of the BRST transformation. If we take the Hamiltonian interpretation of the correlation function and use the BRST invariance of the vacua, we can further compute eq.(2.45) as follows.

$$\begin{aligned}
&\langle \delta_{BRS}\Big(F[\varphi_A]\Big)\ S^2F[\varphi_B]\ SF[\varphi_C]\rangle \\
&= \langle 0|\delta_{BRS}\Big(F[\varphi_A]\Big)\ S^2F[\varphi_B]\ SF[\varphi_C]|0\rangle \\
&= (-)^{G(A)+1}\langle 0|F[\varphi_A]\ \delta_{BRS}\Big(S^2F[\varphi_B]\Big)\ FS[\varphi_C]|0\rangle \\
&\qquad\qquad + (-)^{G(A)+G(B)+1}\langle 0|F[\varphi_A]\ S^2F[\varphi_B]\ \delta_{BRS}\Big(SF[\varphi_C]\Big)|0\rangle \\
&= (-)^{G(A)+1}\langle F[\varphi_A]\ S^2F[\varphi_{QB}]\ SF[\varphi_C]\rangle \\
&\qquad\qquad + (-)^{G(A)+G(B)+1}\langle F[\varphi_A]\ S^2F[\varphi_B]\ SF[\varphi_{QC}]\rangle, \qquad (2.46)
\end{aligned}$$

which turns out to be $(-)\langle 1\ 2\ 3| \left(Q^{(2)}+Q^{(3)}\right) |A\rangle_1 |B\rangle_2 |C\rangle_3$. Thus we obtain the BRST invariance (2.44) of the 3-vertex.

The oscillator representation of the 3-vertex has the form,

$$\langle 1\ 2\ 3|$$

$$= \sum_{\sum_i^3 k_i=0} {}_1\langle k_1|c_{-1}^{(1)}c_0^{(1)} \otimes {}_2\langle k_2|c_{-1}^{(2)}c_0^{(2)} \otimes {}_3\langle k_3|c_{-1}^{(3)}c_0^{(3)}$$
$$\times \exp\left\{\frac{1}{2}\sum_{i,j=1}^{3}\sum_{n,m=0}^{\infty}\sum_{\mu}(-)^{n+m}N_{nm}^{ij}\alpha_n^{\mu(i)}\alpha_{m\mu}^{(j)} + \sum_{i,j=1}^{3}\sum_{n\geq 0,m\geq 1}^{\infty}(-)^{n+m+1}N_{c\ nm}^{ij}b_n^{(i)}c_m^{(j)}\right\}, \tag{2.47}$$

where N_{nm}^{ij} and $N_{c\ nm}^{ij}$ are the Neumann coefficients given by the Fourier components of the two-points functions of X and (b,c) in a suitable coordinate. Their explicit forms can be found in [19, 13]. The oscillator representation clearly shows that the 3-vertex is a grassmann-odd vector and has the ghost number equal to three. In particular it is symmetric under the cyclic permutation of open-string indices,

$$\Big\langle 1\ 2\ 3\Big| = \Big\langle 2\ 3\ 1\Big|. \tag{2.48}$$

This actually means $\Big\langle 1\ 2\ 3\Big|A\rangle_1|B\rangle_2|C\rangle_3 = \Big\langle 2\ 3\ 1\Big|A\rangle_1|B\rangle_2|C\rangle_3$ for any three states. The RHS of this equation is identified with $(-)^{G(A)(G(B)+G(C))}\Big\langle 2\ 3\ 1\Big|B\rangle_2|C\rangle_3|A\rangle_1$. It is equal to $(-)^{G(A)(G(B)+G(C))}\Big\langle 1\ 2\ 3\Big|B\rangle_1|C\rangle_2|A\rangle_3$. Thus eq.(2.48) means

$$\Big\langle 1\ 2\ 3\Big|\Big\{|A\rangle_1|B\rangle_2|C\rangle_3 - (-)^{G(A)(G(B)+G(C))}|B\rangle_1|C\rangle_2|A\rangle_3\Big\} = 0 \tag{2.49}$$

for any three states.

Witten's Φ^3-action of open-string field theory

Open-string field Φ is a vector of the open-string Hilbert space $\mathcal{H}$. It is grassmann-odd and has the ghost number $G(\Phi)=1$. The cubic action of open-string field theory is defined by

Definition 2.1 (Cubic action of open-string field theory)

$$S^{cubic}\Big[\Phi\Big] = \frac{1}{2}\Big\langle\omega_{12}\Big|\Phi\rangle_1\Big(Q^{(2)}|\Phi\rangle_2\Big) + \frac{1}{3}\Big\langle 1\ 2\ 3\Big|\Phi\rangle_1|\Phi\rangle_2|\Phi\rangle_3. \tag{2.50}$$

Selection rules by the ghost number conservation imply that each term in the action does not vanish for a generic Φ.

Guiding principle for the construction [1] of the above action was open-string gauge algebra $(Q,\star)$. Here Q is the BRST charge. $\star$ is a non-commutative associative algebra on $\mathcal{H}$,

$$|A\star B\rangle_1 \equiv \Big\langle 0\ 2\ 3\Big|S_{01}\Big\rangle|A\rangle_2|B\rangle_3, \tag{2.51}$$

where $\star$ is understood as a map from $\mathcal{H}^{(2)}\times\mathcal{H}^{(3)}$ to $\mathcal{H}^{(1)}$. The ghost number of the RHS of eq.(2.51) is equal to $G(A)+G(B)$. Hence $\star$ preserves the ghost number, $G(A\star B)=G(A)+G(B)$. Non-commutativity, $A\star B\neq B\star A$, clearly has its origin in the use of the 3-vertex. Associativity of the algebra,

$$(A\star B)\star C = A\star(B\star C), \tag{2.52}$$

follows from the GGRT theorem [13],[20]. It states in this particular case that 4-vertex $\Big\langle 1\ 2\ 3\ 4\Big|$, which is introduced by $\Big\langle 1\ 2\ 3\ 4\Big| \equiv \Big\langle 1\ 2\ a\Big|\Big\langle a'\ 3\ 4\Big|S_{a'a}\Big\rangle$, becomes symmetric under the cyclic rotation (1,2,3,4) $\mapsto$ (4,1,2,3). Actually one can evaluate the RHS of eq.(2.52) as follows.

$$\begin{aligned}
|A\star(B\star C)\rangle_5 &= \Big\langle 1\ 2\ 3\Big|S_{15}\Big\rangle|A\rangle_2|B\star C\rangle_3 \\
&= \Big\langle 1\ 2\ 3\Big|S_{15}\Big\rangle|A\rangle_2\Big(\Big\langle 4\ 6\ 7\Big|S_{43}\Big\rangle|B\rangle_6|C\rangle_7\Big) \\
&= \Big(\Big\langle 1\ 2\ 3\Big|\Big\langle 4\ 6\ 7\Big|S_{43}\Big\rangle\Big)\Big|S_{15}\Big\rangle|A\rangle_2|B\rangle_6|C\rangle_7 \\
&= \Big\langle 1\ 2\ 6\ 7\Big|S_{15}\Big\rangle|A\rangle_2|B\rangle_6|C\rangle_7 \\
&= \Big\langle 1\ 2\ 3\ 4\Big|S_{15}\Big\rangle|A\rangle_2|B\rangle_3|C\rangle_4.
\end{aligned}$$

A similar computation shows $|(A\star B)\star C\rangle_5 = \Big\langle 4\ 1\ 2\ 3\Big|S_{15}\Big\rangle|A\rangle_2|B\rangle_3|C\rangle_4$. Due to the cyclic symmetry of the above 4-vertex these two become equal. The BRST charge is another important constituent of the gauge algebra. It is nilpotent and acts as the derivation on the $\star$-algebra,

$$Q\Big(A\star B\Big) = QA\star B + (-)^{G(A)}A\star QB. \tag{2.53}$$

This can be seen from the BRST invariance of the 3-vertex as follows.

$$\begin{aligned}
Q^{(1)}|A\star B\rangle_1 &= Q^{(1)}\Big\langle 0\ 2\ 3\Big|S_{01}\Big\rangle|A\rangle_2|B\rangle_3 \\
&= \Big\langle 0\ 2\ 3\Big|\Big((-)Q^{(1)}\Big|S_{01}\Big\rangle\Big)|A\rangle_2|B\rangle_3 \\
&= \Big\langle 0\ 2\ 3\Big|\Big(Q^{(0)}\Big|S_{01}\Big\rangle\Big)|A\rangle_2|B\rangle_3 \\
&= \Big(\Big\langle 0\ 2\ 3\Big|(-)\Big(Q^{(2)}+Q^{(3)}\Big)\Big)\Big|S_{01}\Big\rangle|A\rangle_2|B\rangle_3 \\
&= \Big\langle 0\ 2\ 3\Big|S_{01}\Big\rangle\Big(Q^{(2)}|A\rangle_2\Big)|B\rangle_3 + (-)^{G(A)}\Big\langle 0\ 2\ 3\Big|S_{01}\Big\rangle|A\rangle_2\Big(Q^{(3)}|B\rangle_3\Big) \\
&= |QA\star B\rangle_1 + (-)^{G(A)}|A\star QB\rangle_1.
\end{aligned} \tag{2.54}$$

The action (2.50) was obtained [1] originally in a form by which role of the open-string gauge algebra becomes manifest. Let us rewrite the cubic term in the action as follows.

$$\Big\langle 1\ 2\ 3\Big|\Phi\rangle_1|\Phi\rangle_2|\Phi\rangle_3 = \Big\langle 1\ 2\ 3\Big|\Big({}_1P_a|\Phi\rangle_a\Big)|\Phi\rangle_2|\Phi\rangle_3$$

$$
\begin{aligned}
&= \Big\langle 1\ 2\ 3\Big|\Big\langle\omega_{ab}\Big|S_{b1}\Big\rangle|\Phi\rangle_a|\Phi\rangle_2|\Phi\rangle_3 \\
&= \Big\langle\omega_{ab}\Big|\Phi\rangle_a\Big\langle 1\ 2\ 3\Big|S_{1b}\Big\rangle|\Phi\rangle_2|\Phi\rangle_3 \\
&= \Big\langle\omega_{12}\Big|\Phi\rangle_1|\Phi\star\Phi\rangle_2. \qquad (2.55)
\end{aligned}
$$

By using this expression of the cubic term we can write down the action (2.50) into the following form.

$$
S^{cubic}\Big[\Phi\Big] = \frac{1}{2}\Big\langle\omega_{12}\Big|\Phi\Big\rangle_1\Big\{Q^{(2)}|\Phi\rangle_2 + \frac{2}{3}|\Phi\star\Phi\rangle_2\Big\}. \qquad (2.56)
$$

Analogy with the Chern-Simons gauge theory is very suggestive. The action enjoys the following gauge symmetry.

$$
\delta_\rho\Phi = Q\rho + \Phi\star\rho - \rho\star\Phi, \qquad (2.57)
$$

where ρ is arbitrary vector of $\mathcal{H}$ with $G(\rho)=0$. The invariance of the action can be shown by using the remarks noted in the previous paragraph.

3 Low Energy Theory Vertices

Let us consider a second-quantization of open-string field theory. Schematically it is performed by the path-integral,

$$
\int D\Phi\,\exp\Big\{-S^{cubic}[\Phi]\Big\}. \qquad (3.1)
$$

The path-integral may be computed by treating the cubic term as a perturbation. All connected vacuum Feynman graphs contribute to the path-integral. Vacuum Feynman graph consists of the trivalent vertices connected by the open-string propagators without any external strings. The propagator is determined from the quadratic part of the action (2.50). In the Siegel gauge $b_0\Phi=0$ it formally turns out to be $b_0\frac{1}{L_0}$. This causes the short- and long-distance divergences as world-sheet theory. In this paper we hope to present a renormalization group analysis for the short-distance divergence.

It is convenient to use the Schwinger representation of the propagator,

$$
b_0\frac{1}{L_0} = \int_0^\infty d\tau\ b_0\exp\Big\{-\tau L_0\Big\}. \qquad (3.2)
$$

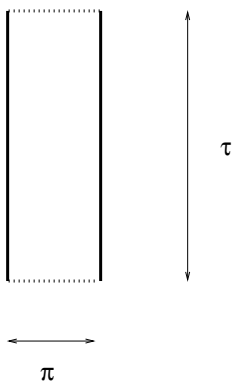

Figure 5: Open-string strip. $\tau(>0)$ is the Schwinger imaginary time. The width of open-string is set to π.

One can interpret $e^{-\tau L_0}$ as the evolution operator for open-string. It is realized by a strip of length τ (Figure 5). Open-string diagrams are metrized trivalent ribbon graphs. Metric of the graph is given by the (imaginary time) parameters τ. To control the short-distance divergence we introduce the cut-off scale parameter ζ (>0) and use the regularized propagator given by

$$\left(b_0 \frac{1}{L_0}\right)^{reg} = \int_{2\zeta}^{\infty} d\tau \; b_0 \exp\Big\{-\tau L_0\Big\}. \tag{3.3}$$

In presence of the cut-off scale ζ, any open-string diagram which has at least one internal strip of length less than 2ζ can not be reproduced from the trivalent vertices by using the regularized propagator. This is because τ in eq.(3.3) is greater than 2ζ. Any internal strip, which connects the trivalent vertices, must have length greater than 2ζ. On the other hand, when all the internal strips have length more than 2ζ, it can be reproduced from the trivalent vertices in this cut-off theory (Figure 6). If we wish to take account of all the diagrams even in presence of the cut-off scale ζ, we need to introduce an action which contains higher interactions beyond the cubic term. These interactions are obtained from graphs which are *one-particle irreducible* with respect to the regularized propagator (3.3). They are those graphs all internal strips of which have length less than 2ζ. If we take the perspective of renormalization group *ala* Wilson [10] or Polchinski [11], the action is nothing but a low energy action obtained by integrating out all the contributions from length scale less than ζ. We denote the low energy action at the cut-off scale ζ by $S^{\zeta}_{eff}[\Phi]$. It becomes the microscopic action $S^{cubic}[\Phi]$ as ζ goes to 0, while, as ζ goes to ∞, it provides a macroscopic description.

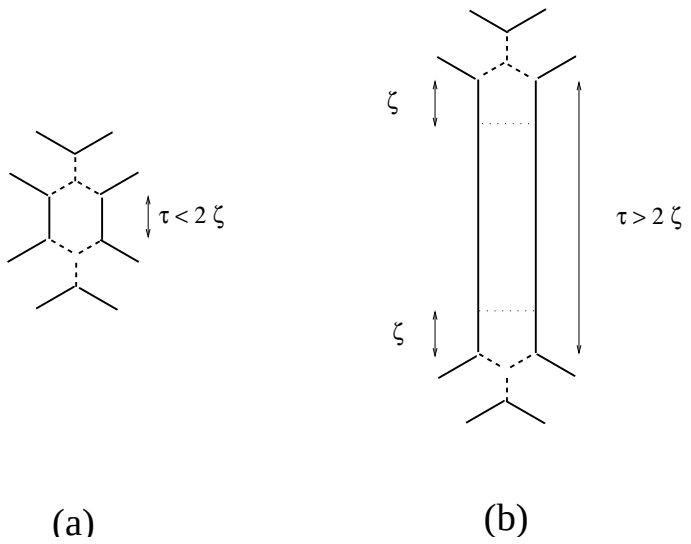

Figure 6: An open-string diagram which contains (a) can not be reproduced from the trivalent vertices at the cut-off scale ζ. On the other hand, when all internal strips are (b) the diagram can be reproduced.

Contribution from open-string loops might play an important role in the low energy description, particularly related with duality between open- and closed-strings [21]. Nevertheless, at the present stage, we do not have a systematic machinery enough to handle with it. In this paper we study the classical part of $S^{\zeta}_{eff}[\Phi]$ which does not include the loop effect.

3.1 Low energy theory vertices ; examples

Since our consideration is limited to the classical case, we call connected open-string diagrams which are tree simply as open-string diagrams. External open-strings of the diagram are clockwise-ordered. Open-string diagram is a metrized connected trivalent tree ribbon graph with clockwise-ordered external ribbons. Open-string n-diagram is a metrized connected trivalent tree ribbon graph with clockwise-ordered n external ribbons.

Let $\mathcal{M}^{\partial}_n$ be the set of clockwise-ordered n different points $(z_1, \cdots, z_n)$ on the boundary of two-disk D (or the upper half-plane) divided by the standard action[2] of $SL_2(\mathbf{R})$. ($n \geq 3$.)

[2] $(z_1, \cdots, z_n) \mapsto \left(\frac{az_1+b}{cz_1+d}, \cdots, \frac{az_n+b}{cz_n+d}\right)$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{R})$.

The dimension of $\mathcal{M}_n^\partial$ is $n-3$. The set of metrized connected trivalent tree ribbon graphs with clockwise-ordered n external ribbons is identified with $\mathcal{M}_n^\partial$. Each ribbon graph has $n-3$ internal strips. Their length play the role of local coordinates of $\mathcal{M}_n^\partial$. Infinities of $\mathcal{M}_n^\partial$ are the configurations in which some of z_i coincide with one another on ∂D. Stable compactification of $\mathcal{M}_n^\partial$ is given, under the above identification, by adding the trivalent tree ribbon graphs at least one internal strip of which has infinite length. We denote this compactification by $\mathcal{CM}_n^\partial$. Topologically $\mathcal{CM}_n^\partial$ becomes a $(n-3)$-dimensional ball B_{n-3}. We fix orientation of $\mathcal{CM}_n^\partial$ by the standard orientation of B_{n-3}.

If we have an open-string n-diagram, we can obtain another one by permuting the labels of external open-strings from $(1,2,\cdots,n-1,n)$ to $(2,3,\cdots,n,1)$. In general, it is different from the original diagram. This permutation of open-string indices gives an automorphism of $\mathcal{CM}_n^\partial$. We denote this automorphism by s. Clearly s generates $\mathbf{Z}_n$ (or its subgroup) action on $\mathcal{CM}_n^\partial$. As we will see in the subsequent discussions, the automorphism s is related with asymmetry of open-string vertex under the cyclic-permutation of open-string indices.

3.1.1 Open-string 4-vertex $\left\langle 1\ 2\ 3\ 4 : \zeta \right|$

The compactification $\mathcal{CM}_4^\partial$ is $[-\infty,\infty]$. We identify the s-channel of four open-string interaction with $[0,\infty]$ and the t-channel with $[-\infty,0]$. (Figure 7). Length of the internal strip is x in the s-channel while it is $|x|$ in the t-channel.

We can construct a state $\langle x| \in (\mathcal{H}^{\otimes 4})^*$ for each $x \in \mathcal{CM}_4^\partial$ by applying the Feymman rule to the corresponding diagram. Explicitly we define $\langle x|^{(1234)}$ as follows.

$$\langle x|^{(1234)} = \begin{cases} \left\langle 1\ 2\ a \right| \left\langle a'\ 3\ 4 \right| \left(b_0 e^{-xL_0} \left| S_{a'a} \right\rangle \right) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ \left\langle 2\ 3\ a \right| \left\langle a'\ 4\ 1 \right| \left((-b_0) e^{xL_0} \left| S_{a'a} \right\rangle \right) & \text{for } x < 0. \end{cases} \tag{3.4}$$

We need not specify insertions of b_0 and L_0 on the inverse reflector $\left|S_{a'a}\right\rangle$ because either way gives the same by the BPZ conjugation. The reason why we attach the superscript (1234) to the state $\langle x|$ will become clear soon. The state $\langle x|^{(1234)}$ is grassmann-even and has the ghost number four. We have a $(\mathcal{H}^{\otimes 4})^*$-valued one-form $\langle \Omega|^{(1234)}$ on $\mathcal{CM}_4^\partial$. It is defined by

$$\Omega(x)^{(1234)} = dx \langle x|^{(1234)}. \tag{3.5}$$

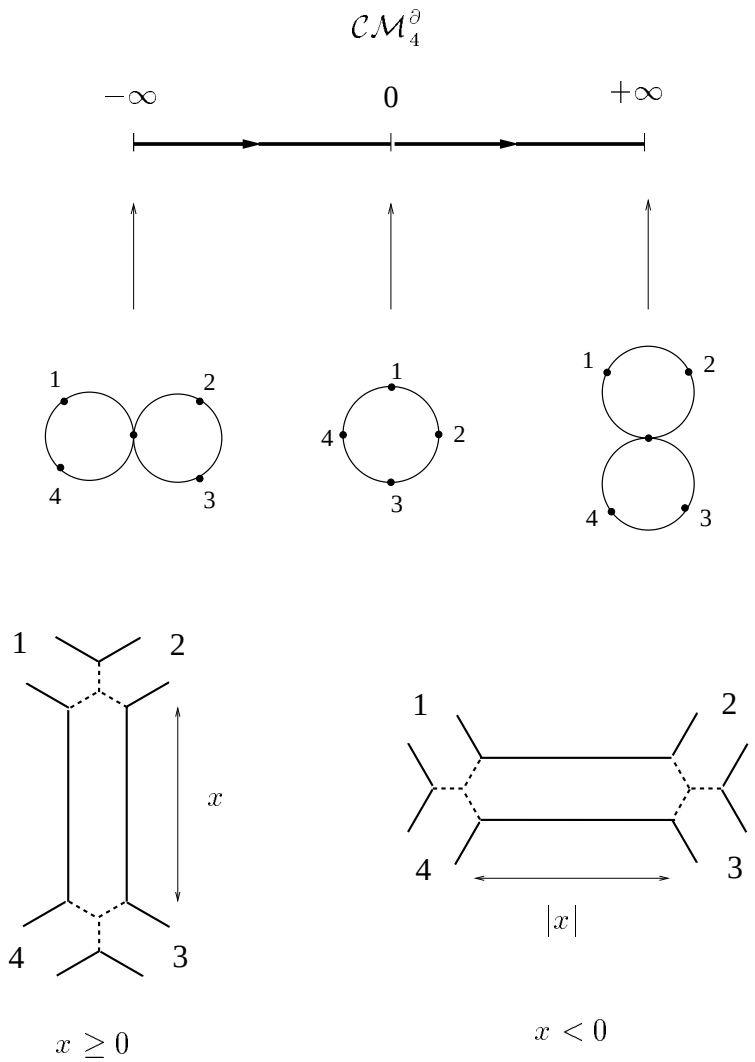

Figure 7: $\mathcal{CM}_4^\partial = [-\infty, +\infty]$. The configurations of four points on ∂D at $x = 0, \pm\infty$ are depicted in the second line. Open-string diagram at $x \in [-\infty, +\infty]$ is on the third line. The s, t-channels are respectively $x > 0$ and $x < 0$.

Let us consider an effect of the cyclic permutation of open-string indices. Another state can be obtained if we permute the open-string indices from $(1,2,3,4)$ to $(2,3,4,1)$ in the RHS of eq.(3.4). We denote this new state by $\langle x|^{(2341)}$. We also let $\langle\Omega|^{(2341)}$ be the one-form $dx\langle x|^{(2341)}$. Two states $\langle x|^{(1234)}$ and $\langle x|^{(2341)}$ are related with each other by $\langle -x|^{(1234)} = -\langle x|^{(2341)}$. This follows from the anti-symmetry of the inverse reflector and the cyclic symmetry of the trivalent vertex besides its odd-grassmannity. The permutation exchanges the s- and t-channels. Thus the automorphism s generates a $\mathbf{Z}_2$-action on $\mathcal{CM}_4^\partial$. It is given by $s(x)=-x$. Therefore $\langle\Omega|^{(2341)}$ is the pull-back of $\langle\Omega|^{(1234)}$ by s.

$$\Omega^{(2341)} = s^*\Omega^{(1234)}. \tag{3.6}$$

We now examine the low energy description at the cut-off scale ζ. We put $\mathcal{V}_4(\zeta)\equiv(-2\zeta,2\zeta)$. Open-string diagrams which belong to $\mathcal{V}_4(\zeta)$ can not be reproduced from the trivalent vertices at this length scale. Any state $\langle x|$ for $x\in\mathcal{V}_4(\zeta)$ must appear in $S^\zeta_{eff}[\Phi]$ as a part of an interaction vertex. It will be a form such as $\frac{1}{4}\int_{-2\zeta}^{2\zeta}\langle\Omega|^{(1234)}|\Phi\rangle_1|\Phi\rangle_2|\Phi\rangle_3|\Phi\rangle_4$. To specify the correct form, we first consider the states $\langle\pm2\zeta|$. They are expected to be obtained simply by joining two 3-vertices at one open-string, since $x=\pm2\zeta$ are the boundaries of $\mathcal{V}_4(\zeta)$. For this to be possible we need to modify slightly the trivalent vertex.

Definition 3.1 (Open-string 3-vertex at ζ) *Open-string 3-vertex at the cut-off scale ζ is defined by*

$$\Big\langle 1\ 2\ 3:\zeta\Big| = \Big\langle 1\ 2\ 3\Big| \prod_{i=1,2,3} e^{-\zeta L_0^{(i)}}. \tag{3.7}$$

The modified 3-vertex (Figure 8) is a grassmann-odd vector which satisfies the BRST invariance and the cyclic symmetry in the same manner as the original one. In (3.7) we attach the propagator of length ζ to each (not a specific) external open-string in order to keep the cyclic symmetry of 3-vertex. States obtained by joining two modified 3-vertices are not $\langle\pm2\zeta|$ but $\langle\pm2\zeta|\prod_{i=1}^4 e^{-\zeta L_0^{(i)}}$. Appearance of the propagators of length ζ is a common phenomenon in the low energy description. Let $\mathcal{V}_n(\zeta)$ be the set of open-string n-diagrams all internal strips of which have length less than 2ζ. They are by no means obtained from lower diagrams at the cut-off scale ζ. Open-string diagram at the boundary of $\mathcal{V}_n(\zeta)$ has at least one internal strip of length equal to 2ζ. Let us take any two open-string diagrams which belong respectively to

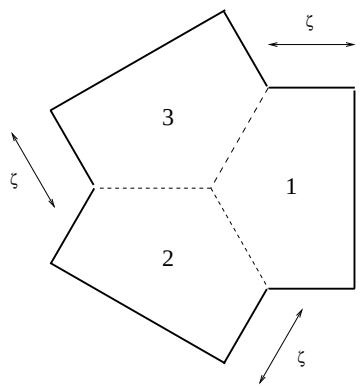

Figure 8: Open-string trivalent vertex at the cut-off scale ζ.

$\mathcal{V}_{n_1+1}(\zeta)$ and $\mathcal{V}_{n_2+1}(\zeta)$, and join them at one external string. As the result we obtain a (n_1+n_2)-diagram. It must be at the boundary of $\mathcal{V}_{n_1+n_2}(\zeta)$. For this to be possible, each external strip needs to have length ζ. This shows that we must insert $e^{-\zeta L_0}$ to each external open-string in order to obtain the vertices at the cut-off scale ζ.

Definition 3.2 (Open-string 4-vertex at ζ) *Open-string 4-vertex at the cut-off scale ζ is defined by an integration of the $(\mathcal{H}^{\otimes 4})^*$-valued one-form $\langle\Omega|$ (3.5) over $\mathcal{V}_4(\zeta)$ multiplied by the external propagators of length ζ,*

$$\Big\langle 1\ 2\ 3\ 4:\zeta\Big| = \int_{-2\zeta}^{2\zeta} \langle\Omega|^{(1234)} \prod_{i=1}^{4} e^{-\zeta L_0^{(i)}}. \tag{3.8}$$

Indices in the 4-vertex are understood to label the Hilbert spaces attached to clockwise-ordered four open-strings on ∂D. Thereby the vertex is regarded as a vector of $(\mathcal{H}^{(1)}\otimes\mathcal{H}^{(2)}\otimes\mathcal{H}^{(3)}\otimes\mathcal{H}^{(4)})^*$.

Open-string 4-vertex (3.8) is grassmann-even and has the ghost number equal to four. When we use the one-form $\Omega^{(2341)}$ instead of $\Omega^{(1234)}$ in the above definition, we obtain another 4-vertex $\Big\langle 2\ 3\ 4\ 1:\zeta\Big|$. But, owing to the relation (3.6), these two are shown to be identical.

$$\begin{aligned}
\Big\langle 2\ 3\ 4\ 1:\zeta\Big| &= \int_{-2\zeta}^{2\zeta} \langle\Omega|^{(2341)} \prod_{i=1}^{4} e^{-\zeta L_0^{(i)}} \\
&= \int_{-2\zeta}^{2\zeta} s^*\langle\Omega|^{(1234)} \prod_{i=1}^{4} e^{-\zeta L_0^{(i)}} \\
&= -\int_{-2\zeta}^{2\zeta} \langle\Omega|^{(1234)} \prod_{i=1}^{4} e^{-\zeta L_0^{(i)}} \\
&= -\Big\langle 1\ 2\ 3\ 4:\zeta\Big|.
\end{aligned} \tag{3.9}$$

We say this as asymmetry of the open-string 4-vertex under the cyclic permutation. The factor $(-)$ in (3.9) originates in the fact that the automorphism s does not preserve the orientation of $\mathcal{V}_4(\zeta)$.

