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On K3-Thurston 7-manifolds and their deformation space: A case study with remarks on general K3T and M-theory compactification

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Abstract

M-theory suggests the study of 11-dimensional space-times compactified on some 7-manifolds. From its intimate relation to superstrings, one possible class of such 7-manifolds are those that have Calabi-Yau threefolds as boundary. In this article, we construct a special class of such 7-manifolds, named as *K3-Thurston* (K3T) 7-manifolds. The factor from the K3 part of the deformation space of these K3T 7-manifolds admits a Kähler structure, while the factor of the deformation space from the Thurston part admits a special Kähler structure. The latter rings with the nature of the scalar manifold of a vector multiplet in an $N = 2$ $d = 4$ supersymmetric gauge theory. Remarks and examples on more general K3T 7-manifolds and issues to possible interfaces of K3T to M-theory are also discussed.

Key words: M-theory, 7-manifold, K3-fibration, hyperbolic 3-manifold of finite volume, bundle-filling, K3-Thurston, deformation space, special Kähler, degenerate K3, Calabi-Yau moduli space.

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0. Introduction and outline.

Introduction.

M-theory anticipates the space-time to be 11-dimensional compactified on a 7-dimensional space. A class of such 7-spaces that have appeared in the literature on M-theory compactification are Joyce manifolds ([Ac1, Ac2] and [Joy]). Many of these manifolds can be realized as K3-fibrations over Euclidean 3-orbifolds ([Li]). A natural extension of this from a 3-dimensional geometor's point of view is to consider 7-manifolds that are K3-fibred over a hyperbolic 3-orbifolds. We call such 7-manifolds *K3-Thurston (K3T)* 7-manifolds. If one also allows these 7-manifolds to have Calabi-Yau threefolds as boundary, then one may allow the base to be hyperbolic 3-orbifolds with $\mathbb{Z} \oplus \mathbb{Z}$-cusps. They form the set of hyperbolic 3-orbifolds of finite volume. In this article, we construct a very simple kind of K3T 7-manifolds and study their deformation space.

To make this paper more accessible to physicists, some essential background or references are collected in Sec. 1. In Sec. 2, using a K3 surface X with antisymplectic involution, representations of a 3-manifold group $\pi_1(M^3)$ to $\mathbb{Z}_2$, and the technique of bundle-filling, we construct a class of K3T 7-manifolds that are either closed or with boundary the Calabi-Yau threefolds $\cup_{i=1}^{h} \mathbb{T}^2 \times X$. Concrete examples of such are given, with base some hyperbolic link complements in S^3. We then turn to the study of the deformation space Def(K3T) of this class of K3T 7-manifolds in Sec. 3. We discuss the Kähler factor of Def(K3T) from the K3 part of K3T and the special Kähler factor of Def(K3T) from the Thurston part of K3T. Remarks on general K3T 7-manifolds, some ingredients to construct them, and some examples are given in Sec. 4. Finally, we discuss in Sec. 5 possible interfaces from K3T to M-theory. The field theoretical contents when M-theory is compactified on such 7-manifolds will await future work.

Convention. Since both real and complex manifolds are involved in this article, to avoid confusion, a *real* n-dimensional manifold will be called an *n-manifold* while a *complex* n-dimensional manifold an *n-fold*.

Outline.

1. Essential mathematical background for physicists.
2. The construction of a class of K3T 7-manifold.
   - 2.1 The two ingredients: K3 surface with involution and hyperbolic 3-manifolds of finite volume.
   - 2.2 The construction of K3T 7-manifolds by bundle-filling.
   - 2.3 Examples from the link complements in S^3.
3. The deformation space.
   - 3.1 The deformation space of K3T 7-manifolds constructed.
   - 3.2 The K3 part of the deformations of K3T.
   - 3.3 The Thurston part of the deformations of K3T.
4. Remarks and examples on general K3T 7-manifolds.
5. Issues on applications to M-theory compactification.

1 Essential mathematical background for physicists.

We collect in this section some essential mathematical background or references for the convenience of physicists and also for the introduction of terminologies and notations. Additional necessary facts are stated in the related sections.

• **K3 surfaces and their deformation space.** Physicists are referred to [As] for a very nice exposition. A standard reference is [B-P-VV]. Degenerations and isolated singularities of a K3 surface are referred to [F-S], [Ku]. [Pe], and [P-P]. A-D-E surface singularities and their monodromy diffeomorphism are referred to [Di]. See also [A-M1,A-M2], [Dol], and [Sc].

• **Hyperbolic geometry.** ([B-P], [C-F-K-P], [C-R], [M-T] and [Th1-Th5].) A *hyperbolic 3-manifold* is a Riemannian 3-manifold of constant negative sectional curvature, usually normalized to -1. Up to isometry, there is a unique complete simply-connected one: the *hyperbolic 3-space* $\mathbb{H}^3$. All other complete hyperbolic 3-manifolds are quotients of $\mathbb{H}^3$. There are several analytic models for $\mathbb{H}^3$; two of them are particularly important to us:

(1) *The upper half-space model*:

$$\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_+ = \{(z,t) \mid z \in \mathbb{C}, t > 0\} \quad \text{with metric} \quad ds^2 = \frac{dz \cdot d\overline{z} + dt^2}{t^2}\,.$$

(2) *The Poincaré ball model*:

$$\mathbb{H}^3 = \{(x_1,x_2,x_3) \in \mathbb{R}^3 | x_1^2+x_2^2+x_3^2 < 1\} \quad \text{with metric} \quad ds^2 = \frac{4\,(\,dx_1^2+dx_2^2+dx_3^2\,)}{(\,1-x_1^2-x_2^2-x_3^2\,)^2}\,.$$

(Cf. FIGURE 1-2(a).)

The group $Isom^+(\mathbb{H}^3)$ of orientation-preserving isometries of $\mathbb{H}^3$ is the same as the Möbius group $PSL(2,\mathbb{C})$ that acts on the *ideal boundary* of $\mathbb{H}^3$

$$S^2_\infty \;=\; \partial_\infty \mathbb{H}^3 \;=\; \mathbb{C}\mathrm{P}^1 \;=\; \mathbb{C} \cup \{\infty\} \;=\; \{(x_1,x_2,x_3) \in \mathbb{R}^3 | x_1^2+x_2^2+x_3^2 = 1\}$$

by linear fractional transformations. Note that $PSL(2,\mathbb{C})$ is isomorphic to $SO^{\uparrow}_+(1,3)$, the identity component of the Lorentz group.

All hyperbolic 3-manifolds discussed in this paper will be assumed to be orientable.

• **Developing map and holonomy.** ([Th1, Th5].) Given a hyperbolic 3-manifold M^3, perhaps with boundary, then one can cover M^3 with a hyperbolic coordinate charts $(U_\alpha, \phi_\alpha : U_\alpha \to \mathbb{H}^3)$ so that all $U_\alpha, U_\alpha \cap U_\beta$ are topologically a ball. The transition function $\phi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ then coincides with the restriction to $\phi_\beta(U_\alpha \cap U_\beta)$ of a unique element, also denoted by $\phi_{\alpha\beta}$, in $Isom(\mathbb{H}^3)$. Fix an initial chart U_0 and a point p_0 in the interior of U_0. Let $\gamma : [0,1] \to M^3$ be a path at p_0 and $(U_0,\phi_0),(U_1,\phi_1)\cdots,(U_n,\phi_n)$ be a sequence of hyperbolic charts along γ, i.e. $\gamma \subset \cup_{i=0}^n U_i$

and $U_i \cap U_{i+1}$ nonempty for $i = 0, \ldots, n-1$. One can form a new sequene of hyperbolic charts (U_i, ϕ_i'), $i = 0, \ldots, n$, by setting

$$\phi_0' \;=\; \phi_0\,, \qquad \phi_1' \;=\; \phi_{01} \circ \phi_1\,, \qquad \cdots\,, \qquad \phi_n' \;=\; \phi_{01} \circ \phi_{12} \circ \cdots \phi_{n-1,n} \circ \phi_n\,,$$

where $\phi_{i,i+1}$ are understood as elements in $Isom(\mathbb{H}^3)$. A key feature of (U_i, ϕ_i') is that the new transition functions $\phi_{i,i+1}'$ have now become the identity map. Thus we shall call the sequence (U_i, ϕ_i'), $i = 0, \ldots, n$, the *analytic continuation* of (U_0, ϕ_0) along γ. Such continuation depends only on the initial chart (U_0, ϕ_0) and the homotopy class $[\gamma]$ of γ relative to its end-points and we will denote the germ associated with ϕ_n' by $\phi_0^{[\gamma]}$. (FIGURE 1-1.)

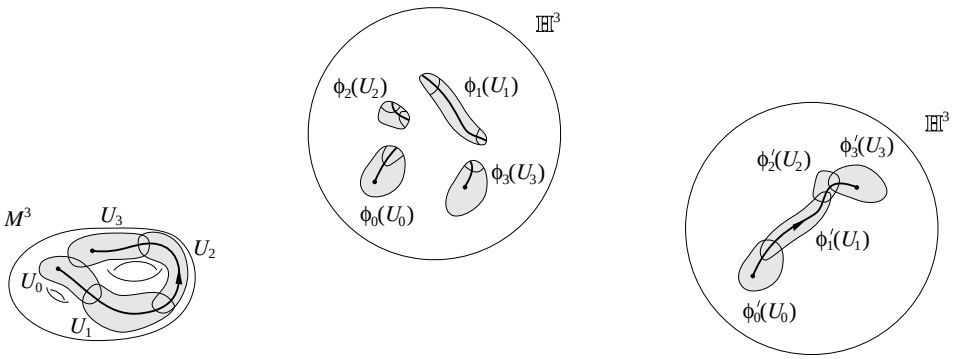


FIGURE 1-1. The analytic continuaion of hyperbolic patches along a path γ (cf. FIG. 3.15 in [Th5]).

Fix a lifting of U_0 to the universal covering $\widetilde{M^3}$ of M^3, with the lifted hyperbolic structure, then a local isometry

$$Dev \;:\; \widetilde{M^3} \;\longrightarrow\; \mathbb{H}^3$$

can be uniquely defined via analytic continuation. This map is called the *developing map* of M^3. It depends only on the choice of (U_0, ϕ_0) and its lifting and, hence, is unique up to a post-composition by an isometry of $\mathbb{H}^3$. Let σ be an element of $\pi_1(M^3, p_0)$ and α be its representative. Then the *holonomy* g_σ of σ is by definition the unigue $g_\sigma \in Isom(\mathbb{H}^3)$ such that $\phi_0^\sigma = g_\sigma \circ \phi_0$. Let T_σ be the deck transformation on $\widetilde{M^3}$ associated to σ, then $Dev \circ T_\sigma = g_\sigma \circ Dev$. From this, one can deduce that the map $\mu : \sigma \mapsto g_\sigma$ is a group homomorphism, called the *holonomy homomorphism*. For M^3 oriented, one has

$$\mu \;:\; \pi_1(M^3) \;\longrightarrow\; PSL(2;\mathbb{C})\,,$$

unique up to the conjugation by an isometry of $\mathbb{H}^3$.

These concepts apply also to general geometric structures e.g. Euclidean, spherical, or affine structures, on a manifold.

• **Ideal tetrahedron.** ([Th1] and [Th4].) An *ideal tetrahedron* in $\mathbb{H}^3$ is a 3-simplex Δ^3 in $\mathbb{H}^3$ inscribed in S^2_∞ such that all its faces are totally geodesic. These faces are by themselves ideal triangles, as indicated in FIGURE 1-2(a). All ideal triangles are isometric to each other. Given an edge e of an oriented ideal tetrahedron Δ^3. Suppose that e is assigned a temporary orientation, then the two faces of Δ^3, with the induced orientation from that of Δ^3, that contains e can be distinguished as "right" if the induced orientation on e from that face is the same as the assigned orientation of e, or "left" if the induced orientation on e from that face is the opposite of the assigned orientation of e. There is then a unique $g \in Isom^+(\mathbb{H}^3)$ that sends the right face to the left face while preserving the two end-points of e. This defines a number $z(e) \in \mathbb{C}$ with $\|z\|$ being the translation distance of g along e and $\arg(z)$ being the angle of rotation of g. If the opposite orientation of e is chosen, then the role of left and right for the two faces that contains e are reversed. Thus g is replaced by g^{-1}, but $z(e)$ remains the same. The complex number $z(e)$ is thus called the *edge invariant* of Δ^3 associated to (unoriented) e. Any of these edge invariants determines the ideal tetrahedron up to an isometry. They correspond to the cross ratio of the four points of inscription when appropriately ordered. We shall denote Δ^3 by $\Delta(z)$, where z is the edge invariant of some edge of Δ^3, and call $\Delta(z)$ *the ideal tetrahedron of modulus* z. (FIGURE 1-2 (a) and (b).)

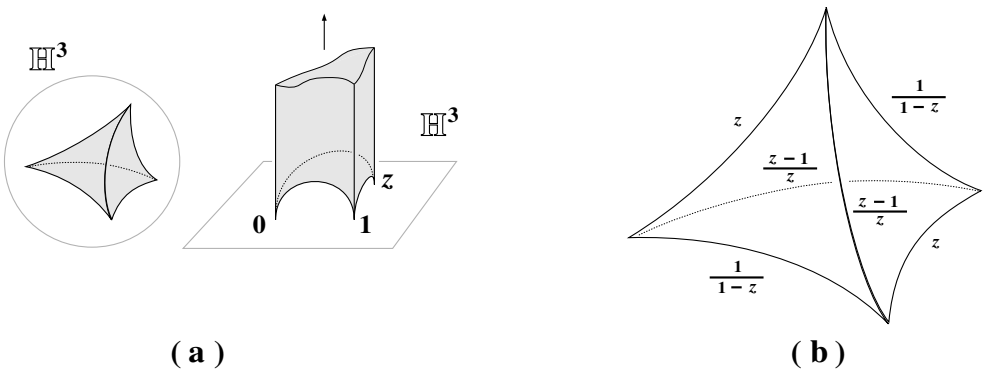


FIGURE 1-2. In (a), an ideal tetrahedron $\Delta(z)$ of modulus z is shown both in the Poincaré ball and the upper half-space model of $\mathbb{H}^3$. In (b), the edge invariants associated to $\Delta(z)$ are indicated.

The importance of understanding ideal tetrahedra first is due to the fact proved by Thurston in [Th4] that *any hyperbolic 3-manifold M^3 of finite volume can be obtained by pasting a collection of ideal tetrahedra along their faces.* We shall call this an *ideal triangulation* of M^3. Since M^3 is assumed to be orientable, one may assume that the moduli of these ideal tetrahedra lie in the upper half-plane $\mathbb{H}_+ = \{z \in \mathbb{C} \,|\, \mathrm{Im}\,(z) > 0\}$. FIGURE 1-3 indicates how an ideal tetrahedron in such triangulation may be embedded in M^3.

• **Dehn filling.** ([Ro] and [Th1].) Given a 3-manifold N^3 with a boundary component

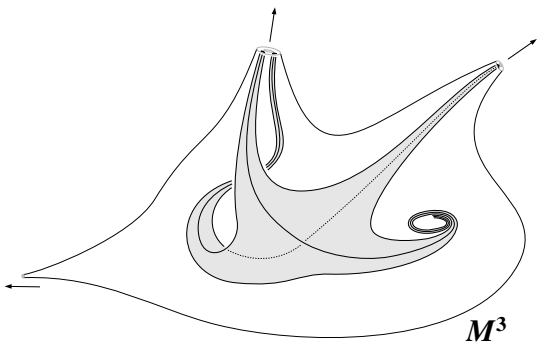


FIGURE 1-3. How an ideal tetrahedron $\Delta(z)$ in an ideal triangulation for a cusped hyperbolic 3-manifold M^3 is indicated. Notice that a leg of $\Delta(z)$ may go into an end of M^3 or wind around a simple closed geodesic in M^3. (A truncated cusp or leg that goes to ∞ is indicated by '$\rightarrow$'.)

