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Effect of the fractional oscillator noise in the overdamped linear oscillator with the presence of a periodic force

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Abstract

The overdamped linear oscillator with periodic external force driven by the fractional oscillator (FO) noise is considered and investigated. The correlation function of the FO noise presents power-law-like function, exponential-like function and oscillatory decays similar to those of the harmonic noise. The amplitude of the stationary state of the first moment is obtained and analyzed in relation to the parameters of the system; it presents non-monotonic behaviors which are related to the resonance phenomenon.

Keywords: Langevin equation, overdamped linear oscillator, stochastic resonance, fractional oscillator noise

1. Introduction

Oscillatory behavior can be found in many systems in nature and it plays a crucial role in many phenomena [1, 2]. In particular, the linear oscillator with periodic external force driven by a noise has also been considered in many works due to its wide application in diverse systems, for instance the RL circuit; for the multiplicative noise, the system may present resonance stochastic phenomenon [3–5]. Besides, the fractional calculus is a ubiquitous tool due to the fact that it has been applied to a wide range of systems, such as physical, chemical and biological systems. For instance, it can describe anomalous diffusion processes [6–8], tissue viscoelasticity [9, 10], electronic circuits [11, 12], biological systems [13–20] and complex phenomena [21, 22].

The aim of this work is to investigate the overdamped linear oscillator with periodic external force driven by a multiplicative colored noise characterized by the fractional oscillator (FO) noise which has been named as the fractional Ornstein–Uhlenbeck noise in [23]. In this approach the procedure to deal with a colored noise is to extend the space of variables so that the noise itself becomes a variable driven by the white noise. The use of the FO noise for investigating the overdamped linear oscillator is due to the fact that it can describe various types of the correlation function; it can describe power-law-like function, exponential-like function and oscillatory decays similar to those of the harmonic noise. The proximity of these functions to the ordinary power-law and exponential functions makes the FO noise interesting to be investigated. It may be employed to improve the descriptions of systems which use the ordinary power-law and exponential functions and oscillatory decay described by the harmonic noise [24]. The amplitude of the stationary state of the first moment is obtained analytically and numerically analyzed in relation to the parameters of the system. The amplitude may show non-monotonic behavior which characterizes the resonance effect. In the next section the FO noise is introduced and its correlation function for the stationary state is shown numerically. In section 3 the overdamped linear oscillator with periodic external force driven by the FO noise is investigated. The amplitude of the stationary state of the first moment is numerically analyzed in relation to the parameters of the system. Conclusion is provided in section 4.

2. FO noise

The FO noise is a generalization of the ordinary Ornstein–Uhlenbeck noise in which the ordinary derivative is replaced by a fractional derivative (see also [25–29]), and it is described by the following Langevin equation:

$$\frac{d^\alpha y}{dt^\alpha} + \Omega_\alpha y = \beta_\alpha L(t), \tag{1}$$

where Ω_α and β_α are real positive numbers, and $d^\alpha y/dt^\alpha$ is the Caputo fractional derivative defined by

$$\frac{d^\xi y}{dt^\xi} = \frac{1}{\Gamma(n-\xi)}\int_0^t d\tau \frac{\frac{d^n y}{d\tau^n}}{(t-\tau)^{\xi+1-n}}, \quad n-1<\xi<n, \tag{2}$$

where n is a positive integer number, $\Gamma[z]$ is the gamma function and $L(t)$ is the Gaussian white noise with the averages given by

$$\langle L(t)\rangle = 0 \tag{3}$$

and

$$\langle L(t_1)L(t_2)\rangle = \delta(t_1 - t_2). \tag{4}$$

The generalization of the Ornstein–Uhlenbeck process described by equation (1) should be distinguished by another generalization in which the ordinary Ornstein–Uhlenbeck process is driven by a fractional Brownian motion (see for instance [30–33] for description and application).

The value of α is restricted to the interval $0<\alpha<2$. Note that the FO process described by equation (1) has been applied to financial time series [26, 27] and it may be associated with the output signal of a supercapacitor [11].