The 4-vertex is not invariant under the action of the BRST charge. Instead we have :

Proposition 3.1 (Q-action on 4-vertex) *Action of the BRST charge on the 4-vertex (3.8) becomes as follows.*

$$\Big\langle 1\ 2\ 3\ 4:\zeta\Big|\Big(\sum_{i=1}^{4} Q^{(i)}\Big) = \Big\langle 1\ 2\ a:\zeta\Big|\Big\langle a'\ 3\ 4:\zeta\Big|S_{a'a}\Big\rangle - \Big\langle 2\ 3\ a:\zeta\Big|\Big\langle a'\ 4\ 1:\zeta\Big|S_{a'a}\Big\rangle. \tag{3.10}$$

Eq.(3.10) has the following geometrical interpretation. Orientation of $\mathcal{V}_4(\zeta)$ is given, as used in eq.(3.8), by the orientation of $\mathcal{CM}_4^{\partial} = [-\infty, \infty]$. Therefore we have

$$\partial\mathcal{V}_4(\zeta) = \Big\{2\zeta\Big\} - \Big\{-2\zeta\Big\}, \tag{3.11}$$

$$\partial\Big\{\pm 2\zeta\Big\} = 0, \tag{3.12}$$

where ∂ is the boundary operator. Similarity between eqs.(3.10) and (3.11) including their signatures should be emphasized. Each term in the RHS of eq.(3.10) is BRST-closed as follows from the BRST invariances of the 3-vertex and the inverse reflector. This corresponds to eq.(3.12). The action of the BRST charge on the 4-vertex becomes a representation of the boundary operator ∂.

The 4-vertex depends on the cut-off scale parameter ζ. It vanishes at $\zeta = 0$. The scale dependence comes from $\mathcal{V}_4(\zeta)$ and $\prod e^{-\zeta L_0^{(i)}}$ in eq.(3.8).

Proposition 3.2 (Scale dependence of 4-vertex) *Scale dependence of the 4-vertex (3.8) is described by*

$$\begin{aligned}\frac{1}{2}\frac{d}{d\zeta}\Big\langle 1\ 2\ 3\ 4:\zeta\Big| &= \Big\langle 1\ 2\ a:\zeta\Big|\Big\langle a'\ 3\ 4:\zeta\Big|\Big(b_0\Big|S_{a'a}\Big\rangle\Big) - \Big\langle 2\ 3\ a:\zeta\Big|\Big\langle a'\ 4\ 1:\zeta\Big|\Big(b_0\Big|S_{a'a}\Big\rangle\Big) \\ &\quad -\frac{1}{2}\Big\langle 1\ 2\ 3\ 4:\zeta\Big|\Big(\sum_{i=1}^{4} L_0^{(i)}\Big).\end{aligned} \tag{3.13}$$

Now we prove Propositions 3.1 and 3.2. The following lemma becomes useful in the proof of Proposition 3.1.

Lemma 3.1 *The BRST charge acts on the state* $\langle x|^{(1234)}$ *as*

$$\langle x|^{(1234)}\Big(\sum_{i=1}^{4}Q^{(i)}\Big)=\begin{cases}\partial_x\left[\Big\langle 1\ 2\ a\Big|\Big\langle a'\ 3\ 4\Big|\left(e^{-xL_0}\Big|S_{a'a}\Big\rangle\right)\right] & for\ \ x>0,\\ \\ \partial_x\left[\Big\langle 2\ 3\ a\Big|\Big\langle a'\ 4\ 1\Big|\left(e^{xL_0}\Big|S_{a'a}\Big\rangle\right)\right] & for\ \ x<0.\end{cases}\tag{3.14}$$

Proof of Lemma 3.1 : We show eq.(3.14) for the case of $x>0$. We rewrite the LHS of eq.(3.14) as follows.

$$\begin{aligned}&\langle x|^{(1234)}\Big(\sum_i Q^{(i)}\Big)\\ &=\Big\{\Big\langle 1\ 2\ a\Big|\Big\langle a'\ 3\ 4\Big|\Big(\sum_i Q^{(i)}\Big)\Big\}\left(b_0e^{-xL_0}\Big|S_{a'a}\Big\rangle\right)\\ &=\Big\{\Big(-\Big\langle 1\ 2\ a\Big|\left(Q^{(1)}+Q^{(2)}\right)\Big)\Big\langle a'\ 3\ 4\Big|+\Big\langle 1\ 2\ a\Big|\Big(\Big\langle a'\ 3\ 4\Big|\left(Q^{(3)}+Q^{(4)}\right)\Big)\Big\}\left(b_0e^{-xL_0}\Big|S_{a'a}\Big\rangle\right).\end{aligned}\tag{3.15}$$

Eq.(3.15) can be further computed by using the BRST invariance of the 3-vertices.

$$\begin{aligned}Eq.(3.15) &= \Big\{\Big(\Big\langle 1\ 2\ a\Big|Q^{(a)}\Big)\Big\langle a'\ 3\ 4\Big|-\Big\langle 1\ 2\ a\Big|\Big(\Big\langle a'\ 3\ 4\Big|Q^{(a')}\Big)\Big\}\left(b_0e^{-xL_0}\Big|S_{a'a}\Big\rangle\right)\\ &= -\Big\langle 1\ 2\ a\Big|\Big\langle a'\ 3\ 4\Big|\Big\{\left(Q^{(a)}+Q^{(a')}\right)\left(b_0e^{-xL_0}\Big|S_{a'a}\Big\rangle\right)\Big\}.\end{aligned}\tag{3.16}$$

Then, by using the BRST invariance of the inverse reflector and the anti-commutation relation, $\{Q,b_0\}=L_0$, eq.(3.16) becomes as follows.

$$\begin{aligned}Eq.(3.16) &= -\Big\langle 1\ 2\ a\Big|\Big\langle a'\ 3\ 4\Big|\left(L_0e^{-xL_0}\Big|S_{a'a}\Big\rangle\right)\\ &= \partial_x\left[\Big\langle 1\ 2\ a\Big|\Big\langle a'\ 3\ 4\Big|\left(e^{-xL_0}\Big|S_{a'a}\Big\rangle\right)\right].\end{aligned}\tag{3.17}$$

This is the RHS of eq.(3.14) for the case of $x\geq 0$.¶

Proof of Proposition 3.1 : The proof is done by a bunch of simple calculations. We compute the LHS of eq.(3.10) by using Lemma 3.1.

$$\begin{aligned}&\Big\langle 1\ 2\ 3\ 4:\zeta\Big|\Big(\sum_i Q^{(i)}\Big)\\ &=\int_0^{2\zeta}\langle\Omega|^{(1234)}\Big(\sum_j Q^{(j)}\Big)\prod_j e^{-\zeta L_0^{(i)}}+\int_{-2\zeta}^{0}\langle\Omega|^{(1234)}\Big(\sum_j Q^{(j)}\Big)\prod_j e^{-\zeta L_0^{(i)}}\\ &=\Big\{\Big\langle 1\ 2\ a\Big|\Big\langle a'\ 3\ 4\Big|\left(e^{-2\zeta L_0}\Big|S_{a'a}\Big\rangle\right)-\Big\langle 1\ 2\ a\Big|\Big\langle a'\ 3\ 4\Big|S_{a'a}\Big\rangle\Big\}\prod_i e^{-\zeta L_0^{(i)}}\\ &\qquad+\Big\{\Big\langle 2\ 3\ a\Big|\Big\langle a'\ 4\ 1\Big|S_{a'a}\Big\rangle-\Big\langle 2\ 3\ a\Big|\Big\langle a'\ 4\ 1\Big|\left(e^{-2\zeta L_0}\Big|S_{a'a}\Big\rangle\right)\Big\}\prod_i e^{-\zeta L_0^{(i)}}\end{aligned}\tag{3.18}$$

Then we arrange eq.(3.18) as follows.

$$\begin{aligned} Eq.(3.18) &= \Big\langle 1\ 2\ a:\zeta\Big|\Big\langle a'\ 3\ 4:\zeta\Big|S_{a'a}\Big\rangle - \Big\langle 2\ 3\ a:\zeta\Big|\Big\langle a'\ 4\ 1:\zeta\Big|S_{a'a}\Big\rangle \\ &\quad - \Big(\Big\langle 1\ 2\ a\Big|\Big\langle a'\ 3\ 4\Big|S_{a'a}\Big\rangle - \Big\langle 2\ 3\ a\Big|\Big\langle a'\ 4\ 1\Big|S_{a'a}\Big\rangle\Big)\prod_i e^{-\zeta L_0^{(i)}} \\ &= \Big\langle 1\ 2\ a:\zeta\Big|\Big\langle a'\ 3\ 4:\zeta\Big|S_{a'a}\Big\rangle - \Big\langle 2\ 3\ a:\zeta\Big|\Big\langle a'\ 4\ 1:\zeta\Big|S_{a'a}\Big\rangle \\ &\quad - \Big(\Big\langle 1\ 2\ \ 3\ 4\Big| - \Big\langle 2\ 3\ 4\ 1\Big|\Big)\prod_i e^{-\zeta L_0^{(i)}}. \end{aligned} \tag{3.19}$$

The last term vanishes identically because of the cyclic symmetry of $\Big\langle 1\ 2\ 3\ 4\Big|$ [13]. (We remind the reader that $\Big\langle 1\ 2\ 3\ 4\Big|$ and our 4-vertex $\Big\langle 1\ 2\ 3\ 4:\zeta\Big|$ are different.) Therefore we obtain the RHS of eq.(3.10).¶

Proof of Proposition 3.2 : The first-order variation of the 4-vertex with respect to the cut-off scale ζ can be computed as follows :

$$\begin{aligned} \delta\Big\langle 1\ 2\ 3\ 4:\zeta\Big| &= \delta\left[\int_{\mathcal{V}_4(\zeta)}\langle\Omega|^{(1234)}\prod_{i=1}^4 e^{-\zeta L_0^{(i)}}\right] \\ &= \delta\left[\int_{\mathcal{V}_4(\zeta)}\langle\Omega|^{(1234)}\right]\prod_{i=1}^4 e^{-\zeta L_0^{(i)}} + \int_{\mathcal{V}_4(\zeta)}\langle\Omega|^{(1234)}\delta\left[\prod_{i=1}^4 e^{-\zeta L_0^{(i)}}\right] \\ &= 2\delta\zeta\left\{\Big(\langle 2\zeta|^{(1234)} + \langle -2\zeta|^{(1234)}\Big)\prod_{i=1}^4 e^{-\zeta L_0^{(i)}} - \frac{1}{2}\Big\langle 1\ 2\ 3\ 4:\zeta\Big|\Big(\sum_{i=1}^4 L_0^{(i)}\Big)\right\}. \end{aligned} \tag{3.20}$$

It is easy to see that eq.(3.20) precisely gives the RHS of eq.(3.13).¶

3.1.2 Open-string 5-vertex $\Big\langle 1\ 2\ 3\ 4\ 5:\zeta\Big|$

The compactification $\mathcal{CM}_5^\partial$ is a two-disk. We fix an identification of $\mathcal{CM}_5^\partial$ with the set of open-string 5-diagrams as follows. Let $\mathcal{U}_i$ be the fan $y_i \geq |x_i| \geq 0$ on the two-plane (x_i, y_i) for $1 \leq i \leq 5$. $\mathcal{CM}_5^\partial$ is understood as $\cup_{i=1}^5 \mathcal{U}_i$. See Figure 9. For each $(x_i, y_i) \in \mathcal{U}_i$ we associate an open-string diagram as given in Figure 10. Open-string indices in the diagrams, say $2i$, are understood modulo 5. Length of the two internal strips are given by x_i (or $|x_i|$) and y_i. Diagrams at the boundary $x_i = y_i$ of $\mathcal{U}_i$ coincide with those at the boundary $x_{i+1} = -y_{i+1}$ of $\mathcal{U}_{i+1}$, while diagrams at the other boundary $x_i = -y_i$ coincide with those at the boundary $x_{i-1} = y_{i-1}$ of $\mathcal{U}_{i-1}$. As the result we can joint these $\mathcal{U}_i$ together in the clockwise order along their boundaries so that they

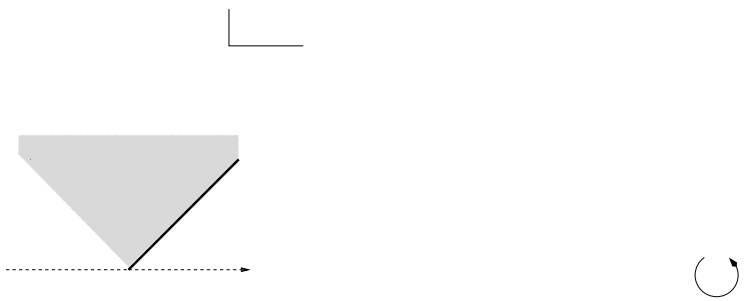

Figure 9: (a) $\mathcal{U}_i$ is given by the shaded region on the (x_i, y_i)-plane. (b) The compactification $\mathcal{CM}_5^{\partial}$. It is understood as a union $\cup_{i=1}^{5}\mathcal{U}_i$. The arrow denotes the orientation of $\mathcal{CM}_5^{\partial}$.

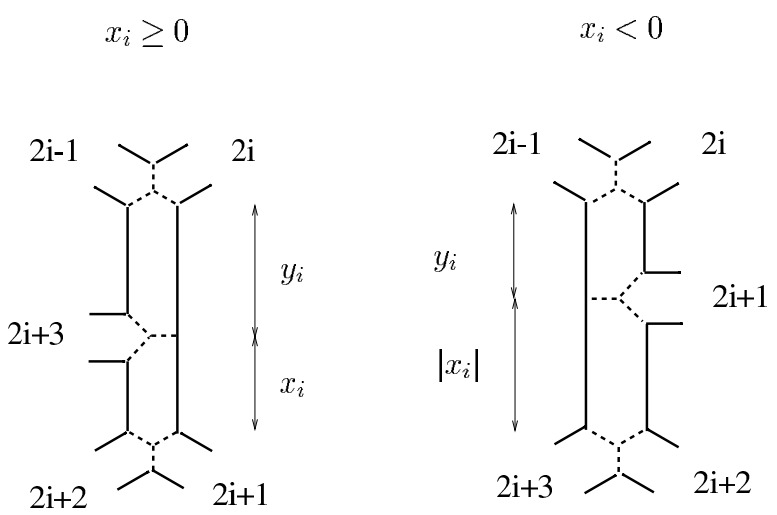

Figure 10: Open-string diagram at $(x_i, y_i) \in \mathcal{U}_i$.

are consistent with the open-string diagrams. Hence this gives an identification of $\mathcal{CM}_5^\partial$ with the set of open-string 5-diagrams.

We have a state $\langle\Sigma| \in (\mathcal{H}^{\otimes 5})^*$ for each $\Sigma \in \mathcal{CM}_5^\partial$. It is given by a patch-wise construction. We introduce a state $\Big\langle (x_i, y_i)\Big|_i^{(12345)} \in (\mathcal{H}^{\otimes 5})^*$ for $(x_i, y_i) \in \mathcal{U}_i$ by applying the Feynman rule to the corresponding open-string diagram.

$$\Big\langle (x_i, y_i)\Big|_i^{(12345)}$$
$$= \begin{cases} \Big\langle 2i{-}1\ 2i\ a\Big|\Big\langle b\ 2i{+}1\ 2i{+}2\Big|\Big\langle a'\ b'\ 2i{+}3\Big| & \\ \qquad \times \Big(b_0 e^{-y_i L_0}\Big|S_{a'a}\Big\rangle\Big)\Big(b_0 e^{-x_i L_0}\Big|S_{b'b}\Big\rangle\Big) & \text{for } \ y_i \geq x_i > 0, \\ \Big\langle 2i{-}1\ 2i\ a\Big|\Big\langle a'\ 2i{+}1\ b\Big|\Big\langle b'\ 2i{+}2\ 2i{+}3\Big| & \\ \qquad \times \Big(b_0 e^{-y_i L_0}\Big|S_{a'a}\Big\rangle\Big)\Big((-)b_0 e^{x_i L_0}\Big|S_{b'b}\Big\rangle\Big) & \text{for } \ y_i \geq -x_i > 0, \\ 0 & \text{for } \ x_i = 0, y_i \geq 0. \end{cases} \tag{3.21}$$

This state is grassmann-odd and has the ghost number equal to five. We then define a $(\mathcal{H}^{\otimes 5})^*$-valued two-form $\langle\Omega|^{(12345)}$. We put it equal to a two-form $dx_i \wedge dy_i \Big\langle (x_i, y_i)\Big|_i^{(12345)}$ on each $\mathcal{U}_i$,

$$\Omega^{(12345)}\Big|_{\mathcal{U}_i} = dx_i \wedge dy_i \Big\langle (x_i, y_i)\Big|_i^{(12345)}. \tag{3.22}$$

We examine an effect of the cyclic permutation of open-string indices. We can obtain another state by permuting the open-string indices from $(1,2,3,4,5)$ to $(2,3,4,5,1)$ in the RHS of eq.(3.21). We denote this new state by $\Big\langle (x_i, y_i)\Big|_i^{(23451)}$. We call the corresponding two-form $\langle\Omega|^{(23451)}$. As one can easily check, the state $\Big\langle (x_i, y_i)\Big|_i^{(23451)}$ turns out to be $\Big\langle (x_i, y_i)\Big|_{i+3}^{(12345)}$. This permutation generates a $\mathbf{Z}_5$-action on $\mathcal{CM}_5^\partial$. It is given by $s(\mathcal{U}_i) = \mathcal{U}_{i+3}$. Therefore $\langle\Omega|^{(23451)}$ is the pull-back of $\langle\Omega|^{(12345)}$ by s.

$$\Omega^{(23451)} = s^*\Omega^{(12345)}. \tag{3.23}$$

We put $\mathcal{U}_i(\zeta) \equiv \Big\{(x_i, y_i) \in \mathcal{U}_i \ \Big|\ y_i < 2\zeta\Big\}$. Any diagram belonging to $\mathcal{U}_i(\zeta)$ can not be reproduced from lower diagrams at the scale ζ. We put $\mathcal{V}_5(\zeta) \equiv \cup_{i=1}^5 \mathcal{U}_i(\zeta)$. (Figure 11.) The state $\Big\langle (x_i, y_i)\Big|_i^{(12345)}$ must appear in $S_{eff}^\zeta[\Phi]$ as a part of an interaction vertex for any $(x_i, y_i) \in \mathcal{V}_5(\zeta)$. We then define open-string 5-vertex at the scale ζ as follows.

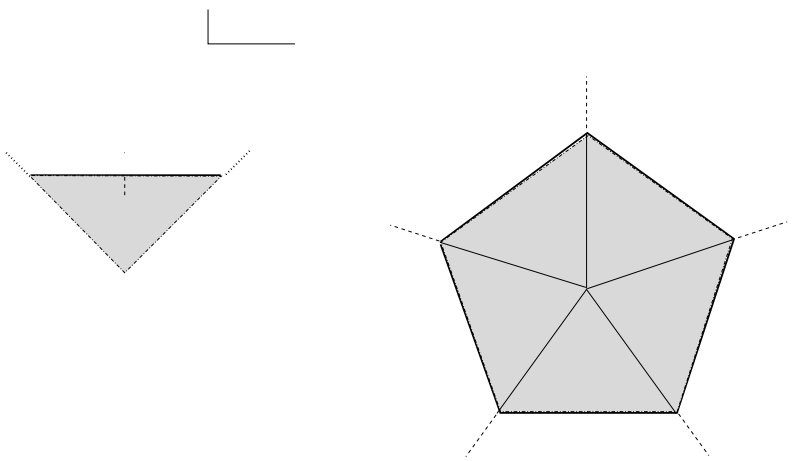

Figure 11: (a) $\mathcal{U}_i(\zeta)$ is the shaded region of $\mathcal{U}_i$. (b) $\mathcal{V}_5(\zeta)$ is a union $\cup_{i=1}^5 \mathcal{U}_i(\zeta)$. It becomes the shaded region of $\mathcal{CM}_5^\partial = B_2$. The arrow denotes the orientation of $\mathcal{V}_5(\zeta)$.

Definition 3.3 (Open-string 5-vertex at ζ) *Open-string 5-vertex at the cut-off scale ζ is defined by an integration of the $(\mathcal{H}^{\otimes 5})^*$-valued two-form $\langle\Omega|$ (3.22) over $\mathcal{V}_5(\zeta)$ multiplied by the external propagators of length ζ.*

$$\Big\langle 1\ 2\ 3\ 4\ 5 : \zeta \Big| = \int_{\mathcal{V}_5(\zeta)} \langle\Omega|^{(12345)} \prod_{k=1}^{5} e^{-\zeta L_0^{(k)}}. \tag{3.24}$$

Indices of the vertex are understood to label the Hilbert spaces $\mathcal{H}^{(i)}$ attached to clockwise-ordered five open-strings on ∂D. Thereby the vertex is regarded as a vector of $(\mathcal{H}^{(1)} \otimes \cdots \otimes \mathcal{H}^{(5)})^*$.

The 5-vertex is grassmann-odd and has the ghost number equal to five. Another 5-valent vertex might be obtained by using the two-form $\Omega^{(23451)}$ instead of $\Omega^{(12345)}$ in the definition. We denote it by $\Big\langle 2\ 3\ 4\ 5\ 1 : \zeta \Big|$. But they eventually become the same due to the relation (3.23).

$$\begin{aligned}
\Big\langle 2\ 3\ 4\ 5\ 1 : \zeta \Big| &= \int_{\mathcal{V}_5(\zeta)} \langle\Omega|^{(23451)} \prod_{k=1}^{5} e^{-\zeta L_0^{(k)}} \\
&= \int_{\mathcal{V}_5(\zeta)} s^* \langle\Omega|^{(12345)} \prod_{k=1}^{5} e^{-\zeta L_0^{(k)}} \\
&= \int_{s(\mathcal{V}_5(\zeta)) = \mathcal{V}_5(\zeta)} \langle\Omega|^{(12345)} \prod_{k=1}^{5} e^{-\zeta L_0^{(k)}} \\
&= \Big\langle 1\ 2\ 3\ 4\ 5 : \zeta \Big|.
\end{aligned} \tag{3.25}$$

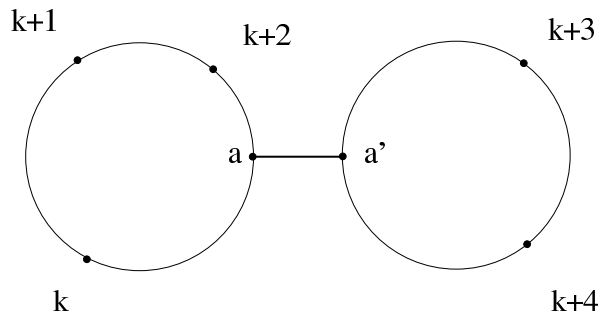

Figure 12: A typical configuration of five points on ∂D which belongs to $\mathcal{V}_{(k\ k+1\ k+2\ a)}(\zeta) \times \mathcal{V}_{(a'\ k+3\ k+4)}(\zeta)$ $(\subset \partial\mathcal{V}_5(\zeta))$. The solid line connecting two points a and a' represents open-string strip of length equal to 2ζ.

We say this as symmetry of the open-string 5-vertex under the cyclic permutation.

The 5-vertex is not invariant under the action of the BRST charge. Similarly to the case of the 4-vertex the BRST transform can be written down using the lower vertices.

Proposition 3.3 (Q-action on 5-vertex) *Action of the BRST charge on the 5-vertex (3.24) becomes as follows.*

$$\Big\langle 1\ 2\ 3\ 4\ 5{:}\zeta\Big|\Big(\sum_{i=1}^{5} Q^{(i)}\Big) = -\sum_{k=1}^{5}\Big\langle k\ k{+}1\ k{+}2\ a{:}\zeta\Big|\Big\langle a'\ k{+}3\ k{+}4{:}\zeta\Big|S_{a'a}\Big\rangle, \tag{3.26}$$

where open-string indices in the RHS are understood modulo 5.

The origin of eq.(3.26) is also in the geometry. The boundary of $\mathcal{V}_5(\zeta)$ is a circle. As a set, it is a sum of $\mathcal{V}_{(k\ k+1\ k+2\ a)}(\zeta) \times \mathcal{V}_{(a'\ k+3\ k+4)}(\zeta)$. Here we write $\mathcal{V}_4(\zeta)$ and $\mathcal{V}_3(\zeta)$ (a single point) as $\mathcal{V}_{(a\ k\ k+1\ k+2)}(\zeta)$ and $\mathcal{V}_{(a'\ k+3\ k+4)}(\zeta)$ in order to show that external open-strings participating in the diagrams are labeled in the clockwise order respectively by $(k, k{+}1, k{+}2, a)$ and $(a', k{+}3, k{+}4)$. (Figure 12). Orientation of $\mathcal{V}_5(\zeta)$ is given by $\mathcal{CM}_5^{\partial}$ (the standard orientation of two-disk). Thereby the circle is oriented. On the other hand, $\mathcal{V}_4(\zeta) \times \mathcal{V}_3(\zeta)$ has the orientation determined from $\mathcal{CM}_4^{\partial} \times \mathcal{CM}_3^{\partial}$. Comparing these orientations we obtain

$$\partial\mathcal{V}_5(\zeta) = -\sum_{k=1}^{5} \mathcal{V}_{(k\ k+1\ k+2\ a)}(\zeta) \times \mathcal{V}_{(a'\ k+3\ k+4)}(\zeta). \tag{3.27}$$

Looking at eq.(3.26) and eq.(3.27) we find a coincidence between the BRST transformation and the boundary operation. Actually the action of the BRST charge on the 5-vertex becomes a representation of the boundary operator.

The 5-vertex depends on the cut-off scale. It vanishes at $\zeta = 0$. The dependence comes from the integration region $\mathcal{V}_5(\zeta)$ besides the propagators $\prod_i e^{-\zeta L_0^{(i)}}$ in eq.(3.24).

Proposition 3.4 (Scale dependence of 5-vertex) *Scale dependence of the 5-vertex (3.24) is described by*

$$\begin{aligned}\frac{1}{2}\frac{d}{d\zeta}\Big\langle 1\ 2\ 3\ 4\ 5{:}\zeta\Big| \;=\; & -\sum_{k=1}^{5}\Big\langle k\ k{+}1\ k{+}2\ a{:}\zeta\Big|\Big\langle a'\ k{+}3\ k{+}4{:}\zeta\Big|\Big(b_0\Big|S_{a'a}\Big\rangle\Big) \\ & -\frac{1}{2}\Big\langle 1\ 2\ 3\ 4\ 5{:}\zeta\Big|\Big(\sum_{k=1}^{5} L_0^{(k)}\Big). \end{aligned} \tag{3.28}$$

We give proofs of Propositions 3.3 and 3.4. The next lemma is useful in the proof of Proposition 3.3.

Lemma 3.2 *The BRST charge acts on the state* $\Big\langle (x_i,y_i)\Big|_i^{(12345)}$ *as follows :*

$$\begin{aligned}&\Big\langle (x_i,y_i)\Big|_i^{(12345)}\Big(\sum_i Q^{(i)}\Big) \\ &= \begin{cases} \Big\langle 2i{-}1\ 2i\ a\Big|\Big\langle b\ 2i{+}1\ 2i{+}2\Big|\Big\langle a'\ b'\ 2i{+}3\Big| & \\ \quad\times\Big\{\Big(b_0 e^{-y_i L_0}\Big|S_{a'a}\Big\rangle\Big)\frac{\partial}{\partial x_i}\Big(e^{-x_i L_0}\Big|S_{b'b}\Big\rangle\Big)+\frac{\partial}{\partial y_i}\Big(e^{-y_i L_0}\Big|S_{a'a}\Big\rangle\Big)\Big(b_0 e^{-x_i L_0}\Big|S_{b'b}\Big\rangle\Big)\Big\} & \textit{for}\ \ x_i>0 \\ \Big\langle 2i{-}1\ 2i\ a\Big|\Big\langle a'\ 2i{+}1\ b\Big|\Big\langle b'\ 2i{+}2\ 2i{+}3\Big| & \\ \quad\times\Big\{\Big(b_0 e^{-y_i L_0}\Big|S_{a'a}\Big\rangle\Big)\frac{\partial}{\partial x_i}\Big(e^{x_i L_0}\Big|S_{b'b}\Big\rangle\Big)+\frac{\partial}{\partial y_i}\Big(e^{-y_i L_0}\Big|S_{a'a}\Big\rangle\Big)\Big((-)b_0 e^{x_i L_0}\Big|S_{b'b}\Big\rangle\Big)\Big\} & \textit{for}\ \ x_i<0 \end{cases}. \end{aligned} \tag{3.29}$$

This lemma can be shown in the same manner as Lemma 3.1. We omit the proof.

