Influence of the Earth's rotation on measurement of gravitational constant G with the time-of-swing method*

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In the measurement of the Newtonian gravitational constant G with the time-of-swing method, the influence of the Earth's rotation has been roughly estimated before, which is far beyond the current experimental precision. Here, we present a more complete theoretical modeling and assessment process. To figure out this effect, we use the relativistic Lagrangian expression to derive the motion equations of the torsion pendulum. With the correlation method and typical parameters, we estimate that the influence of the Earth's rotation on G measurement is far less than 1 ppm, which may need to be considered in the future high-accuracy experiments of determining the gravitational constant G.

Keywords: the Earth's rotation, relativistic Lagrangian, G measurement, time-of-swing method

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1. Introduction

The Newtonian gravitational constant G is one of the most fundamental and universal constants, which is closely related to theoretical physics, astrophysics, and geophysics, while its precision is the lowest so far.[1–5] Although the precision of G has been improved over the past few decades, the values in CODATA 2014 are still in poor agreement because of the extreme weakness and nonshieldability of gravity, which indicates that there may be some systematic errors that have not been discovered or correctly understood in the experiments.[6,7] At present, the best results of G measurement have been given by a group in Huazhong University of Science and Technology (HUST), who reported two independent values of G using torsion pendulum experiments with the time-of-swing method[4,8–10] and the angular acceleration method.[1,11,12] The G values given by these two experiments are 6.674184×10^{-11} $\mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}}$ and 6.674484×10^{-11} $\mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}}$, with relative standard uncertainties of 11.64 ppm and 11.61 ppm, respectively.[13]

The effect of the Earth's rotation[6,14,15] on G measurement has been roughly estimated to be very small before, but it is necessary to conduct a comprehensive and detailed modeling for it, which may be helpful for the G measurement with higher accuracy in the future. In the G measurement with the angular acceleration method, the effect of the Earth's rotation has been fully analyzed, but the analysis of this effect in the time-of-swing method is still missing. This work is to solve this problem. In this paper, we present the derivation of the Lagrangian of the torsion pendulum in the general relativistic frame. Based on this, the motion equations of the torsion pendulum can be obtained. With the correlation method[16–18] and the numerical simulation in MATLAB, we extract the periods of the torsion pendulum before and after adding the perturbation brought by the Earth's rotation, respectively. From the difference of the periods, the effect of the Earth's rotation on measuring G can be obtained. This study shows that the influence of the Earth's rotation mainly contributes to the G measurement by coupling itself with the pendulum motion of the torsion pendulum and if the amplitude of the pendulum motion is controlled at the milliradian level, the Earth's rotation only contributes an uncertainty about 10^{-3} ppm to the G value.

The outline of this paper is as follows. Section 2 briefly introduces the principle of measuring the gravitational constant G with the time-of-swing method. In Section 3, the Lagrangian expression of the torsion pendulum is derived in detail in the relativistic frame. In Section 4, we obtain the motion equations of the torsion pendulum from the Lagrangian, and estimate the influence of the Earth's rotation on G measurement with the correlation method. Finally, the paper is concluded in Section 5.

2. The principle of the time-of-swing method

Let us have a simple review on the principle of the time-of-swing method. The time-of-swing method was proposed by

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Braun in the 1890s and developed by Heyl, Cohen, and Taylor later,[19–22] which has been widely used to measure G now. In this method, a torsion pendulum is suspended by a very thin fiber, and two source masses are placed on opposite sides of the pendulum, as shown in Fig. 1. When the line connecting the source masses is parallel to the pendulum, namely, the near configuration, the attraction of the source masses to the pendulum provides an additional positive restoring torque, so that the total restoring torque increases. This leads to the increase of the torsional oscillation frequency and the decrease of the torsional oscillation period. In contrast, when the line connecting the source masses is vertical to the pendulum, namely, the far configuration, the attraction of the source masses to the pendulum provides an additional negative restoring torque, so that the total restoring torque decreases. This leads to the decrease of the torsional oscillation frequency and the increase of the torsional oscillation period.

