IOP Publishing Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **53** (2020) 155203 (13pp) https://doi.org/10.1088/1751-8121/ab7c8c

Real forms of embeddings of maximal reductive subalgebras of the complex simple Lie algebras of rank up to 8

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Received 24 November 2019, revised 2 March 2020
Accepted for publication 4 March 2020
Published 26 March 2020

Abstract

We consider the problem of determining the noncompact real forms of maximal reductive subalgebras of complex simple Lie algebras. We briefly describe two algorithms for this purpose that are taken from the literature. We discuss applications in theoretical physics of these embeddings. The supplementary material to this paper contains the tables of embeddings that we have obtained for all real forms of the semisimple Lie algebras of rank up to 8.

Keywords: real forms, maximal subalgebras, Maxwell–Einstein theory, magic square

S Supplementary material for this article is available online

1. Introduction

The classification of maximal reductive subalgebras of exceptional and classical complex Lie algebras is well known. A systematic study started with the work of Dynkin [Dyn52a, Dyn52b]. Subsequently, it was developed in a number of works, such as [LG72, Min06, dG11]. For what concerns non-compact real forms, the maximal reductive subalgebras of exceptional and classical Lie algebras which give rise to symmetric embeddings are listed e.g. in table 9.7 of [Gil06], which reports results taken from a rather vast literature (see the references therein). On the other hand, the non-compact real forms of maximal, semisimple (reductive) subalgebras of exceptional Lie algebras which give rise to non-symmetric embeddings are studied in a number of papers scattered in works of mathematics and of theoretical physics, which are not

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easily collected; for instance, in the high-energy physics literature (Maxwell–Einstein-scalar theories, possibly endowed with some amount of local supersymmetry), some maximal subalgebras giving rise to non-symmetric embeddings have been considered in [AFMT08, FMT12, FMZ13]. In the mathematical literature Gray [Gra72] classified the real forms of the maximal reductive subalgebras that arise as fixed point subalgebras of automorhisms of order three (overlooking two cases, see section 3.1). Komrakov [Kom90] also classified all real forms of the maximal (non-symmetric) embedding $A_4 \oplus A_4 \subset E_8$, and he also carried out a classification of the maximal S-subalgebras of exceptional real Lie algebras.

Recently algorithms, together with their implementations, have been developed to compute the real forms of embeddings of complex semisimple Lie algebras, [DFdG15, FdG15]. However, a classificatory and exhaustive approach was not pursued in these papers. In the present paper, by developing and systematically applying these techniques we aim at obtaining the complete list of noncompact real forms of embeddings of maximal reductive subalgebras of the simple complex Lie algebras of ranks up to 8.

Some remarks to clarify our aim are in order. Firstly, with a real form of an embedding of complex Lie algebras we mean a pair of real forms $\mathfrak{a}, \mathfrak{g}$ of complex reductive Lie algebras $\mathfrak{a}^c, \mathfrak{g}^c$ such that $\mathfrak{a}^c \subset \mathfrak{g}^c$ and $\mathfrak{a} \subset \mathfrak{g}$. Secondly, it is possible to have a pair of complex semisimple (or reductive) Lie algebras $\mathfrak{a}^c \subset \mathfrak{g}^c$ with real forms $\mathfrak{a}, \mathfrak{g}$ such that $\mathfrak{a}$ is maximal in $\mathfrak{g}$ without $\mathfrak{a}^c$ being maximal in $\mathfrak{g}^c$. In fact, Komrakov [Kom90] has a list of cases where that happens. We do not quite understand this list (for example, the cases appearing under item a) in the cited paper seem to be maximal reductive subalgebras, and also appear in our tables), nor do we have methods to detect those cases independently. Therefore we restrict to listing the real forms of embeddings of the maximal complex reductive subalgebras of complex semisimple Lie algebras.

The second part of the paper has the tables that we obtained containing the maximal reductive subalgebras of the noncompact simple real Lie algebras of rank up to 8. These subalgebras are also contained, in explicit form, in the latest version of the CoReLG package for GAP4 ([DFdG19]).

The remainder of this paper is divided in a few sections. In section 2 we describe the maximal reductive subalgebras of the simple complex Lie algebras. Section 3 gives an overview of some applications of real embeddings of reductive Lie algebras in high-energy theoretical physics. After that we have section 4 giving a brief overview of the computational methods that we used to obtain real forms of complex embeddings. The supplementary material to this paper has the tables of those real forms for the simple Lie algebras of rank up to 8.

2. Maximal reductive subalgebras of complex semisimple Lie algebras

In this section we describe the complex embeddings of which we compute the real forms.

Let $\mathfrak{g}^c$ be a complex simple Lie algebra. After Dynkin [Dyn52b], a subalgebra $\mathfrak{a}^c \subset \mathfrak{g}^c$ is said to be *regular* if there is a Cartan subalgebra $\mathfrak{h}^c$ of $\mathfrak{g}^c$ with $[\mathfrak{h}^c, \mathfrak{a}^c] \subset \mathfrak{a}^c$. Such a subalgebra is spanned by root spaces with respect to $\mathfrak{h}^c$ along with $\mathfrak{h}^c \cap \mathfrak{a}^c$. A subalgebra of $\mathfrak{g}^c$ is called an *R-subalgebra* if it is contained in a proper regular subalgebra of $\mathfrak{g}^c$. A subalgebra that is not an R-subalgebra is called an S-subalgebra.

Dynkin showed that every non-semisimple subalgebra of $\mathfrak{g}^c$ is an R-subalgebra ([Dyn52b], theorem 7.3). Hence S-subalgebras are necessarily semisimple. So the maximal reductive S-subalgebras coincide with the maximal semisimple S-subalgebras. These have been classified by Dynkin [Dyn52a, Dyn52b], and are also contained in the lists obtained in [dG11].

Table 1. Maximal reductive non-semisimple subalgebras of the simple Lie algebras. Here T_1 denotes a 1-dimensional center.

