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Circle problem and the spectrum of the Laplace operator on closed 2-manifolds

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Abstract. In this survey the circle problem is treated in the broad sense, as the problem of the asymptotic properties of the quantity $P(x)$, the remainder term in the circle problem. A survey of recent results in this direction is presented. The main focus is on the behaviour of $P(x)$ on short intervals. Several conjectures on the local behaviour of $P(x)$ which lead to a solution of the circle problem are presented. A strong universality conjecture is stated which links the behaviour of $P(x)$ with the behaviour of the second term in Weyl's formula for the Laplace operator on a closed Riemannian 2-manifold with integrable geodesic flow.

Bibliography: 43 titles.

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Contents

1. Introduction

The (Gauss) circle problem is one of the most well-known problems in analytic number theory. Here is its statement.

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Let $R(x)$ be the number of integer points in the disc with radius $\sqrt{x}$, and let $P(x)$ be defined by

$$R(x) = \pi x + P(x). \tag{1.1}$$

The circle problem is to establish the estimate

$$P(x) = O(x^{1/4+\varepsilon}) \quad \forall\, \varepsilon > 0 \qquad (x \to \infty). \tag{1.2}$$

Since

$$R(x) = \sum_{n \leqslant x} r(n), \tag{1.3}$$

where $r(n)$ is the number of representations of n as a sum of squares of two integers, the circle problem belongs to the list of problems on the behaviour of sums of arithmetic functions. Another, no less well-known, problem in this list is the (Dirichlet) divisor problem, which is to establish the estimate

$$\Delta(x) = O(x^{1/4+\varepsilon}), \tag{1.4}$$

where $\Delta(x)$ is defined by

$$\sum_{n \leqslant x} d(n) = x(\log x + 2\gamma - 1) + \Delta(x), \tag{1.5}$$

with $d(n)$ the number of divisors of n.

The circle problem, the divisor problem, and the problem of the asymptotic behaviour of the quantity $E(T)$ defined by

$$\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 dt = T \log \frac{T}{2\pi} + (2\gamma - 1)T + E(T), \tag{1.6}$$

where $\zeta(\,\cdot\,)$ is the Riemann zeta function, are test problems verifying whether methods for estimating trigonometric sums are efficient. These methods lead to non-trivial estimates of the form

$$P(x) = O(x^{\theta+\varepsilon}), \quad \Delta(x) = O(x^{\theta+\varepsilon}), \quad E(T) = O(T^{\theta+\varepsilon}) \qquad \left(\theta < \frac{1}{3}\right). \tag{1.7}$$

The history of the gradual improvement of these estimates can be learned from our references. In § 3.1 we present the most recent results in this direction.

We note that after reducing the above three problems to estimates for trigonometric sums, the results obtained in one of them can easily be carried over to the other two.

As in a number of recent papers, this survey treats the circle problem in the broad sense, as the problem of the asymptotic properties of $P(x)$.

A survey of new results on the properties of $\Delta(x)$ and $E(T)$ was presented in [1], where the focus was on the moments of these quantities.

One of our aims here is to present some relatively new results on the behaviour of $P(x)$ on short intervals (we say that an interval $I(T) \subset [T, 2T]$ is short if its length $|I(T)|$ satisfies $|I(T)| \ll T^{\lambda+\varepsilon}$, $\lambda < 1$) and to state a number of conjectures on the local behaviour of $P(x)$. Among these are, for instance, Jutila's conjecture (see (4.32)) and the following result (see the end of § 4). Suppose that $r(n) \neq 0$ and $|P(n)| < Cn^{1/4}$, and let $x = n$ be a local maximum point of $|P(x)|$. If $|P(x)| < B|P(n)|$ (where $B < 1$) for $|x-n| < Cn^{1/2-\varepsilon}$, then $|P(n)| = O(n^{1/4+\varepsilon})$. Note that our assumption about the behaviour of $P(x)$ means that, in a neighbourhood of a maximum, $P(x)$ behaves like a random walk starting from $P(n)$ at time n.

