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Representation of the quantum mechanical wavefunction by orthogonal polynomials in the energy and physical parameters

A D Alhaidari

Saudi Center for Theoretical Physics, PO Box 32741, Jeddah 21438, Saudi Arabia

E-mail: haidari@sctp.org.sa

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Abstract

We present a formulation of quantum mechanics based on the theory of orthogonal polynomials. The wavefunction is expanded over a complete set of square integrable basis where the expansion coefficients are orthogonal polynomials in the energy and physical parameters. Information about the corresponding physical systems (both structural and dynamical) are derived from the properties of these polynomials. We demonstrate that an advantage of this formulation is that the class of *exactly solvable* quantum mechanical problems becomes larger than in the conventional formulation (see, for example, table 3 in the text). We limit our investigation in this work to the Askey classification scheme of hypergeometric orthogonal polynomials and focus on the Wilson polynomial and two of its limiting cases (the Meixner–Pollaczek and continuous dual Hahn polynomials). Nonetheless, the formulation is amenable to other classes of orthogonal polynomials.

Keywords: tridiagonal representation, orthogonal polynomials, recursion relation, asymptotics, energy spectrum, phase shift

1. Introduction

In physics, we are accustomed to writing vector quantities (e.g. force, velocity, electric field, etc) in terms of their components in some conveniently chosen vector space. For example, the force $\vec{F}$ is written in three dimensional space with Cartesian coordinates as $\vec{F} = f_x\hat{x} + f_y\hat{y} + f_z\hat{z}$, where $\{f_x, f_y, f_z\}$ are the components of the force along the unit vectors $\{\hat{x}, \hat{y}, \hat{z}\}$. These components contain all physical information about the quantity whereas the unit vectors (basis) are dummy, but must form a complete set to allow for a faithful physical representation. In fact, we can as well write the same force in another coordinates, say the spherical coordinates with basis $\{\hat{r}, \hat{\theta}, \hat{\varphi}\}$, as $\vec{F} = f_r\hat{r} + f_\theta\hat{\theta} + f_\varphi\hat{\varphi}$, where $\{f_r, f_\theta, f_\varphi\}$ are the new components that contain the same physical information. And so on, where in general we can write $\vec{F} = \sum_n f_n\hat{x}_n$. The basis (unit vectors) $\{\hat{x}_n\}$ is chosen conveniently depending on the symmetry of the problem (e.g. rectangular, spherical, cylindrical, elliptical, etc). It is basic knowledge that if two unit vectors are independent then they can be, but not necessarily, orthogonal (i.e. $\langle\hat{x}_n|\hat{x}_m\rangle = \delta_{nm}$), however they cannot be collinear (i.e. $\langle\hat{x}_n|\hat{x}_m\rangle = \pm 1$ for $n \neq m$). Therefore, the chosen basis set must be complete and consisting of independent elements but they do not have to be orthogonal to each other.

In quantum mechanics we can also think of the wavefunction $\psi(x)$ as a local vector and write it in terms of its components $\{f_n\}$ along some local unit vectors (basis) $\{\phi_n(x)\}$ as $\psi(x) = \sum_n f_n\,\phi_n(x)$. All physical information about the system are contained in the components (expansion coefficients) $\{f_n\}$. On the other hand, the basis set $\{\phi_n(x)\}$ is dummy but, in analogy to the unit vectors in the above example, must be normalizable (square integrable) and complete. If the physical system is also associated with a set of real parameters $\{\mu\}$, then the components of the wavefunction at an energy E could be written as parameterized functions of the energy, $\{f_n^\mu(E)\}$, and the state of the system becomes

$$\psi_E^\mu(x) = \sum_n f_n^\mu(E)\,\phi_n(x). \tag{1}$$

We have shown elsewhere [1] that if we write $f_n^\mu(E) = f_0^\mu(E)P_n^\mu(\varepsilon)$, where ε is some proper function of E and $\{\mu\}$, then completeness of the basis and energy normalization of the density of state make $\{P_n^\mu(\varepsilon)\}$ a complete set of

Table 1. Asymptotics ($n \to \infty$) of most of the continuous polynomials in the Askey scheme of hypergeometric orthogonal polynomials. The polynomials are shown in the second column in their orthonormal version. The asymptotics of the Wilson polynomial is obtained here in section 2 whereas the asymptotics of the continuous Hahn polynomial is derived in [9]. The asymptotics of the Meixner–Pollaczek and continuous dual Hahn polynomials are obtained in [1]. The rest are well known.

Polynomial	Orthonormal version	x	Asymptotics [1, 9]	τ	ξ	$\theta(x)$	$\varphi(x)$
Laguerre	$\sqrt{\frac{n!}{(\nu+1)_n}}\, L_n^{\nu}(x)$	$x \geqslant 0$	$n^{-1/4} A_L(x) \cos\left[2\sqrt{nx} - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right]$	$1/4$	$1/2$	$2\sqrt{x}$	0
Jacobi	$\sqrt{\frac{2n+\mu+\nu+1}{\mu+\nu+1}\frac{n!\,(\mu+\nu+1)_n}{(\mu+1)_n(\nu+1)_n}}$ $\times P_n^{(\mu,\nu)}(\cos x)$	$\pi \geqslant x \geqslant 0$	$A_J(x)\cos\left[\left(n + \frac{\mu+\nu+1}{2}\right)x - \frac{\pi}{2}\left(\mu + \frac{1}{2}\right)\right]$	0	1	x	0
Meixner–Pollaczek	$P_n^{\mu}(x;\,\theta)$	$x \in \mathbb{R}$	$n^{-1/2} A_{MP}(x)\cos[n\theta - x\log n + \delta_{MP}(x)]$	$1/2$	1	θ	$-x$
Continuous Hahn	$\mathscr{P}_n^{\mu}(x;\,\nu;\,a,\,b)$	$x \in \mathbb{R}$	$n^{-1/2} A_H(x)\cos\left[n\frac{\pi}{2} - (2x + a - b)\log n + \delta_H(x)\right]$	$1/2$	1	$\pi/2$	$-(2x + a - b)$
Continuous dual Hahn	$S_n^{\mu}(x^2;\,a,\,b)$	$x \geqslant 0$	$n^{-1/2} A_{dH}(x)\cos[x\log n + \delta_{dH}(x)]$	$1/2$	**	0	x
Wilson	$W_n^{\mu}(x^2;\,\nu;\,a,\,b)$	$x \geqslant 0$	$n^{-1/2} A_W(x)\cos[2x\log n + \delta_W(x)]$	$1/2$	**	0	$2x$

orthogonal polynomials with an associated non-negative weight function $\rho^\mu(\varepsilon) = [f_0^\mu(E)]^2$. That is,

$$\int \rho^\mu(\varepsilon) P_n^\mu(\varepsilon) P_m^\mu(\varepsilon)\,\mathrm{d}\varepsilon = \delta_{n,m}. \tag{2}$$

Therefore, we can rewrite the wavefunction expansion (1) as follows

$$\psi_E^\mu(x) = \sqrt{\rho^\mu(\varepsilon)} \sum_n P_n^\mu(\varepsilon)\,\phi_n(x). \tag{3}$$

In cases where $\phi_n(x) \sim \sigma(x) x^n$, this formula suggests that the wavefunction could also be considered as the generating function of the energy polynomials $\{P_n^\mu(\varepsilon)\}$. In their treatment of quasi-exactly solvable systems, Bender and Dunne were among the first to realize this fact [2].

Building on our past and recent experience in dealing with various physical systems [1, 3–9], we limit the current investigation of the energy polynomials to the Askey classification scheme of hypergeometric orthogonal polynomials. We focus on the Wilson polynomial and two of its limiting cases [10]: the Wilson → continuous dual Hahn (${}_4F_3 \to {}_3F_2$) and the Wilson → Meixner–Pollaczek (${}_4F_3 \to {}_3F_2 \to {}_2F_1$). There is another chain in the Askey diagram that deals with the discrete version of hypergeometric orthogonal polynomials. At the top of this chain sits the Racah polynomial and it includes in the limit other discrete polynomials like the Meixner, Krawtchouk, Hahn, dual Hahn, etc. Now, the connection between scattering and the asymptotics ($n \to \infty$) of continuous orthogonal polynomials is well-stablished [11–13]. Using this connection, scattering information about the system, whose wavefunction is written as shown by equation (3), is readily available from the asymptotics $P_{n\to\infty}^\mu(\varepsilon)$. Table 1 shows a list of the asymptotics of most of the continuous polynomials in the Askey scheme of hypergeometric orthogonal polynomials. The asymptotics of the Wilson polynomial is obtained in section 2 below whereas the asymptotic of the continuous Hahn polynomial is derived in appendix B of [9] (see equation (37) therein). The rest are either well known or derived elsewhere (see, for example, the appendix in [1] where the asymptotics of the Meixner–Pollaczek and continuous dual Hahn polynomials were obtained). We should note that throughout this paper we deal with the orthonormal version of orthogonal polynomials where the right-hand side of the orthogonality relation (2) is $\delta_{n,m}$, the corresponding three-term recursion relation is symmetric, and $P_0^\mu(\varepsilon) = 1$. Due to the prime significance of the Wilson polynomial (being at the top of the Askey scheme), we present the derivation of its asymptotics in section 2. We should note that despite the fact that the Meixner–Pollaczek and continuous dual Hahn polynomials are limiting cases of the Wilson polynomial, we could not use the asymptotics of the Wilson polynomial because we cannot interchange this limit with the asymptotic limit. Table 1 suggests that all orthogonal polynomials relevant to our study are those with the following asymptotic behavior

$$P_n^\mu(\varepsilon) \approx n^{-\tau} A^\mu(\varepsilon)\{\cos[n^\xi\,\theta(\varepsilon) + \varphi(\varepsilon)\log n + \delta^\mu(\varepsilon)] + O(n^{-1})\}, \tag{4}$$

where τ and ξ are real dimensionless constants with the value of τ being dependent on the type of normalization of the polynomial, which we consistently choose as orthonormalization where the right-hand side of the orthogonality relation (2) is $\delta_{n,m}$ and $P_0^\mu(\varepsilon) = 1$. In the asymptotic formula (4), $A^\mu(\varepsilon)$ is the scattering amplitude and $\delta^\mu(\varepsilon)$ is the phase shift. For the polynomials listed in table 1, the values of the parameters τ and ξ belong to the sets $\tau \in \left\{0, \frac{1}{4}, \frac{1}{2}\right\}$, $\xi \in \left\{0, \frac{1}{2}, 1\right\}$ and we observe the following three alternative scenarios:

(1) $\theta(\varepsilon) \neq 0$, $\varphi(\varepsilon) = 0$.
(2) $\theta(\varepsilon) = 0$, $\varphi(\varepsilon) \neq 0$.
(3) $\theta(\varepsilon) \neq 0$, $\varphi(\varepsilon) \neq 0$.

In the third scenario, the simultaneous presence of the logarithmic term ($\log n$) and power term (n^ξ), like in the case of the Meixner–Pollaczek and continuous Hahn polynomials, is very interesting and a source of curiosity. The understanding of this behavior and its physical implication should be very fruitful. Unfortunately, we do not have the needed expertise to address this issue at present.

