

Charged black hole chemistry with massive gravitons

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Abstract

In the subject of black hole chemistry, a broad variety of critical phenomena for charged topological black holes (TBHs) with massive gravitons (within the framework of dRGT massive gravity) is discussed in detail. Since critical behavior and the nature of possible phase transition(s) crucially depend on the specific choice of ensemble, and, in order to gain more insight into criticality in the massive gravity framework, we perform our analysis in both canonical (fixed charge, Q) and grand canonical (fixed potential, Φ) ensembles. It is shown that, for charged TBHs in the grand canonical ensemble, the van der Waals (vdW) phase transition could take place in $d \geq 5$, the reentrant phase transition (RPT) in $d \geq 6$ and the analogue of triple point in $d \geq 7$, which are different from the results of the canonical ensemble. In the canonical ensemble, the vdW phase transition is observed in $d \geq 4$, the vdW type phase transition in $d \geq 6$ and the critical behavior associated with the triple point in $d \geq 6$. In this regard, the appearance of grand canonical $P - V$ criticality and the associated phase transition(s) in black holes with various topologies depend on the effective topological factor $k_{\text{eff}}^{(\text{GC})} \equiv [k + m_g^2 c_0^2 c_2 - 2(d_3/d_2)\Phi^2]$ instead of k in Einstein's gravity, where k is the normalized topological factor ($k_{\text{eff}}^{(\text{C})} \equiv [k + m_g^2 c_0^2 c_2]$ plays this role in the canonical ensemble of TBHs in massive gravity). Such evidence gives the (grand) canonical study of extended phase space thermodynamics with massive gravitons a special significance.

Keywords: black hole chemistry, massive gravity, phase transition

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1. Introduction

Asymptotically anti-de Sitter (AdS) black holes admit certain phase transitions in Einstein's gravity, e.g. Hawking–Page phase transition (a transition between the AdS spacetime and the AdS black hole) [1] is observed for a variety of asymptotically AdS black hole configurations such as Schwarzschild-AdS, Reissner–Nordström-AdS (RN-AdS) and Kerr (-Newman) AdS ones [1–4]. Seeking for a transition from a black hole phase to another phase, the van der Waals (vdW) behavior has been found for charged-AdS [2, 3, 5] and Kerr (-Newman) AdS black holes [4, 6, 7]. Further investigations on the phase transition between black hole phases have been revealed that the reentrant phase transition (RPT) present in liquid crystals and multicomponent fluids [8] can occur for Kerr-AdS black holes in $d \geq 6$ dimensions [6]. In addition, the small/intermediate/large black hole (SBH/IBH/LBH) phase transition associated with the triple point was found for multi-spinning Kerr-AdS black holes (in $d \geq 6$) [9] which is interpreted as solid/liquid/gas phase transition typical of many materials. These critical behaviors have been observed in a vast range of black hole configurations including nontrivial electromagnetic fields [10–12] and higher order curvatures [13–24]. But the story is different in dRGT massive gravity theory [25, 26] since all topological black holes (TBHs) can experience critical behavior and phase transition due to the massive graviton self-interaction potentials [27–33]. From the AdS/CFT perspective [35, 36], it leads to new prospects for investigating critical behavior of TBHs since there exists a dual interpretation in the gauge field theory (CFT) living on the AdS boundary.

These analogies between the standard thermodynamic phase transitions and the black hole ones have been found in the extended phase space (varying cosmological constant as pressure, $\Lambda = -8\pi P$ [37]) [5, 6, 9, 11–14, 16, 18–24, 27, 28, 30–33] and in some cases in the non-extended phase space (fixed cosmological constant) [2–4, 17, 29, 34]. For example, Reissner–Nordström-AdS black holes possess a first order phase transition with swallowtail behavior which closely resembles the well-known vdW phase transition in fluids in both non-extended [2] and extended [5] phase spaces with the same critical exponents as the vdW system. But, these analogies in the non-extended phase space are confusing since some black hole's intensive (extensive) quantities have to be identified with irrelevant extensive (intensive) quantities in the fluid system; for example, the identification between the fluid temperature and the $U(1)$ charge of RN-AdS black holes [2, 3] is really puzzling. By employing the extended phase space, these kinds of mismatches will be eliminated [5]. Motivated by this fact, the extended phase space thermodynamics (first established in [37] and then developed in [38–42]) is of direct interest for the present day, and, from this perspective, black holes can be understood from the viewpoint of standard chemistry, known as Black Hole Chemistry [43].

In Einstein's gravity, charged or rotating AdS black holes only admit phase transition in canonical (fixed Q or fixed J) ensemble [5, 44] whereas, as we will see in this paper, this statement is no longer valid in massive gravity. However, for spherically symmetric black holes in Einstein's gravity coupled with nonlinear electromagnetic sources such as power Maxwell invariant (PMI) electrodynamics, $P - V$ criticality is observed in both canonical (fixed $U(1)$ charge) and grand canonical (fixed $U(1)$ potential) ensembles of the extended phase space [12]. Using the canonical and grand canonical analysis, it was shown that the non-extended phase space of the charged-AdS black holes in dRGT massive gravity accepts first order phase transitions in a way reminiscent of vdW systems [29]. Moreover, AdS black hole solutions within the framework of higher curvature gravities exhibit rich (non) extended phase space with the associated critical behaviors in both canonical and grand canonical ensembles [13–16, 18–22, 33, 45]. Therefore, it seems geometrical modifications of general relativity (such as massive gravity, Lovelock gravity, $F(R)$ gravity, ...) or the presence of nontrivial

energy-momentum tensors (such as PMI or Born–Infeld electrodynamics) are mandatory to have black hole phase transitions in the grand canonical ensemble where electric potential Φ or angular velocity Ω are fixed.

One of the geometrical modifications of general relativity (GR) is the dRGT massive gravity which is regarded as a consistent extension of Einstein’s GR with an explicit mass term for spin-2 gravitons [25, 26]. This alternative theory of gravity modifies GR in the large scales (IR limit) and has some nice properties such as being ghost free [26, 46], in agreement with recent observational data of LIGO collaboration [47, 48], and the ability to explain the current observations related to dark matter [49] and also the accelerating expansion of universe without requiring any dark energy component [50, 51]. On the other hand, the possibility of embedding massive gravity in an ultraviolet-complete theory like string theory is indicated in [52]. In this paper, the authors considered 4-dimensional AdS solutions of IIB string theory in which the lowest spin-2 mode has a tiny mass (m_g), and explicitly showed that massive AdS_4 gravity is a part of the string theory landscape [52]. For these reasons we regard massive gravity as a class of theories that merit further exploration.

Massive gravity theories need an auxiliary reference metric ($f_{\mu\nu}$) to define a mass term for massless gravitons [48, 53], and, in principle, one can construct a special theory of massive gravity for each choice of reference metric [54]. Massive gravity theories in AdS space with a singular (degenerate) reference metric have been particularly useful in the context of gauge/gravity duality, where a finite DC conductivity obtains when studying the dual boundary theory [57, 62, 63], in contrast to massless gravity theories such as Einstein and Gauss–Bonnet ones with infinite DC conductivity [64–67]. On the other hand, AdS black hole solutions in massive gravity theories can effectively describe different phases of condensed matter systems with broken translational symmetry such as solids, liquids, (perfect) fluids etc [61, 68–70]. As reported in [28], one can build a gravitational theory dual system with this property and then see if it can thermodynamically simulate the critical behavior of those condensed matter systems as well. To do that, it is shown that in parallel with everyday thermodynamics, one has to insert a $P - V$ term for the AdS black hole systems and this can be done in the context of BHC [43] by extending the thermodynamic phase space [37]. One of the interesting cases is the chemistry of charged TBHs in massive gravity, since these black objects can be viewed as effective dual field theory of different types of charged condensed matter systems. Hence in this paper, we study charged BHC with massive gravitons with three reasonable reasons/goals: (i) considering TBHs in a more complicated environment via modified gravity, here dRGT massive gravity, leads to more possibilities for investigation. (ii) In the context of BHC via dRGT massive gravity, since holographic phase transitions take place in both canonical and grand canonical ensembles (as will be shown here), we can learn more about the effect of ensemble one is dealing with and keep track the outcomes of critical phenomena in each ensemble, separately. (iii) We intend to generalize the results of neutral massive TBHs to the charged cases and see what happens when a $U(1)$ charge is added. On the other hand, the inclusion of a $U(1)$ charge is equivalent to state that the charged TBHs may resemble those critical behaviors present in charged condensed matter systems.

So far, all studies related to the extended phase space thermodynamics and critical behaviors (only for vdW phase transition) of charged TBHs in massive gravity have been performed in the canonical ensemble. In addition, for the other types of phase transitions such as the RPT or triple point phenomena, only the uncharged (neutral) TBHs have been studied [28, 31]. In this regard, the RPT and triple point phenomena are not yet observed in the charged black holes of massive gravity. Here, we explicitly consider the effects of $U(1)$ charge and potential to investigate the associated $P - V$ criticality and bring out several important results. Since

the specific choice of ensemble has an important role on the criticality and nature of possible phase transition(s), it is of great interest to generalize these studies in the extended phase space of TBHs to the grand canonical ensemble. In fact, as we will explicitly show that a certain critical behavior in a specific d -dimensional spacetime could be present or absent depending on the nature of the ensemble one is dealing with.

Taking the above mentioned motivations into account, we intend to develop the extended phase space of the dRGT black hole solutions in the grand canonical ensemble along with the canonical ensemble by applying suitable boundary conditions. Our purpose is to discover which properties of AdS black holes are universal and which ones show a dependence on the spacetime dimensions and the ensemble. To do that, this paper is structured as follows: first, in section 2, considering the full nonlinear theory of dRGT massive gravity in higher dimensions, we present exact charged black hole solutions with appropriate boundary conditions. Then, in section 3, the subject of charged black hole chemistry in massive gravity is investigated in the canonical ensemble. Afterward, in section 4, the context of charged black hole chemistry in massive gravity is promoted to the grand canonical ensemble, and then, in section 5, we compute the associated critical exponents in both canonical and grand canonical ensembles. Finally, in section 6, we finish our paper with some concluding remarks and summarize the results.

2. Action, field equations, and topological black holes

The total action, \mathcal{I}_G , for a gravitating system consists of three terms as

$$\mathcal{I}_G = \mathcal{I}_b + \mathcal{I}_s + \mathcal{I}_{ct}, \quad (2.1)$$

where \mathcal{I}_b , \mathcal{I}_s and \mathcal{I}_{ct} are called the bulk action, the surface term (boundary action), and the counterterm action, respectively. The bulk action for dRGT massive gravity on the $d = (n + 2)$ -dimensional background manifold \mathcal{M} in the presence of negative cosmological constant ($\Lambda = -\frac{d_1 d_2}{2\ell^2}$, with the AdS radius ℓ) and Maxwell invariance \mathcal{F} is

$$\mathcal{I}_b = -\frac{1}{16\pi G_d} \int_{\mathcal{M}} d^d x \sqrt{-g} [R - 2\Lambda - \mathcal{F} + m_g^2 \sum_{i=1}^{d-2} c_i \mathcal{U}_i(g, f)], \quad (2.2)$$

where $\mathcal{F} \equiv F^{\mu\nu} F_{\mu\nu}$, and $F_{\mu\nu}$ is the Faraday tensor which is constructed using the $U(1)$ gauge field A_ν as $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$. In the above action, m_g is the graviton mass parameter, c_i 's are massive couplings which are arbitrary constants, $g_{\mu\nu}$ is the physical metric, and $f_{\mu\nu}$ is a fixed second rank symmetric tensor as an auxiliary reference metric to define a mass term for massless spin-2 particles (i.e. gravitons). As shown in [25, 26], gravitons become massive without any ghost problem [46] if one adds the interaction potentials \mathcal{U}_i to the Lagrangian density of Einstein's gravity⁴. The graviton interaction terms are symmetric polynomials of the eigenvalues of $d \times d$ matrix $\mathcal{K}^\mu{}_\nu = \sqrt{g^{\mu\lambda} f_{\lambda\nu}}$ and may be written as⁵

$$\mathcal{U}_i = \sum_{y=1}^i (-1)^{y+1} \frac{(i-1)!}{(i-y)!} \mathcal{U}_{i-y}[\mathcal{K}^y], \quad (2.3)$$

⁴ This statement holds in higher dimensions as well. The higher dimensional extension of the massive (bi)gravity, including higher order graviton's self-interactions, is discussed in [55, 56] which confirms the absence of ghost fields using the Cayley–Hamilton theorem.

