

# On Lagrangian and non-Lagrangian conformal-invariant nonlinear electrodynamics

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## Abstract

A general approach is presented to describe nonlinear classical Maxwell electrodynamics with conformal symmetry in flat spacetime. We introduce generalized nonlinear constitutive equations, expressed in terms of constitutive tensors dependent on conformal-invariant functions of the field strengths. This allows a characterization of Lagrangian and non-Lagrangian theories. We obtain a general formula for possible Lagrangian densities in nonlinear conformal-invariant electrodynamics. This generalizes the standard Lagrangian of classical linear electrodynamics so as to preserve the conformal symmetry.

Keywords: nonlinear Maxwell equations, conformal symmetry, Lagrangian theories

(Some figures may appear in colour only in the online journal)

## 1. Introduction

The conformal invariance of Maxwell's equations in Minkowski spacetime  $M^{(4)}$  is well known. Kastrup [1] provides a historical review of conformal symmetry in geometry and physics. Our purpose in the present article is to develop *nonlinear* electrodynamics with conformal symmetry, both Lagrangian and non-Lagrangian.

In earlier work [2, 3], two of us considered general nonlinear Maxwell and Yang–Mills equations in  $M^{(4)}$ , satisfying Lorentz symmetry. The relation between the field strength tensor  $F$  and the displacement tensor  $G$  was provided by nonlinear constitutive equations, which depend only on Lorentz-invariant functions of the field strengths. Broadly speaking, such nonlinear (classical) Maxwell equations may describe electromagnetic fields in matter (e.g. where the electric permittivity  $\epsilon$  and/or the magnetic permeability  $\mu$  may depend on the field strengths), or may provide a phenomenological description of possible fundamental properties

of spacetime (e.g. as described by Born–Infeld or Euler–Kockel Lagrangians). We obtained an explicit condition distinguishing Lagrangian from non-Lagrangian theories.

This approach was subsequently generalized to include supersymmetric classical electrodynamics [4] and found an application to the null-string limit of Born–Infeld theory [5].

Here we focus on the behavior of the Maxwell fields and spacetime coordinates under conformal inversion in flat spacetime. First we review briefly nonlinear Maxwell theories in  $M^{(4)}$  with Lorentz symmetry. We write down the symmetry of Maxwell's equations under conformal inversion, and discuss conformal invariant (or pseudoinvariant) functions of the field strengths in  $M^{(4)}$ .

Then we again introduce generalized nonlinear constitutive equations, expressed in terms of constitutive tensors dependent on conformal-invariant functions. This allows characterization of Lagrangian and non-Lagrangian theories, and leads to a general formula for possible Lagrangian densities in nonlinear conformal-invariant electrodynamics. Our results generalize the standard Lagrangian of classical linear electrodynamics subject to the preservation of conformal

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symmetry. The approach introduced here differs from other work on conformal invariant nonlinear electrodynamics, e.g. [6, 7] and references therein.

## 2. Conformal symmetry of Maxwell's equations

Recall that the conformal group for the Minkowski (3 + 1)-dimensional spacetime  $M^{(4)}$  (described by coordinates  $x^\mu$ ,  $\mu = 0, 1, 2, 3$ ) includes spacetime translations, spatial rotations, Lorentz boosts, and dilations, together with the special conformal transformations [8, 9]. The latter can be obtained as a sequence of transformations: conformal inversion, translation by a 4-vector  $b^\mu$ , and conformal inversion again. The conformal inversion is given by (summing over repeated indices)

$$x'^\mu = \frac{x^\mu}{x_\nu x^\nu}. \tag{1}$$

A more detailed discussion of conformal inversion is provided in [10]. The special conformal transformations are then

$$x'^\mu = \frac{(x^\mu - b^\mu x_\nu x^\nu)}{(1 - 2b_\nu x^\nu + b_\nu b^\nu x_\mu x^\mu)}. \tag{2}$$

Next we write the well-known conformal symmetry of electromagnetism [9], with respect to the transformation (1).