The solution of equation (1) is obtained by using the Laplace transform and it is given by

$$y(t) = \bar{y}_0(t) + \beta_\alpha \int_0^t d\tau\tau^{\alpha-1}E_{\alpha,\alpha}(-\Omega_\alpha t^\alpha)L(\tau), \tag{5}$$

where $\bar{y}_0(t)$ is given by

$$\bar{y}_0(t) = E_{\alpha,1}(-\Omega_\alpha t^\alpha)y_0,\ 0<\alpha<1, \tag{6}$$

$$\bar{y}_0(t) = E_{\alpha,1}(-\Omega_\alpha t^\alpha)y_0 + tE_{\alpha,2}(-\Omega_\alpha t^\alpha)v_0,\ 1<\alpha<2, \tag{7}$$

$y_0 = y(t=0)$, $v_0 = dy(t=0)/dt$ and $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}. \tag{8}$$

The mean value of a noise is usually taken as zero $\langle y(t)\rangle = 0$ which implies that $\langle y_0\rangle = \langle v_0\rangle = 0$.

The correlation function can be calculated from (5) and it yields

$$\langle y(t)y(t+\tau)\rangle = \beta_\alpha^2\int_0^t du u^{\alpha-1}E_{\alpha,\alpha}(-\Omega_\alpha u^\alpha) \times (u+\tau)^{\alpha-1}E_{\alpha,\alpha}(-\Omega_\alpha(u+\tau)^\alpha). \tag{9}$$

The stationary state of the correlation function is obtained from the limit $t\to\infty$. Figure 1 shows the stationary state of the correlation function. For $0<\alpha\leqslant 1$ it decays monotonically, whereas for $1<\alpha<2$ it shows oscillatory behavior. The correlation function diverges at $\tau=0$ for $0<\alpha\leqslant 1/2$, thus it is highly correlated for small time-lag τ; in that case the second moment $\langle y^2(t)\rangle$ diverges at the lower limit of the integral, i.e. $\langle y^2(t)\rangle \sim \lim_{a\to 0}\int_a du u^{2(\alpha-1)} \to \infty$ for $0<\alpha\leqslant 1/2$. All correlations decay as the time-lag τ increases. The oscillatory decay of the correlation function for $1<\alpha<2$ behaves like an underdamped process; in that case the particle may assume higher velocity in the statistical sense. The negative value of the correlation function means that if the particle is at a position in the positive x at this instant, it is more likely to occupy at a position in the negative x in the next instant.

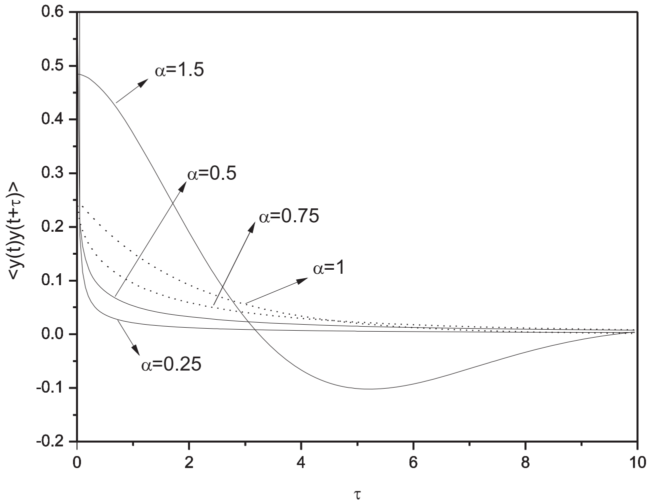

Figure 1. Plots of the correlation function (9) for the FO noise in the stationary regime with $\beta_\alpha = 0.5$ and $\Omega_\alpha = 0.5$, in arbitrary units.

For $0<\alpha\leqslant 1/2$ the correlation function presents a power-law-like function and for $1/2<\alpha\leqslant 1$ it presents an exponential-like function. For $1<\alpha<2$ the correlation function shows oscillatory decays similar to those of the harmonic noise.

3. Overdamped linear oscillator driven by the multiplicative FO noise

The overdamped linear oscillator with periodic external force driven by a multiplicative colored noise has been considered in many works due to its wide application in diverse systems, including the RL circuit. In this work, the system is driven by the FO noise and its first moment is investigated. The system is described by

$$\frac{dx}{dt} + (r_0 + r_1 y(t))x = A_0\sin(\omega t). \tag{10}$$

In particular, for $r_0 = R/L$, $r_1 = R_1/L$ and $A_0 = V_0/L$, where R is the resistance, L is the inductance and V_0 is the voltage, the system (10) describes the RL circuit with the presence of a periodic force and noise.

