The best parametrization for solving the boundary value problem for the system of differential-algebraic equations with delay

M N Afanaseva[1], A N Vasilyev[2], E B Kuznetsov[3] and D A Tarkhov[4]

[1,3]Department of Information Technology and Applied Mathematics of Moscow Aviation Institute (National Research University), 125993, 4, Volokolamskoe shosse, Moscow, Russia

[2,4] Peter the Great St. Petersburg Polytechnic University, 195251, 29, Polytechnicheskaya street, Saint Petersburg, Russia

[1]E-mail: mary.mai.8@yandex.ru

[2]E-mail: a.n.vasilyev@gmail.com

[3]E-mail: kuznetsov@mai.ru

[4]E-mail: dtarkhov@gmail.com

Abstract. In this paper, we considered the numerical approach for solving a nonlinear boundary value problem for the system of differential-algebraic equations with delay argument. The shooting method is used to solve the boundary value problem. The Newton method is used to find the parameter of shooting. To overcome the difficulties associated with the choice of the initial approximation we apply E. Lahaye's parameter continuation method. If the curve of the solution contains limit points, the method diverges. Then to find the parameter we used the method of continuation with respect to the best parameter - the length of the curve of the solution set. The solution is constructed by advancing the sequence of values of the parameter. With a discrete continuation, the initial-value problem is transformed by a finite-difference representation of the derivatives and entering the best argument and the corresponding equation of hypersphere. The resulting system is solved using the Newton method. To find the values of the functions at the delay point Lagrange polynomial with three points is used. An example of the behavior of an elastoviscoplastic rod is considered.

1. Introduction

We consider the numerical solution for solving a nonlinear boundary value problem for the system of differential-algebraic equations with delay. Such systems with boundary conditions model the behavior of complex systems in physics and mechanics, in particular, in the mechanics of a deformable solid. Particularly, such systems describe the behavior of an elastoviscoplastic deformation and creep theory. The availability of the delay argument allows describing the behavior of the analyzed functions not only in the current but also in the previous point in time.

To solve the boundary value problem, the methods for solving the initial value problem and the methods for solving an operator equation can be used [1].

For the numerical solution of such problems, the method of finite differences or the shooting method can be used [2].

In the monograph [2] the linear boundary value problems are considered. For nonlinear problems, it is proposed to linearize the problem first and then to use the orthogonal sweep method.

The solution of linear boundary value problems for differential-difference equations is discussed in the book [3].

The applying of the best parametrization method for solving the boundary value problems for nonlinear ordinary differential equations is given in the work [4].

The solution of the boundary value problem for the system of differential equations with retarded argument was discussed in the publication [5].

The solution of the boundary value problem for differential-algebraic equations without a retarded argument was explored in the work [6].

In this paper we demonstrate the numerical solution of a boundary value problem by using the following methods: shooting method for solving a boundary value problem; combination of the Newton's method, E. Lahaye's parameter continuation method and the method of the best parametrization to find the value of the "shooting" parameter; the Newton's method and the best parameterization method for solving the initial problem at each step of the shooting method.

Let us consider the nonlinear system of equations:

$$\frac{dy}{dt} = f(t, y(t), y(t - \tau), \dot{y}(t - \tau), x(t), x(t - \tau), \dot{x}(t - \tau)) = 0, \tag{1}$$

$$G(t, y(t), y(t - \tau), x(t), x(t - \tau)) = 0, t \in [a, b]$$

with boundary conditions:

$$W(y(a), y(b), x(a), x(b)) = 0, \tag{2}$$

Where:

$$f : \mathbb{R}^1 \times \mathbb{R}^{3s} \times \mathbb{R}^{3r} \to \mathbb{R}^s,$$

$$G : \mathbb{R}^1 \times \mathbb{R}^{2s} \times \mathbb{R}^{2r} \to \mathbb{R}^r,$$

$$W : \mathbb{R}^{2s} \times \mathbb{R}^{2r} \to \mathbb{R}^s$$

and $y(t) : \mathbb{R}^1 \to \mathbb{R}^s, x(t) : \mathbb{R}^1 \to \mathbb{R}^r$, the required functions are included in the system at different values of the argument.