⁵ Our notation is a little different with [25, 26, 46] and is in agreement with [27, 57, 58].

in which the square root of \mathcal{K} stands for matrix square root (i.e. $(\sqrt{\mathcal{K}})^\mu{}_\lambda (\sqrt{\mathcal{K}})^\lambda{}_\nu = \mathcal{K}^\mu{}_\nu$) and the rectangular brackets denote traces, $[\mathcal{K}] = \mathcal{K}^\mu{}_\mu$. Some explicit form of \mathcal{U}_i 's are given as

$$\begin{aligned}\mathcal{U}_1 &= [\mathcal{K}], \\ \mathcal{U}_2 &= [\mathcal{K}]^2 - [\mathcal{K}^2], \\ \mathcal{U}_3 &= [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3], \\ \mathcal{U}_4 &= [\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 8[\mathcal{K}][\mathcal{K}^3] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4], \\ \mathcal{U}_5 &= [\mathcal{K}]^5 - 10[\mathcal{K}]^3[\mathcal{K}^2] + 20[\mathcal{K}]^2[\mathcal{K}^3] - 20[\mathcal{K}^2][\mathcal{K}^3] \\ &\quad + 15[\mathcal{K}][\mathcal{K}^2]^2 - 30[\mathcal{K}][\mathcal{K}^4] + 24[\mathcal{K}^5], \\ &\vdots\end{aligned}\tag{2.4}$$

Using the variational principle, the electromagnetic and gravitational field equations of the bulk action (2.2) can be obtained. Varying the bulk action with respect to the dynamical metric ($g_{\mu\nu}$) and gauge field (A_ν) results

$$\begin{aligned}\delta\mathcal{I}_b &= -\frac{1}{16\pi G_d} \int_{\mathcal{M}} d^d x \sqrt{-g} [G_{\mu\nu} + \Lambda g_{\mu\nu} + m_g^2 \mathcal{X}_{\mu\nu} - T_{\mu\nu}] \delta g^{\mu\nu} \\ &\quad + \frac{1}{8\pi G_d} \int_{\partial\mathcal{M}} d^{d-1} x \sqrt{-h} n^\alpha h^{\mu\nu} \delta g_{\mu\nu, \alpha} \\ &\quad - \frac{1}{4\pi G_d} \int_{\mathcal{M}} d^d x \sqrt{-g} [\nabla_\mu F^{\mu\nu}] \delta A_\nu \\ &\quad + \frac{1}{4\pi G_d} \int_{\partial\mathcal{M}} d^{d-1} x \sqrt{-h} n_\mu F^{\mu\nu} \delta A_\nu,\end{aligned}\tag{2.5}$$

where n_μ is a radial unit vector pointing outwards and $h_{\mu\nu}$ is the induced metric of the boundary ($\partial\mathcal{M}$)⁶. In the above expression, $T_{\mu\nu}$ and $\mathcal{X}_{\mu\nu}$ are the consequences of varying the Maxwell invariant and graviton interaction potentials with respect to the $g_{\mu\nu}$ as below

$$T_{\mu\nu} = -\frac{1}{2} g_{\mu\nu} \mathcal{F} + 2F_{\mu\lambda} F_\nu{}^\lambda,\tag{2.6}$$

$$\mathcal{X}_{\mu\nu} = -\sum_{i=1}^{d-2} \frac{c_i}{2} \left[\mathcal{U}_i g_{\mu\nu} + \sum_{y=1}^i (-1)^y \frac{i!}{(i-y)!} \mathcal{U}_{i-y} \mathcal{K}_{\mu\nu}^y \right],\tag{2.7}$$

where, in our notation, $\mathcal{U}_{i-y} = 1$ if $i = y$. The explicit form of $\mathcal{X}_{\mu\nu}$ can be presented as

$$\begin{aligned}\mathcal{X}_{\mu\nu} &= -\frac{c_1}{2} (\mathcal{U}_1 g_{\mu\nu} - \mathcal{K}_{\mu\nu}) \\ &\quad - \frac{c_2}{2} (\mathcal{U}_2 g_{\mu\nu} - 2\mathcal{U}_1 \mathcal{K}_{\mu\nu} + 2\mathcal{K}_{\mu\nu}^2) \\ &\quad - \frac{c_3}{2} (\mathcal{U}_3 g_{\mu\nu} - 3\mathcal{U}_2 \mathcal{K}_{\mu\nu} + 6\mathcal{U}_1 \mathcal{K}_{\mu\nu}^2 - 6\mathcal{K}_{\mu\nu}^3) \\ &\quad - \frac{c_4}{2} (\mathcal{U}_4 g_{\mu\nu} - 4\mathcal{U}_3 \mathcal{K}_{\mu\nu} + 12\mathcal{U}_2 \mathcal{K}_{\mu\nu}^2 - 24\mathcal{U}_1 \mathcal{K}_{\mu\nu}^3 + 24\mathcal{K}_{\mu\nu}^4) \\ &\quad - \frac{c_5}{2} (\mathcal{U}_5 g_{\mu\nu} - 5\mathcal{U}_4 \mathcal{K}_{\mu\nu} + 20\mathcal{U}_3 \mathcal{K}_{\mu\nu}^2 - 60\mathcal{U}_2 \mathcal{K}_{\mu\nu}^3 + 120\mathcal{U}_1 \mathcal{K}_{\mu\nu}^4 - 120\mathcal{K}_{\mu\nu}^5) + \dots\end{aligned}\tag{2.8}$$

⁶It should be noted that, in the massless limit of massive gravity, some problematic boundary terms induced by the bulk action appear and one should introduce novel boundary counterterms which dominate over the Gibbons–Hawking term and cancel those terms [59]. In the present work, considering the massive graviton case, no further boundary term is needed since all the new fields of massive gravity enter the action with the first derivative, so do not alter the equations of motion.

As a result, the gravitational and electromagnetic field equations of the bulk theory are obtained as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + m_g^2 \mathcal{X}_{\mu\nu} = -\frac{1}{2} g_{\mu\nu} \mathcal{F} + 2F_{\mu\lambda} F_\nu{}^\lambda, \quad (2.9)$$

$$\nabla_\mu F^{\mu\nu} = 0. \quad (2.10)$$

According to the variation of the bulk action (2.5) and asking for a well-defined variational principle, one has to appropriately cancel the boundary terms by use of adding surface term(s), \mathcal{I}_s . To do so, the Gibbons–Hawking action, \mathcal{I}_{GH} , can remove the derivative terms of $g_{\mu\nu}$ normal to the boundary and is given by

$$\mathcal{I}_{\text{GH}} = \frac{1}{8\pi G_d} \int_{\partial\mathcal{M}} d^{d-1}x \sqrt{-h} K, \quad (2.11)$$

where K is the trace of extrinsic curvature of boundary, $\partial\mathcal{M}$. On the other hand, the electromagnetic boundary term has to be eliminated by use of imposing boundary condition or proposing another surface term. There are two possibilities which define the fixed potential and the fixed charge ensembles at infinity as

$$\begin{aligned} \delta A_\nu|_{\partial\mathcal{M}} = 0 &\longleftrightarrow \text{fixed potential ensemble} \\ \mathcal{I}_s = \mathcal{I}_{\text{GH}} + \mathcal{I}_{\text{EM}} &\longleftrightarrow \text{fixed charge ensemble} \end{aligned} \quad (2.12)$$

where \mathcal{I}_{EM} is a new surface term that is needed to remove the electromagnetic boundary term in (2.5), and, consequently, to fix charge on the boundary, with the following explicit form

$$\mathcal{I}_{\text{EM}} = -\frac{1}{4\pi G_d} \int_{\partial\mathcal{M}} d^{d-1}x \sqrt{-h} n_\mu F^{\mu\nu} A_\nu. \quad (2.13)$$

We refer to the such gravitating systems with fixed charge and fixed potential boundary conditions at infinity as the canonical and the grand canonical ensembles, respectively, which is common in literature. In our considerations, these boundary conditions will be imposed separately in order to compare the results of black holes' *PV* criticality in both canonical and grand canonical ensembles.

To find topological (AdS) black holes, we make use of the following $d(=n+2)$ -dimensional line element ansatz

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 h_{ij} dx_i dx_j \quad (i, j = 1, 2, \dots, n), \quad (2.14)$$

where the line element $h_{ij} dx_i dx_j$ is the metric of n -dimensional (unit) hypersurface with the constant curvature $d_1 d_2 k$ and volume ω_n with the following forms

$$h_{ij} dx_i dx_j = \begin{cases} dx_1^2 + \sum_{i=2}^{d-2} \prod_{j=1}^{i-1} \sin^2 x_j dx_i^2 & (k = +1) \\ \sum_{i=1}^{d-2} dx_i^2 & (k = 0) \\ dx_1^2 + \sinh^2 x_1 \sum_{i=2}^{d-2} dx_i^2 \prod_{j=2}^{i-1} \sin^2 x_j & (k = -1) \end{cases} \quad (2.15)$$

in which $\prod_x^y \dots = 1$ if $x > y$. Another line element ansatz is necessary for the reference metric $f_{\mu\nu}$. We are primarily interested in building an effective field theory by use of the gravitational language which could describe some properties of different phases of matter. It has been indicated in a series of papers [57, 68–70] that the black hole solutions of Einstein’s gravity minimally coupled with a number of N scalar fields (ϕ^a) on AdS space can describe different types of matter (such as solids and fluids) in a covariant way. In such theories, the number of independent scalar fields (known as Stückelberg fields) is less than the number of space-time dimensions d (i.e. $N < d$). After the (unitary) gauge fixing $\phi^a = \delta_\mu^a x^\mu$, the field structure of such theories are equivalent to a family of massive gravity on AdS with a singular and (spatial) degenerate reference metric, i.e. a reference metric with vanishing tt and rr entries on the diagonal form. Remarkably as emphasized in [71], the holographic language of massive gravity with a singular reference metric in terms of Stückelberg fields [60, 61, 68–70] is related to the language of gauge field theories for liquid crystals [71, 72], which shows a deep connection between these theories, and it becomes more manifest using BHC as speculated in [28]. On the other hand, in view of gauge/gravity duality, massive gravity on AdS space with a singular (degenerate) reference metric is dual to homogenous and isotropic condensed matter systems which leads to a boundary theory with the finite DC conductivity [57, 62, 63], a desired property for normal conductors that is absent in massless gravity theories [64–67]. For these reasons, massive gravity with a singular reference metric is of direct interest to us. So we make use of the following singular and (spatial) degenerate ansatz [29, 57]

$$f_{\mu\nu} = \text{diag} (0, 0, c_0^2 h_{ij}), \quad (2.16)$$

in which c_0 is a positive constant. Since $f_{\mu\nu}$ depends only on the spatial components h_{ij} of the spacetime metric, the theory do not preserve general covariance in the transverse spatial coordinates x_1, x_2, \dots, x_{d_2} . This choice of the reference metric establishes a subclass of dRGT massive gravity theories, called as the reduced massive gravity [60, 61], that is ghost free and admits exact black hole solutions on AdS [57, 62, 63, 73].

Nevertheless, it is possible to find black hole solutions with a nonsingular reference metric (in which the tt and rr components of the reference metric are non-zero), but the analytic solutions can be found only for some specific values of massive coefficients (see [74–80] for more details). The reference metric in these black hole solutions is assumed to be the Minkowski metric which means that diffeomorphism breaks along all of the temporal and spatial directions. Assuming the Minkowskian reference metric, spherically symmetric black hole solutions were found in [74, 75] and in the limit of vanishing graviton mass they go smoothly to the Schwarzschild and RN black holes on de Sitter space. Asymptotically flat black hole solutions were found in [76] and they are potentially considered as the viable classical solutions for stars and black holes in massive gravity, but the curvature diverges near the horizon of these solutions. However, black hole solutions with non-singular horizon were introduced in [77]. There exist other interesting black hole solutions including charged and rotation parameters that were the subject of [78–80]. It should be noted that all of these solutions have been found in 4-dimensions. In addition, the author of [55, 56] found out Schwarzschild–Tangherlini–(A)dS in five-dimensional massive gravity as well as in massive bi-gravity with assumptions that the reference metric is compatible with (for massive gravity) or proportional (for massive bi-gravity) to the physical one.

Using the ansatz (2.16), the interaction terms \mathcal{U}_i are calculated as

$$\mathcal{U}_i = \left(\frac{c_0}{r}\right)^i \prod_{j=2}^{i+1} d_j, \quad (2.17)$$

in which we have used the convention $d_i = d - i$ (throughout this paper, this convention will be used). Considering the above relation, it is inferred that there are at most $(d - 2)$ potential terms (\mathcal{U}_i) in a d -dimensional spacetime and all the higher-order terms vanish identically. Therefore, the upper bound $(d - 2)$ exists for summation in equation (2.7) in all dimensions.

Solving the Maxwell field equations (2.10) yields the $U(1)$ gauge field as

$$A_\mu = \left(\phi - \frac{q}{d_3 r^{d_3}} \right) \delta_\mu^0, \quad (2.18)$$

in which q is a constant related to the total electric charge of spacetime. The constant ϕ is obtained by the regularity condition [81, 82] at the horizon ($r = r_+$), i.e. $A_t(r_+) = 0$, which leads to $\phi = \frac{q}{d_3 r_+^{d_3}}$. The electric potential Φ of the black hole spacetime is actually electrostatic potential difference between the horizon and the boundary at infinity. We choose the event horizon as the reference (i.e. $\Phi = 0$ as $r \rightarrow r_+$). Thus, the electric potential can be measured at the infinity with respect to the horizon as

$$\Phi = A_\mu \chi^\mu|_{r \rightarrow \infty} - A_\mu \chi^\mu|_{r \rightarrow r_+} = \frac{q}{d_3 r_+^{d_3}}, \quad (2.19)$$

where $\chi = \partial_t$ is the temporal Killing vector. Moreover, the electric charge can be obtained using the Gauss' law as

$$Q = \frac{\omega_n}{4\pi} q. \quad (2.20)$$

Now, the gravitational field equations (2.9) can be solved using the metric ansatz (2.14). To do that, the rr -component of equation (2.9) is enough for our purpose to obtain the metric function $V(r)$. The rr -component of equation (2.9) is given as

$$d_2 d_3 (k - f(r)) - d_2 r \left(\frac{df(r)}{dr} \right) - 2\Lambda r^2 - 2q^2 r^{-2d_3} + m_g^2 \sum_{i=1}^{d-2} \left(\frac{c_0^i c_i}{r^{i-2}} \prod_{j=2}^{i+1} d_j \right) = 0. \quad (2.21)$$

Consequently, the metric function $V(r)$ is obtained as

$$V(r) = k - \frac{2\Lambda r^2}{d_1 d_2} - \frac{m}{r^{d_3}} + m_g^2 \sum_{i=1}^{d-2} \left(\frac{c_0^i c_i}{d_2 r^{i-2}} \prod_{j=2}^i d_j \right) + \frac{2q^2}{d_2 d_3 r^{2d_3}}, \quad (2.22)$$

where m is a positive constant related to the finite mass of spacetime. This solution is valid to all orders in arbitrary dimensions, and, simultaneously, satisfies all components of equation (2.9).