Proof of Proposition 3.3 : We put $I_i \equiv \int_{\mathcal{U}_i(\zeta)} dx_i\wedge dy_i\ \Big\{\Big\langle (x_i,y_i)\Big|_i^{(12345)}\Big(\sum_{k=1}^5 Q^{(k)}\Big)\Big\}$. The LHS of eq.(3.26) is just $\sum_{i=1}^5 I_i$. Due to Lemma 3.2 the integral I_i reduces to an integral on $\partial\mathcal{U}_i(\zeta)$. The boundary $\partial\mathcal{U}_i(\zeta)$ has three components. I_i becomes a sum of the three boundary integrals. The boundary integral along $x_i = y_i$ turns out to equal $(-1)\times$ the boundary integral of I_{i+1} along $-x_{i+1}=y_{i+1}$. Thus their contributions to $\sum_i I_i$ cancel each other. (These two boundaries are identified in $\mathcal{V}_5(\zeta)$.) Net contribution of I_i to the sum becomes the boundary integral along

$y_i = 2\zeta$. Therefore we have

$$
\begin{aligned}
&\Big\langle 1\ 2\ 3\ 4\ 5 : \zeta \Big| \Big(\sum_{i=1}^{5} Q^{(i)} \Big) \qquad \Big(= \sum_{i=1}^{5} I_i \Big) \\
&= \sum_{i=1}^{5} \Bigg\{ \int_0^{2\zeta} dx_i \Big\langle 2i{-}1\ 2i\ a \Big| \Big\langle b\ 2i{+}1\ 2i{+}2 \Big| \Big\langle a'\ b'\ 2i{+}3 \Big| \Big(e^{-2\zeta L_0} \Big| S_{a'a} \Big\rangle \Big) \Big(b_0 e^{-x_i L_0} \Big| S_{b'b} \Big\rangle \Big) \\
&\qquad + \int_{-2\zeta}^{0} dx_i \Big\langle 2i{-}1\ 2i\ a \Big| \Big\langle a'\ 2i{+}1\ b \Big| \Big\langle b'\ 2i{+}2\ 2i{+}3 \Big| \Big(e^{-2\zeta L_0} \Big| S_{a'a} \Big\rangle \Big) \Big((-) b_0 e^{x_i L_0} \Big| S_{b'b} \Big\rangle \Big) \Bigg\} \\
&\qquad\qquad \times \prod_{k=1}^{5} e^{-\zeta L_0^{(k)}} \\
&= -\sum_{i=1}^{5} \Big\langle 2i{-}1\ 2i\ a \Big| \\
&\qquad \times \Bigg\{ \int_0^{2\zeta} dx \Big\langle a'\ 2i{+}1\ b \Big| \Big\langle b'\ 2i{+}2\ 2i{+}3 \Big| \Big(b_0 e^{-xL_0} \Big| S_{b'b} \Big\rangle \Big) \\
&\qquad\quad + \int_{-2\zeta}^{0} dx \Big\langle 2i{+}1\ 2i{+}2\ b \Big| \Big\langle b'\ 2i{+}3\ a' \Big| \Big((-) b_0 e^{xL_0} \Big| S_{b'b} \Big\rangle \Big) \Bigg\} \prod_{k=1}^{5} e^{-\zeta L_0^{(k)}} \\
&\qquad\qquad \times \Big(e^{-2\zeta L_0} \Big| S_{a'a} \Big\rangle \Big)
\end{aligned}
\tag{3.30}
$$

We rewrite eq.(3.30) in terms of the vertices at the scale ζ. It turns out to be

$$
\begin{aligned}
Eq.(3.30) &= -\sum_{i=1}^{5} \Big\langle 2i{-}1\ 2i\ a : \zeta \Big| \Big\langle a'\ 2i{+}1\ 2i{+}2\ 2i{+}3 : \zeta \Big| S_{a'a} \Big\rangle, \\
&= -\sum_{k=1}^{5} \Big\langle k\ k{+}1\ k{+}2\ a : \zeta \Big| \Big\langle a'\ k{+}3\ k{+}4 : \zeta \Big| S_{a'a} \Big\rangle.
\end{aligned}
\tag{3.31}
$$

This is precisely the RHS of eq.(3.26).¶

Proof of Proposition 3.4 : We compute the first-order variation of the 5-vertex with respect to the cut-off scale parameter ζ as follows.

$$
\begin{aligned}
&\delta \Big\langle 1\ 2\ 3\ 4\ 5 : \zeta \Big| \\
&= \delta \Big[\int_{\mathcal{V}_5(\zeta)} \langle \Omega |^{(12345)} \prod_{k=1}^{5} e^{-\zeta L_0^{(k)}} \Big] \\
&= \delta \Big[\int_{\mathcal{V}_5(\zeta)} \langle \Omega |^{(12345)} \Big] \prod_{k=1}^{5} e^{-\zeta L_0^{(k)}} + \int_{\mathcal{V}_5(\zeta)} \langle \Omega |^{(12345)} \delta \Big[\prod_{k=1}^{5} e^{-\zeta L_0^{(k)}} \Big] \\
&= \sum_{i=1}^{5} \int_{\delta \mathcal{U}_i(\zeta)} dx_i \wedge dy_i \Big\langle (x_i, y_i) \Big|_i^{(12345)} \prod_{k=1}^{5} e^{-\zeta L_0^{(k)}} - \delta\zeta \Big\langle 1\ 2\ 3\ 4\ 5 : \zeta \Big| \Big(\sum_{k=1}^{5} L_0^{(k)} \Big).
\end{aligned}
\tag{3.32}
$$

The first term of eq.(3.32) is further evaluated as follows.

$$
\begin{aligned}
&\sum_i \int_{\delta\mathcal{U}_i(\zeta)} dx_i \wedge dy_i \Big\langle (x_i,y_i)\Big|_i^{(12345)} \prod_{k=1}^{5} e^{-\zeta L_0^{(k)}} \\
&= 2\delta\zeta \sum_i \Bigg\{ \int_0^{2\zeta} dx_i \Big\langle 2i{-}1\ 2i\ a \Big| \Big\langle b\ 2i{+}1\ 2i{+}2 \Big| \Big\langle a'\ b'\ 2i{+}3 \Big| \left(b_0 e^{-2\zeta L_0}\Big| S_{a'a}\right\rangle\right) \left(b_0 e^{-x_i L_0}\Big| S_{b'b}\right\rangle\right) \\
&\qquad + \int_{-2\zeta}^{0} dx_i \Big\langle 2i{-}1\ 2i\ a \Big| \Big\langle a'\ 2i{+}1\ b \Big| \Big\langle b'\ 2i{+}2\ 2i{+}3 \Big| \left(b_0 e^{-2\zeta L_0}\Big| S_{a'a}\right\rangle\right) \left((-)b_0 e^{x_i L_0}\Big| S_{b'b}\right\rangle\right) \Bigg\} \\
&\qquad\qquad \times \prod_{k=1}^{5} e^{-\zeta L_0^{(k)}} \\
&= -2\delta\zeta \sum_i \Big\langle 2i{-}1\ 2i\ a \Big| \\
&\qquad\qquad \times \Bigg\{ \int_0^{2\zeta} dx \Big\langle a'\ 2i{+}1\ b \Big| \Big\langle b'\ 2i{+}2\ 2i{+}3 \Big| \left(b_0 e^{-xL_0}\Big| S_{b'b}\right\rangle\right) \\
&\qquad\qquad\qquad + \int_{-2\zeta}^{0} dx \Big\langle 2i{+}1\ 2i{+}2\ b \Big| \Big\langle b'\ 2i{+}3\ a' \Big| \left((-)b_0 e^{xL_0}\Big| S_{b'b}\right\rangle\right) \Bigg\} \\
&\qquad\qquad\qquad \times \prod_{k=1}^{5} e^{-\zeta L_0^{(k)}} \left(b_0 e^{-2\zeta L_0}\Big| S_{a'a}\right\rangle\right). \qquad (3.33)
\end{aligned}
$$

We write eq.(3.33) in terms of the vertices at the scale ζ. It turns out to be

$$
\begin{aligned}
Eq.(3.33) &= -2\delta\zeta \sum_i \Big\langle 2i{-}1\ 2i\ a{:}\zeta \Big| \Big\langle a'\ 2i{+}1\ 2i{+}2\ 2i{+}3{:}\zeta \Big| \left(b_0\Big| S_{a'a}\right\rangle\right), \qquad (3.34) \\
&= -2\delta\zeta \sum_k \Big\langle k\ k{+}1\ k{+}2\ a{:}\zeta \Big| \Big\langle a'\ k{+}3\ k{+}4{:}\zeta \Big| \left(b_0\Big| S_{a'a}\right\rangle\right). \qquad (3.35)
\end{aligned}
$$

Hence we obtain the first term of the RHS of eq.(3.28). The second term in eq.(3.32) gives the second term of the RHS of eq.(3.28). ¶

3.2 Conjecture on higher open-string vertices

We want to obtain low energy open-string n-valent vertices for the cases of $n \geq 6$. These vertices are determined in principle from the sum of the graphs which are one-particle irreducible with respect to the regularized propagator. In the previous subsection explicit constructions are given for $n = 3, 4$ and 5 cases. $\mathcal{V}_3(\zeta), \mathcal{V}_4(\zeta)$ and $\mathcal{V}_5(\zeta)$ are the moduli of the corresponding irreducible graphs. The approach taken there may be generalized.

The compactification $\mathcal{CM}_n^{\partial}$ is B_{n-3}. Each point of $\mathcal{CM}_n^{\partial}$ represents an open-string n-diagram. We can obtain a map $\mathcal{CM}_n^{\partial} \to (\mathcal{H}^{\otimes n})^* : \Sigma \mapsto \langle\Sigma|$ by applying the Feynman rule

to the diagrams. Since they are trivalent ribbon graphs, each diagram consists of $(n-2)$ trivalent vertices and $(n-3)$ internal strips. Two trivalent vertices are connected at one external open-string by inserting there the propagator of the form $b_0 e^{-\tau L_0}|S_{a'a}\rangle$ in the following way.

$$\Big\langle 1\ 2\ a{:}\zeta\Big|\Big\langle a'\ 3\ 4\ {:}\zeta\Big|\Big(b_0\Big|S_{a'a}\Big\rangle\Big)\,. \tag{3.36}$$

The trivalent vertex is a grassmann-odd vector with the ghost number three and the propagator is a grassmann-even vector with the ghost number two. The state $\langle\Sigma|$ acquires the ghost number n and the grassmannity $(-)^n$. We have several ways to identify points of $\mathcal{CM}_n^\partial$ with the diagrams. Different identifications are related with one another by the automorphisms generated by s. Let us take one of them and fix it. We call the state $\langle\Sigma|$ given under this identification as $\langle\Sigma|^{(1\ 2\cdots n-1\ n)}$. If one takes another identification related by the automorphism s, the corresponding state is called $\langle\Sigma|^{(2\ 3\cdots n\ 1)}$.

To have an explicit representation of the map it is convenient to take a patch-wise construction. Let $\mathcal{CM}_n^\partial \equiv \cup_i \mathcal{U}_i$. Each $\mathcal{U}_i$ may be identified with a suitable cone of $\mathbf{R}^{n-3}$. The Euclidean coordinates $(x_1^i,\cdots,x_{n-3}^i)$ of $\mathcal{U}_i$ represent the metric of the underlying ribbon graph. The map $\Sigma \mapsto \langle\Sigma|$ is introduced in a patch-wise manner. We denote the state given at $(x_1^i,\cdots,x_{n-3}^i)$ by $\Big\langle(x_1^i,\cdots,x_{n-3}^i)\Big|_i$.

The next step is to obtain a $(\mathcal{H}^{\otimes n})^*$-valued $(n-3)$-form $\langle\Omega|$. It is given on each $\mathcal{U}_i$ by $dx_1^i\wedge\cdots\wedge dx_{n-3}^i\,\Big\langle(x_1^i,\cdots,x_{n-3}^i)\Big|_i$. Recall $\mathcal{V}_n(\zeta)$ is the set of open-string n-diagrams which are one-particle irreducible with respect to the regularized propagator (3.3). In other word it is the set of open-string n-diagrams all internal strips of which have length less than 2ζ. $\mathcal{V}_n(\zeta)$ also becomes a $(n-3)$-ball. Open-string n-vertex at the cut-off scale ζ is defined by an integration of Ω over the ball $\mathcal{V}_n(\zeta)$.

$$\Big\langle 1\ 2\cdots n{-}1\ n\,{:}\zeta\Big| = \int_{\mathcal{V}_n(\zeta)} \langle\Omega|^{(1\ 2\cdots n-1\ n)} \prod_{i=1}^n e^{-\zeta L_0^{(i)}}.$$

Indices in the vertex are understood to label clockwise-ordered n open-strings on ∂D. The vertex is regarded as a vector of $(\mathcal{H}^{(1)}\otimes\cdots\otimes\mathcal{H}^{(n)})^*$. It has the ghost number n and the grassmannity $(-)^n$. The automorphisms of $\mathcal{CM}_n^\partial$ will provide symmetry of the vertex. Depending on choice of the identifications, we have several Ω. They are expected to be related with one another by the automorphisms.

$$s^*\Omega^{(1\ 2\cdots n-1\ n)} = \Omega^{(2\ 3\cdots n\ 1)}.$$

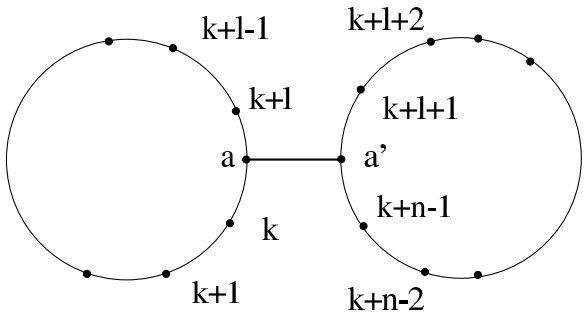

Figure 13: A typical configuration of n points on ∂D which belongs to $\mathcal{V}_{(k\ \cdots\ k+l\ a)}(\zeta) \times \mathcal{V}_{(a'\ k+l+1\ \cdots\ k+n-1)}(\zeta)$ $(\subset \partial\mathcal{V}_n(\zeta))$. The solid line connecting two points a and a' represents open-string strip of length equal to 2ζ.

This implies that the n-vertex becomes symmetric or anti-symmetric under the cyclic permutation when s preserves the orientation of $\mathcal{CM}_n^\partial$ or not.

The boundary $\partial\mathcal{V}_n(\zeta)$ is identified, as a set, with a suitable sum of $\mathcal{V}_{l+2}(\zeta) \times \mathcal{V}_{n-l}(\zeta)$. $\mathcal{V}_n(\zeta)$ is oriented by the standard orientation of B_{n-3}. Thereby $\partial\mathcal{V}_n(\zeta)$ is oriented. On the other hand, $\mathcal{V}_{l+2}(\zeta) \times \mathcal{V}_{n-l}(\zeta)$ is also oriented by $B_{l-1} \times B_{n-l-3}$. Taking account of their orientations we have

$$\partial\mathcal{V}_n(\zeta) = \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n-3}(\pm)\mathcal{V}_{(k\ k+1\ \cdots\ k+l\ a)}(\zeta) \times \mathcal{V}_{(a'\ k+l+1\ \cdots\ k+n-1)}(\zeta).$$

The factor $1/2$ is needed to avoid the double counting of the components. We write $\mathcal{V}_{l+2}(\zeta)$ and $\mathcal{V}_{n-l}(\zeta)$ as $\mathcal{V}_{(k\ k+1\ \cdots\ k+l\ a)}(\zeta)$ and $\mathcal{V}_{(a'\ k+l+1\ \cdots\ k+n-1)}(\zeta)$, in order to show that open-strings participating in the diagrams are labeled in the clockwise order respectively by $(k, k+1, \cdots, k+l, a)$ and $(a', k+l+1, \cdots, k+n-1)$. (Figure 13). The signature $(\pm)$ must be determined by a comparison of the orientations of $\partial\mathcal{V}_n(\zeta)$ and $\mathcal{V}_{l+2}(\zeta) \times \mathcal{V}_{n-l}(\zeta)$. As we have observed in the previous examples, action of the BRST charge on the open-string vertices is expected to provide a representation of the boundary operator ∂. If so, we will have

$$\Big\langle 1\cdots n:\zeta\Big|\Big(\sum_{i=1}^{n}Q^{(i)}\Big) = \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n-3}(\pm)\Big\langle\underbrace{k\ k+1\ \cdots\ k+l\ a}_{l+2}:\zeta\Big|\Big\langle\underbrace{a'\ k+l+1\ \cdots\ k+n-1}_{n-l}:\zeta\Big|S_{a'a}\Big\rangle.$$

The n-vertex vanishes at $\zeta = 0$ since $\mathcal{V}_n(0)$ becomes a point. The scale dependence of open-string vertices follows from the above speculation. The dependence comes from the ball

$\mathcal{V}_n(\zeta)$ and the multiplier $\prod_i e^{-\zeta L_0^{(i)}}$. Variation of the n-vertex with respect to ζ is a sum of the variations of these two. Therefore we have

$$\begin{aligned}\frac{d}{d\zeta}\Big\langle 1\ 2\cdots n:\zeta\Big| \;=\; & \sum_{k=1}^{n}\sum_{l=1}^{n-3}(\pm)\Big\langle \underbrace{k\ k{+}1\ \cdots\ k{+}l\ a}_{l+2}:\zeta\Big|\Big\langle \underbrace{a'\ k{+}l{+}1\ \cdots\ k{+}n{-}1}_{n-l}:\zeta\Big|\Big(b_0\Big|S_{a'a}\Big\rangle\Big) \\ & -\Big\langle 1\ 2\ \cdots\ n:\zeta\Big|\Big(\sum_{i=1}^{n}L_0^{(i)}\Big).\end{aligned}$$

To carry out this program and examine the conjectural properties of open-string vertices, we need a systematic description of $\mathcal{CM}_n^{\partial}$. In particular, it is required so that the two possible orientations of each component appearing in $\partial\mathcal{V}_n(\zeta)$ must be compared with each other in a systematic way. Unfortunately we do not know such a description. We leave it as a future problem. Instead we would like to propose a possible solution in the following conjecture.

Conjecture *Open-string n-vertex $\Big\langle 1\cdots n:\zeta\Big| \in (\mathcal{H}^{\otimes n})^*$ $(n\geq 3)$ can be taken so that it enjoys the following action of the BRST charge,*

$$\begin{aligned}&\Big\langle 1\cdots n:\zeta\Big|\Big(\sum_{i=1}^{n}Q^{(i)}\Big) \\ &\quad=-\frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n-3}(-)^{(n+1)(k+l+1)}\Big\langle k\ k{+}1\ \cdots\ k{+}l\ a:\zeta\Big|\Big\langle a'\ k{+}l{+}1\ \cdots\ k{+}n{-}1:\zeta\Big|S_{a'a}\Big\rangle,\end{aligned} \tag{3.37}$$

besides the following cyclic asymmetry with respect to the open-string indices,

$$\Big\langle 1\ 2\ \cdots\ n{-}1\ n:\zeta\Big| = (-)^{n+1}\Big\langle 2\ 3\ \cdots\ n\ 1:\zeta\Big|. \tag{3.38}$$

Indices in the RHS of eq.(3.37), say $k{+}l$, are understood modulo n.

Open-string indices of the n-vertex are understood to label the Hilbert spaces $\mathcal{H}^{(i)}$ attached to clockwise-ordered n open-strings on ∂D. We regard the n-vertex $\Big\langle 1\ \cdots\ n\ :\zeta\Big|$ as a vector of $(\mathcal{H}^{(1)}\otimes\cdots\otimes\mathcal{H}^{(n)})^*$. It has the ghost number equal to n and grassmannity $(-)^n$. We give a few comments on the above conjecture. First, the asymmetry in eq.(3.38) is consistent with the action of the BRST charge (3.37). Secondly, it can be checked that for the cases of $n=3,4$ and 5, eqs.(3.37) and (3.38) reproduce correctly the corresponding results obtained in the previous subsection.

As we referred in the program at the beginning of this subsection, action of the BRST charge on the vertices is expected to provide a representation of the boundary operator ∂. For this to be realized, the conjectural action of the BRST charge must be nilpotent, at least. To show the nilpotency is a non-trivial test of the conjecture. If we accept eqs.(3.37) and (3.38), we have the following proposition.

Proposition 3.5 *Action of the BRST charge given in (3.37) is nilpotent.*

$$\left\{\left\langle 1\ 2\ \cdots\ n:\zeta\right|\left(\sum_{i=1}^{n}Q^{(i)}\right)\right\}\left(\sum_{i=1}^{n}Q^{(i)}\right)=0. \tag{3.39}$$

To show this proposition, some complicated calculations are required. We provide the proof in Appendix. The cut-off scale dependence of the vertices can be read from the conjecture.

Proposition 3.6 (Scale dependence of higher vertices) *Scale dependence of the open-string n-vertex ($n\geq 3$) is described by*

$$\begin{aligned}\frac{d}{d\zeta}&\left\langle 1\ 2\cdots n:\zeta\right| \\ &=-\sum_{k=1}^{n}\sum_{l=1}^{n-3}(-)^{(n+1)(k+l+1)}\left\langle k\ k{+}1\ \cdots\ k{+}l\ a:\zeta\right|\left\langle a'\ k{+}l{+}1\ \cdots\ k{+}n{-}1:\zeta\right|\left(b_0\left|S_{a'a}\right\rangle\right) \\ &\quad-\left\langle 1\ 2\ \cdots\ n:\zeta\right|\left(\sum_{i=1}^{n}L_0^{(i)}\right).\end{aligned} \tag{3.40}$$

We note that eq.(3.40) reproduces the previous results for the cases of $n=3,4$ and 5.

We proceed on the rest of this paper by assuming the conjecture.

4 A_∞-Algebra In Low Energy Theory

To develop low energy description of classical open-string field theory, we introduce the notion of homotopy associative algebra (A_∞-algebra). A_∞-algebra was introduced in [4] and further investigated in [5, 6] including deformation theory [3]. In string field theory, A_∞ algebra was discussed in [22, 23].

Let C be a **Z**-graded module. A graded module ΠC is defined by shifting the degree, $(\Pi C)^m \equiv C^{m+1}$. We write the shifted degree by ϵ. We also define $B_k\Pi C \equiv (\Pi C)^{\otimes k}$ and

[3]We follow the convention used in [5, 6].

$B\Pi C \equiv \bigoplus_k B_k\Pi C$. We consider a family of maps $m_k : B_k\Pi C \to \Pi C$ of degree $\epsilon=1$. Each m_k induces a map $d_k : B\Pi C \to B\Pi C$ by

$$d_k\Big(x_1,\cdots,x_n\Big) \equiv \sum_{p=0}^{n-k}(-)^{\sum_{j=1}^{p}\epsilon(x_j)}x_1\otimes\cdots\otimes x_p\otimes m_k\Big(x_{p+1}\cdots x_{p+k}\Big)\otimes x_{p+k+1}\otimes\cdots\otimes x_n. \quad (4.1)$$

Then we put $\hat{d}\equiv\sum_k d_k$.

Definition 4.1 (A_∞-algebra) *(C, m_k) for $k=1,2,\cdots$ is said to be a A_∞-algebra if $\hat{d}\hat{d}=0$. (C, m_k) for $k=0,1,2,\cdots$ is said to be a weak A_∞-algebra if $\hat{d}\hat{d}=0$.*

Let (C, m_k) be a A_∞-algebra. The condition $\hat{d}\hat{d}=0$ becomes equivalent to the constraints, $d_kd_1+\cdots+d_1d_k=0$ for $k\geq 1$. These can be written down in terms of m_k as

$$\sum_{l=1}^{k}\sum_{p=0}^{k-l}(-)^{\sum_{i=1}^{p}\epsilon(x_i)}m_{k+1-l}\Big(x_1,\cdots,x_p,m_l(x_{p+1},\cdots,x_{p+l}),x_{p+l+1},\cdots,x_k\Big)=0. \quad (4.2)$$

It may be instructive to comment on a first few series of these relations.

$$m_1\Big(m_1(x)\Big)=0, \quad (4.3)$$

$$m_1\Big(m_2(x,y)\Big)+m_2\Big(m_1(x),y\Big)+(-)^{\epsilon(x)}m_2\Big(x,m_1(y)\Big)=0, \quad (4.4)$$

$$\begin{aligned} &m_2\Big(m_2(x,y),z\Big)+(-)^{\epsilon(x)}m_2\Big(x,m_2(y,z)\Big)+m_1\Big(m_3(x,y,z)\Big) \quad (4.5)\\ &\quad +m_3\Big(m_1(x),y,z\Big)+(-)^{\epsilon(x)}m_3\Big(x,m_1(y),z\Big)+(-)^{\epsilon(x)+\epsilon(y)}m_3\Big(x,y,m_1(z)\Big)=0. \end{aligned}$$

The first equation implies that m_1 is a boundary operator and the second equation shows that m_1 is a derivation with respect to m_2. The third equation is related with an associativity relation. When m_3 vanishes, m_2 defines an associative algebra on C by putting $x\cdot y \equiv (-)^{\epsilon(x)}m_2\Big(x,y\Big)$. When m_3 does not vanish, the algebra "$\cdot$" is not associative. But it induces an associative algebra on the cohomology $H^*(\Pi C : m_1) \equiv \mathrm{Ker}\ m_1/\mathrm{Im}\ m_1$.

Apart from general theory of A_∞-algebra, we concentrate on classical open-string field theory. The open-string Hilbert space $\mathcal{H}$ is naturally $\mathbf{Z}$-graded by the ghost number G. We introduce a (-1)-shifted ghost number operator ϵ by $\epsilon(A)\equiv G(A)-1$. By using the low energy open-string vertices we define a family of maps $m_k : B_k\Pi\mathcal{H}\to\Pi\mathcal{H}$ $(k\geq 1)$ as follows.

Definition 4.2 (m_k at the cut-off scale ζ) *For the case of $k=1$ let $m_1(A)$ be $Q|A\rangle$. For the case of $k\geq 2$ we set* [4]

$$m_k\Big(A_1,A_2,\cdots,A_k:\zeta\Big)=(-)^{\sum_{i=1}^{[k/2]}\epsilon(A_{k+1-2i})}\Big\langle a'\ 1\ 2\cdots k:\zeta\Big|S_{a'a}\Big\rangle|A_1\rangle_1|A_2\rangle_2\cdots|A_k\rangle_k. \quad (4.6)$$

[4] $[x]$ is the maximum integer not greater than x.

Our notation in eq.(4.6) emphasizes that the maps m_k for $k \geq 2$ depend on the cut-off scale ζ through the open-string vertices. Each map has the (-1)-shifted ghost number equal to one. This can be seen as follows. Since the ghost numbers of the $(k+1)$-vertex and inverse reflector are respectively equal to $k+1$ and three, the ghost number of $m_k\big(A_1, \cdots, A_k : \zeta\big)$ becomes $2+\sum_{i=1}^{k} \epsilon(A_i)$ $\Big(= (k+1)+3+\sum_{i=1}^{k} G(A_i)-2(k+1)\Big)$. Hence $\epsilon\Big(m_k(A_1, \cdots, A_k : \zeta)\Big) = 1+\sum_{i=1}^{k} \epsilon(A_i)$. This means $\epsilon(m_k) = 1$. The following is the main result of this paper.

Theorem 4.1 (A_∞-algebra in classical open-string field theory) *$(\mathcal{H}, m_k)$ is a A_∞-algebra. Namely the maps m_k $(k \geq 1)$ given by definition 4.2 satisfy the infinite set of algebraic relations (4.2).*

This theorem tells that the open-string vertices have the structure of A_∞-algebra which is independent of the cut-off scale. In the microscopic description at $\zeta = 0$, all the maps m_k for $k \geq 3$ vanish. The corresponding A_∞-algebra is the open-string gauge algebra $(Q, \star)$. This is a non-commutative associative algebra. On the other hand, in the macroscopic description the higher maps do not vanish and the gauge algebra becomes non-associative. The scale dependence of the A_∞-algebra $(\mathcal{H}, m_k)$ becomes as follows.

Proposition 4.1 (Scale dependence of the A_∞-algebra) *Scale dependence of the maps m_k (4.6) for $k \geq 2$ are described by*

$$\begin{aligned}
&\frac{\partial m_k}{\partial \zeta}\big(A_1, \cdots, A_k : \zeta\big) \\
&\quad = -2 \sum_{l=2}^{k-1} \sum_{p=0}^{k-l} m_{k+1-l}\big(A_1, \cdots, A_p, b_0 m_l(A_{p+1}, \cdots, A_{p+l} : \zeta), A_{p+l+1}, \cdots, A_k : \zeta\big) \\
&\qquad - L_0 m_k\big(A_1, \cdots, A_k : \zeta\big) - \sum_{p=0}^{k-1} m_k\big(A_1, \cdots, A_p, L_0 A_{p+1}, A_{p+2}, \cdots, A_k : \zeta\big). \qquad (4.7)
\end{aligned}$$

In the deformation theory of A_∞-algebra there exists a concept of equivalence relation between two A_∞-algebras. This is called homotopy-equivalence [5, 6]. It is interesting to see whether the A_∞-algebra at $\zeta \neq 0$ is homotopy-equivalent to the gauge algebra $(Q, \star)$ or not. If it is affirmative, integral of eqs.(4.7) will be the A_∞-map [5, 6] which realizes the homotopy-equivalence. To answer this question is important on the physical ground since eq.(4.7) plays an important role in our description of renormalization group of open-string field theory. Resolution of the question may shed light on geometry of the space of boundary field theories in two-dimensions.

