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Noncommutative dynamical variables in magnetohydrodynamics algebraic structure

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Abstract – Magnetohydrodynamics (MHD) describes the behavior of a charged fluid in a strong magnetic field. One way to analyze noncommutativity in MHD is by considering the result of an eternal magnetic field on noncommutative (NC) photon dynamics. In this paper we have introduced a new MHD Lagrangian and we have obtained the Navier-Stokes MHD equation. We have constructed a NC algebra for the dynamical MHD variables and analyzed the mechanical energy variation rate together with the coupling between the vortex and magnetic field. We have calculated the rate of variation of circulation and analyzed each term. We have seen that these terms are connected to noncommutativity which can act as a source of vorticity.

Introduction. – Magnetohydrodynamics (MHD) is the dynamics of an electrically conducting fluid, fully or partially ionized gas or liquid metal, in low-frequency interaction between electrically conducting fluids and electromagnetic fields. The applications of MHD cover a very wide range of physical objects, from liquid metals to cosmic plasmas. In the field of space plasma [1–3], MHD waves play a very important role. For a long time MHD was the most used theory in the description of the dynamics of variation of astrophysical plasma systems such as the formation of the solar corona [4–6]. We have many examples about the application of MHD theory, such as the study of the linear properties of the fast magnetosonic wave propagating in inhomogeneous plasma [7–9], heating and acceleration of the winds from rotating stars due to the damped fast MHD waves for locally strong magnetic fields in stellar atmospheres [10].

A standard path of analysis of MHD is the idea of an action principle for the theory which can lead us to very interesting results. In [11], Newcomb proposed for the first time the action principle for an ideal MHD, in Euler and Lagrange variables, and then he was followed by other authors [12–18]. The Lagrangian structure depends on the choice of basic fields in each case.

In the last few years, an alternative manner to describe fluid dynamics, which is still an open theoretical problem has been proposed. Some of us have investigated recently this issue, already analyzed in the literature [19,20]. The idea is based on the proposal of a reformulation of the equations of motion. The result was a set of Maxwell-type equations to describe the fluid. Starting from this new Maxwell-type structure for the equations of motion [21–24], some of us have shown that a new Lagrangian formulation for a compressible charged fluid embedded in an electromagnetic field can be obtained and the result is a Maxwell-type action for the fluid, where the basic fields are the velocity, vorticity and the total energy.

This new structure for the equations of motion of a fluid represents a new path in the analysis of the system dynamics for compressible and incompressible fluids, as well as for a charged fluid embedded into an electromagnetic field, with important applications in plasma physics [20].

The concept of a NC space time is not new either, and it was first discussed in a published paper by Snyder in

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1947 [25]. His work was motivated by the need to overcome the divergences found in quantum field theory such as quantum electrodynamics, to mention one example. The noncommuting spatial coordinates [25] have well-known consequences in physics. Namely, the quantized movement of particles embedded in a strong magnetic field, so that the projection on the lowest Landau level can be explained, is depicted by noncommuting coordinates on the plane perpendicular to the field [26]. A few years back, this NC phenomenon has been used in several quantum mechanical investigations, involving both theoretical models [27] and phenomenological applications [28]. Simultaneously, we can consider that generalizations to quantum field theory have also been constructed, originating various NC field theories, for example, NC quantum electrodynamics [29].

Since the construction of a NC space-time had its inspiration in quantum phase space, in few words we can say that the coordinate operators $\hat{x}^i$ satisfy a kind of uncertainty relation $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$. The objective was to use a space time coordinate system with a NC structure in small scales and to introduce a cutoff in the ultraviolet regime. However, shortly after Snyder's paper, Yang [30] demonstrated that, although using this NC algebra, the QFT divergences are still there, causing work on noncommutativity to be abandoned for some time. Then, 50 years later, Seiberg and Witten [31] demonstrated that the algebra resulting from a string model embedded into a magnetic field is a NC algebra.