In covariant notation we have the standard electromagnetic tensor fields

$$F_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{c} E_x & \frac{1}{c} E_y & \frac{1}{c} E_z \\ -\frac{1}{c} E_x & 0 & -B_z & B_y \\ -\frac{1}{c} E_y & B_z & 0 & -B_x \\ -\frac{1}{c} E_z & -B_y & B_x & 0 \end{pmatrix},$$

$$G_{\mu\nu} = \begin{pmatrix} 0 & cD_x & cD_y & cD_z \\ -cD_x & 0 & -H_z & H_y \\ -cD_y & H_z & 0 & -H_x \\ -cD_z & -H_y & H_x & 0 \end{pmatrix}; \tag{3}$$

the Hodge dual tensors are  $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$  and  $\tilde{G}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}$ , where  $\epsilon^{\mu\nu\rho\sigma}$  is the totally antisymmetric Levi-Civita tensor with  $\epsilon^{0123} = 1$ . Then Maxwell's equations are

$$\partial_\mu \tilde{F}^{\mu\nu} = 0, \quad \partial_\mu G^{\mu\nu} = j^\nu, \tag{4}$$

where  $j^\mu = (c\rho, \mathbf{j})$  is the 4-current. The first equations in (4) imply that one can set  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ , where  $A_\mu$  is an abelian gauge field; but in general there is no such representation for  $G_{\mu\nu}$ . The field strength tensors  $F_{\mu\nu}$  and  $\tilde{F}^{\mu\nu}$  are physically observable, in that their components can be inferred from measurement of the force  $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$  on a test charge  $q$  moving with velocity  $\mathbf{v}$ . The relation of the tensors  $G_{\mu\nu}$  and  $\tilde{G}^{\mu\nu}$  to observable fields is determined by the properties of the medium.

Under the inversion (1) we obtain the coordinate transformations [11]

$$\partial'_\mu := \frac{\partial}{\partial x'^\mu} = x^2 \partial_\mu - 2x_\mu (x \cdot \partial), \tag{5}$$

$$\square' := \partial'_\mu \partial'^\mu = (x^2)^2 \square - 4x^2 (x \cdot \partial), \tag{6}$$

where  $x^2 = x_\mu x^\mu$  and  $(x \cdot \partial) = x^\alpha \partial_\alpha$ . The transformations of the fields that respect the symmetry are then

$$A'_\mu(x') = x^2 A_\mu(x) - 2x_\mu (x^\alpha A_\alpha(x)), \tag{7}$$

$$F'_{\mu\nu}(x') = (x^2)^2 F_{\mu\nu}(x) - 2x^2 x^\alpha (x_\mu F_{\alpha\nu}(x) + x_\nu F_{\mu\alpha}(x)). \tag{8}$$

Combining this symmetry with that of the Poincaré transformations and dilations, we have the well-known symmetry with respect to the conformal group.

In  $M^{(4)}$  we have two fundamental Poincaré-invariant functions of the field strengths

$$I_1 = \frac{1}{2} F_{\mu\nu}(x) F^{\mu\nu}(x), \quad I_2 = -\frac{c}{4} F_{\mu\nu}(x) \tilde{F}^{\mu\nu}(x). \tag{9}$$

It is known [8], that any other Poincaré-invariant function of the field strengths can be expressed in terms of  $I_1$  and  $I_2$  (see also [12]). Thus, basing our description of relativistic nonlinear electrodynamics on  $I_1$  and  $I_2$  should achieve the maximum level of generality, subject to writing constitutive equations in the form described in the next section. Sometimes  $I_2$  is called a pseudoinvariant, because it changes sign under spatial reflection (parity) [9].

