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Laser entropy: from lasers and masers to Bose condensates and black holes

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Abstract

The entropy of an ordinary (photon) laser and an atom laser (Bose condensate) is calculated. In particular, the nonzero entropy of a single mode laser or maser operating near threshold is obtained. This result is to be compared with incorrect arguments frequently made in the study of the maser heat engine to the effect that at threshold maser radiation is characterized by an infinite temperature and its entropy is zero. Similarly, the entropy of the ground state of a Bose–Einstein condensate (a.k.a. the atom laser) is also calculated for the first time. This is to be compared with the textbook wisdom which holds that: 'The condensed particles ... are condensed in momentum space, a set of stationary particles ... having zero energy and zero entropy.' On the other hand, the BEC total entropy is found to be equal to that of the excited state entropy. This is true because the correlation entropy between the ground state and the excited states is equal and opposite to the ground state entropy. Furthermore, we find that the entropy of the radiation emitted by atoms falling into a black hole (BH) can be easily calculated by a laser type quantum master equation. The entropy thus calculated has much in common with Bekenstein–Hawking BH entropy.

Keywords: laser entropy, BEC entropy, black hole acceleration radiation entropy

(Some figures may appear in colour only in the online journal)

1. Introduction

Studying the entropy of thermal light led Planck to the quantum of action and Einstein to the photon concept. Half a century later the maser/laser appeared on the scene and it was shown that the three level maser could be regarded as a kind of quantum heat engine [1] yielding a quantum equivalent to the Carnot cycle [2]. More recently it has been recognized that quantum coherence in the lasing atoms allows lasing without inversion [3–5]; and by extension that we can extract work from a *single heat bath* (without violating the second law) via vanishing quantum coherence [6]. A clear analysis of the maser as a quantum heat engine has been given [7]. As has an analysis of the Carnot bound on masers without inversion [8] and laser cooling of solids [9]; for a recent review of quantum thermodynamics see [10].

The present work was initially stimulated by the studies of Harris [11] on quantum heat engines and electromagnetically induced transparency [12], in which he shows that:

> '[U]sing the second law, one may easily obtain a result that using [the usual] Maxwell's and Schrödinger's equations takes several pages of calculations.'

In particular, he uses an entropy relation similar to that in [1] for a maser/laser system driven by hot and cold radiation, as in figure 1, given by[1]

$$\delta S_{\mathrm{QHE}} = -\frac{\hbar\nu_h}{T_h} + \delta S_{\mathrm{maser}} + \frac{\hbar\nu_c}{T_c}, \tag{1}$$

[1] Equation (1) describes the entropy changes when a single hot (pump) photon is absorbed and a maser and cold (entropy sink) photon are emitted.