The existence of essential singularity of the spacetime is confirmed by computing the Kretschmann scalar which is

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = \left(\frac{\partial^2 V(r)}{\partial r^2} \right)^2 + 2d_2 \left(\frac{1}{r} \frac{\partial V(r)}{\partial r} \right)^2 + 2d_2 d_3 \left(\frac{V(r) - k}{r^2} \right)^2. \quad (2.23)$$

Taking into account the metric function $V(r)$, it can be found $R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \propto r^{-4d_2}$ near the origin ($r \rightarrow 0^+$) and, for $r \neq 0$, all the curvature scalars are finite. This curvature singularity cannot be eliminated by any coordinate transformation, while can be covered by an event horizon, r_+ ($V(r_+) = 0$). The roots of the metric function $V(r) = g^{rr} = 0$ specify the number of horizons, and, as reported in [33, 58, 83], the metric function $V(r)$ could have more than two roots. So the multi-horizon black hole solutions are found in massive gravity and we assume that r_+ is the event horizon radius of the black hole solutions, i.e. the largest real positive root of $V(r) = 0$.

The Hawking temperature of the black hole spacetimes can be obtained by applying the definition of surface gravity [88] or employing the Euclidean formalism [1, 36]. Considering the latter, by the analytic continuation of the Lorentzian metric (2.14) to Euclidean signature, i.e. $t_E = it$, we get

$$ds_E^2 = V(r)dt_E^2 + \frac{dr^2}{V(r)} + r^2 h_{ij} dx_i dx_j. \quad (2.24)$$

The above Euclidean metric has a conical singularity at the horizon ($r = r_+$), so regularity condition near the horizon requires that the Euclidean time be periodic, i.e. $t_E \sim t_E + \beta$ (otherwise the expansion of the Euclidean spacetime around $r = r_+$ shows a conical singularity). Thus, one obtains the Hawking temperature of the obtained TBHs in the canonical ensemble as

$$\beta^{-1} = T = \frac{1}{4\pi} \frac{\partial V(r)}{\partial r} \Big|_{r=r_+} = \frac{d_2 d_3 k - 2\Lambda r_+^2 - 2q^2 r_+^{-2d_3} + m_g^2 \sum_{i=1}^{d-2} \left(\frac{c_0^i c_i}{r_+^{i-2}} \prod_{j=2}^{i+1} d_j \right)}{4\pi d_2 r_+}, \quad (2.25)$$

and, for the case of grand canonical ensemble (fixed Φ), using $q = d_3 \Phi r_+^{d_3}$, it is given by

$$\beta^{-1} = T = \frac{d_2 d_3 k - 2\Lambda r_+^2 - 2d_3^2 \Phi^2 + m_g^2 \sum_{i=1}^{d-2} \left(\frac{c_0^i c_i}{r_+^{i-2}} \prod_{j=2}^{i+1} d_j \right)}{4\pi d_2 r_+}. \quad (2.26)$$

Working in the Euclidean formulation (for more details see [84–87]), the semi-classical partition functions of TBHs may be evaluated using the following path integral over the dynamical metric ($g \equiv g_{\mu\nu}$) as

$$\mathcal{Z} = \int \mathcal{D}[g, \varphi] e^{-\mathcal{I}_E[g, \varphi]}, \quad (2.27)$$

in which \mathcal{D} denotes integration over all paths, φ is considered as matter fields and \mathcal{I}_E represents the Euclidean version of the Lorentzian action \mathcal{I}_G by implementing the Wick rotation, $t_E = it$. In next sections, we will explicitly evaluate semi-classical black hole partition functions in both canonical and grand canonical ensembles by employing the appropriate boundary conditions. This facilitates the study of thermodynamics of TBHs.

3. Black hole chemistry in canonical ensemble

3.1. Canonical partition function

The gravitational partition function in the canonical (fixed charge) ensemble is defined by following path integral

$$\mathcal{Z}_C = \int \mathcal{D}[g, A] e^{-\mathcal{I}_E[g, A]} \simeq e^{-\mathcal{I}_{\text{on-shell}}(\beta, Q)}, \quad (3.1)$$

in which A and $\mathcal{I}_{\text{on-shell}}$ represent the gauge field (A_μ) and the on-shell gravitational action, respectively. The most dominant contribution of the partition function originates from substituting the classical solutions of the action, i.e. the so-called on-shell action by applying the saddle-point approximation. To do that, one has to first compute the total on-shell action ($\mathcal{I}_{\text{on-shell}}$) in the Euclidean formalism. In the case of the canonical ensemble, we need to add

the electromagnetic surface term \mathcal{I}_{EM} to the action, i.e. $\mathcal{I}_G = \mathcal{I}_b + \mathcal{I}_{GH} + \mathcal{I}_{\text{EM}} + \mathcal{I}_{ct}$. Here, we compute the on-shell action using the Hawking–Witten prescription (the so-called subtraction method [1, 36]), and so, we only need to evaluate the on-shell bulk action plus the electromagnetic surface term (\mathcal{I}_{EM}) as the associated boundary condition with the fixed charge ensemble.

In order to have a finite gravitational partition function, following Hawking–Witten prescription, we subtract the on-shell action of the AdS background without black hole (referred to as \mathcal{I}_{AdS}), i.e. setting $m = Q = 0$ in equation (2.22), from the on-shell action of the black hole spacetime (referred to as \mathcal{I}_{BH}) and compute the explicit form of on-shell action in Euclidean signature. That leads to free energy difference as

$$F \equiv \Delta F = \beta^{-1} (\mathcal{I}_{\text{BH}} - \mathcal{I}_{\text{AdS}}), \quad (3.2)$$

in which the zero point energy of the boundary gauge theory (based on AdS/CFT duality) is canceled. Now, we briefly explain how to compute the on-shell action of charged TBHs in dRGT massive gravity. First, the Ricci scalar (R) is obtained using the gravitational field equations (2.9) as

$$R = \frac{1}{d_2} (2\Lambda d + 2m_g^2 \mathcal{X} + d_4 \mathcal{F}), \quad \mathcal{X} \equiv g^{\mu\nu} \mathcal{X}_{\mu\nu}. \quad (3.3)$$

Using this equation and the explicit form of interaction potentials (2.17), the bulk Lagrangian presented in (2.2) is explicitly calculated as

$$\mathcal{L}_{\text{bulk}} \equiv R - 2\Lambda - \mathcal{F} + m_g^2 \sum_{i=1}^{d-2} c_i \mathcal{U}_i(g, f) = \frac{2}{d_2} (2\Lambda - \mathcal{F}) + m_g^2 \sum_{i=1}^{d-2} (i-2) \frac{c_0^i c_i}{r^i} \prod_{j=3}^{i+1} d_j, \quad (3.4)$$

in which the following identity [28] has been used

$$2 \prod_{j=3}^{i+1} d_j + \sum_{y=1}^i (-1)^y \frac{i!}{(i-y)!} \prod_{j=2}^{i-y+1} d_j = (i-2) \prod_{j=3}^{i+1} d_j, \quad (3.5)$$

where $\prod_x^y \dots = 1$ if $x > y$. Substituting the classical solutions, equations (2.22) and (2.18), the on-shell bulk action for the TBH spacetimes with the fixed charge boundary condition can be computed after a long and tedious calculation as

$$\mathcal{I}_{\text{BH}} = \frac{\beta \omega_n}{16\pi G_d} \left[\frac{2}{\ell^2} r^{d_1} - m_g^2 \sum_{i=1}^{d-2} \frac{(i-2)c_0^i c_i}{d-i-1} r^{d-i-1} \prod_{j=3}^{i+1} d_j - \frac{4q^2}{d_2 r^{d_3}} \right]_{r_+}^R, \quad (3.6)$$

in which ‘ R ’ is an upper cutoff on the radial integrations in order to regularize the action, and will be canceled at the end. In the above calculation, we have used the electromagnetic surface term, \mathcal{I}_{EM} , in the Euclidean signature as

$$\mathcal{I}_{\text{EM}} = -\frac{1}{4\pi G_d} \int_{\partial \mathcal{M}} d^{d-1} x \sqrt{h} m_\mu F^{\mu\nu} A_\nu = \frac{\beta \omega_n}{16\pi G_d} \left[\frac{4q^2}{d_3 r_+^{d_3}} \right]. \quad (3.7)$$

Repeating the same procedure for the AdS background in massive gravity (without any matter or electromagnetic field, i.e. setting $Q = m = 0$), one obtains

$$\mathcal{I}_{\text{AdS}} = \frac{\beta_0 \omega_n}{16\pi G_d} \left[\frac{2}{\ell^2} R^{d_1} - m_g^2 \sum_{i=1}^{d-2} \frac{(i-2)c_0^i c_i}{d-i-1} R^{d-i-1} \prod_{j=3}^{i+1} d_j \right], \quad (3.8)$$

with the period β_0 . Both AdS and black hole spacetimes at $r = R$ must have the same geometry, i.e. $\beta_0 V_0(R)^{1/2} = \beta V(R)^{1/2}$, which leads to

$$\beta_0 = \beta \left(1 - \frac{m\ell^2}{2r^{d-1}} + O(r^{-2(d-1)}) \right). \quad (3.9)$$

Using this fact and the following identity [28]

$$\frac{1}{d_2} \prod_{j=2}^i d_j + \frac{i-2}{d-i-1} \prod_{j=3}^{i+1} d_j = (i-1) \prod_{j=3}^i d_j, \quad (3.10)$$

the renormalized on-shell action in the canonical ensemble is eventually computed as

$$\begin{aligned} \mathcal{I}_{\text{on-shell}} &\equiv \lim_{R \rightarrow \infty} (\mathcal{I}_{\text{BH}} - \mathcal{I}_{\text{AdS}}) \\ &= \frac{\beta \omega_n r_+^{d_3}}{16\pi G_d} \left[k - \frac{r_+^2}{\ell^2} + \frac{2(2d-5)q^2}{d_2 d_3 r_+^{2d_3}} + m_g^2 \sum_{i=1}^{d-2} \left(\frac{(i-1)c_0^i c_i}{r_+^{i-2}} \prod_{j=3}^i d_j \right) \right]. \end{aligned} \quad (3.11)$$

3.2. Thermodynamics

Thermodynamic quantities are straightforwardly extracted from the obtained partition function. Using the canonical partition function, equations (3.1) and (3.11), the mass of the black hole is computed as (setting $G_d = 1$)

$$M = -\frac{\partial}{\partial \beta} \ln \mathcal{Z}_C = \frac{d_2 \omega_n r_+^{d_3}}{16\pi} \left[k + \left(\frac{r_+}{\ell} \right)^2 + m_g^2 \sum_{i=1}^{d-2} \left(\frac{c_0^i c_i}{d_2 r_+^{i-2}} \prod_{j=2}^i d_j \right) + \frac{2q^2}{d_2 d_3 r_+^{2d_3}} \right], \quad (3.12)$$

where, in the extended phase space ($\Lambda = -\frac{d_1 d_2}{2\ell^2} = -8\pi P$), has to be interpreted as the black hole enthalpy, $M \equiv H$. The above black hole mass is in agreement with the ADM mass formula as

$$M = \frac{d_2 \omega_n}{16\pi} m, \quad (3.13)$$

in which the constant m is obtained from $V(r_+) = 0$ (see equation (2.22)), and thus, the same result as equation (3.12) is obtained. Working in the extended phase space, the free energy in the canonical ensemble is obtained as

$$\begin{aligned} G &= \beta^{-1} \ln \mathcal{Z}_C(T, P, Q) = M - TS \\ &= \frac{\omega_n r_+^{d_3}}{16\pi} \left[k - \frac{16\pi P r_+^2}{d_1 d_2} + \frac{(4d-10)q^2}{d_2 d_3 r_+^{2d_3}} + m_g^2 \sum_{i=1}^{d-2} \left(\frac{(i-1)c_0^i c_i}{r_+^{i-2}} \prod_{j=3}^i d_j \right) \right], \end{aligned} \quad (3.14)$$

which, in fact, is the Gibbs free energy of the AdS black hole. Using the obtained free energy, the other thermodynamic variables (i.e. V , Φ and S) can be easily computed (notice that r_+ is understood as a function of P and T according to equation (2.25)). The thermodynamic volume is given by

$$V = \left(\frac{\partial G}{\partial P} \right)_{T, Q} = \frac{\omega_n}{d_1} r_+^{d_1}. \quad (3.15)$$

The electric potential can be calculated using the following thermodynamic relation as

$$\Phi = \left(\frac{\partial G}{\partial Q} \right)_{T,P} = \frac{q}{d_3 r_+^{d_3}}, \quad (3.16)$$

which is in agreement with the equation (2.19). Finally, the black hole entropy is given by

$$S = - \left(\frac{\partial G}{\partial T} \right)_{P,Q} = \frac{\omega_n}{4} r_+^{d_2}, \quad (3.17)$$

which satisfy the so-called area law and is in agreement with the relation $S = \beta(M - G)$ as expected.