Under conformal inversion,  $I_1$  and  $I_2$  are no longer individually invariant. Rather, they transform by

$$I'_1(x') = \frac{1}{2} F'_{\mu\nu}(x') (F')^{\mu\nu}(x') = (x^2)^4 I_1(x), \tag{10}$$

$$I'_2(x') = -\frac{c}{4} F'_{\mu\nu}(x') (\tilde{F}')^{\mu\nu}(x') = -(x^2)^4 I_2(x). \tag{11}$$

So the ratio  $I_2(x)/I_1(x)$  is a pseudoinvariant under conformal inversion

$$\frac{I'_2(x')}{I'_1(x')} = -\frac{I_2(x)}{I_1(x)}. \tag{12}$$

Because the special conformal transformations involve two inversions, the ratio  $I_2(x)/I_1(x)$  is a true invariant under the special conformal group.

## 3. Maxwell's equations and nonlinear constitutive equations

Let us recall that in SI units Maxwell's equations are

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{div } \mathbf{B} = 0,$$

$$\text{curl } \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}, \quad \text{div } \mathbf{D} = \rho, \tag{13}$$

where  $\mathbf{j}$  and  $\rho$  are current and charge densities [8, 9]. We consider only flat spacetime, so that the metric is given by the Minkowski metric tensor is  $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ ,  $x^\mu = (ct, x^i)$ ,  $\mu, \nu, \dots = 0, 1, 2, 3$ ;  $i = 1, 2, 3$ ; with

$(\partial_\mu) = (\partial/\partial x^\mu) = (c^{-1}\partial/\partial t, \nabla)$ . The Lorentz invariants (9) in terms of  $\mathbf{E}$ ,  $\mathbf{B}$ ,  $\mathbf{D}$ ,  $\mathbf{H}$  are

$$I_1 = \mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2, \quad I_2 = \mathbf{B} \cdot \mathbf{E}. \quad (14)$$

The constitutive equations relating  $\mathbf{E}$ ,  $\mathbf{B}$ ,  $\mathbf{D}$  and  $\mathbf{H}$ , taken together with Maxwell's equations (13), determine the symmetry.

General nonlinear constitutive equations with Poincaré symmetry take the form [2, 13, 14]

$$\begin{aligned} \mathbf{D} &= M(I_1, I_2)\mathbf{B} + \frac{1}{c^2}N(I_1, I_2)\mathbf{E}, \\ \mathbf{H} &= N(I_1, I_2)\mathbf{B} - M(I_1, I_2)\mathbf{E}, \end{aligned} \quad (15)$$

where  $M(I_1, I_2)$  and  $N(I_1, I_2)$  are smooth scalar functions of the invariants (14). The constitutive equations in vacuum are  $\mathbf{D} = \varepsilon_0 \mathbf{E}$  and  $\mathbf{B} = \mu_0 \mathbf{H}$ , where  $\varepsilon_0$  and  $\mu_0$  are, respectively, the permittivity and permeability of empty space, with  $\varepsilon_0 \mu_0 = c^{-2}$ ; (so that  $M = 0$ , and  $N = 1/\mu_0 = \varepsilon_0 c^2$ ).

In covariant form, the constitutive equations (15) become

$$G_{\mu\nu} = N(I_1, I_2)F_{\mu\nu} + cM(I_1, I_2)\tilde{F}_{\mu\nu}. \quad (16)$$

Now, because of (10)–(12), the constitutive functions  $M$  and  $N$  in a system with conformal symmetry can depend *only* on the ratio  $I_2(x)/I_1(x)$ . Then let us write  $M(I_1, I_2) = \mathcal{M}(u)$  and  $N(I_1, I_2) = \mathcal{N}(u)$  [13, 14]. For convenience, we denote

$$I_1 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} \equiv X, \quad I_2 = -\frac{c}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} \equiv Y. \quad (17)$$

Then we rewrite the constitutive equations (15) in the covariant form

$$G_{\mu\nu}^{(\text{conformal})} \equiv G_{\mu\nu} = \mathcal{N}(u)F_{\mu\nu} + c\mathcal{M}(u)\tilde{F}_{\mu\nu}, \quad (18)$$

where

$$u = \frac{Y}{cX} \quad (19)$$

is a dimensionless conformal invariant.