$\partial_0 N^3$ a 2-torus $\mathbb{T}^2$. One can fill $\partial_0 N^3$ by sewing a solid torus $D^2 \times S^1$ to N^3 via a homeomorphism h from $\partial(D^2 \times S^1)$ to $\partial_0 N^3$. This procedure is called a *Dehn-filling* of N^3. Define a *meridian* of the solid torus to be any simple loop $\hat{m}$ in $\partial(D^2 \times S^1)$ that bounds in $D^2 \times S^1$. Then the topology of the new manifold $N^3 \cup_h (D^2 \times S^1)$ depends only on the free homotopy class of the image $h(\hat{m})$ in $\partial_0 N^3$. For this reason, suppose that $\partial N^3 = \cup_{i=1}^k \mathbb{T}_i^2$ with fixed basis (m_i, l_i) for $H_1(\mathbb{T}_i^2, \mathbb{Z})$, then a manifold obtained from N^3 by Dehn-filling can be denoted unambiguously by $N^3_{(a_i,b_i;\cdots a_k,b_k)}$, where $(a_i, b_i) = \infty$ if $\mathbb{T}_i^2$ is not filled, or the relatively prime integer pair from the equality $h_i(\hat{m}) = a_i m_i + b_i l_i$ in $H_1(\mathbb{T}_i^2, \mathbb{Z})$ if $\mathbb{T}_i^2$ is filled.

• **Knots and links in S^3.** ([Ki] and [Ro].) Up to free homotopies in $S^3 - K$, the *meridian* m of a knot K in S^3 is the circle that bounds the fiber 2-disk of the normal bundle $\nu(K)$ of K in S^3 and the *longitude* l of K is a knot in S^3 that is parallel to K and generates the 0-framing along K, as indicated in FIGURE 1-4. Note that the linking number $lk(l, K) = 0$, while $lk(m, K) = 1$ when the orientations of m, K, and S^3 are chosen appropriately. All the terminologies and facts used can be found in [Ro].

FIGURE 1-4. The meridian m (thick line) and the longitude l (thin line) of a knot K in S^3 (cf. FIG. 2.2 in [Ki]).

2 The construction of a class of K3T 7-manifolds.

By a *K3T 7-manifold*, we mean a 7-manifold that is fibred over a hyperbolic 3-manifold with generic fiber a K3 surface. In this section, we construct a special class of such 7-manifolds which are either closed or with boundary component the Calabi-Yau threefolds $\mathbb{T}^2 \times$ K3. Some necessary facts for the construction are summarized in Sec. 2.1; the construction is given in Sec. 2.2; and examples are provided in Sec. 2.3.

2.1 The two ingredients: K3 surfaces with involution and hyperbolic 3-manifolds of finite volume.

Some necessary facts for the construction are summarized in this subsection. Readers are referred to [Bo], [G-W], [Ni4], [Vo] for K3 surfaces with antisymplectic involution and to [B-P], [Th1] for hyperbolic 3-manifolds of finite volume.

K3 surfaces with antisymplectic involution..

Let X be a K3 surface with an involution ι that acts by (-1) on the holomorphic 2-form of X. Let Σ be the set of fixed points of ι. Then Σ is a disjoint union of smooth complex curves in X that fall into one of the following three situations classified by Nikulin ([Ni4]):

(1) Σ is empty;

(2) $\Sigma = C_1 \cup C_1'$, where C_1 and C_1' are both elliptic curves;

(3) $\Sigma = C_g + E_1 + \cdots + E_k$. where C_g is a curve of genus g and E_i are rational curves.

And these complex curves descend isomorphically to complex curves (also denoted by Σ) in X/ι. In all cases, the quotient X/ι is a smooth complex surface. The deformation space of the complex structures of such K3 surface was discussed in [Dol] and [G-W] and will be addressed in Sec. 3.2.

A quick tour on hyperbolic 3-manifolds of finite volume.

The Mostow's rigidity theorem says that a complete hyperbolic 3-manifold M^3 of finite volume with a fixed fundamental group is unique up to isometries. However, it turns out to be more fruitful to think of M^3 as obtained from the canonical filling of a compact hyperbolic 3-manifold N^3 with toroidal boundary. For example, one may take N^3 to be the thick part

$$M^3_{[\varepsilon,\infty)} \;=\; \{\, p \in M^3 \,|\, \text{there is a ball of radius } \varepsilon \text{ embedded in } M^3 \text{ with center at } p \,\}$$

of M^3 for some ε small enough. While the hyperbolic structure of M^3 does not allow deformation, the hyperbolic structure on N^3 does and the deformation space has complex

dimension the number of components of ∂N^3 (if one neglects the boundary behaviors). Furthermore, when N^3 is deformed to another hyperbolic 3-manifold $N^{3\prime}$ with boundary, $N^{3\prime}$ can be canonically filled and may lead to another complete hyperbolic 3-manifold $M^{3\prime}$ of finite volume with different topology. This gives a certain hierarchical structure on the space $\mathcal{H}$ of complete hyperbolic 3-manifolds of finite volume. With an appropriate topology defined on $\mathcal{H}$, the volume function *vol* on $\mathcal{H}$ is continuous and finite-to-one. Its image is a countable well-ordered subset (i.e. every subset has a smallest element) in $\mathbb{R}_{>0}$. These are results in the so-called *Jørgensen-Thurston theory* of hyperbolic 3-manifold of finite volume. For our purpose, let us explain more detail of the canonical filling. The deformation space of hyperbolic structures on N^3 will be addressed in Sec. 3.3.

Recall the upper half-space model $\mathbb{C} \times \mathbb{R}_+$ for $\mathbb{H}^3$ from Sec. 1. Let N^3 be a compact hyperbolic 3-manifold with toroidal boundary. The hyperbolic structure on N^3 induces a holonomy map

$$\mu \ : \ \pi_1(N^3) \ \longrightarrow \ PSL(2;\mathbb{C}) \, .$$

Let $\mathbb{T}^2$ be a boundary component of N^3. Then $\mu(\pi_1(\mathbb{T}))$ is an abelian subgroup in $PSL(2;\mathbb{C})$. Thus elements in $\mu(\pi_1(\mathbb{T}^2))$ must share the same set of fixed points on S^2_∞. Furthermore the number of fixed points can only be either one or two.

If $\mu(\pi_1(\mathbb{T}^2))$ has only one fixed point on S^2_∞, after conjugation one may assume that it is ∞. Then $\mu(\pi_1(\mathbb{T}^2))$ is a rank 2 lattice in the translation group of $\mathbb{C}$. The corresponding boundary of N^3 can be filled by the infinite cusp $\mathbb{T}^2 \times \mathbb{R}_+$, geometrically modelled on the quotient

$$\{(z,t) \,|\, z \in \mathbb{C}, t \geq h\}/z \sim z + c_1, z \sim z + c_2$$

where $\{z \sim z + c_1, z \sim z + c_2\}$ is a set of generators of $\mu(\pi_1(\mathbb{T}^2))$ and h depends by the hyperbolic structure of N^3 around that boundary (FIGURE 2-2(a)). Note that this gives an Euclidean structure on the corresponding $\mathbb{T}^2$-boundary of N^3.

If $\mu(\pi_1(\mathbb{T}^2))$ has two fixed points on S^2_∞, after conjugation one may assume that they are 0 and ∞. Then $\mu(\pi_1(\mathbb{T}^2))$ is an abelian subgroup in $GL(1;\mathbb{C}) = \mathbb{C}^\times$. The lifted holonomy map of $\mu|_{\pi_1(\mathbb{T}^2)}$

$$\widetilde{\mu} \ : \ \pi_1(\mathbb{T}^2) \ \longrightarrow \ \widetilde{\mathbb{C}^\times} \, ,$$

where $\widetilde{\mathbb{C}^\times}$ is the universal covering group of $\mathbb{C}^\times$, is now discrete and injective and gives an complex affine structure on the corresponding $\mathbb{T}^2$-boundary of N^3. Let $\mathcal{N}_\varepsilon$ be the closed ε-neighborhood of the t-axis in $\mathbb{H}^3$ with the t-axis deleted and $\widetilde{\mathcal{N}_\varepsilon}$ be the universal covering of $\mathcal{N}_\varepsilon$, equipped with the lifted hyperbolic structure. Then $\widetilde{\mu}(\pi_1(\mathbb{T}^2))$ acts freely on $\widetilde{\mathcal{N}_\varepsilon}$ as a group of isometries. This action extends to the metric completion $\overline{\widetilde{\mathcal{N}_\varepsilon}}$ of $\widetilde{\mathcal{N}_\varepsilon}$. The corresponding boundary of N^3 can then be filled by $\overline{\widetilde{\mathcal{N}_\varepsilon}}/\widetilde{\mu}(\pi_1(\mathbb{T}^2))$ for some ε depending on the hyperbolic structure of N^3 around that boundary. (FIGURE 2-1.)

After performing the above canonical filling to every boundary of N^3, the resulting 3-space W^3 is hyperbolic of finite volume and metrically complete. However, it may acquire some non-manifold point or curvature defect from the second type of filling. Let us next turn to these possible singularities.

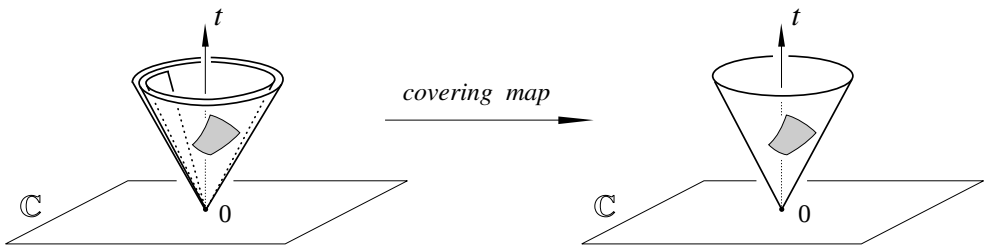


FIGURE 2-1. The deleted neifhboorhood $\mathcal{N}_\varepsilon$ of t-axis and its universal covering $\widetilde{\mathcal{N}_\varepsilon}$, (for clarity, only their boundary is shown). A fundamental domain of $\pi_1(\mathbb{T}^2)$-action on $\partial\widetilde{\mathcal{N}_\varepsilon}$ via $\tilde{\mu}$ and its image on $\partial\mathcal{N}_\varepsilon$ are indicated.

The completion $\overline{\widetilde{\mathcal{N}_\varepsilon}}$ of $\widetilde{\mathcal{N}_\varepsilon}$ is obtained by adding the (lifted) t-axis. The induced action of $\widetilde{\pi_1(\mathbb{T}^2)}$ on the t-axis induces a group homomorphism

$$\hat{\mu} \ : \ \pi_1(\mathbb{T}^2) \ \longrightarrow \ (\mathbb{R}_{>0}, \times) \, .$$

There are two situations:

- *Case (i)* : $\hat{\mu}(\pi_1(\mathbb{T}^2))$ *is dense in* $\mathbb{R}_{>0}$. Then $\overline{\widetilde{\mathcal{N}_\varepsilon}}/\tilde{\mu}(\pi_1(\mathbb{T}^2))$ is the one-point compactification of $\widetilde{\mathcal{N}_\varepsilon}/\tilde{\mu}(\pi_1(\mathbb{T}^2))$. The corresponding boundary in N^3 is filled by a cone over $\mathbb{T}^2$ (FIGURE 2-2(b)).

- *Case (ii)* : $\hat{\mu}(\pi_1(\mathbb{T}^2))$ *is discrete in* $\mathbb{R}_{>0}$. Then the completion $\overline{\widetilde{\mathcal{N}_\varepsilon}}/\tilde{\mu}(\pi_1(\mathbb{T}^2))$ is topologically a solid torus. The difference $\overline{\widetilde{\mathcal{N}_\varepsilon}}/\tilde{\mu}(\pi_1(\mathbb{T}^2)) - \widetilde{\mathcal{N}_\varepsilon}/\tilde{\mu}(\pi_1(\mathbb{T}^2))$ is the loop $\gamma = t\text{-axis}\,/\hat{\mu}(\pi_1(\mathbb{T}^2))$. Let λ_1 generate the kernel of $\hat{\mu}$ and λ_2 generate the image of $\hat{\mu}$. Then γ has length $|\log|\lambda_2||$ and the normal cross section of γ in the completion is a 2-dimensional hyperbolic cone D^2_θ of cone angle $\theta = |\arg\lambda_1|$ (FIGURE 2-2(c) and (d)). Hence, when $\theta \neq 2\pi$, there are curvature defects along γ.

Remark 2.1.1 [hyperbolic Dehn surgery]. Recall the procedure

$$M^3 \ \rightarrow \ N^3 \ \rightarrow \ \text{deforming } N^3 \ \rightarrow \ \text{canonical filling} \ \rightarrow \ M^{3\prime} \, .$$

from a complete hyperbolic M^3 of finite volume at the start of the tour. Assume that the deformed N^3 is filled either by infinite cusps or solid torus in Case (ii) with $\theta = 2\pi$. Then the underlying topology of $M^{3\prime}$ is a 3-manifold obtained from N^3 by performing a Dehn filling on the boundary. Such $M^{3\prime}$ is said to be obtained from M^3 by a hyperbolic surgery. The following fact is due to Thurston ([Th1], see also [B-P]):

Fact 2.1.1.1 [hyperbolic surgery theorem]. *Let M^3 be a complete hyperbolic 3-manifold of finite volume with h cusps. Fixed a basis (m_i, l_i) for $H_1(\mathbb{T}^2_i; \mathbb{Z})$, where $\mathbb{T}^2_i$ is*

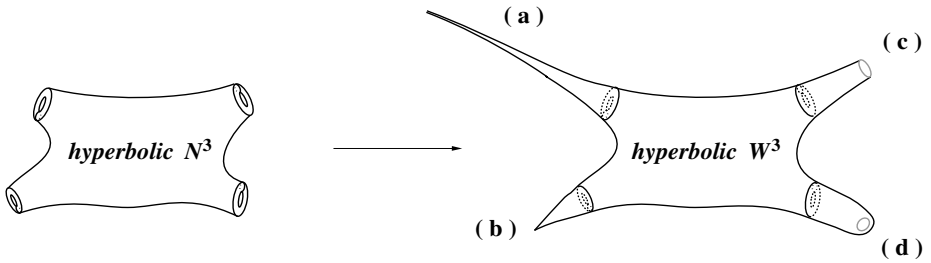


FIGURE 2-2. The boundary of a compact hyperbolic 3-manifold N^3 with toroidal boundary can be canonically filled by (a) a cusp, (b) a cone, (c) a solid torus with curvature defects along the core, or (d) a nice hyperbolic solid torus. The result is denoted by W^3 in the figure. In Cases (c) and (d), the core loop is indicated by a grey loop.

the 2-torus associated to the i-th cusp. Let $M^3_{(p_1,q_1;\cdots;p_h,q_h)}$ *be the 3-manifold obtained by performong a* (p_i,q_i)*-Dehn surgery on the i-th cusp, where* (p_i,q_i) *is a pair of coprime integers or the symbol* ∞ *if the i-th cusp is left unsurgered. Then* $M^3_{(p_1,q_1;\cdots;p_h,q_h)}$ *admits a hyperbolic structure if all* (p_i,q_i) *are close to* ∞ *in* $(\mathbb{Z}\oplus\mathbb{Z})\cup\{\infty\}$.

This is an important theorem that gives a hierarchical structure on th space $\mathcal{H}$ of hyperbolic 3-manifolds of finite volume and is good to keep in mind when considering such 3-manifolds.