In conclusion, the obtained thermodynamic quantities satisfy analytically the first law of thermodynamics in the Gibbs energy representation, i.e. $dG = -SdT + \Phi dQ + VdP$. Furthermore, one can use the Legendre transform $G = M - TS$ to write down the first law in the enthalpy representation as $dM = TdS + \Phi dQ + VdP$, in which the temperature and thermodynamic volume are respectively obtained using $T = (\partial M / \partial S)_{P,Q}$ and $V = (\partial M / \partial P)_{S,Q}$ in agreement with equations (2.25) and (3.15). Finally, using these ingredients, it can be derived that obtained thermodynamic quantities obey the extended Smarr formula as

$$(d-3)M = (d-2)TS + (d-3)\Phi Q - 2PV + \sum_{i=1}^{d-2} (i-2)C_i c_i, \quad (3.18)$$

where the conjugate potentials (C_i) corresponding to the massive couplings (c_i) are given by

$$C_i = \left(\frac{\partial M}{\partial c_i} \right)_{S,P,Q,c_j \neq i} = \frac{\omega_n}{16\pi} m_g^2 c_0^i r_+^{d_{i+1}} \prod_{j=2}^i d_j. \quad (3.19)$$

As a result, the extended Smarr relation suggests that one should take into account c_i 's as the new thermodynamic variables. This leads to the extended first law as $dM = TdS + \Phi dQ + VdP + \sum_{i=1}^{d-2} C_i dc_i$. Comparing equation (3.18) with [37, 89], the same result for the Smarr formula in the extended phase space is obtained if one invokes the method of scaling argument as

$$(d-3)M = (d-2) \left(\frac{\partial M}{\partial S} \right) S + (d-3) \left(\frac{\partial M}{\partial Q} \right) Q - 2 \left(\frac{\partial M}{\partial P} \right) P + \sum_{i=1}^{d-2} (i-2) \left(\frac{\partial M}{\partial c_i} \right) c_i, \quad (3.20)$$

with the following scalings for the thermodynamic quantities

$$[M] = L^{d-3}, \quad [c_i] = L^{i-2}, \quad [P] = L^{-2}, \quad [S] = L^{d-2}, \quad [Q] = L^{d-3}. \quad (3.21)$$

3.3. Holographic phase transitions

Using equation (2.25), the canonical equation of state of charged TBHs is calculated in the extended phase space as

$$P = \frac{d_2 \tilde{T}}{4r_+} - \frac{d_2 d_3 k_{\text{eff}}^{(C)}}{16\pi r_+^2} - \frac{m_g^2}{16\pi} \sum_{i=3}^{d-2} \left(\frac{c_0^i c_i}{r_+^i} \prod_{j=2}^{i+1} d_j \right) + \frac{q^2}{8\pi r_+^{2d_2}}, \quad (3.22)$$

in which the effective topological factor $k_{\text{eff}}^{(C)}$ has been introduced as

$$k_{\text{eff}}^{(C)} \equiv [k + m_g^2 c_0^2 c_2], \quad (3.23)$$

and \tilde{T} is the shifted Hawking temperature [28, 30, 31, 33] with the following explicit form

$$\tilde{T} = T - \frac{m_g^2 c_0 c_1}{4\pi} = \frac{d_2 d_3 k - 2\Lambda r_+^2 - 2q^2 r_+^{-2d_3} + m_g^2 \sum_{i=2}^{d-2} \left(\frac{c_0^i c_i}{r_+^{i-2}} \prod_{j=2}^{i+1} d_j \right)}{4\pi d_2 r_+}. \quad (3.24)$$

The inflection point(s) of isothermal curves in $P - v$ diagrams determine the critical point(s), i.e. by applying the following relations

$$\begin{aligned} \left(\frac{\partial P}{\partial v} \right)_T = 0 &\iff \left(\frac{\partial P}{\partial r_+} \right)_T = 0, \\ \left(\frac{\partial^2 P}{\partial v^2} \right)_T = 0 &\iff \left(\frac{\partial^2 P}{\partial r_+^2} \right)_T = 0, \end{aligned} \quad (3.25)$$

where the specific volume v is proportional to r_+ as $v = 4r_+ \ell_P^{d_2} / d_2$ [5, 11, 28, 30, 31, 33]. So, because of the dependency between r_+ , v and the thermodynamic volume V , criticality in one of $P - r_+$, $P - v$ or $P - V$ planes indicates criticality in the others. On the other hand, since the essential information of the phase transitions encodes in the $G - T$ diagrams, so we mainly discuss based on the isobaric curves of these diagrams in the next sections.

Implementing equation (3.25) for the equation of state (3.22) leads to

$$d_3 k_{\text{eff}}^{(C)} r_+^{2d_3} + m_g^2 \sum_{i=3}^{d-2} \left(i(i-1) c_0^i c_i r_+^{2d-i-4} \prod_{j=3}^{i+1} d_j \right) - 2(2d-5)q^2 = 0. \quad (3.26)$$

As will be shown, the number of physical critical point(s) of the above equation specifies the type of phase transition. In the following, we will analyze the canonical ensemble holographic phase transitions in massive gravity theory case by case with detail. In advance, it should be noted that our considerations are for all types of charged TBHs (with $k = 0, \pm 1$).

In fact, according to the given equations in this section, the combination $k_{\text{eff}}^{(C)} \equiv [k + m_g^2 c_0^2 c_2]$ as an effective topological factor always appears. If we find a set of parameters related to a critical behavior in a charged-AdS black hole system with a specific event horizon geometry, one can always obtain the same critical behavior in another black hole system with different horizon geometry. The only necessity is that the same value must be provided for $k_{\text{eff}}^{(GC)}$, which is always possible by varying the massive constant c_2 . Consequently, the same critical points with the same critical behavior are found for the case of the spherical, planar, and hyperbolic black holes. This is a remarkable property of TBHs in massive gravity (first indicated in [28]).

3.3.1. van der Waals (vdW) phase transition. The vdW phase transition in the context of massive gravity is widely discussed before in [27, 29, 30]. But here, we try to generalize the analytical results to higher dimensions. To have the vdW phase transition, one (physical) critical point is needed. This phenomenon can commence to appear in $d \geq 4$ dimensions. In a 4-dimensional spacetime, only the first two massive couplings (c_1 and c_2) are present. So, we can simply assume that the first two massive couplings are non-zero and the others vanish in higher dimensions. Of course, according to equation (3.26), one can always find one (physical) critical point associated with the vdW behavior by use of an appropriate fine tuning of massive parameters. Here, using equation (3.26), we apply the first approach which leads to the following critical point equation

$$d_3 k_{\text{eff}}^{(C)} r_+^{2d_3} - 2(2d - 5)q^2 = 0. \quad (3.27)$$

The critical horizon radius is obtained easily as

$$r_c = \left(\frac{2(2d - 5)q^2}{d_3 k_{\text{eff}}^{(C)}} \right)^{\frac{1}{2d_3}}, \quad (3.28)$$

in which the constraint $k_{\text{eff}}^{(C)} > 0$ must be satisfied. According to the later constraint, there is no limitation on the value of the $U(1)$ charge in Einstein ($m_g = 0$) or massive gravity. Evidently, in the massless limit of gravitons ($m_g = 0$) which leads to the Einstein gravity, no phase transition and critical behavior take place for charged TBHs with Ricci flat or hyperbolic horizon geometries (i.e. $k = 0, -1$).

Regarding equation (3.28), the critical pressure and temperature are computed as

$$P_C = \frac{d_3^2 k_{\text{eff}}^{(C)}}{16\pi r_c^2}, \quad (3.29)$$

and

$$\tilde{T}_C = \frac{d_3^2 k_{\text{eff}}^{(C)}}{(2d - 5)\pi r_c}. \quad (3.30)$$

Now, using these critical quantities, we can easily plot the vdW behavior of phase transition for a set of TBHs. In (figure 1), $G - T$ diagrams for the isobaric curves near the critical point are depicted. As seen, for the range $P < P_C$, the characteristic swallowtail form is observed which closely resembles the vdW phase transition in fluids. For the range $P > P_C$, the isobars correspond to the ideal gas with a single phase. Moreover, the presented example is a generic feature of all types of TBHs.

Using $v = 4r_+/d_2$ in the geometric units, the obtained thermodynamic quantities at the critical point satisfy the following universal ratio

$$\frac{P_C r_c}{\tilde{T}_C} = \frac{2d - 5}{16} \iff \frac{P_C v_c}{\tilde{T}_C} = \frac{2d - 5}{4d_2}, \quad (3.31)$$

in which the shifted Hawking temperature ($\tilde{T} = T - m_g^2 c_0 c_1 / 4\pi$) is used and the constraint $k_{\text{eff}}^{(C)} > 0$ which imposed from equation (3.28) can ensure the positivity of \tilde{T}_C . Interestingly, the obtained critical ratio does not depend on the topology of the event horizon. This exactly matches with the universal ratio of RN-AdS black holes in Einstein's gravity, and in $d = 4$ dimensions, the universal ratio of vdW fluid, i.e. $3/8$, is obtained too. Obviously, in terms of the standard Hawking temperature (T), the universal ratio will be a function of graviton's mass (m_g). Regarding this case, one finds the standard universal ratio as

$$\frac{P_C v_c}{T_C} = \frac{(2d - 5)d_3^2 k_{\text{eff}}^{(C)}}{d_2 \left(4d_3^2 k_{\text{eff}}^{(C)} + (2d - 5)m_g^2 c_0 c_1 r_c \right)}. \quad (3.32)$$

It is interesting to note that expanding the above universal ratio around the infinitesimal values of the graviton's mass (m_g) yields

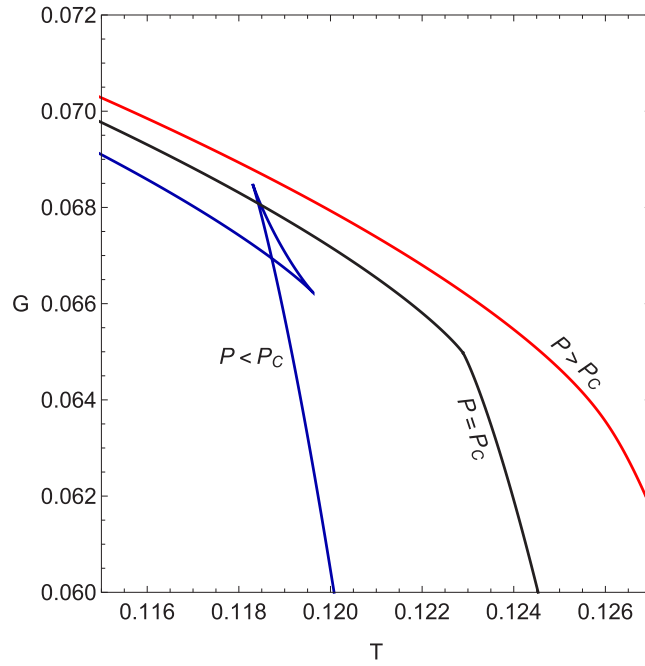


Figure 1. vdW phase transition in the canonical (fixed charge) ensemble: $G - T$ diagram for a spherical black hole with $[k = +1, c_2 = 0]$, or a planar black hole with $[k = 0, c_2 = 1]$, or a hyperbolic black hole with $[k = -1, c_2 = 2]$. The other parameters have been set as $d = 4$, $m_g = 1$, $c_0 = 1$, $c_1 = 1$, and $q = 1$. Critical data: $(T_C = 0.122\,894, P_C = 0.003\,316)$.

$$\frac{P_C v_c}{T_C} = \frac{2d-5}{4d_2} \left\{ 1 - \frac{(2d-5)m_g^2 c_0 c_1}{4d_3^2 k} \left(\frac{2(2d-5)q^2}{d_3 k} \right)^{\frac{1}{2d_3}} + O(m_g^4) \right\}, \quad (3.33)$$

which only is true for the case of spherical symmetry ($k = +1$). According to equation (3.28), when the geometry of event horizon is planar or hyperbolic (i.e. $k = 0, -1$), the zero limit ($m_g \rightarrow 0$) of critical radius does not exist anymore and thus one is not allowed to expand the universal ratio $P_C v_c / T_C$ around $m_g = 0$. This shows the drastic effect of event horizon's geometry in the massless limit ($m_g = 0$) of massive gravity as expected. It is natural since we know that there is no criticality for planar or hyperbolic black holes in Einstein's gravity. Consequently, since the expansion around $m_g = 0$ is not permissible (for $k = 0, -1$), it can be inferred from equation (3.28) that a lower mass bound is needed to have a positive definite critical radius and the subsequent critical behavior for Ricci flat or hyperbolic black holes.

3.3.2. Reentrant phase transition (RPT). We now discuss the reentrant behavior of phase transition (first seen in [6, 11]) in the canonical ensemble. As stated in [33], the RPT phenomenon can appear when the equation of state could give birth to two critical points in which the associated pressures and temperatures are positive definite, but, it is seen that only one of these critical points referred to as (T_C, P_C) , is physical and the other one is unphysical since it cannot minimize the Gibbs free energy. By further studying the phase space, it will be evident that a virtual triple point (T_{Tr}, P_{Tr}) and another critical point (T_Z, P_Z) emerge. As a result, three separate phases of black holes appear. The critical points (T_Z, P_Z) and (T_C, P_C) are, respectively, the

endpoints of the zeroth-order and first order coexistence lines. Moreover, the first order and the zeroth-order coexistence lines join at the virtual triple point $(T_{\text{Tr}}, P_{\text{Tr}})$.

In the context of massive gravity, it seems that the phenomenon of RPT may be observed when higher order interactions of massive gravitons up to the third interaction terms are considered (which is possible in $d \geq 5$ dimensions). But we speculate this is not the case. Using equation (3.26), the equation of critical point(s) in five-dimensional spacetime reads (setting $c_0 = 1$)

$$m_g^2 c_2 r_+^4 + 3m_g^2 c_3 r_+^3 - 5q^2 = 0, \quad (3.34)$$

in which, without loss of generality, it is assumed that $k = 0$ (so we have $k_{\text{eff}}^{(C)} = m_g^2 c_0^2 c_2$). Also, m_g^2 can be absorbed into the massive couplings c_1 and c_2 . Analyzing equation (3.34) by mathematica software [90] shows that this equation could possess two critical points according to the following conditions

$$0 < q^2 < -\frac{3^7}{5 \times 2^8} \frac{c_3^4}{c_2^3}, \quad c_2 < 0, \quad c_3 > 0, \quad (3.35)$$

in which it is assumed that $r_+ > 0$. But, the pressures corresponding to the obtained critical roots must be positive definite, i.e. $P > 0$, and this later condition cannot be satisfied for both roots simultaneously. In fact, by combining equation (3.34) with the condition $P > 0$, a critical radius is obtained with the following fine tuning

$$0 < q^2 < -\frac{3^4}{5 \times 2^4} \frac{c_3^4}{c_2^3}, \quad c_2 < 0, \quad c_3 > 0, \quad (3.36)$$

and another one can also be found as

$$c_2 > 0, c_3 > 0, \quad \text{or} \quad c_2 > 0, c_3 < 0. \quad (3.37)$$

Examining the above conditions, it is seen that the first root requires $c_2 < 0$ while the second root requires $c_2 > 0$. So there are no combinations of the parameters at the same time where both roots are solutions. Thus, the phenomenon of RPT cannot take place in $d = 5$ dimensions. Now, following this procedure, we can discuss RPT in 6-dimensions. As summarized in table 1, in a six-dimensional spacetime, there are eight possibilities for the sign variations of massive coupling constants c_2 , c_3 and c_4 . Only in the cases of (1) and (5), two roots can be found. But, the same as RPT in five-dimensions, one cannot find any region in the phase space where both roots and the associated pressures are positive definite (but, interestingly, a region with three physical critical points are found which is the subject of section 3.3.4). Thus, in 6-dimensions, the RPT phenomenon does not take place in the canonical ensemble of massive gravity's charged TBHs. This analytical procedure does not work in the case of higher dimensions, i.e. $d \geq 7$, since the critical point equation has mathematically a more complicated structure. So we applied numerical analysis and did not find any evidence that shows spacetime dimensions with the range $d = 7, 8, 9$ can exhibit reentrant behavior for phase transition as well. Motivated by this, we speculate that charged TBHs may not exhibit RPT in the canonical ensemble of charged TBHs in massive gravity (at least in $d = 4, 5, 6$ and possibly in $d = 8, 9$ dimensions). Whether or not such phenomenon exists for massive gravity's TBHs in higher dimensions ($d \geq 7$) remains an open question.