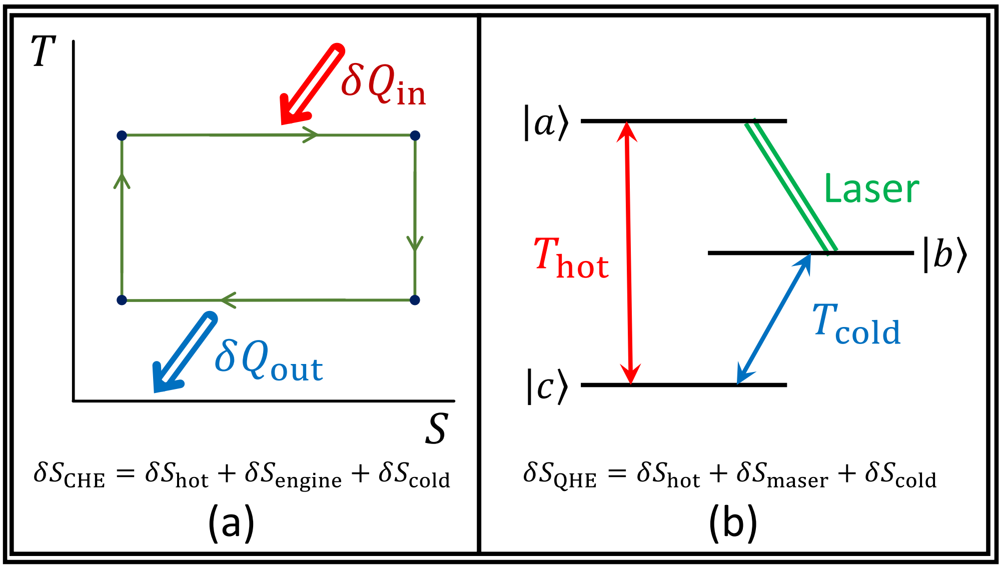

Figure 1. (a) Classical Carnot heat engine (CHE) operates between high temperature energy source and low temperature energy sink. The entropy change for a complete cycle is the sum of contributions from the hot energy source and the cold entropy sink together with the entropy coming from the engine due to e.g. friction. (b) The laser driven by hot and cold thermal reservoirs is a quantum heat engine (QHE). The entropy change for this QHE is the sum of single photon entropy changes due to the hot and cold light together with the contribution associated with the entropy change due to a single photon added to the laser/maser.

where ν_h, T_h [ν_c, T_c] are the frequency and temperature of the hot [cold] monochromatic radiation resonant with the $c \to a$ [$c \to b$] transition, and δS_{maser} is the maser entropy change associated with a change in the average photon number of one.

The physics behind equation (1) is similar to the textbook treatment of the classical Carnot heat engine (CHE) of figure 1; in which the entropy change after a complete cycle δS_{CHE} as determined by drawing energy δQ_{in} from a high temperature energy source and dumping energy δQ_{out} into a low tempeature entropy sink is given by

$$\delta S_{\text{CHE}} = \frac{\delta Q_{\text{in}}}{T_h} + \delta S_{\text{engine}} + \frac{\delta Q_{\text{out}}}{T_c}, \tag{2}$$

where δS_{engine} is the entropy generated by engine inefficiency, e.g. friction. By conservation of energy the work $\delta W = \delta Q_{\text{in}} - \delta Q_{\text{out}}$ and so we have the famous Carnot efficiency

$$\frac{\delta W}{\delta Q_{\text{in}}} \leqslant 1 - \frac{T_c}{T_h}. \tag{3}$$

Equation (1) is similar in spirit to equation (2) and as was argued in [2], the change in entropy corresponding to a single hot (energy source) photon absorbed and a maser photon emitted together with a cold (entropy sink) photon is given by equation (1). Now at threshold, the populations in $|a\rangle$ and $|b\rangle$ are equal so the entropy change per photon of the maser $\delta S_m = \hbar\nu_m/T_m$ is said [1, 2] to vanish since $T_m = \infty$. In such a case equation (1) yields the Carnot quantum efficiency

$$\frac{\hbar\nu_m}{\hbar\nu_h} \leqslant 1 - \frac{T_c}{T_h}, \tag{4}$$

where we have used the fact that $\hbar\nu_m = \hbar\nu_h - \hbar\nu_c$. This result is a good example in support of Harris' point since the derivation of equation (4) by conventional density matrix techniques [13] takes a bit of algebra. Equation (1) 'clearly' applies below threshold when the emitted 'laser' light is essentially thermal. But what if we are above threshold? One often encounters statements such as: 'because the maser radiation is in a pure state, its entropy is zero.' But the maser/laser radiation is not in a pure state. And, as is shown in section 2 and the appendix, the entropy of maser light is not zero but is determined by the density matrix formulation of the quantum theory of the laser [14] in which the photon and atom laser statistics is calculated from the master equation

$$\begin{aligned}\dot{\rho}_{n,n} = &-G(n+1)\,(n+1)\,\rho_{n,n} + G(n)\,n\rho_{n-1,n-1}\\ &- L\,n\rho_{n,n} + L\,(n+1)\rho_{n+1,n+1},\end{aligned} \tag{5}$$

where the gain and loss coefficients, $G(n)$ and L, for the photon and atom lasers are given in sections 2 and 3.