The idea of introducing noncommutativity in MHD is not new. In [32], the authors used the Moyal product to analyze noncommutativity in electrodynamics to construct the theory that describes a charged fluid in a strong magnetic field. This result leads the fluid particles into their lowest Landau level. The noncommutativity appears in the charged fluid density, which does not commute with itself. However, in this paper we extended this new structure to MHD, where we have a low-frequency interaction between electrically conducting fluid, the electromagnetic field and infinite conductivity. In this paper we have proposed a MHD Lagrangian and we developed a noncommutative (NC) generalization of the theory. The NC variables were introduced in the theory from the canonical Poisson brackets. We will see that the NC algebra will be accomplished in the velocities phase space. The motivation for introducing a velocities NC algebra in fluid dynamics is to obtain its effects in the same way they were obtained in classical mechanics. In classical mechanics a symplectic structure is assumed for the phase space. It is consistent with the commutative algebra of NC quantum mechanics. Hence, after Jackiw *et al.* [33] introduced NC fluid dynamics and its effects as an extension of MHD relevant to quark gluon plasma, we asked about the consequences of introducing NC fluid velocities. This paper is motivated by this question.

We have followed here an organization of the subjects such that, in the next section, we have provided the reader with a very brief review of MHD. In the third section, we proposed a new MHD Lagrangian and its NC version. The circulation analysis is given in the fourth section. The conclusions were described in the last section.

Basic review of MHD. – MHD is concerned with the low-frequency interactions between electrically conducting fluids and electromagnetic fields. At low frequencies, the Maxwell displacement current is usually neglected. The non relativistic mechanical motion is described in terms of a single conducting fluid with the usual hydrodynamics variables of density, velocity and pressure with electromagnetic forces. This combined system of equations describes the MHD formalism [1,2].

Let us consider the idealization of an incompressible, "perfectly conducting" fluid in the absence of gravity, but in an external magnetic field. By "perfectly conducting" we mean that the fluid has an infinite conductivity, $\bar{\sigma} \to \infty$. So, the basic equations of MHD with the dissipative processes are

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}), \tag{1}$$

and

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla \left(p + \frac{B^2}{8\pi} \right) + \frac{1}{4\pi\rho} (\vec{B} \cdot \nabla) \vec{B} + \nu \nabla^2 \vec{v}, \tag{2}$$

or

$$\frac{\partial \vec{v}}{\partial t} + \vec{l} = -\nabla \left(\frac{1}{\rho} p + \frac{1}{2} v^2 + \frac{B^2}{8\pi\rho} \right) + \frac{1}{4\pi\rho} (\vec{B} \cdot \nabla) \vec{B} + \nu \nabla^2 \vec{v}, \tag{3}$$

where $\vec{l} = \vec{\omega} \times \vec{v}$ is the Lamb vector, p is the pressure and $\nu = \eta/\rho$ is the kinematic viscosity. Equation (2) shows that the magnetic force is equivalent to a "magnetic hydrostatic pressure" $B^2/8\pi$, which is called the "magnetic pressure."

The infinite conductivity implies that

$$\vec{E} + \vec{v} \times \vec{B} = 0, \tag{4}$$

which can be used to eliminate $\vec{E}$ in Faraday's equations

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} \tag{5}$$

yielding eq. (1). This nonlinear coupling between fluid velocity and magnetic field leads to very interesting dynamic effects, in addition to the purely hydrodynamic nonlinearity $\vec{v} \cdot \nabla \vec{v}$.

The action principle for MHD. – Recently [21–24], some of us have shown that a Lagrangian formulation for a compressible charged fluid embedded in an electromagnetic field can be obtained and the result is a Maxwell-type action for the fluid, given by

$$\mathcal{L} = \frac{1}{2} \left(-\frac{\partial \vec{v}}{\partial t} - \nabla \Omega + \vec{k} + \frac{e}{m} \vec{E} \right)^2 - \frac{1}{2} \left(\nabla \times \vec{v} + \frac{e}{m} \vec{B} \right)^2, \tag{6}$$

where e is the charge and m is the mass of the charge, $\Omega = h + \frac{1}{2}v^2$ is the total energy, h is the enthalpy per mass unit. The vector $\vec{k} = T\nabla S + \frac{1}{\rho}\nabla\sigma$ means the contribution due to both viscosity and statistical features; here T is the temperature, S is the entropy per unit mass, and

$$\sigma_{ij} = \eta\epsilon_{ij} + \xi_{ij}D, \tag{7}$$

where $D = \partial_i v_i$, is the viscosity stress, η and ξ are the well-known coefficients of viscosity [34].