On the set $E_0 = \{T < a \mid \exists t > a, t - \tau = T\}$, sufficiently smooth functions $\varphi_{1,2}(T)$, $\psi_{1,2}(T), T \in [a - \tau, a)$ are given such that:

$$\begin{aligned} y_\tau &= y(t - \tau) = \varphi_1(T), \dot{y}_\tau = \dot{y}(t - \tau) = \varphi_2(T), \\ x_\tau &= x(t - \tau) = \psi_1(T), \dot{x}_\tau = \dot{x}(t - \tau) = \psi_2(T), \end{aligned} \tag{3}$$

The boundary conditions (2) satisfy the consistency conditions:

$$G(a, y(a), y(a - \tau), x(a), x(a - \tau)) = 0,$$
$$G(b, y(b), y(b - \tau), x(b), x(b - \tau)) = 0. \tag{4}$$

2. Transformation of the problem

The problem is transformed by the finite-difference representation of the derivatives and entering the best argument λ and the corresponding equation of hypersphere [7]-[9]:

$$\begin{cases} (y_{(i+1)} - y_{(i)}) - f_{(i+1)}(t, y, y_\tau, \dot{y}_\tau, x, x_\tau, \dot{x}_\tau)(t_{(i+1)} - t_{(i)}) = 0, \\ G_{(i+1)}(t, y, y_\tau, x, x_\tau) = 0, \\ (t_{(i+1)} - t_{(i)})^2 + (y_{(i+1)} - y_{(i)})^2 + (x_{(i+1)} - x_{(i)})^2 - \Delta\lambda = 0, \\ W(y(a), y(b), x(a), x(b)) = 0, \\ y_\tau = \varphi_1(T), x_\tau = \psi_1(T), T \in [a - \tau, a) \\ \dot{y}_\tau = \varphi_2(T), \dot{x}_\tau = \psi_2(T), T \in [a - \tau, a) \end{cases} \tag{5}$$

Now functions t, y and x are functions of the best argument λ.

3. Solution of the boundary value problem

To solve the boundary value problem, we use the shooting method. So the condition at the end point of the integration interval is replaced by parameter p:

$$y_l(a) = p_l, l = \overline{1,s},$$
$$x_m(a) = p_{s+m}, m = \overline{1,r}. \tag{6}$$

Now the solution of the problem (6) depends on p:

$$y = y(\lambda, p), x = x(\lambda, p). \tag{7}$$

And for functions (7) the boundary conditions (2) must be satisfied:

$$F(p) = W(y(a, p), y(b, p), x(a, p), x(b, p)) = 0. \tag{8}$$

To find the parameter p from the system (8), we use the Newton method as it has the highest degree of convergence:

$$p^{(k+1)} = p^{(k)} - \left[\frac{F(p^{(k)}) - F(p^{(k-1)})}{p^{(k)} - p^{(k-1)}}\right]^{-1} F(p^{(k)}), \tag{9}$$

$$p^{(0)} = p_0.$$

For the successful convergence of the Newton method (9), it is necessary to choose the initial approximation p_0 close to the root. To overcome the difficulties associated with the choice of the initial approximation the system is transformed by introducing a new parameter $\mu \in [0, 1]$ such that for $\mu = 0$ the solution of the system is known, and for $\mu = 1$ the original equation with the desired solution is obtained [10]:

$$\Phi(p, \mu) = F(p) - (1 - \mu)F(p_0) = 0, \tag{10}$$

p_0 - the solution when $\mu = 0$.