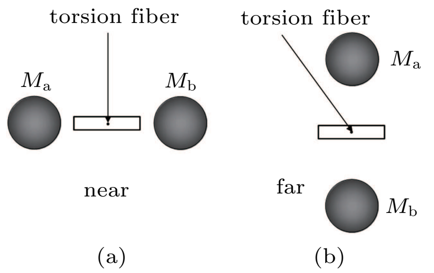

Fig. 1. The (a) near and (b) far configurations of G measurement with the time-of-swing method, where M_a and M_b are the source masses.

In vacuum, when the source masses are placed around, the motion equation of the torsion pendulum can be written as[23,24]

$$I\ddot{\theta}+k\theta+GC_g\theta=0, \tag{1}$$

where I denotes the moment of inertial of the torsion pendulum, θ is the deflection angle, k is the fiber torsion constant, GC_g is the effective gravitational torsion constant, and C_g is the coupling constant determined by the mass distributions and positions of the pendulum and source masses. From Eq. (1), we can obtain the period of the torsion pendulum as

$$\tau=2\pi\sqrt{I/(k+GC_g)}. \tag{2}$$

Based on the two configurations mentioned above, two different periods τ_n and τ_f can be obtained, respectively. Therefore, G can be determined by

$$G=\frac{4\pi^2 I}{C_g}\left(\frac{1}{\tau_n^2}-\frac{1}{\tau_f^2}\right). \tag{3}$$

3. The Lagrangian expression of the torsion pendulum in the general relativistic frame

Here, we consider the influence of the Earth's rotation on G measurement with the time-of-swing method. This section focuses on obtaining the motion equations of the torsion pendulum containing the Earth's rotation. In order to achieve this, the Lagrangian expression of the torsion pendulum should be obtained first. We start from the relativistic expression of the Lagrangian density in any curvilinear coordinate system

$$L=c^2\left(1-\frac{1}{c}\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}}\right), \tag{4}$$

with $g_{\mu\upsilon}$ being the four-dimensional metric tensor and c being the speed of light. In this work, we adopt the following convention: Greek (space–time) indices including μ, υ, α, ... will run from 0 to 3, and small Roman (spatial) indices i, j, k, ... will run from 1 to 3; that is, x^0 is considered as the time coordinate and x^i as the spatial one. In weak field approximation up to the Newtonian order, $g_{\mu\upsilon}$ is usually split into the 3+1 formalism as

$$g_{00}\approx-1+g_{00}^{(2)},\quad g_{0i}\approx g_{0i}^{(1)},\quad g_{ij}\approx\delta_{ij}. \tag{5}$$

By substituting Eq. (5) into Eq. (4), the Lagrangian can be further expressed as

$$L=\frac{\boldsymbol{v}^2}{2}+\frac{1}{2}g_{00}^{(2)}c^2+g_{0i}^{(1)}c\boldsymbol{v}. \tag{6}$$

Usually, the external gravitational field of the Earth described by the Schwarzschild space–time is written in the spherical coordinates as[6]

$$\begin{aligned}\mathrm{d}s^2 =& -\left(1+\frac{2\phi}{c^2}\right)(c\,\mathrm{d}t)^2\\ &+(\mathrm{d}r^2+r^2\,\mathrm{d}\theta^2+r^2\sin^2\theta\,\mathrm{d}\varphi^2),\end{aligned} \tag{7}$$

where $\phi=-GM_e/r$ is the Newtonian gravitation potential with Earth's mass M_e. With the coordinate transformation $\varphi\to\Omega_e t+\varphi$, in which Ω_e is the angular speed of the Earth's rotation, we can directly obtain the metric in the rotating frame as

$$\begin{aligned}\mathrm{d}s^2 =& -\left(1+\frac{2\phi}{c^2}-\frac{\Omega_e^2 r^2\sin^2\theta}{c^2}\right)(c\,\mathrm{d}t)^2\\ &+2\Omega_e r^2\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\varphi\\ &+(\mathrm{d}r^2+r^2\,\mathrm{d}\theta^2+r^2\sin^2\theta\,\mathrm{d}\varphi^2).\end{aligned} \tag{8}$$