Among $P(x)$, $\Delta(x)$, and $E(T)$ the quantity $P(x)$ is distinguished by its possible interpretation as the second term in Weyl's formula for the Laplace operator on a flat torus. This lets us consider the properties of $P(x)$ in the context of the theory of quantum chaos [2], [3]. Some of the central objects under investigation in this theory are the statistical properties of the discrete spectra $\{\lambda_i \geqslant 0\}$ of the Laplace operator on closed Riemannian 2-manifolds. The conjecture of universality stated in the theory of quantum chaos asserts that the statistical properties of the spectra $\{\lambda_i\}$ on short intervals ($x < \lambda_i < x + O(x^{1/2-\varepsilon})$) depend only on whether the geodesic flow is integrable; if it is, then the eigenvalues on a short interval must have a Poisson distribution. Note that even for a flat torus, when $\lambda_i = e^2 + m^2$ ($e, m \in \mathbb{Z}$), this has not yet been proved. A stronger conjecture can also be made: if the geodesic flow is integrable, then on short intervals the second term in Weyl's formula behaves like $P(x)$. In this case the results in §§ 3–6 become conjectures about the behaviour of the second term in Weyl's formula.

At the end of § 7 we briefly mention the corresponding questions in the case of a non-integrable geodesic flow. In this case, the role of the circle problem (in the universality conjecture) can presumably be taken on by the problem of the behaviour of the second term in Weyl's formula for the Riemann surface $\Gamma\backslash H$, where H is the upper half-plane with the Poincaré metric and Γ is the strictly hyperbolic group corresponding to a generic point in the Teichmüller space.

We devote §§ 2–6 to the properties of the quantity $P(x)$, giving no proofs but only presenting references to the corresponding works. While the results there have been stated for $E(T)$ or $\Delta(x)$, we present only the corresponding result for $P(x)$. The connections between $P(x)$ and the spectral properties of the Laplace operator are discussed in § 7.

Throughout what follows, $\varepsilon > 0$ is a quantity which can be taken to be arbitrarily small, and C denotes absolute constants whose values can be specified.

2. Starting formulae

By Definition (1.1)

$$P(x) = \sum_{0 \leqslant n \leqslant x} r(n) - \pi x, \tag{2.1}$$

where $r(n)$ is the number of representations of n as a sum of two squares. Thus, $P(x)$ is a piecewise linear function with discontinuities of the first kind for $x \in K = \{n \geqslant 0,\ r(n) \neq 0\}$:

$$\begin{gathered} P(n_\alpha) - P(n_\alpha - 0) = r(n_\alpha), \qquad n_\alpha \in K. \\ P^{(1)}(x) = -\pi, \qquad x \notin K. \end{gathered} \tag{2.2}$$

The following two formulae were proved in [4] and [5]:

$$P(x) = -8 \sum_{j \leqslant \sqrt{x/2}} \psi(\sqrt{x-j^2}) + O(1), \tag{2.3}$$

$$P(x) = 4 \sum_{d<\sqrt{x}} \left\{ \psi\left(\frac{x}{4d+1}\right) - \psi\left(\frac{x}{4d+3}\right) + \psi\left(\frac{x-3}{4d}\right) - \psi\left(\frac{x-1}{4d}\right) \right\} + O(1). \tag{2.4}$$

Here

$$\psi(x) = x - [x] - \frac{1}{2} = \{x\} - \frac{1}{2}.$$

Although the best known estimates of the form (1.7) have been obtained using (2.3) and (2.4), the truncated Voronoi formula is better suited for an analysis of the properties of $P(x)$.

For $x \notin K$ Voronoi's formula has the form

$$P(x) = x^{1/4} \sum_{j=1}^{\infty} \frac{r(j)}{\sqrt{j}} J_1(2\pi\sqrt{jx}\,), \tag{2.5}$$

where $J_\nu(\,\cdot\,)$ is the Bessel function. Vornoi [6] discovered the formula (2.5) in 1904, and it was proved by Hardy [7] in 1915. A proof of (2.5) strictly in the framework of real analysis was given in [8] (where it was called Hardy's identity).