Proof of Theorem 4.1 : Let us abbreviate the vector $\Big\langle a'\ 1\ 2\cdots k{:}\zeta\Big|S_{a'a}\Big\rangle|A_1\rangle_1|A_2\rangle_2\cdots|A_k\rangle_k$ by $\widetilde{m}_k\Big(A_1,A_2,\cdots,A_k{:}\zeta\Big)$. We first rewrite $Q\ \widetilde{m}_k$ $(k\geq 2)$ as follows.

$$\begin{aligned}
&Q\ \widetilde{m}_k\Big(A_1,A_2,\cdots,A_k{:}\zeta\Big)\\
&=(-)^{k+1}\Big\langle a'\ 1\ 2\cdots k{:}\zeta\Big|\ \Big(Q^{(a)}\Big|S_{a'a}\Big\rangle\Big)\ |A_1\rangle_1|A_2\rangle_2\cdots|A_k\rangle_k\\
&=(-)^{k}\Big\langle a'\ 1\ 2\cdots k{:}\zeta\Big|S_{a'a}\Big\rangle\ \Big\{\Big(\sum_{j=1}^{k}Q^{(j)}\Big)|A_1\rangle_1|A_2\rangle_2\cdots|A_k\rangle_k\Big\} \qquad (4.8)\\
&\quad+(-)^{k}\Big\{\Big\langle a'\ 1\ 2\cdots k{:}\zeta\Big|\Big(Q^{(a')}+\sum_{j=1}^{k}Q^{(j)}\Big)\Big\}\ \Big|S_{a'a}\Big\rangle|A_1\rangle_1|A_2\rangle_2\cdots|A_k\rangle_k, \qquad (4.9)
\end{aligned}$$

where the BRST invariance of the inverse reflector besides its odd-grassmannity are used to show the second equality. We treat two terms (4.8) and (4.9) separately. The first term is written down easily to a form expressed in terms of $\widetilde{m}_k$.

$$\begin{aligned}
&Eq.(4.8)\\
&=(-)^{k}\sum_{p=0}^{k-1}(-)^{\sum_{i=1}^{p}G(A_i)}\Big\langle a'\ 1\cdots k{:}\zeta\Big|S_{a'a}\Big\rangle|A_1\rangle_1\cdots|A_p\rangle_p\ \Big(Q^{(p+1)}|A_{p+1}\rangle_{p+1}\Big)\ |A_{p+2}\rangle_{p+2}\cdots|A_k\rangle_k\\
&=(-)^{k}\sum_{p=0}^{k-1}(-)^{p+\sum_{i=1}^{p}\epsilon(A_i)}\widetilde{m}_k\Big(A_1,\cdots,A_p,QA_{p+1},A_{p+2},\cdots,A_k\Big). \qquad (4.10)
\end{aligned}$$

On the other hand we need some care to evaluate the second term. We compute the second term using the conjecture. By eq.(3.37) the second term becomes

$$\begin{aligned}
&Eq.(4.9)\\
&=(-)^{k}\Big\{\Big\langle 0\ 1\ 2\cdots k{:}\zeta\Big|\Big(\sum_{j=0}^{k}Q^{(j)}\Big)\Big\}\ \Big|S_{0a}\Big\rangle|A_1\rangle_1|A_2\rangle_2\cdots|A_k\rangle_k\\
&=(-)^{k+1}\Big\{\sum_{l=2}^{k-1}\sum_{p=0}^{k-l}(-)^{kp}\Big\langle\underbrace{p{+}l{+}1\ \cdots\ k\ 0\ 1\ \cdots\ p\ b}_{k+2-l}{:}\zeta\Big|\Big\langle\underbrace{b'\ p{+}1\ \cdots\ p{+}l}_{l+1}{:}\zeta\Big|S_{b'b}\Big\rangle\Big\}\\
&\quad\times\Big|S_{0a}\Big\rangle|A_1\rangle_1\cdots|A_k\rangle_k. \qquad (4.11)
\end{aligned}$$

We then permute open-string indices of the first vertices according to eq.(3.38) as follows.

$$\begin{aligned}
&Eq.(4.11)\\
&=(-)^{k+1}\Big\{\sum_{l=2}^{k-1}\sum_{p=0}^{k-l}(-)^{kp+(k-l+1)p}\Big\langle\underbrace{0\ 1\ \cdots\ p\ b\ p{+}l{+}1\ \cdots\ k}_{k+2-l}{:}\zeta\Big|\Big\langle\underbrace{b'\ p{+}1\ \cdots\ p{+}l}_{l+1}{:}\zeta\Big|S_{b'b}\Big\rangle\Big\}
\end{aligned}$$

$$\times \Big|S_{0a}\Big\rangle |A_1\rangle_1 \cdots |A_k\rangle_k. \tag{4.12}$$

Finally we arrange eq.(4.12), taking account of the grassmannities, so that it is expressed in terms of $\widetilde{m}_k$.

$$\begin{aligned}
&Eq.(4.12)\\
&= (-)^{k+1} \sum_{l=2}^{k-1} \sum_{p=0}^{k-l} (-)^{p+l\left(1+\sum_{i=1}^{p}\epsilon(A_i)\right)} \Big\langle \underbrace{0\ 1\ \cdots\ p\ b\ p{+}l{+}1\ \cdots\ k}_{k+2-l} : \zeta \Big| S_{0a} \Big\rangle |A_1\rangle_1 \cdots |A_p\rangle_p\\
&\quad \times \Big\langle \underbrace{b'\ p{+}1\ \cdots\ p{+}l}_{l+1} : \zeta \Big| S_{b'b} \Big\rangle |A_{p+1}\rangle_{p+1} \cdots |A_{p+l}\rangle_{p+l}\\
&\quad\quad \times |A_{p+l+1}\rangle_{p+l+1} \cdots |A_k\rangle_k\\
&= (-)^{k+1} \sum_{l=2}^{k-1} \sum_{p=0}^{k-l} (-)^{p+l\left(1+\sum_{i=1}^{p}\epsilon(A_i)\right)} \widetilde{m}_{k+1-l}\Big(A_1, \cdots, A_p, \widetilde{m}_l(A_{p+1}, \cdots, A_{p+l}), A_{p+l+1}, \cdots, A_k\Big).
\end{aligned} \tag{4.13}$$

The sum of eqs.(4.8) and (4.9) is $Q\widetilde{m}_k\Big(A_1, \cdots, A_k\Big)$. Using the expressions (4.10) and (4.13) we obtain the following relation of $\widetilde{m}_k$.

$$\begin{aligned}
&Q\ \widetilde{m}_k\Big(A_1, A_2, \cdots, A_k\Big) + \sum_{p=0}^{k-1} (-)^{k+p+1+\sum_{i=1}^{p}\epsilon(A_i)} \widetilde{m}_k\Big(A_1, \cdots, QA_{p+1}, \cdots, A_k\Big)\\
&+ \sum_{l=2}^{k-1} \sum_{p=0}^{k-l} (-)^{k+p+l\left(1+\sum_{i=1}^{p}\epsilon(A_i)\right)} \widetilde{m}_{k+1-l}\Big(A_1, \cdots, A_p, \widetilde{m}_l(A_{p+1}, \cdots, A_{p+l}), A_{p+l+1}, \cdots, A_k\Big)\\
&= 0.
\end{aligned} \tag{4.14}$$

This relation can be rewritten in terms of m_k if one recalls the correspondence,

$$\widetilde{m}_k\Big(A_1, A_2, \cdots, A_k : \zeta\Big) = (-)^{\sum_{i=1}^{[k/2]}\epsilon(A_{k+1-2i})} m_k\Big(A_1, A_2, \cdots, A_k : \zeta\Big).$$

After straightforward calculations we finally find out that eq.(4.14) is rephrased to

$$\sum_{l=1}^{k} \sum_{p=0}^{k-l} (-)^{\sum_{i=1}^{p}\epsilon(A_i)} m_{k+1-l}\Big(A_1, \cdots, A_p, m_l(A_{p+1}, \cdots, A_{p+l} : \zeta), A_{p+l+1}, \cdots, A_k : \zeta\Big) = 0.$$

This is nothing but the relation (4.2). We omit the last part of the proof since the calculations, though they are straightforward, are needed to be done case-by-case and require some spaces.¶

Proof of Proposition 4.1 : (We use the same notation as in the proof of Theorem 4.1.) We first compute $\partial\widetilde{m}_k/\partial\zeta$ by using Proposition 3.6.

$$
\begin{aligned}
&\frac{\partial\widetilde{m}_k}{\partial\zeta}\big(A_1, A_2, \cdots, A_k\colon \Lambda\big) \\
&= \left(\frac{d}{d\zeta}\Big\langle 0\ 1\ 2\ \cdots\ k\colon\zeta\Big|\right)\Big|S_{0a}\Big\rangle|A_1\rangle_1|A_2\rangle_2\cdots|A_k\rangle_k \\
&= -2\left(\sum_{l=2}^{k-1}\sum_{p=0}^{k-l}(-)^{kp}\Big\langle\underbrace{p{+}l{+}1\ \cdots\ k\ 0\ 1\ \cdots\ p\ b}_{k+2-l}\colon\zeta\Big|\Big\langle\underbrace{b'\ p{+}1\ \cdots\ p{+}l}_{l+1}\colon\zeta\Big|\Big(b_0\Big|S_{b'b}\Big\rangle\Big)\right) \qquad (4.15) \\
&\qquad\times\Big|S_{0a}\Big\rangle|A_1\rangle_1\cdots|A_k\rangle_k \\
&\quad-\Big\langle 0\ 1\cdots k\colon\zeta\Big|\left(\Big(\sum_{i=0}^{k}L_0^{(i)}\Big)\Big|S_{0a}\Big\rangle|A_1\rangle_1\cdots|A_k\rangle_k\right). \qquad (4.16)
\end{aligned}
$$

We arrange the first term (4.15) so that it is expressed in terms of $\widetilde{m}_k$. For this purpose we permute open-string indices of the first vertices by using the asymmetry (3.38) as follows.

$$
\begin{aligned}
&Eq.(4.15) \\
&= -2\left(\sum_{l=2}^{k-1}\sum_{p=0}^{k-l}(-)^{kp+(k-l+1)p}\Big\langle\underbrace{0\ 1\ \cdots\ p\ b\ p{+}l{+}1\ \cdots\ k}_{k+2-l}\colon\zeta\Big|\Big\langle\underbrace{b'\ p{+}1\ \cdots\ p{+}l}_{l+1}\colon\zeta\Big|\Big(b_0\Big|S_{b'b}\Big\rangle\Big)\right) \\
&\qquad\times\Big|S_{0a}\Big\rangle|A_1\rangle_1\cdots|A_k\rangle_k. \qquad (4.17)
\end{aligned}
$$

Then we rewrite eq.(4.17), taking account of the grassmannities of $|A_i\rangle$, into the following form.

$$
\begin{aligned}
&Eq.(4.17) \\
&= -2\sum_{l=2}^{k-1}\sum_{p=0}^{k-l}(-)^{(l+1)\sum_{i=1}^{p}\epsilon(A_i)}\Big\langle\underbrace{0\ 1\ \cdots\ p\ b\ p{+}l{+}1\ \cdots\ k}_{k+2-l}\colon\zeta\Big|S_{0a}\Big\rangle|A_1\rangle_1\cdots|A_p\rangle_p \\
&\qquad\qquad\times\Big(b_0^{(b)}\Big\langle\underbrace{b'\ p{+}1\ \cdots\ p{+}l}_{l+1}\colon\zeta\Big|S_{b'b}\Big\rangle|A_{p+1}\rangle_{p+1}\cdots|A_{p+l}\rangle_{p+l}\Big) \\
&\qquad\qquad\qquad\times|A_{p+l+1}\rangle_{p+l+1}\cdots|A_k\rangle_k \\
&= -2\sum_{l=2}^{k-1}\sum_{p=0}^{k-l}(-)^{(l+1)\sum_{i=1}^{p}\epsilon(A_i)}\widetilde{m}_{k+1-l}\Big(A_1,\cdots,A_p,b_0\widetilde{m}_l(A_{p+1},\cdots,A_{p+l}),A_{p+l+1},\cdots,A_k\Big). \\
&\qquad (4.18)
\end{aligned}
$$

While this, we can easily write down the second term (4.16) by means of $\widetilde{m}_k$ as follows.

$$
Eq.(4.16) \;=\; -L_0\widetilde{m}_k\Big(A_1,A_2,\cdots,A_k\Big)-\sum_{i=1}^{k}\widetilde{m}_k\Big(A_1,\cdots,L_0A_i,\cdots,A_k\Big). \qquad (4.19)
$$

Eqs.(4.18) and (4.19) give the following expression of $\partial\widetilde{m}_k/\partial\zeta$ in terms of $\widetilde{m}_k$.

$$\begin{aligned}
&\frac{\partial\widetilde{m}_k}{\partial\zeta}\Big(A_1,A_2,\cdots,A_k\Big)\\
&\quad=-2\sum_{l=2}^{k-1}\sum_{p=0}^{k-l}(-)^{(l+1)\sum_{i=1}^{p}\epsilon(A_i)}\widetilde{m}_{k+1-l}\Big(A_1,\cdots,A_p,b_0\widetilde{m}_l(A_{p+1},\cdots,A_{p+l}),A_{p+l+1},\cdots,A_k\Big)\\
&\qquad-L_0\widetilde{m}_k\Big(A_1,A_2,\cdots,A_k\Big)-\sum_{i=1}^{k}\widetilde{m}_k\Big(A_1,\cdots,L_0A_i,\cdots,A_k\Big)\quad. \qquad (4.20)
\end{aligned}$$

This describes the scale dependence of $\widetilde{m}_k$. By rewriting this equation in terms of m_k, we find out, after some calculations, that it becomes eq.(4.7).¶

The maps m_k are introduced by the open-string vertices. Conversely the open-string vertices can be written in terms of m_k as follows.

Proposition 4.2 *Let $k\geq 3$. We have the following identities.*

$$\begin{aligned}
\Big\langle 1\cdots k:\zeta\Big|A_1\rangle_1\cdots|A_k\rangle_k &= (-)^{\sum_{i=1}^{[k/2]}\epsilon(A_{k+1-2i})}\Big\langle\omega_{ab}\Big|A_1\rangle_a\Big|m_{k-1}\Big(A_2,\cdots,A_k:\zeta\Big)\Big\rangle_b \qquad (4.21)\\
&= (-)^{\sum_{i=1}^{[k/2]}\epsilon(A_{k-2i})}\Big\langle\omega_{ab}\Big|m_{k-1}\Big(A_1,\cdots,A_{k-1}:\zeta\Big)\Big\rangle_a|A_k\rangle_b. \qquad (4.22)
\end{aligned}$$

Remark 4.1 *This proposition becomes useful in the subsequent discussions.*

Proof of Proposition 4.2 : We give a proof only for the case when k is odd. The other case can be shown in the same manner. We rewrite $\Big\langle\omega_{ab}\Big|A_1\rangle_a\Big|m_{2p}\Big(A_2,\cdots,A_{2p+1}\Big)\Big\rangle_b$ in terms of the open-string vertices as follows.

$$\begin{aligned}
&\Big\langle\omega_{ab}\Big|A_1\rangle_a\Big|m_{2p}\Big(A_2,\cdots,A_{2p+1}:\zeta\Big)\Big\rangle_b\\
&=\Big\langle\omega_{ab}\Big|A_1\rangle_1\Big((-)^{\sum_{i=1}^{p}\epsilon(A_{2i})}\Big\langle 0\ 2\ \cdots\ 2p{+}1:\zeta\Big|S_{0b}\Big\rangle|A_2\rangle_2\cdots|A_{2p+1}\rangle_{2p+1}\Big)\\
&=(-)^{1+\sum_{i=1}^{p}\epsilon(A_{2i})}\Big\langle 0\ 2\ \cdots\ 2p{+}1:\zeta\Big|\Big\langle\omega_{ab}\Big|S_{0b}\Big\rangle|A_1\rangle_a|A_2\rangle_2\cdots|A_{2p+1}\rangle_{2p+1}, \qquad (4.23)
\end{aligned}$$

where we use odd grassmannities of the $(2p+1)$-vertex and the (inverse) reflector to show the second equality. Since we know $\Big\langle\omega_{12}\Big|S_{23}\Big\rangle={}_3P_1$, eq.(4.23) can be evaluated into

$$\begin{aligned}
Eq.(4.23) &= (-)^{1+\sum_{i=1}^{p}\epsilon(A_{2i})}\Big\langle 0\ 2\ \cdots\ 2p{+}1:\zeta\Big|\Big(-{}_0P_a\Big)|A_1\rangle_a|A_2\rangle_2\cdots|A_{2p+1}\rangle_{2p+1}\\
&= (-)^{\sum_{i=1}^{p}\epsilon(A_{2i})}\Big\langle 1\ 2\ \cdots\ 2p{+}1:\zeta\Big|A_1\rangle_1|A_2\rangle_2\cdots|A_{2p+1}\rangle_{2p+1}. \qquad (4.24)
\end{aligned}$$

Hence we obtain eq.(4.21) with $k=2p+1$. Similarly, by rewriting $\left\langle \omega_{ab} \middle| m_{2p}\left(A_1, \cdots, A_{2p}\right)\right\rangle_a |A_{2p+1}\rangle_{2p+1}$ in terms of the open-string vertices, we obtain eq.(4.22) with $k=2p+1$.¶

5 Low Energy Description Of Open-String Field Theory

We use the Batalin-Vilkovisky (BV) formalism [7], [24] for a low energy description of open-string field theory. This formalism was elegantly used in quantizing open-string field theory [8, 9]. Although the BV formalism was originally intended to quantize gauge invariant field theories which we already knew, it was used in closed-string field theory [2] to find the unknown theory.

5.1 Odd symplectic structure of open-string field theory

In order to develop the BV formalism of open-string field theory we need to introduce an odd symplectic structure ω on the open-string Hilbert space $\mathcal{H}$. We define ω as follows.

Definition 5.1 (Odd symplectic structure of $\mathcal{H}$) *Let ω be an element of $(\mathcal{H}^{\otimes 2})^*$ which is given by*

$$\omega(A,B) = (-)^{\epsilon(A)} \left\langle \omega_{12} \middle| A\right\rangle_1 |B\rangle_2, \tag{5.1}$$

where $\epsilon(A)$ is the (-1)-shifted ghost number of A.

The above bilinear form is non-degenerate on $\mathcal{H}$ because of the non-degeneracy of the BPZ pairing. The symmetry of the reflector implies $\omega(A,B) = -(-)^{\epsilon(A)\epsilon(B)}\omega(B,A)$. The selection rule of the BPZ pairing gives $\omega(A,B)\neq 0 \Rightarrow \epsilon(A)+\epsilon(B)=1$.

Let $\left\{\phi_a\right\}_a$ be bases of $\mathcal{H}$. We take the conjugate bases $\left\{\phi^a\right\}_a$ such that they satisfy $\omega(\phi_a, \phi^b)=\delta_a^b$. The (-1)-shifted ghost number of ϕ^a is $\epsilon(\phi^a)=1-\epsilon(\phi_a)$. We put $\omega_{ab} \equiv \omega(\phi_a, \phi_b)$ and $\omega^{ab} \equiv \omega(\phi^a, \phi^b)$. These matrix elements enjoy

$$\sum_c \omega^{ac}\omega_{cb} = \sum_c \omega_{bc}\omega^{ca} = -\delta_b^a. \tag{5.2}$$

This may need a proof : It can be seen that these two bases are related with each other by $\phi_a = \sum_b \phi^b \omega_{ba}$ or equivalently $\phi^a = -\sum_b \phi_b \omega^{ba}$. By using these relations we obtain $\phi_a = \sum_b \phi_b \left(\sum_c (-)\omega^{bc}\omega_{ca}\right)$ and $\phi^a = \sum_b \phi^b \left(\sum_c (-)\omega_{bc}\omega^{ca}\right)$. These imply eq.(5.2).

Open-string field Φ is a vector of $\mathcal{H}$ with $\epsilon(\Phi)=0$. Hence Φ is restricted on a subspace of $\mathcal{H}$. In the BV formalism this restriction is removed [8, 9] by introduction of fields and anti-fields. Let us expand open-string field by the bases ϕ_a,

$$\Phi = \sum_a t^a \phi_a. \tag{5.3}$$

Each coefficient t^a is required to have the ghost number $G(t^a)=-\epsilon(\phi_a)$ and the grassmannity $(-)^{G(t^a)}$ so that Φ is still grassmann-odd with $\epsilon(\Phi)=0$. We can regard these coefficients t^a as coordinates of a super-manifold (infinite-dimensional super-vector space). This manifold is endowed with the odd-symplectic structure ω if one identifies the open-string Hilbert space $\mathcal{H}$ with its tangent space. Let $F(\Phi)$ be a functional of open-string field Φ. We introduce hamiltonian vector $\left|\partial F/\partial\Phi\right\rangle \in \mathcal{H}$ as follows.

Definition 5.2 (Hamiltonian vector) *Hamiltonian vector* $\left|\partial F/\partial\Phi\right\rangle$ *of a functional* $F(\Phi)$ *is a vector of* $\mathcal{H}$*. It is defined by the following first-order variation of* F *with respect to* Φ*.*

$$\delta F(\Phi) = \omega\left(\delta\Phi, \left|\frac{\partial F}{\partial\Phi}\right\rangle\right), \tag{5.4}$$

where $\delta\Phi$ *is arbitrary vector with* $\epsilon(\delta\Phi)=0$*.*

The ghost number of hamiltonian vector becomes $G\left(\left|\partial F/\partial\Phi\right\rangle\right)=G(F)+2$. Expansion of the vector in terms of ϕ^a turns out to have the forms,

$$\left|\frac{\partial F}{\partial\Phi}\right\rangle = \sum_a |\phi^a\rangle \partial_a^L F \qquad \left(= \sum_a F\partial_a^R |\phi^a\rangle\right). \tag{5.5}$$

We regard F as a function of t^a in the RHS of the above equations. We also introduce left- and right-differentials $\partial_a^L \equiv \partial^L/\partial t^a$ and $\partial_a^R \equiv \partial^R/\partial t^a$. $\partial_a^L F$ and $F\partial_a^R$ are defined by the first-order variation of F with respect to t^a.

$$\delta F = \delta t^a \left(\partial_a^L F\right) \qquad \left(= \left(F\partial_a^R\right)\delta t^a\right). \tag{5.6}$$

Let us derive eqs.(5.5). We put $\delta\Phi=\sum_a \delta t^a\phi_a$ and write down eq.(5.4) as follows.

$$\delta F = \sum_a \delta t^a \omega\left(\phi_a, \left|\frac{\partial F}{\partial\Phi}\right\rangle\right) \qquad \left(= -\sum_a \omega\left(\left|\frac{\partial F}{\partial\Phi}\right\rangle, \phi_a\right)\delta t^a\right). \tag{5.7}$$

These expressions mean

$$\partial_a^L F = \omega\left(\phi_a, \left|\frac{\partial F}{\partial\Phi}\right\rangle\right), \qquad F\partial_a^R = -\omega\left(\left|\frac{\partial F}{\partial\Phi}\right\rangle \phi_a\right). \tag{5.8}$$

While this, any vector $A \in \mathcal{H}$ can be expanded in the following forms.

$$|A\rangle = \sum_a |\phi^a\rangle \omega(\phi_a, A) \qquad \left(= -\sum_a \omega(A, \phi_a)|\phi^a\rangle \right). \tag{5.9}$$

Let A be $\left|\partial F/\partial\Phi\right\rangle$ and compare eqs.(5.9) with eqs.(5.8). Then we obtain eqs.(5.5).

Definition 5.3 (Anti-bracket { , }) *Let F and K be functionals of open-string field Φ. Their anti-bracket $\{F, K\}$ is defined by*

$$\Big\{F, K\Big\} = \omega\left(\left|\frac{\partial F}{\partial \Phi}\right\rangle, \left|\frac{\partial K}{\partial \Phi}\right\rangle\right). \tag{5.10}$$

Remark 5.1 *By eqs.(5.5) the anti-bracket (5.10) has the following expression.*

$$\Big\{F, K\Big\} = \sum_{a,b} F \partial_a^R \omega^{ab} \partial_b^L K. \tag{5.11}$$

Proposition 5.1 *The anti-bracket { , } defined by eq.(5.10) enjoys the following properties.*

$$G(\{F, K\}) = G(F) + G(K) + 1, \tag{5.12}$$

$$\{F, K\} = -(-)^{\epsilon(F)\epsilon(K)}\{K, F\}, \tag{5.13}$$

$$\Big\{F, \{K, M\}\Big\} = \{F, K\}M + (-)^{\epsilon(F)(\epsilon(K)+1)}K\{F, M\}, \tag{5.14}$$

$$0 = (-)^{\epsilon(F)\epsilon(M)}\Big\{\{F, K\}, M\Big\} + \textit{cyclic permutations}. \tag{5.15}$$

Eq.(5.14) means that the anti-bracket is a derivation with respect to itself. Eq.(5.15) is the Jacobi identity. Thus the anti-bracket defines a super Lie algebra. This proposition can be shown by standard calculations and we omit the proof.

5.2 Low energy action $S[\Phi : \zeta]$

Low energy action $S^{\zeta}_{eff}[\Phi]$ of open-string field Φ can be obtained by integrating out all the contributions from length scale less than ζ. It includes higher open-string interactions. These interactions are obtained from graphs which are one-particle irreducible with respect to the regularized propagator. We denote the classical part of $S^{\zeta}_{eff}[\Phi]$ by $S[\Phi : \zeta]$. This gives a low energy description of classical open string field theory. All the open-string vertices investigated previously become the elementary interactions at the cut-off scale ζ and contribute to $S[\Phi : \zeta]$. Thus we arrive at the following definition of $S[\Phi : \zeta]$.