In the MHD approximation, considering the incompressible fluid, we can rewrite the Lagrangian in eq. (6), using the condition in eq. (4), as

$$\begin{aligned}\mathcal{L}_{MHD} = &\frac{1}{2}\left[-\frac{\partial\vec{v}}{\partial t} - \nabla\left(\frac{1}{\rho}p + \frac{1}{2}v^2\right) + \nu\nabla^2\vec{v} - \frac{e}{m}\vec{v}\times\vec{B}\right]^2 \\ &-\frac{1}{2}\left(\nabla\times\vec{v} + \frac{e}{m}\vec{B}\right)^2,\end{aligned} \tag{8}$$

in the incompressible fluid $\nabla\sigma = \nu\nabla^2\vec{v}$. From the Lagrangian in eq. (6) we can write the equations of motion for the electromagnetic fields, so we have that

$$\begin{aligned}\nabla\cdot\vec{E} + \frac{e}{m}\nabla\cdot\vec{l} = 0 &\Longrightarrow \nabla\cdot\tilde{\vec{E}} = 0, \\ \nabla\times\vec{B} - \frac{\partial\vec{E}}{\partial t} &= \frac{m}{e}\vec{J}_l,\end{aligned} \tag{9}$$

where $\tilde{\vec{E}} = \vec{E} + \frac{m}{e}\vec{l}$ and $\vec{J}_l = \frac{\partial\vec{l}}{\partial t} - \nabla\times\vec{\omega}$. Equations (9) are the MHD equations, namely, they are the MHD modified Maxwell equations. The other two are the remaining Maxwell equations that were not affected by the MHD terms.

Moreover, for the low frequency, where the displacement current is neglected, the Ampère law is

$$\vec{j} = \rho_\epsilon\vec{v} = \frac{1}{4\pi}\nabla\times\vec{B}, \tag{10}$$

where ρ_ϵ is the charge density. So, we have from eq. (8) that

$$\begin{aligned}\mathcal{L}_{MHD} = &\frac{1}{2}\left(-\frac{\partial\vec{v}}{\partial t} - \nabla\phi + \nu\nabla^2\vec{v} + \frac{1}{4\pi\rho}\vec{B}\cdot\nabla\vec{B}\right)^2 \\ &-\frac{1}{2}\left(\nabla\times\vec{v} + \frac{e}{m}\vec{B}\right)^2,\end{aligned} \tag{11}$$

where

$$\phi = \frac{1}{\rho}p + \frac{1}{2}v^2 + \frac{B^2}{8\pi\rho} \tag{12}$$

and using the identity

$$\vec{B}\times\nabla\times\vec{B} = \frac{1}{2}\nabla(\vec{B}\cdot\vec{B}) - \vec{B}\cdot\nabla\vec{B}.$$

An interesting consequence of the formalism, eq. (11), concerning MHD, is given by the conjugated momenta associated with the velocity

$$\vec{\pi} = \frac{\delta\mathcal{L}_{MHD}}{\delta\dot{\vec{v}}} = -\frac{\partial\vec{v}}{\partial t} - \nabla\phi + \nu\nabla^2\vec{v} + \frac{1}{4\pi\rho}\vec{B}\cdot\nabla\vec{B} = -\vec{l} \tag{13}$$

when it is compared with eq. (3). Thus, we have obtained the Navier-Stokes equation (2) for the MHD.

Now, from the Legendre transformation $\mathcal{H} = \vec{\pi}\cdot\dot{\vec{v}} - \mathcal{L}$ and eq. (13), we obtain the Hamiltonian density

$$\begin{aligned}\mathcal{H} = &\frac{1}{2}\pi^2 - \vec{\pi}\cdot\nabla\phi + \nu\,\vec{\pi}\cdot\nabla^2\vec{v} + \frac{1}{4\pi\rho}\,\vec{\pi}\cdot(\vec{B}\cdot\nabla)\vec{B} \\ &+\frac{1}{2}\Big(\nabla\times\vec{v} + \frac{e}{m}\vec{B}\Big)^2.\end{aligned} \tag{14}$$

In this Hamiltonian formulation, the canonical Poisson Bracket structure is given by

$$\{v_i(\vec{x}), v_j(\vec{y})\} = 0 \quad \text{and} \quad \{v_i(\vec{x}), \pi_j(\vec{y})\} = \delta_{ij}\delta^3(\vec{x} - \vec{y}). \tag{15}$$