#### 4. Lagrangian and non-Lagrangian theories

When the constitutive equations are nonlinear, one may have either a Lagrangian or a non-Lagrangian theory. In the first case, the equations of motion are derived from a Lagrangian density  $L(X, Y)$ . Then the constitutive equations are

$$G_{\mu\nu} = -\frac{\partial L}{\partial F^{\mu\nu}} = -2\frac{\partial L(X, Y)}{\partial X}F_{\mu\nu} + c\frac{\partial L(X, Y)}{\partial Y}\tilde{F}_{\mu\nu}. \quad (20)$$

The functions  $M$  and  $N$  in (16) may then be written

$$N(X, Y) = \frac{1}{\mu} + N_L(X, Y) = -2\frac{\partial L(X, Y)}{\partial X}, \quad (21)$$

$$M(X, Y) = M_L(X, Y) = \frac{\partial L(X, Y)}{\partial Y}. \quad (22)$$

We singled out the constant  $1/\mu$  in (21) so that the choice  $N_L(X, Y) = M_L(X, Y) = 0$  and  $\mu = \mu_0$  yields the standard Lagrangian of linear electrodynamics in vacuum,  $L_0(X, Y) = -(1/2\mu_0)X$  [9]. Then a necessary and sufficient condition for the theory to be Lagrangian is that the cross-derivatives be equal; i.e.

$$-\frac{1}{2}\frac{\partial N(X, Y)}{\partial Y} = \frac{\partial M(X, Y)}{\partial X}, \quad (23)$$

where both sides must equal  $\partial^2 L(X, Y)/\partial X \partial Y$ . If  $M$  and  $N$  in (16) violate (23), then the theory cannot be Lagrangian [3].

#### 5. Conformal-invariant Lagrangian nonlinear electrodynamics

Now let us consider conformal-invariant nonlinear electrodynamics, with the constitutive equations (18). If the theory is Lagrangian, we use the subscript  $L$  and write  $M_L(X, Y) = \mathcal{M}_L(u)$  and  $N_L(X, Y) = \mathcal{N}_L(u)$  (so that in the Lagrangian case  $\mathcal{N} = [1/\mu] + \mathcal{N}_L$  and  $\mathcal{M} = \mathcal{M}_L$ ), to remind us of the dependence of these functions on the specific choice of Lagrangian  $L$ . Then (23) takes the form

$$\frac{d\mathcal{N}_L(u)}{du} - 2cu\frac{d\mathcal{M}_L(u)}{du} = 0, \quad u = \frac{Y}{cX}. \quad (24)$$

Integrating (24), we can express  $\mathcal{N}_L(u)$  as

$$\mathcal{N}_L(u) = 2c \int u \frac{d\mathcal{M}_L(u)}{du} du = 2cu\mathcal{M}_L(u) - 2c \int \mathcal{M}_L(u) du, \quad (25)$$

with the constant of integration subsumed into the term  $1/\mu$ .

Thus we obtain one form of the general constitutive equations respecting conformal symmetry,

$$\begin{aligned} G_{\mu\nu} &= \left( \frac{1}{\mu} + 2uc\mathcal{M}_L(u) - 2c \int \mathcal{M}_L(u) du \right) F_{\mu\nu} \\ &\quad + c\mathcal{M}_L(u)\tilde{F}_{\mu\nu}. \end{aligned} \quad (26)$$

The constitutive equations as represented by (26) depend on one arbitrary function  $\mathcal{M}_L(u)$ , a function of the ratio of relativistic invariants  $u = Y/cX$ .