Integrating equations (1) and (10) yields

$$x(t) = x_0 e^{-r_0 t - r_1\int_0^t d\tau y(\tau)} + A_0\int_0^t d\tau\sin(\omega\tau)e^{-r_0(t-\tau)-r_1\int_\tau^t d\tau_1 y(\tau_1)} \tag{11}$$

and

$$y(t) = \bar{y}_0 + \beta_\alpha\int_0^t d\tau\tau^{\alpha-1}E_{\alpha,\alpha}(-\Omega_\alpha t^\alpha)L(\tau), \tag{12}$$

where $\bar{y}_0$ is given by equations (6) and (7).

Taking the average of equation (11) one obtains the following expression for the first moment:

$$\langle x(t)\rangle = x_0 e^{-r_0 t}\left\langle e^{-r_1\int_0^t d\tau y(\tau)}\right\rangle + A_0\int_0^t d\tau\sin(\omega\tau)e^{-r_0(t-\tau)}\left\langle e^{-r_1\int_\tau^t d\tau_1 y(\tau_1)}\right\rangle. \tag{13}$$

The steady state of the first moment is obtained from the limit $t \to \infty$.

The average $\left\langle e^{-r_1 \int_\tau^t d\tau_1 y(\tau_1)} \right\rangle$ can be calculated by using the higher correlation functions for $L(t)$ given by [34]

$$\langle L(t_1)L(t_2)\ldots L(t_{2n-1})\rangle = 0 \tag{14}$$

and

$$\langle L(t_1)L(t_2)\ldots L(t_{2n})\rangle = \left[\sum_{P_d} \delta(t_{i_1} - t_{i_2})\delta(t_{i_2} - t_{i_3})\ldots\delta(t_{i_{2n-1}} - t_{i_{2n}})\right], \tag{15}$$

where the sum indicates $(2n)!/(2^n n!)$ permutations which lead to different expressions for $\delta(t_{i_1} - t_{i_2})\delta(t_{i_2} - t_{i_3})\ldots\delta(t_{i_{2n-1}} - t_{i_{2n}})$. Expanding the exponential function in Taylor series and using the higher moments for $L(t)$ yields

$$\left\langle e^{-r_1 \int_\tau^t d\tau_1 y(\tau_1)} \right\rangle = e^{-\frac{r_1^2\beta_\alpha^2}{2}\int_\tau^t d\tau_1 \times \int_0^{\tau_1} du_1 \int_\tau^t d\tau_2 \int_0^{\tau_2} du_2 \delta(u_1-u_2)H(\tau_1-u_1)H(\tau_2-u_2)}, \tag{16}$$

where

$$H(t) = t^{\alpha-1}E_{\alpha,\alpha}(-\Omega_\alpha t^\alpha). \tag{17}$$

The multiple integral in equation (16) can be rewritten as follows:

$$\begin{aligned}
&\int_\tau^t d\tau_1 \int_0^{\tau_1} du_1 \int_\tau^t \\
&\times d\tau_2 \int_0^{\tau_2} du_2 \delta(u_1 - u_2)H(\tau_1 - u_1)H(\tau_2 - u_2) \\
&= \left[\int_0^t du_1 \int_0^t du_2 \int_{u_1}^t d\tau_1 \int_{u_2}^t d\tau_2 \right.\\
&- 2\int_0^t du_1 \int_0^\tau du_2 \int_{u_1}^t d\tau_1 \int_{u_2}^\tau d\tau_2 \\
&\left. + \int_0^t du_1 \int_0^\tau du_2 \int_{u_1}^\tau d\tau_1 \int_{u_2}^\tau d\tau_2 \right] \\
&\times \delta(u_1 - u_2)H(\tau_1 - u_1)H(\tau_2 - u_2).
\end{aligned} \tag{18}$$

Note that the orders of the multiple integrals in equation (18) have been changed. In equation (16) the variables u_1 and u_2 are firstly integrated, while in equation (18) the variables τ_1 and τ_2 are firstly integrated. The integrals in equation (18) are easier to be performed than those in equation (16). The result of equation (18) after substituting it into equation (16) is given by

$$\begin{aligned}
&\left\langle \exp\left[-r_1 \int_\tau^t d\tau_1 y(\tau_1)\right]\right\rangle = \exp\left[\frac{r_1^2\beta_\alpha^2}{2\Omega_\alpha^2}\right.\\
&\times \left\{\int_0^t du[E_{\alpha,1}(-\Omega_\alpha(t - u)^\alpha) - 1]^2\right.\\
&+ \int_0^\tau du[E_{\alpha,1}(-\Omega_\alpha(\tau - u)^\alpha) - 1]\\
&\times [E_{\alpha,1}(-\Omega_\alpha(\tau - u)^\alpha)\\
&\left.\left. - 2E_{\alpha,1}(-\Omega_\alpha(t - u)^\alpha) + 1]\right\}\right].
\end{aligned} \tag{19}$$