From the Hamiltonian density and the Poisson bracket structure we obtain that

$$\begin{aligned}\dot{\vec{v}}(\vec{x}) &= \int \mathrm{d}^3\vec{y}\{\vec{v}(\vec{x}), \mathcal{H}(\vec{y})\} \\ &= \vec{\pi} - \nabla\phi - \nu\nabla^2\vec{v} + \frac{1}{4\pi\rho}(\vec{B}\cdot\nabla)\vec{B}\end{aligned} \tag{16}$$

or

$$\vec{\pi} = \frac{\partial\vec{v}}{\partial t} + \nabla\phi - \nu\nabla^2\vec{v} - \frac{1}{4\pi\rho}(\vec{B}\cdot\nabla)\vec{B} = -\vec{l}, \tag{17}$$

which is the Navier-Stokes equation written in eq. (2). And the dynamical equation for $\vec{\pi}$ is given by

$$\begin{aligned}\dot{\vec{\pi}}(\vec{x}) &= \int \mathrm{d}^3\vec{y}\{\vec{\pi}(\vec{x}), \mathcal{H}(\vec{y})\} \\ &= -\nu\nabla^2\vec{\pi} + \nabla\times\vec{\omega} + \frac{e}{m}\nabla\times\vec{B}\end{aligned} \tag{18}$$

which shows the dependence on both the magnetic field and the vorticity.

From eq. (2), a useful equation related to the conservation of kinetic energy can be obtained. Taking the scalar product of $\vec{v}$ with the Navier-Stokes equation in eq. (2)

$$\begin{aligned}&\vec{v}\cdot\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\Big[(\vec{v}\cdot\nabla)\vec{v}\Big] + \frac{1}{\rho}\vec{v}\cdot\nabla P - \nu\vec{v}\cdot\nabla^2\vec{v} \\ &-\frac{1}{4\pi\rho}\vec{v}\cdot[(\nabla\times\vec{B})\times\vec{B}] = 0;\end{aligned} \tag{19}$$

using that $\nabla\cdot\vec{v} = 0$, the incompressible fluid condition, we have that

$$\begin{aligned}&\frac{1}{2}\frac{\partial v^2}{\partial t} + \frac{1}{2}(\vec{v}\cdot\nabla)v^2 + \frac{1}{\rho}\nabla\cdot(P\vec{v}) - \eta\nabla\cdot[\vec{v}\cdot(\nabla\vec{v})] \\ &+\eta(\nabla\vec{v})^2 - \frac{1}{4\pi\rho}\vec{v}\cdot[(\nabla\times\vec{B})\times\vec{B}] = 0\end{aligned} \tag{20}$$

or

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \nabla\cdot\left(\rho\vec{v}\frac{v^2}{2} + P\vec{v} - \nu\vec{v}\cdot(\nabla\vec{v})\right) + \nu(\nabla\vec{v})^2 \\ &-\frac{1}{4\pi\rho}\vec{v}\cdot[(\nabla\times\vec{B})\times\vec{B}] = 0.\end{aligned} \tag{21}$$

The last term can be rewritten as

$$\begin{aligned}&\frac{1}{4\pi\rho}\vec{v}\cdot\left[(\nabla\times\vec{B})\times\vec{B}\right]=\frac{1}{4\pi\rho}(\nabla\times\vec{B})\cdot(\vec{v}\times\vec{B})\\&=\frac{1}{4\pi\rho}(\nabla\times\vec{B})\cdot\vec{E}\\&=\frac{1}{4\pi}\vec{B}\cdot(\nabla\times\vec{B})-\frac{1}{4\pi}\nabla\cdot(\vec{E}\times\vec{B}).\end{aligned}\tag{22}$$

But, from eq. (1), we have that

$$\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}=\frac{1}{2}\frac{\partial B^2}{\partial t}=-\vec{B}\cdot(\nabla\times\vec{E})$$

or

$$\vec{B}\cdot(\nabla\times\vec{E})=\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{1}{2}B^2\right).\tag{23}$$

So, substituting eq. (23) into eq. (22) we have from eq. (21) that

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\frac{1}{8\pi}B^2\right)+\nabla\cdot\left[\vec{v}\left(\frac{1}{2}\rho v^2+P\right)\right.\\&\left.+\frac{\vec{E}\times\vec{B}}{4\pi}-\eta\vec{v}\cdot\left(\nabla\vec{v}\right)\right]=-\eta(\nabla\vec{v})^2.\end{aligned}\tag{24}$$

Note that $\vec{E}\times\vec{B}/4\pi$ is the Poynting vector, namely the electromagnetic energy flux density. The total energy is therefore not conserved because of the right term in eq. (24) which can be interpreted as an energy lost due to viscous dissipation. Note that in contrast to the energy loss due to the term $\eta\vec{v}\cdot(\nabla\vec{v})$ in eq. (24), the energy loss due to $-\eta(\nabla\vec{v})^2$ does not go into a neighboring fluid element, because this term has always the same sign. We will now extend all this structure presented in MHD by introducing the NC algebra.