3.3.3. Triple point and small/intermediate/large black hole (SBH/IBH/LBH) phase transition. Here, we present the first explicit demonstration of the triple point, typical of many materials in nature, in charged TBHs with massive gravitons which takes place in $d \geq 6$

Table 1. The sign variations in $d = 6$ and the associated critical roots (points) of equation (3.26). In this table, by ‘physical root’ we mean that both temperature and pressure associated with the critical root are positive definite.

Case	c_2	c_3	c_4	Roots	Physical roots
(1)	+	+	+	2	1
(2)	+	+	−	1	1
(3)	+	−	+	3	1 or 3
(4)	+	−	−	1	1
(5)	−	+	+	2	1
(6)	−	+	−	1	1
(7)	−	−	+	1	1
(8)	−	−	−	0	0

dimensions. The analogue of triple point in the neutral black holes of massive gravity was reported in [28], in which this phenomenon takes place in dimensions with the range $d \geq 7$, and necessarily one needs to consider up to the five graviton self-interaction potentials. But here, in the canonical ensemble, we explicitly show that this critical behavior can be observed for charged TBHs in spacetimes with $d \geq 6$ in which only the first four graviton self-interaction terms are present. According to the table 1, in order to have the analogue of triple point in massive charged TBHs in the canonical ensemble, we have to consider the sign variations presented in the case (3). We assume that the first four potential terms are nonzero and the rest of them vanish, i.e. $c_i = 0$ for $i \geq 5$. This leads to the following critical point equation

$$d_3 k_{\text{eff}}^{(C)} r_+^2 + 3d_3 d_4 m_g^2 c_0^3 c_3 r_+ + 6d_3 d_4 d_5 m_g^2 c_0^4 c_4 - 2(2d - 5)q^2 r_+^{-2d_4} = 0, \quad (3.38)$$

which predicts three physical critical points. By physical we just mean that the associated pressures and temperatures are positive definite (but always, one of those critical points cannot minimize the Gibbs free energy). This is shown in figure 2, where we have depicted the $G - T$ diagram for various isobaric curves. For pressures in the range $P < P_{C_1}$, we first observe the swallowtail (vdW) behavior which indicates the first-order phase transition. This is the transition from the LBH region to the SBH one. Then, for the isobars with $P_{\text{Tr}} < P < P_{C_2}$, two swallowtails indicating the appearance of two first-order phase transitions is observed which implies three-phase behavior. This is the SBH/IBH/LBH phase transition that resembles the solid/liquid/gas phase transition in usual substances. Finally, as displayed in figure 2, the two swallowtails eventually merge at the gravitational triple point $(T_{\text{Tr}}, P_{\text{Tr}})$ by further decreasing the pressure.

3.3.4. vdW type phase transition. As stated in the previous section, the equation of state (3.38) could possess three critical points in which the associated pressures and temperatures are positive definite. The three-phase behavior associated with the triple point takes place when two critical points referred to as (T_{C_1}, P_{C_1}) and (T_{C_2}, P_{C_2}) , can minimize the Gibbs free energy but the third one cannot. Interestingly, another critical phenomenon can happen when two critical points cannot minimize the Gibbs free energy but the other critical point can do this. In fact, this situation had already appeared in section 3.3.2, the case (3) for the sign variations of massive coefficients in table 1. This kind of phase structure yields a critical behavior which here is referred to as vdW type phase transition and can be obtained by use of varying the electric charge of TBHs. This situation is similar to that seen in [91] for the spherically symmetric charged-AdS black holes (in the canonical ensemble) within the framework of

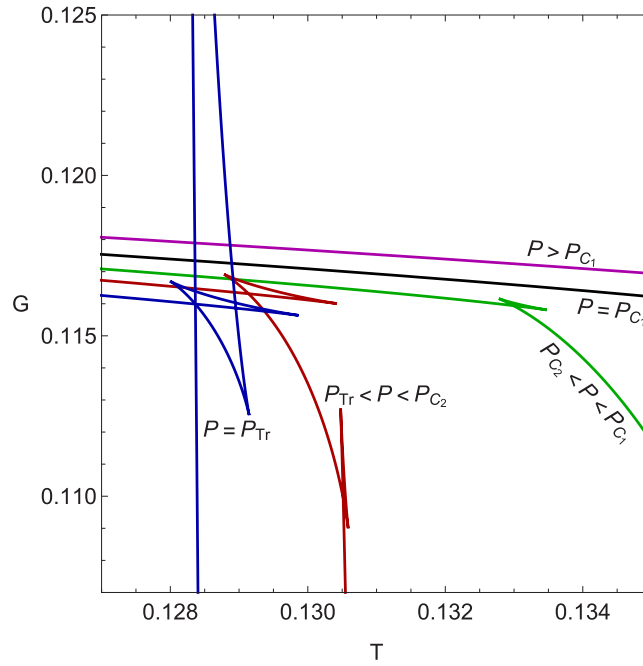


Figure 2. Triple point in the canonical (fixed charge) ensemble: $G - T$ diagram for a spherical black hole with $[k = +1, c_2 = -0.1]$, or a planar black hole with $[k = 0, c_2 = 0.9]$, or a hyperbolic black hole with $[k = -1, c_2 = 1.9]$. The other parameters have been set as $d = 6, m_g = 1, c_0 = 1, c_1 = 1, c_3 = -0.8, c_4 = 0.5$ and $q = 0.8$. Critical data: $(T_{C_1} = 0.138\,126, P_{C_1} = 0.011\,428)$, $(T_{C_2} = 0.131\,100, P_{C_2} = 0.004\,501)$ and $(T_{Tr} = 0.128\,365, P_{Tr} = 0.003\,801)$.

Gauss–Bonnet gravity, exactly in $d = 6$ dimensions. We confirm that this phenomenon happens in the context of massive gravity as well, and, interestingly it starts to appear in $d \geq 6$ dimensions for all types of TBHs⁷.

Our investigations show that, for a certain range of parameters, there is a lower bound for the electric charge (Q_{b_1}) and there may also be an upper bound for it (Q_{b_2}), where for $Q_{b_1} < Q < Q_{b_2}$, the triple point behavior can be observed. For $Q < Q_{b_1}$ and $Q > Q_{b_2}$, one of the physical critical points changes to an unphysical critical point which cannot minimize the Gibbs free energy and consequently the triple point behavior is replaced by the vdW type phase transition. Now, we illustrate this situation for a set of TBHs in figure 3. For this purpose, we have altered the electric charge parameter of the previous subsection (related to figure 2) from $q = 0.8$ to $q = 0.5$. Obviously, a first order phase transition takes place for pressures in the range $P_{C_2} < P < P_{C_1}$ and by further decreasing the pressure, i.e. $P_{C_3} < P < P_{C_2}$, an anomaly appears in the shape of the standard swallowtail behavior. A close-up of this anomaly is depicted in figure 3, which looks like the reentrance of phase transition. This anomaly does not lead to any new BH phase (or equivalently a phase transition) since it cannot minimize the Gibbs free energy, hence we only observe the standard SBH/LBH phase transition. This can be verified by studying the corresponding $P - T$ diagram. In figure 4, the coexistence line

⁷ It should be noted that the authors of [91] have studied the $P - v$ criticality of charged black holes for the case of $\alpha_{GB} = 1$, which is very limited. As they stated, in the case of arbitrary Gauss–Bonnet coupling constant, one may find this phenomenon in higher dimensions as well.

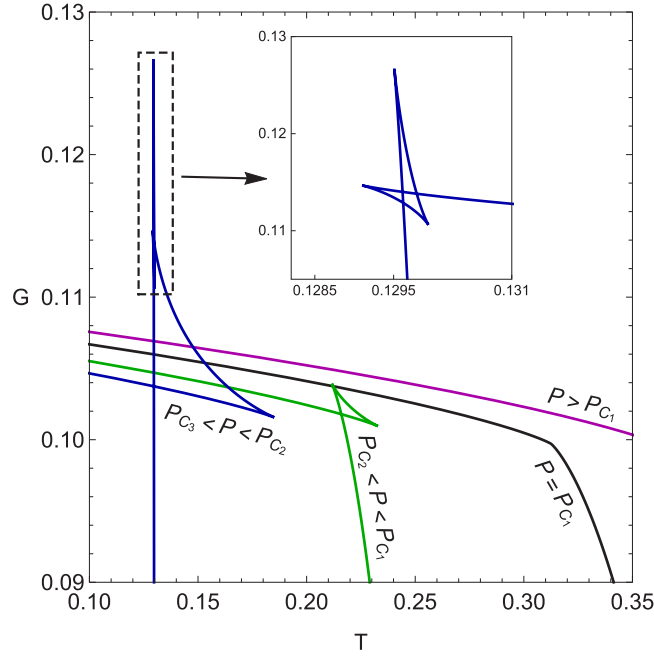


Figure 3. vdW type phase transition in the canonical (fixed charge) ensemble: the $G - T$ diagram for a spherical black hole with $[k = +1, c_2 = -0.1]$, or a planar black hole with $[k = 0, c_2 = 0.9]$, or a hyperbolic black hole with $[k = -1, c_2 = 1.9]$. The other parameters have been set as $d = 6, m_g = 1, c_0 = 1, c_1 = 1, c_3 = -0.8, c_4 = 0.5$ and $q = 0.5$. Critical data: $(T_{C_1} = 0.312\,557, P_{C_1} = 0.177\,989, r_{C_1} = 0.810\,774)$, $(T_{C_2} = 0.131\,126, P_{C_2} = 0.004\,508, r_{C_2} = 3.341\,090)$ and $(T_{C_3} = 0.126\,626, P_{C_3} = 0.002\,673, r_{C_3} = 1.933\,764)$.

of the $P - T$ diagram corresponding with the figure 3 is depicted, which proves that only the two-phase behavior exists. But, for pressures in the range $P_{C_3} < P < P_{C_2}$, the coexistence line is curved more than any other area. In the standard $P - T$ diagrams associated with the vdW behavior, the coexistence curve is thoroughly smooth and uniform.

4. Black hole chemistry in grand canonical ensemble

4.1. Grand canonical partition function

The grand partition function (\mathcal{Z}_{GC}) of the gravitational system could be defined by a Euclidean path integral over the tensor field $g_{\mu\nu}$ and vector field A_μ as follows

$$\mathcal{Z}_{GC} = \int \mathcal{D}[g, A] e^{-\mathcal{I}_E[g, A]} \simeq e^{-\mathcal{I}_{\text{on-shell}}(\beta, \Phi)}, \quad (4.1)$$

where β is the inverse of Hawking temperature in terms of the fixed potential, equation (2.26). In the grand canonical ensemble, the electric charge (Q) fluctuates, but the associated potential (Φ) is fixed at infinity. By imposing the fixed potential boundary condition, $\delta A_\nu|_{\partial\mathcal{M}} = 0$, this ensemble is established. Thus the so-called grand potential referred to as G_Φ , can be defined using the grand partition function. The same as before, we will utilize the subtraction method in order to evaluate the grand canonical on-shell action. Following the approach presented in section 3.1, the on-shell action for the bulk theory of TBHs is computed as

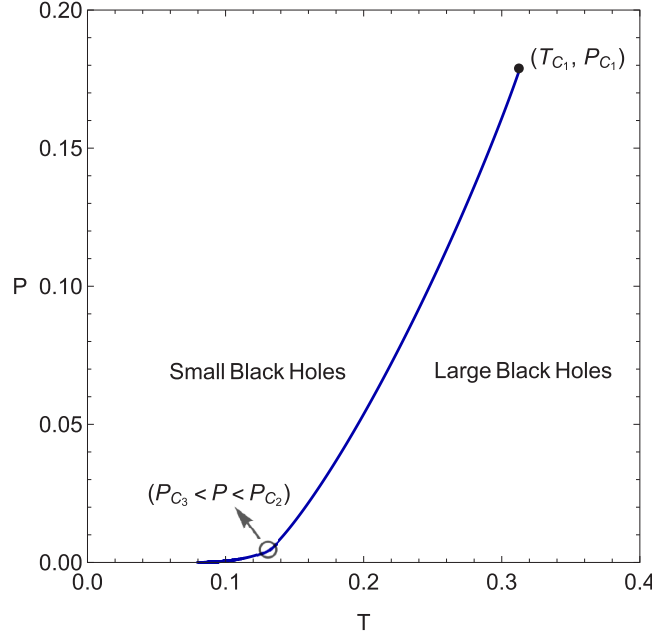


Figure 4. vdW type phase transition in the canonical (fixed charge) ensemble: $P - T$ diagram for a spherical black hole with $[k = +1, c_2 = -0.1]$, or a planar black hole with $[k = 0, c_2 = 0.9]$, or a hyperbolic black hole with $[k = -1, c_2 = 1.9]$. The other parameters have been set as $d = 6, m_g = 1, c_0 = 1, c_1 = 1, c_3 = -0.8, c_4 = 0.5$ and $q = 0.5$. Critical data: $(T_{C1} = 0.312\,557, P_{C1} = 0.177\,989, r_{c1} = 0.810\,774), (T_{C2} = 0.131\,126, P_{C2} = 0.004\,508, r_{c2} = 3.341\,090)$ and $(T_{C3} = 0.126\,626, P_{C3} = 0.002\,673, r_{c3} = 1.933\,764)$.