Definition 5.4 (Low energy action of classical open-string field theory) *Action of classical open-string field at the cut-off scale ζ is given by*

$$S\big[\Phi:\zeta\big] = \frac{1}{2}\big\langle\omega_{12}\big|\Phi\big\rangle_1\left(Q^{(2)}|\Phi\rangle_2\right) + \sum_{k\geq 3}\frac{1}{k}\big\langle 1\ 2\ \cdots\ k:\zeta\big|\Phi\big\rangle_1|\Phi\rangle_2\cdots|\Phi\rangle_k. \tag{5.16}$$

This action reduces to the microscopic action (2.50) as $\zeta\to 0$. In this subsection we examine the low energy description from the perspective of the BV formalism. For this purpose it is convenient to provide another form of the action by which the underlying BV structure becomes manifest. We rewrite the interactions in (5.16) by using Proposition 4.2.

$$\begin{aligned}\sum_{k\geq 3}\frac{1}{k}\big\langle 1\ 2\cdots k:\zeta\big|\Phi\big\rangle_1|\Phi\rangle_2\cdots|\Phi\rangle_k &= \sum_{k\geq 2}\frac{1}{k+1}\big\langle\omega_{12}\big|\Phi\big\rangle_1\big|m_k\big(\Phi^k:\zeta\big)\big\rangle_2 \\ &= \sum_{k\geq 2}\frac{1}{k+1}\omega\left(\Phi,m_k\big(\Phi^k:\zeta\big)\right),\end{aligned} \tag{5.17}$$

where $m_k\big(\Phi^k:\zeta\big)$ is the abbreviation for $m_k\big(\underbrace{\Phi,\cdots,\Phi}_{k}:\zeta\big)$. We will use similar abbreviation frequently. For instance, $m_{k_1+k_2+1}\big(\Phi^{k_1},A,\Phi^{k_2}:\zeta\big)$ stands for $m_{k_1+k_2+1}\big(\underbrace{\Phi,\cdots,\Phi}_{k_1},A,\underbrace{\Phi,\cdots,\Phi}_{k_2}:\zeta\big)$. The quadratic part of the action is just $\frac{1}{2}\omega\left(\Phi,m_1\big(\Phi\big)\right)$. Thus we obtain another expression for $S[\Phi:\zeta]$,

$$S\big[\Phi:\zeta\big] = \omega\left(\Phi,\sum_{k\geq 1}\frac{1}{k+1}m_k\big(\Phi^k:\zeta\big)\right). \tag{5.18}$$

Proposition 5.2 (Equation of motion for Φ) *First-order variation of the action (5.16) with respect to Φ becomes as follows.*

$$\delta S\big[\Phi:\zeta\big] = \omega\left(\delta\Phi,\ \sum_{k\geq 1}m_k\big(\Phi^k:\zeta\big)\right). \tag{5.19}$$

Therefore the equation of motion $\delta S=0$ is given by

$$\sum_{k\geq 1}m_k\big(\Phi^k:\zeta\big) = 0. \tag{5.20}$$

We give a few comments on some relation with deformation theory [5, 6] of A_∞-algebra : Let (C,m_k) be a A_∞-algebra. Let $b\in(\Pi C)^0$. Define a family of maps m_k^b $(k=0,1,\cdots)$ by

$$m_k^b\big(x_1,\cdots,x_k\big) = \sum_{q_1,\cdots,q_{k+1}\geq 0} m_{k+q_1+\cdots+q_{k+1}}\big(\underbrace{b,\cdots,b}_{q_1},x_1,\underbrace{b,\cdots,b}_{q_2},\cdots,\underbrace{b,\cdots,b}_{q_k},x_k,\underbrace{b,\cdots,b}_{q_{k+1}}\big). \tag{5.21}$$

It was shown in [5, 6] that (C, m_k^b) for $k = 0, 1, \cdots$ becomes a *weak* A_∞-algebra. If $m_0^b(1)$ vanishes, it becomes a A_∞-algebra. So $m_0^b(1)$ is an obstruction in the deformation theory. Let us consider the particular case of open-string field theory. For any $\Phi \in (\Pi\mathcal{H})^0$ we obtain a *weak* A_∞-algebra $(\mathcal{H}, m_k^\Phi)$. The obstruction is $m_0^\Phi\big(1:\zeta\big)$. Now we become aware that the equation of motion (5.20) states vanishing the obstruction. For any classical solution $\Phi_\sharp$, even apart from the trivial one, we still have a A_∞-algebra $(\mathcal{H}, m_k^{\Phi_\sharp})$. In the BV formalism variational formula (5.19) determines hamiltonian vector of the action S. We also recognize that the hamiltonian vector is nothing but the obstruction of deformation.

$$\left|\frac{\partial S[\Phi:\zeta]}{\partial\Phi}\right\rangle = m_0^\Phi\big(1:\zeta\big). \tag{5.22}$$

Remark 5.2 *The variational formula (5.19) is mod $O((\delta\Phi)^2)$. By calculations similar to the proof of eq.(5.19) (which we give below), the exact result can be found out to be*

$$S\big[\Phi+\delta\Phi:\zeta\big] = S\big[\Phi:\zeta\big] + \omega\left(\delta\Phi, \sum_{k\geq 0}\frac{1}{k+1}m_k^\Phi\big(\delta\Phi:\zeta\big)\right). \tag{5.23}$$

Proof of Proposition 5.2 : The variation can be evaluated as follows.

$$\begin{aligned}
&\delta S\big[\Phi:\zeta\big] \\
&= \frac{1}{2}\big\langle\omega_{12}\big|\left\{|\delta\Phi\rangle_1\left(Q^{(2)}|\Phi\rangle_2\right)+|\Phi\rangle_1\left(Q^{(2)}|\delta\Phi\rangle_2\right)\right\}+\sum_{k\geq 3}\frac{1}{k}\big\langle 1\cdots k:\zeta\big|\left(\sum_{i=1}^{k}|\Phi\rangle_1\cdots|\delta\Phi\rangle_i\cdots|\Phi\rangle_k\right) \\
&= \big\langle\omega_{12}\big|\delta\Phi\rangle_1\left(Q^{(2)}|\Phi\rangle_2\right)+\sum_{k\geq 3}\frac{1}{k}\left\{\sum_{i=1}^{k}\big\langle i\ i{+}1\cdots i{+}k{-}1:\zeta\big|\delta\Phi\rangle_i|\Phi\rangle_{i+1}\cdots|\Phi\rangle_{i+k-1}\right\} \\
&= \big\langle\omega_{12}\big|\delta\Phi\rangle_1\left(Q^{(2)}|\Phi\rangle_2\right)+\sum_{k\geq 3}\big\langle 1\ 2\cdots k:\zeta\big|\delta\Phi\rangle_1|\Phi\rangle_2\cdots|\Phi\rangle_k \\
&= \sum_{k\geq 1}\big\langle\omega_{12}\big|\delta\Phi\rangle_1\big|m_k\big(\Phi^k:\zeta\big)\big\rangle_2 \\
&= \omega\left(\delta\Phi, \sum_{k\geq 1}m_k\big(\Phi^k:\zeta\big)\right).
\end{aligned} \tag{5.24}$$

We use the asymmetry (3.38) of open-string vertices and the symmetry of the reflector to show the second equality in the above computation. We also use Proposition 4.2 to show the fourth equality.¶

In the previous section, in order to emphasize the perspective of renormalization group, open-string vertices are constructed by using the propagator in the Siegel gauge. These vertices are used in the definition of $S[\Phi:\zeta]$. Nevertheless it turns out that the action is a covariant classical action of open-string field.

Definition 5.5 (Gauge transformation of Φ) *For any vector $\rho\in\mathcal{H}$ with $\epsilon(\rho)=-1$ we define an infinitesimal transformation of Φ by* [5]

$$\delta_\rho\Phi = m_1^\Phi\big(\rho:\zeta\big). \tag{5.25}$$

When $\zeta=0$, this transformation reduces to $\delta_\rho\Phi = Q\rho+\Phi\star\rho-\rho\star\Phi$. This is precisely the gauge symmetry of the microscopic action (2.50). In the case of $\zeta\neq 0$, it is unclear whether the transformation (5.25) is still gauge symmetry of $S[\Phi:\zeta]$. We have the following proposition.

Proposition 5.3 (Gauge invariance of the action) *The action (5.16) is invariant under the infinitesimal transformation (5.25) of Φ.*

Proof of Proposition 5.3 : By using the variational formula (5.19) we rewrite $\delta_\rho S$ to the form,

$$\begin{aligned}\delta_\rho S\big[\Phi:\zeta\big] &= \omega\left(\delta_\rho\Phi,\ m_0^\Phi\big(1:\zeta\big)\right)\\ &= \omega\left(m_1^\Phi\big(\rho:\zeta\big),\ m_0^\Phi\big(1:\zeta\big)\right).\end{aligned} \tag{5.26}$$

We further rewrite eq.(5.26) by using Proposition 4.2 as follows.

$$\begin{aligned}&Eq.(5.26)\\ &=\sum_{q_1,q_2}\omega\left(m_{q_1+q_2+1}\big(\Phi^{q_1},\rho,\Phi^{q_2}\big),\ m_0^\Phi(1)\right)\\ &=\sum_{q_1,q_2}\Big\langle\omega_{ab}\Big|\ m_{q_1+q_2+1}\big(\Phi^{q_1},\rho,\Phi^{q_2}\big)\Big\rangle_a\Big|\ m_0^\Phi(1)\Big\rangle_b\\ &=\sum_{q_1,q_2}(-)^{q_2}\Big\langle 1\cdots q_1+q_2+2:\zeta\Big|\underbrace{|\Phi\rangle_1\cdots|\Phi\rangle_{q_1}}_{q_1}|\rho\rangle_{q_1+1}\underbrace{|\Phi\rangle_{q_1+2}\cdots|\Phi\rangle_{q_1+q_2+1}}_{q_2}\Big|m_0^\Phi(1)\Big\rangle_{q_1+q_2+2}.\end{aligned} \tag{5.27}$$

[5] We follow the convention of the deformation theory of A_∞-algebra.

We replace the open-string fields by taking account of the asymmetry (3.38) of the open-string vertices and then rewrite eq.(5.27), again by using Proposition 4.2 as follows.

$$
\begin{aligned}
&Eq.(5.27)\\
&= -\sum_{q_1,q_2} \Big\langle 1\cdots q_1{+}q_2{+}2 : \zeta \Big| \underbrace{|\Phi\rangle_1 \cdots |\Phi\rangle_{q_2}}_{q_2} \Big| m_0^\Phi(1) \Big\rangle_{q_2+1} \underbrace{|\Phi\rangle_{q_2+2} \cdots |\Phi\rangle_{q_1+q_2+1}}_{q_1} |\rho\rangle_{q_1+q_2+2}\\
&= -\sum_{q_1,q_2} \omega\Big(m_{q_1+q_2+1}\big(\Phi^{q_2}, m_0^\Phi(1), \Phi^{q_1}\big),\ \rho\Big)\\
&= -\omega\Big(m_1^\Phi\big(m_0^\Phi(1)\big),\ \rho\Big)\,. \qquad (5.28)
\end{aligned}
$$

Therefore $\delta_\rho S$ turns out to be

$$\delta_\rho S\Big[\Phi:\zeta\Big] = \omega\Big(\rho, m_1^\Phi\big(m_0^\Phi(1:\zeta):\zeta\big)\Big)\,. \qquad (5.29)$$

Recall $(\mathcal{H}, m_k^\Phi)$ is a *weak* A_∞-algebra. Especially it follows from Definition 4.1 that $m_1^\Phi\big(m_0^\Phi(1)\big) = 0$. Thus $\delta_\rho S = 0$.¶

In the BV formalism [7], [24] quantum master equation is the criterion for consistency of a quantum gauge theory. Its classical limit is called classical master equation. It is simply given by $\Big\{S, S\Big\} = 0$, where S is the classical BV action of the gauge theory. Classical master equation itself is not sufficient to ensure consistency of the quantum theory, but it must be satisfied at the classical level in order to obtain a consistent quantum gauge theory.

Proposition 5.4 (Classical master equation) *The action (5.16) satisfies classical master equation,*

$$\Big\{S\Big[\Phi:\zeta\Big],\ S\Big[\Phi:\zeta\Big]\Big\} = 0. \qquad (5.30)$$

The classical master equation (5.30) eventually reduces to the A_∞-algebra $(\mathcal{H}, m_k)$ as we will find in the proof of this proposition. Physical significance of classical master equation is easy to see if we take another form of this equation. The anti-bracket in eq.(5.30) is equal to $\omega\big(m_0^\Phi\big(1:\zeta\big), m_0^\Phi\big(1:\zeta\big)\big)$, where we identify the hamiltonian vector with the obstruction of the deformation theory.

Definition 5.6 (BRST transformation of Φ) *BRST transformation $\delta_{\mathbf{BRS}}$ of open-string field Φ is defined by the hamiltonian vector (5.22).*

$$\delta_{\mathbf{BRS}}\Phi = m_0^\Phi\big(1:\zeta\big). \qquad (5.31)$$

$\delta_{\mathbf{BRS}}\Phi$ has the ghost number two. Hence we can attach the ghost number one to $\delta_{\mathbf{BRS}}$. We can also see $\delta_{\mathbf{BRS}}$ is grassmann-odd. Physical significance of classical master equation may be observed in the following propositions.

Proposition 5.5 (Nilpotency of BRST transformation)

$$\delta_{\mathbf{BRS}} \cdot \delta_{\mathbf{BRS}} = 0. \tag{5.32}$$

Proposition 5.6 (BRST invariance of the action) *The variational formula (5.19) can be written as*

$$\delta S\Big[\Phi:\zeta\Big] = \omega\Big(\delta\Phi, \delta_{\mathbf{BRS}}\Phi\Big). \tag{5.33}$$

In particular, if one takes $\delta_{\mathbf{BRS}}\Phi$ as $\delta\Phi$, we obtain

$$\delta_{\mathbf{BRS}} S\Big[\Phi:\zeta\Big] = 0. \tag{5.34}$$

Eq.(5.33) is a simple reinterpretation of eq.(5.19). The BRST invariance of the action follows from Proposition 5.4 by using the formula (5.33).

Proof of Proposition 5.4 : We apply the strategy taken in [2] for a proof of the quantum master equation of closed-string field theory. We first write down the anti-bracket $\omega\left(m_0^{\Phi}(1), m_0^{\Phi}(1)\right)$ as follows.

$$\begin{aligned}
&\omega\left(m_0^{\Phi}(1), m_0^{\Phi}(1)\right) \\
&= \sum_{k\geq 2}\left\{\sum_{l=1}^{k-1}\omega\left(m_{k-l}\Big(\Phi^{k-l}\Big), m_l\Big(\Phi^l\Big)\right)\right\} \\
&= \sum_{k\geq 2}\left\{\sum_{l=1}^{k-1}\frac{k-l}{k}\omega\left(m_{k-l}\Big(\Phi^{k-l}\Big), m_l\Big(\Phi^l\Big)\right)\right\} + \sum_{k\geq 2}\left\{\sum_{l=1}^{k-1}\frac{l}{k}\omega\left(m_{k-l}\Big(\Phi^{k-l}\Big), m_l\Big(\Phi^l\Big)\right)\right\} \\
&= \sum_{k\geq 2}\frac{2}{k}\left\{\sum_{l=1}^{k-1}(k-l)\times\omega\left(m_{k-l}\Big(\Phi^{k-l}\Big), m_l\Big(\Phi^l\Big)\right)\right\}.
\end{aligned} \tag{5.35}$$

To proceed on the computation we need a suitable identity : We rewrite $\omega\left(m_{k-l}\Big(\Phi^{k-l}\Big), m_l\Big(\Phi^l\Big)\right)$, which appears in eq.(5.35), into the following form by using Proposition 4.2.

$$\begin{aligned}
\omega\left(m_{k-l}\Big(\Phi^{k-l}\Big),\ m_l\Big(\Phi^l\Big)\right) &= -\Big\langle\omega_{ab}\Big|\ m_{k-l}\Big(\Phi^{k-l}\Big)\Big\rangle_a\Big|\ m_l\Big(\Phi^l\Big)\Big\rangle_b \\
&= -\Big\langle 1\ \cdots\ k-l+1:\zeta\Big|\underbrace{\Phi\rangle_1\cdots|\Phi\rangle_{k-l}}_{k-l}\Big|\ m_l(\Phi^l)\Big\rangle_{k-l+1}.
\end{aligned} \tag{5.36}$$

We permute $(k-l)$ open-string fields in this equation by taking account of the asymmetry of the vertex and then rewrite the equation in terms of m_k by using Proposition 4.2 as follows.

$$
\begin{aligned}
&Eq.(5.36)\\
&= (-)^i \Big\langle i\cdots k-l+1 \;\; 1\cdots i-1 : \zeta \Big| \underbrace{|\Phi\rangle_i \cdots |\Phi\rangle_{k-l}}_{k-l-i+1} \Big| \; m_l(\Phi^l) \Big\rangle_{k-l+1} \underbrace{|\Phi\rangle_1 \cdots |\Phi\rangle_{i-1}}_{i-1}\\
&= \Big\langle \omega_{ab} \Big| \; m_{k-l}\Big(\Phi^{k-l-i+1}, m_l(\Phi^l), \Phi^{i-2}\Big) \Big\rangle_a |\Phi\rangle_b\\
&= \omega\Big(m_{k-l}\Big(\Phi^{k-l-i+1}, m_l(\Phi^l), \Phi^{i-2}\Big), \;\Phi\Big)\,. \qquad (5.37)
\end{aligned}
$$

Hence we obtain the following identity independent of i.

$$
\omega\Big(m_{k-l}\Big(\Phi^{k-l-i}, m_l(\Phi^l), \Phi^{i-1}\Big), \Phi\Big) = \omega\Big(m_{k-l}\Big(\Phi^{k-l}\Big), m_l\Big(\Phi^l\Big)\Big)\,. \qquad (5.38)
$$

Now we proceed on the computation of eq.(5.35). By using the above identity it becomes as follows.

$$
\begin{aligned}
Eq.(5.35) &= \sum_{k\geq 2} \frac{2}{k} \sum_{l=1}^{k-1} \left\{ \sum_{i=1}^{k-l} \omega\Big(m_{k-l}\Big(\Phi^{k-l-i}, m_l(\Phi^l), \Phi^{i-1}\Big), \;\Phi\Big) \right\}\\
&= \sum_{k\geq 1} \frac{2}{k+1} \omega\left(\sum_{l=1}^{k} \sum_{i=0}^{k-l} m_{k+1-l}\Big(\Phi^i, m_l(\Phi^l), \Phi^{k-l-i}\Big), \;\Phi \right). \qquad (5.39)
\end{aligned}
$$

This vanishes identically due to the relation (4.2) of the A_∞-algebra $(\mathcal{H}, m_k)$. Thus we obtain $\omega\Big(m_0^\Phi(1), m_0^\Phi(1)\Big) = 0$.¶

Proof of Proposition 5.5 : We show $\delta_{\mathbf{BRS}} \cdot \delta_{\mathbf{BRS}} \Phi = 0$. We first rewrite $\delta_{\mathbf{BRS}} \cdot \delta_{\mathbf{BRS}} \Phi$ as follows.

$$
\begin{aligned}
\delta_{\mathbf{BRS}} \cdot \delta_{\mathbf{BRS}} \Phi &= \delta_{\mathbf{BRS}}\Big[m_0^\Phi(1)\Big]\\
&= \sum_{k\geq 1} \delta_{\mathbf{BRS}}\Big[m_k(\Phi^k)\Big]\\
&= \delta_{\mathbf{BRS}}\Big[m_1(\Phi)\Big] + \sum_{k\geq 2} \delta_{\mathbf{BRS}}\Big[\Big\langle 0 \; 1\cdots \; k : \zeta \Big| S_{0a}\Big\rangle |\Phi\rangle_1 \cdots |\Phi\rangle_k\Big]\\
&= -m_1\Big(\delta_{\mathbf{BRS}}\Phi\Big) + \sum_{k\geq 2} (-)^k \Big\langle 0 \; 1\cdots \; k : \zeta \Big| S_{0a}\Big\rangle \delta_{\mathbf{BRS}}\Big[|\Phi\rangle_1 \cdots |\Phi\rangle_k\Big]. \qquad (5.40)
\end{aligned}
$$

The first term of eq.(5.40) is equal to $-\sum_{k\geq 1} m_1\Big(m_k(\Phi^k)\Big)$. As for the second term we further compute by using Proposition 4.2 as follows.

$$
\sum_{k\geq 2} (-)^k \Big\langle 0 \; 1\cdots \; k : \zeta \Big| S_{0a}\Big\rangle \delta_{\mathbf{BRS}}\Big[|\Phi\rangle_1 \cdots |\Phi\rangle_k\Big]
$$

$$
\begin{aligned}
&= \sum_{k\geq 2}\left\{\sum_{i=0}^{k-1}(-)^{k+i}\Big\langle 0\ 1\cdots\ k:\zeta\Big|S_{0a}\Big\rangle|\Phi\rangle_1\cdots|\Phi\rangle_i\Big|\delta_{\mathbf{BRS}}\Phi\Big\rangle_{i+1}|\Phi\rangle_{i+2}\cdots|\Phi\rangle_k\right\}\\
&= \sum_{l\geq 1}\sum_{k\geq 2}\left\{\sum_{i=0}^{k-1}(-)^{k+i}\Big\langle 0\ 1\cdots\ k:\zeta\Big|S_{0a}\Big\rangle|\Phi\rangle_1\cdots|\Phi\rangle_i\Big|m_l(\Phi^l)\Big\rangle_{i+1}|\Phi\rangle_{i+2}\cdots|\Phi\rangle_k\right\}\\
&= -\sum_{l\geq 1}\sum_{k\geq 2}\sum_{i=1}^{k} m_k\Big(\Phi^i, m_l(\Phi^l), \Phi^{k-1-i}\Big)\\
&= -\sum_{k\geq 1}\left\{\sum_{l=1}^{k-1}\sum_{i=0}^{k-l} m_{k+1-l}\Big(\Phi^i, m_l(\Phi^l), \Phi^{k-l-i}\Big)\right\}. \qquad (5.41)
\end{aligned}
$$

By putting these two terms together, we eventually obtain the following expression of $\delta_{\mathbf{BRS}}\cdot\delta_{\mathbf{BRS}}\Phi$.

$$
\begin{aligned}
\delta_{\mathbf{BRS}}\cdot\delta_{\mathbf{BRS}}\Phi &= -\sum_{k\geq 1} m_1\Big(m_k(\Phi^k)\Big) - \sum_{k\geq 1}\left\{\sum_{l=1}^{k-1}\sum_{i=0}^{k-l} m_{k+1-l}\Big(\Phi^i, m_l(\Phi^l), \Phi^{k-l-i}\Big)\right\}\\
&= -\sum_{k\geq 1}\left\{\sum_{l=1}^{k}\sum_{i=0}^{k-l} m_{k+1-l}\Big(\Phi^i, m_l(\Phi^l), \Phi^{k-l-i}\Big)\right\}. \qquad (5.42)
\end{aligned}
$$

This actually vanishes by the A_∞-relation.¶

6 Renormalization Group Of Open-String Field Theory

So far, open-string field Φ is simply a vector of the open-string Hilbert space $\mathcal{H}$. In the presence of the short-distance cut-off scale parameter ζ, classical dynamics of open-string field is governed by the action $S[\Phi:\zeta]$. As concerns the classical dynamics, any consideration on *off-shell* open-string field is irrelevant. In particular, we can not find any relation between *off-shell* open-string fields at different scales. On the other hand, if one takes the renormalization group perspective, namely if one regards $S[\Phi:\zeta]$ as the classical part of $S^\zeta_{eff}[\Phi]$ which could be obtained integrated out all the contributions from length scale less than ζ, *off-shell* open-string fields at different scales should be related with one another by renormalization group flow. In this section we study renormalization group of classical open-string field theory. Our formulation of renormalization group equation (RG equation) is based on Proposition 4.1.

Now open-string field Φ is allowed to depend on the cut-off scale parameter ζ. Equivalently the super-coordinates t^a which appear in the expansion by the bases ϕ_a, depend on ζ.

$$
\Phi(\zeta) = \sum_a t^a(\zeta)\phi_a. \qquad (6.1)
$$

RG equation determines the scale dependence of open-string field Φ or equivalently a flow $t^a(\zeta)$ on the super-manifold.

Definition 6.1 (RG equation of classical open-string field) *Let $S[\Phi:\zeta]$ be the action (5.16). RG equation of open-string field Φ is defined by*

$$\frac{d}{d\zeta}S\Big[\Phi(\zeta):\zeta\Big]=0. \tag{6.2}$$

The total derivation in the RG equation (6.2) can be factorized into a sum of two terms. One is a derivation through the coordinates $t^a(\zeta)$. Using the variational formula (5.19) it is given by $\omega\Big(d\Phi/d\zeta,\Big|\partial S/\partial\Phi\Big\rangle\Big)$. The other is a derivation through the open-string vertices in the action. It is given by $\partial S/\partial\zeta$. The factorized form of the RG equation becomes as follows.

$$\omega\left(\frac{d\Phi}{d\zeta},\ \left|\frac{\partial S[\Phi:\zeta]}{\partial\Phi}\right\rangle\right)=-\frac{\partial S}{\partial\zeta}\Big[\Phi:\zeta\Big]. \tag{6.3}$$

The RHS of this equation can be calculated using Proposition 4.1. Let $S_{int}[\Phi:\zeta]$ be the interaction part of the action (5.16).

$$S_{int}\Big[\Phi:\zeta\Big]\equiv\omega\left(\Phi,\sum_{k\geq 2}\frac{1}{k+1}m_k\Big(\Phi^k:\zeta\Big)\right). \tag{6.4}$$

The corresponding hamiltonian vector becomes

$$\left|\frac{\partial S_{int}[\Phi:\zeta]}{\partial\Phi}\right\rangle=\sum_{k\geq 2}m_k\Big(\Phi^k:\zeta\Big). \tag{6.5}$$

With this notation the scale dependence of the action is described as follows.

Proposition 6.1 *Derivation of the action (5.16) with respect to the cut-off scale parameter ζ has the following form.*

$$\frac{\partial S}{\partial\zeta}\Big[\Phi:\zeta\Big]=-\omega\left(b_0\left|\frac{\partial S_{int}[\Phi:\zeta]}{\partial\Phi}\right\rangle+L_0\Phi,\ \left|\frac{\partial S_{int}[\Phi:\zeta]}{\partial\Phi}\right\rangle\right). \tag{6.6}$$

Due to this proposition the RG equation (6.3) becomes

$$\omega\left(\frac{d\Phi}{d\zeta},\ \left|\frac{\partial S[\Phi:\zeta]}{\partial\Phi}\right\rangle\right)=\omega\left(b_0\left|\frac{\partial S_{int}[\Phi:\zeta]}{\partial\Phi}\right\rangle+L_0\Phi,\ \left|\frac{\partial S_{int}[\Phi:\zeta]}{\partial\Phi}\right\rangle\right). \tag{6.7}$$

Eq.(6.7) is a classical analogue of the Polchinski's RG equation [11, 10]. RG equation *ala* Wilson or Polchinski is $dZ\big(t(\zeta):\zeta\big)/d\zeta=0$, where Z is a partition function of a quantum field theory regularized by the short distance cut-off scale ζ. Roughly speaking, correspondence with the RG equation (6.2) can be seen if one puts $Z=e^{-S}$. The precise relation may be established by using the Legendre transformation as considered in [25]. There appears a quantum correction in the original equation. It has the form $\sim \partial^2 S_{int}/\partial\phi\partial\phi$. In our case quantum correction appears as the higher loop contribution of open-string. Description of RG equation of string-field along the line presented in (6.7) was first given for closed-string field theory [12]. Similarity with the BV master equation was emphasized there. Resemblance between the two was further investigated in [26].

As we stated earlier, the RG equation (6.2) is introduced to determine the ζ-evolution of open-string field Φ. Let us solve the RG equation (6.7) in such a form by which it becomes manifest. For this purpose we first rewrite the RHS of eq.(6.7) as follows.

$$\begin{aligned}
&\omega\left(b_0\left|\frac{\partial S_{int}}{\partial\Phi}\right\rangle + L_0\Phi,\ \left|\frac{\partial S_{int}}{\partial\Phi}\right\rangle\right)\\
&\qquad=\omega\left(b_0\left|\frac{\partial S}{\partial\Phi}\right\rangle + Qb_0\Phi,\ \left|\frac{\partial S_{int}}{\partial\Phi}\right\rangle\right)\\
&\qquad=\omega\left(b_0\left|\frac{\partial S}{\partial\Phi}\right\rangle,\ \left|\frac{\partial S_{int}}{\partial\Phi}\right\rangle\right)+\omega\left(Qb_0\Phi,\ \left|\frac{\partial S}{\partial\Phi}\right\rangle\right)\\
&\qquad=\omega\left(b_0\left|\frac{\partial S_{int}}{\partial\Phi}\right\rangle,\ \left|\frac{\partial S}{\partial\Phi}\right\rangle\right)+\omega\left(Qb_0\Phi,\ \left|\frac{\partial S}{\partial\Phi}\right\rangle\right)\\
&\qquad=\omega\left(b_0\left|\frac{\partial S_{int}}{\partial\Phi}\right\rangle + Qb_0\Phi,\ \left|\frac{\partial S}{\partial\Phi}\right\rangle\right),
\end{aligned}\tag{6.8}$$

where the anti-commutation relation, $\{Q,b_0\}=L_0$, is used to show the first equality. The second equality follows from $Q^2=0$ and the BRST invariance of the reflector. To show the third equality we use the asymmetry of ω besides eq.(2.18). The RG equation (6.7) acquires the following expression by eq.(6.8).

$$\omega\left(\frac{d\Phi}{d\zeta}-\left\{b_0\left|\frac{\partial S_{int}[\Phi:\zeta]}{\partial\Phi}\right\rangle + Qb_0\Phi\right\},\ \left|\frac{\partial S[\Phi:\zeta]}{\partial\Phi}\right\rangle\right)=0.\tag{6.9}$$

Thus we obtain

Proposition 6.2 *Open-string field $\Phi(\zeta)$ obeying the following first-order differential equation*

is a solution of the RG equation (6.2).

$$\frac{d\Phi}{d\zeta} = b_0 \left| \frac{\partial S_{int}[\Phi:\zeta]}{\partial \Phi} \right\rangle + Q b_0 \Phi. \tag{6.10}$$

We do not know whether eq.(6.10) is equivalent to the RG equation (6.2) or not since equivalence between eqs.(6.9) and (6.10) is obscure. But this does not cause any problem. Skeptical reader can adopt eq.(6.10) as a definition of RG equation of open-string field and keep eq.(6.2) as a proposition. Eq.(6.10) will be also called RG equation of open-string field.