Type	Max reductive	Type	Max reductive
A_n	$A_k + A_{n-1-k} + T_1$	D_n	$A_{n-1} + T_1$
	$1 \leqslant k \leqslant n-1$		$D_{n-1} + T_1$
B_n	$B_{n-1} + T_1$	E_6	$D_5 + T_1$
C_n	$A_{n-1} + T_1$	E_7	$E_6 + T_1$

Now we consider the regular maximal reductive subalgebras. Let Δ be a set of simple roots of the root system of $\mathfrak{g}^c$ with respect to a Cartan subalgebra $\mathfrak{h}^c \subset \mathfrak{g}^c$. Furthermore, let δ denote the highest root of the root system and write

$$\delta = \sum_{\alpha \in \Delta} n_\alpha \alpha$$

where the n_α are positive integers.

Let e_α, f_α (for α in the root system of $\mathfrak{g}^c$) be elements spanning the root spaces corresponding to α and $-\alpha$ respectively. Let h_α be a scalar multiple of $[e_\alpha, f_\alpha]$ such that $[h_\alpha, e_\alpha] = 2e_\alpha$. Let $\alpha \in \Delta$. Then by $\mathfrak{g}^c(\alpha)$ we denote the subalgebra generated by e_β, f_β for $\beta \in \Delta \backslash \{\alpha\}$ along with e_δ, f_δ. By $\mathfrak{g}^c[\alpha]$ we denote the subalgebra generated by e_β, f_β for $\beta \in \Delta \backslash \{\alpha\}$ along with e_α, h_α. Then $\mathfrak{g}^c(\alpha)$ is a maximal subalgebra if and only if n_α is prime (this statement goes back to [BDS49], see also [GG78], chapter 8). The $\mathfrak{g}^c(\alpha)$ are semisimple, so they are maximal reductive subalgebras. On the other hand, a maximal reductive, non semisimple subalgebra is contained in a $\mathfrak{g}^c[\alpha]$. The latter has a unique (up to conjugation) maximal reductive subalgebra, which we denote by $\mathfrak{g}^c[\alpha]'$; it is generated by e_β, f_β for $\beta \in \Delta \backslash \{\alpha\}$ along with h_α. However, it is possible that $\mathfrak{g}^c[\alpha]'$ is contained in a $\mathfrak{g}^c(\beta)$. This situation can be characterized as follows. Consider the Dynkin diagram of the set of roots $\Delta \cup \{-\delta\}$ (this is called the extended Dynkin diagram). Now $\mathfrak{g}^c[\alpha]'$ is contained in a $\mathfrak{g}^c(\beta)$ if and only if the diagram obtained by removing the node corresponding to α is not the Dynkin diagram of $\mathfrak{g}^c$. (Indeed, if the latter diagram is not the Dynkin diagram of $\mathfrak{g}^c$ then $\mathfrak{g}^c(\alpha)$ is not equal to $\mathfrak{g}^c$ and $\mathfrak{g}^c[\alpha]' \subset \mathfrak{g}^c(\alpha)$. Conversely, if the mentioned diagram is the Dynkin diagram of $\mathfrak{g}^c$ then none of the $\mathfrak{g}(\beta)$ contain the semisimple part of $\mathfrak{g}^c[\alpha]'$.) By inspection this then leads to the maximal reductive R-subalgebras listed in table 1.

Remark 2.1. The subalgebras of table 1 are also given by Dynkin in [Dyn52b], table 12a. This table lists the regular semisimple subalgebras $\mathfrak{a}^c \subset \mathfrak{g}^c$ such that no semisimple regular subalgebra $\hat{\mathfrak{a}}^c$ exists with $\mathfrak{a}^c \subsetneq \hat{\mathfrak{a}}^c \subsetneq \mathfrak{g}^c$. However, in one case this appears to be not quite correct. In fact, for the embeddings $B_{n-1} \subset B_n$ we have the chain

$$B_{n-1} \subset D_n \subset B_n.$$

Here both subalgebras B_{n-1} and D_n are regular, but they are not normalized by the same Cartan subalgebra.

We also have the dual chain $D_{n-1} \subset B_{n-1} \subset D_n$. But here, obviously, B_{n-1} is not regular in D_n.

Remark 2.2. Here we point out two oversights that are present in some places in the literature. Firstly, in various works (such as [MP81, Sla81, Yam15]), the subalgebra $A_1 \oplus A_1 \oplus A_1$ is reported to be maximal in D_6, while actually it is not. We have that $A_1 \oplus A_1 \oplus A_1$ is a maximal (non-symmetric) subalgebra in $A_1 \oplus C_3$, which in turn is maximal (and non-symmetric) in D_6. This has been noted in the tables in [LG72].

Secondly, the maximal S-subalgebra C_3 in C_7 (giving rise to a non-symmetric embedding) is not listed in some works (such as [MP81, Sla81, Yam15]), whereas it is considered in [Dyn52a], [LG72], table 7. All in all, the existence of a maximal C_3 in C_7 is a consequence of the anti-self-conjugation (i.e., symplecticity) of the irreducible representation $\mathbf{14}' = \wedge_0^3 \mathbf{6}$ of C_3. It is here worth recalling that the action of C_3 on the $14'$ is 'of type E_7' [Bro69]. In [Kac80] it has been proved that such action has a finite number of nilpotent orbits, with one-dimensional ring of invariant polynomials generated by a quartic homogeneous polynomial. The latter is related to the square of the Bekenstein–Hawking entropy of extremal black hole solutions to the 'magic' Maxwell–Einstein $\mathcal{N} = 2$ supergravity having the split real form $\mathfrak{sp}(6, \mathbb{R})$ of C_3 as electric-magnetic duality symmetry [GST83] (see [BFKM08] for a review and a list of references).