This equation (10) can be solved by the continuation method on a parameter in the form of E. Lahaye (discrete continuation) [11] or the form of D. F. Davidenko (continuous continuation) [12] - [14]. For finding solutions to the system (10), E. Lahaye's parameter continuation method is used [11]. The interval on which the parameter μ changes is divided into m equal parts:

$$0 < \mu_1 < \ldots < \mu_{m-1} < \mu_m = 1.$$

For each μ_k we calculate p_k by the Newton method:

$$p_{(k)}^{(i+1)} = p_{(k)}^{(i)} - \left[\frac{\Phi(p_{(k)}^{(i)}, \mu_{(k)}) - \Phi(p_{(k)}^{(i-1)}, \mu_{(k)})}{p_{(k)}^{(i)} - p_{(k)}^{(i-1)}} \right]^{-1} \Phi(p_{(k)}^{(i)}, \mu_{(k)}), \tag{11}$$

$$p_{(k+1)}^{(0)} = p_{(k)}^{(r_k)}, i = 1, 2, \ldots, r_{k-1}.$$

However, this approach is effective only in the case when the solution monotonously depends on the parameter μ.

Then the algorithm is transformed by entering a new parameter ν - the length of the curve of the solution set [4], that ensures that all possible solutions of problem (6) are found.

The curve of the solution set is divided into l intervals with a constant step:

$$\nu_0 = 0 < \nu_1 < \nu_2 < \ldots < \nu_l = L.$$

Now all variables of the system (11) are functions of the parameter ν and can change non-monotonously. The values of p and μ can be found from the system:

$$\Psi_{k+1}(z) = \begin{cases} F(p) - (1-\mu)F(p_0) = 0, \\ (p - p_k^{(r_k)})^2 + (\mu - \mu_k^{(r_k)})^2 - \Delta\nu^2 = 0, \end{cases} \tag{12}$$

where $z = (p, \mu), \nu_{k+1} = \nu_k + \Delta_\nu, \nu_0 = 0, \nu_l = L$.

For every newly calculated value of the parameter ν the initial-value problem is solved by the Newton method:

$$z_k^{(i+1)} = z_k^{(i)} - \left[\frac{\partial \Psi_k(z_k^{(i)})}{\partial z} \right]^{-1} \Psi_k(z_k^{(i)}), i = 1, 2, \ldots r_{k-1}, \tag{13}$$

$$z_k^{(0)} = 2z_{k-1}^{(r_{k-1})} - z_{k-2}^{(r_{k-2})}.$$

For each calculated value of the parameter p the Cauchy problem (5) is solved by the Newton method. A feature of the problem under consideration is the presence of the delay parameter τ. The values of the functions at the deviation point, where the values are not defined by conditions of the problem, are calculated by using the Lagrange polynomial with three points [15], [16]:

$$P_2(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}.$$

4. Numerical experiment

We demonstrate the work of the algorithm with numerical examples. Some examples of nonlinear systems that model processes of a deformation and creep theory are illustrated in works [17]-[20].

4.1. Example 1

Let us consider the solution of the boundary value problem for the system of differential-algebraic equations with delay for non-dimensional functions $y_1(t)$, $y_2(t)$, $x(t)$:

$$\begin{cases} \dfrac{dy_1}{dt} = -y_2(t) + 2x(t) - 2y_1(t), \\ \dfrac{dy_2}{dt} = 2x(t) - 3y_1(t) - 2\sin t - \cos t + y_1(t-0.05), \\ x(t) - y_1(t) - \sin t = 0, \\ y_2(0) = 2, \\ x(1) = 4, \\ y_1(T) = 1, T \in [-0.5, 0), \end{cases}$$

deviation parameter $\tau = 0.05$.

The transformed problem by the finite-difference representation of the derivatives and end entering the “shooting” parameter *p* will take the following form:

$$\begin{cases} (y_{1i} - y_{1*}) - (t_i - t_*)(-y_{2i} + 2x_i - 2y_{1i}) = 0, \\ (y_{2i} - y_{2*}) - (t_i - t_*)(2x_i - 3y_{1i} - 2\sin t_i - \cos t_i + y_{1\tau i}) = 0, \\ x_i - y_{1i} - \sin t_i = 0, \\ y_2(0) = 2, \\ x(0) = p, \\ y_1(T) = 1, T \in [-0.5, 0), \end{cases}$$

The second boundary condition of the considered boundary value problem is realized when $p = 3.6708$. This value is calculated at values $\Delta\lambda = 0.01$ and $\Delta\nu = 0.01$.

The solutions at finding the value of the “shooting” parameter are illustrated in figures 1-3.