In section 2 we sketch the calculation of the laser/maser entropy from the density matrix formulation of the quantum theory of the optical maser [14] and compare it to the entropy of high temperature single mode thermal light. Although the laser/maser physics is semiequivalent, the numerical parameters and operating characteristics are not. This distinction is to be kept in mind as we sort out the deep physics from some conceptual inconsistencies contained in [1, 2].

In section 3, the entropy of the analogous ground state of a Bose–Einstein [15] condensate, a.k.a. the atom laser is presented. A summary and discussion is given in section 4.

2. Laser entropy

In the quantum theory of the optical maser the gain coefficient $G(n)$ and the loss rate L of equation (5) are given by

$$G(n) = \frac{\alpha}{1 + \frac{\beta}{\alpha}n} \quad \text{and} \quad L = \gamma, \tag{6}$$

in terms of the laser parameter: α= linear gain, β = nonlinear saturation coefficient, and the cavity loss rate $\gamma = \nu / Q$ is governed by the cavity Q factor. As is shown in [14], the steady state solution to (5) yields the n photon probability distribution which can be written as

$$\rho_{nn} = \frac{1}{Z}\frac{B!A^n}{(n+B)!}, \tag{7a}$$

where the normalization is given in terms of the confluent hypergeometric function

$$Z = {}_1F_1(1; B+1; A), \tag{7b}$$

and

$$A = \frac{\alpha^2}{\beta\gamma} \quad \text{and} \quad B = \frac{\alpha}{\beta}. \tag{7c}$$

Just above threshold, e.g. $\alpha \sim 1.1\gamma$, we may write ρ_{nn} in the appealing form

$$\rho_{nn} = \frac{A^{n+B}}{(n+B)!}\mathrm{e}^{-A}, \tag{8}$$

which allows the photon distribution to be approximated by the Gaussian

$$\rho_{n,n} \cong \frac{1}{\sqrt{2\pi A}}\exp\left[-\frac{(n-\bar{n})^2}{2A}\right], \tag{9}$$

where $\bar{n} = A - B = \frac{\alpha}{\gamma}\frac{\alpha-\gamma}{\gamma}$.

Plugging ρ_{nn} given by equations (8) or (9) into the von Neumann entropy equation

$$S_m = -k_{\mathrm{B}}\sum_n \rho_{nn}\ln\rho_{nn}, \tag{10}$$

we obtain the maser entropy

$$S_m \cong k_{\mathrm{B}}\ln\sqrt{2\pi\frac{\alpha}{\alpha-\gamma}\bar{n}_m} + \frac{k_{\mathrm{B}}}{2}. \tag{11}$$

We note that the entropy change implied by equation (11) due to a change $\delta\bar{n}_m$ in the average photon number is given by

$$\delta S_m = \frac{k_{\mathrm{B}}\bar{n}_m}{2\bar{n}_m}. \tag{12}$$

The preceding is to be compared with monochromatic thermal light characterized by the density matrix

$$\rho_{nn} = \frac{\bar{n}_t^n}{(\bar{n}_t+1)^{(n+1)}}, \tag{13}$$

where $\bar{n}_t$ is the Planck function

$$\bar{n}_t = \frac{1}{\exp(\hbar\nu/k_{\mathrm{B}}T) - 1}. \tag{14}$$

In this case, equations (10) and (13) yield the thermal blackbody entropy

$$S_t = k_{\mathrm{B}}(\bar{n}_t+1)\ln(\bar{n}_t+1) - k_{\mathrm{B}}\bar{n}_t\ln\bar{n}_t, \tag{15}$$

and the entropy change associated with $\delta\bar{n}_t$ is given by

$$\delta S_t = \frac{\hbar\nu\delta\bar{n}_t}{T}. \tag{16}$$

Finally, we note that for a maser/laser well below threshold $G = \alpha$ and S is given by equation (15) with $\bar{n}_m = [(\alpha/\gamma) - 1]^{-1}$, and δS_m takes the form