3. Physical applications of real embeddings of reductive Lie algebras

3.1. Super-Ehlers embeddings, and their non-supersymmetric versions

The non-compact real form $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R}) \subset E_{8(8)}$ of the maximal (non-symmetric) embedding $A_4 \oplus A_4 \subset E_8$ is known in supergravity as an example of 'super-Ehlers' embedding, concerning the maximally supersymmetric Einstein gravity in 7 space-time dimensions. Super-Ehlers embeddings, which unify the Ehlers gravity embeddings with the global electric-magnetic duality symmetries of Einstein–Maxwell theories (at least in the cases with symmetric scalar manifolds), have been introduced and studied, in presence of underlying (local) supersymmetry, in [FMT12]; in particular, in the appendices of [FMT12] a general proof of existence of such regular and rank-preserving embeddings, which are non-symmetric in most cases, is given, within an approach completely different from the ones employed in the present paper. The general structure of super-Ehlers embeddings is the following (with $3 \leqslant D \leqslant 11$ denoting the number of Lorentzian space-time dimensions):

$$\mathfrak{g}_{D,\mathcal{N}} \oplus \mathfrak{sl}(D-2, \mathbb{R}) \subset \mathfrak{g}_{3,\mathcal{N}}; \tag{3.1}$$

where $\mathfrak{g}_{D,\mathcal{N}}$ is the electric-magnetic duality Lie algebra of D-dimensional Maxwell–Einstein theories endowed with $2\mathcal{N}$ local supersymmetries (corresponding, in $D = 3$, to $\mathcal{N}$-extended supergravity), and $\mathfrak{sl}(D-2, \mathbb{R})$ is the Ehlers symmetry Lie algebra in D Lorentzian dimensions. In presence of supersymmetry, super-Ehlers embeddings are listed and classified in [FMT12] (for the $D = 5$ case, see also [FMZ13]). Note that

$$E_{6(6)} \oplus \mathfrak{sl}(3, \mathbb{R}) \subset E_{8(8)},$$
$$E_{6(-26)} \oplus \mathfrak{sl}(3, \mathbb{R}) \subset E_{8(-24)}$$

are super-Ehlers embeddings for $\mathcal{N} = 16$ and $\mathcal{N} = 4$ supergravity theories in $D = 5$, respectively. (These two embeddings have been overlooked in [Gra72].) On the other hand, non-supersymmetric Maxwell–Einstein theories (coupled to non-linear sigma model of scalar fields) in various dimensions are not considered in [FMT12], but nevertheless they display some 'non-supersymmetric Ehlers embeddings'; whose some examples list as follows:

$$\mathfrak{so}(6, 6) \oplus \mathfrak{sl}(2, \mathbb{R}) \subset E_{7(7)}; \tag{3.2}$$

$$\mathfrak{sl}(6, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R}) \subset E_{7(7)}; \tag{3.3}$$

$$\mathfrak{sl}(6,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \subset E_{6(6)}; \tag{3.4}$$

$$\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R}) \subset E_{6(6)}. \tag{3.5}$$

These embeddings have been discussed in the so-called non-supersymmetric magic Maxwell–Einstein theories in [MPRR17] (see also [MR19]), in which the role of the Ehlers symmetry and related truncations has been highlighted.

3.2. Cubic Jordan algebras, their symmetries and related embeddings

Various embeddings reported in the tables of the present paper have an interpretation in terms of symmetries of cubic Jordan algebras. Such symmetries are given by the $\mathfrak{der}$ (derivations), $\mathfrak{str}_0$ (reduced structure), $\mathfrak{conf}$ (conformal) and $\mathfrak{qconf}$ (quasi-conformal) Lie algebras associated to a given cubic Jordan algebra. The quasi-conformal realizations of non-compact groups were discovered by the authors of [GKN01] and were further developed and applied in [GNPW06, GP10, GP09, GNPP08, GNPW07]. The conformal groups associated with Jordan algebras were studied much earlier in [Gun75, Gun93]. Over the reals $\mathbb{R}$, the Lie algebras $\mathfrak{str}_0$, $\mathfrak{conf}$ and $\mathfrak{qconf}$ respectively correspond to the electric-magnetic duality (U-duality[5]) Lie algebras of some Maxwell–Einstein supergravity theories in $D = 5, 4, 3$ Lorentzian space-time dimensions (cfr. e.g. [GP10], and the references therein; in $D = 3$ all vectors need to be dualized into scalars); such symmetries are non-linearly realized on the scalars, while vectors do sit in some linear representations of them. Jordan algebras, such as $J_3^{\mathbb{H}_s}$, $J_3^{\mathbb{C}_s}$ and $\mathbb{R} \oplus \Gamma_{m,n}$ with m (or n) $\neq 1$ and 5, are associated to non-supersymmetric models [MPRR17, MR19]. When considering simple cubic Jordan algebras, all the aforementioned related Lie algebras fill the first ($\mathfrak{der}$), second ($\mathfrak{str}_0$), third ($\mathfrak{conf}$) and fourth ($\mathfrak{qconf}$) rows of the relevant magic square of Freudenthal–Rozenfeld–Tits [Fre63, Tit66, Roz56], and the following maximal embeddings hold (see [CCM15], and references therein):

$$\mathfrak{der} \subset \mathfrak{str}_0; \tag{3.6}$$

$$\mathfrak{der} \oplus \mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{conf}; \tag{3.7}$$

$$\mathfrak{str}_0 \oplus \mathfrak{so}(1,1) \subset \mathfrak{conf}; \tag{3.8}$$

$$\mathfrak{conf} \oplus \mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{qconf}; \tag{3.9}$$

$$\mathfrak{str}_0 \oplus \mathfrak{sl}(3,\mathbb{R}) \subset \mathfrak{qconf}; \tag{3.10}$$

$$\mathfrak{der} \oplus G_{2(2)} \subset \mathfrak{qconf}. \tag{3.11}$$

Within the physical interpretation of such symmetries as U-duality Lie algebras, the commuting $\mathfrak{so}(1,1)$ in (3.8) can be regarded as the Kaluza–Klein compactification radius of the S^1-reduction $D = 5 \to 4$; alternatively, such an $\mathfrak{so}(1,1)$ can also be conceived as the Lie algebra of the pseudo-Kähler connection of the pseudo-special Kähler (and pseudo-Riemannian) symmetric coset[6] $\frac{\mathrm{Conf}}{\mathrm{Str}_0 \otimes SO(1,1)}$, obtained from $\frac{\mathrm{Str}_0}{\mathrm{mcs}(\mathrm{Str}_0)}$ by applying the inverse R^*-map pertaining to a timelike compactification $D = s + t = 4 + 1 \to 4 + 0$ where s and t respectively denote the number of spacelike and timelike dimensions [dWVVP93, AC09, CLL + 98]. On the other hand, the commuting $\mathfrak{sl}(2,\mathbb{R})$ in (3.9) can be identified with the Ehlers