Selecting the coordinate origin on the surface of the Earth with the transformation $\boldsymbol{r}\to\boldsymbol{R}_e+\boldsymbol{r}$, we can rewrite the metric of the local laboratory frame in the rectangular coordinates as

$$\begin{aligned}\mathrm{d}s^2 =& -\left(1-2\frac{\boldsymbol{g}\cdot\boldsymbol{r}}{c^2}\right)(c\,\mathrm{d}t)^2+2\boldsymbol{\Omega}_e\times(\boldsymbol{R}_e+\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{r}\,\mathrm{d}t\\ &+(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2).\end{aligned} \tag{9}$$

Here, $\boldsymbol{R}_e$ is the radius of the Earth and $\boldsymbol{g}=-GM_e\boldsymbol{R}_e/R_e^3-\boldsymbol{\Omega}_e\times\boldsymbol{\Omega}_e\times\boldsymbol{R}_e$ is the gravity acceleration on the surface of the Earth. The linear speed of the Earth's rotation is $\boldsymbol{V}_e=$

$\boldsymbol{\Omega}_{\mathrm{e}} \times \boldsymbol{R}_{\mathrm{e}}$. Based on Eqs. (6) and (9), the Lagrangian for a torsion pendulum in the local laboratory frame can be expressed as

$$L_m = \int \left(\frac{\boldsymbol{v}^2}{2} - gz + \boldsymbol{V}_{\mathrm{e}} \cdot \boldsymbol{v} + \boldsymbol{\Omega}_{\mathrm{e}} \cdot \boldsymbol{r} \times \boldsymbol{v} \right) \mathrm{d}m, \tag{10}$$

where $\mathrm{d}m$ denotes the infinitesimal mass and the z direction is assumed to be along the plumb line with $\boldsymbol{g} = -g\boldsymbol{e}_z$. Since the suspension point of the torsion fiber is fixedly connected with the laboratory, the local laboratory frame can actually be regarded as the suspension point frame.

When we only consider the angle deflection θ, the position vector of the infinitesimal mass $\mathrm{d}m$ can be expressed as $(x, y, z) = (L_{\mathrm{p}} \cos\theta, L_{\mathrm{p}} \sin\theta, -l)$ in the suspension point frame, in which L_{p} is the distance between an arbitrary point on the torsion pendulum and the center of the pendulum, and l is the length of the torsion fiber. However, in real experiments, due to the vibrational noise from seismicity or control systems, the torsion pendulum has a pendulum motion in addition to the horizontal rotation. Therefore, the position vector of $\mathrm{d}m$ will change. For simplicity, we set $\delta\varphi_x(t)$ and $\delta\varphi_y(t)$ as the angular displacements of the fiber from the vertical position in y and x directions, respectively. With the coordinate transform shown below

$$\begin{cases} x \to z \sin\delta\varphi_y + x \cos\delta\varphi_y, \\ y \to y \cos\delta\varphi_x - z \sin\delta\varphi_x, \\ z \to y \sin\delta\varphi_x + z \cos\delta\varphi_x + z \cos\delta\varphi_y - x \sin\delta\varphi_y - z, \end{cases} \tag{11}$$

the position vector of $\mathrm{d}m$ turns out to be

$$\begin{aligned} &(L\cos\theta\cos\delta\varphi_y - l\sin\delta\varphi_y, \quad L\sin\theta\cos\delta\varphi_x + l\sin\delta\varphi_x, \\ &L(\sin\theta\sin\delta\varphi_x - \cos\theta\sin\delta\varphi_y) \\ &\quad + l(1 - \cos\delta\varphi_x - \cos\delta\varphi_y)). \end{aligned} \tag{12}$$

By substituting Eq. (12) into Eq. (10), we can obtain a Lagrangian function related to θ, $\delta\varphi_x$, and $\delta\varphi_y$. Here, we set $I_1 = \int l^2 \mathrm{d}m$ as the moment of inertia of the torsion pendulum related to the pendulum motion and $I_2 = \int L_{\mathrm{p}}^2 \mathrm{d}m$ as the moment of inertia of the torsion pendulum related to the rotation motion. Because the torsion fiber goes through the center of the pendulum and the pendulum is symmetric, the relations of $\int \boldsymbol{L}_{\mathrm{p}} \mathrm{d}m = 0$, $\int xy \mathrm{d}m = 0$, $\int zx \mathrm{d}m = 0$, and $\int zy \mathrm{d}m = 0$ are satisfied. As the angular displacements $\delta\varphi_x(t)$ and $\delta\varphi_y(t)$ are very small, the approximation of $\sin\delta\varphi \approx \delta\varphi$ and $\cos\delta\varphi \approx 1$ is reasonable. Therefore, the final simplified Lagrangian in the suspension point frame can be expressed as