There may be a constant term in  $\mathcal{M}_L$ . However, as was earlier remarked in [2], adding a constant  $\kappa$  to  $M$  (i.e. to  $\mathcal{M}_L$ ) does not change the observable physics. Indeed, referring back to (13)–(15), a result of adding  $\kappa$  to  $M$  is to add a term  $\kappa \mathbf{B}$  to  $\mathbf{D}$ . But since  $\text{div } \mathbf{B} = 0$ , the value of  $\rho = \text{div } \mathbf{D}$  is unchanged. Likewise, a term  $-\kappa \mathbf{E}$  is added to  $\mathbf{H}$ . But the resulting term in the equation for  $\mathbf{j}$  is offset by the term that was added to  $\mathbf{D}$ . Hence the system  $\mathbf{E}$ ,  $\mathbf{B}$ ,  $\rho$ ,  $\mathbf{j}$  is unaffected by  $\kappa$ ; but it is these fields which describe all the observable forces produced by and acting on electric charges and currents.

Equivalently to (24), one may write

$$\mathcal{M}_L = \frac{1}{2c} \int \frac{1}{u} \frac{d\mathcal{N}_L}{du} du \quad (27)$$

and (26) becomes

$$G_{\mu\nu} = \left( \frac{1}{\mu} + \mathcal{N}_L(u) \right) F_{\mu\nu} + \left( \frac{1}{2} \int \frac{1}{u} \frac{d\mathcal{N}_L}{du} du \right) \tilde{F}_{\mu\nu}, \quad (28)$$

in which  $\mathcal{N}_L(u)$  is taken to be the arbitrary function of  $u$ .

The general Lagrangian density  $L = L_{\text{nonlin}}^{\text{conform}}(X, Y)$  for conformal-invariant nonlinear electrodynamics can now be written in several equivalent forms; e.g.

$$L = L_0 + cX \int \mathcal{M}_L(u) du = L_0 + Y \left( \frac{1}{u} \int \mathcal{M}_L(u) du \right), \quad (29)$$

or

$$L = L_0 + Y \left( \frac{1}{2c} \int \frac{1}{u^2} \mathcal{N}_L(u) du \right), \quad (30)$$

where  $L_0 = -(1/2\mu) X$  describes standard linear electrodynamics with  $\mu = \mu_0$  and  $G_{\mu\nu} = (1/\mu_0) F_{\mu\nu}$ . As noted after (26), if  $\mathcal{M}_L(u) = \kappa$  (a constant), then

$$L_{\text{lin}} = -\frac{1}{2\mu} X + \kappa Y$$

describes electrodynamics physically equivalent to that described by  $L_0$ . Otherwise, the general conformal-invariant electrodynamics described by (29) or (30) is nonlinear.

The following examples illustrate some of the many possibilities.

**Example 1.** Let  $\mathcal{M}_L(u) = \lambda_1 u$ , where the coefficient  $\lambda_1$  (with the dimensionality of  $\epsilon_0 c$ ) controls the magnitude of the nonlinearity. Then  $\mathcal{N}_L(u) = \lambda_1 c u^2$ , and

$$L = -\frac{1}{2\mu} X + \lambda_1 \frac{Y^2}{2cX}, \quad (31)$$

$$G_{\mu\nu} = \left( \frac{1}{\mu} + \lambda_1 \frac{Y^2}{cX^2} \right) F_{\mu\nu} + \lambda_1 \frac{Y}{X} \tilde{F}_{\mu\nu}. \quad (32)$$

**Example 2.** More generally, let  $\mathcal{M}_L(u) = \lambda_n u^n$ ,  $n \neq -1$  (where again,  $\lambda_n$  has the dimensionality of  $\epsilon_0 c$ ). Then  $\mathcal{N}_L(u) = \lambda_n [2cn/(n+1)] u^{n+1}$ ; and

$$L = -\frac{1}{2\mu} X + \lambda_n \frac{1}{(n+1)} \frac{Y^{n+1}}{c^n X^n}, \quad (33)$$

$$G_{\mu\nu} = \left( \frac{1}{\mu} + \lambda_n \frac{2n}{(n+1)} \frac{Y^{n+1}}{c^n X^{n+1}} \right) F_{\mu\nu} + \lambda_n \frac{Y^n}{c^{n-1} X^n} \tilde{F}_{\mu\nu}. \quad (34)$$

Note that if  $n < 0$ , the model is singular when  $Y = 0$ ; i.e. when  $\mathbf{B} \cdot \mathbf{E} = 0$ .