Noncommutative dynamical variables in the MHD algebra. – We have generalized the canonical formulation of the MHD in the last section. Now let us extend the Poisson brackets for the velocity field such that

$$\{v_i(\vec{x}),v_j(\vec{y})\}=\theta_{ij}\delta^3(\vec{x}-\vec{y}),\tag{25}$$

where θ_{ij} is an anti-symmetric and constant parameter tensor, which transforms the velocity algebra into a NC one. So, by keeping the form of the Hamiltonian in eq. (14) unaltered, we obtain the NC generalized Navier-Stokes equation,

$$\begin{aligned}\dot{v}_i(\vec{x})&=\int\mathrm{d}^3\vec{y}\,\{v_i(\vec{x}),\mathcal{H}(\vec{y})\}\\&=\pi_i-\partial_i\phi+\nu\partial^2v_i+\frac{1}{4\pi\rho}(B_j\partial_j)B_i\\&\quad+\nu\theta_{ij}\partial^2\pi_j-\theta_{ik}\epsilon_{jnk}\epsilon_{jlm}\partial_n\partial_lv_m\\&\quad-\frac{e}{m}\theta_{ik}\epsilon_{jnk}\partial_nB_j+\theta_{ik}v_k\partial_jp_j,\end{aligned}\tag{26}$$

which can be rewritten as

$$\begin{aligned}&\frac{\partial v_i}{\partial t}+\partial_i\phi-\nu\partial^2v_i-\frac{1}{4\pi\rho}(B_j\partial_j)B_i=-l_i-\nu\theta_{ij}\partial^2l_i\\&-\theta_{ik}\epsilon_{jnk}\epsilon_{jlm}\partial_n\partial_lv_m-\frac{e}{m}\theta_{ik}\epsilon_{jnk}\partial_nB_j-\theta_{ik}v_k\partial_jl_j\end{aligned}\tag{27}$$

or

$$\begin{aligned}&\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}=-\frac{1}{\rho}\nabla\left(p+\frac{B^2}{8\pi}\right)+\nu\nabla^2\vec{v}\\&+\frac{1}{4\pi\rho}(\vec{B}\cdot\nabla)\vec{B}+\nu\vec{\theta}\times\nabla^2\vec{l}\\&+\vec{\theta}\times\left[\nabla\times\left(\vec{\omega}+\frac{e}{m}\vec{B}\right)\right]+(\vec{\theta}\times\vec{v})\nabla\cdot\vec{l},\end{aligned}\tag{28}$$

and we can see that when $\theta^{ij}=0$ we recover the canonical equation for MHD in eq. (4). Let us analyze the contribution to both the mechanical energy and the conservation of circulation in the NC space. In the same way we did to obtain eq. (24), taking the scalar product of $\vec{v}$ with eq. (28), we have that

$$\begin{aligned}&v_i\frac{\partial v_i}{\partial t}+v_i(v_j\partial_j)v_i=-\frac{1}{\rho}v_i\partial_ip+\nu v_i\partial^2v_i\\&+\frac{1}{4\pi\rho}v_i[\epsilon_{inm}\epsilon_{mjk}B_n(\partial_jB_k)]+\nu\theta_ijv_i\nabla^2l_j\\&-\theta_{ik}v_i\epsilon_{jnk}\partial_n\left(\omega_j+\frac{e}{m}B_j\right)+\theta_{ik}v_iv_k\partial_jl_j.\end{aligned}\tag{29}$$

Notice that the last term in eq. (29) $\theta_{ik}v_iv_k\partial_jl_j=\vec{\theta}\cdot(\vec{v}\times\vec{v})\nabla\cdot\vec{l}=0$. So, after a few simple calculations and using eqs. (22) and (23), we have that