It should be noted that despite the divergence of the correlation function (9) for $0 < \alpha \leqslant 1/2$ at $\tau = 0$ the above calculation does not pose any definition problem due to the fact that it does not involve the second moment $\langle y^2(t)\rangle$ of the noise directly; it does not present any definition problem for the system analyzed in [23] too. Substituting equation (19) into (13) yields

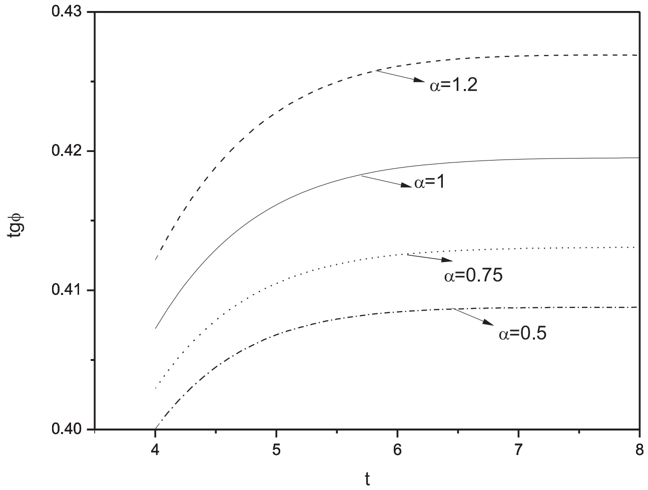

Figure 2. Plots of the phase $tg\phi$ (27) versus time t for $r_0 = 1.25$, $r_1 = 0.5$, $\beta_\alpha = 0.5$, $\omega = 0.5$ and $\Omega_\alpha = 0.5$, in arbitrary units.

$$\begin{aligned}
\langle x(t)\rangle &= x_0 \exp\left[-r_0 t + \frac{r_1^2\beta_\alpha^2}{2\Omega_\alpha^2}\right.\\
&\left.\times \int_0^t du[E_{\alpha,1}(-\Omega_\alpha(t - u)^\alpha) - 1]^2\right]\\
&+ A_0 \int_0^t d\tau \sin(\omega\tau)\exp\left[-r_0(t - \tau) + \frac{r_1^2\beta_\alpha^2}{2\Omega_\alpha^2}\right.\\
&\times \left\{\int_0^t du[E_{\alpha,1}(-\Omega_\alpha(t - u)^\alpha) - 1]^2\right.\\
&+ \int_0^\tau du[E_{\alpha,1}(-\Omega_\alpha(\tau - u)^\alpha) - 1]\\
&\times [E_{\alpha,1}(-\Omega_\alpha(\tau - u)^\alpha)\\
&\left.\left. - 2E_{\alpha,1}(-\Omega_\alpha(t - u)^\alpha) + 1]\right\}\right].
\end{aligned} \tag{20}$$

The first term of the right-hand side of equation (20) gives the condition for the convergence of the first moment in the steady state. Using the asymptotic expansion of the generalized Mittag-Leffler function [35]

$$E_{\alpha,\gamma}(z) \sim -\sum_{n=1}^{\infty} \frac{z^{-n}}{\Gamma[\gamma - n\alpha]}, \quad |\arg(-z)| < \left(1 - \frac{\alpha}{2}\right)\pi, \tag{21}$$

one obtains the following inequality:

$$r_0 > \frac{(r_1\beta_\alpha)^2}{2\Omega_\alpha^2}. \tag{22}$$

One can see that the condition (22) does not depend on the parameter α explicitly.