Bound states, if they exist, occur at a set (infinite or finite) of energies that make the scattering amplitude vanish. That is, the kth bound state occurs at an energy $E_k = E(\varepsilon_k)$ such that $A^\mu(\varepsilon_k) = 0$ and the corresponding bound state wavefunction is written as[1]

$$\psi_k^\mu(x) = \sqrt{\omega^\mu(\varepsilon_k)} \sum_n Q_n^\mu(\varepsilon_k)\,\phi_n(x), \tag{5}$$

where $\{Q_n^\mu(\varepsilon_k)\}$ are the discrete version of the polynomials $\{P_n^\mu(\varepsilon)\}$ and $\omega^\mu(\varepsilon_k)$ is the associated discrete weight function. That is, $\sum_k \omega^\mu(\varepsilon_k) Q_n^\mu(\varepsilon_k) Q_m^\mu(\varepsilon_k) = \delta_{n,m}$. Sometimes, the quantum system consists of continuous as well as discrete energy spectra simultaneously. In that case, the total wavefunction is written as follows

$$\Psi^\mu(x, t) = \int \mathrm{e}^{-\mathrm{i}Et} \psi_E^\mu(x)\mathrm{d}E + \sum_k \mathrm{e}^{-\mathrm{i}E_k t} \psi_k^\mu(x), \tag{6}$$

and the appropriate polynomial orthogonality becomes

$$\int \rho^\mu(\varepsilon) P_n^\mu(\varepsilon) P_m^\mu(\varepsilon)\,\mathrm{d}\varepsilon + \sum_k \omega^\mu(\varepsilon_k) P_n^\mu(\varepsilon_k) P_m^\mu(\varepsilon_k) = \delta_{n,m}. \tag{7}$$

Now, the type of orthogonal polynomials associated with a given physical system depends on the structure of its energy spectrum: purely continuous, purely discrete or a mix of both and on whether the discrete energy spectrum is infinite or finite. Table 2 summarizes this association.

In the conventional (textbook) formulation of quantum mechanics, the potential function plays a central role in

[1] There is an alternative, but equivalent, method for obtaining the energy spectrum from the phase shift by calculating the poles of the corresponding scattering matrix. That is, the phase shift angle at those energies becomes half-odd integer of π making the tangent of the phase angle diverge. Frequently, such condition occurs when the argument of the gamma function that appears in the phase shift becomes a negative integer (or zero) corresponding to the energy level in the spectrum. This alternative method is addressed in the book by Landau and Lifshitz [14] and recently by Chen *et al* in [15, 16], etc.

Table 2. The orthogonal polynomial(s) associated with a given physical system as a function of the structure of its energy spectrum. Note the matching of the spectra of the physical system and that of its corresponding orthogonal polynomial except when the physical system has a mix of a continuous and an infinite discrete energy spectrum then there is a need for two polynomials to represent the system: a continuous one and a discrete one with infinite spectrum.

Energy spectrum			Polynomial spectrum		
	Discrete			Discrete	
Continuous	Finite	Infinite	Continuous	Finite	Infinite
✓	✗	✗	✓	✗	✗
✗	✗	✓	✗	✗	✓
✓	✓	✗	✓	✓	✗
✓	✗	✓	✓	✗	✗
			✗	✗	✓

providing physical information about the system. Hence, in the present formulation, the set of orthogonal polynomials replaces the potential function in this role. In fact, it is more than just that. As we shall see below, the orthogonal polynomials also carry kinematic information (e.g. the angular momentum) whereas the potential function does not.

Since the polynomials that are relevant to our work satisfy the general orthogonality relation (7) and have the asymptotic behavior (4) then Favard's theorem [17] dictates that such polynomials must satisfy a three-term recursion relation of the form $\varepsilon P_n^\mu(\varepsilon)=a_n^\mu P_n^\mu(\varepsilon)+b_{n-1}^\mu P_{n-1}^\mu(\varepsilon)+c_n^\mu P_{n+1}^\mu(\varepsilon)$, where $b_n^\mu c_n^\mu>0$ for all n. Of course, not all polynomials satisfy three-term recursion relations. In fact, some satisfy higher order recursions (e.g. four-term and five-term). However, such polynomials are not in the scope of our present study. Consequently, the three-term recursion relation dictates that the basis set $\{\phi_n(x)\}$ must produce a tridiagonal matrix representation for the corresponding wave operator. As such, the matrix wave equation becomes equivalent, and amenable to the said recursion relation. In this paper, we leave out technical details but include the most relevant information concerning orthogonal polynomials in the Appendices. Interested readers may consult cited references for the derivation of applicable results, especially [1, 3].

In the following section and due to the prime significance of the Wilson polynomial that sits at the top of the Askey tree of the hypergeometric class of orthogonal polynomials, we derive its asymptotics using the Darboux's method and show that it agrees with the original work of Wilson [18] and with the second scenario of formula (4). In sections 3–5, we present several examples of orthogonal polynomials from the Askey scheme of the hypergeometric type and derive properties of the corresponding physical systems. Finally, we conclude in section 6 by making relevant comments and discussing related issues. Throughout the paper, we adopt the atomic units, $\hbar=M=1$.

2. Asymptotics of the Wilson polynomial

We write the Wilson polynomial in a different notation from that which is given by equation (1.1.1) on page 24 of [10] as follows

$$\tilde{W}_n^\mu(z^2;\nu;a,b)=\frac{(\mu+a)_n(\mu+b)_n}{(a+b)_n n!}\times{}_4F_3\left(\left.\begin{matrix}-n,n+\mu+\nu+a+b-1,\mu+\mathrm{i}z,\mu-\mathrm{i}z\\ \mu+\nu,\mu+a,\mu+b\end{matrix}\right|1\right),\tag{8}$$

where $z\geqslant 0$. We can relate this notation to that in [10] as $\tilde{W}_n^\mu(z^2;\nu;a,b)=W_n(z^2;\mu,\nu,a,b)/n!\,(\mu+\nu)_n(a+b)_n$. If $\mathrm{Re}(\mu,\nu,a,b)>0$ and non-real parameters occur in conjugate pairs, then the generating function of these Wilson polynomials becomes (see equation (1.1.12) in [10])

$$\sum_{n=0}^\infty \tilde{W}_n^\mu(z^2;\nu;a,b)\,t^n={}_2F_1\left(\left.\begin{matrix}\mu+\mathrm{i}z,\nu+\mathrm{i}z\\ \mu+\nu\end{matrix}\right|t\right)\times{}_2F_1\left(\left.\begin{matrix}a-\mathrm{i}z,b-\mathrm{i}z\\ a+b\end{matrix}\right|t\right).\tag{9}$$

Now, we apply the Darboux method to this generating function to obtain the asymptotics of $\tilde{W}_n^\mu(z^2;\nu;a,b)$. The method is described in section 9 of chapter 8 in [19]. In addition, we also need the contiguous relation (equation (7) of Ex. 21 in [20])

$$(a+b-c)_2F_1\left(\left.\begin{matrix}a,b\\ c\end{matrix}\right|z\right)=a(1-z)_2F_1\left(\left.\begin{matrix}a+1,b\\ c\end{matrix}\right|z\right)-(c-b)_2F_1\left(\left.\begin{matrix}a,b-1\\ c\end{matrix}\right|z\right),\tag{10}$$

and the Euler transformation [20]

$${}_2F_1\left(\left.\begin{matrix}a,b\\ c\end{matrix}\right|z\right)=(1-z)^{c-a-b}{}_2F_1\left(\left.\begin{matrix}c-a,c-b\\ c\end{matrix}\right|z\right).\tag{11}$$

Moreover, we will also employ the Gauss sum (Theorem 18 in section 32 of [20])

$${}_2F_1\left(\left.\begin{matrix}a,b\\ c\end{matrix}\right|1\right)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\quad \mathrm{Re}\,(c-a-b)>0.\tag{12}$$

The contiguous relation (10) makes the first ${}_2F_1$ on the right side of the generating function (9) equal to

$$\frac{1}{2\mathrm{i}z}\left[(\mu+\mathrm{i}z)(1-t)_2F_1\left(\left.\begin{matrix}\mu+1+\mathrm{i}z,\nu+\mathrm{i}z\\ \mu+\nu\end{matrix}\right|t\right)-(\mu-\mathrm{i}z)_2F_1\left(\left.\begin{matrix}\mu+\mathrm{i}z,\nu-1+\mathrm{i}z\\ \mu+\nu\end{matrix}\right|t\right)\right].\tag{13}$$

We use the Euler transformation (11) to rewrite the first term inside the square bracket above as

$$(\mu+\mathrm{i}z)(1-t)^{-2\mathrm{i}z}{}_2F_1\left(\left.\begin{matrix}\nu-1-\mathrm{i}z,\,\mu-\mathrm{i}z\\ \mu+\nu\end{matrix}\right|t\right). \tag{14}$$

Therefore, the first ${}_2F_1$ term on the right side of the generating function (9) becomes

$$\frac{(1-t)^{-\mathrm{i}z}}{2\mathrm{i}z}\left[(\mu+\mathrm{i}z)(1-t)^{-\mathrm{i}z}{}_2F_1\left(\left.\begin{matrix}\mu-\mathrm{i}z,\,\nu-1-\mathrm{i}z\\ \mu+\nu\end{matrix}\right|t\right)\right.$$
$$\left.-\text{ complex conjugate}\right]. \tag{15}$$