2.2 The construction of K3T 7-manifolds by bundle-filling.

Let M^3 be a complete hyperbolic 3-manifold of finite volume, X be a K3 surface with antisymplectic involution ι, and Σ be the set of fixed points of ι. Let $\Gamma = \pi_1(M^3)$ and $\rho_X : \Gamma \to \langle\,\iota\,\rangle = \mathbb{Z}_2$ be a representation. Then the universal covering $\widetilde{M^3}$ of M^3 is $\mathbb{H}^3$ and $(\Gamma, \rho_X(\Gamma))$ acts freely on $\mathbb{H}^3 \times X$ by the diagonal action. Consequently, one obtains a K3T 7-manifold

$$\pi \,:\, M^7 \,=\, M^3 \times_{\rho_X} X \,=\, (\mathbb{H}^3 \times X)/(\Gamma, \rho_X(\Gamma)) \,\longrightarrow\, M^3\,.$$

In this case, π is a K3-bundle over M^3. Note that ρ_X can be regarded as an element in $H^1(M^3;\mathbb{Z}_2)$ and vice versa. The isometry group of M^3 (in general it is trivial) acts on $H^1(M^3;\mathbb{Z}_2)$, the quotient labels then a class of K3T 7-manifolds with fiber X that are naturally associated to M^3. Thurston's work on hyperbolic 3-manifolds of finite volume suggests a way to construct a family of K3T descendants from any such π by "bundle-filling", as we shall now explain.

The basic idea.

Regard M^3 as a filling of some compact N^3 with toroidal boundary. The hyperbolic structure on N^3 can be regarded as a collection of local hyperbolic charts

$$\{\, \varphi_\alpha \,:\, U_\alpha \,\longrightarrow\, \mathbb{H}^3 \,|\, \alpha \in \text{ some index set } I \,\}\,,$$

where $\{U_\alpha \,|\, \alpha \in I\}$ is a covering of N^3. Deformation of hyperbolic structures on N^3 can then be thought of as shufflings of these local charts. This leads to alteration of the transition functions $\{\varphi_{\alpha\beta}\}$ between charts and also the corresponding representation $\pi_1(N^3) = \Gamma \to PSL\,(2;\mathbb{C})$.

Now choose $\{U_\alpha \,|\, \alpha \in I\}$ to be also a local trivialization of π. Let $(\varphi_{\alpha\beta}, h_{\alpha\beta})$ be the transition function between $U_\alpha \times X$ and $U_\beta \times X$. Then, by construction, one can choose $h_{\alpha\beta}$ to be either the identity map Id_X or the involution ι on X. When one deforms the hyperbolic structure on N^3, $\varphi_{\alpha\beta}$ get changed but the corresponding $h_{\alpha\beta}$ can be left the same. This leads to a new bundle π' topologically equivalent to $\pi|_{N^3}$. Let $M^{3\prime}$ be the filling of the deformed N^3, following Thurston. As will be explained below, the filling of N^3 to $M^{3\prime}$ induces a filling of $\pi|_{N^3}$ to a K3-fibration $\pi' : W^7 \to M^{3\prime}$ in a way that depends only on the hyperbolic structure on N^3 and ρ_X. By understanding these fillings and how the resulting singularities of W^7 can be resolved, one obtains a K3T 7-manifold $M^{7\prime}$ that descends from π.

The fibration over a boundary $\mathbb{T}^2$ of N^3 and its filling.

With notations from earlier discussions, let $\mathbb{T}^2$ be a boundary component of the deformed N^3. Recall μ, $\widetilde{\mu}$, and $\hat{\mu}$ from Sec. 2.1.

If $\mu(\pi_1(\mathbb{T}^2))$ has only one fixed point on S^2_∞, this boundary $\mathbb{T}^2$ is filled to a cusp $\mathcal{C}' = \mathbb{T}^2 \times \mathbb{R}_+$ in $M^{3\prime}$. Let $\mathcal{C}$ be the corresponding cusp in M^3, then the restriction of π' to a collar of $\mathbb{T}^2$ in N^3 extends to $\pi'|_{\mathcal{C}'}$ that is the pullback of $\pi|_{\mathcal{C}}$ via a quasi-isometry from $\mathcal{C}'$ to $\mathcal{C}$. In this case, $\pi'|_{\mathcal{C}'}$ is topologically equivalent to $\pi|_{\mathcal{C}}$.

If $\mu(\pi_1(\mathbb{T}^2))$ has two fixed points on S^2_∞, then the restriction of π' over a collar of $\mathbb{T}^2$ in N^3 extends to

$$\pi'|_{\overline{\widetilde{\mathcal{N}_\varepsilon}}/\widetilde{\mu}(\pi_1(\mathbb{T}^2))} \;:\; \{\overline{\widetilde{\mathcal{N}_\varepsilon}} \times X\}/(\widetilde{\mu}(\pi_1(\mathbb{T}^2)),\, \rho_X(\pi_1(\mathbb{T}^2))) \;\longrightarrow\; \overline{\widetilde{\mathcal{N}_\varepsilon}}/\widetilde{\mu}(\pi_1(\mathbb{T}^2))\,,$$

where the group action is diagonal. Corresponding to Cases (i) and (ii) in Sec. 2.1, one has respectively

- *Case (i')*: $\hat{\mu}(\pi_1(\mathbb{T}^2))$ *is dense in* $\mathbb{R}_{>0}$. Recall that $\overline{\widetilde{\mathcal{N}_\varepsilon}}/\widetilde{\mu}(\pi_1(\mathbb{T}^2))$ is the one-point compactification of $\widetilde{\mathcal{N}_\varepsilon}/\widetilde{\mu}(\pi_1(\mathbb{T}^2))$ by, say, p_*. Then the fiber of π' over p_* is $X_* = X/\rho_X(\pi_1(\mathbb{T}^2))$. If $\rho_X(\pi_1(\mathbb{T}^2)) = \langle\iota\rangle$, then $X_* = X/\iota$ is an exceptional fiber of multiplicity 2. If $\rho_X(\pi_1(\mathbb{T}^2)) = \{Id_X\}$, then the extended π' over $\overline{\widetilde{\mathcal{N}_\varepsilon}}/\widetilde{\mu}(\pi_1(\mathbb{T}^2))$ is a trivial fibration and $X_* = X$ is a regular fiber.

- *Case (ii′)*: $\hat{\mu}(\pi_1(\mathbb{T}^2))$ *is discrete in* $\mathbb{R}_{>0}$. Recall the core loop γ of $\overline{\widetilde{\mathcal{N}_\varepsilon}}/\tilde{\mu}(\pi_1(\mathbb{T}^2))$. Then the fiber of π' over γ is $X_c = X/\rho_X(\lambda_1)$. If $\rho_X(\lambda_1) = \iota$, then $X_c = X/_\iota$ is an exceptional fiber of multiplicity 2. If $\rho_X(\lambda_1) = Id$, then $X_c = X$ is a regular fiber.

Thus, after filling $\pi'|_{N^3}$ over all the boundary $\mathbb{T}^2$ of N^3, one obtains a flat K3-fibration

$$\pi' : W^7 \longrightarrow M^{3\prime}.$$

By construction, if one regards M^3 as embedded in $M^{3\prime}$, then the monodromy of $\pi'|_{M^3}$ is the same as the monodromy of π. This W^7 in general is not a manifold. Let us now turn to its singularities.

The singular locus of W^7 and its resolution.

Let S be the singular set S of W^7, that consists of all the non-manifold points of W^7. Then, from previous discussion, S sits only over the difference $M^{3\prime} - M^3$, which is a disjoint union of finitely many points p_* and simple loops γ obtained from filling the deformed hyperbolic structure of N^3. Let S_0 be a component of S and $\nu(\,\cdot\,)$ be the tubular neighborhood of a subset in W^7. We shall discuss first the topology of $\nu(\pi'^{-1}(p_*))$ and $\nu(\pi'^{-1}(\gamma))$, and then S_0 and how S_0 may be resolved.

In the following discussions, $\mathbb{T}^2$ is the boundary component of N^3 involved. Also, we shall denote *the cone over a base* B by $Cone(B)$ and the 2*-dimensional hyperbolic cone with cone angle* θ by D^2_θ.

The two cases are as follows:

Case (a): *Over* p_* *in Case (i′) above.* There are two situations:

1. $\rho_X(\pi_1(\mathbb{T}^2)) = \{Id_X\}$. Then $\pi'^{-1}(p_*) = X$ and $\nu(\pi'^{-1}(p_*)) = X \times Cone(\mathbb{T}^2)$. Thus $S_0 = \pi'^{-1}(p_*) = X$.

2. $\rho_X(\pi_1(\mathbb{T}^2)) = \langle\iota\rangle$. Then $\pi'^{-1}(p_*) = X/_\iota$. Let $\lambda_0 \in \pi_1(\mathbb{T}^2)$ generate the image of ρ_X. Then there is a double covering $\kappa : \hat{\mathbb{T}}^2 \to \mathbb{T}^2$ such that $\rho_X \circ \kappa_*(\pi_1(\hat{\mathbb{T}}^2))$ is trivial. Let h be the nontrivial deck transformation on $\hat{\mathbb{T}}^2$ associated to κ. Then $\partial\nu(\pi'^{-1}(p_*)) = (\hat{\mathbb{T}}^2 \times X)/(h,\iota)$, which is fibered over $\pi'^{-1}(p_*)$ with generic fiber $\hat{\mathbb{T}}^2$ and exceptional fiber $\mathbb{T}^2 = \hat{\mathbb{T}}^2/h$ of multiplicity 2 over Σ. Consequently, $\nu(\pi'^{-1}(p_*))$ is a $Cone(\hat{\mathbb{T}}^2)$-fibration over $X/_\iota$. with exceptional $Cone(\mathbb{T}^2)$-fiber of multiplicity 2 over Σ. The monodromy of the fibration around the loop $[\iota]$, which generates the orbifold fundamental group $\pi_1^{\rm orb}(S_0)$, is the extension of h on $\hat{\mathbb{T}}^2$ to $Cone(\hat{\mathbb{T}}^2)$. Thus $S_0 = \pi'^{-1}(p_*) = X/_\iota$.

Case (b): *Over* γ *in Case (ii′) above.* There are three situations:

1. $\rho_X(\pi_1(\mathbb{T}^2)) = \{Id_X\}$: Then $\pi'^{-1}(\gamma) = X \times \gamma$ and $\nu(\pi'^{-1}(\gamma)) = (X \times \gamma) \times D^2_{|\arg\lambda_1|}$. Thus S_0 is empty.

2. $\rho_X(\lambda_1) = \iota$ and $\rho_X(\lambda_2) = Id_X$: Then $\pi'^{-1}(\gamma) = X/_\iota \times \gamma$. Let $\mathcal{C} = \overline{\widetilde{\mathcal{N}_\varepsilon}}/\widetilde{\mu}(\pi_1(\mathbb{T}^2))$. Then there is a double branched covering $\kappa : \hat{\mathcal{C}} \to \mathcal{C}$ branched over the core γ of $\mathcal{C}$ such that the pullback fibration $\kappa^*(\pi'|_{\mathcal{C}})$ is trivial over $\hat{\mathcal{C}}$. The total space of $\kappa^*(\pi'|_{\mathcal{C}})$ is $\hat{\mathcal{C}} \times X$. Let h be the nontrivial deck transformation on $\hat{\mathcal{C}}$ associated to κ. Since $\nu(\pi'^{-1}(\gamma))$ is the total space of $\pi'|_{\mathcal{C}}$, one has $\nu(\pi'^{-1}(\gamma)) = (\hat{\mathcal{C}} \times X)/(h, \iota)$, where the action is diagonal. Thus $\nu(\pi'^{-1}(\gamma))$ fibers over $\pi'^{-1}(\gamma)$ with generic fiber $D^2_{2|\arg\lambda_1|}$. The exceptional fibers lie over $\Sigma \times \gamma$ and each is isometric to $D^2_{|\arg\lambda_1|}$, with multiplicity 2. This shows that $\nu(\pi'^{-1}(\gamma))$ has transverse A_1-singularities along $\Sigma \times \gamma$ and that $S_0 = \Sigma \times \gamma$.

3. $\rho_X(\lambda_1) = Id_X$ and $\rho_X(\lambda_2) = \iota$: Then $\pi'^{-1}(\gamma) = X_\iota$, the mapping torus of X associated to ι. Analogous to the previous situation, there is a double covering $\kappa : \hat{\mathcal{C}} \to \mathcal{C}$, induced by a double covering $S^1 \times \gamma$, such that the pullback fibration $\kappa^*(\pi'|_{\mathcal{C}})$ is trivial over $\hat{\mathcal{C}}$. The total space of $\kappa^*(\pi'|_{\mathcal{C}})$ is $\hat{\mathcal{C}} \times X$. Let h be the nontrivial deck transformation on $\hat{\mathcal{C}}$ associated to κ. Then $\nu(\pi'^{-1}(\gamma)) = (\hat{\mathcal{C}} \times X)/(h, \iota)$, where the action is diagonal. Thus $\nu(\pi'^{-1}(\gamma))$ is a regular fiberation over $\pi'^{-1}(\gamma)$ with fiber $D^2_{|\arg\lambda_1|}$ and S_0 is empty.

   Note that, for the situation $\rho_X(\lambda_1) = \rho_X(\lambda_2) = \iota$, one can replace λ_2 by $\lambda_2 - \lambda_1$ and render it Situation (2) above.

Thus, if Case (i′) happens for some boundary of N^3, then, from Case (a), W^7 can never be a manifold. Nor is there any known standard way to resolve such singularities.

On the other hand, if Case (ii′) happens for all the boundary components of N^3, both Case (b-1) and Case (b-3) above lead only to manifold-points in W^7, while the A_1-singularities in Case (b-2) can be resolved by transverse blowups along $S_0 = \Sigma \times \gamma$ (cf. Remark 2.2.1). After, resolving the singularities, one obtains then a K3T 7-manifolds

$$\widetilde{\pi}' \ : \ M^{7'} \ \longrightarrow \ M^{3'}.$$

When Case (b-2) happens, the exceptional fiber of $\widetilde{\pi}'$ over a point in the corresponding γ is then $X/_\iota \cup (\Sigma \times \mathbb{C}\mathrm{P}^1)$, where the two components intersect along Σ. Over the complement of such γ, $\widetilde{\pi}'$ is the same as π'.

Remark 2.2.1 [hierarchy of K3T]. For a fixed (X, ι) and ρ_X, the space W^7 obtained by bundle-filling is determined by the hyperbolic structure on N^3. Thus the associated space of K3T 7-manifolds also exhibit a hierarchical structure inherited from that on the space of hyperbolic 3-manifolds of finite volume. However, notice that, when Case (b-2) happens, the set of isotopy classes of identifications of $\hat{\mathcal{C}}$ with $\gamma \times \mathbb{C}$ is parametrized by $\pi_1(GL(1;\mathbb{C})) = \mathbb{Z}$. Non-isotopic identifications may lead to non-homeomorphic $M^{7'}$ from the same W^7.

2.3 Examples from the link complements in S^3.

In [Th4], Thurston showed that the interior of a compact 3-manifold M^3 is hyperbolic if and only if M^3 is prime, homotopically atoroidal, and not homeomorphic to the quotient $(\mathbb{T}^2 \times I)/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ acts on $\mathbb{T}^2$ as the covering group of $\mathbb{T}^2$ over the Klein bottle and on the interval $I = [0,1]$ by the reflection with respect to $\frac{1}{2}$. A corollary of this is that a knot K in S^3 is hyperbolic if and only if K is neither a satellite nor a torus knot ([Th2]). Combined with Fact 2.1.1.1 in Sec. 2.1 and the construction in Sec. 2.2, this shows that there are abundant of nontrivial closed K3T 7-manifolds.

To construct nontrivial K3T 7-manifolds that have the Calabi-Yau threefold $\mathbb{T}^2 \times X$ as a boundary component, let L be a link of k many components in S^3 such that its complement $S^3 - L$ admits a complete hyperbolic structure. From the theorem of Thurston mentioned in the beginning of the previous paragraph, there are plenty of such L in S^3 and their complement provide us with basic examples of complete hyperbolic 3-manifolds of finite volume. It is a basic fact, following the Alexander duality and the universal coefficient theorem [Mun], that $H^1(S^3 - L; \mathbb{Z}_2) = \oplus_k \mathbb{Z}_2$ is generated by taking the $\mathbb{Z}_2$-reduction of the linking number with respect to a component of L. Thus there are 2^k-many homomorphisms ρ_X from $\pi_1(S^3 - L)$ to $\langle \iota \rangle$. From the Mostow's rigidity theorem, unless $S^3 - L$ admits a nontrivial group of isometries, distinct ρ_X gives rise to non-isomorphic K3-bundles

$$\pi \ : (S^3 - L) \times_{\rho_X} X \ \longrightarrow \ M^3 = S^3 - L \, .$$

Applying the deformation and filling in Sec. 2.2 to π, one generates then many other examples of K3T 7-manifolds. By choosing ρ_X so that its restriction to the unsurgered $\mathbb{T}^2$-boundary of $S^3 - L$ is trivial, one then obtains many examples of nontrivial K3T 7-manifolds with boundary component $\mathbb{T}^2 \times X$.