In order to analyze the first moment equation (20) (without the first term) is rewritten as follows:

$$\langle x(t)\rangle = A(t)\sin(\omega t - \phi(t)), \tag{23}$$

where

$$\begin{aligned}
A(t) &= A_0 \exp\left[\frac{r_1^2\beta_\alpha^2}{2\Omega_\alpha^2}\int_0^t du[E_{\alpha,1}(-\Omega_\alpha(t - u)^\alpha) - 1]^2\right]\\
&\times \sqrt{B^2(t) + C^2(t)},
\end{aligned} \tag{24}$$

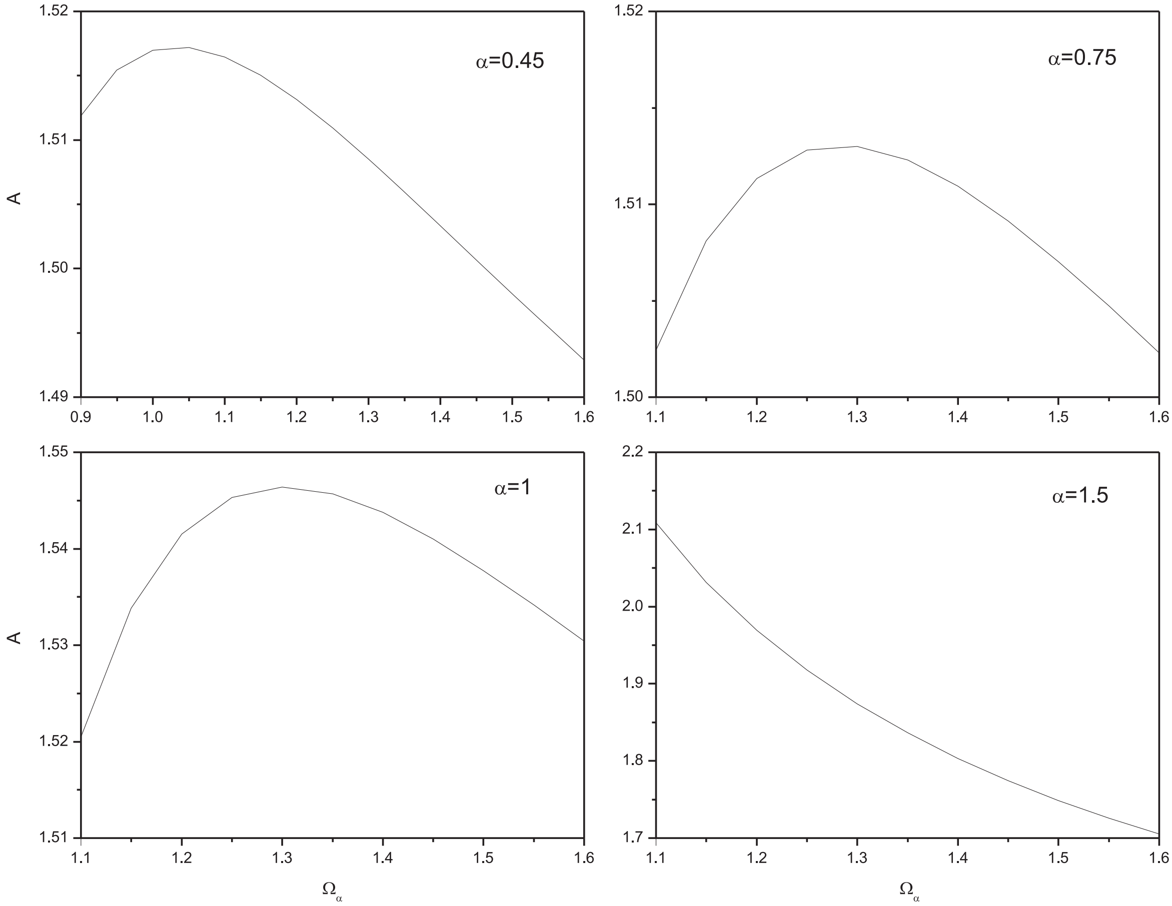

Figure 3. Plots of the amplitude A (24) in the stationary state versus the parameter of the FO noise Ω_α for $A_0 = 1$, $r_0 = 0.5$, $r_1 = 1.8$, $\beta_\alpha = 0.5$ and $\omega = 0.5$, in arbitrary units.