$$\mathcal{I}_{\text{BH}} = \frac{\beta \omega_{d-2}^k}{16\pi G_d} \left(\left[\frac{2}{\ell^2} r^{d-1} - m_g^2 \sum_{i=1}^{d-2} \frac{(i-2)c_0^i c_i}{d-i-1} r^{d-i-1} \prod_{j=3}^{i+1} d_j \right]_{r_+}^{R=\infty} - 4 \frac{d_3}{d_2} \Phi^2 r_+^{d_3} \right) \quad (4.2)$$

in which ‘ R ’ is an upper cutoff. In the above relation, the fixed potential boundary condition at infinity, i.e. $A_t(r = \infty) = \Phi(r_+)$, is used. The on-shell action of AdS background within the massive gravity framework (without any matter or electromagnetic field) is obtained the same as before in equation (3.8), which is repeated below for convenience

$$\mathcal{I}_{\text{AdS}} = \frac{\beta_0 \omega_n}{16\pi G_d} \left[\frac{2}{\ell^2} R^{d_1} - m_g^2 \sum_{i=1}^{d-2} \frac{(i-2)c_0^i c_i}{d-i-1} R^{d-i-1} \prod_{j=3}^{i+1} d_j \right]. \quad (4.3)$$

Demanding both spacetimes have the same Hawking temperature (or equivalently the same geometry) at $r = R$, i.e. $\beta_0 V_0(R)^{1/2} = \beta V(R)^{1/2}$, then subtracting the on-shell action of the AdS background from the on-shell action of the TBHs, one obtains

$$\begin{aligned} \mathcal{I}_{\text{on-shell}} &\equiv \lim_{R \rightarrow \infty} (\mathcal{I}_{\text{BH}} - \mathcal{I}_{\text{AdS}}) \\ &= \frac{\beta \omega_n r_+^{d_3}}{16\pi G_d} \left[k - \frac{r_+^2}{\ell^2} - 2 \frac{d_3}{d_2} \Phi^2 + m_g^2 \sum_{i=1}^{d-2} \left(\frac{(i-1)c_0^i c_i}{r_+^{i-2}} \prod_{j=3}^i d_j \right) \right]. \end{aligned} \quad (4.4)$$

4.2. Extended phase space thermodynamics

Working in the extended phase space, the grand canonical potential G_Φ (sometimes is called Gibbs free energy in grand canonical ensemble) which depends on T , P , and Φ , is calculated as

$$G_\Phi = \beta^{-1} \ln \mathcal{Z}_{GC}(T, P, \Phi) = M - TS - \Phi Q$$

$$= \frac{\omega_n r_+^{d_3}}{16\pi} \left[k - \frac{16\pi P r_+^2}{d_1 d_2} - 2 \frac{d_3}{d_2} \Phi^2 + m_g^2 \sum_{i=1}^{d-2} \left(\frac{(i-1)c_0^i c_i}{r_+^{i-2}} \prod_{j=3}^i d_j \right) \right], \quad (4.5)$$

in which we have set $G_d = 1$. Moreover, it is seen that the Gibbs free energy in the canonical ensemble is related to the grand potential using a Legendre transform as

$$G_\Phi(T, P, \Phi) = G(T, P, Q) - \Phi Q. \quad (4.6)$$

The above relation is a guideline to find the correct first law of thermodynamics as $dG_\Phi = -SdT + VdP - Qd\Phi$. This can be confirmed via the standard thermodynamic relations which will be briefly explained. The thermodynamic variables Q , V and S are extracted from the grand potential G_Φ as

$$Q = - \left(\frac{\partial G_\Phi}{\partial \Phi} \right)_{T, P} = \frac{\omega_n}{4\pi} q, \quad (4.7)$$

$$V = \left(\frac{\partial G_\Phi}{\partial P} \right)_{T, \Phi} = \frac{\omega_{d_2}}{d_1} r_+^{d_1}, \quad (4.8)$$

and

$$S = - \left(\frac{\partial G_\Phi}{\partial T} \right)_{P, \Phi} = \frac{\omega_n}{4} r_+^{d_2}. \quad (4.9)$$

The above thermodynamic variables are in full agreement with the previous results, i.e. equations (2.20), (3.15) and (3.17), that we have obtained before. Therefore, we deduce that all intensive and corresponding extensive variables satisfy the first law of black hole thermodynamics in the grand potential representation. The extended Smarr formula does not depend on the ensemble one is dealing with, thus the same relation as before in equation (3.18) is obtained again. Moreover, according to section 3.2, if one considers the massive coupling constants as the new thermodynamic variables, the first law in the grand potential representation is generalized as $dG_\Phi = -SdT + VdP - Qd\Phi + \sum_{i=1}^{d-2} C_i dc_i$.

4.3. Holographic phase transitions

The grand canonical equation of state is obtained as

$$P = \frac{d_2 \tilde{T}}{4r_+} - \frac{d_2 d_3 k_{\text{eff}}^{(\text{GC})}}{16\pi r_+^2} - \frac{m_g^2}{16\pi} \sum_{i=3}^{d-2} \left(\frac{c_0^i c_i}{r_+^i} \prod_{j=2}^{i+1} d_j \right), \quad (4.10)$$

where, in this ensemble, the effective topological factor $k_{\text{eff}}^{(\text{GC})}$ and the shifted Hawking temperature \tilde{T} are given by

$$k_{\text{eff}}^{(\text{GC})} \equiv [k + m_g^2 c_0^2 c_2 - 2(d_3/d_2)\Phi^2] \quad (4.11)$$

and

$$\tilde{T} = T - \frac{m_g^2 c_0 c_1}{4\pi} = \frac{d_2 d_3 k - 2\Lambda r_+^2 - 2(d_3 \Phi)^2 + m_g^2 \sum_{i=2}^{d-2} \left(\frac{c_0^i c_i}{r_+^{i-2}} \prod_{j=2}^{i+1} d_j \right)}{4\pi d_2 r_+}. \quad (4.12)$$

Obviously, in the Einstein limit, i.e. $m_g \rightarrow 0$, we cannot observe any critical behavior, which means that there does not exist criticality in the grand canonical ensemble of AdS black holes in GR with or without the linear electromagnetic Maxwell fields.

The critical point occurs at the spike like divergence of specific heat at constant pressure i.e. an inflection point in the $P - v$ (or equivalently $P - r_+$) diagram and can be found by using equation (3.25) which for the grand canonical equation of state (4.10) leads to

$$2k_{\text{eff}}^{(\text{GC})} r_+^{d_4} + m_g^2 \sum_{i=3}^{d-2} \left(i(i-1) c_0^i c_i r_+^{d-i-2} \prod_{j=4}^{i+1} d_j \right) = 0. \quad (4.13)$$

Now, to be more specific, following section 3, we analyze the equations of state and holographic phase transitions for case by case of TBHs with detail. As before, we generally concentrate on the isobaric curves of $G - T$ diagrams since all the essential information about the critical behaviors can be extracted from them. In addition, equations (4.10) and (4.13) help us to find that the combination $k_{\text{eff}}^{(\text{C})} \equiv [k + m_g^2 c_0^2 c_2]$ in the canonical ensemble is replaced by the $k_{\text{eff}}^{(\text{GC})} \equiv [k + m_g^2 c_0^2 c_2 - 2(d_3/d_2)\Phi^2]$ in the grand canonical ensemble. So the same critical behavior with the same critical points would be found for the case of the spherical, planar, and hyperbolic black holes if the same value for $k_{\text{eff}}^{(\text{GC})}$ are provided. In this view, TBHs in massive gravity at their critical point may be indistinguishable.

4.3.1. vdW phase transition. In order to observe the vdW behavior in a given black hole spacetime, one physical critical point must exist in the thermodynamic phase space which minimize the Gibbs free energy. Within the framework of the grand canonical ensemble, this can be obtained in the spacetime dimensions with the range $d \geq 5$ ⁸. In $d = 5$, only the first three massive couplings (c_1 , c_2 and c_3) appears. Regarding equation (4.13) and following the approach presented in section 3, the critical point of the massive charged TBHs can be obtained from the root of following relation

$$k_{\text{eff}}^{(\text{GC})} r_+ + 3d_4 m_g^2 c_0^3 c_3 = 0, \quad (4.14)$$

in which we have assumed that, in higher dimensions, the other massive couplings c_i ($i \geq 4$) vanish. This is the simplest way to find the vdW behavior in arbitrary dimensions, and, of course, it is permissible to consider higher order graviton self-interaction terms and then observe a vdW phase transition by using a fine tuning of massive couplings. Considering equation (4.14), the critical radius is easily obtained as

$$r_c = \frac{3d_4 m_g^2 c_0^3 c_3}{k_{\text{eff}}^{(\text{GC})}}, \quad (4.15)$$

with the following constraints on the parameters

⁸ Comparing with the vdW critical behavior in the canonical ensemble, it is seen that, in $d = 4$, only canonical vdW phase transition can take place.

$$c_3 > 0 \leftrightarrow \Phi^2 < \frac{d_2[k + m_g^2 c_0^2 c_2]}{2d_3}, \quad (4.16)$$

and

$$c_3 < 0 \leftrightarrow \Phi^2 > \frac{d_2[k + m_g^2 c_0^2 c_2]}{2d_3}. \quad (4.17)$$

As seen, there are two strict limitations on the value of the $U(1)$ potential, Φ . According to these constraints, one has to assume $[k + m_g^2 c_0^2 c_2] > 0$ when c_3 is positive definite, while there is no such constraint when c_3 is negative. As we will see in a moment, the first constraint (4.16) does not lead to vdW phase transition since the associated pressure and temperature are negative definite. The thermodynamic pressure and temperature at the critical point (4.15) are given as

$$P_C = -\frac{d_2 d_3 d_4 m_g^2 c_0^3 c_3}{16\pi r_C^3}, \quad (4.18)$$

and

$$\tilde{T}_C = -\frac{3d_3 d_4 m_g^2 c_0^3 c_3}{4\pi r_C^2}. \quad (4.19)$$

According to the above relations, the massive coupling c_3 must be negative definite, so the condition (4.16) does not lead to criticality at all.

In figure 5, according to equations (4.15)–(4.17), the massive coupling coefficients and $U(1)$ potential Φ have been adjusted in a way to produce a vdW behavior. As the canonical ensemble case, the swallowtail behavior is observed for pressures in the range $P < P_C$ which indicates the existence of two-phase behavior. As seen, the single phase behavior takes place for pressures in the range $P > P_C$.

The obtained critical data, equations (4.15), (4.18) and (4.19), satisfy the grand canonical universal ratio as

$$\frac{P_C r_C}{\tilde{T}_C} = \frac{d-2}{12} \iff \frac{P_C v_c}{\tilde{T}_C} = \frac{1}{3}. \quad (4.20)$$

Interestingly, the universal ratio at critical point when is written down in terms of the critical specific volume (v_c) and the shifted Hawking temperature (\tilde{T}_C), i.e. $P_C v_c / \tilde{T}_C$, is constant and does not depend on the spacetime dimensions (d). To our knowledge, till now, this case and the case of charged-AdS BHs in the PMI-Einstein gravity (see [12]) are the only examples of such a BH spacetime with a constant universal ratio. At this stage, the standard universal ratio may also be written as

$$\frac{P_C v_c}{T_C} = \frac{d_3 d_4 c_0^3 c_3}{3d_3 d_4 c_0^3 c_3 - c_0 c_1 r_c^2}. \quad (4.21)$$

Expanding the above universal ratio around the small values of graviton mass yields

$$\frac{P_C v_c}{T_C} = \frac{1}{3} + \frac{m_g^4 c_0^4 c_1 c_3 d_2^2 d_4}{d_3 (d_2 k - 2d_3 \Phi^2)^2} + O(m_g^6). \quad (4.22)$$

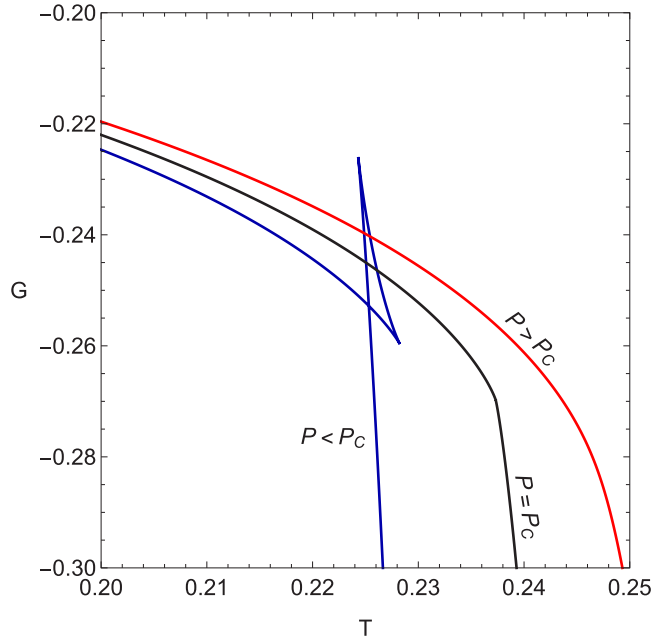


Figure 5. vdW phase transition in the grand canonical (fixed potential) ensemble: the $G - T$ diagram for a spherical black hole with $[k = +1, c_2 = 2]$, or a planar black hole with $[k = 0, c_2 = 3]$, or a hyperbolic black hole with $[k = -1, c_2 = 4]$. The other parameters have been set as $d = 5$, $m_g = 1$, $c_0 = 1$, $c_1 = 1$, $c_3 = -3$ and $\Phi = 0.1$. Critical data: $(T_C = 0.237321, P_C = 0.013086)$.

But, it does not mean that the massless limit of massive gravity in the grand canonical ensemble leads to the outcome of Einstein's gravity. In fact, according to equation (4.15), the critical radius vanishes in the massless limit ($m_g = 0$), as expected, since there exists no grand canonical criticality for TBHs in Einstein's gravity. So the massless limit ($m_g = 0$) of equation (4.22) does not appear at all since there is not any critical point at this limit.