Let us now give some examples to illuminate the discussions. The detail of the hyperbolic structure of the knot/link complements that appear in these examples is in [Th1].

Example 2.3.1 [closed K3T via figure-8 knot]. Let K be the *figure-8 knot* in S^3, as shown in FIGURE 2-3-1.

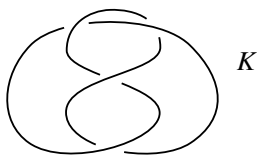


FIGURE 2-3-1. The figure-8 knot in S^3.

Then $S^3 - K$ is hyperbolic and $H^1(S^3 - K; \mathbb{Z}_2) = \mathbb{Z}_2$. Let m be the meridian and l be the longitude of K. Thurston proves that (Theorem 4.7 in [Th1]) every manifold obtained by Dehn surgery along figure-8 knot has a hyperbolic structure, except the six manifolds $(S^3 - K)_{(\pm a, \pm b)}$ where $(a,b) = (1,0), (0,1), (1,1), (2,1), (3,1)$, or $(4,1)$.

The two elements in $H^1(S^3 - K; \mathbb{Z}_2)$ are

$$\begin{aligned} \rho_0 &: m \longmapsto Id_X\,, \quad (\, l \longmapsto Id_X\,) \quad \text{and} \\ \rho_1 &: m \longmapsto \iota\,, \quad (\, l \longmapsto Id_X\,)\,. \end{aligned}$$

For ρ_0, $(S^3 - K) \times_{\rho_0} X = (S^3 - K) \times X$ and the deform-and-fill procedure applied to $(S^3 - K) \times X$ only yields K3T 7-manifolds in the product form $M^{3\prime} \times X$.

For ρ_1, if $a \equiv 0 \pmod 2$, then either Case (b-1) or Case (b-3) happens and the deform-and-fill procedure applied to $(S^3 - K) \times_{\rho_1} X$ yields directly closed K3T 7-manifolds. If $a \equiv 1 \pmod 2$, then Case (b-2) happens and the deform-and-fill applied to $(S^3 - K) \times_{\rho_1} X$ yields K3T 7-spaces with A_1-singularities. After blowups, this yields also closed K3T 7-manifolds.

□

Example 2.3.2 [K3T with boundary $\mathbb{T}^2 \times X$ via Whitehead link]. Let $L = K_1 \cup K_2$ be the *Whitehead link* in S^3, as shown in FIGURE 2-3-2. Then $S^3 - L$ is hyperbolic and

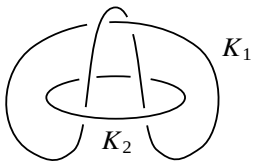


FIGURE 2-3-2. The Whitehead link in S^3.

$H^1(S^3 - L; \mathbb{Z}_2) = \mathbb{Z}_1 \oplus \mathbb{Z}_2$. From Fact 2.1.1.1, $(S^3 - K)_{(a_1,b_2;a_2,b_2)}$ is hyperbolic for $a_1^2 + b_1^2$, $a_2^2 + b_2^2$ large enough.

The elements in $H^1(S^3 - L; \mathbb{Z}_2)$ are

$$\rho_{\epsilon_1\epsilon_2} \,:\, m_i \longmapsto \iota^{\epsilon_i}\,,\ (\, l_i \longrightarrow Id_X\,)\,, \quad i = 1, 2\,,$$

where ϵ_1, ϵ_2 is 0 or 1. To obtain a K3T 7-manifold with boundary $\mathbb{T}^2 \times X$, one considers either $(S^3 - L) \times_{\rho_{01}} X$ with $(a_1, b_1) = \infty$ or $(S^3 - L) \times_{\rho_{10}} X$ with $(a_2, b_2) = \infty$. In the former case, if $a_2 \equiv 0 \pmod 2$, then either Case (b-1) or Case (b-3) happens and the deform-and-fill procedure applied to $(S^3 - K) \times_{\rho_{01}} X$ yields directly a K3T 7-manifold with boundary $\mathbb{T}^2 \times X$. If $a_2 \equiv 1 \pmod 2$, then Case (b-2) happens and the deform-and-fill applied to $(S^3 - K) \times_{\rho_{01}} X$ yields a K3T 7-space with A_1-singularities. After blowups, this yields also a K3T 7-manifold with boundary $\mathbb{T}^2 \times X$. Similarly for the latter case.

□

Example 2.3.3 [K3T hierarchy]. Let K be the figure-8 knot in Example 2.3.1 and L be the Whitehead link in Example 2.3.2. With the same notations as in the corresponding examples, $(S^3 - K)_{(a,b)}$ is homeomorphic to $(S^3 - L)_{(a,b;1,-1)}$ ([Ro]), cf. FIGURE 2-3-3. Consequently, $(S^3 - K) \times_{\rho_1} X$ is bundle-isomorphic to the bundle-filling of $(S^3 - L) \times_{\rho_{10}} X$

with the Dehn surgery coefficient $(a, b; \infty)$. Any K3T 7-manifold constructed from $(S^3 - K) \times_{\rho_1} X$ by deform-and-fill can also be constructed from $(S^3 - L) \times_{\rho_{10}} X$ by deform-and-fill. Thus the K3T 7-manifold $(S^3 - K) \times_{\rho_1} X$ and its associated K3T family are all descendants of $(S^3 - L) \times_{\rho_{10}} X$.

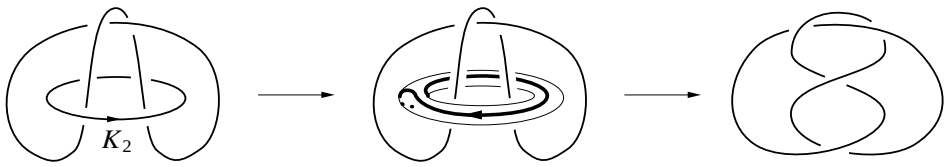


FIGURE 2-3-3. The figure-8 knot complement in S^3 is obtainable from the the Whitehead link complement in S^3 by the $(1, -1)$-surgery along K_2.

□

Example 2.3.4 [general K3T in our class]. Let $L = K_1 \cup \cdots \cup K_i \cup \cdots \cup K_k$ be a link in S^3 such that $S^3 - L$ is hyperbolic. Examples of such links from [Th1] are illustrated in FIGURE 2-3-4. Let (m_i, l_i) be the meridian and the longitute pair of K_i and $C = [\, c_{ij} \,]_{ij} = [\, lk(K_i, K_j) \,]_{ij}$ be the linking matrix of L (with respect to some orientation of L and S^3, which does not enter the discussion after mod 2). Then the manifold $(S^3 - L)_{(a_1,b_1;\cdots;a_i,b_i;\cdots;a_k,b_k)}$ is hyperbolic if all $a_i^2 + b_i^2$ are large.

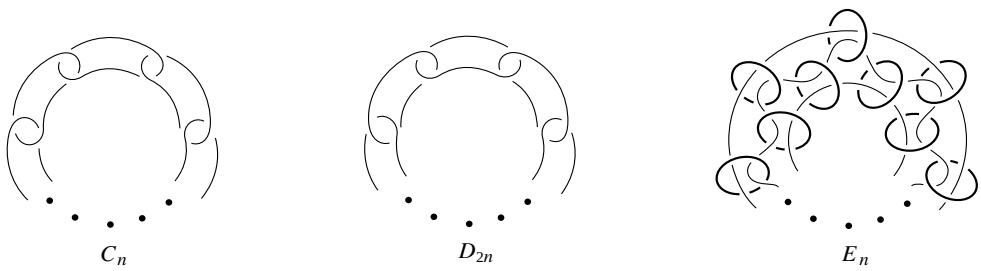


FIGURE 2-3-4. Some examples of hyperbolic links in S^3 from [Th1]. For the notation, the link C_n (resp. D_{2n}, E_n) has n (resp. $2n$, $2n + 3$) components.

The elements in $H^1(S^3 - L; \mathbb{Z}_2)$ are given by

$$\rho_{\epsilon_1 \cdots \epsilon_k} \,:\, m_i \longmapsto \iota^{\epsilon_i} \,,\ (\, l_i \longrightarrow \iota^{\Sigma_{j=1}^k c_{ij}\epsilon_j} \,) \,, \quad i = 1, \ldots, k \,,$$

where $\epsilon_1, \cdots, \epsilon_k$ are either 0 or 1. Let $C_{(2)}$ be $C \pmod 2$, $\mathcal{V}$ be the module $\oplus_k \mathbb{Z}_2$, $\mathcal{V}_{i_1,\cdots,i_h}$, $1 \le i_1 < \cdots < i_h \le k$, be the codimension h submodule that consists of points in $\mathcal{V}$ whose i_s-coordinate, $s = 1, \ldots h$ are 0, and $\mathbb{T}_i$ be the i-th boundary component of

$S^3-\nu(L)$ associated to K_i. Then, the condition that there is a nontrivial $\rho_{\epsilon_1,\cdots\epsilon_k}$ such that $\rho_{\epsilon_1,\cdots\epsilon_k}(\pi_1(\mathbb{T}_j))$ is trivial for $j\in\{i_1,\cdots i_h\}$ is that $\mathcal{V}_{i_1,\cdots,i_h}\cap C_{(2)}\,\mathcal{V}_{i_1,\cdots,i_h}\neq(0,\cdots,0)$. For such $\rho_{\epsilon_1\cdots\epsilon_k}$, one can choose (a_i,b_i) to be ∞ for $i\in\{i_1,\cdots i_h\}$ and some integer pair with large enough $a_i^2+b_i^2$ otherwise. The corresponding deform-and-fill procedure applied to π gives then a nontrivial K3T 7-space

$$\pi' \,:\, W^7 \;\longrightarrow\; (S^3-L)_{(a_1,b_1;\cdots;a_k,b_k)}\,.$$

Since $\rho_{\epsilon_1,\cdots\epsilon_k}(a_im_i+b_il_i)=\iota^{a_i\epsilon_i+b_i\sum_{j=1}^n c_{ij}\epsilon_j}$, one has:

- if $a_i\epsilon_i+b_i\sum_{j=1}^n c_{ij}\epsilon_j\equiv 0\ (\mathrm{mod}\,2)$, then either Case (b-1) or Case (b-3) happens and it contributes no singularity to W^7; and
- if $a_i\epsilon_i+b_i\sum_{j=1}^n c_{ij}\epsilon_j\equiv 1\ (\mathrm{mod}\,2)$, then Case (b-2) happens and this contributes A_1-singularities to W^7, which can be resolved by blowups.

In the end, this yields a K3T 7-manifold

$$\widetilde{\pi}' \,:\, M^{7'} \;\longrightarrow\; (S^3-L)_{(a_1,b_1;\cdots;a_k,b_k)}$$

with Calabi-Yau boundary $\cup_h\,\mathbb{T}^2\times X$.

For the given three series of hyperbolic links C_n, D_{2n}, and E_n, with respect to appropriate labelling of components of L, their $\mathbb{Z}_2$ linking matrices $C_{(2)}$ are respectively

$$\begin{bmatrix} 0&1&0&\cdots&0&1\\ 1&0&1& & &0\\ 0&1&0& & & \\ \vdots& &\ddots&\ddots& &\vdots\\ & & & &1&0\\ 0& &\cdots&1&0&1\\ 1&0&\cdots&0&1&0 \end{bmatrix}_{n\times n},\quad \begin{bmatrix} 0&1&0&\cdots&0&1\\ 1&0&1& & &0\\ 0&1&0& & & \\ \vdots& &\ddots&\ddots& &\vdots\\ & & & &1&0\\ 0& &\cdots&1&0&1\\ 1&0&\cdots&0&1&0 \end{bmatrix}_{2n\times 2n},\quad \text{and}$$

$$\left[\begin{array}{ccc|ccccccc} & & & 1&\cdots&1&0&\cdots&0\\ \multicolumn{3}{c|}{\mathrm{O}_{3\times 3}} & 1&\cdots&1&1&\cdots&1\\ & & & 0&\cdots&0&1&\cdots&1\\ \hline 1&1&0 & & & & & & \\ \vdots&\vdots&\vdots & & & & & & \\ 1&1&0 & & & & & & \\ 0&1&1 & & & \multicolumn{4}{c}{\mathrm{O}_{2n\times 2n}} \\ \vdots&\vdots&\vdots & & & & & & \\ 0&1&1 & & & & & & \end{array}\right],$$

where $\mathrm{O}_{s\times s}$ is the $s\times s$ zero-matrix. By choosing n large enough, one can guarantee the existence of nontrivial $\mathbb{Z}_2$-solution to the system of homogeneous equations on $(\epsilon_1,\cdots\epsilon_k)$: $\epsilon_i\equiv\sum_{j=1}^k c_{ij}\epsilon_j\equiv 0\ (\mathrm{mod}\,2)$ with $i\in\{i_1,\cdots,i_h\}$. Following the previous discussion, one then obtains nontrivial K3T 7-manifolds with boundary $\cup_h\,\mathbb{T}^2\times X$ for arbitrary h.

□

3 The deformation space of K3T 7-manifolds.

Having in mind the potential application of K3T 7-manifolds to the compactification of M-theory, which involves 11-dimensional space-time, we study in this section the deformation space of the K3T 7-manifolds constructed in Sec. 2. If the application of K3T to M-theory can be carried out solidly, the deformation space of K3T should be related to the scalar fields or their moduli space in the associated effective 4-dimensional theory.

3.1 The deformation space of K3T 7-manifolds constructed.

Recall the K3T 7-manifold

$$\widetilde{\pi}' \,:\, M^7 \;\longrightarrow\; M^{3'}\,.$$

By construction, $\widetilde{\pi}'$ admits a flat connection over the complement of its set of critical values. The complex structure of X determines a complex structure on a generic fiber of $\widetilde{\pi}'$. The horizontal foliation by the flat connection provides a transverse hyperbolic structure with respect to $\widetilde{\pi}'$, determined by the hyperbolic structure of $N^3 \subset M^{3'}$. Deformations of the complex structures on X and the hyperbolic structures on N^3 induce deformations of the fiber complex structures and the tranverse hyperbolic structures of $\widetilde{\pi}'$. Since these two deformations are independent from each other, the deformation space of the K3T 7-manifolds constructed in Sec. 2 exhibits thus a decomposition

$$Def(\,\mathrm{K3T}\,) \;=\; Def(X,\iota) \,\times\, Def(N^3) \,\times H^1(N^3,\mathbb{Z}_2)\,,$$

where the last component $H^1(N^3,\mathbb{Z}_2)$ corresponds to the twistings by ρ_X.

It is known that $Def(X,\iota)$ admits a Kähler structure. And, as a corollary of works by many people, $Def(N^3)$ admits a special Kähler structure. In view of the goal that motivates us the current work - compactification of M-theory on a K3T 7-manifold -, the appearance of these structures is a very welcome feature.

Let us now discuss these structures in more detail.

3.2 The K3 part of deformations of K3T.

The deformation space for K3 surfaces has been studied intensively by several authors. In this section, we summarize the results that are related to the current work. Readers are referred to [As], [Bo], [B-P-VV], [Dol], [G-W], [Ni4], [Ti], and [Vo] for more details.