Proof of Proposition 6.1 : We first write down the partial derivation $\partial S[\Phi:\zeta]/\partial\zeta$ by using Proposition 4.1.

$$\begin{aligned}
\frac{\partial S}{\partial \zeta}\big[\Phi:\zeta\big] &= \sum_{k\geq 2} \frac{1}{k+1} \omega\left(\Phi,\ \frac{\partial m_k}{\partial \zeta}\big(\Phi^k:\zeta\big)\right) \\
&= -\sum_{k\geq 2} \frac{1}{k+1} \omega\left(\Phi,\ L_0 m_k\big(\Phi^k\big) + \sum_{p=0}^{k-1} m_k\big(\Phi^p, L_0\Phi, \Phi^{k-1-p}\big)\right) \qquad (6.11) \\
&\quad -2\sum_{k\geq 2} \frac{1}{k+1} \omega\left(\Phi,\ \sum_{l=2}^{k-1}\sum_{p=0}^{k-l} m_{k+1-l}\big(\Phi^p, b_0 m_l(\Phi^l), \Phi^{k-l-p}\big)\right). \qquad (6.12)
\end{aligned}$$

We treat two terms (6.11) and (6.12) separately. We rewrite the first term by using Proposition 4.2 as follows.

$$\begin{aligned}
&\omega\left(\Phi,\ L_0 m_k\big(\Phi^k\big) + \sum_{p=0}^{k-1} m_k\big(\Phi^p, L_0\Phi, \Phi^{k-1-p}\big)\right) \\
&= \Big\langle \omega_{ab}\Big|\Phi\Big\rangle_a \Big(L_0^{(b)}\Big| m_k\big(\Phi^k\big)\Big\rangle_b\Big) + \sum_{p=0}^{k-1}\Big\langle \omega_{ab}\Big|\Phi\Big\rangle_a \Big| m_k\big(\Phi^p, L_0\Phi, \Phi^{k-1-p}\big)\Big\rangle_b \\
&= \Big\langle \omega_{ab}\Big|\Phi\Big\rangle_a \Big(L_0^{(b)}\Big| m_k\big(\Phi\big)\Big\rangle_b\Big) \\
&\quad + \sum_{p=0}^{k-1}\Big\langle 1\ \cdots\ k{+}1:\zeta\Big| \underbrace{|\Phi\rangle_1 \cdots |\Phi\rangle_{p+1}}_{p+1} \Big(L_0^{(p+2)}|\Phi\rangle_{p+2}\Big) \underbrace{|\Phi\rangle_{p+3} \cdots |\Phi\rangle_{k+1}}_{k-p-1}. \qquad (6.13)
\end{aligned}$$

Due to the property (2.18) of the reflector we have $\Big\langle \omega_{12}\Big|\Big(L_0^{(1)}|A\rangle_1\Big)|B\rangle_2. = \Big\langle \omega_{12}\Big|A\rangle_1\Big(L_0^{(2)}|B\rangle_2\Big)$. We permute open-string fields in the second term of eq.(6.13) by taking account of the asymmetry (3.38) of the vertices, and then rewrite eq.(6.13) again in terms of m_k by using Proposition 4.2 as follows.

$$Eq.(6.13) \ = \ \Big\langle \omega_{ab}\Big|\Big(L_0^{(a)}|\Phi\rangle_a\Big)\Big| m_k\big(\Phi^k\big)\Big\rangle_b$$

$$
\begin{aligned}
&+\sum_{p=0}^{k-1}\Big\langle p{+}1\ p{+}2\ \cdots\ p{:}\zeta\Big|\Big(L_0^{(p+1)}|\Phi\rangle_{p+1}\Big)\underbrace{|\Phi\rangle_{p+2}\cdots|\Phi\rangle_{p}}_{k}\\
=\ &\Big\langle\omega_{ab}\Big|\Big(L_0^{(a)}|\Phi\rangle_a\Big)\Big|m_k\Big(\Phi^k\Big)\Big\rangle_b\\
&+k\times\Big\langle 1\ 2\ \cdots\ k{+}1{:}\zeta\Big|\Big(L_0^{(1)}|\Phi\rangle_1\Big)\underbrace{|\Phi\rangle_2\cdots|\Phi\rangle_{k+1}}_{k}\\
=\ &(k{+}1)\times\omega\Big(L_0\Phi,m_k\Big(\Phi^k\Big)\Big)\,. \qquad (6.14)
\end{aligned}
$$

As the result we obtain the following expression of eq.(6.11).

$$
\begin{aligned}
Eq.(6.11)\ &=\ -\sum_{k\geq 2}\omega\Big(L_0\Phi,\ m_k\Big(\Phi^k\Big)\Big)\\
&=\ -\omega\left(L_0\Phi,\ \left|\frac{\partial S_{int}}{\partial\Phi}\right\rangle\right). \qquad (6.15)
\end{aligned}
$$

Nextly we examine the second term. By using Proposition 4.2 we rewrite eq.(6.12) as follows.

$$
\begin{aligned}
&\omega\left(\Phi,\ \sum_{l=2}^{k-1}\sum_{p=0}^{k-l}m_{k+1-l}\Big(\Phi^p,b_0m_l(\Phi^l),\Phi^{k-l-p}\Big)\right)\\
&=\sum_{l=2}^{k-1}\sum_{p=0}^{k-l}\Big\langle\omega_{ab}\Big|\Phi\rangle_a\Big|\ m_{k+1-l}\Big(\Phi^p,b_0m_l(\Phi^l),\Phi^{k-l-p}\Big)\Big\rangle_b\\
&=\sum_{l=2}^{k-1}\sum_{p=0}^{k-l}\Big\langle 1\cdots k{+}2{-}l{:}\zeta\Big|\underbrace{\Phi\rangle_1\cdots|\Phi\rangle_{p+1}}_{p+1}\Big(b_0\Big|m_l\Big(\Phi^l\Big)\Big\rangle_{p+2}\Big)\underbrace{|\Phi\rangle_{p+3}\cdots|\Phi\rangle_{k+2-l}}_{k-l-p}\\
&=\sum_{l=2}^{k-1}(k{+}1{-}l)\times\Big\langle 1\cdots k{+}2{-}l{:}\zeta\Big|\Big(b_0\Big|m_l\Big(\Phi^l\Big)\Big\rangle_1\Big)\underbrace{|\Phi\rangle_2\cdots|\Phi\rangle_{k+2-l}}_{k+1-l}\\
&=\sum_{l=2}^{k-1}(k{+}1{-}l)\times\omega\Big(b_0m_l\Big(\Phi^l\Big),\ m_{k+1-l}\Big(\Phi^{k+1-l}\Big)\Big)\,. \qquad (6.16)
\end{aligned}
$$

In this computation, in addition to the use of Proposition 4.2, particularly to show the third equality we use the asymmetry (3.38) of the vertex. We can rewrite eq.(6.16) in a convenient form. We notice the equality,

$$
\omega\Big(b_0m_l\Big(\Phi^l\Big),\ m_{k+1-l}\Big(\Phi^{k+1-l}\Big)\Big)=\omega\Big(b_0m_{k+1-l}\Big(\Phi^{k+1-l}\Big),\ m_l\Big(\Phi^l\Big)\Big)\,. \qquad (6.17)
$$

By using this equality we write down eq.(6.16) to the following form.

$$
\sum_{l=2}^{k-1}(k{+}1{-}l)\times\omega\Big(b_0m_l\Big(\Phi^l\Big),\ m_{k+1-l}\Big(\Phi^{k+1-l}\Big)\Big)
$$

$$= \sum_{l=2}^{k-1} \frac{k+1-l}{2} \omega\left(b_0 m_l\left(\Phi^l\right),\ m_{k+1-l}\left(\Phi^{k+1-l}\right)\right) + \sum_{l=2}^{k-1} \frac{l}{2} \omega\left(b_0 m_{k+1-l}\left(\Phi^{k+1-l}\right),\ m_l\left(\Phi^l\right)\right)$$
$$= \sum_{l=2}^{k-1} \frac{(k+1-l)+l}{2} \omega\left(b_0 m_l\left(\Phi^l\right),\ m_{k+1-l}\left(\Phi^{k+1-l}\right)\right)$$
$$= \frac{k+1}{2} \sum_{l=2}^{k-1} \omega\left(b_0 m_l\left(\Phi^l\right),\ m_{k+1-l}\left(\Phi^{k+1-l}\right)\right) \tag{6.18}$$

Hence the second term is found out to be

$$\begin{aligned} Eq.(6.12) &= -\sum_{k\geq 2} \sum_{l=2}^{k-1} \omega\left(b_0 m_l\left(\Phi^l\right),\ m_{k+1-l}\left(\Phi^{k+1-l}\right)\right) \\ &= -\omega\left(b_0 \left|\frac{\partial S_{int}}{\partial \Phi}\right\rangle,\ \left|\frac{\partial S_{int}}{\partial \Phi}\right\rangle\right). \end{aligned} \tag{6.19}$$

Sum of (6.11) and (6.12) is equal to $\partial S/\partial\zeta$. Expressions (6.15) and (6.19) show that it is precisely eq.(6.6).¶

7 Analysis Of RG Flow In Siegel Gauge

If we impose the Siegel gauge condition $b_0\Phi = 0$ on open-string field, the RG equation (6.10) obtains a simple form,

$$\frac{d\Phi}{d\zeta} = b_0 \left|\frac{\partial S_{int}[\Phi:\zeta]}{\partial \Phi}\right\rangle. \tag{7.1}$$

Clearly the evolution of Φ governed by eq.(7.1) preserves the gauge condition. In this section we investigate the RG equation (7.1). Throughout this section the Siegel gauge condition is imposed on open-string field.

Let $\mathcal{H}^S$ be subspace of the open-string Hilbert space $\mathcal{H}$ consisting of states which satisfy the Siegel gauge condition. Any vector $\phi \in \mathcal{H}^S$ does not contain c_0. The odd symplectic structure ω becomes null on $\mathcal{H}^S$. Instead of ω we introduce another bilinear form g.

Definition 7.1 (Bilinear form g on $\mathcal{H}^S$) *For any vectors $A, B \in \mathcal{H}^S$, we define $g(A,B)$ by*

$$g(A,B) = \left\langle \omega_{12} \middle| c_0^{(1)} |A\rangle_1 |B\rangle_2. \right. \tag{7.2}$$

The bilinear form g is non-degenerate on $\mathcal{H}^S$. $\langle\omega_{12}| = \langle\omega_{21}|$ implies $g(A,B) = (-)^{\epsilon(A)\epsilon(B)+\epsilon(A)+\epsilon(B)}$ $g(B,A)$. Selection rule (2.16) of the BPZ pairing gives $g(A,B) \neq 0 \Rightarrow \epsilon(A) + \epsilon(B) = 0$.

Let $\{\phi_i\}_i$ be bases of $\mathcal{H}^S$. We can take the conjugate bases $\{\phi^i\}_i$ such that they satisfy $g(\phi^i, \phi_j) = \delta^i_j$. The (-1)-shifted ghost number of ϕ^i is $\epsilon(\phi^i) = -\epsilon(\phi_i)$. We put $g_{ij} \equiv g(\phi_i, \phi_j)$ and $g^{ij} \equiv (-)^{\epsilon(\phi_j)} g(\phi^i, \phi^j)$. It can be seen that two bases are related with each other by $\phi^i = \sum_j g^{ij}\phi_j$ or equivalently by $\phi_i = \sum_j g_{ij}\phi^j$. Hence g^{ij} is the inverse of g_{ij}.

$$\sum_k g^{ik} g_{kj} = \sum_k g_{jk} g^{ki} = \delta^i_j. \tag{7.3}$$

In the Siegel gauge, open-string field Φ has an expansion of the form $\Phi = \sum_i t^i \phi_i$. Each coefficient t^i has the ghost number $G(t^i) = -\epsilon(\phi_i)$ and the grassmannity $(-)^{G(t^i)}$. They are coordinates of the super-submanifold constrained by the gauge condition. We write the ζ-evolution of Φ as

$$\Phi(\zeta) = \sum_i t^i(\zeta)\phi_i. \tag{7.4}$$

RG flow $t^i(\zeta)$ on the super-manifold is determined from eq.(7.1) by using this expansion. We choose ϕ_i to be the eigenvectors of L_0.

$$L_0|\phi_i\rangle = \Delta_i|\phi_i\rangle. \tag{7.5}$$

7.1 Classical solution and RG equation

Classical solutions of open-string field theory are stationary configurations of classical open-string field action. In our description of open-string field theory, we have different classical actions depending on the cut-off scale one considers. The classical solutions or equivalently solutions of the equation of motion (5.20) are expected to depend on the cut-off scale parameter ζ. It becomes unclear whether there exists any relation between classical solutions at different scales. We start our discussion by resolving this question.

Let $\Phi_\sharp$ be a solution of the equation motion at a given scale ζ_0,

$$m_0^\Phi\big(1:\zeta_0\big)\Big|_{\Phi=\Phi_\sharp} = 0. \tag{7.6}$$

We examine the RG flow of this classical solution. Let $\Phi_\sharp(\zeta)$ be the solution of the RG equation (7.1) which satisfies the initial value condition $\Phi_\sharp(\zeta_0) = \Phi_\sharp$. We obtain the following proposition.

Proposition 7.1 *Suppose we have a classical solution $\Phi_\sharp$ at a given scale ζ_0. The RG flow of $\Phi_\sharp$ generates a family of classical solutions $\{\Phi_\sharp(\zeta)\}_\zeta$. Namely, at any value of ζ, $\Phi_\sharp(\zeta)$ satisfies the equation of motion, $m_0^{\Phi_\sharp(\zeta)}(1:\zeta)=0$.*

To explain an implication of this proposition, we rewrite the RG equation (7.1), by using the anti-commutation relation $\{Q,b_0\}=L_0$, to the form,

$$\frac{d\Phi(\zeta)}{d\zeta} = -L_0\Phi(\zeta) + b_0 m_0^{\Phi(\zeta)}(1:\zeta). \tag{7.7}$$

Then we recognize that the RG flow of a classical solution is described by the simple equation,

$$\frac{d\Phi_\sharp(\zeta)}{d\zeta} = -L_0\Phi_\sharp(\zeta), \tag{7.8}$$

because $\Phi_\sharp(\zeta)$ satisfies the equation of motion as stated in the above proposition. This evolution equation is easily integrated and provides the following relation.

$$\Phi_\sharp(\zeta) = e^{-\zeta L_0}\Phi_\sharp(0). \tag{7.9}$$

This means that any classical solution in the microscopic description ($\zeta=0$) still continues to be a classical solution in the low energy description ($\zeta>0$) modulo (7.9). The classical action of $\Phi_\sharp(\zeta)$ is independent of ζ. We have $S[\Phi_\sharp(\zeta):\zeta]=S^{cubic}[\Phi_\sharp(0)]$.

In terms of the super-coordinates t^i we can write eq.(7.9) as

$$t^i_\sharp(\zeta)e^{\zeta\Delta_i} = t^i_\sharp(0). \tag{7.10}$$

Hence any classical solution in the microscopic description gives rise to a trivial flow on the super-manifold. It may sound unpleasant since we hope to interpret it as a fixed point of the RG flow. We can save this situation by introducing rescaled coordinates instead of t^i : Let Φ be open-string field at the cut-off scale ζ. We expand it in the form, $\Phi=\sum_i T^i e^{-\zeta\Delta_i}\phi_i$. The coefficients T^i define the rescaled coordinates. The RG flow $\Phi(\zeta)$ determines a flow $T^i(\zeta)$. It is related with $t^i(\zeta)$ by

$$T^i(\zeta) = t^i(\zeta)e^{\zeta\Delta_i}. \tag{7.11}$$

Eq.(7.10) becomes $T^i_\sharp(\zeta)=T^i_\sharp(0)$. Classical solutions are fixed points of the flow of T^i.

Now let us show Proposition 7.1. The following lemma becomes useful in the proof.

Lemma 7.1 *Let Φ be a open-string field. The scale dependence of $m_0^\Phi(1:\zeta)$ is described by*

$$\frac{\partial m_0^\Phi}{\partial\zeta}\big(1:\zeta\big) = m_1^\Phi\big(L_0\Phi:\zeta\big) - 2m_1^\Phi\left(b_0 m_0^\Phi(1:\zeta):\zeta\right) + (L_0 - 2b_0Q)m_0^\Phi\big(1:\zeta\big). \tag{7.12}$$

Proof of Lemma 7.1 : We first compute $\partial m_0^\Phi/\partial\zeta$ by using Proposition 4.1.

$$\begin{aligned}
\frac{\partial m_0^\Phi}{\partial\zeta}\big(1:\zeta\big) &= \sum_{k\geq 2}\frac{\partial m_k}{\partial\zeta}\left(\Phi^k:\zeta\right) \\
&= -\sum_{k\geq 2}\left\{L_0 m_k\left(\Phi^k:\zeta\right) + \sum_{p=0}^{k-1} m_k\left(\Phi^p, L_0\Phi, \Phi^{k-1-p}:\zeta\right)\right\} \qquad (7.13)\\
&\quad -2\sum_{k\geq 2}\sum_{l=2}^{k-1}\sum_{p=0}^{k-l} m_{k+1-l}\big(\Phi^p, b_0 m_l(\Phi^l:\zeta), \Phi^{k-l-p}:\zeta\big). \qquad (7.14)
\end{aligned}$$

We treat two terms (7.13) and (7.14) separately. We write down the first term into the following form.

$$\begin{aligned}
Eq.(7.13) &= -\left\{L_0\left(\sum_{k\geq 1} m_k\big(\Phi^k\big) - Q\Phi\right) + \left(\sum_{k\geq 1}\sum_{p=0}^{k-1} m_k\big(\Phi^p, L_0\Phi, \Phi^{k-p-1}\big) - QL_0\Phi\right)\right\} \\
&= -L_0\left(m_0^\Phi(1) - Q\Phi\right) - \left(m_1^\Phi(L_0\Phi) - QL_0\Phi\right) \\
&= -L_0 m_0^\Phi(1) - m_1^\Phi(L_0\Phi) + 2L_0Q\Phi. \qquad (7.15)
\end{aligned}$$

As for the second term, we arrange it as follows.

$$\begin{aligned}
Eq.(7.14) &= -2\sum_{k\geq 2}\sum_{l=1}^{k-1}\sum_{p=0}^{k-l} m_{k+1-l}\big(\Phi^p, b_0 m_l(\Phi^l), \Phi^{k-l-p}\big) + 2\sum_{k\geq 2}\sum_{p=0}^{k-1} m_k\big(\Phi^p, b_0Q\Phi, \Phi^{k-1-p}\big) \\
&= -2\left\{m_1^\Phi\big(b_0 m_0^\Phi(1)\big) - Qb_0 m_0^\Phi(1)\right\} + 2\left\{m_1^\Phi\big(b_0Q\Phi\big) - Qb_0Q\Phi\right\} \\
&= -2m_1^\Phi\big(b_0 m_0^\Phi(1)\big) + 2m_1^\Phi\big(L_0\Phi\big) + 2Qb_0 m_0^\Phi(1) - 2L_0Q\Phi. \qquad (7.16)
\end{aligned}$$

Using expressions (7.15) and (7.16), sum of two terms (7.13) and (7.14) becomes the RHS of eq.(7.12).¶

Proof of Proposition 7.1 : Let $\Phi(\zeta)$ be a solution of the RG equation (7.1). We compute $dm_0^{\Phi(\zeta)}/d\zeta$ as follows.

$$\frac{d}{d\zeta}m_0^{\Phi(\zeta)}\big(1:\zeta\big) = \sum_{k\geq 1}\frac{d}{d\zeta}m_k\big(\Phi(\zeta)^k:\zeta\big)$$

$$
\begin{aligned}
&= \sum_{k\geq 1} m_k\Big(d\Phi^k/d\zeta : \zeta\Big) + \frac{\partial m_0^{\Phi}}{\partial\zeta}\Big(1:\zeta\Big) \\
&= m_1^{\Phi(\zeta)}\Big(d\Phi/d\zeta : \zeta\Big) + \frac{\partial m_0^{\Phi}}{\partial\zeta}\Big(1:\zeta\Big). \qquad (7.17)
\end{aligned}
$$

We further evaluate eq.(7.17) by using Lemma 7.1 as follows.

$$
\begin{aligned}
Eq.(7.17) &= m_1^{\Phi(\zeta)}\Big(d\Phi/d\zeta : \zeta\Big) + m_1^{\Phi(\zeta)}\Big(L_0\Phi : \zeta\Big) \\
&\qquad - 2m_1^{\Phi(\zeta)}\Big(b_0 m_0^{\Phi(\zeta)}(1:\zeta) : \zeta\Big) + (L_0 - 2b_0Q)m_0^{\Phi(\zeta)}\Big(1:\zeta\Big) \\
&= (L_0 - 2b_0Q)m_0^{\Phi(\zeta)}\Big(1:\zeta\Big) - m_1^{\Phi(\zeta)}\Big(b_0 m_0^{\Phi(\zeta)}(1:\zeta) : \zeta\Big), \qquad (7.18)
\end{aligned}
$$

where we use the RG equation (7.7) to show the last equality. Thus we find out

$$
\frac{d}{d\zeta} m_0^{\Phi(\zeta)}\Big(1:\zeta\Big) = (L_0 - 2b_0Q)m_0^{\Phi(\zeta)}\Big(1:\zeta\Big) - m_1^{\Phi(\zeta)}\Big(b_0 m_0^{\Phi(\zeta)}(1:\zeta) : \zeta\Big). \qquad (7.19)
$$

Now we consider the particular case of $\Phi_\sharp(\zeta)$. At ζ_0 it satisfies $m_0^{\Phi_\sharp(\zeta_0)}(1:\zeta_0) = 0$. The RHS of eq.(7.19) vanishes there. This means $dm_0^{\Phi(\zeta)}(1:\zeta)/d\zeta\,|_{\zeta=\zeta_0} = 0$. Thus, for any ζ sufficiently close to ζ_0, we have $m_0^{\Phi(\zeta)}(1:\zeta) = 0$. For such ζ, again by eq.(7.19), $m_0^{\Phi(\zeta)}(1:\zeta)$ vanishes. Repeating this argument, we finally obtain $m_0^{\Phi(\zeta)}(1:\zeta) = 0$ for arbitrary ζ.¶

7.2 Beta functions

Let us express the RG equation (7.1) as a flow equation of the super-coordinates T^i. We first introduce the free energy F as a function of T^i and ζ.

Definition 7.2 (Free energy)

$$
F(T,\zeta) \equiv S[\Phi:\zeta]\Big|_{\Phi=\sum_i T^i e^{-\zeta\Delta_i}\phi_i} \qquad (7.20)
$$

Proposition 7.2 (Beta function) *The RG equation (7.1) is equivalent to the following evolution equations of T^i.*

$$
\frac{dT^i}{d\zeta} = \beta^i(T,\zeta), \qquad (7.21)
$$

where beta functions β^i are given by

$$
\beta^i(T,\zeta) = \sum_j g_{ren}^{ij}(\zeta)\frac{\partial^L F(T,\zeta)}{\partial T^j}. \qquad (7.22)
$$

Here $g_{ren}^{ij}(\zeta)$ in eqs.(7.22) is the inverse of the renormalized bilinear form, $g_{ij}^{ren}(\zeta) \equiv e^{-\zeta(\Delta_i+\Delta_j)}g_{ij}$.

Fixed points of the evolution equations (7.21) are zeros of the beta functions. According to eqs.(7.22) these zeros are solutions of $\partial^L F/\partial T^i = 0$ since g^{ij} is non-degenerate on $\mathcal{H}^S$. The free energy F is given by eq.(7.20). Therefore zeros of the beta functions are nothing but classical solutions of open-string field theory.

Perturbative expansions (expansions at $T=0$) of the beta functions can be read from the perturbative expansion of $F(T,\zeta)$. They have the following forms (cf. proof of proposition 7.2).

$$\beta^i(T,\zeta) = \Delta_i T^i + \sum_{j,k,l} g^{ij}_{ren}(\zeta) C^{ren}_{jkl}(\zeta) T^k T^l + O(T^3). \tag{7.23}$$

Here $C^{ren}_{ijk}(\zeta)$ are the renormalized three points functions, $C^{ren}_{ijk}(\zeta) \equiv e^{-2\zeta(\Delta_i+\Delta_j+\Delta_k)} C_{ijk}$, where we put $C_{ijk} = (-)^{G(T^i)+G(T^j)+G(T^j)G(T^k)} \Big\langle 1\ 2\ 3 \Big| \phi_i \rangle_1 |\phi_j\rangle_2 |\phi_k\rangle_3$. Contrary to our naive expectation, the beta functions given by eqs.(7.22) depend on the cut-off scale parameter ζ *explicitly*. In a quantum field theory, a standard argument to show that beta functions of the theory have no explicit dependence on the cut-off scale is based on a simple dimensional analysis. In this argument it is assumed that there is at most only one dimensionful parameter whatever regularization scheme one chooses. The regularization we choose for open-string field theory does not satisfy this property. We formulate open-string field theory from the perspective of *two-dimensional* chiral CFT. But the regularization we choose is simply to put a restriction on length of open-string evolution. It is a regularization of *one-dimension*. Actually we have two length scales. The missing scale is length of open-string itself. If we restore string length scale l_s in the argument, there appear two dimensionful parameters ζl_s and l_s in our regularization.

Remark 7.1 *Dependence of the beta functions on the cut-off scale parameter can be seen as follows : We put* $\Phi(T,\zeta) \equiv \sum_i T^i e^{-\zeta\Delta_i}\phi_i$. *By using the RG equation (7.1) we obtain the following expression for the beta functions.*

$$\sum_i \beta^i \phi_i = e^{\zeta L_0} b_0 m_0^{\Phi(T,\zeta)}\big(1:\zeta\big). \tag{7.24}$$

We partial-differentiate eq.(7.24) with respect to ζ. *By using Lemma 7.1 it turns out to be*

$$\sum_i \frac{\partial \beta^i}{\partial \zeta}\phi^i = -2e^{\zeta L_0} b_0 \sum_{k_1+k_2\geq 1} m_{k_1+k_2+1}\big(\Phi^{k_1}, e^{-\zeta L_0}\sum_j \beta^j\phi_j, \Phi^{k_2}:\zeta\big). \tag{7.25}$$

This remark or proposition 7.1 shows that zeros of the beta functions (7.22) are independent of ζ. Let us suppose that 0 and $T_c = (T_c^i)_i$ ($\neq 0$) be zeros of the beta functions at $\zeta = 0$. Namely

$\beta^i(T,\zeta=0)\Big|_{T=0,T_c}=0$. Since they are the zeros even at $\zeta>0$, the beta functions must have the following forms.

$$\beta^i(T,\zeta)=\sum_{N,M}\beta^i_{N,M}(\zeta)T^N(T-T_c)^M, \tag{7.26}$$

where $N=(n_i)_i$ and $M=(m_i)_i$ are multi-indices taking values in $\mathbf{Z}_{\geq 1}$. This means that ζ-dependence of the beta functions is absorbed into the coefficients $\beta^i_{N,M}$.

The RG equation of open-string field is originally introduced by eq.(6.2) in Definition 6.1. In the Siegel gauge we can write down this equation as the Gell-Mann$-$Low equation for the free energy.

$$\frac{\partial F(T,\zeta)}{\partial\zeta}+\sum_i\beta^i\frac{\partial^L F(T,\zeta)}{\partial T^i}=0. \tag{7.27}$$

This is an equation which determines the evolution of T^i. The RG flow $T(\zeta)$ is a solution of this equation. By using eqs.(7.22) we can rewrite the Gell-Mann$-$Low equation (7.27) as follows.

$$\frac{\partial F(T,\zeta)}{\partial\zeta}=-\sum_{i,j}\beta^i g^{ren}_{ij}\beta^j. \tag{7.28}$$

By a slight look of this expression besides (7.22) of the beta functions, one may feel a similarity between the free energy F (7.20) and Zamolodochikov's c-function. But it is merely a formal resemblance [6].

In a conventional approach to open-string field theory, presumably assuming a suitable decoupling of the ghost sector, one uses mostly the matter Hilbert space $\mathcal{H}_{matter}$ rather than the open-string Hilbert space. We identify $\mathcal{H}_{matter}$ with a subspace of $\mathcal{H}^S$ by the map $i:\ \mathcal{H}_{matter}\to\mathcal{H}^S$, where $i(O)\equiv c_1O$. To present a physical application of the proposition, we follow the conventional approach although the restriction on $\mathcal{H}_{matter}$ might cause a problem on the RG equation. Practically we assume that the RG flow or the ζ-evolution of Φ is closed on $\mathcal{H}_{matter}$ [7].

The bilinear form g, when restricted on $\mathcal{H}_{matter}$, turns out to have the form.

$$g(c_1O,c_1O')=\Big\langle I[\varphi_O](\infty)\varphi_{O'}(0)\Big\rangle, \tag{7.29}$$

[6]Nevertheless, possibility of taking classical open-string field action as an analogue of c-function, which one calls g-function, was discussed in [17]. The argument there was based on the assumption that the free energy does not depend on ζ explicitly. A discrepancy between the two is also discussed recently in [27].

[7]This might be shown by a detailed analysis of the symmetric 3-vertex.

where the RHS is the standard BPZ pairing of the matter CFT. Eq.(7.29) can be shown if one computes the LHS by using the oscillator representation (2.23) of the reflector. Let $\left\{O_I\right\}_I$ be bases of $\mathcal{H}_{matter}$. O_I are chosen as the eigenvectors of L_0^{matter} : $L_0^{matter}O_I = \Delta_I O_I$. We put $g_{IJ} \equiv g(c_1 O_I, c_1 O_J)$. These define a positive-definite symmetric bilinear form on $\mathcal{H}_{matter}$. Open-string field is now supposed to be a vector of $\mathcal{H}_{matter}$. We put $\Phi = \sum_I T^I e^{-\zeta(\Delta_I - 1)} c_1 O_I$. If one takes the σ-model viewpoint of the matter theory, Φ describes a generic perturbation $\sum_I T^I \varphi_{O_I}$ of the σ-model and β^I are the beta functions associated with this perturbation. Under this circumstance the free energy F can be thought as a generating function of all correlation functions of the matter theory. Eqs.(7.22) can be written down as follows.

$$\frac{\partial F(T,\zeta)}{\partial T^I} = \sum_I g_{IJ}^{ren} \beta^I. \tag{7.30}$$

Closed-string version of eq.(7.30) can be found in [28], where it was stated as a conjecture.