The regularized Voronoi formula

$$\sum_{n=0}^{\infty} r(n)g(n) = \pi \sum_{n=0}^{\infty} r(n) \int_0^{\infty} g(t) J_0(2\pi\sqrt{nt}\,)\, dt, \tag{2.6}$$

which holds for each smooth, sufficiently rapidly decaying function $g(\,\cdot\,)$, follows from the two-dimensional Poisson formula. Formally, we obtain (2.5) from (2.6) by taking

$$g(n) = \begin{cases} 1, & n \leqslant x, \\ 0, & n > x. \end{cases}$$

The generating Dirichlet series for $r(n)$ has the form

$$\sum_{n=1}^{\infty} \frac{r(n)}{n^s} = 4\zeta(s)L(s \mid \chi_4), \quad \text{where } L(s \mid \chi_4) = \sum_{n=1}^{\infty} \frac{\chi_4(n)}{n^s}, \tag{2.7}$$

and $\chi_4(\,\cdot\,)$ is the primitive Dirichlet character modulo 4. We obtain the truncated Voronoi formula

$$P(x) = P_N(x) + \Delta_N P(x), \tag{2.8}$$

$$P_N(x) = -\frac{x^{1/4}}{\pi} \sum_{1 \leqslant j \leqslant N} \frac{r(j)}{j^{3/4}} \cos\left(2\pi\sqrt{jx} + \frac{\pi}{4}\right), \tag{2.9}$$

$$|\Delta_N P(x)| \leqslant C(\varepsilon)\left(N^{\varepsilon} + \frac{x^{1/2+\varepsilon}}{\sqrt{N}}\right), \tag{2.10}$$

from (2.7), using Perron's formula [9]. The estimate (2.10) can be refined:

$$\Delta_N P(x) \leqslant C\biggl(\overline{r}(N)\log N + \frac{x^{1/2}}{\sqrt{N}}\overline{r}(x)\log x\biggr). \tag{2.11}$$

Here we have used that

$$r(n) \leqslant \overline{r}(n), \quad \text{where } \overline{r}(x) = \exp\biggl(\frac{\log x}{\log_2 x}\biggr), \quad x > n_0, \tag{2.12}$$

and

$$\log_k x = \underbrace{\log\dots\log}_{k \text{ times}} x.$$

The precise formula for

$$P_1(x) = \int_0^x P(t)\,dt \tag{2.13}$$

is called Landau's formula (see [8]) and has the form

$$P_1(x) = \frac{x}{\pi}\sum_{j=1}^{\infty}\frac{r(j)}{j}\,J_2\bigl(2\pi\sqrt{jx}\,\bigr). \tag{2.14}$$

All the results below (with the exception of the ones in § 3.1) were obtained using the truncated Voronoi formula and Landau's formula.

3. Classical results

3.1. Estimates of the form $P(x)=O(x^{\theta+\varepsilon})$. The conjectural estimate (1.2) is the content of the circle problem and has not yet been proved.

For an appropriate choice of N, the formulae (2.8)–(2.10) yield (1.7) with $\theta=1/3$. All the estimates with $\theta < 1/3$ have been derived from non-trivial estimates of trigonometric sums (see [5]). By 1942, developments of methods of van der Corput had led to proofs of (1.7) for $\theta = 13/40 = 0.3250$. Further progress was connected with a new method proposed in [10] for estimating trigonometric sums. With its use, the estimate (1.7) was established in [11] for

$$\theta = \frac{7}{22} = 0.318181\dots. \tag{3.1}$$

The Bombieri–Iwaniec method produces the best known results for sums of the form $\sum_{N\leqslant n\leqslant 2N}\psi(f(n))$. Its applications to the problem of estimating the number of integer points inside a contour are the subject of the book [12]. In [13] the estimate (1.7) was proved for

$$\theta = \frac{23}{73} = 0.3151\dots, \tag{3.2}$$

and in [14] it was proved for

$$\theta = \frac{131}{416} = 0.3149\dots. \tag{3.3}$$

It was proposed in [15] to modify one step in the application of the complicated Bombieri–Iwaniec method, thus enabling the authors to prove (1.7) for

$$\theta = \frac{517}{1648} = 0.31371\dots\ . \tag{3.4}$$

So far, this is the best result. On the other hand, it was mentioned in [13] that $\theta = 5/16 = 0.3125$ marks perhaps a bound below which the method will not work.