Let L be the lattice $(-E_8)\oplus(-E_8)\oplus H\oplus H\oplus H$ of signature $(3,19)$ and X be a K3 surface. Then the K3 lattice $H^2(X;\mathbb{Z})$ with the cup product is isomorphic to L. The choice of an isometry $\phi : H^2(X;\mathbb{Z}) \to L$ determines a point in $P(L_{\mathbb{C}})$, corresponding to the complex line $\phi_{\mathbb{C}}(H^{2,0}(X;\mathbb{C}))$. This point is called the period point of the marked K3 surface (X,ϕ) and the set of all period points for a class of marked K3 surfaces is called the period domain of that class. It gives the deformation space of the complex structures of the K3 surfaces in that class.

Assume from now on that X admits an antisymplectic involution ι, then ι induces an isometry ι^* on $H^2(X;\mathbb{Z})$. Let M be the Picard lattice $H^{1,1}(X,\mathbb{R})\cap H^2(X;\mathbb{Z})$ of X and $T = M^\perp \subset H^2(X;\mathbb{Z})$ be the transcendental lattice. Then M coincides with $H^2(X;\mathbb{Z})^+$, the fixed lattice of ι^*, and T coincides with $H^2(X;\mathbb{Z})^-$, the sublattice, on which ι^* acts by -1. Fix an embedding $i : M \to L$ and identify $i(M)$ with M. Let

$$D_M \;=\; \{\,\mathbb{C}\Omega \in P(T\otimes\mathbb{C}) \,|\, \Omega\wedge\Omega = 0,\, \Omega\wedge\overline{\Omega} > 0\,\}$$

and

$$\Delta_M \;=\; \{\,\mathbb{C}\Omega \in D_M \,|\, \text{there exists a nonzero } \alpha\in T \text{ such that } \alpha\cdot\Omega = 0\,\}\,.$$

Then ([Vo]) the period domain for the marked K3 surfaces (X, ϕ) with involution acting on $H^2(x;\mathbb{Z})$ as i^* is given by $D_M - \Delta_M$. A point in Δ_M corresponds to a K3 surface X' with an involution ι' whose induced isometry ι'^* on $H^2(X';\mathbb{Z}) = H^2(X;\mathbb{Z})$ is different from ι^*. Presumably such X' has a larger Picard lattice than M.

Let us now turn to the geometry of the deformation space. By the Hodge index theorem ([B-P-VV], [Dol]), M has signature $(1,t)$ for some t and, hence, T has signature $(2,r) = (2, 19-t)$. A $\mathbb{C}\Omega \in D_M$ can be identified with the oriented positive-definite 2-plane in $T_{\mathbb{R}}$ spanned by $(\mathrm{Re}\,\Omega, \mathrm{Im}\,\Omega)$. This leads to the identification

$$D_M \;\cong\; O(2,r)/SO(2)\times O(r)\,.$$

Since $O(2,r)$ has four components, D_M has two isomorphic components.

The inner product Q on T induces an $O(2,r)$-invariant Kähler metric ds^2 on D_M as follows. First Q induces a symmetric 2-tensor, still denoted by Q, on $T_{\mathbb{C}}$. This defines a Hermitian inner product of signature $(2,r)$ on $T_{\mathbb{C}}$ by setting $\langle v, w\rangle = Q(v,\overline{w})$. Let ω_0 be the associated Kähler form. Let $(z_1, z_2, z_3, \cdots, z_{r+2})$ be the complex coordinates for $(T_{\mathbb{C}}, \langle\,\cdot\,,\cdot\,\rangle)$ with respect to an orthonormal basis. Let

$$K \;=\; \log Q(v,\overline{v}) \;=\; \log(z_1\overline{z}_1 + z_2\overline{z}_2 - z_3\overline{z}_3 - \cdots - z_{n+2}\overline{z}_{n+2})$$

and

$$\omega \;=\; \sqrt{-1}\,\partial\overline{\partial}K \;=\; \sqrt{-1}\,\frac{Q(v,\overline{v})\,\omega_0 - Q(dv,\overline{v})\wedge Q(v,d\overline{v})}{Q(v,\overline{v})^2}\,,$$

where $v = (z_1,\cdots,z_{r+2})$ and $dv = (dz_1,\cdots,dz_{r+2})$ in terms of the given coordinates. Note that ω is a homogeneous 2-form on $T_{\mathbb{C}} - \{0\}$ and hence descends to a 2-form, also denoted by ω, on $P(T_{\mathbb{C}})$. It is invariant under the induced $O(2,r)$-action. Since this action is transitive on D_M, by choosing a point in D_M, one can determine the signature of the induced hermitian metric on D_M. For example, take $p = [\,1 : i : 0 : \cdots : 0\,] \in D_M$; then, in terms of the local coordinates

$$(\,w_0,\, w_1,\,\cdots,\, w_r\,) \;=\; (\,\tfrac{z_2}{z_1},\,\tfrac{z_3}{z_1},\,\cdots,\,\tfrac{z_{r+2}}{z_1}\,)\,,$$

D_M around p is decribed by $1 + w_0^2 - w_1^2 - \cdots - w_n^2 = 0$, and the complex cotangent space $T_p^*D_M$ is given by $dw_0 = 0$. Consequently,

$$\omega|_{T_pD_M} \;=\; -\tfrac{\sqrt{-1}}{2}\sum_{i=1}^{r} dw_i\wedge d\overline{w}_i\,.$$

Thus $-\omega$ induces a positive-definite Hermitian metric ds^2 on D_M. This is the Weil-Petersson metric on D_M ([Ti]). It has negative holomorphic sectional curvatures that are bounded away from zero ([Gr-S], [Ti]).

The above discussion also shows that (D_M, ds^2) is complete as a Riemannian manifold. Since Δ_M has complex codimension at least 1, we shall take D_M as the deformation space for our K3 surfaces.

3.3 The Thurston part of deformations of K3T.

The deformation space $Def(M^3)$ of hyperbolic structures associated to a complete hyperbolic 3-manifold M^3 of finite volume was explored by Thurston in [Th1]. Many details of his method were later studied further in [N-Z] and [Yo1, Yo2]. With the anticipation of relating $Def(M^3)$ to some 4-dimensional $N = 2$ supersymmetric nonlinear σ-model, we shall rephrase their results in terms of the complex symplectic language. A corollary from their work is that there is a natural special Kähler structure in a neighborhood of the complete $M^3 \in Def(M^3)$. The associated integrable system and Seiberg-Witten-like 1-form can also be constructed, following [Fr1] and [D-M1]. We shall now explain this in some details.

$Def(M^3)$ from the complex symplectic viewpoint.

(a) An isotropic embedding χ of $Def(M^3)$ in $(\mathbb{C}^{2n},\omega)$.

Recall from Sec. 1 the upper half-plane $\mathbb{H}_+ = \{z' \in \mathbb{C} \,|\, \mathrm{Im}\,(z') > 0\}$ and the ideal tetrahedron $\Delta(z)$ of modulus $z \in \mathbb{H}_+$. Let M^3 be a complete hyperbolic 3-manifold of finite volume and

$$M^3 \;=\; \Delta(z_1^0) \,\cup\, \cdots \cup\, \Delta(z_n^0)$$

be an ideal triangulation of M^3. Since the Euler characteristic of M^3 is zero, the number of edges in the triangulation is the same as the number n of the tetrahedra. Deformations of the ideal tetrahedra lead then to deformations of the hyperbolic structures on M^3. Requiring the geometric angle around each edge of the triangulation from the piecewise hyperbolic geometry be exactly 2π leads to a system of rational constraint equations:

$$\prod_{\nu=1}^{n} (z_\nu)^{r'_{j\nu}} (1-z_\nu)^{r''_{j\nu}} \;=\; \prod_{\nu=1}^{n} (z_\nu^0)^{r'_{j\nu}} (1-z_\nu^0)^{r''_{j\nu}} \;(=\, \pm e^{2\pi i}) \;, \quad j \;=\; 1, \ldots, n \;,$$

where $z_i \in \mathbb{H}_+$ and the product in each equation is taken in the universal covering group $\widetilde{\mathbb{C}^\times}$ of $\mathbb{C}^\times = (\mathbb{C} - \{0\}, \times)$. There are some additional constraint equations coming from the meridian to simple closed geodesics, around which the ends of ideal tetrahedra wind. These equations are of the same form as the constraint equations from edges. All together, j runs from 1 to some $n' \geq n$. The system of equations define $Def(M^3)$ as an affine variety in $(\mathbb{H}_+)^n \subset \mathbb{C}^n$. This variety has a distinguished point $[M_0^3]$ corresponding to the complete M^3. It is known from [N-Z] and [Th1] that the complex dimension of $Def(M^3)$ around $[M_0^3]$ coincides with the number h of cusps of M^3.

Let $\mathbb{C}^{2n} = (\mathbb{C}^{2n},\omega)$ be the complex vector space with complex coordinates $(w_1, \cdots w_{2n})$ and the complex symplectic form $\omega = \Sigma_{i=1}^n dw_i \wedge dw_{n+i}$. Consider the following holomorphic inclusion map

$$\begin{array}{rccc} \Theta \;: & (\mathbb{H}_+)^n & \longrightarrow & \mathbb{C}^{2n} \\ & (z_1\,, \cdots, z_n) & \longmapsto & (\log\left(\frac{z_1}{z_1^0}\right)\,, \cdots, \log\left(\frac{z_n}{z_n^0}\right)\,; \log\left(\frac{1-z_1}{1-z_1^0}\right)\,, \cdots, \log\left(\frac{1-z_n}{1-z_n^0}\right)) \,, \end{array}$$

where we require that $\mathrm{Im}\,(\log z) \in (0,\pi)$ and $\mathrm{Im}\,(\log(1-z)) \in (-\pi,0)$. Then the image $Im\Theta$ of Θ is an embedded complex Lagrangian submanifold in $(\mathbb{C}^{2n},\omega_0)$. Let R be the matrix $[R',R''] = [(r'_{j\nu}),(r''_{j\nu})]_{n'\times 2n}$ regarded as a linear map φ_R from $\mathbb{C}^{2n}$ to $\mathbb{C}^{n'}$ by matrix multiplication, then

$$Def(M^3) = Im\Theta \cap \varphi_R^{-1}(0)\,.$$

We shall denote this embedding of $Def(M^3)$ in $(\mathbb{C}^{2n},\omega)$ by χ. Note that χ send $[M_0^3]$ to the origin of $\mathbb{C}^{2n}$.

In this way, one realizes $Def(M^3)$ as an isotropic analytic variety in $(\mathbb{C}^{2n},\omega)$ and one can choose a neighborhood U of $[M_0^3] \in Def(M^3)$ that forms a complex h-dimensional isotropic submanifold in $(\mathbb{C}^{2n},\omega)$. In the following discussion, we shall choose U topologically a complex ball.

(b) The tautological embedding ψ of U in $H^1(\partial M^3;\mathbb{C})$.

Recall from [Th1] (cf. Sec. 1) that the hyperbolic structure on M^3 associated to $(z_1,\cdots,z_n)$ in $Def(M^3)$ induces a complex affine structure on ∂M^3 and hence determines a holonomy representation from $H_1(\partial M^3;\mathbb{Z})$ to the complex affine group $Aff(1,\mathbb{C})$. The derivative of the holonomy (i.e. ignoring the translation part of the honomomy) gives then a holonomy map μ from $H_1(\partial M^3;\mathbb{Z})$ to $\mathbb{C}^\times$. Explicitly, given a class $[\gamma]$ in $H_1(\partial M^3;\mathbb{Z})$ represented by a simplicial edge-loop γ with respect to the induced triangulation on ∂M^3 from the ideal triangulation of M^3, μ can be expressed in terms of z_ν's as ([N-Z])

$$\mu([\gamma]) = \prod_{\nu=1}^{n} \left(\frac{z_\nu}{z_\nu^0}\right)^{c'_\nu} \left(\frac{1-z_\nu}{1-z_\nu^0}\right)^{c''_\nu}\,,$$

where c'_ν and c''_ν are some integers and the factors appearing in the above product are from the moduli of the triangle vertices touching γ from the right with respect to the orientation of γ and ∂M^3 (FIGURE 3-1). This gives an embedding of the neighborhood U of $[M_0^3]$ from Part (a) into $H^1(\partial M^3;\mathbb{C}^\times) = Hom\,(H_1(\partial M^3;\mathbb{Z}),\mathbb{C}^\times)$ with $[M_0^3]$ mapped to the element $(\cdot \to 1)$ in $H^1(\partial M^3;\mathbb{Z})$. Since log is a local embedding from $(\mathbb{C}^\times,1)$ to $(\mathbb{C},0)$, one can lift the image of U in $H^1(\partial M^3;\mathbb{C}^\times)$ to $H^1(\partial M^3;\mathbb{C}) = Hom(H_1(\partial M^3;\mathbb{Z}),\mathbb{C})$ by taking logarithm of the coefficient group.

We shall denote the latter embedding of U by ψ. Note that $[M_0^3]$ is mapped to the zero element of $H^1(M^3;\mathbb{C})$.

(c) ψ is Lagrangian.

First observe that, when tensored with $\mathbb{C}$, the cup product on $H^1(\partial M^3;\mathbb{Z})$ induces a complex symplectic structure ω^+ on $H^1(\partial M^3;\mathbb{C})$ and the intersecting pairing on $H_1(\partial M^3;\mathbb{Z})$ induces a complex symplectic structure ω_- on $H_1(\partial M^3;\mathbb{C})$. The symplectic dual between $H^1(\partial M^3;\mathbb{C})$ and $H_1(\partial M^3;\mathbb{C})$ coincides with the Poincaré dual between the two.

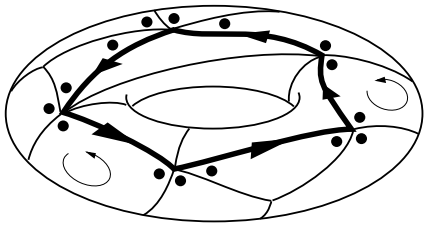


FIGURE 3-1. The triangle vertices that contribute to the holonomy of the induced complex affine structure along a loop γ (indicated by the thick line) on ∂M^3 are indicated by $\bullet$ and the orientation of the boundary $\mathbb{T}^2$ is indicated by $\circlearrowleft$. (Cf. FIG. 3 in [N-Z].)

Now let $[\gamma] \in H_1(\partial M^3;\mathbb{Z})$. Then $[\gamma]$ gives rise to a linear map $\varphi_{[\gamma]}$ from $\mathbb{C}^{2n}$ to $\mathbb{C}$ defined by

$$\varphi_{[\gamma]}(w_1, \cdots, w_n\,;\, w_{n+1}, \cdots, w_{2n}) \;=\; c'_1 w_1 + \cdots c'_n w_n + c''_1 w_{n+1} + \cdots c''_n w_{2n}\,,$$

where c'_ν, c''_ν are obtained as in Part (b). The correspondence $[\gamma] \mapsto \varphi_{[\gamma]}$ gives a homomorphism from $H_1(\partial M^3;\mathbb{Z})$ to the dual space $(\mathbb{C}^{2n^*},\omega^*)$ of $(\mathbb{C}^{2n},\omega)$. This then induces a homomorphism $\xi : H_1(\partial M^3;\mathbb{C}) \to \mathbb{C}^{2n^*}$. From Theorem 2.2 in [N-Z] and the discussion therearound, one has

Fact 3.3.1. ([N-Z]) *ξ is injective and symplectic.*

After taking the symplectic dual, one thus realizes $H_1(\partial M^3;\mathbb{C})$ as a complex $2h$-dimensional symplectic subspace $\mathcal{S}$ in $(\mathbb{C}^{2n},\omega)$.