In accordance with the Darboux method, the dominant term in a comparison function for the t-expansion of this factor becomes its limiting value at $t=1$. The Gauss sum (12) evaluates the above ${}_2F_1$ at $t=1$ as $\frac{\Gamma(\mu+\nu)\Gamma(1+2\mathrm{i}z)}{\Gamma(\mu+1+\mathrm{i}z)\Gamma(\nu+\mathrm{i}z)}=\frac{2\mathrm{i}z}{\mu+\mathrm{i}z}\frac{\Gamma(\mu+\nu)\Gamma(2\mathrm{i}z)}{\Gamma(\mu+\mathrm{i}z)\Gamma(\nu+\mathrm{i}z)}$. Thus, expression (15) near $t=1$ becomes

$$\begin{aligned}(1-t)^{-\mathrm{i}z}&\left[(1-t)^{-\mathrm{i}z}\frac{\Gamma(\mu+\nu)\Gamma(2\mathrm{i}z)}{\Gamma(\mu+\mathrm{i}z)\Gamma(\nu+\mathrm{i}z)}\right.\\ &\left.+\text{ complex conjugate}\right]\\ &=(1-t)^{-\mathrm{i}z}\frac{\Gamma(\mu+\nu)|\Gamma(2\mathrm{i}z)|}{|\Gamma(\mu+\mathrm{i}z)\Gamma(\nu+\mathrm{i}z)|}\\ &\left.\times[(1-t)^{-\mathrm{i}z}\mathrm{e}^{\mathrm{i}\alpha}+\mathrm{c.\,c.}\right],\end{aligned} \tag{16}$$

where $\alpha=\arg[\Gamma(2\mathrm{i}z)/\Gamma(\mu+\mathrm{i}z)\Gamma(\nu+\mathrm{i}z)]$. Repeating the same treatment on the second ${}_2F_1$ on the right side of the generating function (9), we obtain the same result as (16) but with the replacement $z\to-z$, $\mu\to a$ and $\nu\to b$ giving

$$\begin{aligned}(1-t)^{+\mathrm{i}z}&\left[(1-t)^{+\mathrm{i}z}\frac{\Gamma(a+b)\Gamma(-2\mathrm{i}z)}{\Gamma(a-\mathrm{i}z)\Gamma(b-\mathrm{i}z)}\right.\\ &\left.+\text{ complex conjugate}\right]\\ &=(1-t)^{+\mathrm{i}z}\frac{\Gamma(a+b)|\Gamma(2\mathrm{i}z)|}{|\Gamma(a+\mathrm{i}z)\Gamma(b+\mathrm{i}z)|}\\ &\left.\times[(1-t)^{-\mathrm{i}z}\mathrm{e}^{\mathrm{i}\beta}+\mathrm{c.\,c.}\right],\end{aligned} \tag{17}$$

where $\beta=\arg[\Gamma(2\mathrm{i}z)/\Gamma(a+\mathrm{i}z)\Gamma(b+\mathrm{i}z)]$. Multiplication of (16) and (17) gives

$$\begin{aligned}&\frac{\Gamma(\mu+\nu)\Gamma(a+b)\ |\Gamma(2\mathrm{i}z)|^2}{|\Gamma(\mu+\mathrm{i}z)\Gamma(\nu+\mathrm{i}z)\Gamma(a+\mathrm{i}z)\Gamma(b+\mathrm{i}z)|}\\ &\quad\times[(1-t)^{-2\mathrm{i}z}\mathrm{e}^{\mathrm{i}(\alpha+\beta)}+\mathrm{e}^{\mathrm{i}(\alpha-\beta)}+\mathrm{c.\,c.}].\end{aligned} \tag{18}$$

Aside from t-independent factors, the comparison function near $t=1$ of the above expression is $(1-t)^{-2\mathrm{i}z}$. The expansion of this term is

$$(1-t)^{-2\mathrm{i}z}=\sum_{n=0}^{\infty}\frac{(2\mathrm{i}z)_n}{\Gamma(n+1)}t^n. \tag{19}$$

Therefore, applying the Darboux method to the generating function (9) gives the following asymptotics for the Wilson polynomial

$$\begin{aligned}&\tilde{W}_n^{\mu}(z^2;\nu;a,b)\\ &\approx\frac{\Gamma(\mu+\nu)\Gamma(a+b)|\Gamma(2\mathrm{i}z)|}{|\Gamma(\mu+\mathrm{i}z)\Gamma(\nu+\mathrm{i}z)\Gamma(a+\mathrm{i}z)\Gamma(b+\mathrm{i}z)|}\\ &\quad\times[n^{2\mathrm{i}z-1}\mathrm{e}^{-\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}(\alpha+\beta)}+\mathrm{c.\,c.}],\end{aligned} \tag{20}$$

where $\gamma=\arg[\Gamma(2\mathrm{i}z)]$ and we have used $(z)_n=\frac{\Gamma(n+z)}{\Gamma(z)}$ and $\frac{\Gamma(n+a)}{\Gamma(n+b)}\approx n^{a-b}$. Now, using $\arg(a)+\arg(b)=\arg(ab)$, $\arg(a)-\arg(b)=\arg(a/b)$ and $a^{\mathrm{i}b}=\mathrm{e}^{\mathrm{i}b\ln a}$, we obtain

$$\begin{aligned}\tilde{W}_n^{\mu}(z^2;\nu;a,b)&\approx\frac{2}{n}\Gamma(\mu+\nu)\Gamma(a+b)|\mathscr{A}(\mathrm{i}z)|\\ &\times\cos\{2z\ln(n)+\arg[\mathscr{A}(\mathrm{i}z)]\}+O(n^{-1}),\end{aligned} \tag{21}$$

where $\mathscr{A}(z)=\Gamma(2z)/\Gamma(\mu+z)\Gamma(\nu+z)\Gamma(a+z)\Gamma(b+z)$. This finding agrees with Wilson's result [18][2], which was obtained using a convexity argument which is especially well suited to estimating certain hypergeometric series and their integral analogs. The asymptotics of the orthonormal version of the polynomial, which is given in appendix C by equation (C1), could easily be obtained from (21) as

$$\begin{aligned}W_n^{\mu}(z^2;\nu;a,b)&\approx B(\mu,\nu,a,b)\sqrt{\frac{2}{n}}\{2|\mathscr{A}(\mathrm{i}z)|\\ &\times\cos[2z\ln n+\arg\mathscr{A}(\mathrm{i}z)]+O(n^{-1})\},\end{aligned} \tag{22}$$

where

$$B(\mu,\nu,a,b)=\sqrt{\frac{\Gamma(\mu+\nu)\Gamma(a+b)\Gamma(\mu+a)\Gamma(\mu+b)\Gamma(\nu+a)\Gamma(\nu+b)}{\Gamma(\mu+\nu+a+b)}}$$

and we have again used the asymptotic identity $\frac{\Gamma(n+a)}{\Gamma(n+b)}\approx n^{a-b}$. Comparing this asymptotics to formula (4) leads to the following scattering amplitude and phase shift

$$A^{\mu}(\varepsilon)=\frac{2\sqrt{2}B(\mu,\nu,a,b)}{|\Gamma(\mu+\mathrm{i}z)\Gamma(\nu+\mathrm{i}z)\Gamma(a+\mathrm{i}z)\Gamma(b+\mathrm{i}z)/\Gamma(2\mathrm{i}z)|}, \tag{23}$$

$$\begin{aligned}\delta^{\mu}(\varepsilon)=\arg\Gamma(2\mathrm{i}z)-\arg[&\Gamma(\mu+\mathrm{i}z)\Gamma(\nu+\mathrm{i}z)\\ &\times\Gamma(a+\mathrm{i}z)\Gamma(b+\mathrm{i}z)].\end{aligned} \tag{24}$$

The scattering amplitude shows that a discrete finite spectrum occur if $\mu+\mathrm{i}z=-k$, where $k=0,1,2,..,N$ and N is the largest integer less than or equal to $-\mu$. Thus, the spectrum formula associated with the Wilson polynomial is

$$z_k^2=-(k+\mu)^2. \tag{25}$$

[2] There is a typo in [18] by which the + sign inside the argument of the cosine in equation (21) is replaced by a − sign (private communication with Wilson on July 2016).

Despite the fact that the Meixner–Pollaczek and continuous dual Hahn polynomials are limiting cases of the Wilson polynomial, their asymptotics could not be obtained using the asymptotics of the Wilson polynomial calculated above simply because we cannot interchange this limit with the asymptotic limit. Therefore, those asymptotics have to be derived independently. This is done elsewhere (see, for example, the appendix in [1]).

3. The Meixner–Pollaczek polynomial class of problems

In this section, we consider the class of quantum systems whose continuum scattering states are described by the wavefunction (3) where the expansion coefficients are the Meixner–Pollaczek polynomials $P_n^{\mu}(z, \theta)$ shown in appendix A by equation (A1).

3.1. The Coulomb problem

We start by choosing the physical parameters in the polynomial as: $\mu = \ell + 1$, $\cos\theta = \frac{2E - (\lambda/2)^2}{2E + (\lambda/2)^2}$ and $z = Z/\sqrt{2E}$, where ℓ is a non-negative integer, Z is a real number and λ is a positive length scale parameter. Depending on the range of values of the physical parameters, this system can have a continuous or discrete energy spectrum. For example, if E is negative then z becomes pure imaginary and $|\cos\theta| > 1$. As explained in appendix A, this is equivalent to the replacement of $z \to \mathrm{i}z$ and $\theta \to \mathrm{i}\theta$, which turns the Meixner–Pollaczek polynomials into its discrete version, the Meixner polynomials. Substituting these parameters in equation (A7) gives the following scattering phase shift

$$\delta(E) = \arg\Gamma(\ell + 1 + \mathrm{i}Z/\sqrt{2E}). \tag{26}$$

Whereas, substitution in equation (A8) gives the following energy spectrum formula

$$E_k = -\frac{1}{2}\frac{Z^2}{(k + \ell + 1)^2}. \tag{27}$$

These results are identical to those of the well-known Coulomb interaction in three dimensions, $V(r) = Z/r$, where r is the radial coordinate, Z is the electric charge and ℓ is the angular momentum quantum number. The discrete polynomial in the bound states wavefunction expansion (5) is the Meixner polynomial shown in appendix A by equation (A9). The requirement on the basis to support a tridiagonal matrix representation of the Schrödinger wave operator, $-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\ell(\ell+1)}{2r^2} + \frac{Z}{r} - E$, gives (see section II.A.2 in [3])

$$\phi_n(r) = \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+2\ell+2)}}\,(\lambda r)^{\ell+1}\mathrm{e}^{-\lambda r/2}L_n^{2\ell+1}(\lambda r), \tag{28}$$

where $L_n^{\nu}(z)$ is the Laguerre polynomial. This is not the only exactly solvable problem in the Meixner–Pollaczek polynomial class. Next, we give two other examples; one with only an infinite bound states spectrum and another with a continuous as well as discrete finite spectrum (however, in this class it is exactly solvable only for the discrete bound states).

3.2. The oscillator problem

For the second example in this class, we take the polynomial parameters as $\mu = \frac{1}{2}\left(\ell + \frac{3}{2}\right)$ and $z = \mathrm{i}E/2\omega$, where ω is a real number. The choice of z as pure imaginary mandates the replacement $\theta \to \mathrm{i}\theta$ so that reality is maintained for the polynomial (A1) and its recursion relation (A4). As seen in appendix A, this leads only to bound states and the discrete Meixner polynomial. The infinite energy spectrum formula is obtained from equation (A8) as

$$E_k = \omega\left(2k + \ell + \tfrac{3}{2}\right). \tag{29}$$

This corresponds to the well-known energy spectrum of the isotropic oscillator $V(r) = \frac{1}{2}\omega^2 r^2$ with oscillator frequency ω. The corresponding eigenstates are written as in equation (5) in terms of the discrete Meixner polynomial. Moreover, the requirement that the basis yield a tridiagonal matrix representation for the wave operator, $-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\ell(\ell+1)}{2r^2} + \frac{1}{2}\omega^2 r^2 - E$, gives (see section II.A. 1 in [3])

$$\phi_n(r) = \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\ell+3/2)}}\,(\lambda r)^{\ell+1}\mathrm{e}^{-\lambda^2 r^2/2}L_n^{\ell+1/2}(\lambda^2 r^2), \tag{30}$$

where λ is a length scale parameter such that $\lambda^2 \leqslant 4\omega$. Additionally, the parameter β in the Meixner polynomial (A9) is obtained as $\beta = \mathrm{e}^{-2\theta}$ where $\cosh\theta = \frac{\omega^2 + (\lambda/2)^4}{\omega^2 - (\lambda/2)^4}$.