Proof of Proposition 7.2 : Let us put $\Phi = \sum_i T^i e^{-\zeta\Delta_i}\phi_i$. We can express the coefficients T^i by $T^i = e^{\zeta\Delta_i} g(\phi^i, \Phi)$. Then $dT^i/d\zeta$ acquires the following form.

$$\frac{dT^i}{d\zeta} = e^{\zeta\Delta_i} g\left(\phi^i, \frac{d\Phi}{d\zeta}\right) + \Delta_i T^i. \tag{7.31}$$

We evaluate the RHS of eq.(7.31) by using the RG equation (7.1) as follows.

$$\frac{dT^i}{d\zeta} = e^{\zeta\Delta_i} g\left(\phi^i, b_0 \left|\frac{\partial S_{int}}{\partial \Phi}\right\rangle\right) + \Delta_i T^i. \tag{7.32}$$

We treat two terms of eq.(7.32) separately. The bilinear form in the first term can be expressed as a pairing by ω as follows.

$$\begin{aligned} g\left(\phi^i, b_0 \left|\frac{\partial S_{int}}{\partial \Phi}\right\rangle\right) &= \left\langle \omega_{12} \right| \left(c_0^{(1)} |\phi^i\rangle_1\right) \left(b_0^{(2)} \left|\frac{\partial S_{int}}{\partial \Phi}\right\rangle_2\right) \\ &= (-)^{\epsilon(\phi^i)} \left\langle \omega_{12} \right| \left(b_0^{(1)} c_0^{(1)} |\phi^i\rangle_1\right) \left|\frac{\partial S_{int}}{\partial \Phi}\right\rangle_2 \\ &= \omega\left(\phi^i, \left|\frac{\partial S_{int}}{\partial \Phi}\right\rangle\right), \end{aligned} \tag{7.33}$$

where the last equality follows from the gauge condition, $b_0\phi^i = 0$ besides the anti-commutation relation, $\{c_0, b_0\} = 1$. We remark that ϕ^i is expressed in terms of Φ as follow.

$$\phi^i = \sum_j e^{\zeta\Delta_j} g^{ij} \frac{\partial^L \Phi}{\partial T^j}. \tag{7.34}$$

By using this expression for ϕ^i in eq.(7.33), it follows from Definition 5.2 that the first term of eq.(7.32) becomes

$$e^{\zeta\Delta_i} g\left(\phi^i, b_0 \left| \frac{\partial S_{int}}{\partial \Phi} \right\rangle\right) = \sum_j e^{\zeta(\Delta_i+\Delta_j)} g^{ij} \frac{\partial^L}{\partial T^j} \left\{ S_{int}\Big[\Phi : \Lambda\Big] \Big|_{\Phi=\sum_k T^k e^{-\zeta\Delta_k}\phi_k} \right\}. \tag{7.35}$$

Nextly we examine the second term of eq.(7.32). In the Siegel gauge, the BRST charge Q contributes to $\omega(\Phi, Q\Phi)$ as an operator $c_0 L_0$. Thus we have the following identity.

$$\omega\Big(\Phi, Q\Phi\Big) = g\Big(\Phi, L_0\Phi\Big). \tag{7.36}$$

This means that the quadratic term of the open-string field action has the following expression in terms of T^i.

$$\omega\left(\Phi, \frac{1}{2}Q\Phi\right)\Big|_{\Phi=\sum_k T^k e^{-\zeta\Delta_k}\phi_k} = \frac{1}{2}\sum_{i,j} T^i e^{-\zeta(\Delta_i+\Delta_j)} g_{ij}\Delta_j T^j. \tag{7.37}$$

Then we write down the second term of eq.(7.32) as a differentiation of $\omega(\Phi, Q\Phi)$ by T.

$$\Delta_i T^i = \sum_j e^{\zeta(\Delta_i+\Delta_j)} g^{ij} \frac{\partial^L}{\partial T^j} \left\{ \omega\left(\Phi, \frac{1}{2}Q\Phi\right)\Big|_{\Phi=\sum_k T^i e^{-\zeta\Delta_k}\phi_k} \right\}. \tag{7.38}$$

By using the expressions (7.35) and (7.38) we proceed on the computation of eq.(7.32) as follows.

$$\begin{aligned}
\frac{dT^i}{d\zeta} &= \sum_j e^{\zeta(\Delta_i+\Delta_j)} g^{ij} \frac{\partial^L}{\partial T^j} \left[\left\{ S_{int}\Big[\Phi : \zeta\Big] + \omega\left(\Phi, \frac{1}{2}Q\Phi\right) \right\} \Big|_{\Phi=\sum_k T^i e^{-\zeta\Delta_k}\phi_k} \right] \\
&= \sum_j e^{\zeta(\Delta_i+\Delta_j)} g^{ij} \frac{\partial^L}{\partial T^j} \left[S\Big[\Phi : \zeta\Big] \Big|_{\Phi=\sum_k T^i e^{-\zeta\Delta_k}\phi_k} \right] \\
&= \sum_j e^{\zeta(\Delta_i+\Delta_j)} g^{ij} \frac{\partial^L F(T,\zeta)}{\partial T^j}.
\end{aligned} \tag{7.39}$$

Thus we obtain eq.(7.22).¶

7.3 Perspective of world-sheet boundary theory

To close this section we hint an interpretation of the previous results in terms of world-sheet boundary theories. In particular we clarify the role of the cut-off scale parameter ζ. In the next section we further develop this perspective.

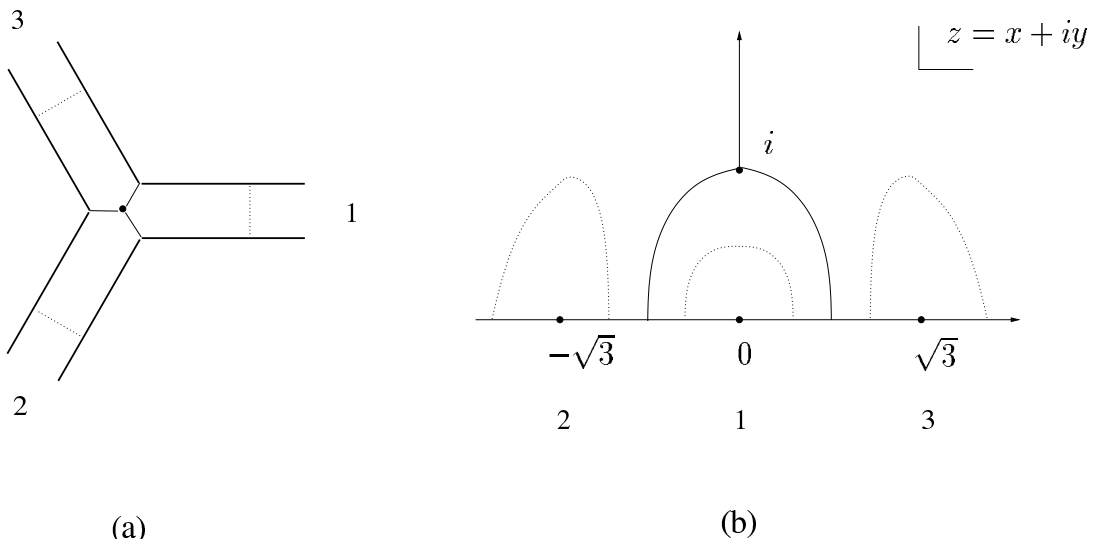

Figure 14: (a) Trivalent open-string diagram. (b) The image of trivalent diagram on the upper half-plane.

(Tree) open-string diagrams can be mapped holomorphically to the upper half-plane Im$z >$ 0. These maps are called Mandelstam maps. External open-strings in n-diagram are mapped to n different points on the real line. The inverse of Mandelstam maps are given by using quadratic differentials on $\mathbf{CP}_1$. There exists one-to-one correspondence [29] between the sets of open-string tree diagrams and quadratic differentials on $\mathbf{CP}_1$ which satisfy suitable conditions.

Let us explain the use of quadratic differentials by a simple example. Trivalent open-string diagram (Figure 14 (a)) can be mapped to the upper half-plane as depicted in Figure 14 (b). Adding the mirror image we obtain Figure 2. There $\mathbf{CP}_1$ is exactly covered by $f_i(V_i)$ $(i = 1, 2, 3)$. Recall V_i is the unit disk $|v_i| \leq 1$ and f_i is the holomorphic map (2.34). The quadratic differential associated with Figure 14 (b) (or Figure 2) is

$$\varphi = \frac{9(1+z^2)}{z^2(z^2-3)^2} dz^{\otimes 2}. \tag{7.40}$$

It has second-order poles at 0 and $\pm\sqrt{3}$ and first-order zeros at $\pm i$. Also it has the following expansion in the neighborhood of each pole.

$$\varphi = \left\{ \frac{1}{(z-z_P)^2} + \cdots \right\} dz^{\otimes 2}, \tag{7.41}$$

where P denotes the pole. Let $\sqrt{\varphi}^{(i)}$ be a differential on $f_i(V_i)$ which satisfies $(\sqrt{\varphi}^{(i)})^{\otimes 2} = \varphi$. Particularly we can choose $\sqrt{\varphi}^{(i)}$ such that

$$f_i^* \sqrt{\varphi}^{(i)} = \frac{dv_i}{v_i}. \tag{7.42}$$

We then introduce an analytic function ρ_i on $f_i(V_i)$ by an integral of $\sqrt{\varphi}^{(i)}$ in the following manner.

$$\rho_1(z) - \rho_1(0) = \int_0^z \sqrt{\varphi}^{(1)}. \tag{7.43}$$

Due to the property (7.42) we can regard e^{ρ_i} as a holomorphic map from $f_i(V_i)$ to V_i. It is the inverse of f_i.

$$e^{\rho_i} \circ f_i = 1. \tag{7.44}$$

The i-th strip in Figure 14 (a) is parametrized by $\rho_i = \tau_i + i\sigma_i$.

Consider the s-channel diagram which appears in $\partial \mathcal{V}_4(\zeta)$. See Figure 15 (a). It can be mapped to the upper half-plane. The image is depicted in Figure 15 (b). The four open-strings are mapped to $\pm a$ and $\pm 1/a$ on the real line. The mid-points of the trivalent vertices are mapped to ib and i/b. $(0 < a, b < 1)$. a and b are suitable functions of ζ. These functions are determined from a consideration on a quadratic differential associated with Figure 15 (b).

The quadratic differential needs to have second-order poles at $\pm a$ and $\pm 1/a$ and first-order zeros at $\pm ib$ and $\pm i/b$. In the neighborhood of each pole it must have the expansion (7.41). These conditions restrict the quadratic differential to the following form.

$$\varphi = \frac{4a^2(a^4-1)^2}{(a^2+b^2)(1+a^2b^2)} \frac{(z^2+b^2)(1+b^2z^2)}{(z^2-a^2)^2(1-a^2z^2)^2} dz^{\otimes 2}. \tag{7.45}$$

Let $\sqrt{\varphi}$ be a differential defined locally (in each image of the open-string strip and its mirror image). We further need to impose the condition which fixes width of the internal strip to π.

$$\int_{|z|=1} \sqrt{\varphi} = 2\pi i. \tag{7.46}$$

This allows us to solve b in terms of a. Thus obtained quadratic differentials describe open-string 4-diagrams. Finally we need to relate a with ζ so that φ describes Figure 15 (a). It can be achieved by imposing the condition,

$$\int_{ib}^{i} \sqrt{\varphi} = \zeta, \tag{7.47}$$

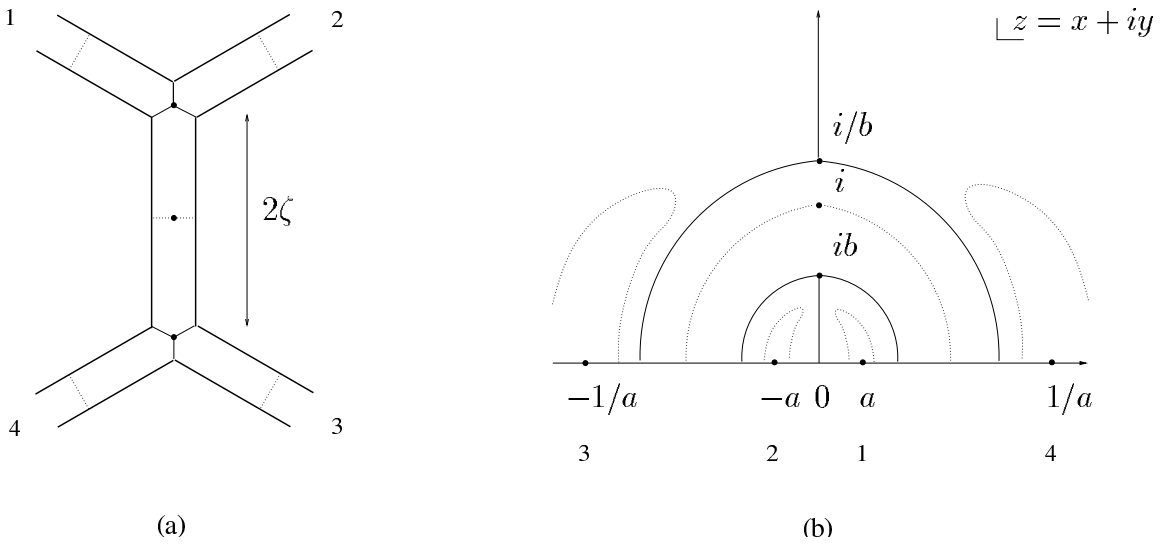

Figure 15: (a) The s-channel diagram which appears in $\partial\mathcal{V}_4(\zeta)$. (b) The s-channel diagram is mapped holomorphically to the upper half-plane. The mid-points of the trivalent vertices are mapped to ib and i/b. The external open-strings $1, 2, 3$ and 4 are mapped respectively to $\pm a$ and $\pm 1/a$ on the real line.

where the integral is taken along the imaginary axis.

The asymptotic forms of $a(\zeta)$ and $b(\zeta)$ can be written down explicitly. They are expected to approach to zero as ζ goes to ∞. When a is nearly zero we can solve the condition (7.46) in the following form.

$$b(a) = \sqrt{3}a + O(a^2). \tag{7.48}$$

Therefore we approximate the quadratic differential (7.45) by $\frac{z^2+3a^2}{(z^2-a^2)^2}dz^{\otimes 2}$. The condition (7.47) will be replaced by the following one.

$$\int_{i\sqrt{3}a}^{i} \frac{\sqrt{z^2+3a^2}}{z^2-a^2}dz = \zeta. \tag{7.49}$$

This integral is easily evaluated and we obtain

$$a(\zeta) = \frac{2}{3\sqrt{3}}e^{-\zeta} + O(e^{-3\zeta}). \tag{7.50}$$

The approximation used above becomes consistent when a is sufficiently small. It is when ζ is sufficiently large.

Each strip of an open-string diagram has a kähler metric $d\rho d\overline{\rho}$. ($\rho = \tau + i\sigma$ is a holomorphic coordinate of strip.) They induce a kähler metric on the upper half-plane. We call it bulk metric d^2s_{bulk}. If one extends the bulk metric to the real line, it becomes singular at the points where the open-strings are inserted. From the perspective of world-sheet boundary theories we should rescale the bulk metric near the real line such that the rescaled metric provides a smooth metric on the real line. World-sheet boundary theory will be constructed on the real line by using such a smooth metric $d^2s_{boundary}$. A convenient choice of a smooth metric on the boundary is

$$d^2s_{boundary} = dxdx. \tag{7.51}$$

If we use the metric (7.51) on the real line, the regularization employed so far in open-string field theory corresponds to a point-splitting regularization of short-distance behavior of a boundary theory. The estimation (7.50) shows that the point-splitting is prescribed by the boundary length scale δl with the order

$$\delta l \sim e^{-\zeta}. \tag{7.52}$$

The low energy action $S[\Phi : \zeta]$ can be understood as an analogue of a generating function of all correlation functions of a world-sheet boundary theory regularized by the point-splitting (7.52). ζ needs to be sufficiently large. In the next section we identify it with the so-called boundary open-string field theory.

8 Boundary Open-String Field Theory

In this section we discuss a relation between two different formulations of classical open-string field theory. One is the formulation given in the previous sections of this paper. It is based on the Wilson renormalization group of world-sheet theory in which the microscopic action is identified with the cubic action (2.50) and the cut-off scale parameter is introduced by length of open-string strip (which appears in the Schwinger representation of the open-string propagator). The other is the so-called boundary open-string field theory. It is proposed in [18] intended for the background independent description of open-string field theory. To discuss their relation we

need to introduce boundary states. These are states of the closed-string Hilbert space. String vertices which describe interactions of open-strings with a single closed-string are required. We start this section by a consideration of the moduli problem relevant to constructions of these vertices. Our discussion in this section is not rigorous although it is physically motivated. We believe that perspectives presented in this section become useful in our understanding of duality of open- and closed- strings or holography in string theory.

Let $\widehat{\mathcal{M}}^1_n$ be the set of clockwise-ordered n different points $(z_1, \cdots, z_n)$ on the boundary of *one-punctured* two-disk. Let z_0 be the puncture.

$$\widehat{\mathcal{M}}^1_n = \left\{ (z_0; z_1, \cdots, z_n) \,\middle|\, \begin{array}{c} z_i \in \partial D \;\; (1 \leq i \leq n), \;\; z_0 \in D, \\ z_i \neq z_j \;\; \text{if} \;\; i \neq j. \end{array} \right\}. \tag{8.1}$$

We also let $\widehat{\mathcal{M}}^0_n$ be the set of clockwise-ordered n different points $(z_1, \cdots, z_n)$ on the boundary of two-disk.

$$\widehat{\mathcal{M}}^0_n = \left\{ (z_1, \cdots, z_n) \,\middle|\, \begin{array}{c} z_i \in \partial D \;\; (1 \leq i \leq n), \\ z_i \neq z_j \;\; \text{if} \;\; i \neq j. \end{array} \right\}. \tag{8.2}$$

$SL_2(\mathbf{R})$ acts on the both sets in the standard manner. We put $\mathcal{M}^1_n \equiv \widehat{\mathcal{M}}^1_n / SL_2(\mathbf{R})$ for $n \geq 1$, and $\mathcal{M}^0_n \equiv \widehat{\mathcal{M}}^0_n / SL_2(\mathbf{R})$ for $n \geq 3$. In the previous sections we denote $\mathcal{M}^0_n$ by $\mathcal{M}^\partial_n$. We can always put the puncture at the origin of two-disk by using the $SL_2(\mathbf{R})$. Thereby $\mathcal{M}^1_n$ is identified with $\widehat{\mathcal{M}}^0_n / U(1)$, where $U(1)$ is the rotation group of ∂D.

To describe infinities of $\mathcal{M}^1_n$, we introduce a pair of multi-indices (I, J), where the multi-indices $I = (i_1, \cdots, i_p)$ $(p \geq 0)$ and $J = (j_1, \cdots, j_q)$ $(q \geq 2)$ satisfy the conditions : $I \cap J = \emptyset$ and $(i_1, \cdots, i_p, j_1, \cdots, j_q) = (1, \cdots, n)$ mod cyclic permutations. At infinities of $\mathcal{M}^1_n$ there appear $\mathcal{M}^1_{(I,s)} \times \mathcal{M}^0_{(s,J)}$ for all the pairs (I, J). "s" denotes the singular point in the configuration at the infinities. See Figure 16. Stable compactification of $\mathcal{M}^1_n$, which we call $\mathcal{CM}^1_n$, is defined inductively by adding $\mathcal{CM}^1_{(I,s)} \times \mathcal{CM}^0_{(s,J)}$ to the infinities of $\mathcal{M}^1_n$. [8]

$$\mathcal{CM}^1_n = \mathcal{M}^1_n \cup \left\{ \bigcup_{(I,J)} \mathcal{CM}^1_{(I,s)} \times \mathcal{CM}^0_{(s,J)} \right\}, \tag{8.3}$$

[8] $\mathcal{CM}^0_{(s,J)}$ is $\mathcal{CM}^\partial_{(s,J)}$ in the previous notation.

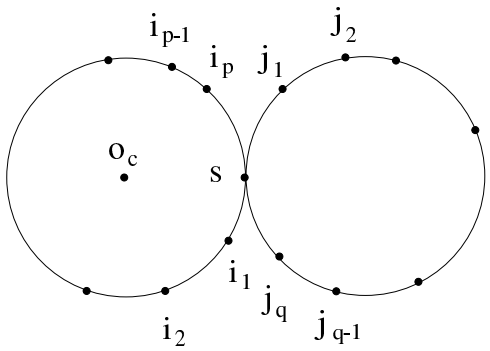

Figure 16: A typical configuration at the infinities $\mathcal{M}^1_{(I,s)} \times \mathcal{M}^0_{(s,J)}$ of $\mathcal{M}^1_n$.

where $\mathcal{CM}^1_1 \equiv \mathcal{M}^1_1$ is a point. Topologically $\mathcal{CM}^1_n$ becomes a $(n-1)$-dimensional ball B_{n-1}. This can be seen from the identification of $\mathcal{M}^1_n$ with $\widehat{\mathcal{M}}^0_n/U(1)$. We fix orientation of $\mathcal{CM}^1_n$ by the standard orientation of B_{n-1}. Taking account of the orientations we obtain

$$\partial\mathcal{CM}^1_n = \sum_{(I,J)} (\pm)\mathcal{CM}^1_{(I,s)} \times \mathcal{CM}^0_{(s,J)}. \tag{8.4}$$

The signature $(\pm)$ must be determined by a comparison of the orientations of $\partial\mathcal{CM}^1_n$ and $\mathcal{CM}^1_{(I,s)} \times \mathcal{CM}^0_{(s,J)}$. Later we will discuss about this problem.

In the previous sections we introduced $\mathcal{V}^0_n(\zeta) \equiv \mathcal{V}_n(\zeta)$, which is the subset of $\mathcal{CM}^0_n$, in order to obtain open-string vertices at the cut-off scale ζ. Analogous consideration is also possible at the present situation : We identify $\mathcal{M}^1_n$ with $\widehat{\mathcal{M}}^0_n/U(1)$. Consider the configurations near the infinities. In these configurations some of n points (on ∂D) are getting "close" to one another. We can measure how close they are by length of open-string strips. Length of open-string strips provide a local coordinates at least near the boundary of $\mathcal{CM}^1_n$. If we put a restriction on their length by the scale parameter ζ, the set of forbidden configurations provides a neighborhood of the infinities. Let us denote this neighborhood by $\mathcal{P}^1_n(\zeta)$. We put $\mathcal{V}^1_n(\zeta) \equiv \mathcal{CM}^1_n \setminus \mathcal{P}^1_n(\zeta)$. This plays the same role as $\mathcal{V}^0_n(\zeta)$ in the construction of string vertex. $\mathcal{V}^1_n(\zeta)$ is topologically B_{n-1}. Thus obtained $\mathcal{V}^1_n(\zeta)$ for $\zeta > 0$ satisfies

$$\partial\mathcal{V}^1_n(\zeta) = \sum_{(I,J)} (\pm)\mathcal{V}^1_{(I,s)}(\zeta) \times \mathcal{V}^0_{(s,J)}(\zeta). \tag{8.5}$$

This equation plays an important role of our understanding of boundary states.

Boundary states are introduced as states of closed-string. For their description we prepare some terminology of closed-string field theory. We follow the convention used in [2]. Let $\mathcal{H}^c_{matter}$ and $\mathcal{H}^c_{ghost}$ be respectively the matter and ghost Hilbert spaces of critical bosonic string. We put $\mathcal{H}_{aux} \equiv \mathcal{H}^c_{matter} \otimes \mathcal{H}^c_{ghost}$. The SL_2-invariant vacuum $|0\rangle$ of the auxiliary Hilbert space $\mathcal{H}_{aux}$ is set to have no ghost number. The SL_2-invariant vacuum $\langle 0|$ of the dual Hilbert space $\mathcal{H}^*_{aux}$ is also set to have no ghost number. Both states are grassmann-even. Dual pairing between $\mathcal{H}_{aux}$ and $\mathcal{H}^*_{aux}$ is prescribed based on $\langle 0|c_{-1}\bar{c}_{-1}c_0\bar{c}_0c_1\bar{c}_1|0\rangle = 1$, where $c_{\pm 1,0}$ and $\bar{c}_{\pm 1,0}$ are respectively the ghost zero modes of chiral and anti-chiral parts on $\mathbf{CP}_1$. Closed-string Hilbert space $\mathcal{H}_c$ consists of vectors ψ of $\mathcal{H}_{aux}$ which satisfy the conditions, $b_0^-\psi = L_0^-\psi = 0$ [9]. Closed-string field Ψ is a grassmann-even vector of $\mathcal{H}_c$ and has the ghost number $G(\Psi) = 2$.

Closed-string Hilbert space has an odd symplectic structure ω^c. We write it in the form, $\omega^c(A,B) = \langle \omega^c_{12}|A\rangle_1|B\rangle_2$ for $A, B \in \mathcal{H}_c$. $\langle \omega^c_{12}|$ is an element of $(\mathcal{H}_c^{\otimes 2})^*$. It is defined by the restriction of $\langle R^c_{12}|c_0^{-(2)}$ on $\mathcal{H}_c$. Here $\langle R^c_{12}|$ is the reflector of $\mathcal{H}_{aux}$. It is given by using the BPZ pairing as follows. $\langle R^c_{12}|A\rangle_1|B\rangle_2 \equiv \langle A^T|B\rangle$ for any $A, B \in \mathcal{H}_{aux}$. The reflector is a grassmann-even vector and has the ghost number equal to six. Hence the odd symplectic form $\langle \omega^c_{12}|$ is a grassmann-odd vector and has the ghost number equal to seven. The selection rule of ω^c becomes : $\omega^c(A,B) \neq 0 \Longrightarrow G(A) + G(B) = 5$. The inverse of the odd symplectic form is called sewing ket. We denote it by $|S^c_{12}\rangle$. It is an element of $\mathcal{H}_c^{\otimes 2}$ and satisfies $\langle \omega^c_{12}|S^c_{23}\rangle = {}_3P_1$. The sewing ket is grassmann-odd and has the ghost number equal to five.

We want to gain ultimately a field theory which describes interactions of open- and closed-strings. If such a field theory consistently exists, we need to have all the string vertices for these interactions. Recent attempts in this direction can be found in [23, 30]. Nevertheless the construction is still beyond completion. One of the reasons is that we still do not have a suitable physical perspective of closed-strings by open-strings or open-strings by closed-strings. We also wish to find such a physical perspective in the following discussion. Our discussion is based on several assumptions. They are physically reasonable and acceptable if a field theory of interacting open- and closed-strings exists consistently.