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\frac{1}{8\pi}B^2\right)+\partial_k\left[v_k\left(\frac{1}{2}\rho v^2+P\right)+\frac{1}{4\pi}\epsilon_{ijk}E_iB_j\right.\\&\left.-\eta v_j\partial_jv_k+\eta\theta_{ij}v_i\partial_kl_j+\rho\theta_{im}\epsilon_{jkm}v_i\left(\omega_j+\frac{e}{m}B_j\right)\right]=\\&-\eta(\partial_kv_i)^2+\eta\theta_{ij}(\partial_kv_i)(\partial_kl_j)+\rho\theta_k(\partial_kv_j)\left(\omega_j+\frac{e}{m}B_j\right).\end{aligned}\tag{30}$$

We can also see that in this case not only the dissipation contributes to the non-conservation of the mechanical energy, we have one term due exclusively to noncommutativity. Making $\eta\to0$ we obtain that

$$\begin{aligned}&\frac{\partial\mathcal{E}}{\partial t}+\partial_k\left[v_k\left(\frac{1}{2}\rho v^2+p\right)+\frac{1}{4\pi}\epsilon_{ijk}E_iB_j\right.\\&\left.+\rho\theta_{im}\epsilon_{jkm}v_i\left(\omega_j+\frac{e}{m}B_j\right)\right]=\\&+\rho\theta_k(\partial_kv_j)\left(\omega_j+\frac{e}{m}B_j\right),\end{aligned}\tag{31}$$

where the energy is

$$\mathcal{E}=\frac{1}{2}\rho v^2+\frac{1}{8\pi}B^2.\tag{32}$$

It is interesting to note that the NC term responsible for the non-conservation of energy in this limit has a typical minimum coupling between both the vortex and magnetic field, as already observed in [22].

The circulation. The scalar functional of considerable importance in the description of vortex flows is the circulation $\mathcal{C}$ around a simple curve γ, defined as the line integral around a closed curve of the velocity field

$$\mathcal{C} = \oint_\gamma \vec{v}\cdot \mathrm{d}\vec{x}, \tag{33}$$

where $\vec{v}$ is the fluid velocity of a small element of a defined curve and $\mathrm{d}\vec{x}$ is the differential length of the mentioned small element. The circulation, associated with a physical quantity, calculated along the loop, may be zero or finite depending on whether this physical quantity is an exact differential or not. For example, if this physical quantity is $T\mathrm{d}S$ (T = temperature; S = entropy), the circulation is generally finite and measures the heat gained in a quasistatic thermodynamic cycle. The vorticity is the circulation per unit area calculated around an infinitesimal loop. On the other hand, we can say that the flux of vorticity is the circulation. The circulation around a closed contour is equal to the total vorticity enclosed within it. We now calculate the rate of change of this circulation as the curve moves with the fluid [35]

$$\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\oint_\gamma \vec{v}\cdot\mathrm{d}\vec{x} = \oint_\gamma \left[\frac{\partial\vec{v}}{\partial t} + (\nabla\times\vec{v})\times\vec{v}\right]\cdot\mathrm{d}\vec{x}. \tag{34}$$

Substituting eq. (28) into eq. (34) we have that the rate of variation of the circulation $\mathcal{C}$ is

$$\begin{aligned}\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} &= -\oint_\gamma \partial_i\mathcal{E}\mathrm{d}x_i + \nu\oint_\gamma \partial^2 v_i\mathrm{d}x_i + \frac{1}{2\pi\rho}\oint_\gamma(\vec{B}\cdot\nabla)B_i\mathrm{d}x_i\\ &\quad -\nu\theta_{ij}\oint_\gamma\partial^2 l_j\mathrm{d}x_i - \theta_{ik}\epsilon_{jnk}\oint_\gamma\partial_n\left(\omega_j+\frac{e}{m}B_j\right)\mathrm{d}x_i\\ &\quad -\theta_{ik}\oint_\gamma v_k\partial_j l_j\mathrm{d}x_i, \end{aligned}\tag{35}$$

where

$$\mathcal{E} = \frac{1}{\rho}P + \frac{1}{2}\rho v^2 + \frac{1}{8\pi}B^2. \tag{36}$$