**Example 3.** Let  $\mathcal{N}_L(u) = \alpha u$  (where  $\alpha$  has the dimensionality of  $1/\mu_0$  or  $\epsilon_0 c^2$ ). Then  $\mathcal{M}_L(u) = (\alpha/2c) \ln |u|$ , and we obtain

$$L = -\frac{1}{2\mu} X + \alpha \frac{Y}{2c} \ln \left| \frac{Y}{cX} \right|, \quad (35)$$

$$G_{\mu\nu} = \left( \frac{1}{\mu} + \alpha \frac{Y}{cX} \right) F_{\mu\nu} + \frac{\alpha}{2} \ln \left| \frac{Y}{cX} \right| \tilde{F}_{\mu\nu}. \quad (36)$$

This model also is singular when  $u = 0$ ; i.e. when  $\mathbf{B} \cdot \mathbf{E} = 0$ .

**Example 4.** Constitutive equations based on more general functions of  $u$  (other than polynomial or logarithmic) are likewise compatible with conformal symmetry. For instance, we may let  $\mathcal{M}_L(u) = \lambda \sin bu$ , where  $b \neq 0$  is an additional dimensionless parameter. Then  $\mathcal{N}_L(u) = 2\lambda c (u \sin bu + b^{-1} \cos bu)$ ; and now

$$L = -\frac{1}{2\mu} X - \lambda \frac{c}{b} X \cos \left( \frac{bY}{cX} \right), \quad (37)$$

$$G_{\mu\nu} = \left[ \frac{1}{\mu} + 2\lambda \left( \frac{Y}{X} \right) \sin \left( \frac{bY}{cX} \right) + \frac{c}{b} \cos \left( \frac{bY}{cX} \right) \right] F_{\mu\nu} + \lambda c \sin \left( \frac{bY}{cX} \right) \tilde{F}_{\mu\nu}. \quad (38)$$

One can obtain similar equations for the examples given by  $\mathcal{M}_L(u) = \lambda \cos bu$ ,  $\mathcal{M}_L(u) = \lambda \sinh bu$ , and  $\mathcal{M}_L(u) = \lambda \cosh bu$ .

## 6. Conclusion

We have described an approach to nonlinear classical Maxwell electrodynamics with conformal symmetry, based on generalized constitutive equations. The latter are expressed in terms of constitutive tensors that depend on conformal-invariant functions of the field strengths. Our general description includes both Lagrangian and non-Lagrangian theories. The latter would of course include nonconservative, or dissipative systems.

A straightforward criterion distinguishes the Lagrangian case, which leads to a general formula for the Lagrangian density. We present several examples, illustrating a variety of possibilities for nonlinear conformal-invariant electrodynamics. These may occur as nonlinear perturbations of the usual, linear Maxwell theory.

By requiring conformal symmetry, we thus distinguish a particular class of nonlinear Lagrangian theories. Some choices may be candidates for the phenomenological description of conservative electrodynamics in the presence of matter, or even for describing possible fundamental properties of spacetime. We remark here that the Born-Infeld and Euler-Kockel Lagrangians are not of the form of (29) or (30), and thus break the conformal symmetry.

Elsewhere we have suggested further generalization based on the well-known identification of conformally compactified Minkowski spacetime with the projective light cone in a  $(4+2)$ -dimensional spacetime  $Y^{(6)}$  [11]. The conformal symmetry is then expressed through (ordinary and hyperbolic) rotations in  $Y^{(6)}$ . Writing the 6-dimensional analog of Maxwell's equations, there are now two independent conformal-invariant functions of the field strengths. Dimensional reduction then permits one to recover additional possibilities in Minkowski spacetime  $M^{(4)}$ . This is a topic of our ongoing research.

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