3.3. The Morse problem

The final problem in the Meixner–Pollaczek polynomial class corresponds to the parameter assignments: $\mu = \frac{1}{2} + \sqrt{-\varepsilon}$ and $z = \mathrm{i}u_1/2\sqrt{u_0}$, where all parameters are real with $\varepsilon < 0$, $u_0 > 0$ and $u_1 < 0$. Thus, it is required that $\theta \to \mathrm{i}\theta$ in equations (A1) and (A4) turning the polynomial into one of its two discrete versions. Formula (A8) gives the energy spectrum as

$$\varepsilon_k = -\left(k + \tfrac{1}{2} + u_1/2\sqrt{u_0}\right)^2, \tag{31}$$

where $k = 0, 1, .., N$ and N is the largest integer less than or equal to $-u_1/2\sqrt{u_0} - \frac{1}{2}$. If we introduce an inverse length parameter α and write $E = \frac{1}{2}\alpha^2\varepsilon$ and $V_i = \frac{1}{2}\alpha^2 u_i$, then we can rewrite the spectrum formula (31) as follows

$$E_k = -\tfrac{1}{2}\alpha^2\left(k + \tfrac{1}{2} + V_1/\alpha\sqrt{2V_0}\right)^2. \tag{32}$$

This, in fact, is the energy spectrum formula of the one-dimensional Morse potential $V(x)=V_0 e^{2\alpha x}+V_1 e^{\alpha x}$ where $-\infty<x<+\infty$ [21]. The corresponding bound states are written as in equation (5) in terms of the discrete version of the Meixner–Pollaczek polynomial with a finite spectrum, which is the Krawtchouk polynomial not the Meixner polynomial. The orthonormal version of this polynomial is given in appendix A by equation (A11). The requirement that the corresponding basis gives a tridiagonal matrix representation for the wave operator, $-\frac{1}{2}\frac{d^2}{dx^2}+V_0 e^{2\alpha x}+V_1 e^{\alpha x}-E$, results in (see section II.A.3 in [3])

$$\phi_n(x)=\sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\nu+1)}}\,y^{\nu/2}e^{-y/2}L_n^{\nu}(y), \tag{33}$$

where $y(x)=e^{\alpha x}$ and $\nu=\frac{2}{\alpha}\sqrt{-2E}$. Moreover, the parameter $\gamma=e^{-2\theta}$ in the Krawtchouk polynomial is obtained from $\cosh\theta=\frac{2V_0+(\alpha/2)^2}{2V_0-(\alpha/2)^2}$ with $V_0\geqslant\alpha^2/8$.

In this class, we were able to obtain full solutions for two problems, the Coulomb and the isotropic oscillator. The latter has only discrete bound states whereas the former has both discrete bound states as well as continuum scattering states. Additionally, we were able to obtain only partial solution to the 1D Morse oscillator. We could find only the discrete bound states solution but not the continuum scattering states. In the following section, we will remedy that.

4. The continuous dual Hahn polynomial class of problems

In this section, we consider the class of problems whose continuum scattering states are described by the wavefunction (3) where the expansion coefficients are the continuous dual Hahn polynomial $S_n^{\mu}(z^2;a,b)$ shown in appendix B by equation (B1). Throughout this section, we restrict our investigation to the special case where the two polynomial parameters a and b are equal.

4.1. The Morse problem

We start by choosing the polynomial parameters as: $\mu=\rho+\frac{1}{2}$, $a=b=\frac{\nu+1}{2}$ and $z^2=\varepsilon$. If $\rho>-\frac{1}{2}$, then μ is positive and we obtain only a continuous spectrum corresponding to scattering states with the phase shift given in appendix B by equation (B7) as

$$\delta^{\mu}(\varepsilon)=\arg\Gamma(2i\sqrt{\varepsilon})-\arg\Gamma\left(\rho+\frac{1}{2}+i\sqrt{\varepsilon}\right)-2\arg\Gamma\left(\frac{\nu+1}{2}+i\sqrt{\varepsilon}\right). \tag{34}$$

If we introduce an inverse length parameter α and write $2E=\alpha^2\varepsilon$ and $2V=\alpha^2\rho$, then this scattering phase shift reads as follows

$$\delta^{\mu}(\varepsilon)=\arg\Gamma\left(\frac{2i}{\alpha}\sqrt{2E}\right)-\arg\Gamma\left(\frac{2V}{\alpha^2}+\frac{1}{2}+\frac{i}{\alpha}\sqrt{2E}\right)-2\arg\Gamma\left(\frac{\nu+1}{2}+\frac{i}{\alpha}\sqrt{2E}\right). \tag{35}$$

This is identical to the scattering phase shift associated with the 1D Morse potential, $V(x)=\frac{1}{2}\left(\frac{\alpha}{2}\right)^2 e^{2\alpha x}+Ve^{\alpha x}$, with $-\infty<x<+\infty$ [22, 23]. Therefore, the continuous energy scattering states are written as the infinite bounded sum of equation (3) with the expansion coefficients as the continuous dual Hahn polynomials $S_n^{\rho+\frac{1}{2}}\left(\varepsilon;\frac{\nu+1}{2},\frac{\nu+1}{2}\right)$. The requirement that the corresponding basis gives a tridiagonal matrix representation for the wave operator, $-\frac{1}{2}\frac{d^2}{dx^2}+\frac{\alpha^2}{8}e^{2\alpha x}+Ve^{\alpha x}-E$, results in (see section II.B.1 in [3])

$$\phi_n(x)=\sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\nu+1)}}\,y^{\frac{\nu+1}{2}}e^{-y/2}L_n^{\nu}(y), \tag{36}$$

where $y(x)=e^{\alpha x}$. On the other hand, if $\rho<-\frac{1}{2}$ (i.e. $V<-\alpha^2/4$) then μ is negative and the problem has both continuous as well as discrete energy states and the corresponding wavefunction is given by equation (6). The discrete energy spectrum is obtained using formula (B8) as

$$E_k=-\frac{1}{2}\alpha^2\left(k+\frac{1}{2}+2V/\alpha^2\right)^2, \tag{37}$$

for $k=0,1,..,N$ and N is the larger integer less than or equal to $-(2V/\alpha^2)-\frac{1}{2}$. This spectrum formula is identical to (32) above with $V_0=\alpha^2/8$ and $V_1=V$. Thus, we obtained here a full solution to the 1D Morse problem for both scattering and bound states whereas the solution obtained in the previous section was only for bound states.

4.2. The oscillator problem

Next, we make the parameter assignments: $\mu=\frac{1}{2}-\varepsilon$, $a=b=\frac{\nu+1}{2}$ and $z=\frac{i}{2}\left(\ell+\frac{1}{2}\right)$. Thus, we obtain only bound states whose energy spectrum is given by formula (B8) as

$$E_k=\alpha^2\left(2k+\ell+\frac{3}{2}\right), \tag{38}$$

where we have chosen an inverse length parameter α and wrote $2E=\alpha^2\varepsilon$. This spectrum is identical to that of the 3D isotropic oscillator (29) above with oscillator frequency $\omega=\alpha^2$ and angular momentum quantum number ℓ. The corresponding wavefunction is written as equation (5) in terms of the discrete dual Hahn polynomial. The requirement that the basis elements should result in a tridiagonal matrix representation for the wave operator, $-\frac{1}{2}\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{2r^2}+\frac{1}{2}\alpha^4r^2-E$, gives (see section II.B.2 in [3])

$$\phi_n(r)=\sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\nu+1)}}\,(\alpha r)^{\nu+\frac{3}{2}}e^{-\alpha^2r^2/2}L_n^{\nu}(\alpha^2r^2). \tag{39}$$

4.3. The Coulomb problem

Finally, if we choose the parameters as: $\mu=\frac{1}{2}+\frac{\rho}{\sqrt{-\varepsilon}}$, $a=b=\frac{\nu+1}{2}$ and $z=i\left(\ell+\frac{1}{2}\right)$ where ε and ρ are negative, then we obtain only bound states whose energy spectrum is given by formula (B8) as

$$E_k = -Z^2/2(k + \ell + 1)^2, \tag{40}$$

where we have chosen an inverse length positive parameter λ and wrote $2E = \lambda^2\varepsilon$, $\rho = Z/\lambda$. This is identical to the energy spectrum of the Coulomb problem given by equation (27) above with a negative electric charge Z and angular momentum quantum number ℓ. However, the solution here is only for bound states. The tridiagonal requirement on the basis functions results in the following realization (see section II.B. 3 in [3])

$$\phi_n(r) = \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\nu+1)}}\,(\lambda r)^{1+\frac{\nu}{2}}e^{-\lambda r/2}L_n^{\nu}(\lambda r), \tag{41}$$

with $\lambda^2 = -8E_k$.

5. The Wilson polynomial class of problems

In this section, we consider the class of problems whose continuum scattering states are described by the wavefunction (3) where the expansion coefficients are the Wilson polynomials $W_n^{\mu}(z^2;\nu;a,b)$ shown in appendix C by equation (C1). Throughout this section, we consider the special case where the two polynomial parameters a and b are equal and fixed by certain physical constraints (e.g. the number of bound states).

5.1. The hyperbolic Pöschl–Teller potential

We start by choosing the polynomial parameters as follows: $\nu + \mu = 1 + \sqrt{\frac{1}{4} + 2u_1}$, $\nu - \mu = \sqrt{\frac{1}{4} - 2u_0}$ and $z^2 = \frac{1}{2}\varepsilon$, where all the parameters $\{\varepsilon, u_i\}$ are dimensionless and real with $u_0 \leqslant \frac{1}{8}$ and $u_1 \geqslant -\frac{1}{8}$. If $\sqrt{\frac{1}{4} - 2u_0} < 1 + \sqrt{\frac{1}{4} + 2u_1}$, then μ is positive and the system consists only of continuous energy scattering states that are written in terms of $W_n^{\mu}\left(\frac{1}{2}\varepsilon;\nu;a,a\right)$. The phase shift associated with these scattering states is obtained using formula (24) as

$$\begin{aligned}\delta(\kappa) = {}&\arg\Gamma(2i\kappa/\lambda) - \arg\Gamma\Bigg(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{4} + 2V_1/\lambda^2} \\ &- \frac{1}{2}\sqrt{\frac{1}{4} - 2V_0/\lambda^2} + i\frac{\kappa}{\lambda}\Bigg) - \arg \\ &\times \Gamma\Bigg(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{4} + 2V_1/\lambda^2} + \frac{1}{2}\sqrt{\frac{1}{4} - 2V_0/\lambda^2} + i\frac{\kappa}{\lambda}\Bigg) \\ &- 2\arg\Gamma\left(a + i\frac{\kappa}{\lambda}\right),\end{aligned} \tag{42}$$

where we have introduced an inverse length scale parameter λ and wrote $E = \lambda^2\varepsilon$, $V_i = \lambda^2 u_i$, and defined the wave number $\kappa = \sqrt{2E}$. On the other hand, if $\sqrt{\frac{1}{4} - 2u_0} > 1 + \sqrt{\frac{1}{4} + 2u_1}$ then μ is negative and the system consist of both scattering as well as bound states and the corresponding wavefunction is written as shown by equation (6). The spectrum formula (25) results in the following bound states energy eigenvalues