3.2. $\boldsymbol{\Omega}$-theorems. Ω-theorems answer, in particular, the question of how accurate the conjectural estimate (1.2) can be. Roughly speaking, they show that we cannot take $\varepsilon = 0$ in (1.2).

Recall that the relation $P(x) = \Omega(f(x))$ means that there exists a sequence $x_k \to \infty$ such that $|P(x_k)| > Cf(x_k)$ $(k \geqslant k_0)$. The first Ω-theorems were proved by Hardy in 1916, and then subsequently improved many times (see [16] and [17]). It was shown in [18] that

$$P(x) = \Omega\bigl(x^{1/4}(\log x)^{1/4}(\log_2 x)^{3(2^{1/3}-1)/4}(\log_3 x)^{-5/8}\bigr). \tag{3.5}$$

This is apparently the best result obtained so far.

3.3. Moments. We will use the following notation for moments:

$$m_k(T) = \int_0^T P^k(x)\,dx, \qquad \overline{m}_k(T) = \int_0^T |P(x)|^k\,dx. \tag{3.6}$$

It was shown in [19] that

$$\overline{m}_k(T) = O(T^{1+k/4+\varepsilon}) \quad \text{for } k \leqslant A_0, \quad A_0 = \frac{35}{4}. \tag{3.7}$$

This result uses an estimate obtained in [19] for the number of large deviations. Let

$$1 \leqslant x_1 < x_2 < \dots < x_R \leqslant T,$$
$$|x_i - x_j| \geqslant V \quad (i \neq j), \qquad |P(x_i)| \geqslant V \quad (V \geqslant T^{7/32}).$$

Then

$$R \leqslant CT^{\varepsilon}(TV^{-3} + T^{15/4}V^{-12}). \tag{3.8}$$

For $k > 35/4$ it was shown in [19] that

$$\overline{m}_k(T) = O\bigl(T^{(35k+38)/108+\varepsilon}\bigr). \tag{3.9}$$

By another method, based on the existence of a distribution function for $P(x)$ (see § 5), it was proved in [20] that the limits

$$\begin{gathered} \lim_{T\to\infty} T^{-(1+k/4)}\overline{m}_k(T) \qquad (k \leqslant 9), \\ \lim_{T\to\infty} T^{-(1+k/4)}m_k(T) \qquad (k = 1, 3, 5, 7, 9) \end{gathered} \tag{3.10}$$

exist. In contrast to [19], the estimates (1.7) with $\theta = 7/22$ (see (3.1)) were used, leading to a proof of (3.10) for $\overline{m}_k(T)$ with $k \leqslant A_0 = 28/3 = 9.33\dots$.

We note that *if* (3.7) *holds for all* k, *then* (1.2) *also holds.*

For $k \leqslant 9$ formulae of the form

$$m_k(T) = A_k T^{1+k/4} + O(T^{1+k/4-\delta_k}), \qquad \delta_k > 0, \tag{3.11}$$

have also been obtained. It was shown in [21] that

$$m_2(T) = A_2 T^{3/2} + O(T(\log T)^2), \tag{3.12}$$

$$\text{where} \quad A_2 = \frac{1}{3\pi^2} \sum_{n=1}^{\infty} \frac{r^2(n)}{n^{3/2}} = 1.69396\dots .$$

This result was slightly improved in [22], where it was shown that we can replace $O(T(\log T)^2)$ by $O(T(\log T)^{3/2} \log_2 T)$ on the right-hand side of (3.12). For $k = 3, 4$ the formula (3.11) was obtained in [23], and for $k = 5, 6, 7, 8, 9$ it was proved in [24].

4. Behaviour of $P(x)$ on short intervals

Everywhere below we assume that $x \in [T, 2T]$.