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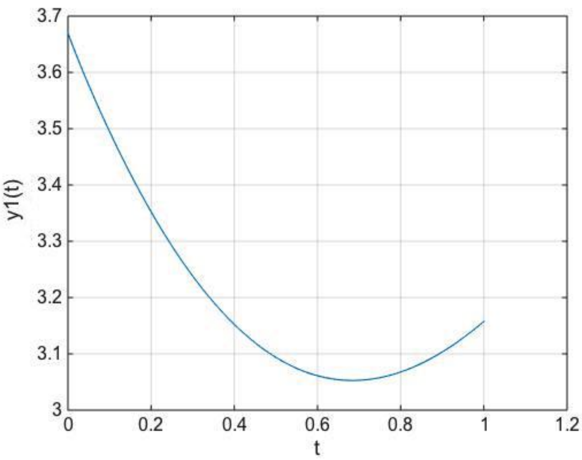

Figure 1. The solution $y_1(t)$ of the problem.

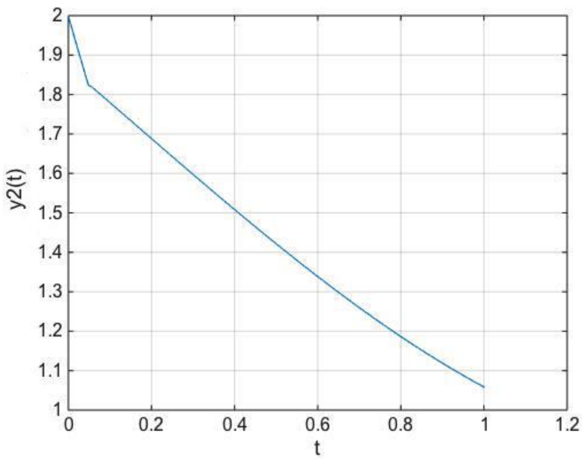

Figure 2. The solution $y_2(t)$ of the problem.

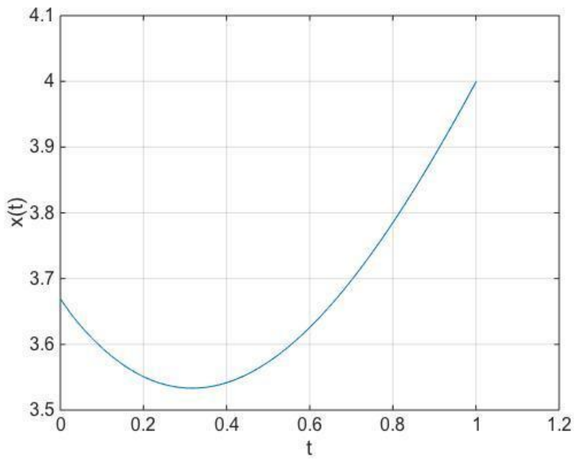

Figure 3. The solution $x(t)$ of the problem.

4.2. Example 2

Let us consider a modified nonlinear boundary value problem that models the behavior of an elastoviscoplastic rod of a finite length under the creep conditions:

$$\begin{cases} \dfrac{dy_1}{dt} = y_1(t) - 2y_2(t) - \dfrac{x(t)}{y_1^2(t-1)}, \\ \dfrac{dy_2}{dt} = y_2(t-1) - 2y_1(t) + x(t), \\ x(t) - y_1(t-1) + t^2 - 4 = 0, \\ y_2(0) = 0, \\ y_1(1) = 4, \\ y_1(T) = \sqrt[3]{2T+1}, T \in [-1, 0), \\ \qquad y_2(T) = 0, \end{cases}$$

deviation parameter $\tau = 1$.