Recall the matrix R from Part (a). Its row vectors span a complex subspace in $\mathbb{C}^{2n^*}$. Let $\mathcal{R}$ be its symplectic dual in $\mathbb{C}^{2n}$. Then Proposition 2.3 of [N-Z] can be rephrased as

Fact 3.3.2. ([N-Z]) *$\mathcal{R}$ is isotropic and is contained in the symplectic orthogonal complement $\mathcal{S}^\perp$ of $\mathcal{S}$. Furthermore, $\varphi_R^{-1}(0) = \mathcal{R} \oplus \mathcal{S}$.*

One can now identify $\mathcal{S}$ with $H^1(\partial M^3;\mathbb{C})$ canonically since both are symplectic dual to $H_1(\partial M^3;\mathbb{C})$. Let $\pi_{\mathcal{S}}$ be the projection from $\varphi_R^{-1}(0)$ to $\mathcal{S}$. Then, by chasing the definitions of the maps involved, one has

Lemma 3.3.3. $\pi_{\mathcal{S}} \circ \chi = \psi$.

This shows that

Corollary 3.3.4. *ψ is a Lagrangian embedding of U in $(H^1(\partial M^3;\mathbb{C}),\omega^+)$.*

The relation of all the maps and spaces involved are indicated in FIGURE 3-2 to make the above discussion more transparent.

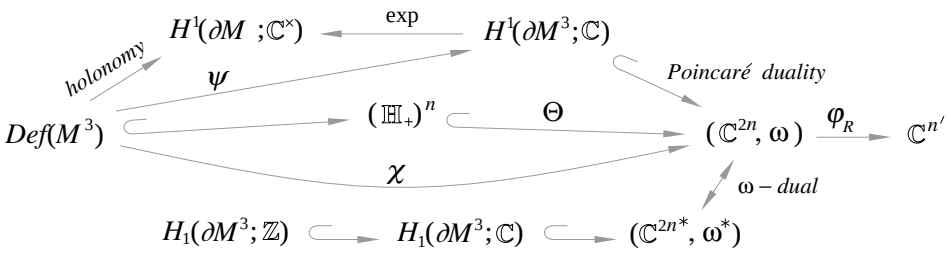


FIGURE 3-2. The relation of various maps and spaces.

The special Kähler structure on U via ψ and the associated integrable system.

With the preparation above, we can now spell out the natural special Kähler structure on U. Let $\partial M^3 = \cup_h \mathbb{T}^2_i$ and (α_i, β_i) be a canonical basis of $H_1(\mathbb{T}^2_i;\mathbb{Z})$ with $\alpha_i \cdot \beta_i = 1$. Then together $(\alpha_1, \cdots, \alpha_h, \beta_1, \cdots, \beta_j)$ forms a canonical basis of $H_1(\partial M^3;\mathbb{Z})$. Let

$$\mathcal{A}^* = Span_{\mathbb{C}}(\alpha_1, \cdots, \alpha_h) \quad \text{and} \quad \mathcal{B}^* = Span_{\mathbb{C}}(\beta_1 \cdots, \beta_h)$$

be the isotropic framed subspaces in $\mathbb{C}^{2n^*}$ spanned by $(\alpha_1, \cdots \alpha_h)$ and $(\beta_1, \cdots \beta_h)$ respectively and $\mathcal{A}$, $\mathcal{B}$ be their symplectic dual in $\mathcal{S} \subset \mathbb{C}^{2n}$. Note that $(\mathcal{B}, \mathcal{A})$ forms a transverse pair of framed Lagrangian subspaces in $\mathcal{S}$ and one can identify $\mathcal{S}$ as $T^*\mathcal{B} = \mathcal{B} \oplus (-\mathcal{A})$, where T^*B is the holomorphic cotangent bundle of $\mathcal{B}$ and $-\mathcal{A}$ is $\mathcal{A}$ with framing induced from $(-\alpha_1, \cdots, -\alpha_h)$.

Remark 3.3.5. As maps on $\mathcal{S}$, $\mathcal{A}^*$ corresponds the projection map $\mathrm{pr}_{\mathcal{B}}$ from $\mathcal{S}$ onto $\mathcal{B}$, while $\mathcal{B}^*$ corresponds to the projection map $\mathrm{pr}_{-\mathcal{A}}$ from $\mathcal{S}$ onto $-\mathcal{A}$. This explains why $\mathcal{S} = T^*\mathcal{B} = \mathcal{B} \oplus (-\mathcal{A})$.

With U identified with $\psi(U)$ in $\mathcal{S}$, $T_{[M_0^3]}U$, the holomorphic tangent space of U at $[M_0^3]$, is another Lagrangian subspace in $\mathcal{S}$.

Lemma 3.3.6 [transversality]. *$T_{[M_0^3]}U$ is transverse to both $\mathcal{B}$ and $-\mathcal{A}$.*

Proof. Let u_i be the element in $\mathbb{C}^{2n^*}$ associated to α_i and v_i be the element in $\mathbb{C}^{2n^*}$ associated to β_i. With notations from Remark 3.3.5, $\mathrm{pr}_{\mathcal{B}} = (u_1, \cdots, u_h)$ and $\mathrm{pr}_{-\mathcal{A}} = (v_1, \cdots, v_h)$. From [N-Z], when restricted to $U \subset \mathcal{S}$, both $(u_1, \cdots, u_h)$ and $(v_1, \cdots, v_h)$ form holomorphic coordinate charts around $[M_0^3] \in U$ with the coordinates for $[M_0^3]$ being $(0, \cdots, 0)$. This concludes the lemma. □

Since U is Lagrangian in $\mathcal{S} = T^*\mathcal{B}$, the restriction to U of the complex canonical 1-form $\alpha = \sum v_i\, du_i$ on $T^*\mathcal{B}$ vanishes. Thus U can be realized as the graph of the holomorphic 1-form $d\mathfrak{F} = \alpha|_U$ on $\mathcal{B}$ for some holomorphic function $\mathfrak{F}$ on $\mathcal{B}$. Pulling back to U via $\mathrm{pr}_{\mathcal{B}}$, one

can regard $\mathfrak{F}$ as defined on U. Lemma 3.3.6 implies that if one chooses the neighborhood U of $[M_0^3] \in Def(M^3)$ appropriately, then U is transverse to both $\mathrm{pr}_{\mathcal{B}}$ and $\mathrm{pr}_{-\mathcal{A}}$. For such U, $\mathfrak{F}$ then serves as the prepotential that determines a special Kähler structure ω_U on U (cf. [Fr1]; also Theorem 2 and remarks in Sec. 4 of [Hi2]). By specifying $\mathfrak{F}(0, \cdots 0) = 0$, $\mathfrak{F}$ is unique on U. Let K_U be the Kähler potential for ω_U, then in terms of the u-coordinates on U

$$v_i \;=\; \frac{\partial \mathfrak{F}}{\partial u_i}, \qquad\qquad K_U \;=\; \tfrac{1}{2} \sum \mathrm{Im}\,(v_i\, \overline{u_i}) \;=\; \tfrac{1}{2} \sum \mathrm{Im}\,(\frac{\partial \mathfrak{F}}{\partial u_i}\, \overline{u_i}),$$

and

$$\omega_U \;=\; \sqrt{-1}\,\partial\overline{\partial}\, K_U \;=\; \tfrac{\sqrt{-1}}{2} \sum \mathrm{Im}\,(\frac{\partial v_i}{\partial u_j})\, du_i \wedge d\overline{u_j} \;=\; \tfrac{\sqrt{-1}}{2} \sum \mathrm{Im}\,(\frac{\partial^2 \mathfrak{F}}{\partial u_i\, \partial u_j})\, du_i \wedge d\overline{u_j}.$$

By Lemma 4.1 in [N-Z], at $u = (0, \cdots, 0) \in U$ the Jacobian matrix $[\partial v_i / \partial u_j]_{ij}$ is the diagonal matrix $Diag\,(\tau_1^0, \cdots, \tau_h^0)$, where τ_i^0 is the modulus of the complex structure on $\mathbb{T}_i^2$ associated to the i-th cusp of M^3. Since every τ_i^0 lies in the upper half plane $\mathbb{H}_+$ and U is connected, the Kähler metric on U associated to ω_U is positive-definite.

Remark 3.3.7 [independence of choices of (α_i, β_i)]. In terms of the terminology in [Fr1], each $(\mathcal{A}^*, \mathcal{B}^*)$-pair provides a conjugate pair of special holomorphic coordinate systems on U. Under different choices of (α_i, β_i), the $(u_1, \cdots, u_h, v_1, \cdots, v_h)$ transforms under $\Pi_h SL(2, \mathbb{Z}) \subset Sp\,(2h, \mathbb{Z})$, where the i^{th} $SL(2, \mathbb{Z})$ in the product acts only on (u_i, v_i). Thus the Kähler structure ω_U on U constructed above is independent of the choices of (α_i, β_i). The twisted real part

$$(x_1, \cdots, x_h, y_1, \cdots y_h) \;=\; (\mathrm{Re}\, u_1, \cdots, \mathrm{Re}\, u_h, -\mathrm{Re}\, v_1, \cdots, -\mathrm{Re}\, v_h)$$

of $(u_1, \cdots, u_h, v_1, \cdots, v_h)$ provides a real Darboux coordinate system for (U, ω_U). Since the transition functions among such real coordinate systems are also integral, this defines a flat torsion free symplectic connection ∇ on U.

Remark 3.3.8 [naturality of $\mathfrak{F}$]. ([N-Z] and [Yo1]) For $u = (u_1, \cdots u_h) \in U$ representing a hyperbolic 3-manifold M_u^3 obtained by a Dehn filling on some cusps of M^3, the value $\mathfrak{F}(u)$ is related to the volume $vol(M_u^3)$ and the Chern-Simons invariant $CS(M_u^3)$ of M_u^3 by the identity

$$e^{\frac{2}{\pi} vol\,(M_u^3) + i\, CS(M_u^3)} \prod_i e^{length\,(\gamma_i) + i\, torsion\,(\gamma_i)} \;=\; e^{\frac{2}{\pi} vol\,(M^3) + i\, CS(M^3)}\; e^{-\frac{1}{2\pi i}(-2 + \sum u_i \frac{\partial}{\partial u_i})\, \mathfrak{F}},$$

where γ_i are the simple geodesic loops assiciated to the Dehn-filling and $length\,(\gamma_i)$, $torsion\,(\gamma_i)$ are the length and the torsion of γ_i (cf. [Me], [N-Z], [Th2], and [Yo1]). This indicates that $\mathfrak{F}$ is a very natural holomorphic function on U.

Now recall from [D-M2] and [Fr1] that

Definition 3.3.9 [algebraic integrable system]. An *algebraic integrable system* is a holomorphic map $\pi : Y \to M$, where Y is a complex symplectic manifold with a holomorphic symplectic form η, such that

(1) the fiber $Y_m = \pi^{-1}(m)$ for all $m \in M$ is a compact Lagrangian submanifols, hence an affine torus;

(2) there is a smooth family $[\rho]$ of cohomology classes $[\rho_m] \in H^{1,1}(Y_m) \cap H^2(Y_m; \mathbb{Z})$ such that $[\rho_m]$ is a positive polarization of Y_m.

Following previous discussions and notations, the holomorphic 1-forms $du_1, \cdots, du_h, dv_1, \cdots, dv_h$ generate a lattice Λ of complex Lagrangian sections in T^*U that are flat with respect to ∇. Let T^*U be the holomorphic cotangent bundle of U. From Theorem 3.4 in [Fr1], $Y = T^*U/\Lambda \to U$ is an algebraic integrable system over U with η from the canonical holomorphic 2-form on T^*U and ρ from the dual of ω_U. Since Λ is independent of the choices of (α_i, β_i) that defines (u_i, v_i), Y is canonically associated to U.

The Seiberg-Witten-like 1-form λ on Y.

Recall (e.g. [Don] and [D-M1]) that, for the integrable system associated to the Seiberg-Witten theory of a gauge group or to a complete family of Calabi-Yau threefolds, there is a natural 1-form λ on the total space of the integrable system, whose periods along the fiber tori provide a conjugate pair of holomorphic coordinate systems for the special Kähler or the projective special Kähler geometry on its base. Something similar happens also to the integrable system (Y, η, ρ) over U.

Lemma 3.3.10 [Seiberg-Witten-like 1-form]. *There exists a smooth 1-form λ on Y such that, if $(u_1, \cdots, u_h)$ and $(v_1, \cdots, v_h)$ are a conjugate pair of holomorphic coordinate systems on U as discussed earlier and $C'_1, \cdots, C'_h, C''_1, \cdots, C''_h$ are the 1-cycles on the fiber of Y associated to $du_1, \cdots, du_h, dv_1, \cdots, dv_h$ respectively, then λ satisfies the following properties:*

(1) the restriction of λ to the fiber of Y is closed and the period of λ with respect to $(C'_1, \cdots, C'_h, C''_1, \cdots, C''_h)$ is $(u_1, \cdots, u_h, v_1, \cdots, v_h)$;

(2) $d\lambda = \eta$ and hence $\int_{C'_i} \eta = du_i$ and $\int_{C''_i} \eta = dv_i$.

Proof. Take $du_1, \cdots, du_h, dv_1, \cdots, dv_h$ as a real basis for fibers of T^*U and consider the following smooth complex-valued function on T^*U

$$\begin{array}{rccc} f & : & T^*U & \longrightarrow & \mathbb{C} \\ & & \sum c'_i\, du_i + \sum c''_i\, dv_i & \longmapsto & \sum c'_i\, u_i + \sum c''_i\, v_i\,. \end{array}$$

Notice that f is independent of choices of the (u, v)-coordinates used. Its differential

$$df \;=\; (\sum u_i\, dc'_i + \sum v_i\, dc''_i) + (\sum c'_i\, du_i + \sum c''_i\, dv_i)$$

is a complex-valued smooth 1-form on T^*U. Let t be an element in the lattice Λ of sections generated by $du_1, \cdots, du_h, dv_1, \cdots, dv_h$ and $\mathcal{O}_t$ be the fiberwise translation of T^*U generated by t, Then

$$\mathcal{O}_t^* df - df = t .$$

Observe that the summand $(\sum c_i' du_i + \sum c_i'' dv_i)$ of df is exactly the canonical holomorphic 1-form α on T^*U, which satisfies also $\mathcal{O}_t^*\alpha - \alpha = t$. Consequently, the difference $df - \alpha$ is invariant under $\mathcal{O}_t$ and hence descends to a smooth complex-valued 1-form λ on Y. That λ satisfies both Property (1) and Property (2) above follows by construction. This concludes the proof.

□

Remark 3.3.11 [relation to some natural 1-forms supported on the cusps of M^3]. Recall from Sec. 2.1 the notations and the fact that each deformed cusp $\mathcal{C}$ of M^3, if finite, is geometrically the quotient $\widetilde{\mathcal{N}_\varepsilon}/\widetilde{\mu}(\pi_1(\mathbb{T}^2))$. The group of the induced $\widetilde{\mu}(\pi_1(\mathbb{T}^2))$-action on the ideal boundary $\partial_\infty \mathbb{H}^3 = \mathbb{C} \cup \infty$ is a subgroup of the group of Möbius transformations that fix 0 and ∞. Since $d \log z = \frac{dz}{z}$ is invariant under the latter group, one may first lift $\frac{dz}{z}$ to $\widetilde{\mathcal{N}_\varepsilon}$ via the projection $(z,t) \mapsto z$ and the covering map, and then project the resulting 1-form on $\widetilde{\mathcal{N}_\varepsilon}$ to a complex-valued smooth 1-form $\xi_{\mathcal{C}}$ on $\mathcal{C}$. The discussion is similar for $\mathcal{C}$ infinite. The periods of these $\xi_{\mathcal{C}}$ along a canonical basis $(\alpha_1, \beta_1; \cdots; \alpha_h, \beta_h)$ for $H_1(\partial M^3; \mathbb{Z})$ gives then a conjugate pair of holomorphic coordinates $(u_1, \cdots, u_h)$ and $(v_1, \cdots, v_h)$ that appear in earlier discussions. This suggests another natural flat bundle $\mathcal{T}$ over U whose fiber $\mathcal{T}_u$ over $u \in U$ is the product of the toroidal components of ∂M^3 with the complex affine structure determined by u. From the product structure, one can lift $\xi_{\mathcal{C}}$ to $\mathcal{T}_u$ and sum them together to form a 1-form along a fiber of $\mathcal{T}$. Using the flat connection on $\mathcal{T}$, one obtains then a smooth complex-valued 1-form ξ on $\mathcal{T}$. By identifying $\mathcal{T}$ with the quotient $(U \times H_1(\partial M^3; \mathbb{R}))/H_1(\partial M^3, \mathbb{Z})$ and $H_1(\partial M^3; \mathbb{Z})$ with Λ, one has a bundle isomorphism

$$\varphi : Y \longrightarrow \mathcal{T} .$$

The pullback 1-form $\varphi^*\xi$ is then cohomologous to λ constructed in Lemma 3.3.10. In some way, φ^{-1} resembles the Abel-Prym map in the usual Seiberg-Witten theory (cf. [Don]).