$$\begin{aligned} & L(\theta, \delta\varphi_x, \delta\varphi_y) \\ = {} & I_1 \left[\frac{1}{2}(\delta\dot{\varphi}_x^2 + \delta\dot{\varphi}_y^2) + \delta\dot{\varphi}_x \Omega_x + \delta\dot{\varphi}_y \Omega_y \right. \\ & \left. - \delta\dot{\varphi}_x \Omega_z \delta\varphi_y + \delta\dot{\varphi}_y \Omega_z \delta\varphi_x \right] \\ & + I_2 \left[\frac{1}{2}(\delta\dot{\varphi}_x^2 \sin^2\theta + \delta\dot{\varphi}_y^2 \cos^2\theta) \right. \\ & + \frac{1}{2}\dot{\theta}^2 + \dot{\theta}(\delta\dot{\varphi}_x \delta\varphi_y \sin^2\theta - \delta\dot{\varphi}_y \delta\varphi_x \cos^2\theta + \Omega_z \\ & \left. + \Omega_x \delta\varphi_y - \Omega_y \delta\varphi_x) + \delta\dot{\varphi}_x \Omega_x \sin^2\theta + \delta\dot{\varphi}_y \Omega_y \cos^2\theta \right] \\ & + \int (gl) \mathrm{d}m + \int [\delta\dot{\varphi}_x (V_y l + V_z l \delta\varphi_x) \\ & + \delta\dot{\varphi}_y (-V_x l + V_z l \delta\varphi_y)] \mathrm{d}m, \end{aligned} \tag{13}$$

in which only the first order terms of $\delta\varphi_x(t)$ and $\delta\varphi_y(t)$ are kept.

4. Evaluating the influence of the Earth's rotation

Based on the Lagrangian expression in Eq. (13), the motion equations of torsion pendulum's rotation and pendulum motion can be respectively obtained. For the rotation of the torsion pendulum, the following equation is satisfied:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = -GC_g\theta - k\theta. \tag{14}$$

Substituting Eq. (13) into Eq. (14), we can further obtain

$$\begin{aligned} I_2\ddot{\theta} + k\theta = {} & -GC_g\theta - I_2\dot{\Omega}_z - I_2[\dot{\Omega}_x \delta\varphi_y + \Omega_x \delta\dot{\varphi}_y - \dot{\Omega}_y \delta\varphi_x \\ & - \Omega_y \delta\dot{\varphi}_x - 2\theta(\Omega_x \delta\dot{\varphi}_x - \Omega_y \delta\dot{\varphi}_y)]. \end{aligned} \tag{15}$$

Due to the high stability of the Earth's rotation, Ω changes very little with time and affects the motion of the torsion pendulum mainly by coupling itself with the pendulum motion. Here, we set $\dot{\Omega} = 0$, equation (15) can be rewritten as

$$\begin{aligned} I_2\ddot{\theta} + k\theta = {} & -GC_g\theta - I_2[\Omega_x \delta\dot{\varphi}_y - \Omega_y \delta\dot{\varphi}_x \\ & - 2\theta(\Omega_x \delta\dot{\varphi}_x - \Omega_y \delta\dot{\varphi}_y)]. \end{aligned} \tag{16}$$