$$\delta S_m = \frac{k_{\mathrm{B}}\delta\bar{n}_m}{\bar{n}_m}. \tag{17}$$

3. Bose–Einstein condensation (BEC) (a.k.a. 'Atom Laser') entropy

BEC has been dubbed the 'atom laser' [16] and it has been shown that the density matrix treatment for the photons in a laser cavity given by equation (5) also applies to the ground state of the BEC. In this case the index n is replaced by n_0 denoting the number of atoms in the lowest state having energy ϵ_0. Einstein taught us that for N atoms in a box the average number in the condensate is $\overline{n_0} = N[1 - (T/T_c)^3]$ where T_c is the critical temperature[2].

We are here interested in the probability of having n_0 out of N in the ground state of a parabolic trap for which $\overline{n_0} = N[1 - (T/T_c)^3]$. This probability is given by the diagonal elements of the ground state density matrix ρ_{n_0,n_0} which obeys equation (5) with gain

$$G(n_0) = \kappa(N - n_0) \tag{18}$$

describing the rate of addition of atoms (gain) to the ground state due to the excited atoms (ϵ_k, $k \neq 0$) colliding with the walls having temperature T and falling into the ground state at a rate κ. Likewise atoms are removed (lost) from the ground state due to interaction with the hot walls (temperature T) at a rate

$$L(n_0) = \kappa N(T/T_c)^3. \tag{19}$$

The master equation for ρ_{n_0,n_0} obtained from equations (5), (18), (19) has the steady state ($\dot{\rho}_{n_0,n_0} = 0$) solution given by [17]

$$\rho_{n_0,n_0} = \frac{\mathcal{H}^{N-n_0}}{(N-n_0)!}\mathrm{e}^{-\mathcal{H}}, \tag{20}$$

where $\mathcal{H} = N(T/T_c)^3$.

The BEC ground state entropy obtained by inserting (20) into equation (10) can be plotted as a function of T/T_c; the result is found to be in good agreement with the ground state entropy obtained from exact numerical calculations for a mesoscopic condensate of say 10^3 atoms.

Here we will simply note that for low-enough temperatures the variance of the BEC atom distribution equation (20) is governed to a reasonable approximation by $N(T/T_C)^3$ [17]

[2] Einstein considered the case of a square well trap for which the $(T/T_c)^3$ factor becomes $(T/T_c)^{3/2}$.

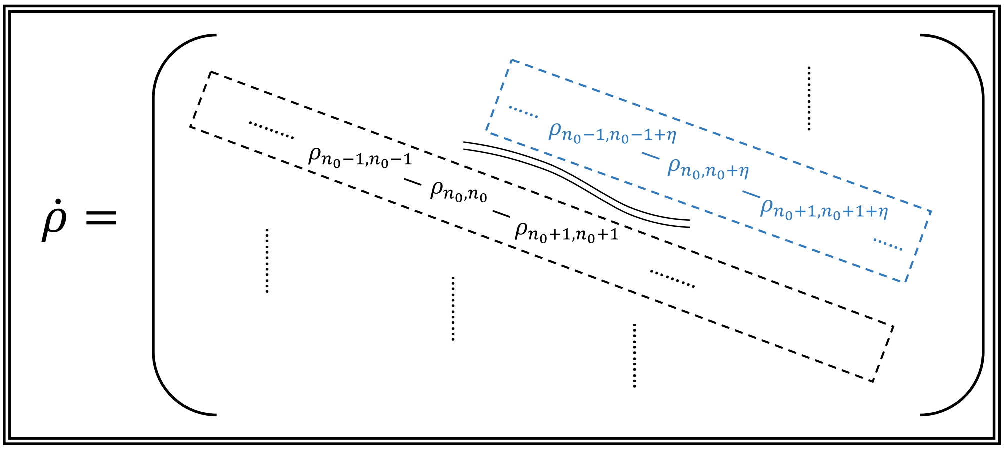

Figure 2. The density matrix equation of motion couples only elements of equal off diagonality η. For example in equation (5) for the photon statistics $\eta = 0$.

and the BEC ground state entropy for a parabolic trap is found to be[3]

$$S_g = k_{\mathrm{B}} \ln \sqrt{2\pi N (T/T_C)^3} + \frac{k_{\mathrm{B}}}{2}. \tag{21}$$

4. Laser entropy discussion

We now turn to a discussion and summary of our results.