[5] Here U-duality is referred to as the 'continuous' symmetries of [CJ78, CJ79]. Their discrete versions are the U-duality non-perturbative string theory symmetries introduced in [HT95].
[6] As intuitive, the names starting with a capital letter denote the Lie groups whose Lie algebra is the same name starting lowercase. Moreover, 'mcs' denotes the maximal compact subgroup.

Table 2. The *single-split* MS $\mathcal{L}_3(\mathbb{A}_s, \mathbb{B})$ [GST83] (see [CCM15] for details)

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$SO(3)$	$SU(3)$	$USp(3)$	$F_{4(-52)}$
$\mathbb{C}_s$	$SL(3,\mathbb{R})$	$SL(3,\mathbb{C})$	$SL(3,\mathbb{H})$	$E_{6(-26)}$
$\mathbb{H}_s$	$Sp(6,\mathbb{R})$	$SU(3,3)$	$SO^*(12)$	$E_{7(-25)}$
$\mathbb{O}_s$	$F_{4(4)}$	$E_{6(2)}$	$E_{7(-5)}$	$E_{8(-24)}$

Table 3. The *double-split* MS $\mathcal{L}_3(\mathbb{A}_s, \mathbb{B}_s)$ [BS03] (see [CCM15] for further details)

	$\mathbb{R}$	$\mathbb{C}_s$	$\mathbb{H}_s$	$\mathbb{O}_s$
$\mathbb{R}$	$SO(3)$	$SL(3,\mathbb{R})$	$Sp(3,\mathbb{R})$	$F_{4(4)}$
$\mathbb{C}_s$	$SL(3,\mathbb{R})$	$SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$	$SL(6,\mathbb{R})$	$E_{6(6)}$
$\mathbb{H}_s$	$Sp(6,\mathbb{R})$	$SL(6,\mathbb{R})$	$SO(6,6)$	$E_{7(7)}$
$\mathbb{O}_s$	$F_{4(4)}$	$E_{6(6)}$	$E_{7(7)}$	$E_{8(8)}$

symmetry arising from the S^1-reduction $D = 4 \to 3$; such an $\mathfrak{sl}(2,\mathbb{R})$ can also be interpreted as the connection of the para-quaternionic (and pseudo-Riemannian) symmetric coset $\frac{\mathrm{QConf}}{\mathrm{Conf} \otimes SL(2,\mathbb{R})}$, obtained from $\frac{\mathrm{Conf}}{\mathrm{mcs(Conf)}}$ by applying the inverse c^*-map pertaining to a timelike compactification $D = (3,1) \to (3,\ 0)$ [BMG88, CFG89, CLL + 98]. As the embeddings (3.6)–(3.11) are obtained by moving along the columns of the relevant (real form of the) Magic Square (for a fixed row entry), another class of embeddings can be obtained by moving along the rows of the relevant Magic Square (for a fixed column entry). In the symmetric (real forms of the) rank-3 Magic Square, as the double-split $\mathcal{L}_3(\mathbb{A}_s, \mathbb{B}_s)$ given in table 3, these embeddings trivially coincide with (3.6)–(3.11), but their intepretation corresponds to the restriction from one (division $\mathbb{A}$ or split $\mathbb{A}_s$) algebra to a smaller one. For the non-symmetric, single-split Magic Square $\mathcal{L}_3(\mathbb{A}_s, \mathbb{B})$ reported in table 2, different maximal embeddings hold true, which are different from the ones given in (3.6)–(3.11); namely ($G_2^{\mathrm{cpt}} \equiv G_{2(-14)}$):

$$\mathfrak{d}(J_3^{\mathbb{R}}) \subset \mathfrak{d}(J_3^{\mathbb{C}}) \tag{3.12}$$

$$\mathfrak{su}(2) \oplus \mathfrak{d}(J_3^{\mathbb{R}}) \subset \mathfrak{d}(J_3^{\mathbb{H}}) \tag{3.13}$$

$$\mathfrak{u}(1) \oplus \mathfrak{d}(J_3^{\mathbb{C}}) \subset \mathfrak{d}(J_3^{\mathbb{H}}) \tag{3.14}$$

$$\mathfrak{su}(2) \oplus \mathfrak{d}(J_3^{\mathbb{H}}) \subset \mathfrak{d}(J_3^{\mathbb{O}}) \tag{3.15}$$

$$\mathfrak{su}(3) \oplus \mathfrak{d}(J_3^{\mathbb{C}}) \subset \mathfrak{d}(J_3^{\mathbb{O}}) \tag{3.16}$$

$$G_2^{\mathrm{cpt}} \oplus \mathfrak{d}(J_3^{\mathbb{R}}) \subset \mathfrak{d}(J_3^{\mathbb{H}}) \tag{3.17}$$

(where $\mathfrak{d}$ can be each of $\mathfrak{der}$, $\mathfrak{str}_0$, $\mathfrak{conf}$, $\mathfrak{qconf}$).

Similar embeddings holds for simple Lorentzian cubic Jordan algebras (see [CCM15]).