4 Remarks and examples on general K3T 7-manifolds.

In this section, we discuss two ways in which the construction in Sec. 2.2 can be generalized. This will provide us with many other classes of K3T 7-manifolds.

Flat K3T 7-manifolds from general $(X, Aut(X))$.

The construction in Sec. 2.2 has an immediate generalization: one may replace the K3 surface with involution by a K3 surface X with nontrivial group of automorphisms $Aut(X)$. Here an *automorphism* means a diffeomorphism that preserves the complex structure.

Recall that every K3 surface is Kähler. The *Strong Torelli Theorem* ([B-P-VV] and [L-P]) for K3 surfaces can be stated as: *The group $Aut(X)$ of automorphisms of a K3 surface X coincides with the group of Hodge isometries of $H^2(X,\mathbb{Z})$ that preserve the Kähler cone of X.* When X is algebraic, let S_X be the Picard lattice of X and W_X the group on $H^2(X;\mathbb{Z})$ generated by the Picard-Lefschetz reflections $s_d : x \mapsto x + (x,d)d$ associated to elements d in S_X with $(d,d) = -2$. Let $Isom(S_X)$ be the group of isometries of S_X. Then the quotient of $Aut(X)$ by the finite normal subgroup that consists of automorphisms whose induced map on S_X are trivial contains a subgroup of finite index that is isomorphic to a subgroup of finite index in the quotient group $Isom(S_X)/W_X$ ([PS-S]). These fundamental facts allow one to convert the study of $Aut(X)$ to the study of lattices with a bilinear form and the fundamental polyhedron of the W_X-action (cf. [Ni1-5]).

Examples of K3 surfaces with an infinite automorphism group are provided by exceptional K3 surfaces. These are algebraic K3 surfaces whose rank of S_X equals the maximal possible number 20. They are all realizable as an elliptic pencil with infinitely many sections ([S-I]). Once a section is fixed as the identity section, the fiber then has an abelian group structure. The fiberwise translation by a section gives then an automorphism of X. This special subgroup of $Aut(X)$ is called the *Mordell-Weil group* of X.

On the other hand, K3 surfaces with finite automorphism group have been studied extensively by Nikulin ([Ni1-5]) and others. Finite groups G that can act on a K3 surface effectively and leave the holomorphic 2-form fixed were completely worked out by Mukai ([Muk] and [Ni2]). He also gave the K3 surfaces on which such G act, as in TABLE 4-1. In particular, exactly the following fifteen groups can be realized as such automorphism group for some K3 surface ([Ni2] and its Added-in-Proof):

$$(\mathbb{Z}_2)^k\,,\ 0 \le k \le 4\,; \quad \mathbb{Z}_3\,; \quad \mathbb{Z}_4\,; \quad \mathbb{Z}_5\,; \quad \mathbb{Z}_6\,; \quad \mathbb{Z}_7\,; \quad \mathbb{Z}_8\,; \quad \mathbb{Z}_2 \oplus \mathbb{Z}_4\,; \quad \mathbb{Z}_2 \oplus \mathbb{Z}_6\,;$$
$$\mathbb{Z}_3 \oplus \mathbb{Z}_3\,; \quad \mathbb{Z}_4 \oplus \mathbb{Z}_4\,.$$

and the quotient orbifolds have A-singularities.

A program along this line, following the deform-and-fill procedure in Sec. 2.2, involves the study of the following pieces:

1. K3 surface X with nontrivial $Aut(X)$, how $Aut(X)$ acts on X, and the deformation space of K3 surfaces that share isomorphic $Aut(X)$. (Cf. FIGURE 4-1.)

2. Representations ρ_X of $\pi_1(M^3)$ into $Aut(X)$. (A computer code may be helpful here. Cf. e.g. [Fo], [GA], [Joh], [L-R], and [Ri1].)

3. Singularities on W^7 obtained from $N^3 \times_{\rho_X} X$ by the deform-and-fill procedure and their resolutions.

4. Structures on the deformation space of the resulting K3T 7-manifolds

$$Def(\text{K3T}) \;=\; Def(X, Aut(X)) \times Def(N^3) \times Hom(\pi_1(N^3), Aut(X))$$

(cf. Sec. 3.1).

n^o	G	order	K3 surface
1	$L_2(7)$	168	$x^3y + y^3z + z^3x + t^4 = 0$ in $\mathbb{C}\mathrm{P}^3$
2	Alt_6	360	$\sum_{i=1}^6 x_i = \sum_{i=1}^6 x_i^2 = \sum_{i=1}^6 x_i^3 = 0$ in $\mathbb{C}\mathrm{P}^5$
3	Sym_5	120	$\sum_{i=1}^5 x_i = \sum_{i=1}^6 x_i^2 = \sum_{i=1}^5 x_i^3 = 0$ in $\mathbb{C}\mathrm{P}^5$
4	$M_{20} = 2^4 Alt_5$	960	$x^4 + y^4 + z^4 + t^4 + 12xyzt = 0$ in $\mathbb{C}\mathrm{P}^3$
5	$F_{384} = 4^2 Sym_4$	384	$x^4 + y^4 + z^4 + t^4 = 0$ in $\mathbb{C}\mathrm{P}^3$
6	$Alt_{4,4} = 2^4 Alt_{3,3}$	288	$\begin{cases} x^2 + y^2 + z^2 = \sqrt{3}u^2 \\ x^2 + \omega y^2 + \omega^2 z^2 = \sqrt{3}v^2 \\ x^2 + \omega^2 y^2 + \omega z^2 = \sqrt{3}w^2 \end{cases}$ in $\mathbb{C}\mathrm{P}^5$
7	$T_{192} = (Q_8 * Q_8) \rtimes Sym_3$	192	$x^4 + y^4 + z^4 + t^4 - 2\sqrt{-3}(x^2y^2 + z^2t^2) = 0$ in $\mathbb{C}\mathrm{P}^3$
8	$H_{192} = 2^4 D_{12}$	192	$\begin{cases} x_1^2 + x_3^2 + x_5^2 = x_2^2 + x_4^2 + x_6^2 \\ x_1^2 + x_4^2 = x_2^2 + x_5^2 = x_3^2 + x_6^2 \end{cases}$ in $\mathbb{C}\mathrm{P}^5$
9	$N_{72} = 3^2 D_8$	72	$x_1^3 + x_2^3 + x_3^3 + x_4^3 = x_1x_2 + x_3x_4 + x_5^2 = 0$ in $\mathbb{C}\mathrm{P}^4$
10	$M_9 = 3^2 Q_8$	72	double covering of $\mathbb{C}\mathrm{P}^2$ branched over $x^6 + y^6 + z^6 - 10(x^3y^3 + y^3z^3 + z^3x^3) = 0$
11	$T_{48} = Q_8 \rtimes Sym_3$	48	double covering of $\mathbb{C}\mathrm{P}^2$ branched over $xy(x^4 + y^4) + z^6 = 0$

TABLE 4-1. Mukai's table of K3 surfaces with finite symplectic automorphism group G. Readers are referred to [Muk] for the notation of various G and more details.

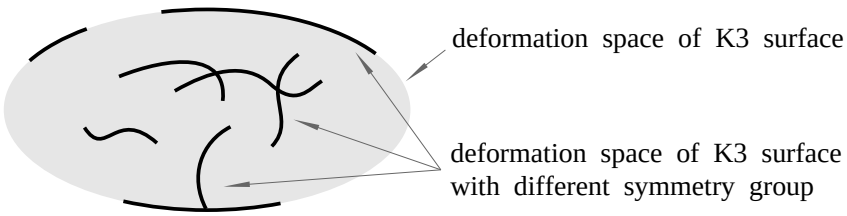


FIGURE 4-1. A schematic diagram for the deformation space of the complex structures of K3 surfaces with symmetry.

With the information provided here, one can construct more classes of K3T 7-manifolds. The following example of K3-fibration, though perhaps with nonhyperbolic base, is enough to illuminate the idea.

Example 4.1 [K3-fibration via K3 with other symmetry]. Let X be the K3 surface given by the nonsingular complete intersection of the quadric $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2=0$ and the cubic $x_1^3+x_2^3+x_3^3+x_4^3+x_5^3=0$ in $\mathbb{C}\mathrm{P}^4$ with homogeneous coordinates $[x_1:x_2:x_3:x_4:x_5]$. Then X admits an action of the symmetric group Sym_5 of degree 5 induced from the action of Sym_5 on $\mathbb{C}\mathrm{P}^4$ by permutations of the homogeneous coordinates. Let K be the trefoil knot in S^3 (FIGURE 4-2), whose fundamental group $\pi_1(S^3-K)$ has a presentation $\langle a,b \,|\, aba=bab\rangle$.

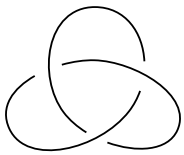


FIGURE 4-2. The trefoil knot in S^3. (Its complement admits non-hyperbolic geometric structure.)

Consider the representation ρ from $\pi_1(S^3-K)$ to Sym_5 generated by

$$a \longmapsto (12345)\,, \qquad b \longmapsto (13542)$$

([Fo], also [Ri1]; up to conjugation, this is the unique nonabelian representation from $\pi_1(S^3-K)$ to Sym_5 that sends the generators a and b to 5-cycles). Thus $\pi_1(S^3-K)$ acts on X through ρ. Now let W^7 be the quotient $\widetilde{S^3-\nu(K)}\times X/\pi_1(S^3-K)$, where $\widetilde{S^3-\nu(K)}$ is the universal covering of $S^3-\nu(K)$ and the group action is diagonal. Since the $(1,1)$-loop along K is sent by ρ to the identity in Sym_5, the boundary of this 7-manifold is diffeomorphic to $X_{\rho(a)}\times S^1$, where $X_{\rho(a)}$ is the mapping torus $X\times[0,1]/X\times\{0\}\overset{\rho(a)}{\sim}X\times\{1\}$.

To see how this boundary can be filled as in Sec. 2.2, notice that $\rho(a)$ acts on X with four fixed points of order 5:

$$[1:\zeta:\zeta^2:\zeta^3:\zeta^4]\,,\quad [1:\zeta^2:\zeta^4:\zeta:\zeta^3]\,,\quad [1:\zeta^3:\zeta:\zeta^4:\zeta^2]\,,\quad [1:\zeta^4:\zeta^3:\zeta^2:\zeta]\,,$$

where ζ is a fifth root of 1 and the quotient has 4 A_4-singularities. Consequently, the action of the cyclic group $\mathbb{Z}_5=\langle f\rangle$ on $D^2\times X$ generated by

$$f\ :\ (z,x)\longmapsto (\zeta z\,,\,\rho(a)(x))$$

has fixed points only on $\{O\}\times X$, which are exactly the fixed points of $\rho(a)$ on X. By construction, the quotient complex singular threefold has boundary $X_{\rho(a)}$. To resolve the singularity, observe that the induced linear map of f on $T'_p(D^2\times X)$, where p is a fixed point, can be put into the diagonal form $Diag\{\zeta;\zeta,\zeta^{-1}\}$ if one chooses the holomorphic

coordinates of X around p appropriately. Hence, if one blows up $D^2 \times X$ at these fixed points and denotes the new complex threefold by $\overline{Z}$, then $\langle f \rangle$ acts on $\overline{Z}$ freely. The quotient $Z = \overline{Z}/\langle f \rangle$ is smooth and fibred over $D^2/\langle f \rangle$, which can be smoothened to a smooth 2-disk again, with central fiber $\{X \sharp 4\,(\overline{\mathbb{CP}^2} \cup_{\mathbb{CP}^1} \mathbb{CP}^2)\}/\langle f \rangle$. Consequently, $Z \times S^1$ fills ∂W^7. Let M^3 be the 3-manifold $(S^3 - K)_{(1,1)}$. After filled and smoothened, the resulting M^7 is K3-fibred over M^3 with exceptional fiber $\{X \sharp 4\,(\overline{\mathbb{CP}^2} \cup_{\mathbb{CP}^1} \mathbb{CP}^2)\}/\langle f \rangle$ over the core curve of the filling solid torus.

□

Remark 4.2. Any K3T (or just K3-fibred) 7-manifold thus constructed admits a natural flat connection over the complement of the set of critical values of the K3-fibration. Also its generic K3 fibers are all biholomorphic. Thus, in some sense, it is in the category of *flat analytic-K3T 7-manifolds*, whose precise definition will be left to the future after we gain better feeling about K3T.

From analytic-K3T to smooth-K3T.

In contrast with the flat analytic-K3T 7-manifolds so far discussed, let us give the reader some idea about general smooth-K3T 7-manifolds before ending.

Though no precise definition for a smooth-K3T 7-manifold is attempted in this paper, it is helpful to recall the following definition in a well-understood case from [F-M2]:

Definition 4.3 [smooth-elliptic fibration]. A *C^∞-elliptic surface* is a smooth map $\pi : S \to C$ from a closed, smooth, oriented 4-manifold S to a smooth oriented 2-manifold C such that for each point $p \in C$ there is an open disk $\Delta \subset C$ of p, a complex elliptic surface $Z \to \Delta$ and a smooth orientation-preserving diffeomorphism $\pi^{-1}(\Delta) \to Z$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(\Delta) & \stackrel{\cong}{\longrightarrow} & Z \\ \downarrow \pi & & \downarrow \\ \Delta & = & \Delta \end{array} .$$

In other words, S is the total space of pastings of local elliptic fibrations $Z \to \Delta$ with smooth transition functions. In the following, we shall use the word "smooth" to mean C^r for some r.

For our situation, the base manifold of the K3-fibration has real dimension 3; and there is no theory of a real family of complex manifolds developed yet. So it is not clear what wild things may arise if one follows the above definition to define smooth-K3T 7-manifolds. However, there are situations for which one can still have control of the degenerate fiber: e.g. one may requires that locally the fibration is diffeomorphic to a K3-fibration of the form $\pi : Z \to \Delta \times [0,1]$ with π restricted to each $\Delta \times \{t\}$ the ususal K3-fibration from a complex 1-parameter family of K3 surfaces, or of the form $\pi : Z \to \Delta^3$ that comes

from the restriction to a real 3-disk of a K3-fibration associated to a complex 2-parameter family of K3 surfaces.

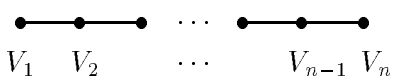
Confined to such, there are two kinds of complex 1-parameter degenerate K3 surfaces that have played roles in the string literature (e.g. [As] and [A-M2]):

- *Orbifold degenerations* (*A-D-E singularities*): ([B-P-VV] and [Di]) When a K3 surface admits disjoint A-D-E chains of (-2) curves, one may contract them through a complex 1-parameter family via deformation of complex structures. The result is then a degenerate K3 surface with A-D-E singularities. A huge class of such examples are provided by elliptic K3 surfaces with a section. The Kodaira's table of singular elliptic fibers that can appear in an elliptic fibration of a complex surface contains various extended A-D-E Dynkin diagrams. For a K3 surface with a section, one can then contract the irreducible components of the singular fibers that are disjoint from the section and obtains a K3 surface with A-D-E singularities. From the work of Miranda and Persson ([M-P]), one knows that, while the total number of singular fibers of an elliptic K3 surface cannot exceed the Euler number 24 of a K3 surface, there are *more than a thousand* of combinations of them that can truely happen. Via contraction, this provides then a big collection of degenerate K3 surfaces with various combinations A-D-E singularities. The nature of A-D-E singularities makes the resulting complex surface a complex orbifold.