1. Back to the quantum heat engine: As per section 3 the change in entropy due to a single photon addition or subtraction is obtained from equations (12) and using the replacement $\delta n = \dot{\bar{n}} \delta t = \pm 1$ to obtain

$$\delta S_{\mathrm{maser}} = \frac{k_{\mathrm{B}}}{2\bar{n}_m} \quad \text{and} \quad \delta S_{\mathrm{thermal}} = \frac{k_{\mathrm{B}}}{\bar{n}_{\mathrm{high}}},$$

where $\bar{n}_{\mathrm{high}} = k_{\mathrm{B}} T / \hbar\nu$ is the number of thermal photons in the high temperature limit.

Hence, for a laser very near threshold with $\bar{n} \sim 10^6$ say, then $\delta S_{\mathrm{laser}} \sim 10^{-6} k_{\mathrm{B}}$. If this is compared with the thermal entropy change expression for high temperature $k_{\mathrm{B}}/\bar{n}_{\mathrm{high}}$, and $\bar{n}_{\mathrm{high}} = k_{\mathrm{B}} T/\hbar\nu \sim 1$ for $k_{\mathrm{B}} T$ and $\hbar\nu$ both around 1 eV, we see that in this case $\delta S_{\mathrm{laser}}$ is negligible, and the Carnot efficiency result of equation (4) is valid above threshold.

However, for a (microwave) maser with $\hbar\nu \sim 10^{-6}$ eV and $k_{\mathrm{B}} T \sim 1$ eV, $\bar{n}_{\mathrm{high}} \sim 10^6$. So if $\bar{n}_{\mathrm{maser}} \sim 10^6$ then $\delta S_{\mathrm{maser}}$ is comparable to the entropy change $\delta S_{\mathrm{thermal}}$. In such a case $\delta S_{\mathrm{maser}}$ is *not negligible*. Equation (4) assumes $\delta S_{\mathrm{maser}}$ is small to $\delta S_{\mathrm{thermal}}$ but this will not generally be the case.

$$\dot{S}_h + \dot{S}_m + \dot{S}_c \geqslant 0, \tag{22}$$

which in view of equations (12) reads. Thus we are again led to equation (4) even though we are now above threshold. But for some problems such as the micromaser [18] $\bar{n}_m$ is not a large number. We leave this as an open problem to be treated elsewhere.

2. The entropy is not given by a simple $S = k_{\mathrm{B}} \ln W$ type expression given by equation (11) in the threshold region. Well above threshold i.e. for $(\alpha - \gamma)/\gamma \gtrsim 0.1$ the simple form given this simple form is correct but not for $(\alpha - \gamma)/\gamma \sim 10^{-3}$ as will be discussed elsewhere.

3. Laser Entropy and the laser linewidth: Indeed, the source of the maser entropy is presaged by the insightful statement of Morse [19] who says: 'during a spontaneous process $\cdots$the entropy always increases.' In fact it is precisely the spontaneous (as opposed to stimulated) emission events which are the source of the Schawlow–Townes optical maser linewidth; and which are the source of the time dependence of the laser radiation density matrix given by [17]

$$\rho_{n,n+\eta}(t) = \rho_{n,n+\eta}(0) \mathrm{e}^{-\eta^2 D t}, \tag{23}$$

where η measures the degree of off-diagonality as per figure 2. Equation (23) implies the electric field

$$\langle \hat{E}(t) \rangle = \sum_n \mathscr{E}_0 \, \rho_{n,n+1}(0) \sqrt{n+1} \exp(\mathrm{i}\nu t - Dt), \tag{24}$$