3.3. Semisimple subalgebras of simple Jordan algebras, and their symmetries

Another remarkable class of embedding stems from the relation between simple cubic Jordan algebras [JvNW34] $J_3^{\mathbb{A}}$ or $J_3^{\mathbb{A}_s}$ and some elements of the (bi-parametric) infinite sequence of semi-simple Jordan algebras $\mathbb{R} \oplus \mathbf{\Gamma}_{m,n}$ mentioned above, exploiting the Jordan-algebraic isomorphisms $J_2^{\mathbb{A}} \cong \mathbf{\Gamma}_{1,q+1}\ \left(\cong \mathbf{\Gamma}_{q+1,1}\right)$ and $J_2^{\mathbb{A}_s} \cong \mathbf{\Gamma}_{q+2+1,q/2+1}$, where $q := \dim_R A = 8,4,2,1$ for $\mathbb{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}$, and $q := \dim_R \mathbb{A}_s = 8,4,2$ for $\mathbb{A}_s = \mathbb{O}_s, \mathbb{H}_s, \mathbb{C}_s$ (see e.g. appendix A of [GP10]—and references therein—for an introduction to division and split algebras). Indeed, the following (maximal, rank-preserving) Jordan-algebraic embeddings hold:

$$J_3^{\mathbb{A}} \supset \mathbb{R} \oplus J_2^{\mathbb{A}} \cong \mathbb{R} \oplus \mathbf{\Gamma}_{1,q+1}; \tag{3.18}$$

$$J_3^{\mathbb{A}_s} \supset \mathbb{R} \oplus J_2^{\mathbb{A}_s} \cong \mathbb{R} \oplus \mathbf{\Gamma}_{q/2+1,q/2+1}. \tag{3.19}$$

Thus, one can consider their consequences at the level of symmetries of cubic Jordan algebras defined over the corresponding algebras, obtaining:

$$\mathfrak{d}(\mathbb{R} \oplus J_2^{\mathbb{A}}) \oplus \mathcal{A}_q \subset \mathfrak{d}(J_3^{\mathbb{A}}) \tag{3.20}$$

$$\mathfrak{d}(\mathbb{R} \oplus J_2^{\mathbb{A}_s}) \oplus \widetilde{\mathcal{A}}_q \subset \mathfrak{d}(J_3^{\mathbb{A}_s}) \tag{3.21}$$

(where as above $\mathfrak{d}$ can be each of $\mathfrak{der}$, $\mathfrak{str}_0$, $\mathfrak{conf}$, $\mathfrak{qconf}$).

Note the maximal nature of the embeddings (3.20)–(3.21), as well as the presence of the commuting algebras $\mathcal{A}_q$ and $\widetilde{\mathcal{A}}_q$, defined as follows:

$$\mathcal{A}_q := \ \mathfrak{tri}\,(q) \ominus so(q) = \varnothing,\ \mathfrak{so}(3),\ \mathfrak{so}(2),\ \varnothing \quad \text{for } q = 8,4,2,1; \tag{3.22}$$

$$\widetilde{\mathcal{A}}_q := \ \widetilde{\mathfrak{tri}}\,(q) \ominus \widetilde{so}(q) = \varnothing,\ \mathfrak{sl}(2,\mathbb{R}),\ \mathfrak{so}(1,1) \quad \text{for } q = 8,4,2, \tag{3.23}$$

where $\mathfrak{tri}$ and $\mathfrak{so}$ respectively denote the triality and orthogonal (norm-preserving) symmetries (and they are tilded when pertaining to split algebras; see e.g. [CFMZ10, CCM13], and references therein). Analogous results hold for Lorentzian Jordan algebras. Within the physical (U-duality) interpretation, $\mathcal{A}_q$ and $\widetilde{\mathcal{A}}_q$ are consistent with the properties of spinors in $q+2$ dimensions, with Lorentzian signature $(t,s) = (1, q+1)$ respectively Kleinian signature $(t,s) = \left(q/2+1, q/2+1\right)$; indeed, the electric-magnetic (U-duality) symmetry Lie algebra in $D = 6$ (Lorentzian) space-time dimensions is $\mathfrak{so}\,(1,q+1) \oplus \mathcal{A}_q$ for $\mathbb{A}$-based theories (which are endowed with minimal, chiral $(1,0)$ supersymmetry) and $\mathfrak{so}\,\left(q/2+1, q/2+1\right) \oplus \widetilde{\mathcal{A}}_q$ for $\mathbb{A}_s$ -based theories (which are non-supersymmetric for $q = 2,4$, and endowed with maximal, non-chiral $(2,2)$ supersymmetry for $q = 8$); cfr. e.g. [KT83], [GSS11] (and references therein) and [MNY11] for further discussion.

4. Computational methods

In this section we briefly describe the computational methods that we used to construct the real forms of the maximal semisimple (or reductive) subalgebras of the simple Lie algebras of ranks up to 8. We have one procedure for constructing the regular subalgebras (taken from [DFdG15]) and one procedure for the S-subalgebras (from [FdG15]). First we recall some general notation, and subsequently describe the methods in two subsections. Our main reference for the general theory is [Oni04].

By $\iota \in \mathbb{C}$ we denote the imaginary unit. Let $\mathfrak{g}^c$ be a complex simple Lie algebra. An *anti-involution* of $\mathfrak{g}^c$ is a map $\eta : \mathfrak{g}^c \to \mathfrak{g}^c$ with $\eta(x+y) = \eta(x) + \eta(y)$, $\eta(\mu x) = \bar{\mu} x$, $\eta([x,y]) = [\eta(x), \eta(y)]$, $\eta(\eta(x)) = x$ for all $x, y \in \mathfrak{g}^c$, $\mu \in \mathbb{C}$. If $\eta : \mathfrak{g}^c \to \mathfrak{g}^c$ is any map then we set $\mathfrak{g}^c_\eta = \{x \in \mathfrak{g}^c | \eta(x) = x\}$.