$$E_k = -\frac{\lambda^2}{2}\left(2k + 1 + \sqrt{\frac{1}{4} + 2V_1/\lambda^2} - \sqrt{\frac{1}{4} - 2V_0/\lambda^2}\right)^2, \tag{43}$$

where $k = 0, 1, .., N$ and N is the largest integer less than or equal to $\frac{1}{2}\sqrt{\frac{1}{4} - 2V_0/\lambda^2} - \frac{1}{2}\sqrt{\frac{1}{4} + 2V_1/\lambda^2} - \frac{1}{2}$. These results are identical to those of the 1D hyperbolic Pöschl–Teller potential $V(x) = \frac{V_0}{\cosh^2(\lambda x)} + \frac{V_1}{\sinh^2(\lambda x)}$ with $x \geqslant 0$ [23, 24]. The requirement that the basis $\{\phi_n(x)\}$ must produce a tridiagonal and symmetric representation of the Schrödinger wave operator with this potential gives (see section III.B.1 in [3][3])

$$\phi_n(x) = A_n(1-y)^{\alpha}(1+y)^{\beta}P_n^{(2a-1,\mu+\nu-1)}(y), \tag{44}$$

where

$$A_n = \sqrt{\frac{2n+\mu+\nu+2a-1}{2^{\mu+\nu+2a-1}}\frac{\Gamma(n+1)\Gamma(n+\mu+\nu+2a-1)}{\Gamma(n+2a)\Gamma(n+\mu+\nu)}},$$

$y(x) = 2\tanh^2(\lambda x) - 1$, $\alpha = a$, $2\beta = \mu + \nu - \frac{1}{2}$ and $P_n^{(\sigma,\tau)}(y)$ is the Jacobi polynomial.

5.2. The trigonometric Scarf potential

For this problem, we choose $\nu + \mu = 1 + \sqrt{u_+ + u_- + \frac{1}{4}}$, $\nu - \mu = 2\sqrt{\varepsilon - u_0}$ and $z^2 = -\frac{1}{4}\left(u_+ - u_- + \frac{1}{4}\right)$ with $(u_+ \pm u_-) \geqslant -\frac{1}{4}$. Thus, z is pure imaginary and the system consists only of bound states. The spectrum formula (25) gives

$$\begin{aligned}\varepsilon_k = u_0 + \Bigg(&k + \frac{1}{2} + \frac{1}{2}\sqrt{u_+ + u_- + \frac{1}{4}} \\ &+ \frac{1}{2}\sqrt{u_+ - u_- + \frac{1}{4}}\Bigg)^2,\end{aligned} \tag{45}$$

where $k = 0, 1, 2, .., \infty$ since μ is always negative for any k. Thus, the bound states are written as in equation (5) with the discrete version of the Wilson polynomial, which is the Racah polynomials, $R_n^N(k;\alpha,\beta,\gamma)$, defined in appendix C as expansion coefficients. If we introduce the inverse length scale parameter λ and write $\varepsilon = 2E/\lambda^2$ and $u_i = 2V_i/\lambda^2$, then this becomes the energy spectrum of the 1D trigonometric Scarf potential $V(x) = V_0 + \frac{V_+ - V_- \sin(\lambda x)}{\cos^2(\lambda x)}$ where $|x| \leqslant \pi/2\lambda$ [21]. The associated basis that produces a tridiagonal matrix representation for the Schrödinger wave operator with this potential is given by equation (44) with $y(x) = \sin(\lambda x)$, $\alpha = a + \frac{1}{4}$ and $2\beta = \mu + \nu - \frac{1}{2}$ (see section III.B.2 in [3]).

5.3. The hyperbolic Eckart potential

Now, we make the following choice of polynomial parameters: $\nu + \mu = 1 + \sqrt{4u_1 + 1}$, $\nu - \mu = 2\sqrt{-\varepsilon - u_0}$ and $z^2 = \varepsilon$. If the parameters are such that μ is positive then the system consists only of continuous energy scattering states that are written in terms of $W_n^{\mu}(\varepsilon;\nu;a,a)$. The corresponding

[3] The polynomial $G_n^{(\mu\nu)}(z;\sigma)$ defined in this paper is identical to the Wilson polynomial defined here in appendix C as $W_n^{\frac{\nu+1}{2}-\sqrt{-\sigma}}\left(\frac{1}{2}z;\frac{\nu+1}{2}+\sqrt{-\sigma};\frac{\mu+1}{2},\frac{\mu+1}{2}\right)$

phase shift is obtained from formula (24) as

$$\delta(\varepsilon) = \arg\Gamma(\mathrm{i}2\sqrt{\varepsilon}) - \arg\Gamma\left(\tfrac{1}{2} + \sqrt{u_1 + \tfrac{1}{4}} - \mathrm{i}\sqrt{\varepsilon + u_0} + \mathrm{i}\sqrt{\varepsilon}\right) - \arg\Gamma\left(\tfrac{1}{2} + \sqrt{u_1 + \tfrac{1}{4}} + \mathrm{i}\sqrt{\varepsilon + u_0} + \mathrm{i}\sqrt{\varepsilon}\right) - 2\arg\Gamma(a + \mathrm{i}\sqrt{\varepsilon}). \quad (46)$$

On the other hand, if μ is negative then the system consists of both scattering as well as bound states and the corresponding wavefunction is written as shown by equation (6). The spectrum formula (25) results in the following energy spectrum

$$\varepsilon_k = -\frac{1}{4}\left[k + \frac{\mu+\nu+1}{2} - \frac{u_0}{k + \frac{\mu+\nu+1}{2}}\right]^2 - u_0. \quad (47)$$

If we introduce the inverse length scale parameter λ and write $\varepsilon = 2E/\lambda^2$ and $u_i = 2V_i/\lambda^2$, then these results become identical to those of the 1D hyperbolic Eckart potential [23, 25]

$$V(x) = \frac{1}{\mathrm{e}^{\lambda x} - 1}\left[V_0 + \frac{V_1}{1 - \mathrm{e}^{-\lambda x}}\right] = \frac{V_1/4}{\sinh^2(\lambda x/2)} + \frac{V_0/2}{\tanh(\lambda x/2)} - \frac{V_0}{2}, \quad (48)$$

with $x \geqslant 0$. The basis functions corresponding to this problem are those given by equation (44) with $y(x) = 1 - 2\mathrm{e}^{-\lambda x}$, $\alpha = a$ and $2\beta = \mu + \nu$ (see section III. B.3 in [3]).

5.4. The hyperbolic Rosen–Morse potential

The fourth and final problem corresponds to the following selection of polynomial parameters: $\nu + \mu = B - A + \frac{1}{2}$, $\nu - \mu = B + A + \frac{1}{2}$ and $z^2 = \varepsilon$, where A and B are real dimensionless parameters. Thus, if A is negative then the system is in the continuum energy state but if A is positive then the system is a mix of finite discrete bound states and a continuous energy of scattering states. The corresponding phase shift and energy spectrum are obtained as follows

$$\delta(\varepsilon) = \arg\Gamma(\mathrm{i}2\sqrt{\varepsilon}) - \arg\Gamma(-A + \mathrm{i}\sqrt{\varepsilon}) - \arg\Gamma\left(B + \tfrac{1}{2} + \mathrm{i}\sqrt{\varepsilon}\right) - 2\arg\Gamma(a + \mathrm{i}\sqrt{\varepsilon}), \quad (49)$$

$$\varepsilon_k = -(k - A)^2, \quad (50)$$

where $k = 0, 1, .., N$ and N is the largest integer less than or equal to $-\mu = A$. If we introduce an inverse length parameter λ and write $\varepsilon = 2E/\lambda^2$, then these results correspond to the hyperbolic Rosen–Morse potential [23]

$$\frac{2}{\lambda^2}V(x) = \frac{(B^2 + A^2 + A) - B(2A + 1)\cosh(\lambda x)}{\sinh^2(\lambda x)}, \quad (51)$$

where $x \geqslant 0$. The basis functions corresponding to this problem are those given by equation (44) with $y(x) = \cosh(\lambda x)$.

6. Conclusion and discussion

We have shown that by writing the wavefunction in terms of orthogonal polynomials as shown in one of the three appropriate forms given by equations (3), (5) or (6), then all physical information about the system is obtained from the properties of these polynomials. However, such polynomials are required to have the asymptotic behavior shown in equation (4). We found that the Coulomb, oscillator and Morse potentials are associated with the Meixner–Pollaczek and continuous dual Hahn polynomial classes. We also found that other well-known exactly solvable problems correspond to the Wilson polynomial class that includes it discrete version, the Racah polynomial. These latter problems include, but not limited to, the hyperbolic Pöschl–Teller, Eckart, Rosen–Morse and trigonometric Scarf potentials. We conclude by making an important remark and discussing two relevant issues:

- In the process of identifying the quantum mechanical system associated with a given polynomial, it might seem that we have made an arbitrary choice of polynomial parameters. However, those choices are, in fact, unique and were made carefully to correspond to the conventional class of exactly solvable potentials. Had we made an alternative choice of parameters, then the corresponding quantum system would have been different and might not belong to the well-known class of exactly solvable problems. Nonetheless, such alternative choices must respect any constraints on the parameters (e.g. reality and ranges). For example, if we choose the Meixner–Pollaczek polynomial parameters as: $\cosh\theta = \frac{\kappa - \mu\lambda}{\kappa + \mu\lambda}$ and $z = \mathrm{i}\ln(\kappa/\lambda)$, where $\kappa^2 = 2E$ and $\mu < 0$ then we would have obtained the following bound states energy spectrum using formula (A8)

$$E_k = \tfrac{1}{2}\lambda^2\mathrm{e}^{-2(k+\mu)}, \quad (52)$$

  where $k = 0, 1, .., N$ and N is the largest integer such that $\mathrm{e}^{-(N+\mu)} > -\mu$. The polynomial that enters in the bound state wavefunction expansion (5) is the discrete version of the Meixner–Pollaczek polynomial with finite spectrum, which is the Krawtchouk polynomial. The energy spectrum (52) does not correspond to any of the known exactly solvable potentials making such a system very appealing and motivates us to search for the associated potential function. That is, doing the inverse quantum mechanical problem: finding the potential function starting from the energy spectrum. Such a problem is highly non-trivial. However, the tridiagonal representation requirement results in a sever restriction on the space of solutions of this problem making it tractable. A procedure to accomplish that and obtain the potential function analytically or numerically was developed and applied in [4].
- There is another class of four-parameter orthogonal polynomials that does not belong to the Askey scheme of orthogonal polynomials, which was not treated in the mathematics or physics literature, but it corresponds to

Table 3. Partial list of exactly solvable potential functions in the polynomial class associated with $H_n^{(\mu,\nu)}(z;\alpha,\theta)$. The coordinate transformation $y(x)$ enters in the basis (44) that supports a tridiagonal matrix representation for the corresponding wave operator. The presence of the V_1 term in all of these potentials inhibits exact solvability of the Schrödinger wave equation in the standard formulation of quantum mechanics.