where $\mathscr{E}_0$ is the electric field per photon and $D \equiv \alpha/4\bar{n}$ is the laser phase diffusion coefficient [20]. The Fourier transform of equation (24) is a Lorentzian centered at ν_l and with a full width at half max given by

$$\Delta\nu = 2D = \frac{\alpha}{2\bar{n}}. \tag{25}$$

The physics behind the linewidth (25) is (partially) illustrated by writing the equations of motion for a maser below threshold as

$$\dot{\bar{n}} = \alpha(\bar{n} + 1) - \gamma\bar{n}, \tag{26a}$$

$$\dot{\bar{E}} = \frac{1}{2}(\alpha - \gamma)\bar{E}, \tag{26b}$$

where we here use the notation $\bar{E} = \langle E \rangle$. Then, below threshold, the steady state relation (26a) yields $\alpha - \gamma = \alpha/\bar{n}$ and using this in (26b) yields $\dot{\bar{E}} = -(\alpha/2\bar{n})\bar{E}$; which implies a phase diffusion coefficient $D' = \alpha/2\bar{n}$ and a below threshold linewidth

[3] Equation (21) applies when we are not too close to $T = 0$. When we are at $T = 0$, $\rho = \delta_{N,n_0}$ and therefore $S(T = 0)$ vanishes.

Table 1. The spontaneously generated entropy flux for hot thermal light ($k_B T \gg \hbar\nu$) having average photon number $\bar{n}_h = k_B T/\hbar\nu$ with the flux of a laser above threshold having average photon number $\bar{n}_l$ where κ is defined following equation (12). This is compared with the laser linewidth below and above threshold which are well known to differ by a historically bothersome factor of two, the origin of which is clear in the maser entropy flux.

	Below theshold	Above threshold
Laser linewidth	$\frac{\nu/Q}{\bar{n}_h}$	$\frac{\nu/Q}{2\bar{n}_l}$
Entropy flux	$\frac{\kappa}{\bar{n}_h}$	$\frac{\kappa}{2\bar{n}_l}$

$$\Delta\nu' = 2D' = \frac{\alpha}{\bar{n}}. \tag{27}$$

Concluding this linewidth review we note that α is essentially $\gamma = \nu/Q$ in steady-state; and we compare the proceeding linewidth discussion with the 'spontaneously generated' laser entropy flux below and above threshold in table 1. Another way in which the laser linewidth and laser entropy are related is discussed in the next paragraph.

4. Off-diagonality and more: several points should be noted concerning the off-diagonal nature of the maser density matrix and its entropy, a few of these are:

(i) As is seen from equation (11), the maser entropy well above threshold takes the form of the famous Boltzmann microcanonical entropy but is quite different; for example, the entropy of a gas is extensive (i.e. goes as the number of gas atoms) but the maser entropy is not an extensive variable.

(ii) The factor of 2 in the linewidth encountered in going from below to above threshold is a well known, if a bit subtle, aspect of laser physics. On the other hand the laser entropy flux, factor of 2 in passing through threshold is due to the laser entropy going from $\ln \bar{n}$ below threshold to $\ln \sqrt{\bar{n}}$ above threshold.

(iii) The degree of off-diagonality η as it appears in equation (23) can be large i.e. $0 \leqslant \eta \leqslant n$ where n can be of order (or greater than) $\bar{n}$. Hence such off-diagonal character of the laser density matrix vanishes rapidly as is shown by equation (23). The paper by Chen and Fan [21] treats the off-diagonal term but uses a linear gain-loss master equation.

5. On the entropy of the BEC ground state entropy: We note that in the thermodynamic limit the entropy of a Bose gas [22] has the dependence

$$S \approx 3.6 k_B N \left(\frac{T}{T_C}\right)^3. \tag{28}$$

For a macroscopic Bose gas of say 10^{23} atoms $S_g \sim \ln N$ is negligible compared to S. But for a mesoscopic BEC of 10^3 atoms $N(T/T_c)^3 = 1$ when $T/T_c \cong 0.1$; and in such a case $S \sim 4k_B$ and $S_g \sim k_B$ are of the same order.