The transformed problem looks as follows:

$$\begin{cases} (y_{1i} - y_*) - (t_i - t_*)(y_{1i} - 2y_{2i} - \dfrac{x_i}{y_{1\tau i}^{\;2}}) = 0, \\ (y_{2i} - y_*) - (t_i - t_*)(y_{2\tau i} - 2y_{1i} + x_i) = 0, \\ x_i - y_{1\tau i} + t_i^{\;2} - 4 = 0, \\ y_2(0) = 0, \\ y_1(0) = p, \\ y_1(T) = \sqrt[3]{2T+1}, T \in [-1, 0), \\ \qquad y_2(T) = 0, \end{cases}$$

Graphs of the solution for non-dimensional functions $y_1(t)$, $y_2(t), x(t)$ of the boundary value problem for the calculated value of the parameter $p = 5.10765$, initial value $p_0 = 2$, $\Delta\lambda = 0.01$ and $\Delta\nu = 0.01$ are presented in figures 4-6.

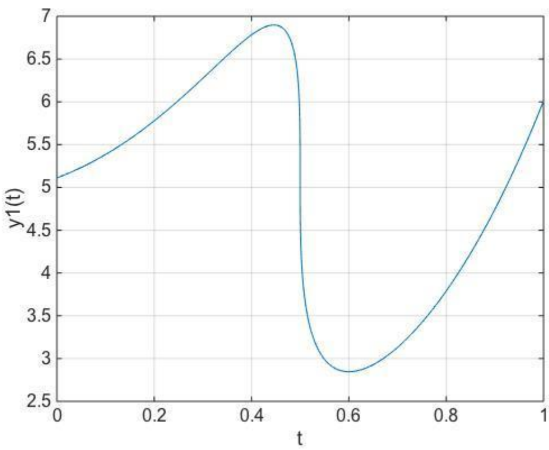

Figure 4. The solution $y_1(t)$ of the problem.

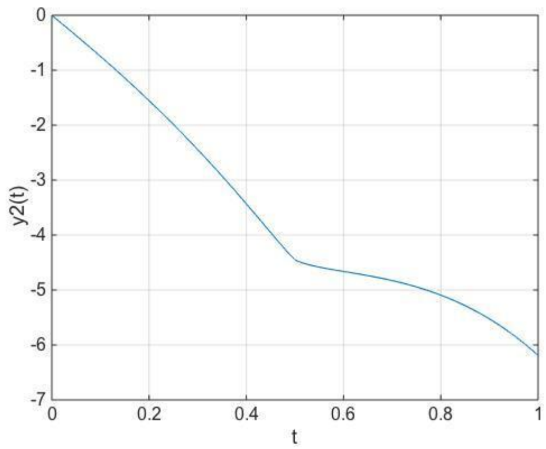

Figure 5. The solution $y_2(t)$ of the problem.

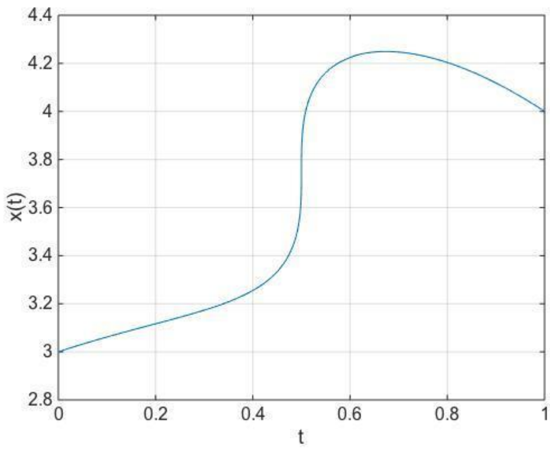

Figure 6. The solution $x(t)$ of the problem.

This problem has a feature at the point $t = 0.5$: the right side of the first equation contains a denominator of zero.

The graphs of functions $y_1(t)$ and $x(t)$ illustrate that in the vicinity of the point $t = 0.5$ the tangent is orthogonal to the axis t. Therefore, some numerical methods may diverge on passing through this point.

5. Conclusion

Using the best parameterization method in the construction of the shooting algorithm makes it possible to find all solutions of the problem under consideration. And the best parameterization method allows calculating the solutions of the initial value problem at the founded value of the “shooting” parameter even when the curve of the solution has an irregularity.

Numerical studies have shown that the application of the best parameterization method and the shooting method in conjunction with the solution of the initial-value problem allows us to find possible solutions to the boundary value problem effectively.

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