$V(x)$	x	$y(x)$
$V_0+\frac{V_+-V_-\sin(\pi x/L)}{\cos^2(\pi x/L)}+V_1\sin(\pi x/L)$	$-\frac{1}{2}L\leqslant x\leqslant+\frac{1}{2}L$	$\sin(\pi x/L)$
$\frac{1}{1-(x/L)^2}\left\{V_0+\frac{V_+}{(x/L)^2}+\frac{V_-}{1-(x/L)^2}+V_1[2(x/L)^2-1]\right\}$	$0\leqslant x\leqslant L$	$2(x/L)^2-1$
$\frac{1}{e^{\lambda x}-1}\left[V_0+V_-e^{\lambda x}+\frac{V_+}{1-e^{-\lambda x}}+V_1(1-2e^{-\lambda x})\right]$	$x\geqslant 0$	$1-2e^{-\lambda x}$
$V_-+\frac{V_+}{\sinh^2(\lambda x)}+\frac{V_0+V_1[2\tanh^2(\lambda x)-1]}{\cosh^2(\lambda x)}$	$x\geqslant 0$	$2\tanh^2(\lambda x)-1$
$V_+-V_-\tanh(\lambda x)+\frac{V_0+V_1\tanh(\lambda x)}{\cosh^2(\lambda x)}$	$-\infty<x<+\infty$	$\tanh(\lambda x)$
$V_0+\frac{V_+}{\sin^2(\pi x/L)}+\frac{V_-}{\cos^2(\pi x/L)}-V_1\cos(2\pi x/L)$	$0\leqslant x\leqslant\frac{1}{2}L$	$2\sin^2(\pi x/L)-1$
$V_0+\frac{V_+-V_-\cosh(\lambda x)}{\sinh^2(\lambda x)}+V_1\cosh(\lambda x)$	$x\geqslant 0$	$\cosh(\lambda x)$

physical problems that are solvable using our formulation. The class consists of three polynomials, one with a continuous spectrum designated as $H_n^{(\mu,\nu)}(z;\alpha,\theta)$ and two of its discrete version with finite and infinite spectra. So far, it is defined by its three-term recursion relation that reads as follows [6, 26]

$$\begin{aligned}(\cos\theta)H_n^{(\mu,\nu)}(z;\alpha,\theta)=&\left\{z\sin\theta\left[\left(n+\tfrac{\mu+\nu+1}{2}\right)^2+\alpha\right]\right.\\&\left.+\frac{\nu^2-\mu^2}{(2n+\mu+\nu)(2n+\mu+\nu+2)}\right\}H_n^{(\mu,\nu)}(z;\alpha,\theta)\\&+\frac{2(n+\mu)(n+\nu)}{(2n+\mu+\nu)(2n+\mu+\nu+1)}H_{n-1}^{(\mu,\nu)}(z;\alpha,\theta)\\&+\frac{2(n+1)(n+\mu+\nu+1)}{(2n+\mu+\nu+1)(2n+\mu+\nu+2)}H_{n+1}^{(\mu,\nu)}(z;\alpha,\theta),\end{aligned}\tag{53}$$

where $0\leqslant\theta\leqslant\pi$. It is a polynomial of degree n in z and α, which is obtained for all n staring with $H_0^{(\mu,\nu)}(z;\alpha,\theta)=1$ and $H_1^{(\mu,\nu)}(z;\alpha,\theta)$, which is computed from (53) by setting $n=0$ and $H_{-1}^{(\mu,\nu)}(z;\alpha,\theta)\equiv0$. Setting $z\equiv0$ turns (53) into the recursion relation of the Jacobi polynomial $P_n^{(\mu,\nu)}(\cos\theta)$. Physical requirements dictate that μ and ν are greater than -1 and $z\in\mathbb{R}$. In section III.A of [3], this polynomial class was used in solving several physical problems. These were associated with either *new* potential functions or generalizations of exactly solvable potentials. Here, the term *new* means that all of these potentials could never be treated or solved exactly in the literature prior to the present formulation. Table 3 is a partial list of potential functions associated with this class showing also the coordinate transformation $y(x)$ that enters in the basis elements (44) which supports a tridiagonal matrix representation for the corresponding wave operator. Note that the addition of the V_1 term in all of these potential functions prevents them from being exactly solvable in the standard formulation of quantum mechanics.

- The same formulation outlined above could be extended to the wavefunction of several variables as long as the corresponding problem is completely separable. For example, in three-dimensional configuration space with spherical coordinates, if we can write the total wavefunction as $\psi(r,\theta,\varphi)=\frac{1}{r}R(r)\Theta(\theta)\Phi(\varphi)$ then we can apply the same formulation and write equation (3) as follows

$$R_E^\mu(r)=\sqrt{\rho^\mu(\varepsilon)}\sum_n P_n^\mu(\varepsilon)\,\phi_n(r),\tag{54a}$$

$$\Theta_{E_\theta}^\mu(\theta)=\sqrt{\tilde\rho^\mu(\varepsilon_\theta)}\sum_n\tilde P_n^\mu(\varepsilon_\theta)\,\tilde\phi_n(\theta),\tag{54b}$$

$$\Phi_{E_\varphi}^\mu(\varphi)=\sqrt{\hat\rho^\mu(\varepsilon_\varphi)}\sum_n\hat P_n^\mu(\varepsilon_\varphi)\,\hat\phi_n(\varphi),\tag{54c}$$

where $\{\phi_n(r),P_n^\mu(\varepsilon)\}$, $\{\tilde\phi_n(\theta),\tilde P_n^\mu(\varepsilon_\theta)\}$, and $\{\hat\phi_n(\varphi),\hat P_n^\mu(\varepsilon_\varphi)\}$ are the radial, angular, and azimuthal basis and associated polynomials, respectively. Each basis set must produce a tridiagonal matrix representation for the corresponding wave operator

$$\left[-\frac{1}{2}\frac{d^2}{dr^2}+\frac{E_\theta}{r^2}+V_r(r)-E\right]R_E^\mu(r)=0,\tag{55a}$$

$$\left[-\frac{1}{2}(1-x^2)\frac{d^2}{dx^2}+x\frac{d}{dx}+\frac{E_\varphi}{1-x^2}+V_\theta(\theta)-E_\theta\right]\times\Theta_{E_\theta}^\mu(\theta)=0,\tag{55b}$$

$$\left[-\frac{1}{2}\frac{d^2}{d\varphi^2}+V_\varphi(\varphi)-E_\varphi\right]\Phi_{E_\varphi}^\mu(\varphi)=0,\tag{55c}$$

where $x=\cos\theta$. The corresponding 3D separable potential function is $V(r,\theta,\varphi)=V_r(r)+\frac{1}{r^2}\left[V_\theta(\theta)+\frac{1}{1-x^2}V_\varphi(\varphi)\right]$. As illustration, one may consult [27] where we have applied this procedure to obtain an exact solution for a problem of this type and we wrote two alternative solutions to equation (55b); one in terms of $H_n^{(\mu,\nu)}(z;\alpha,\theta)$ and another in terms of the Wilson polynomial. In [28, 29], the same was done in two dimensions for another problem.

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Appendix A. The two-parameter Meixner–Pollaczek polynomial class

The orthonormal version of this polynomial is written as follows (see pages 37–38 of [10])

$$P_n^\mu(z,\theta)=\sqrt{\tfrac{(2\mu)_n}{n!}}\,e^{in\theta}{}_2F_1\left(\begin{matrix}-n,\,\mu+iz\\2\mu\end{matrix}\middle|\,1-e^{-2i\theta}\right),\quad\text{(A1)}$$

where $(a)_n=a(a+1)(a+2)\ldots(a+n-1)=\frac{\Gamma(n+a)}{\Gamma(a)}$, z is the whole real line, $\mu>0$ and $0<\theta<\pi$. This is a polynomial in z which is orthonormal with respect to the measure $\rho^\mu(z,\theta)dz$. That is,

$$\int_{-\infty}^{+\infty}\rho^\mu(z,\theta)P_n^\mu(z,\theta)P_m^\mu(z,\theta)dz=\delta_{nm},\quad\text{(A2)}$$

where the normalized weight function is

$$\rho^\mu(z,\theta)=\tfrac{1}{2\pi\,\Gamma(2\mu)}(2\sin\theta)^{2\mu}\,e^{(2\theta-\pi)z}\,|\Gamma(\mu+iz)|^2.\quad\text{(A3)}$$

These polynomials satisfy the following symmetric three-term recursion relation

$$\begin{aligned}(z\sin\theta)P_n^\mu(z,\theta)&=-[(n+\mu)\cos\theta]P_n^\mu(z,\theta)\\&+\tfrac12\sqrt{n\,(n+2\mu-1)}\,P_{n-1}^\mu(z,\theta)\\&+\tfrac12\sqrt{(n+1)(n+2\mu)}\,P_{n+1}^\mu(z,\theta).\end{aligned}\quad\text{(A4)}$$

The asymptotics ($n\to\infty$) is obtained as follows (see, for example, the appendix in [1])

$$\begin{aligned}P_n^\mu(z;\theta)\approx\frac{2n^{-1/2}e^{\left(\frac12\pi-\theta\right)z}}{(2\sin\theta)^\mu\,|\Gamma(\mu+iz)|}\Big\{\cos\Big[n\theta+\arg\Gamma(\mu+iz)\\-\mu\tfrac{\pi}{2}-z\ln(2n\sin\theta)\Big]+O(n^{-1})\Big\},\end{aligned}\quad\text{(A5)}$$

which is in the required general form given by equation (4). Therefore, the scattering amplitude and phase shift are obtained as follows

$$A^\mu(\varepsilon)=2e^{\left(\frac12\pi-\theta\right)z}/(2\sin\theta)^\mu\,|\Gamma(\mu+iz)|,\quad\text{(A6)}$$

$$\delta^\mu(\varepsilon)=\arg\Gamma(\mu+iz).\quad\text{(A7)}$$

The scattering amplitude (A6) shows that a discrete infinite spectrum occurs if $\mu+iz=-k$, where $k=0,1,2,\ldots$ Thus, the spectrum formula associated with this polynomial is

$$z_k^2=-(k+\mu)^2,\quad\text{(A8)}$$

and bound states are written as in equation (5) where the discrete version of the Meixner–Pollaczek polynomial is obtained by the substitution $z=i(k+\mu)$ and $\theta\to i\theta$ in equation (A1). The latter substitution is needed to maintain reality of the recursion relation (A4). In fact, and as expected, the substitution $\theta\to i\theta$ makes the asymptotics of (A1) vanish due to the decaying exponential $e^{-n\theta}$. Making these substitutions in (A1) gives the discrete version as the following orthonormal Meixner polynomial (see pages 45–46 in [10])

$$M_n^\mu(k;\beta)=\sqrt{\tfrac{(2\mu)_n}{n!}}\,\beta^{n/2}{}_2F_1\left(\begin{matrix}-n,\,-k\\2\mu\end{matrix}\middle|\,1-\beta^{-1}\right),\quad\text{(A9)}$$

where $\beta=e^{-2\theta}$ with $\theta>0$ making $0<\beta<1$. The substitution $z=i(k+\mu)$ and $\theta\to i\theta$ in the recursion relation (A4) together with $2\cosh\theta=\frac{1}{\sqrt\beta}+\sqrt\beta$ and $2\sinh\theta=\frac{1}{\sqrt\beta}-\sqrt\beta$ transform the recursion relation (A4) to

$$\begin{aligned}(\beta-1)k\,M_n^\mu(k;\beta)&=-[n\,(1+\beta)+2\mu\beta]M_n^\mu(k;\beta)\\&+\sqrt{n\,(n+2\mu-1)\beta}\,M_{n-1}^\mu(k;\beta)\\&+\sqrt{(n+1)(n+2\mu)\beta}\,M_{n+1}^\mu(k;\beta).\end{aligned}\quad\text{(A10)}$$