We emphasize that the present BEC entropy analysis is approximate but as will be further discussed elsewhere, it gives a good account of the ground state entropy. This is to be compared with conventional wisdom which one often hears saying that [19]:

> As expected, the n_0 particles constituting the 'condensate' do not contribute to the entropy of the system, while the $N - n_0$ particles that constitute the normal part do contribute.

The ground state entropy of a mesoscopic BEC yields many interesting questions. For example, the relation between the correlation entropy and the ground state entropy is an open question.

6. Summary: The quantum entropy of a laser below, at, and above threshold is well described by the quantum theory of the maser. The entropy flux of the maser is not 'zero' and this can be important for a complete analysis of the Carnot bound of maser operation. A similar analysis of the quantum theory of the 'atom laser' yields a nonvanishing BEC ground state entropy.

The present paper poses many open questions and several have already been noted. A few others are:

(i) The threshold $\alpha = \gamma$ region is interesting and should be further investigated.

(ii) It would be interesting to extend the laser entropy—linewidth discussion to include the noise generated correlated emission laser and lasing in the presence of squeezed light.

(iii) The treatment of the full nonlinear master equation [17] is challenging, and its application to the study of the time evolution of laser entropy is a challenging problem.

(iv) The ground state entropy is *not* simply the total entropy minus the excited state entropy, as will be shown elsewhere.

5. Entropy of radiation emitted by atoms falling into a black hole (BH)

We consider [23] an atomic cloud consisting of two-level atoms emitting acceleration radiation as it falls into a BH. We find that the quantum master equation technique, as developed in the quantum theory of the laser, provides a useful tool for the analysis of BH acceleration radiation and the associated entropy. In particular, we derive the equation of motion for the density matrix of the emitted radiation where the density matrix is given by equation (30).

We find that once we have cast the acceleration radiation problem in the language of quantum optics and cavity quantum electrodynamics, the entropy follows directly. That is, once we calculate $\dot{\rho}$ for the field produced by accelerating atoms, we can use the von Neumann entropy relation to write

$$\dot{S}_p = -k_B \operatorname{Tr}(\dot{\rho} \ln \rho), \tag{29}$$

where S_p is the entropy of the emitted photons.

We obtain an evolution equation for the radiation following the approach used in the quantum theory of the laser

[14]. As is further discussed in [23], the coarse-grained time rate of change of the radiation field density matrix for a particular field mode is found to be

$$\frac{1}{R}\frac{\mathrm{d}\rho_{n,n}}{\mathrm{d}t} = -\frac{\kappa g^2}{\omega^2}\mathrm{e}^{-\xi}[(n+1)\rho_{n,n} - n\rho_{n-1,n-1}] - \frac{\kappa g^2}{\omega^2}\mathrm{e}^{\xi}[n\rho_{n,n} - (n+1)\rho_{n+1,n+1}], \tag{30}$$

where g is the atomfield coupling constant, $\xi = 2\pi\nu r_g/c$,

$$R = \frac{\xi}{\sinh(\xi)}, \tag{31}$$

and ν is the photon frequency far from the BH. Using equations (29) and (30), we find that the von Neumann entropy generation rate is

$$\dot{S}_p = \frac{4\pi k_B r_g}{c}\sum_\nu \dot{\bar{n}}_\nu \nu, \tag{32}$$

where r_g is the Schwarzschild radius, and $\dot{\bar{n}}_\nu$ is the flux of photons with frequency ν propagating away from the BH. This result is extended in [23] to obtain

$$\dot{S}_p = \frac{k_B c^3}{4\hbar G}\dot{A}_p, \tag{33}$$

here $\dot{A}_p = (2\dot{m}_p/M)A$ is the rate of change of the BH area due to photon emission which we calculate using laser theory techniques.