A real form $\mathfrak{g}$ of $\mathfrak{g}^c$ is given by three maps, $\tau, \sigma, \theta : \mathfrak{g}^c \to \mathfrak{g}^c$ where

- τ, σ are anti-involutions and θ is an involution,
- $\theta = \tau\sigma = \sigma\tau$,
- $\mathfrak{g}^c_\tau$ is compact and $\mathfrak{g}^c_\sigma = \mathfrak{g}$,
- setting $\mathfrak{k} = \{x \in \mathfrak{g} | \theta(x) = x\}$, $\mathfrak{p} = \{x \in \mathfrak{g} | \theta(x) = -x\}$; then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$ (note that $\theta(\mathfrak{g}) = \mathfrak{g}$ because θ commutes with σ); the restriction of θ to $\mathfrak{g}$ is called a *Cartan involution* of $\mathfrak{g}$,
- set $\mathfrak{u} = \mathfrak{g}^c_\tau$, then θ leaves $\mathfrak{u}$ invariant so that $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_{-1}$ (the eigenspaces of θ with eigenvalues ± 1) and $\mathfrak{k} = \mathfrak{u}_1$, $\mathfrak{p} = \imath\mathfrak{u}_{-1}$.

Let $\mathfrak{a} \subset \mathfrak{g}$ be a semisimple subalgebra and $\mathfrak{a}^c = \mathfrak{a} + \imath\mathfrak{a}$, which is a semisimple subalgebra of $\mathfrak{g}^c$. Then $\mathfrak{a}$ is a real form of the complex subalgebra $\mathfrak{a}^c$. So $\mathfrak{a}$ has a Cartan decomposition $\mathfrak{a} = \mathfrak{k}_\mathfrak{a} \oplus \mathfrak{p}_\mathfrak{a}$. It follows from the Karpelevich–Mostow theorem ([Oni04], corollary 1 of section 6) that $\mathfrak{a}$ is conjugate by an inner automorphism to a subalgebra $\mathfrak{a}'$ such that $\mathfrak{k}_{\mathfrak{a}'} \subset \mathfrak{k}$ and $\mathfrak{p}_{\mathfrak{a}'} \subset \mathfrak{p}$. Equivalently, θ maps $\mathfrak{a}'$ to itself, and its restriction to $\mathfrak{a}'$ is a Cartan involution of $\mathfrak{a}'$. So we may restrict to finding θ-stable subalgebras of $\mathfrak{g}$.

4.1. Real forms of regular subalgebras

We say that a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is regular if it is normalized by a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. If we want to stress the particular Cartan subalgebra that we are referring to we say that $\mathfrak{a}$ is $\mathfrak{h}$-regular. In that case $\mathfrak{a}^c$ is spanned by root spaces of $\mathfrak{g}^c$ (relative to $\mathfrak{h}^c$) along with $\mathfrak{a}^c \cap \mathfrak{h}^c$. By $\Psi(\mathfrak{h}^c, \mathfrak{a}^c)$ we denote the set of roots involved in this. It is a subset of the root system of $\mathfrak{g}^c$ with respect to $\mathfrak{h}^c$.

Let G^c denote the adjoint group of $\mathfrak{g}^c$. This group can be characterized in several ways. Firstly it is the connected algebraic subgroup of $GL(\mathfrak{g}^c)$ with Lie algebra $\mathrm{ad}\mathfrak{g}^c$. Secondly it is the group generated by all elements $\exp(\mathrm{adx})$ where $x \in \mathfrak{g}^c$ is nilpotent. Thirdly, it is the identity component of $\mathrm{Aut}(\mathfrak{g}^c)$. By G we denote the group consisting of all $g \in G^c$ such that $g(\mathfrak{g}) = \mathfrak{g}$. If we represent elements of G^c by their matrices with respect to a basis of $\mathfrak{g}$ then $G = G^c(\mathbb{R})$, the group of all elements of G^c with coefficients in $\mathbb{R}$. Let G_0 be the identity component of G. In this section we outline how to obtain the regular semisimple subalgebras of $\mathfrak{g}$ up to conjugacy by G_0.

First of all, Dynkin ([Dyn52b]) devised an algorithm to obtain the regular semisimple subalgebras of $\mathfrak{g}^c$ up to conjugacy by G^c (see [Gra17], section 5.9 for a recent account). We can use this algorithm to obtain all $\mathfrak{h}^c$-regular semisimple subalgebras up to G^c-conjugacy, for a fixed Cartan subalgebra $\mathfrak{h}^c$ of $\mathfrak{g}^c$. We have that two $\mathfrak{h}^c$-regular semisimple subalgebras $\mathfrak{a}^c_1$, $\mathfrak{a}^c_2$ of $\mathfrak{g}^c$ are G^c-conjugate if and only if $\Psi(\mathfrak{h}^c, \mathfrak{a}^c_1)$, $\Psi(\mathfrak{h}^c, \mathfrak{a}^c_2)$ are $W(\mathfrak{g}^c, \mathfrak{h}^c)$-conjugate, where the latter denotes the Weyl group of the root system of $\mathfrak{g}^c$ with respect to $\mathfrak{h}^c$.

If one is only interested in subalgebras of $\mathfrak{g}^c$ then it suffices to consider just one Cartan subalgebra as they are all conjugate under G^c. In general the real form $\mathfrak{g}$ has more Cartan subalgebras that are non-conjugate. Sugiura ([Sug59]) proved that $\mathfrak{g}$ has a finite number of Cartan subalgebras up to G_0-conjugacy. In [DFG13] Sugiura's method was made into an algorithm for listing the Cartan subalgebras of $\mathfrak{g}$ up to conjugacy by G_0.

Let $\mathfrak{a}$ be a regular semisimple subalgebra of $\mathfrak{g}$. Then the normalizer $\mathfrak{n}_\mathfrak{g}(\mathfrak{a}) = \{x \in \mathfrak{g} | [x, \mathfrak{a}] \subset \mathfrak{a}\}$ is reductive. Therefore it has a *unique* maximally noncompact Cartan subalgebra (that is, a Cartan subalgebra whose intersection with $\mathfrak{p}$ has maximal dimension, (see [Kna02], proposition 6.61). We say that $\mathfrak{a}$ is *strongly* $\mathfrak{h}$*-regular* if $\mathfrak{h}$ is a maximally noncompact Cartan subalgebra of $\mathfrak{n}_\mathfrak{g}(\mathfrak{a})$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Set $N = \{g \in G_0 | g(\mathfrak{h}) = \mathfrak{h}\}$ and $Z = \{g \in G_0 | g(x) = x \text{ for all } x \in \mathfrak{h}\}$. Then $W(\mathfrak{g}, \mathfrak{h}) = N/Z$ is called the *real Weyl group* of $\mathfrak{h}$. It can naturally be identified with a subgroup of $W(\mathfrak{g}^c, \mathfrak{h}^c)$, (see [Kna02], section 7.8). There are algorithms to compute $W(\mathfrak{g}, \mathfrak{h})$, see [DG19].