- *Semistable degenerations* (*Kulikov degenerations of type II and III*): ([F-S] and [Ku].) This is given by a family $\pi : X \to D^2$ of K3 surfaces over the unit disk $D^2 = \{t \in \mathbb{C} \,|\, |t| \leq 1\}$ such that X is smooth with trivial canonical bundle, all fibers $X_t = \pi^{-1}(t)$ are smooth K3 for $t \in D^2 - \{0\}$, and the central fiber X_0 is a reduced divisor in X with normal crossing singularities. There are only three situations:

  (I) X_0 is a smooth K3 surface.

  (II) X_0 is a chain of elliptic ruled surfaces $V_1 + \cdots + V_n$ whose associated simplicial complex of intersections is given by

  with rational surfaces on either end; the double curves $V_i \cap V_{i+1}$ that appear in the intersections are all elliptic curves.

  (III) X_0 is a union of rational surfaces; the double curves on each irreducible component of X_0 form a cycle of rational curves; and the associated simplicial complex of intersections is a triangulation of S^2.

  (Note that the *simplicial complex of intersections* associated to a decomposition $X_0 = \cup_i V_i$ by irreducible components is defined by assigning to each V_i a vertex, to each non-empty $V_i \cap V_j$ an edge e_{ij} connectiog V_i and V_j, to each non-empty $V_i \cap V_j \cap V_k$ a face with boundary $e_{ij} \cup e_{jk} \cup e_{ki}$, etc.)

From Mumford's semistable reduction theorem ([K-K-M-SD]), an orbifold - indeed any - degeneration $\pi : X \to D^2$ of algebraic K3 surfaces can always be converted into a semistable degeneration $\pi' : X' \to D^{2'}$ after blowups at the degenerate fiber $X_0 = \pi^{-1}(0)$, a base change $D^{2'} \to D^2$ with $t' \mapsto t = t'^k$ for some positive integer k, and then blowups again at the new central degenerate fiber of $X \times_{D^2} D^{2'}$ over $D^{2'}$. When this happens, the monodromy diffeomorphism τ' of K3 associated to π' is the power τ^k, where τ is the monodromy diffeomorphism of K3 associated to π.

Combined with the construction in Sec. 2.2, this immediately provides us with another big class of K3T 7-manifolds.

Example 4.4 [smooth-K3T]. Let E_i, $i = 1, \cdots, n$, be a disjoint collection of A-D-E chains of embedded 2-spheres of self-intersection -2 in a K3 surface X and τ_i be the associated monodromy diffeomorphism supported in a small neighborhood of E_i. One can choose these neighborhoods to be disjoint from each other so that τ_i commute with each other. Consequently, they generate an abelian subgroup $\langle\, \tau_1, \cdots, \tau_n \,\rangle$ in $\mathit{Diff}(\mathrm{K3})$.

Let $L = K_1 \cup \cdots \cup K_i \cup \cdots \cup K_k$ be a hyperbolic link in S^3. As in Example 2.3.4, let (m_i, l_i) be the meridian and the longitute pair of K_i, $C = [\,c_{ij}\,]_{ij} = [\,lk(K_i, K_j)\,]_{ij}$ be the linking matrix of L with respect to some orientation of L and S^3, and the manifold from surgery $(S^3 - L)_{(a_1,b_1;\cdots;a_i,b_i;\cdots;a_k,b_k)}$ be hyperbolic. Since $\langle\, \tau_1, \cdots, \tau_n \,\rangle$ is abelian, a representation from $\pi_1(S^3 - L)$ to $\langle\, \tau_1, \cdots, \tau_n \,\rangle$ is determined by a representation ρ_X from $H_1(S^3 - L; \mathbb{Z})$, which is a free abelian group generated by $\{m_1, \cdots, m_k\}$, to $\langle\, \tau_1, \cdots, \tau_n \,\rangle$. Let

$$\rho_X(m_i) \;=\; \tau_1^{\alpha_{i1}} \cdots \tau_n^{\alpha_{in}}\,,$$

then

$$\rho_X(l_i) \;=\; \tau_1^{\sum_{j=1}^k c_{ij}\alpha_{j1}} \cdots \tau_n^{\sum_{j=1}^k c_{ij}\alpha_{jn}}$$

and, for $(a_i, b_i) \neq \infty$,

$$\rho_X(a_i m_i + b_i l_i) \;=\; \tau_1^{a_i\alpha_{i1} + b_i\sum_{j=1}^k c_{ij}\alpha_{j1}} \cdots \tau_n^{a_i\alpha_{in} + b_i\sum_{j=1}^k c_{ij}\alpha_{jn}}\,.$$

For such boundary component of $S^3 - \nu(L)$, let us investigate how the K3-bundle

$$\pi \;:\; (S^3 - \nu(L)) \times_{\rho_X} X \;\longrightarrow\; S^3 - \nu(L)\,.$$

can be filled.

Consider the solid torus $V = \Delta \times S^1$, where Δ is a compact disk in $\mathbb{C}$. Let $\hat{m}$ and $\hat{l}$ be a meridian and a longitude of V respectively. A representation ρ from $\pi_1(\partial V) = H_1(\partial V; \mathbb{Z})$ to $\langle\, \tau_1, \cdots, \tau_n \,\rangle$ with

$$\rho(\hat{m}) \;=\; f_{\hat{m}} \;=\; \tau_1^{k_1} \cdots \tau_n^{k_n} \qquad \text{and} \qquad \rho(\hat{l}) \;=\; f_{\hat{l}} \;=\; \tau_1^{k'_1} \cdots \tau_n^{k'_n}\,,$$

defines a flat K3-bundle:

$$\mathrm{pr}_1 \;:\; \partial V \times_\rho X \;=\; (\mathbb{R}^2 \times X)/(\pi_1(\partial V), \rho(\pi_1(\partial V))) \;\longrightarrow\; \partial V\,,$$

where the group action on $\mathbb{R}^2 \times X$ is diagonal. Let $\mathrm{pr}_2 : Z \to \Delta$ be a C^∞-K3 fibration (cf. Definition 4.3) with degenerate fibers X_{ij} over s_{ij} in the interior of Δ the singular K3 surface with $E_i \subset X$ pinched to a point, where $i \in \{i' | k_{i'} \neq 0\}$ and $j = 1, \ldots, |k_i|$, such that the monodromy of the fibration along $\partial\Delta$ (with induced orientation from that of Δ) is precisely $f_{\hat{m}}$. Such fibration can be easily constructed by a fiber sum of the corresponding complex 1-parameter family degenerations $\pi_{ij} : Z_{ij} \to (\Delta, 0)$ that give each individual isolated A-D-E singularity, as indicated in FIGURE 4-3. Since τ_i can act on

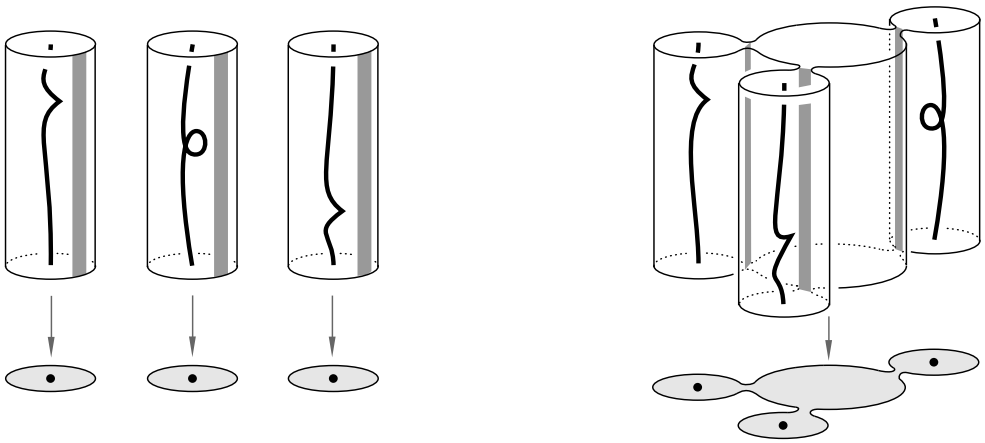


FIGURE 4-3. A fiber sum of basic K3-fibrations associated to complex 1-parameter family degenerations $\pi_{ij} : Z_{ij} \to (\Delta, 0)$, realized as the fiber sum of each fibration to the trivial fibration $\Delta \times X \to \Delta$. The degenerate fiber of π_{ij} is shown and the occurrence of fiber sum is indicated by grey shade.

Z_{ij} as a fibration C^1-automorphism and Z is constructed by fiber sum, τ_i acts also on Z. Together with the fact that $\tau_1, \cdots, \tau_n$ commute, this implies that $f_{\hat{l}}$ can act on Z as a fibration C^1-automorphism that projects to the identity map on Δ under pr_2 and commutes with $f_{\hat{m}}$.

By taking the mapping torus $Z_{f_{\hat{l}}}$ of Z with respect to $f_{\hat{l}}$, one then obtains a K3-fibration

$$\mathrm{pr} \ : \ Z_{f_{\hat{l}}} \ \longrightarrow \ V \ ,$$

whose restriction to $\mathbb{T}^2 = \partial V$ is equivalent to pr_1 and hence can be used to bundle-fill pr_1. Applying such bundle-filling to π for each boundary $\mathbb{T}^2$ of $S^3 - \nu(L)$ with $(a_i.b_i) \neq \infty$, one obtains thus a smooth-K3T 7-manifold

$$\pi' \ : \ M^7 \ \longrightarrow \ (S^3 - L)_{(a_1,b_1;\cdots;a_i,b_i;\cdots;a_k,b_k)} \, .$$

This concludes the example.

□

Remark 4.5 [flat smooth-K3T]. Similar to the analytic category, one can also consider $(X, \mathit{Diff}(\mathrm{K3}))$ for a K3 surface X and apply the construction in Sec. 2.2 to obtain flat smooth-K3T 7-manifolds. A general theory of such involves in particular two issues:

(1) the structure of $\mathit{Diff}(\mathrm{K3})$ and/or the mapping class group $MCG(\mathrm{K3})$ of a K3 surface and

(2) representations of a 3-manifold group $\pi_1(N^3)$ into $\mathit{Diff}(\mathrm{K3})$ or $MCG(\mathrm{K3})$,

which by themselves are already very challenging. As is well-known, the mapping class group of a compact Riemann surface is finitely generated by the Dehn twists along a system of simple loops (e.g. [Bi] and [H-T]); it is interesting to know if something similar may happen for K3 surfaces.

Remark 4.6 However, in the smooth category, it is not clear to us what kind of finite dimensional deformation space can be associated to a K3T 7-manifold.

5 Issues on applications to M-theory compactification.

K3T 7-manifolds are constructed as natural interpolating 7-manifolds among some K3-fibred Calabi-Yau threefolds. This is meant to reflects the physical fact that M-theory interpolates various string theories. In this section, we list some issues for further study in order to understand the role of K3T 7-manifolds in M-theory.

• **K3T and supergravity.** M-theory has 11-dimensional supergravity as its low-enery limit ([H-W1]). The field contents of the latter [C-J-S] consist of the 11-bein e^a_μ (graviton), a Majorana spin-$\frac{3}{2}$ field ψ_μ (gravitino), and a 3-form $A_{\mu\nu\rho}$ as gauge tensor. In a local chart, the bosonic part of the field equations in the theory reads (cf. [Fre], summation convention for repeating indices assumed)

$$\begin{aligned} R_{mn} - \frac{1}{2} g_{mn} R &= -\frac{1}{3}(H_{mpql} H_n{}^{pql} - \frac{1}{8} g_{mn} H^2) \\ \nabla_m H^{mpql} &= -\frac{1}{576} \varepsilon^{m_1 \cdots m_8 pql} H_{m_1 \cdots m_4} H_{m_5 \cdots m_8}\,, \end{aligned}$$

where $g_{mn} = e^p_m e^q_n \eta_{pq}$ with $\eta_{pq} = \mathit{Diag}(-1, 1, \cdots, 1)$ the standard metric on the 11-dimensional Minkowskian space-time $\mathbb{R}^{1+10}$, R_{mn} and R are the Ricci and scaler curvature of g, ∇ the covariant derivative associated to g, $H = dA$, and $\varepsilon = \varepsilon_{m_1 \cdots m_{11}}$ is the standard volume-form on $\mathbb{R}^{1+10}$. The imposition of various ansatz and the requirement of the residual supersymmetry when compactified to 4 dimensions lead to constraints on the underlying geometry, for example, existence of Killing spinors (e.g. [Fre]) and many exact solutions (e.g. [C-R-W], [D-N-P1], and [vN-W]) to the above equations. Though related to Calabi-Yau threefolds by taking boundary and to K3-fibred Joyce manifolds as a generalization via 3-dimensional geometry, unlike either kind of manifolds, general K3T 7-manifolds do not seem to satisfy these known constraints, nor is it clear where there

exists a K3T 7-manifold that supports a solution to the general field equations above. Perhaps one has to look for other possible interfaces between K3T and M.

• **Deformation space of K3T vs. deformation space of Calabi-Yau.** One such candidate interface is via deformation space. This is rooted at the fundamental theorem on hyperbolic 3-manifolds ([Ber1, Ber2], [McM], and [Su]), which states roughly that, given a hyperbolic 3-manifold M^3, the Teichmüller space of complete hyperbolic structures on M^3 is isomorphic to the Teichmüller space of complex structures on its ideal boundary $\partial_\infty M^3$ by sending a hyperbolic structure on M^3 to the conformal structure it induces on $\partial_\infty M^3$. We already saw this kind of "*boundary-dictate-interior*" behavior in Sec. 3, where hyperbolic structures on N^3 is controlled by the complex affine structures on its toroidal ∂N^3. For analytic K3T 7-manifolds with Calabi-Yau boundary, this suggests a close relation between Def(K3T) and Def(Calabi-Yau), perhaps via fibration, embedding, or limit (cf. [B-D-F-P-T-P-Z] and [Le]).

For the class of K3T 7-manifolds constructed in Sec. 2, let $(u_1, \cdots, u_h)$ and $(v_1, \ , v_h)$ be the conjugate coordinates systems for $U \subset Def(M^3)$ in Sec. 3, then an immediate map relating Def(K3T) and Def(Calabi-Yau) is given by

$$\begin{array}{ccccc} Def(\,\mathrm{K3T}\,) & \longrightarrow & Def_c(\,\partial(\mathrm{K3T})\,) & \subset & Def(\,\cup_{i=1}^h \mathbb{T}^2 \times X\,) \\ (w\,;\,u_1,\cdots,u_h) & \longmapsto & \Big(\,(\frac{v_1}{u_1}, w)\,,\cdots,\,(\frac{v_h}{u_h}, w)\,\Big)\,, & & \end{array}$$

where w is the complex structure of X and $Def_c(\partial(\mathrm{K3T}))$ is the deformation space of the complex structures on the boundary $\cup_i \mathbb{T}^2 \times X$. Details of this map and what Kähler deformations of the Calabi-Yau boundary are translated to on the K3T side remain to be explored.

• **K3T, dilogarithm, and CFT.** The appearance of hyperbolic 3-manifolds of finite volume as a key ingredient in K3T 7-manifolds has a mysterious feature. We already saw in Sec. 3 that the deformation space of a such 3-manifold M^3 has a special Kähler structure with a prepotential $\mathfrak{F}$ related to the volume function *vol* (cf. Remark 3.3.8). It turns out that vol is related to the dilogarithm function ([N-Z] and [Za]) defined by the analytic continuation of the following power series

$$Li_2(z) \;=\; \sum_{n=1}^{\infty} \frac{z^2}{n^2} \qquad \text{on} \qquad \{\, z \,|\, \|z\| < 1 \,\} \;\subset\; \mathbb{C}\,.$$

Some role of this function in conformal field theory (CFT) was explored in [Na] and [N-R-T]. Can one deepen their work and, from which, find yet another way K3T and string/M-theory may be related?

We conclude this paper with these unresolved issues for future pursuit.

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