We assume an existence of string vertices $\left\langle \mathbf{0_c}; 1 \ \cdots \ n : \zeta \right|$ for $n \geq 1$. Here the index $\mathbf{0_c}$ represents a closed-string on D and the indices i $(1 \leq i \leq n)$ label clockwise-ordered n open-strings on ∂D. The vertices describe interactions between a single closed-string and n

[9] We put $L_0^{\pm} \equiv L_0 \pm \bar{L}_0$, $b_0^{\pm} \equiv b_0 \pm \bar{b}_0$ and $c_0^{\pm} \equiv (c_0 \pm \bar{c}_0)/2$.

open-strings at the cut-off scale ζ. If one takes the closed-string picture, an incoming closed-string is reflected at ∂D and at the same time decaying into open-strings. The interaction term $\Big\langle \mathbf{0_c}; 1\ \cdots\ n:\zeta\Big|\Psi\rangle_{\mathbf{0_c}}|\Phi\rangle_1\cdots|\Phi\rangle_n$ should not vanish. Recalling the ghost numbers $G(\Psi)=2$ and $G(\Phi)=1$, this requires that the ghost number of the vertex $\Big\langle \mathbf{0_c}; 1\ \cdots\ n:\zeta\Big|$ is equal to $n+4$. Also for the non-vanishing, odd-grassmannity of Φ requires the following cyclic asymmetry with respect to the open-string indices.

$$\Big\langle \mathbf{0_c}; 1\ 2\ \cdots\ n{-}1\ n:\zeta\Big| = (-)^{n+1}\Big\langle \mathbf{0_c}; 2\ 3\ \cdots\ n\ 1:\zeta\Big|. \tag{8.6}$$

Explicit construction of the vertices was examined in [31] for the cases of $n=1,2$ by a slightly different formulation. In our formulation it will be generalized based on a conjectural map (or a section of the Hilbert bundle on $\mathcal{CM}^1_n$),

$$\begin{array}{ccc} \mathcal{CM}^1_n & \longrightarrow & (\mathcal{H}_c\times\mathcal{H}_o^{\otimes n})^* \\ \Sigma & \mapsto & \langle\Sigma| \end{array} \quad . \tag{8.7}$$

This map gives rise to a $(\mathcal{H}_c\times\mathcal{H}_o^{\otimes n})^*$-valued $(n-1)$-form $\langle\Omega|$ on $\mathcal{CM}^1_n$. The vertex will be defined as an integration of thus obtained $\langle\Omega|$ on $\mathcal{V}^1_n(\zeta)$.

$$\Big\langle \mathbf{0_c}; 1\ \cdots\ n:\zeta\Big| = \int_{\mathcal{V}^1_n(\zeta)}\langle\Omega|^{(1\ \cdots\ n)}\prod_{i=1}^n e^{-\zeta L_0^{(i)}}. \tag{8.8}$$

As is the case of open-string field theory, action of the BRST charge on the vertices is expected to be a representation of the boundary operator ∂. Since it is an interacting theory of closed- and open-strings, the BRST charge in question should be a sum of the closed-string BRST charge Q_c and the open-string BRST charge Q_o. Hence we arrive at the following conjectural action of the BRS charge.

$$\Big\langle \mathbf{0_c}; 1\ \cdots\ n:\zeta\Big|\Big(Q_c^{(\mathbf{0_c})}+\sum_{i=1}^n Q_o^{(i)}\Big) = \sum_{(I,J)}(\pm)\Big\langle \mathbf{0_c}; I, a:\zeta\Big|\Big\langle a', J:\zeta\Big|S^o_{a'a}\Big\rangle, \tag{8.9}$$

where we denote the open-string inverse reflector by $\Big|S^o_{a'a}\Big\rangle$.

Boundary states

So far, our consideration of the string vertices in this section excludes the case of $n=0$. In this particular case we have the state $\Big\langle \mathbf{0_c}\Big| \in \mathcal{H}_c^*$. (It is different from the SL_2-invariant vacuum

$\langle 0|$.) This is closed-string vertex which describes a simple reflection of a single closed-string at ∂D. It is a BRST invariant state and has the ghost number four. If we take the dual by using the sewing ket of closed-string, it acquires a familiar form [32].

$$\Big|B\Big\rangle \equiv \Big\langle \mathbf{0_c}\Big|S^c_{\mathbf{0c*c}}\Big\rangle. \tag{8.10}$$

This is a boundary state of closed-string and is called the Ishibashi state. It satisfies

$$Q_c\Big|B\Big\rangle = 0, \qquad G\Big(\Big|B\Big\rangle\Big) = 3. \tag{8.11}$$

Due to the correct ghost number we have the non-vanishing coupling $\omega^c(\Psi, |B\rangle)$ with closed-string field. Strictly speaking, the Ishibashi state is not a state of $\mathcal{H}_c$ because its norm turns out to be divergent. Without any contradiction there is no local field operator in $2D$ CFT which corresponds to this state. For a suitable class of $2D$ CFT, physically acceptable bases of solutions of $Q_c|B\rangle = 0$ are found in [33] and called the Cardy bases. Roughly speaking, they are characterized by the reflection conditions of closed-string at ∂D. Of course it can be rephrased as the boundary conditions of open-strings when they interact with the closed-string via the vertex (8.8). For simplicity we impose the Neumann condition on the reflection of closed-string.

Let us consider the following boundary state in the presence of an open-string field Φ.

$$\Big|B[\Phi:\zeta]\Big\rangle \equiv e^{-\zeta L_0^{+(*c)}} \left\{ \Big\langle \mathbf{0_c}\Big|S^c_{\mathbf{0c*c}}\Big\rangle + \sum_{k\geq 1} \frac{1}{k} \Big\langle \mathbf{0_c}; 1\cdots k : \zeta\Big|S^c_{\mathbf{0c*c}}\Big\rangle |\Phi\rangle_1 \cdots |\Phi\rangle_k \right\}. \tag{8.12}$$

The ghost number of this state becomes three. This means that we have a non-vanishing coupling with closed-string field Ψ of the form, $\omega^c\Big(\Psi, \Big|B[\Phi:\zeta]\Big\rangle\Big)$. If one takes the closed-string picture, it describes an interaction of a closed-string with a reservoir of open-strings. Due to this interaction or a possible decay to open-strings the boundary state (8.12) is not invariant under the action of the closed-string BRST charge Q_c. Our first investigation is about an interpretation of this action from the open-string viewpoint.

Role of Q_c in open-string field theory

Let us examine the boundary state (8.12) from the open-string picture. We regard the boundary state as a $\mathcal{H}_c$-valued function of Φ. Then the corresponding hamiltonian vector $\Big|\partial B[\Phi]/\partial\Phi\Big\rangle$ is an element of $\mathcal{H}_c \times \mathcal{H}_o$. It is given by the variational formula (5.4),

$$\delta\Big|B[\Phi:\zeta]\Big\rangle = \omega^o\left(\delta\Phi, \ \left|\frac{\partial B[\Phi:\zeta]}{\partial\Phi}\right\rangle\right), \tag{8.13}$$

where we denote the open-string odd symplectic structure by ω^o. After a little calculation we find out the following expression of the hamiltonian vector.

$$\left|\frac{\partial B[\Phi:\zeta]}{\partial\Phi}\right\rangle = e^{-\zeta L_0^{+(*c)}}\sum_{k\geq 1}\left\langle \mathbf{0_c}; 1\ \cdots\ k{-}1\ k:\zeta\right|\left|S^c_{\mathbf{0_c}*\mathbf{c}}\right\rangle|\Phi\rangle_1\cdots|\Phi\rangle_{k-1}\left|S^o_{k*_o}\right\rangle. \tag{8.14}$$

We now compute action of the closed-string BRST charge on the boundary state (8.12). By a simple evaluation it can be rewritten in the following form.

$$\begin{aligned} Q_c\left|B[\Phi:\zeta]\right\rangle &= \omega^o\left(Q_o\Phi,\ \left|\frac{\partial B[\Phi:\zeta]}{\partial\Phi}\right\rangle\right) \\ &\quad + e^{-\zeta L_0^{+(*c)}}\sum_{k\geq 1}\frac{(-)^{k+1}}{k}\left\langle \mathbf{0_c}; 1\ \cdots\ k:\zeta\right|\left(Q_c^{(\mathbf{0_c})}+\sum_{i=1}^{k}Q_o^{(i)}\right)\left|S^c_{\mathbf{0_c}*\mathbf{c}}\right\rangle|\Phi\rangle_1\cdots|\Phi\rangle_k. \end{aligned} \tag{8.15}$$

To proceed computation of the second term of eq.(8.15) we need to use the conjectural form (8.9). The result is a suitable weighted sum of the quantities,

$$\omega^o\left(m_p\left(\Phi^p:\zeta\right),\ e^{-\zeta L_0^{+(*c)}}\left\langle \mathbf{0_c}; 1\ \cdots\ q\ 0_o:\zeta\right|\left|S^c_{\mathbf{0_c}*\mathbf{c}}\right\rangle|\Phi\rangle_1\cdots|\Phi\rangle_q\left|S^o_{0_o*_o}\right\rangle\right), \tag{8.16}$$

for $p\geq 2$ and $q\geq 0$. Their weights are determined by the signatures in eq.(8.9) or equivalently in eq.(8.5). So our computation may stop here. What we can learn from eq.(8.16) is the appearance of the A_∞-algebra. If one recalls that hamiltonian vector of the classical open-string field action (5.16) is the sum of m_k, one may infer that the weighted sum eventually gives rise to the following result.

$$Q_c\left|B[\Phi:\zeta]\right\rangle = \omega^o\left(\left|\frac{\partial S[\Phi:\zeta]}{\partial\Phi}\right\rangle,\ \left|\frac{\partial B[\Phi:\zeta]}{\partial\Phi}\right\rangle\right), \tag{8.17}$$

where $\left|\partial S[\Phi:\zeta]/\partial\Phi\right\rangle$ is the hamiltonian vector given by (5.22).

Eq.(8.17) must be tested. First of all, the above Q_c-action needs to be nilpotent, $Q_c\left(Q_c\left|B[\Phi]\right\rangle\right) = 0$. Let us show this : By using the open-string anti-bracket we rewrite the RHS of eq.(8.17) as $\left\{S[\Phi], |B[\Phi]\rangle\right\}$. We then compute $Q_c\left(Q_c\left|B[\Phi]\right\rangle\right)$ as follows.

$$\begin{aligned} Q_c\left(Q_c\left|B[\Phi]\right\rangle\right) &= (-)Q_c\left\{\left|B[\Phi]\right\rangle, S[\Phi]\right\} \\ &= (-)\left\{Q_c\left|B[\Phi]\right\rangle, S[\Phi]\right\} \\ &= \left\{\left\{\left|B[\Phi]\right\rangle, S[\Phi]\right\}, S[\Phi]\right\} \\ &= -\frac{1}{2}\left\{\left\{S[\Phi], S[\Phi]\right\}, \left|B[\Phi]\right\rangle\right\}, \end{aligned} \tag{8.18}$$

where we use the Jacobi identity (5.15) to show the last equality. Since we have the classical master equation $\{S,S\}=0$ by Proposition 5.4, eq.(8.18) vanishes identically. Another consistency may be obtained if we can find the signatures in eq.(8.9) so that they provide a representation of the boundary operator ∂ and also give rise to eq.(8.17). As such a solution we finally find out the following ones.

$$\begin{aligned}&\Big\langle \mathbf{0_c}; 1\ \cdots\ n:\zeta\Big|\Big(Q_c^{(\mathbf{0_c})}+\sum_{i=1}^{n}Q_o^{(i)}\Big)\\ &=\sum_{l=0}^{n-2}\sum_{k=1}^{n}(-)^{k(n+1)+(l+1)n}\Big\langle \mathbf{0_c}; \underbrace{k\ \cdots\ k+l-1\ a}_{l+1}:\zeta\Big|\Big\langle \underbrace{a'\ k+l\ \cdots\ k+n-1}_{n+1-l}:\zeta\Big|S^o_{a'a}\Big\rangle. \quad (8.19)\end{aligned}$$

These are consistent with the asymmetry (8.6).

Eq.(8.17) itself was alluded previously from a different perspective in [31, 23]. These authors intended to construct new symmetries of open-string field theory. Their argument is as follows. Let $\psi\in\mathcal{H}_c$ be a BRST-closed state ($Q_c\psi=0$) with $G(\psi)=2$. Associated with such a state we can introduce a non-vanishing function $F_\psi(\Phi)\equiv\omega^c\Big(\psi,|B(\Phi)\rangle\Big)$. Then, since $Q_c\psi$ vanishes, eq.(8.17) implies $\Big\{S[\Phi],F_\psi(\Phi)\Big\}=0$. This means that F_ψ generates a symmetry of the theory. As a trivial example we can consider the massless $U(1)$ gauge field a_μ (of open-string) in presence of the massless anti-symmetric tensor field $b_{\mu\nu}$ (of closed-string). In this simple case infinitesimal transformations, $\delta a_\mu=\eta_\mu$ and $\delta b_{\mu\nu}=\partial_\mu\eta_\nu-\partial_\nu\eta_\mu$, correspond to a symmetry in question. η_μ is identified with a suitable component of $\psi=Q_c\eta$.

Significance of the Q_c-action (8.17) should be stressed. The BRST transformation $\delta_{\mathbf{BRS}}$ of open-string field Φ is defined by eq.(5.31). It is just the hamiltonian vector of $S[\Phi:\zeta]$. By using the variational formula (5.4) we can rewrite the Q_c-action into

$$Q_c\Big|B[\Phi:\zeta]\Big\rangle=\delta_{\mathbf{BRS}}\Big|B[\Phi:\zeta]\Big\rangle. \quad (8.20)$$

Thus the closed-string BRST charge induces the BRST transformation of open-string field. Eq.(8.20) becomes a key in the construction of boundary open-string field theory.

Boundary open-string field theory

A framework for background independent open-string field theory was proposed in [18] based on the BV formalism. It is conventionally called boundary open-string field theory. This formulation was further investigated in [35, 34, 36]. Construction based on the BV formalism

requires a triple $((\mathcal{X},\omega),V)$. $\mathcal{X}$ is a super-manifold. ω is an odd symplectic structure of $\mathcal{X}$ and V is a nilpotent fermionic vector field on $\mathcal{X}$. Having such a triple, the BV action, if it exists, is obtained as the hamiltonian function of V. The nilpotency of V ensures the classical master equation. And the BRST transformation of the theory is given by V. In order to construct boundary open-string field theory, a hypothetical "space of all open-string world-sheet theories" is taken [18] as $\mathcal{X}$. ω is determined by correlation functions of world-sheet theories. V is practically identified with the *closed*-string BRST charge Q_c.

We want to interpret the macroscopic open-string field theory presented in this paper as a boundary open-string field theory.

Our formulation is based on the Wilson renormalization group. We start at the trivial classical solution $\Phi=0$. This solution describes a flat Minkowski space $\mathbf{R}^{(1,25)}$ and is a conformal point of the above $\mathcal{X}$. We regards it as a UV fixed point. The open-string Hilbert space $\mathcal{H}$ used in this paper is the tangent of $\mathcal{X}$ at $\Phi=0$. A local coordinate patch of $\mathcal{X}$ centered at this point will be given by the tangent space via the exponential map. Our naive expectation is that, excluding the critical points, theory at $\Phi\neq 0$ on this patch is described by the classical action $S[\Phi:\zeta]$ (or the fluctuation given in Remark 5.2). To pursue this naive picture we need to interpret the scale parameter ζ correctly.

The regularization employed in this paper corresponds to a point-splitting regularization of short-distance on the boundary when ζ is sufficiently large. The point-splitting is prescribed by the boundary length scale $a\sim e^{-\zeta}$ measured by the boundary metric $dxdx$. Consider a space of all open-string world-sheet cut-off theories which cut-off length scale is a. The tangent space at $\Phi=0$, which we denote by $\mathcal{Y}(a)$, is identified with the open-string Hilbert space $\mathcal{H}$. The RG flow in the previous sections defines a map

$$\mathcal{R}^{\zeta}_{\zeta'} \qquad \mathcal{Y}(e^{-\zeta}) \longrightarrow \mathcal{Y}(e^{-\zeta'}), \tag{8.21}$$

where $\zeta>\zeta'$. The previous construction of the RG flow is the reverse of the above map. This is due to the different interpretation of the scale parameter ζ. On the boundary $e^{-\zeta}$ plays the role of the short distance cut-off. Both spaces $\mathcal{Y}(e^{-\zeta})$ and $\mathcal{Y}(e^{-\zeta'})$ are identified with $\mathcal{H}$. The RG flow becomes the flow on $\mathcal{H}$ generated by

$$\frac{d\Phi}{d\zeta}=-\left\{b_0\left|\frac{\partial S_{int}[\Phi:\zeta]}{\partial\Phi}\right\rangle+Qb_0\Phi\right\}. \tag{8.22}$$

The classical action $S[\Phi:\zeta]$ defines a boundary open-string field theory constructed on $\mathcal{Y}(e^{-\zeta})$. By Proposition 5.6, the action $S[\Phi:\zeta]$ satisfies $\delta S[\Phi:\zeta]=\omega\Big(\delta\Phi,\delta_{\mathbf{BRS}}\Phi\Big)$. Thus it is the hamiltonian function of $\delta_{\mathbf{BRS}}$. We now regards $\delta_{\mathbf{BRS}}$ as a nilpotent fermionic vector field on $\mathcal{Y}(e^{-\zeta})$. As we find in eq.(8.20), it can be identified with the *closed*-string BRST charge Q_c. Therefore geometrical ingredients of both theories become same.

In order to define a "space of all open-string world-sheet theories", we first need to take a continuum limit by letting $\zeta\to\infty$. The existence of infinitely many irrelevant operators [10] causes serious problems. The hypothetical space $\mathcal{X}$ is supposed to be as follows in the original description [18]. Let I_0 be a two-dimensional Lagrangian (on D) which describes a fixed bulk closed-string background. One considers two-dimensional Lagrangians of the form,

$$I=I_0+I_{\partial D}, \tag{8.23}$$

where $I_{\partial D}$ is a suitable boundary term describing the coupling to external open-strings.

$$I_{\partial D}=\int_{\partial D}d\theta\ \mathcal{V}(X,b,c), \tag{8.24}$$

where $d\theta$ is the standard line element of S^1. The space $\mathcal{X}$ is roughly introduced as the space of Lagrangians I with I_0 fixed and $I_{\partial D}$ allowed to vary. In order to obtain $\mathcal{X}$ as such, one needs to define $I_{\partial D}$ correctly by a suitable regularization and then take a continuum limit. In [36] [37] such a prescription was partially examined. It is amusing to reconsider the results of [36] [37] from the perspective of our formulation.

[10] States ϕ_i of $\mathcal{H}^S$ which satisfy $\Delta_i>0$.

A Appendix

In this appendix we give a proof of Proposition 3.5 presented in the text. We first compute the LHS of eq.(3.39) by using eq.(3.37) as follows.

$$
\begin{aligned}
&\Big\{\Big\langle 1\ 2\ \cdots\ n:\zeta\Big|\Big(\sum_{i=1}^{n}Q^{(i)}\Big)\Big\}\Big(\sum_{i=1}^{n}Q^{(i)}\Big)\\
&=-\frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n-3}(-)^{(n+1)(k+l+1)}\Big\langle\underbrace{k\ \cdots\ k+l\ a}_{l+2}:\zeta\Big|\Big\langle\underbrace{a'\ k+l+1\ \cdots\ k+n-1}_{n-l}:\zeta\Big|S_{a'a}\Big\rangle\Big(\sum_{j=1}^{n}Q^{(j)}\Big)\\
&=\frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n-3}(-)^{(n+1)(k+l+1)+n-l}\left\{\Big\langle k\ \cdots\ k+l\ a:\zeta\Big|\Big(Q^{(a)}+\sum_{j=k}^{k+l}Q^{(j)}\Big)\right\}\\
&\qquad\qquad\times\Big\langle a'\ k+l+1\ \cdots\ k+n-1:\zeta\Big|S_{a'a}\Big\rangle \qquad\text{(A.1)}\\
&\quad+\frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n-3}(-)^{(n+1)(k+l+1)}\Big\langle k\ \cdots\ k+l\ a:\zeta\Big|\\
&\qquad\qquad\times\left\{\Big\langle a'\ k+l+1\ \cdots\ k+n-1:\zeta\Big|\Big(Q^{(a')}+\sum_{j=k+l+1}^{k+n-1}Q^{(j)}\Big)\right\}\Big|S_{a'a}\Big\rangle, \qquad\text{(A.2)}
\end{aligned}
$$

where we use the BRST invariance (2.28) of the inverse reflector to show the last equality. We rewrite (A.1) by replacing two vertices in the equation. Taking account of the asymmetries (3.38) and the grassmannities, it becomes as follows.

$$
\begin{aligned}
(A.1)\ &=\ \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n-3}(-)^{(n+1)k}\Big\langle\underbrace{k+l+1\ \cdots\ k+n-1}_{n-l}:\zeta\Big|\\
&\qquad\times\left\{\Big\langle\underbrace{a'\ k\ \cdots\ k+l}_{l+2}:\zeta\Big|\Big(Q^{(a')}+\sum_{j=k}^{k+l}Q^{(j)}\Big)\right\}\Big|S_{a'a}\Big\rangle. \qquad\text{(A.3)}
\end{aligned}
$$

After a slight change of the open-string indices eq.(A.3) turns out to be the second term (A.2). Thus we obtain the following expression for the LHS of (3.39).

$$
\begin{aligned}
&\Big\{\Big\langle 1\ 2\ \cdots\ n:\zeta\Big|\Big(\sum_{i=1}^{n}Q^{(i)}\Big)\Big\}\Big(\sum_{i=1}^{n}Q^{(i)}\Big)\\
&=\sum_{k=1}^{n}\sum_{l=1}^{n-3}(-)^{(n+1)(k+l+1)}\Big\langle\underbrace{k\ \cdots\ k+l\ a}_{l+2}:\zeta\Big|\\
&\qquad\times\left\{\Big\langle\underbrace{a'\ k+l+1\ \cdots\ k+n-1}_{n-l}:\zeta\Big|\Big(Q^{(a')}+\sum_{j=k+l+1}^{k+n-1}Q^{(j)}\Big)\right\}\Big|S_{a'a}\Big\rangle. \qquad\text{(A.4)}
\end{aligned}
$$

In order to compute the action of the BRST charge in the RHS of eq.(A.4), the following lemma becomes useful.

Lemma A.1 *The action (3.37) of the BRST charge can be written as follows.*

$$\Big\langle 1\ \cdots\ n:\zeta\Big|\Big(\sum_{i=1}^{n}Q^{(i)}\Big)$$
$$=-\sum_{l=1}^{n-3}\sum_{k=1}^{n-l-1}(-)^{(n+1)(k+1)}\Big\langle\underbrace{k{+}l{+}2\ \cdots\ n\ 1\ 2\cdots k\ a}_{n-l}:\zeta\Big|\Big\langle\underbrace{a'\ k{+}1\ \cdots\ k{+}l{+}1}_{l+2}:\zeta\Big|S_{a'a}\Big\rangle. \tag{A.5}$$

We omit the proof of this lemma. By using this lemma our computation goes as follows.

$$\begin{aligned}
&Eq.(A.4)\\
&=\sum_{k=1}^{n}\sum_{l=1}^{n-3}(-)^{(n+1)(k+l+1)}\Big\langle\underbrace{k\ \cdots\ k{+}l\ a}_{l+2}:\zeta\Big|\\
&\quad\times\Bigg\{\sum_{l'=1}^{n-l-3}\sum_{p=1}^{n-l-l'-1}(-)^{(n-l+1)(p+1)+1}\Big\langle\underbrace{k{+}l{+}l'{+}p{+}1\cdots a'\ \cdots\ k{+}l{+}p{-}1\ b}_{n-l-l'}:\zeta\Big|\\
&\qquad\times\Big\langle\underbrace{b'\ k{+}l{+}p\ \cdots\ k{+}l{+}l'{+}p}_{l'+2}:\zeta\Big|S_{b'b}\Big\rangle\Bigg\}\Big|S_{a'a}\Big\rangle\\
&=\sum_{k=1}^{n}\sum_{l=1}^{n-3}\sum_{l'=1}^{n-l-3}\sum_{p=1}^{n-l-l'-1}(-)^{(n+1)(k+p+l)+l(l+p)+1}\Big\langle\underbrace{k\ \cdots\ k{+}l\ a}_{l+2}:\zeta\Big|\\
&\qquad\times\Big\langle\underbrace{k{+}l{+}l'{+}p{+}1\ \cdots\ a'\ \cdots\ k{+}l{+}p{-}1\ b}_{n-l-l'}:\zeta\Big|\\
&\qquad\times\Big\langle\underbrace{b'\ k{+}l{+}p\ \cdots\ k{+}l{+}l'{+}p}_{l'+2}:\zeta\Big|S_{b'b}\Big\rangle\Big|S_{a'a}\Big\rangle.
\end{aligned}\tag{A.6}$$

Terms appearing in eq.(A.6) are conveniently depicted in Figure 17-(a).

We want to show that eq.(A.6) vanishes identically. For this purpose we exchange the first and third vertices in eq.(A.6) taking account of the grassmannities, and then rewrite eq.(A.6) by using the asymmetries (3.38) of the vertices. We obtain the following expression for eq.(A.6).

$$\begin{aligned}
&Eq.(A.6)\\
&=\sum_{k=1}^{n}\sum_{l=1}^{n-3}\sum_{l'=1}^{n-l-3}\sum_{p=1}^{n-l-l'-1}(-)^{(n+1)(k+l')+l'(l+l'+p)}\Big\langle\underbrace{k{+}p{+}l\ \cdots\ k{+}p{+}l{+}l'\ a}_{l'+2}:\zeta\Big|
\end{aligned}$$

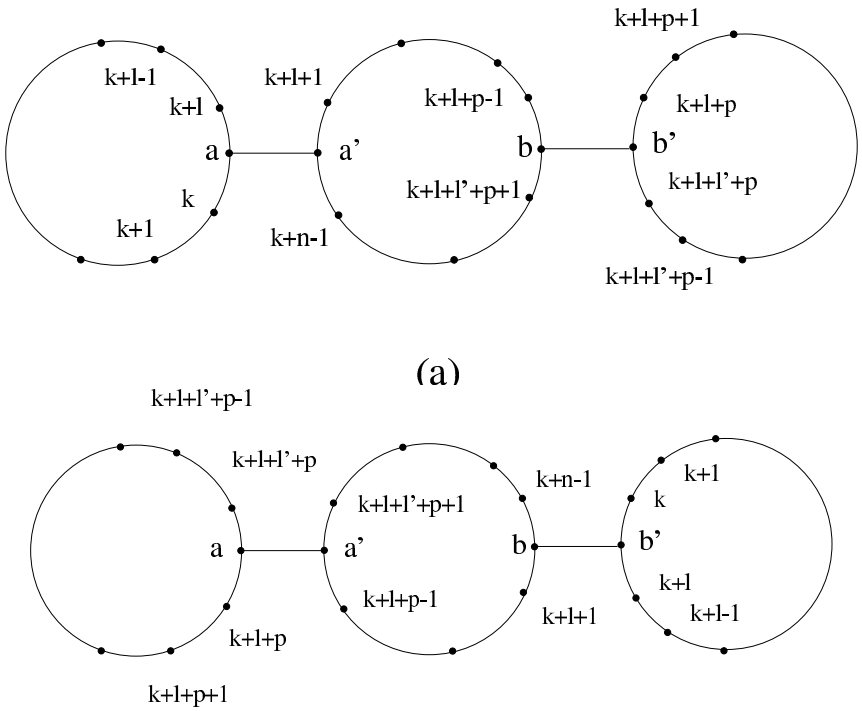

Figure 17: (a) Open-string diagram which appears in eq.(A.6). (b) Open-string diagram which appears in eq.(A.7).

$$\times \Big\langle \underbrace{k{+}l{+}1 \ \cdots \ a' \ \cdots \ k{+}n{-}1 \ b}_{n-l-l'} : \zeta \Big| \times \Big\langle \underbrace{b' \ k \ \cdots \ k{+}l}_{l+2} : \zeta \Big| S_{b'b} \Big\rangle \Big| S_{a'a} \Big\rangle . \tag{A.7}$$

Terms appearing in eq.(A.7) are conveniently depicted in Figure 17-(b).

To compare two expressions (A.6) and (A.7) of the LHS of eq.(3.39), we replace the open-string indices in eq.(A.7) so that they match with those in eq.(A.6). We first exchange $l \leftrightarrow l'$ and then put $k' \equiv k+p+l'$ and $p' \equiv n-p-l-l'$. With these replacements eq.(A.7) becomes as follows.

$$Eq.(A.7) = \sum_{k=1}^{n} \sum_{l=1}^{n-3} \sum_{l'=1}^{n-l-3} \sum_{p=1}^{n-l-l'-1} (-)^{(n+1)(k+l)+l(l+l'+p)} \Big\langle \underbrace{k{+}p{+}l' \ \cdots \ k{+}p{+}l{+}l' \ a}_{l+2} : \zeta \Big|$$

$$\times \Big\langle \underbrace{k{+}l'{+}1 \ \cdots \ a' \ \cdots \ k{+}n{-}1 \ b}_{n-l-l'} : \zeta \Big|$$
$$\times \Big\langle \underbrace{b' \ k \ \cdots \ k{+}l'}_{l'+2} : \zeta \Big| S_{b'b} \Big\rangle \Big| S_{a'a} \Big\rangle$$
$$= \sum_{k'=1}^{n} \sum_{l=1}^{n-3} \sum_{l'=1}^{n-l-3} \sum_{p'=1}^{n-l-l'-1} (-)^{(n+1)(k'+p'+l)+l(l+p')} \Big\langle \underbrace{k' \ \cdots \ k'{+}l \ a}_{l+2} : \zeta \Big|$$
$$\times \Big\langle \underbrace{k'{+}l{+}l'{+}p'{+}1 \ \cdots \ a' \ \cdots \ k'{+}l{+}p'{-}1 \ b}_{n-l-l'} : \zeta \Big|$$
$$\times \Big\langle \underbrace{b' \ k'{+}l{+}p' \ \cdots \ k'{+}l{+}l'{+}p'}_{l'+2} : \zeta \Big| S_{b'b} \Big\rangle \Big| S_{a'a} \Big\rangle . \tag{A.8}$$

If one compares this expression with eq.(A.6), one finds that it is equal to $(-1)\times$eq.(A.6). This means that eq.(A.6)$= (-1)\times$eq.(A.6). Therefore eq.(A.6) must vanish. We complete the proof of Proposition 3.5.¶

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