We have the following criterion: let $\mathfrak{a}_1, \mathfrak{a}_2$ be two strongly $\mathfrak{h}$-regular semisimple subalgebras of $\mathfrak{g}$. They are G_0-conjugate if and only if $\Psi(\mathfrak{h}^c, \mathfrak{a}_1^c)$, $\Psi(\mathfrak{h}^c, \mathfrak{a}_2^c)$ are conjugate under $W(\mathfrak{g}, \mathfrak{h})$ (see [DFdG15], proposition 24). This gives an immediate method for deciding whether $\mathfrak{a}_1$, $\mathfrak{a}_2$ are G_0-conjugate.

Now the algorithm for obtaining all regular semisimple subalgebras of $\mathfrak{g}$ (up to G_0-conjugacy) runs as follows:

(a) Compute the Cartan subalgebras of $\mathfrak{g}$ (up to G_0-conjugacy). For each obtained Cartan subalgebra $\mathfrak{h}$ perform the following steps:
   (1) Compute the $\mathfrak{h}^c$-regular subalgebras of $\mathfrak{g}^c$ using Dynkin's algorithm. For each obtained subalgebra $\mathfrak{a}^c$ perform the following steps:
      (i) Compute the stabilizer S of $\Psi(\mathfrak{h}^c, \mathfrak{a}^c)$ in $W = W(\mathfrak{g}^c, \mathfrak{h}^c)$. Compute a list of representatives $w_1, \ldots, w_s$ of the double cosets $W(\mathfrak{g}, \mathfrak{h})w_i S$ in W.
      (ii) Construct the $\mathfrak{h}^c$-regular subalgebras $\mathfrak{a}_i^c$ of $\mathfrak{g}^c$ with $\Psi(\mathfrak{h}^c, \mathfrak{a}_i^c) = w_i \cdot \Psi(\mathfrak{h}^c, \mathfrak{a}^c)$.
      (iii) Throw away the $\mathfrak{a}_i^c$ that are not σ-stable.
      (iv) Of the remaining ones compute a basis of $\mathfrak{a}_i = \mathfrak{a}_i^c \cap \mathfrak{g}$, and throw away those that are not strongly $\mathfrak{h}$-regular. Add the remaining ones to the final list.

As before, let θ denote the Cartan involution of $\mathfrak{g}$. The Cartan subalgebras found in the first step are θ-stable (see the algorithm in [DFG13]). Therefore the subalgebras constructed by the above procedure are automatically θ-stable as well ([DFdG15], proposition 21).

Note that the above algorithm is correct. Indeed, each $\mathfrak{h}^c$-regular semisimple subalgebra $\mathfrak{b}^c$ of $\mathfrak{g}^c$ that is G^c-conjugate to $\mathfrak{a}^c$ has $\Psi(\mathfrak{h}^c, \mathfrak{b}^c) = w \cdot \Psi(\mathfrak{h}^c, \mathfrak{a}^c)$ for some $\mathfrak{a}^c$ from the initial list. Furthermore, note that the $w_i \cdot \Psi(\mathfrak{h}^c, \mathfrak{a}^c)$ exhaust the images under $W(\mathfrak{g}^c, \mathfrak{h}^c)$ of $\Psi(\mathfrak{h}^c, \mathfrak{a}^c)$ up to conjugacy by $W(\mathfrak{g}, \mathfrak{h})$.

4.2. Real forms of S-subalgebras

From section 2 we recall that a subalgebra of $\mathfrak{g}^c$ is called an *S-subalgebra* if it is not contained in a proper regular subalgebra. For these subalgebras we have no method to list all real forms in $\mathfrak{g}$ up to conjugacy by G_0. Therefore we are more modest and consider the following question. Let $\varepsilon : \mathfrak{g}^c \hookrightarrow \tilde{\mathfrak{g}}^c$ be an embedding of semisimple complex Lie algebras and let $\mathfrak{g}$ be a real form of $\mathfrak{g}^c$. *Find (up to isomorphism) the real forms $\tilde{\mathfrak{g}}$ of $\tilde{\mathfrak{g}}^c$ such that* $\varepsilon(\mathfrak{g}) \subset \tilde{\mathfrak{g}}$. Because two real forms $\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}'$ of $\tilde{\mathfrak{g}}^c$ are isomorphic if and only if there is a $\phi \in \mathrm{Aut}(\tilde{\mathfrak{g}}^c)$ with $\phi(\tilde{\mathfrak{g}}) = \tilde{\mathfrak{g}}'$ ([Oni04], section 2, proposition 1) we may replace the given embedding ε by $\phi\varepsilon$ for any $\phi \in \mathrm{Aut}(\tilde{\mathfrak{g}}^c)$.

Let $\theta, \tilde{\theta}$ be Cartan involutions of $\mathfrak{g}, \tilde{\mathfrak{g}}$ respectively. A classical theorem of Karpelevich states that $\varepsilon(\mathfrak{g}) \subset \tilde{\mathfrak{g}}$ if and only if $\varepsilon \circ \theta = \tilde{\theta} \circ \varepsilon$ (see [OV94], theorem 3.7 of chapter 6). Here we consider θ and $\mathfrak{g}$ to be given, and hence have to construct real forms $\tilde{\mathfrak{g}}$ with Cartan involution $\tilde{\theta}$ satisfying the mentioned condition. Our method is based on proposition 4.2.