In vectorial form, we can write eq. (35) as

$$\begin{aligned}\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} &= -\oint_\gamma\nabla\mathcal{E}\cdot\mathrm{d}\vec{x} + \nu\oint_\gamma\nabla^2\vec{v}\cdot\mathrm{d}\vec{x}\\ &\quad+\frac{1}{2\pi\rho}\oint_\gamma(\vec{B}\cdot\nabla)\vec{B}\cdot\mathrm{d}\vec{x} - \nu\oint_\gamma\nabla^2(\vec{l}\times\vec{\theta})\cdot\mathrm{d}\vec{x}\\ &\quad-\oint_\gamma(\nabla\cdot\vec{l})(\vec{v}\times\vec{\theta})\cdot\mathrm{d}\vec{x} - \oint_\gamma(\mathrm{d}\vec{x}\cdot\nabla)\left[\vec{\theta}\cdot\left(\vec{\omega}+\frac{e}{m}\vec{B}\right)\right]\\ &\quad+\oint_\gamma(\vec{\theta}\cdot\nabla)\left[\left(\vec{\omega}+\frac{e}{m}\vec{B}\right)\cdot\mathrm{d}\vec{x}\right],\end{aligned}\tag{37}$$

where we have used that $\theta_{ij} = \epsilon_{ijk}\theta_k$.

We can observe from eq. (37) that the first term of the right-hand side contributes to the rate of change of the circulation if

$$\oint_\gamma\nabla\mathcal{E}\cdot\mathrm{d}\vec{x} = \int_{\partial\gamma}\mathrm{d}\mathcal{E}\neq 0,$$

namely, if the fluid-dynamic force derived from the energy density $\mathcal{E}$ is not an exact differential.

Besides, we have six other contributions concerning the rate of variation of circulation. The second term on the right-hand side of eq. (37) is also a source term for the contribution due to the viscosity of the fluid, and represents a flow through the surface $\partial\gamma$. The third term on the right-hand side of eq. (37), which is a contribution due to the magnetic field, also acts as a source term for the vorticity.

The last four terms of eq. (37) represent the NC contributions for the rate of variation of circulation. Three of them are functions of the Lamb vector, which acts as a vortex force. Considering the second NC term in eq. (37), the one with the divergence of $\vec{l}$ if we consider a Navier-Stokes flow, the Lamb vector divergence is the source term in a Poisson equation for the Bernoulli function, *i.e.*, noncommutativity introduces a turbulent charge density $n(\vec{x},t)(\nabla\cdot\vec{l} = -\nabla^2\Phi = n(\vec{x},t))$, if we consider a high Reynolds number flow. Hence, the Lamb vector divergence term in eq. (37) puts the Bernoulli function into the rate of variation of circulation.

For irrotational flows, where $\vec{\omega} = \vec{l} = 0$, only the NC terms having $\vec{B}$ survive. Namely, noncommutativity introduces no turbulent charge density. Having said that, the terms that are related to NC variables, act like sources that could, in principle, generate vorticity. However, the third NC term, that is negative definite, acts to reduce the rate.

Conclusions. – We can say that there exist two ways of introducing noncommutativity into a theory. One way is to use directly the Moyal product in order to obtain the NC extension of the original theory. In the other way, it is more appropriate to apply it inside the fluid formalism, which means to begin with the Lagrangian fluid model, to introduce the NC space coordinates and, after that, to use the map that acts like a bridge between the Lagrangian framework and an Euler Hamiltonian structure to yield the NC effects into the fluid field theory.

In this paper we have followed the second approach, but instead of NC space coordinates, we described a NC phase space with NC dynamical variables. Namely, we constructed a NC algebra between the velocity and its conjugated momentum and, with this new NC algebra, we have constructed the NC version of the MHD equation. We calculated the NC extension of the rate of mechanical energy where the NC term, that breaks the energy conservation, has a typical minimum coupling between both the vortex and magnetic field. This result motivated us to calculate the rate of variation of the circulation, which describes the vortex flow. The terms due to the noncommutativity have the Lamb vector inside and so, act as a kind of source that could generate vorticity although one has negative sign and reduce the rate of variation of circulation.

On the other hand, in eq. (28), the NC version of the MHD Navier-Stokes equation, *i.e.*, the equation for the acceleration of the flow, we can see clearly that the NC terms accelerate the flow. We can conjecture if these terms act as a vortex force. We can see clearly that when $\theta = 0$, we recover the Navier-Stokes MHD equation.

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