The associated normalized discrete weight function is $\omega_k^\mu(\beta)=(1-\beta)^{2\mu}\frac{\Gamma(k+2\mu)\;\beta^k}{\Gamma(2\mu)\Gamma(k+1)}$. That is, $\sum_{k=0}^\infty\omega_k^\mu(\beta)\,M_n^\mu(k;\beta)M_m^\mu(k;\beta)=\delta_{n,m}$. Due to the exchange symmetry of n and k in ${}_2F_1\left(\begin{matrix}-n,\,-k\\2\mu\end{matrix}\middle|\,1-\beta^{-1}\right)$, the Meixner polynomial is self-dual satisfying the dual orthogonality relation $\sum_{k=0}^\infty M_k^\mu(n;\beta)M_k^\mu(m;\beta)=\delta_{n,m}/\omega_n^\mu(\beta)$. Now, if we further take $2\mu=-N$, where N is a non-negative integer, then the indices n and k in (A9) cannot be larger than N otherwise the hypergeometric function blows up. Thus, the discrete Meixner polynomial with an infinite spectrum becomes the discrete Krawtchouk polynomial with a finite spectrum whose orthonormal version reads (see pages 46–47 in [10]):

$$K_n^N(k;\gamma)=\sqrt{\tfrac{N!}{n!(N-n)!}}\left(\tfrac{\gamma}{1-\gamma}\right)^{n/2}{}_2F_1\left(\begin{matrix}-n,\,-k\\-N\end{matrix}\middle|\,\gamma^{-1}\right),\quad\text{(A11)}$$

where we wrote $\gamma^{-1}=1-\beta^{-1}$ with $0<\gamma<1$ and $n,k=0,1,..,N$. In writing (A11), we have used $(-N)_n=\frac{\Gamma(n-N)}{\Gamma(-N)}=(-1)^n\frac{\Gamma(N+1)}{\Gamma(N-n+1)}$. These substitutions change the recursion relation (A10) to

$$\begin{aligned}k\,K_n^N(k;\gamma)&=[N\gamma+n(1-2\gamma)]K_n^N(k;\gamma)\\&-\sqrt{n\,(N-n+1)\gamma(1-\gamma)}\,K_{n-1}^N(k;\gamma)\\&-\sqrt{(n+1)(N-n)\gamma(1-\gamma)}\,K_{n+1}^N(k;\gamma).\end{aligned}\quad\text{(A12)}$$

The associated normalized discrete weight function is $\omega_k^N(\gamma) = (1-\gamma)^{N-k}\frac{\Gamma(N+1)\ \gamma^k}{\Gamma(N-k+1)\Gamma(k+1)}$, which is easily obtained from that of the Meixner polynomial by the substitution $2\mu = -N$ and $\beta = -\gamma/(1-\gamma)$. Then, $\sum_{k=0}^N \omega_k^N(\gamma)\, K_n^N(k;\gamma) K_m^N(k;\gamma) = \delta_{n,m}$. The Krawtchouk polynomial is also self-dual and satisfy the dual orthogonality relation $\sum_{k=0}^N K_k^N(n;\gamma) K_k^N(m;\gamma) = \delta_{n,m}/\omega_n^N(\gamma)$.

Appendix B. The three-parameter continuous dual Hahn polynomial class

The orthonormal version of this polynomial is (see pages 29–31 of [10])

$$S_n^\mu(z^2;a,b) = \sqrt{\frac{(\mu+a)_n(\mu+b)_n}{n!\ (a+b)_n}}\ {}_3F_2\left(\left.\begin{matrix}-n,\ \mu+\mathrm{i}z,\ \mu-\mathrm{i}z\\ \mu+a,\ \mu+b\end{matrix}\ \right|\ 1\right), \tag{B1}$$

where ${}_3F_2\left(\left.\begin{matrix}a,\ b,\ c\\ d,\ e\end{matrix}\ \right|\ z\right) = \sum_{n=0}^\infty \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n}\frac{z^n}{n!}$ is the generalized hypergeometric function, $z>0$ and $\mathrm{Re}(\mu,a,b)>0$ with non-real parameters occurring in conjugate pairs. This is a polynomial in z^2 which is orthonormal with respect to the measure $\rho^\mu(z;a,b)\mathrm{d}z$ where the normalized weight function reads as follows

$$\rho^\mu(z;a,b) = \frac{1}{2\pi}\frac{|\Gamma(\mu+\mathrm{i}z)\Gamma(a+\mathrm{i}z)\Gamma(b+\mathrm{i}z)/\Gamma(2\mathrm{i}z)|^2}{\Gamma(\mu+a)\Gamma(\mu+b)\Gamma(a+b)}. \tag{B2}$$

That is,

$$\int_0^\infty S_n^\mu(z^2;a,b)S_m^\mu(z^2;a,b)\rho^\mu(z;a,b)\mathrm{d}z = \delta_{nm}.$$

However, if the parameters are such that $\mu<0$ and $a+\mu$, $b+\mu$ are positive or a pair of complex conjugates with positive real parts, then the polynomial will have a continuum spectrum as well as a finite size discrete spectrum and the polynomial satisfies the following generalized orthogonality relation (equation 1.3.3 in [10])

$$\begin{aligned}&\int_0^\infty \rho^\mu(z;a,b)\, S_n^\mu(z^2;a,b)\, S_m^\mu(z^2;a,b)\,\mathrm{d}z\\ &-2\frac{\Gamma(a-\mu)\Gamma(b-\mu)}{\Gamma(a+b)\Gamma(1-2\mu)}\sum_{k=0}^N(-1)^k(k+\mu)\\ &\times\frac{(\mu+a)_k(\mu+b)_k(2\mu)_k}{(\mu-a+1)_k(\mu-b+1)_k k!}\, S_n^\mu(k;a,b)\, S_m^\mu(k;a,b) = \delta_{n,m},\end{aligned} \tag{B3}$$

where $S_n^\mu(k;a,b)\equiv S_n^\mu(-(k+\mu)^2;a,b)$ and N is the largest integer less than or equal to $-\mu$. It satisfies the following symmetric three-term recursion relation

$$\begin{aligned}z^2S_n^\mu(z^2;a,b) &= [(n+\mu+a)(n+\mu+b)\\ &+n(n+a+b-1)-\mu^2]S_n^\mu(z^2;a,b)\\ &-\sqrt{n(n+a+b-1)(n+\mu+a-1)(n+\mu+b-1)}\\ &\times S_{n-1}^\mu(z^2;a,b)\\ &-\sqrt{(n+1)(n+a+b)(n+\mu+a)(n+\mu+b)}\\ &\times S_{n+1}^\mu(z^2;a,b).\end{aligned} \tag{B4}$$

The asymptotics ($n\to\infty$) is (see, for example, the appendix in [1])

$$\begin{aligned}S_n^\mu(z^2;a,b) &\approx \frac{2}{\sqrt{n}}\frac{\sqrt{\Gamma(\mu+a)\Gamma(\mu+b)\Gamma(a+b)}\,|\Gamma(2\mathrm{i}z)|}{|\Gamma(\mu+\mathrm{i}z)\Gamma(a+\mathrm{i}z)\Gamma(b+\mathrm{i}z)|}\\ &\times\{\cos(z\ln n+\arg[\Gamma(2\mathrm{i}z)/\Gamma(\mu+\mathrm{i}z)\Gamma(a+\mathrm{i}z)\\ &\times\Gamma(b+\mathrm{i}z)])+O(n^{-1})\}.\end{aligned} \tag{B5}$$

This is in the required general form given by equation (4). Therefore, the scattering amplitude and phase shift are obtained as follows

$$A^\mu(\varepsilon) = \frac{2\sqrt{\Gamma(\mu+a)\Gamma(\mu+b)\Gamma(a+b)}}{|\Gamma(\mu+\mathrm{i}z)\Gamma(a+\mathrm{i}z)\Gamma(b+\mathrm{i}z)/\Gamma(2\mathrm{i}z)|}, \tag{B6}$$

$$\begin{aligned}\delta^\mu(\varepsilon) &= \arg\Gamma(2\mathrm{i}z)-\arg\Gamma(\mu+\mathrm{i}z)\\ &-\arg\Gamma(a+\mathrm{i}z)-\arg\Gamma(b+\mathrm{i}z).\end{aligned} \tag{B7}$$

The scattering amplitude (B6) shows that a discrete finite spectrum occur if $\mu+\mathrm{i}z=-k$, where $k=0,1,2,..,N$ and N is the largest integer less than or equal to $-\mu$. Thus, the spectrum formula associated with this polynomial is

$$z_k^2 = -(k+\mu)^2. \tag{B8}$$

Substituting $z=\mathrm{i}(k+\mu)$ in equation (B1) and redefining the parameters as $2\mu=\alpha+\beta+1$, $\mu+a=\alpha+1$, $\mu+b=-N$, we obtain the discrete version of this polynomial as the orthonormal dual Hahn polynomial (see pages 34–36 in [10])

$$\begin{aligned}R_n^N(z_k^2;\alpha,\beta) &= \sqrt{\frac{(\alpha+1)_n(N-n+1)_n}{n!\ (N+\beta-n+1)_n}}\\ &\times{}_3F_2\left(\left.\begin{matrix}-n,\ -k,\ k+\alpha+\beta+1\\ \alpha+1,\ -N\end{matrix}\ \right|\ 1\right),\end{aligned} \tag{B9}$$

where $z_k = k+\frac{\alpha+\beta+1}{2}$, $n,k=0,1,2,..,N$ and either $\alpha,\beta>-1$ or $\alpha,\beta<-N$. The same substitution in (B4) yields the following recursion relation

$$\begin{aligned}\left(k+\tfrac{\alpha+\beta+1}{2}\right)^2R_n^N &= [(n+\alpha+1)(N-n)\\ &+n(N+\beta+1-n)+\tfrac14(\alpha+\beta+1)^2\Big]R_n^N\\ &+\sqrt{n(n+\alpha)(N-n+1)(N-n+\beta+1)}\,R_{n-1}^N\\ &+\sqrt{(n+1)(n+\alpha+1)(N-n)(N-n+\beta)}\,R_{n+1}^N.\end{aligned} \tag{B10}$$