Let $\sigma : \mathfrak{g}^c \to \mathfrak{g}^c$ be the conjugation with respect to the real form $\mathfrak{g}$. Let $\mathfrak{u}$ be a fixed compact form of $\mathfrak{g}^c$ with corresponding conjugation $\tau : \mathfrak{g}^c \to \mathfrak{g}^c$ with $\sigma\tau = \tau\sigma$. Set $\theta = \sigma\tau$. Then the restriction of θ to $\mathfrak{g}$ is a Cartan involution of $\mathfrak{g}$. From [FdG15] we have the following result.

Proposition 4.1. *Let $\tilde{\mathfrak{g}} \subset \tilde{\mathfrak{g}}^c$ be a real form of $\tilde{\mathfrak{g}}^c$ such that $\varepsilon(\mathfrak{g}) \subset \tilde{\mathfrak{g}}$. Then there are a compact form $\tilde{\mathfrak{u}} \subset \tilde{\mathfrak{g}}^c$ of $\tilde{\mathfrak{g}}^c$, with conjugation $\tilde{\tau} : \tilde{\mathfrak{g}}^c \to \tilde{\mathfrak{g}}^c$, and an involution $\tilde{\theta}$ of $\tilde{\mathfrak{g}}^c$ such that*

(*a*) $\varepsilon(\mathfrak{u}) \subset \tilde{\mathfrak{u}}$,
(*b*) $\varepsilon\theta = \tilde{\theta}\varepsilon$,
(*c*) $\tilde{\theta}\tilde{\tau} = \tilde{\tau}\tilde{\theta}$,
(*d*) *there is a Cartan decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}$, such that the restriction of $\tilde{\theta}$ to $\tilde{\mathfrak{g}}$ is the corresponding Cartan involution, and $\tilde{\mathfrak{u}} = \tilde{\mathfrak{k}} \oplus \imath\tilde{\mathfrak{p}}$.*

Conversely, if $\tilde{\mathfrak{u}} \subset \tilde{\mathfrak{g}}$ is a compact form, with corresponding conjugation $\tilde{\tau}$, and $\tilde{\theta}$ is an involution of $\tilde{\mathfrak{g}}^c$ such that (a), (b) and (c) hold, then $\tilde{\theta}$ leaves $\tilde{\mathfrak{u}}$ invariant, and setting $\tilde{\mathfrak{k}} = \tilde{\mathfrak{u}}_1$, $\tilde{\mathfrak{p}} = \imath\tilde{\mathfrak{u}}_{-1}$ (where $\tilde{\mathfrak{u}}_k$ is the k-eigenspace of $\tilde{\theta}$), we get that $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}$ is a real form of $\tilde{\mathfrak{g}}^c$ with $\varepsilon(\mathfrak{g}) \subset \tilde{\mathfrak{g}}$.

To construct the real forms $\tilde{\mathfrak{g}}$ the strategy is now the following. First we fix a compact form $\tilde{\mathfrak{u}}$ of $\tilde{\mathfrak{g}}^c$ and replace ε by $\phi\varepsilon$ for a $\phi \in \mathrm{Aut}(\tilde{\mathfrak{g}}^c)$ in order to have $\varepsilon(\mathfrak{u}) \subset \tilde{\mathfrak{u}}$. Secondly we construct the space

$$\mathcal{A} = \{A \in \mathrm{End}(\tilde{\mathfrak{g}}^c) | A(\mathrm{ad}(\varepsilon\theta(y))) = (\mathrm{ad}\varepsilon(y))A \quad \text{for all } y \in \mathfrak{g}^c\}.$$

(Note that a basis of $\mathcal{A}$ can be computed by solving a set of linear equations; [FdG15] also has a more efficient method to construct $\mathcal{A}$.) Let $\tilde{\theta}$ be an involution of $\tilde{\mathfrak{g}}^c$. Then $\varepsilon\theta = \tilde{\theta}\varepsilon$ if and only if $\tilde{\theta} \in \mathcal{A}$ and

$$\tilde{\theta}(\mathrm{ad}x)\tilde{\theta} = \mathrm{ad}\tilde{\theta}(x) \quad \text{for all } x \in \tilde{\mathfrak{g}}^c \tag{4.1}$$

([FdG15], proposition 3.6). Hence $\mathcal{A}$ contains the maps $\tilde{\theta}$. The conditions that $\tilde{\theta}$ be an involution, (4.1), (c) are translated to polynomial equations on the coefficients of $\tilde{\theta}$ with respect to a basis of $\mathcal{A}$. These polynomial equations are then studied and solved by means of the technique of Gröbner bases (see [CLO15]).

For the details of this procedure we refer to [FdG15]. But we do remark that if $\varepsilon(\mathfrak{g}) \subset \tilde{\mathfrak{g}}$ for some real form $\tilde{\mathfrak{g}}$ of $\tilde{\mathfrak{g}}^c$ then we get just one subalgebra of $\tilde{\mathfrak{g}}$. However, there may be more subalgebras $\mathfrak{a} \subset \tilde{\mathfrak{g}}$ such that $\mathfrak{a}^c$ is G^c-conjugate to $\varepsilon(\mathfrak{g}^c)$, without $\mathfrak{a}$ being G_0-conjugate to $\varepsilon(\mathfrak{g})$. This method does not detect such a situation (unlike the method for the regular case). Furthermore, there are cases in type D_n for n even, where there are subalgebras $\mathfrak{a}_1^c, \mathfrak{a}_2^c \subset \tilde{\mathfrak{g}}^c$ that are not conjugate under G^c, but are conjugate under an outer automorphism. In these cases we just consider one of the corresponding embeddings in $\tilde{\mathfrak{g}}^c$ because changing the embedding ε to a $\phi\varepsilon$ may identify the subalgebras $\mathfrak{a}_1^c$, $\mathfrak{a}_2^c$.

4.3. Implementation

We have implemented these methods in the computer algebra system GAP4 ([GAP18]) using the packages SLA ([dGGT19]) and CoReLG ([DFdG19]). The implementation of the algorithm of section 4.1 is quite straightforward. For the procedure indicated in section 4.2 we remark that the package SLA contains tables with the semisimple subalgebras of the simple complex Lie algebras. The embeddings ε are simply given by those tables.

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