4.3.2. Reentrant phase transition (RPT). As stated in [33], in order to have a RPT phenomenon, the critical point equation must admit two positive critical radii in which the associated pressures and temperatures are positive definite, while only one of the critical points referred to as (T_C, P_C) , can minimize the Gibbs free energy. Simply, a real and inhomogeneous polynomial equation of second-degree of r_+ can produce two critical radii. This is permissible in $d \geq 6$ dimensions which implies the first four massive couplings (c_1, c_2, c_3 and c_4) have to be nonzero. Assuming that these massive couplings (c_1, c_2, c_3 and c_4) are nonzero and the other couplings vanish in higher dimensions, one gets

$$k_{\text{eff}}^{(\text{GC})} r_+^2 + 3d_4 m_g^2 c_0^3 c_3 r_+ + 6d_4 d_5 m_g^2 c_0^4 c_4 = 0. \quad (4.23)$$

The above equation of critical point can be solved simply as

$$r_{c_1}, r_{c_2} = \frac{-3d_4 m_g^2 c_0^3 c_3 \pm \sqrt{\Delta}}{2k_{\text{eff}}^{(\text{GC})}}, \quad \Delta > 0, \quad (4.24)$$

$$\Delta = -24k_{\text{eff}}^{(\text{GC})} (d_4 d_5 m_g^2 c_0^4 c_4) + (3d_4 m_g^2 c_0^3 c_3)^2.$$

It is obvious that the above relation can predict one or at most two positive critical radii for the equation of state of TBHs. Since the reentrant behavior of phase transition takes place whenever the critical point equation possesses two positive roots, so looking for this case, the following conditions should be satisfied

$$r_{c_1} + r_{c_2} = \frac{-3d_4 m^2 c_0^3 c_3}{k_{\text{eff}}^{(\text{GC})}} > 0, \quad r_{c_1} r_{c_2} = \frac{6d_4 d_5 m^2 c_0^4 c_4}{k_{\text{eff}}^{(\text{GC})}} > 0. \quad (4.25)$$

According to the above constraints, when the effective topological factor $k_{\text{eff}}^{(\text{GC})}$ is positive definite, two critical points can be found assuming that $c_3 < 0$ and $c_4 > 0$, and when $k_{\text{eff}}^{(\text{GC})} < 0$, one has to assume $c_3 > 0$ and $c_4 < 0$.

Now, using the obtained information, we illustrate a typical example of the RPT phenomenon in the grand canonical ensemble. In figure 6, the $G - T$ diagrams for a set of charged TBHs are depicted. As seen, two new critical points referred to as (T_Z, P_Z) and $(T_{\text{Tr}}, P_{\text{Tr}})$, emerge in the thermodynamic phase space. For pressures in the range $P_Z < P < P_C$, a first order phase transition occurs as temperature decreases. For $P_{\text{Tr}} < P < P_Z$, as temperature decreases monotonically, a first-order phase transition is initially observed, and then, a finite jump (discontinuity) appears in the global minimum of the Gibbs free energy, which displays the zeroth-order phase transition. This behavior is exactly the standard RPT in the subject of black hole chemistry seen in many black hole spacetimes before. This phenomenon reminds us of those critical behaviors present in multicomponent fluids and liquid crystals [8]. In addition, comparing with [28], this phenomenon occurs qualitatively in the same dimensions as neutral TBHs.

4.3.3. Triple point and small/intermediate/large black hole (SBH/IBH/LBH) phase transition. In the grand canonical ensemble, the analogue of triple point may be found whenever the TBH equation of state is supplemented by higher order interacting potentials of massive gravitons up to the fifth interaction terms. Hence, assuming that the only first five massive couplings are nonzero, equation of critical point (4.10) reduces to the following polynomial

$$k_{\text{eff}}^{(\text{GC})} r_+^3 + 3d_4 m_g^2 c_0^3 c_3 r_+^2 + 6d_4 d_5 m_g^2 c_0^4 c_4 r_+ + 10d_4 d_5 d_6 m_g^2 c_0^5 c_5 = 0. \quad (4.26)$$

Investigation of the exact solutions of three critical points is not possible analytically, and so we apply the numerical techniques. To do that, we have suitably tuned the massive couplings to produce three critical points, in which two of them are physical and one of them cannot minimize the Gibbs free energy. The corresponding critical behavior via the $G - T$ diagram is depicted in figure 7. Obviously, a critical triple point $(T_{\text{Tr}}, P_{\text{Tr}})$ emerges, and, consequently, three-phase behavior appears. Qualitatively, this critical behavior is the same as its counterpart in the canonical ensemble. But, in $d = 6$ dimensions, the triple point behavior can solely take place in the canonical ensemble. In addition, this phenomenon occurs qualitatively in the same dimensions as neutral TBHs.

4.3.4. vdW type phase transition. Here in parallel with section 3.3.4, we discuss the vdW type phase transition in the grand canonical ensemble. For this phenomenon, the essential requirement is the existence of three (possible) critical points for the equation of state (4.10) in which only one of them minimizes the Gibbs free energy. Elementally, this can happen by varying the parameter space of the theory in spacetime dimensions which triple point phenomenon takes place (since for the case of triple point, the equation of state admits three possible critical points). So one can draw a conclusion that the vdW type behavior and triple point

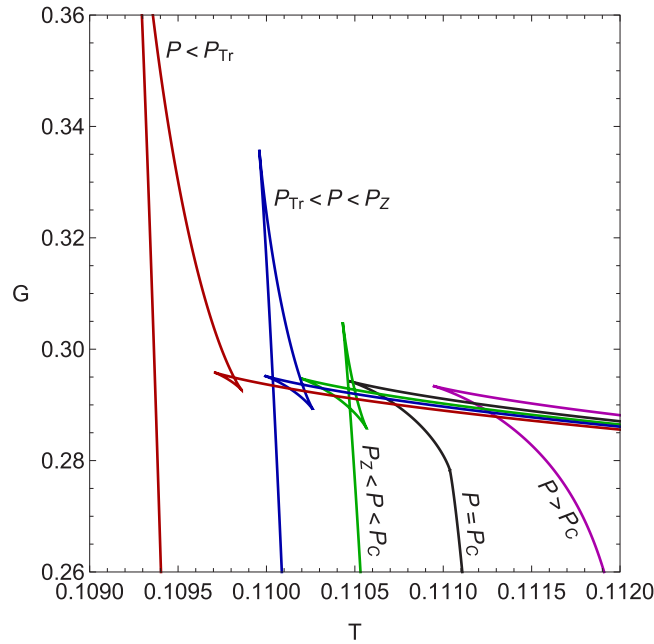


Figure 6. RPT in the grand canonical (fixed potential) ensemble: the $G - T$ diagram for a spherical black hole with $[k = +1, c_2 = -0.2]$, or a planar black hole with $[k = 0, c_2 = 0.8]$, or a hyperbolic black hole with $[k = -1, c_2 = 0.2]$. The other parameters have been set as $d = 6, m_g = 1, c_0 = 1, c_1 = 1, c_3 = -1, c_4 = 0.9$ and $\Phi = 0.1$. Critical data: $(T_C = 0.237321, P_C = 0.013086)$, $(T_Z = 0.1100131, P_Z = 0.0017463)$ and $(T_{Tr} = 0.1099575, P_{Tr} = 0.001724)$.

phenomenon always show up in the same spacetime dimensions. To see this, we can alter the electric potential parameter (Φ) of the previous example in section 4.3.3 (related to figure 7) from $\Phi = 0.2$ to $\Phi = 0.3$. In this case, the critical point equation (4.13) still admits three (positive) critical radii in which the associated temperatures and pressures are positive definite. However, only one of the critical points referred to (T_{C1}, P_{C1}) , is physical since it is the only critical point that minimizes the Gibbs free energy. The corresponding critical phenomenon can be understood using the $G - T$ and $P - T$ diagrams illustrated, respectively, in figures 8 and 9 explicitly confirm the vdW type phase transition in the grand canonical ensemble. This phenomenon persists in higher dimensions ($d \geq 7$) as well.

5. Critical exponents

Critical exponents (α, β, γ and δ) determine the behavior of thermodynamic quantities in the neighborhood of critical points, so various critical exponents imply different behaviors in the phase diagrams. So far, in the realm of Statistical Mechanics, it has been confirmed that these exponents do not depend on the microscopic details of a physical system, and they are highly affected by spatial dimensions, symmetries and the range of interactions [92, 93]. Let us now compute them in the subject of charged black hole chemistry in massive gravity following the approach of [5, 11].

In order to obtain the critical exponents, hereafter, we work with the thermodynamic quantities in terms of the following variables

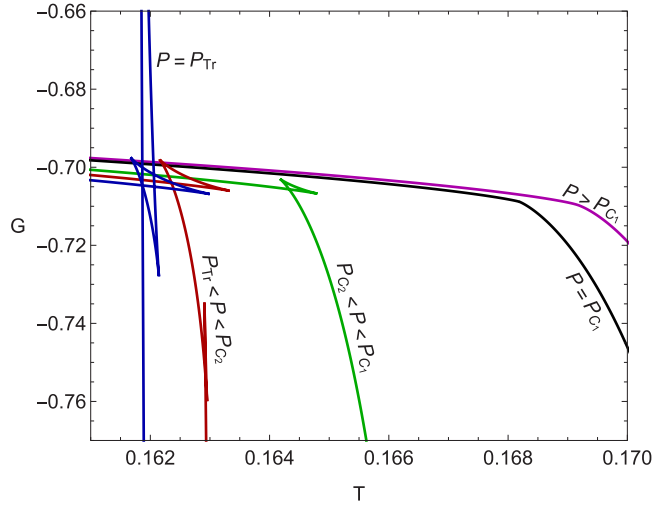


Figure 7. Triple point in the grand canonical (fixed potential) ensemble: the $G - T$ diagram for a spherical black hole with $[k = +1, c_2 = 0.8]$, or a planar black hole with $[k = 0, c_2 = 1.8]$, or a hyperbolic black hole with $[k = -1, c_2 = 2.8]$. The other parameters have been set as $d = 7$, $m_g = 1$, $c_0 = 1$, $c_1 = 1$, $c_3 = -1.8$, $c_4 = 1.3$, $c_5 = -0.7$ and $\Phi = 0.2$. Critical data: $(T_{C_1} = 0.168188, P_{C_1} = 0.009558)$, $(T_{C_2} = 0.1632424, P_{C_2} = 0.604562)$ and $(T_{Tr} = 0.161867, P_{Tr} = 0.570678)$.

$$p \equiv \frac{P}{P_C}, \quad \tau \equiv \frac{T - T_C}{T_C}, \quad w \equiv \frac{v - v_c}{v_c}, \quad \rho_c \equiv \frac{P_C v_c}{T_C}. \quad (5.1)$$

The exponent α specifies the behavior of the specific heat at constant volume as

$$C_v = T \left(\frac{\partial S}{\partial T} \right)_v \propto |\tau|^{-\alpha}. \quad (5.2)$$

Using the relations of the entropy in different ensembles, equations (3.17) and (4.9), it is inferred that $C_v = 0$ since the entropy does not depend on T , and so, we have $\alpha = 0$ in both canonical and grand canonical ensembles.

The exponent β determines the behavior of the order parameter η on the isotherms as

$$\eta = v_l - v_g \propto |\tau|^\beta. \quad (5.3)$$

To compute this exponent, first, we expand the equation of states in the canonical and grand canonical ensembles, equations (3.22) and (4.10), near the critical point

$$p = 1 + \frac{\tau}{\rho_c} (1 - w) + h(v, c_i, q) w^3 + O(\tau w^2, w^4), \quad (5.4)$$

where the function $h(v, c_i, q)$ takes different forms in different ensembles. This function in the canonical ensemble is given by

$$h \equiv \frac{m_g^2 c_0 c_1 - 4\pi T_C}{4\pi P_C v_c} - \frac{4^{2d} (2d - 3) d_1 d_2^5 q^2}{\pi P_C v_c^{2d_2}} + \frac{4d_3 k_{\text{eff}}^{(C)}}{d_2 \pi P_C v_c^2} + \frac{40m_g^2 c_0^3 c_3 d_3 d_4}{d_2^2 \pi P_C v_c^3} + \frac{320m_g^2 c_0^4 c_4 d_3 d_4 d_5}{d_2^3 \pi P_C v_c^4} + O\left(\frac{1}{v_c^5}\right), \quad (5.5)$$

while in the grand canonical ensemble it reads as

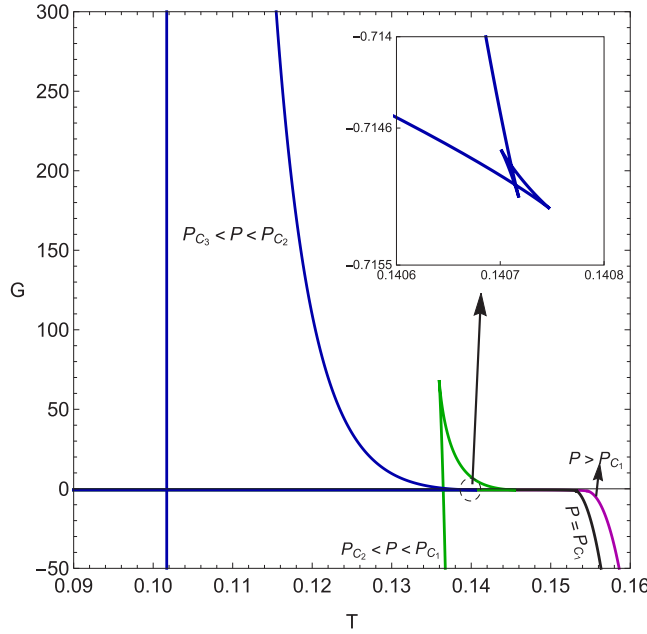


Figure 8. vdW type phase transition in the grand canonical (fixed potential) ensemble: the $G - T$ diagram for a spherical black hole with $[k = +1, c_2 = 0.8]$, or a planar black hole with $[k = 0, c_2 = 1.8]$, or a hyperbolic black hole with $[k = -1, c_2 = 2.8]$. The other parameters have been set as $d = 7, m_g = 1, c_0 = 1, c_1 = 1, c_3 = -1.8, c_4 = 1.3, c_5 = -0.7$ and $\Phi = 0.3$. Critical data: $(T_{C_1} = 0.153\,181, P_{C_1} = 0.004\,8008, r_{c_1} = 5.452\,78), (T_{C_2} = 0.141\,182, P_{C_2} = 0.000\,604\,27, r_{c_2} = 1.976\,26)$ and $(T_{C_3} = 0.140\,511, P_{C_3} = 0.000\,212\,91, r_{c_3} = 2.353\,57)$.