The associated normalized discrete weight function is

$$\omega_k^N(\alpha,\beta)=(\beta+1)_N\frac{(2k+\alpha+\beta+1)(\alpha+1)_k(N-k+1)_k}{(k+\alpha+\beta+1)_{N+1}(\beta+1)_k k!}. \quad \text{(B11)}$$

Therefore, the orthogonality reads:

$$\sum_{k=0}^{N}\omega_k^N(\alpha,\beta)\,R_n^N(k;\alpha,\beta)\,R_m^N(k;\alpha,\beta)=\delta_{n,m}.$$

It also satisfies the dual orthogonality

$$\sum_{k=0}^{N}R_k^N(n;\alpha,\beta)\,R_k^N(m;\alpha,\beta)=\delta_{n,m}/\omega_n^N(\alpha,\beta). \quad \text{(B12)}$$

and the polynomial satisfies the following generalized orthogonality relation (equation 1.1.3 in [10])

$$\begin{aligned}&\int_0^\infty \rho^\mu(z;\nu;a,b)W_n^\mu(z^2;\nu;a,b)\;W_m^\mu(z^2;\nu;a,b)\,dz\\&-2\frac{\Gamma(\mu+\nu+a+b)\Gamma(\nu-\mu)\Gamma(a-\mu)\Gamma(b-\mu)}{\Gamma(-2\mu+1)\Gamma(a+b)\Gamma(a+\nu)\Gamma(b+\nu)}\\&\times\sum_{k=0}^{N}(k+\mu)\frac{(2\mu)_k(\mu+\nu)_k(\mu+a)_k(\mu+b)_k}{(\mu-\nu+1)_k(\mu-a+1)_k(\mu-b+1)_k k!}\\&\times W_n^\mu(k;\nu;a,b)W_m^\mu(k;\nu;a,b)=\delta_{n,m},\end{aligned} \quad \text{(C3)}$$

where $W_n^\mu(k;\nu;a,b)\equiv W_n^\mu(-(k+\mu)^2;\nu;a,b)$ and N is the largest integer less than or equal to $-\mu$. The associated symmetric three-term recursion relation is:

$$\begin{aligned}z^2W_n^\mu=&\left[\frac{(n+\mu+\nu)(n+\mu+a)(n+\mu+b)(n+\mu+\nu+a+b-1)}{(2n+\mu+\nu+a+b)(2n+\mu+\nu+a+b-1)}+\frac{n(n+\nu+a-1)(n+\nu+b-1)(n+a+b-1)}{(2n+\mu+\nu+a+b-1)(2n+\mu+\nu+a+b-2)}-\mu^2\right]W_n^\mu\\&-\frac{1}{2n+\mu+\nu+a+b-2}\sqrt{\frac{n\,(n+\mu+\nu-1)(n+a+b-1)(n+\mu+a-1)(n+\mu+b-1)(n+\nu+a-1)(n+\nu+b-1)(n+\mu+\nu+a+b-2)}{(2n+\mu+\nu+a+b-3)(2n+\mu+\nu+a+b-1)}}\,W_{n-1}^\mu\\&-\frac{1}{2n+\mu+\nu+a+b}\sqrt{\frac{(n+1)\,(n+\mu+\nu)(n+a+b)(n+\mu+a)(n+\mu+b)(n+\nu+a)(n+\nu+b)(n+\mu+\nu+a+b-1)}{(2n+\mu+\nu+a+b-1)(2n+\mu+\nu+a+b+1)}}\,W_{n+1}^\mu.\end{aligned} \quad \text{(C4)}$$

Appendix C. The four-parameter wilson polynomial class

The orthonormal version of the Wilson polynomial could be written in terms of $\tilde{W}_n^\mu(z^2;\nu;a,b)$ which is given by equation (8) as follows

$$\begin{aligned}&W_n^\mu(z^2;\nu;a,b)\\&=\sqrt{\left(\frac{2n+\mu+\nu+a+b-1}{n+\mu+\nu+a+b-1}\right)\frac{(\mu+\nu)_n(a+b)_n(\mu+\nu+a+b)_n\,n!}{(\mu+a)_n(\mu+b)_n(\nu+a)_n(\nu+b)_n}}\\&\tilde{W}_n^\mu(z^2;\nu;a,b)\\&=\sqrt{\left(\frac{2n+\mu+\nu+a+b-1}{n+\mu+\nu+a+b-1}\right)\frac{(\mu+a)_n(\mu+b)_n(\mu+\nu)_n(\mu+\nu+a+b)_n}{(\nu+a)_n(\nu+b)_n(a+b)_n n!}}\\&\times{}_4F_3\left(\left.\begin{matrix}-n,n+\mu+\nu+a+b-1,\mu+iz,\mu-iz\\ \mu+\nu,\mu+a,\mu+b\end{matrix}\right|1\right).\end{aligned} \quad \text{(C1)}$$

If $\mathrm{Re}(\mu,\nu,a,b)>0$ and non-real parameters occur in conjugate pairs, then the orthogonality relation becomes $\int_0^\infty W_n^\mu(z^2;\nu;a,b)W_m^\mu(z^2;\nu;a,b)\rho^\mu(z;\nu;a,b)dz=\delta_{\mathrm{nm}}$, where the normalized weight function is

$$\rho^\mu(z;\nu;a,b)=\frac{1}{2\pi}\frac{\Gamma(\mu+\nu+a+b)\,|\Gamma(\mu+iz)\Gamma(\nu+iz)\Gamma(a+iz)\Gamma(b+iz)/\Gamma(2iz)|^2}{\Gamma(\mu+\nu)\Gamma(a+b)\Gamma(\mu+a)\Gamma(\mu+b)\Gamma(\nu+a)\Gamma(\nu+b)}. \quad \text{(C2)}$$

However, if the parameters are such that $\mu<0$ and $\mu+\nu$, $\mu+a$, $\mu+b$ are positive or a pair of complex conjugates with positive real parts, then the polynomial will have a continuum spectrum as well as a finite size discrete spectrum

The asymptotics of this polynomial is derived in section 2 above and is given by formula (22). The scattering amplitude and phase shift are shown in equations (23) and (24), respectively.

Substituting the zeros of the scattering amplitude, which reads $z=i(k+\mu)$, into equation (8) and redefining the polynomial parameters as $\mu=\frac{1}{2}(\gamma+\delta+1)$, $\nu=\beta+\frac{1}{2}(\delta-\gamma+1)$, $a=\alpha-\frac{1}{2}(\gamma+\delta-1)$ and $b=\frac{1}{2}(\gamma-\delta+1)$ then we obtain the following discrete version of the Wilson polynomial

$$\begin{aligned}\tilde{R}_n^N(k;\alpha,\beta,\gamma)=&\frac{(\alpha+1)_n(\gamma+1)_n}{(\alpha+\beta+N+2)_n n!}\\&\times{}_4F_3\left(\left.\begin{matrix}-n,-k,n+\alpha+\beta+1,k-\beta+\gamma-N\\ \alpha+1,\gamma+1,-N\end{matrix}\right|1\right),\end{aligned} \quad \text{(C5)}$$

which is a renormalized Racah polynomial (see pages 26–29 of [10]) and the parameter δ is related to the integer N as $\delta=-(N+\beta+1)$. The same substitution results in the following three-term recursion relation

$$\frac{1}{4}(N+\beta-\gamma-2k)^2\tilde{R}_n^N = \left[\frac{1}{4}(N+\beta-\gamma)^2 - \frac{(n-N)(n+\alpha+1)(n+\gamma+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} - \frac{n(n+\beta)(n+\alpha+\beta-\gamma)(n+N+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}\right]\tilde{R}_n^N + \frac{(n+\alpha)(n+\beta)(n+\gamma)(n+\alpha+\beta-\gamma)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}\tilde{R}_{n-1}^N + \frac{(n+1)(n-N)(n+\alpha+\beta+1)(n+N+\alpha+\beta+2)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}\tilde{R}_{n+1}^N. \tag{C6}$$

The discrete orthogonality reads as follows

$$\sum_{k=0}^N \frac{2k+\gamma-\beta-N}{k+\gamma-\beta-N}\frac{(-N)_k(\alpha+1)_k(\gamma+1)_k(\gamma-\beta-N+1)_k}{(-\beta-N)_k(\gamma-\beta+1)_k(\gamma-\alpha-\beta-N)_k k!} \times \bar{R}_n^N(k;\alpha,\beta,\gamma)\bar{R}_m^N(k;\alpha,\beta,\gamma) = \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}\frac{(-\alpha-\beta-N-1)_N(\gamma-\beta-N+1)_N}{(-\beta-N)_N(\gamma-\alpha-\beta-N)_N} \times \frac{(\beta+1)_n(\alpha+\beta-\gamma+1)_n(\alpha+\beta+N+2)_n n!}{(-N)_n(\alpha+1)_n(\gamma+1)_n(\alpha+\beta+2)_n}\delta_{n,m} \tag{C7}$$

which is formula (1.2.2) in [10] and where

$$\bar{R}_n^N(k;\alpha,\beta,\gamma) = {}_4F_3\left(\begin{matrix}-n,\,-k,\,n+\alpha+\beta+1,\,k-\beta+\gamma-N\\ \alpha+1,\,\gamma+1,\,-N\end{matrix}\middle|\,1\right). \tag{C8}$$

Therefore, the discrete normalized weight function is

$$\rho^N(k;\alpha,\beta,\gamma) = \frac{(-\beta-N)_N(\gamma-\alpha-\beta-N)_N}{(-\alpha-\beta-N-1)_N(\gamma-\beta-N+1)_N} \times \frac{2k+\gamma-\beta-N}{k+\gamma-\beta-N} \times \frac{(-N)_k(\alpha+1)_k(\gamma+1)_k(\gamma-\beta-N+1)_k}{(-\beta-N)_k(\gamma-\beta+1)_k(\gamma-\alpha-\beta-N)_k k!} \tag{C9}$$

and the orthonormal version of the discrete Racah polynomial is

$$R_n^N(k;\alpha,\beta,\gamma) = \sqrt{\frac{2n+\alpha+\beta+1}{n+\alpha+\beta+1}\frac{(-N)_n(\alpha+1)_n(\gamma+1)_n(\alpha+\beta+2)_n}{(\beta+1)_n(\alpha+\beta-\gamma+1)_n(\alpha+\beta+N+2)_n n!}}\; {}_4F_3\left(\begin{matrix}-n,\,-k,\,n+\alpha+\beta+1,\,k-\beta+\gamma-N\\ \alpha+1,\,\gamma+1,\,-N\end{matrix}\middle|\,1\right). \tag{C10}$$

Thus,

$$\sum_{k=0}^N \rho^N(k;\alpha,\beta,\gamma)R_n^N(k;\alpha,\beta,\gamma)R_m^N(k;\alpha,\beta,\gamma) = \delta_{n,m}. \tag{C11}$$

ORCID iDs

A D Alhaidari https://orcid.org/0000-0002-8648-9040

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