For the pendulum motion of the torsion pendulum, it can be decomposed into separate x and y components. For the y component, we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \delta\dot{\varphi}_x} - \frac{\partial L}{\partial \delta\varphi_x} = -I_1\frac{g}{l}\delta\varphi_x, \tag{17}$$

and further derive the motion equation as

$$\begin{aligned} & I_1(\delta\ddot{\varphi}_x - 2\Omega_z \delta\dot{\varphi}_y) + I_2(\delta\ddot{\varphi}_x \sin^2\theta \\ & + \ddot{\theta}\delta\varphi_y \sin^2\theta + \dot{\theta}\delta\dot{\varphi}_x \sin 2\theta + \dot{\theta}\delta\dot{\varphi}_y \\ & + \dot{\theta}^2 \delta\varphi_y \sin 2\theta + \dot{\theta}\Omega_x \sin 2\theta + \dot{\theta}\Omega_y) \\ & + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \delta\dot{\varphi}_x}\left\{ \int [\delta\dot{\varphi}_x (V_y l + V_z l \delta\varphi_x)] \mathrm{d}m \right\} \\ & - \frac{\partial}{\partial \delta\varphi_x}\left\{ \int [\delta\dot{\varphi}_x (V_y l + V_z l \delta\varphi_x)] \mathrm{d}m \right\} = -I_1\frac{g}{l}\delta\varphi_x. \end{aligned} \tag{18}$$

Similarly, the motion equation of the x component is

$$\begin{aligned} & I_1(\delta\ddot{\varphi}_y + 2\Omega_z \delta\dot{\varphi}_x) + I_2(\delta\ddot{\varphi}_y \cos^2\theta \\ & - \ddot{\theta}\delta\varphi_x \cos^2\theta - \dot{\theta}\delta\dot{\varphi}_y \sin 2\theta - \dot{\theta}\delta\dot{\varphi}_x \\ & + \dot{\theta}^2 \delta\varphi_x \sin 2\theta - \dot{\theta}\Omega_y \sin 2\theta - \dot{\theta}\Omega_x) \end{aligned}$$

$$+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\delta\dot{\varphi}_y}\left\{\int[\delta\dot{\varphi}_y(-V_x l+V_z l\delta\varphi_y)]\mathrm{d}m\right\}$$
$$-\frac{\partial}{\partial\delta\varphi_y}\left\{\int[\delta\dot{\varphi}_y(-V_x l+V_z l\delta\varphi_y)]\mathrm{d}m\right\}=-I_1\frac{g}{l}\delta\varphi_y. \quad (19)$$

Since the pendulum motion has little effect on the rotation of the torsion pendulum, only the main effects need to be considered, while some small coupling effects can be ignored. Thus, equations (18) and (19) can be approximated as

$$I_1\left(\delta\ddot{\varphi}_x+\frac{g}{l}\delta\varphi_x\right)=0, \quad (20)$$
$$I_1\left(\delta\ddot{\varphi}_y+\frac{g}{l}\delta\varphi_y\right)=0, \quad (21)$$

and the corresponding solutions to Eqs. (20) and (21) can be derived as

$$\delta\varphi_x(t)=A_x\cos(\omega_f t+\varphi_x), \quad (22)$$
$$\delta\varphi_y(t)=A_y\cos(\omega_f t+\varphi_y), \quad (23)$$

where $\omega_f=\sqrt{g/l}\approx\sqrt{9.8/0.9}\approx 3.3\ \mathrm{rad\cdot s^{-1}}$ denotes the oscillation frequency, A_x (A_y) and φ_x (φ_y) are the amplitude and initial phase, respectively.

Substituting Eqs. (22) and (23) into Eq. (16), we can obtain

$$\begin{aligned} & I_2\ddot{\theta}+k\theta+GC_g\theta \\ = & -I_2\{-\Omega_x A_y\omega_f\sin(\omega_f t+\varphi_y)+\Omega_y A_x\omega_f\sin(\omega_f t+\varphi_x) \\ & +2\theta[\Omega_x A_x\omega_f\sin(\omega_f t+\varphi_x)-\Omega_y A_y\omega_f\sin(\omega_f t+\varphi_y)]\}, \end{aligned} \quad (24)$$

and the corresponding general solution is

$$\theta_0(t)=A_0\sin(\omega_0 t+\varphi_0), \quad (25)$$

with the angular frequency $\omega_0=\sqrt{(k+GC_g)/I_2}$. This solution describes the free oscillation of the torsion pendulum. By substituting Eq. (25) into the right side of Eq. (24), it can be rewritten as

$$\begin{aligned} & I_2\ddot{\theta}+k\theta+GC_g\theta \\ = & I_2\omega_f\{[\Omega_x A_y+2\Omega_y A_0 A_y\sin(\omega_0 t+\varphi_0)]\sin(\omega_f t+\varphi_y) \\ & -[\Omega_y A_x+2\Omega_x A_0 A_x\sin(\omega_0 t+\varphi_0)]\sin(\omega_f t+\varphi_x)\}. \end{aligned} \quad (26)$$