$$B(t) = \int_0^t \mathrm{d}z \cos(\omega z) \mathrm{e}^{-r_0 z} \rho(t, z), \tag{25}$$

$$C(t) = \int_0^t \mathrm{d}z \sin(\omega z) \mathrm{e}^{-r_0 z} \rho(t, z), \tag{26}$$

$$\phi(t) = \arctan\left(\frac{C(t)}{B(t)}\right), \tag{27}$$

and

$$\rho(t, z) = \exp\left[\frac{r_1^2 \beta_\alpha^2}{2\Omega_\alpha^2} \int_z^t \mathrm{d}u [E_{\alpha,1}(-\Omega_\alpha (t-u)^\alpha) - 1] \times [E_{\alpha,1}(-\Omega_\alpha (t-u)^\alpha) - 2E_{\alpha,1}(-\Omega_\alpha (t+z-u)^\alpha) + 1]\right]. \tag{28}$$

From equation (23) one can analyze the amplitude of the response of the periodic external force with respect to the parameters of the system. Equations (24) and (27) show that the amplitude $A(t)$ and the phase $\phi(t)$ depend on time, however they converge to the stationary state values in the long-time limit $(t \to \infty)$ which can be shown numerically.

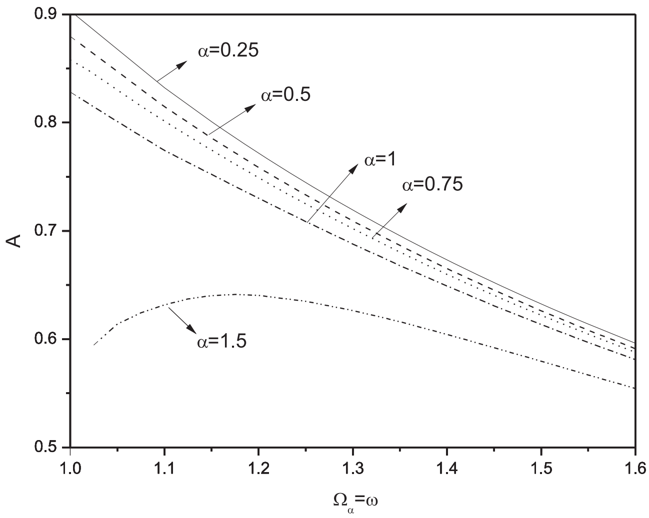

Figure 4. Plots of the amplitude A (24) in the stationary state versus the parameter of the FO noise $\Omega_\alpha = \omega$ for $A_0 = 1$, $r_0 = 0.5$, $r_1 = 1.8$ and $\beta_\alpha = 0.5$, in arbitrary units.

Figure 2 shows the phase $tg\phi(t)$ versus time for different values of α. It shows that the phase converges to a fixed value in the long-time limit. The result for $\alpha = 1$ has also been compared with that one given in [4, 5] and it is in excellent agreement.

Figure 3 shows the stationary state of the amplitude $A(t)$ versus the parameter Ω_α for different values of α. The result for $\alpha = 1$ has also been compared with that one given in [4, 5] and it is in excellent agreement. The curves show non-monotonic decays for $0 < \alpha \leqslant 1$ which are related to resonance effect and monotonic decay for $1 < \alpha < 2$. It means that the effect of the resonance appears for $0 < \alpha \leqslant 1$ which is related to the monotonic behavior of the correlation function of the noise.

Figure 4 shows the stationary state of the amplitude $A(t)$ versus the parameter Ω_α for different values of α. Moreover, the parameter ω has the same variation of the parameter Ω_α. The result for $\alpha = 1$ has also been compared with that one given in [4, 5] and it is in excellent agreement. The behaviors described in figure 4 are different from those given in figure 3; for $0 < \alpha \leqslant 1$ the amplitude decays monotonically, whereas for $1 < \alpha < 2$ it shows non-monotonic decay.

4. Conclusion

In this work the overdamped linear oscillator with periodic external force driven by the FO noise (1) has been analyzed. Analytical result for the first moment has been obtained in terms of the response of the periodic external force, equation (23). Equations (24) and (27) show that the amplitude $A(t)$ and the phase $\phi(t)$ depend on time, however they converge to the stationary state values in the long-time limit ($t \to \infty$) which have been shown numerically (figures 2–4). Figures 3 and 4 show different behaviors for different values of α and they can describe non-monotonic decays. Figure 3 shows a usual plot of the amplitude in terms of a parameter of the system, whereas in figure 4 the amplitude is plotted in terms of the same variations of two parameters of the system. In a wider sense, it is hoped that the FO noise may be useful to describe systems which demand colored noises and it may also be employed to improve the descriptions of the systems which use the exponential and power-law correlation functions and harmonic noise such as nonlinear stochastic systems [36].

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