$$h \equiv \frac{m_g^2 c_0 c_1 - 4\pi T_C}{4\pi P_C v_c} + \frac{4d_3 k_{\text{eff}}^{(\text{GC})}}{d_2 \pi P_C v_c^2} + \frac{40m_g^2 c_0^3 c_3 d_3 d_4}{d_2^2 \pi P_C v_c^3} + \frac{320m_g^2 c_0^4 c_4 d_3 d_4 d_5}{d_2^3 \pi P_C v_c^4} + O\left(\frac{1}{v_c^5}\right). \quad (5.6)$$

As will be clear, the final results do not depend on the function $h(v_c, c_i, q)$ and so the details of this function is not important at all. Differentiating the expansion (5.4) for a fixed $\tau < 0$ yields

$$dp = \left(-\frac{\tau}{\rho_c} + 3h(v, c_i, q)w^2\right)dw. \quad (5.7)$$

Using the above relation and Maxwell's equal area law ($\oint vdP = 0$), one finds

$$\int_{w_l}^{w_s} w dp = \frac{-\tau}{2\rho_c}(w_s^2 - w_l^2) + \frac{3h}{4}(w_s^4 - w_l^4) = 0, \quad (5.8)$$

where w_l and w_s denote the volume of large and small black holes, respectively. Now, using equation (5.4) and the fact that the pressure of different black hole phases keeps unchanged at the critical point, we get

$$1 + \frac{\tau}{\rho_c}(1 - w_s) + h(v, c_i, q)w_s^3 = 1 + \frac{\tau}{\rho_c}(1 - w_l) + h(v, c_i, q)w_l^3. \quad (5.9)$$

Equations (5.8) and (5.9) admit a unique nontrivial solution as $w_s = -w_l = \sqrt{\frac{-\tau}{\rho_c h}}$. Therefore, the behavior of the order parameter η is obtained simply as

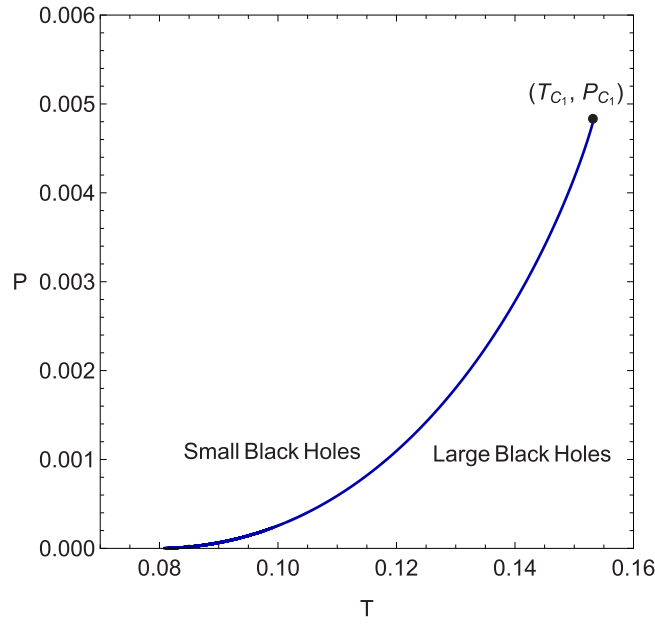


Figure 9. vdW type phase transition in the grand canonical (fixed potential) ensemble: the $P - T$ diagram for a spherical black hole with $[k = +1, c_2 = 0.8]$, or a planar black hole with $[k = 0, c_2 = 1.8]$, or a hyperbolic black hole with $[k = -1, c_2 = 2.8]$. The other parameters have been set as $d = 7, m_g = 1, c_0 = 1, c_1 = 1, c_3 = -1.8, c_4 = 1.3, c_5 = -0.7$ and $\Phi = 0.3$. *Critical data:* $(T_{C1} = 0.153\,181, P_{C1} = 0.004\,8008, r_{c1} = 5.452\,78)$, $(T_{C2} = 0.141\,182, P_{C2} = 0.000\,604\,27, r_{c2} = 1.976\,26)$ and $(T_{C3} = 0.140\,511, P_{C3} = 0.000\,212\,91, r_{c3} = 2.353\,57)$.

$$\eta = v_l - v_s = v_c(w_l - w_s) = 2v_c \sqrt{\frac{-\tau}{\rho_c h}} \propto |\tau|^{1/2}, \quad (5.10)$$

which, according to equation (5.3), confirms that $\beta = 1/2$ for both canonical and grand canonical ensembles.

The exponent γ is extracted from the definition of the isothermal compressibility near the critical point given by

$$\kappa_T = -\frac{1}{v} \frac{\partial v}{\partial P} \Big|_T \propto |\tau|^{-\gamma}. \quad (5.11)$$

Considering both ensembles, we differentiate the expansion of equations of state (5.4) to get

$$\frac{\partial P}{\partial v} \Big|_T = \frac{-P_c \tau}{\rho_c v_c} + O(\tau w, w^2), \quad (5.12)$$

and then, by using $\frac{\partial v}{\partial P} \Big|_T = \left(\frac{\partial P}{\partial v} \Big|_T \right)^{-1}$, we obtain

$$\kappa_T = -\frac{1}{v} \frac{\partial v}{\partial P} \Big|_T \propto \frac{\rho_c v_c}{P_c} \frac{1}{\tau}. \quad (5.13)$$

According to equation (5.11), this relation confirms that $\gamma = 1$ for both canonical and grand canonical ensembles.

Table 2. Holographic phase transitions in canonical (fixed charge) and grand canonical (fixed potential) ensembles.

Phase transition (PT)	Canonical ensemble	Grand canonical ensemble
vdW PT	in $d \geq 4$	in $d \geq 5$
RPT	not seen	in $d \geq 6$
Triple Point	in $d \geq 6$	in $d \geq 7$
vdW type PT	in $d \geq 6$	in $d \geq 7$

Table 3. Summary of universal ratios in the canonical and grand canonical ensembles. For the canonical ensemble, the results are derived up to two interaction potentials, $O(\mathcal{U}_2)$. But in the case of grand canonical ensemble, the results are up to $O(\mathcal{U}_3)$.

	Universal ratio in terms of the shifted Hawking temperature	Universal ratio in terms of the standard Hawking temperature
Canonical ensemble	$\frac{P_C r_c}{T_C} = \frac{2d-5}{16} \iff \frac{P_C v_c}{T_C} = \frac{2d-5}{4d_2}$	$\frac{P_C v_c}{T_C} = \frac{(2d-5)d_5^2 k_{\text{eff}}^{(C)}}{d_2 (4d_5^2 k_{\text{eff}}^{(C)} + (2d-5)m_\phi^2 c_0 c_1 r_c)}$
Grand canonical ensemble	$\frac{P_C r_c}{T_C} = \frac{d-2}{12} \iff \frac{P_C v_c}{T_C} = \frac{1}{3}$	$\frac{P_C v_c}{T_C} = \frac{d_3 d_4 c_0^3 c_3}{3d_3 d_4 c_0^3 c_3 - c_0 c_1 r_c^2}$

The exponent δ , which specifies the shape of the critical isotherm, is defined as

$$|P - P_c| \propto |v - v_c|^\delta. \quad (5.14)$$

This exponent obtains easily by putting $\tau = 0$ in the expansion (5.4), which leads to

$$P - P_c = \frac{P_C h(v, c_i, q)}{v_c^3} (v - v_c)^3. \quad (5.15)$$

Comparing with equation (5.14), we find $\delta = 3$ for both canonical and grand canonical ensembles.

To sum up, the critical exponents are obtained as $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$ and $\delta = 3$ for both canonical and grand canonical ensembles, the same as van der Waals fluid. In addition, the same results for the critical exponents in the (grand) canonical ensemble would be obtained if one uses the definition of w in terms of thermodynamic volume V as $w \equiv \frac{V - V_C}{V_C}$.

6. Conclusion

In the context of gauge/gravity duality, AdS BH solutions of dRGT massive gravity are dual to homogeneous and isotropic condensed matter systems with broken translational invariance [57, 62, 63]. More importantly, as indicated in a series of papers [57, 68–70], AdS BH solutions in massive gravity theories can effectively describe certain properties of different types (phases) of matter (solids, liquids etc) such as elasticity. Motivated by these facts, we extensively explored the chemistry of charged BH solutions of dRGT massive gravity with a suitable degenerate reference metric in the extended phase space and remarkably found a range of novel phase transitions in various ensembles close to realistic ones in real world.

To be more specific, we introduced the $U(1)$ charged TBHs in arbitrary dimensions by considering the full nonlinear theory of dRGT massive gravity with all the higher order graviton self-interactions. We evaluated the renormalized on-shell action in both canonical (fixed

charge) and grand canonical (fixed potential) ensembles with appropriate boundary conditions to obtain the corresponding semi-classical partition functions. By extracting the thermodynamic quantities from the partition functions in both canonical and grand canonical ensembles, we have shown these quantities satisfy the extended first law of thermodynamics in different representations. In addition, the validity of the Smarr formula in the extended phase space has been checked for this class of charged TBHs.

Next, since critical behaviors and nature of possible phase transition(s) are crucially dependent on the specific choice of ensemble, we focused on holographic phase transitions in various ensembles. In this regard, we explicitly demonstrated that vdW phase transition (in $d \geq 4$), vdW type phase transition (in $d \geq 6$) and the SBH/IBH/LBH phase transition associated with the triple point (in $d \geq 6$) are present in the canonical ensemble. Here, the absence of the phenomenon of RPT in this ensemble is interesting (we proved this claim in $d = 5, 6$ analytically, and, by using numerical investigation in $d = 7, 8, 9$ dimensions, this phenomenon did not find too. Whether or not such phenomenon exists for higher dimensional TBHs ($d \geq 7$) in the canonical ensemble remains an open question.). In the case of the grand canonical ensemble, we observe the vdW critical behavior (in $d \geq 5$), RPT (in $d \geq 6$), vdW type phase transition (in $d \geq 7$) and triple point (in $d \geq 7$) in contrast to Einstein's gravity which only phase transition takes place in the canonical ensemble of charged or rotating BHs. So, these critical phenomena may commence appearing in diverse dimensions depending on the ensemble one is dealing with. Here for convenience, we have summarized the final results in table 2.

These results show that, within the framework of massive gravity, critical behavior and phase transition(s) of charged-TBHs in the grand canonical ensemble are qualitatively the same as the uncharged (neutral) black holes. In fact, in the case of the grand canonical ensemble, the scalar potential (Φ) is absorbed into the effective topological factor $k_{\text{eff}}^{(\text{GC})} \equiv [k + m^2 c_0^2 c_2 - 2(d_3/d_2)\Phi^2]$, and thus, for a certain range of Φ , holographic phase transitions are obtained in the same dimensions as neutral black holes [28] in massive gravity. Here, it should be emphasized that the critical behavior of charged TBHs in massive gravity at the critical point are indistinguishable if the effective topological factor $k_{\text{eff}}^{(\text{C})} \equiv [k + m^2 c_0^2 c_2]$ in the canonical ensemble or $k_{\text{eff}}^{(\text{GC})}$ in the grand canonical ensemble have the same value while keeping other parameters fixed.

Motivated by the fact that some characteristic features of universality class of phase transitions such as the critical exponents or universal ratio may depend on the ensemble or the spacetime dimensions, we discussed the universal ratio of critical phenomena at the critical point and also their critical exponents in both ensembles. In the canonical ensemble up to two interaction potentials $O(\mathcal{U}_2)$ (or equivalently $O(c_2)$), it is found that the universal ratio belongs to the universality class presented in equation (3.31) which only depends on the spacetime dimensions whenever it is written down in terms of the shifted Hawking temperature. In $d = 4$, one arrives at $3/8$ for this ratio, exactly the same as vdW fluid. This result is the same as Einstein's gravity, but holds for all types of massive gravity's TBHs in the same manner, in contrast to the Einstein gravity in which only spherical black holes admit criticality. In the grand canonical ensemble up to three interaction potentials $O(\mathcal{U}_3)$ (or equivalently $O(c_3)$), the critical ratio belongs to another universality class (4.20) whenever it is written down in terms of the shifted Hawking temperature. Interestingly for this case, the universal ratio is constant for all types of TBHs, i.e. it is independent of spacetime dimensions or any other parameter. Note that, according to equations (3.30) and (4.19), the shifted Hawking temperature at the critical point is always positive. On the other hand, the universal ratio in both ensembles is a function of massive gravity's parameters (m_g and c_i) and spacetime dimensions (d) whenever

it is written down in terms of the standard Hawking temperature (see equations (3.32) and (4.21)). So, the universal ratio depends on the specific choice of ensemble. These results are summarized in (table 3) for convenience.

Furthermore, considering all the higher-order self-interaction potentials of massive gravitons in arbitrary dimensions, we examined the associated critical exponents in the grand canonical ensemble and proved that they match to those of charged TBHs in the canonical ensemble (i.e. $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$ and $\delta = 3$, exactly the same as vdW fluids). So, all kinds of TBHs in massive gravity have the same critical exponents in arbitrary dimensions which indicates the universality class independent of the spacetime dimensions and also the ensemble one is dealing with.

In conclusion, the nature of TBH phase transitions depends on the ensemble and also spacetime dimensions. The RPT phenomenon only appeared in the grand canonical ensemble, while the rest of the critical phenomena (vdW, triple point and vdW type) appear in both ensembles, but they commence to show up in different dimensions (summarized in table 2). The universal ratio also depends on the ensemble and spacetime dimensions (summarized in table 3). However, the critical exponents, which are the same as the exponents of vdW fluid, depend on neither spacetime dimensions nor ensemble.

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