To simply make the order estimate, equation (26) can be approximately expressed as

$$\begin{aligned} & I_2\ddot{\theta}+k\theta+GC_g\theta \\ = & I_2\omega_f(\Omega_x A_y+2\Omega_y A_0 A_y)\sin(\omega_f t+\varphi_y). \end{aligned} \quad (27)$$

Assume that the form of the particular solution of Eq. (27) is $\theta_f(t)=A_f\sin(\omega_f t+\varphi_y)$ with

$$A_f=\frac{I_2\omega_f(\Omega_x A_y+2\Omega_y A_0 A_y)}{k+GC_g-I_2\omega_f^2}, \quad (28)$$

which includes the perturbation brought by the Earth's rotation. Eventually, the complete solution of Eq. (24) is approximate to

$$\theta(t)=A_0\sin(\omega_0 t+\varphi_0)+A_f\sin(\omega_f t+\varphi_y). \quad (29)$$

Based on the typical parameters $A_0\sim 3$ mrad, $A_y\sim 1$ mrad, $I_2\sim 4.8\times10^{-5}\ \mathrm{kg\cdot m^2}$, $\Omega_x\approx\Omega_y\sim 7.2\times10^{-5}\ \mathrm{rad\cdot s^{-1}}$, $k\sim 1.1\times10^{-8}\ \mathrm{N\cdot m\cdot rad^{-1}}$, $C_g\sim 0.9\ \mathrm{kg^2\cdot m^{-1}}$, $G\sim 6.674\times10^{-11}\ \mathrm{m^3\cdot kg^{-1}\cdot s^{-2}}$, we can obtain $A_f\approx -2.4\times10^{-8}$ rad.

To extract the effective oscillation frequency of the motion described by Eq. (29), the correlation method[16–18] is usually used, which determines the frequency by comparing the measured signal with a strictly sinusoidal reference signal. Here, we generate two groups of data with sampling time of 840 s by MATLAB. One group is a strictly sinusoidal signal with amplitude A_0 and period $\tau_0=2\pi/\omega_0\approx 420$ s. The other group is based on the first group, and additionally includes a perturbative signal with amplitude A_f and period $\tau_f=2\pi/\omega_f\approx 1.9$ s. With the correlation method, the period difference between the first and second data groups is $\Delta\tau\approx -6.5\times10^{-9}$ s. The contribution to the uncertainty of G measurement with the time-of-swing method can be expressed as

$$\frac{\Delta G}{G}=\sqrt{2}\frac{\Delta\tau}{\tau_n-\tau_f}. \quad (30)$$

Taking $\tau_n-\tau_f\approx 1.5$ s, we can obtain $\Delta G/G\approx -6.1\times10^{-3}$ ppm. This means that if the amplitude of the pendulum motion of the torsion pendulum is controlled at the level of milliradian, the uncertainty brought by the Earth's rotation will be far less than 1 ppm.

5. Summary

At present, the highest precision of G value is given by the angular acceleration method and the time-of-swing method. However, there are still some systematic errors that need detailed modeling and analysis, such as the Earth's rotation effect. For the angular acceleration method, the relevant analysis of the Earth's rotation effect has been completed, but in experiments of the time-of-swing method, we only roughly estimated the magnitude of the effect before. Therefore, we present a more complete analysis and assessment process of this effect here. We derive the motion equations of the torsion pendulum with the Lagrangian expression in the general relativistic frame. After the calculation and simulation, we find that the main effect of the Earth's rotation contributes to G measurement by coupling itself with the pendulum motion of the torsion pendulum. This effect is far less than 1 ppm, as long as we control the amplitude of the pendulum motion at the level of milliradian. The model we put forward is applicable to other similar gravitational experiments with torsion pendulum, in which the influence of the Earth's rotation may need to be carefully considered.

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