Thus we find that atoms falling into a BH emit acceleration radiation which looks much like (but is different from) Hawking BH radiation. We find the entropy of the acceleration radiation via a simple laser-like analysis, which we call horizon brightened acceleration radiation entropy to distinguish it from the BH entropy of Bekenstein and Hawking [24]. More on this problem is to be found in [23]. See also [25]. This brief summary given here will hopefully alert the laser physics communities to this interesting blackhole connection.

Acknowledgments

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Dedication

Professor Wolfgang Schleich has made outstanding contributions to many fields of quantum physics ranging from quantum fluctuations in ring-laser gyroscopes [26] and the correlated emission laser [27] via interference in phase space [28] leading to his celebrated textbook [29] and tests [30] of and general relativity subtleties [31], to the foundations of quantum mechanics [32, 33] and the Riemann Hypothesis [34], as well as cold atoms in microgravity [35]. His search for deeper understanding of quantum mechanics in general and quantum optics in particular has yielded new field opening insights. It is a pleasure to dedicate this article to Professor Wolfgang Schleich.

Appendix. Laser entropy details

From the quantum theory of the laser [5, 14] we have

$$\rho_{nn} = \frac{1}{Z}\frac{\left(\frac{\alpha}{\beta}\right)!\left(\frac{\alpha^2}{\beta\gamma}\right)^n}{\left(n+\frac{\alpha}{\beta}\right)!}, \tag{A.1}$$

where the normalization is expressed in terms of the confluent hypergeometric function as

$$Z = {}_1F_1\left(1; \frac{\alpha}{\beta}+1; \frac{\alpha^2}{\beta\gamma}\right). \tag{A.2}$$

The α, β, γ laser parameters are defined in the text following equation (6). In the usual laser limit of large $\alpha^2/\beta\gamma$ we have

$${}_1F_1\left(1; \frac{\alpha}{\beta}+1; \frac{\alpha^2}{\beta\gamma}\right) \Rightarrow \left(\frac{\alpha}{\beta}\right)!\mathrm{e}^{\alpha^2/\beta\gamma}\left(\frac{\alpha^2}{\beta\gamma}\right)^{-\frac{\alpha}{\beta}}, \tag{A.3}$$

which inserted into (A.1) yields equation (8).

To calculate the entropy we write the entropy

$$S = -k_B\sum_n \rho_{nn}\ln\rho_{nn},$$

using equation (8) as

$$= -k_B\sum_n \rho_{nn}[(n+B)\ln A - A - \ln(n+B)!]. \tag{A.4}$$

Making use of Stirling's approximation and noting that $A = B + \bar{n}$ we have

$$= k_B\sum_n \rho_{nn}[\ln\sqrt{2\pi(n+B)} + (n+B)(\ln(n+B) - \ln A)], \tag{A.5}$$

and expanding the logarithms to second order

$$\ln(n+B) = \ln(A+n-\bar{n}) \cong \ln A + \frac{n-\bar{n}}{A} - \frac{(n-\bar{n})^2}{2A^2}, \tag{A.6}$$

we arrive at

$$S \cong k_B\ln\sqrt{2\pi A} + \frac{k_B}{2}. \tag{A.7}$$

Similarly, as will be discussed in detail elsewhere, the BEC ground state density matrix can be written as

$$\rho_{n_0,n_0} = \frac{\mathscr{H}^{N-n_0}}{(N-n_0)!}\mathrm{e}^{-\mathscr{H}}, \tag{A.8}$$

which leads to a ground state entropy

$$S_g = k_{\mathrm{B}} \ln W_g + \frac{k_{\mathrm{B}}}{2}, \qquad W_g = \sqrt{2\pi\mathscr{H}}, \tag{A.9}$$

where $\mathscr{H} = N(T/T_c)^3$. Equation (A.9) is correct over a wide range of temperature, however equation (A.8) shows that $\rho_{n_0,n_0} \sim \delta_{N,n_0}$ and the entropy vanishes at $T = 0$. It should be noted that the BEC ground state entropy (A.9) is not the total entropy minus the excited state entropy as will be discussed at length elsewhere.

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