4.1. Local moments. We start by considering the local moments

$$m_k(H,T) = \int_T^{T+H} P^k(x)\,dx, \quad \overline{m}_k(H,T) = \int_T^{T+H} |P(x)|^k\,dx, \qquad H \ll T. \tag{4.1}$$

It was proved in [25] that

$$\lim_{T\to\infty} \frac{m_k(H,T)}{HT^{k/4}} = \left(1 + \frac{k}{4}\right) A_k, \qquad k = 2, 3, 4, \tag{4.2}$$

provided that

$$\frac{T^{1/2}\log T}{H} \to 0, \quad k = 2, \quad \text{or} \quad \frac{T^{3/4+\varepsilon}}{H} \to 0, \quad k = 3, 4, \tag{4.3}$$

as $T \to \infty$. Conditions on k and $H = H(T)$ ensuring that the limit

$$\lim_{T\to\infty} \frac{\overline{m}_k(H,T)}{HT^{k/4}} \tag{4.4}$$

exists were considered in [26] under the assumption that $|P(x)| \ll x^{\beta+\varepsilon}$. It was shown there that the limit (4.4) exists if $H \gg T^{\lambda}$ and

$$\begin{gathered} k \leqslant \min\left\{11, \frac{8\beta}{4\beta - 1}\right\}, \\ \text{where} \quad \lambda > \frac{1}{2} + \max\left\{0, (k-2)\left(\beta - \frac{1}{4}\right)\right\}. \end{gathered} \tag{4.5}$$

If k is an odd integer, then the same conditions (4.5) ensure that the limit

$$\lim_{T\to\infty} \frac{m_k(H,T)}{HT^{k/4}} \tag{4.6}$$

exists. Setting $\beta = 131/416$ (see (3.3)), we deduce that the limit (4.4) exists for $k \leqslant 9.7$, and the limit (4.6) exists for $k = 1, 3, 5, 7, 9$, provided that

$$H \gg T^{\lambda}, \qquad \lambda > \frac{1}{2} + \max\left\{0, (k-2)\frac{27}{416}\right\}, \tag{4.7}$$

and in particular, $\lambda > 0.5649$ ($k = 3$) and $\lambda > 0.6298$ ($k = 4$). In addition, it was shown in [26] that if (1.2) holds, then the limit (4.4) exists for all $k > 0$, and the limit (4.6) exists for all odd k. It was proved in [27] that for some $\delta > 0$ we have

$$\begin{aligned} m_3(H,T) &= A_3[(T+H)^{7/4} - T^{7/4}](1 + O(T^{-\delta})) \quad (T^{7/12+\varepsilon} \leqslant H \leqslant T), \\ m_4(H,T) &= A_4[(T+H)^2 - T^2](1 + O(T^{-\delta})) \qquad\quad (T^{3/4+\varepsilon} \leqslant H \leqslant T). \end{aligned} \tag{4.8}$$

This is much weaker than (4.7), but estimates of the form $P(x) = O(x^{\beta+\varepsilon})$ were not used in establishing this result. In [28] the problem of estimating $|P(n)|$ for $T - H < n < T + H$ was considered under the assumption that estimates are known for the quantity

$$F_p(n,H,T) = \frac{1}{2H}\int_{n-H}^{n+H} |P(x)|^p \, dx. \tag{4.9}$$

It was shown there that

$$|P(n)| \leqslant C_p\big(H\overline{r}(n)F_p\big)^{1/(p+1)}, \tag{4.10}$$

provided that

$$F_p \leqslant C_p\big(H\overline{r}(n)\big)^p \quad \text{and} \quad |P(n)| \leqslant CH\overline{r}(n). \tag{4.11}$$

In particular, if for $H \leqslant Cn^{\alpha+\varepsilon}$ we have

$$F_4 \leqslant Cn, \tag{4.12}$$

then

$$|P(n)| \leqslant Cn^{(\alpha+1)/5+\varepsilon}. \tag{4.13}$$

For $\alpha = 1/2$ this leads to the estimate $|P(n)| \leqslant Cn^{3/10+\varepsilon}$, which is better than all the known estimates (see § 3.1).

In [26] a local version of (3.8) was proved: if

$$T < x_1 < x_2 < \cdots < x_R < T + H, \qquad T^{1/2} \ll H \ll T,$$

then under the same assumptions as for (3.8), we have

$$R \ll T^{\varepsilon}(TV^{-3} + HT^{11/4}V^{-12}). \tag{4.14}$$

With use of this estimate it was shown there that for $2 \leqslant k \leqslant 11$ and $|P(x)| \leqslant x^{\beta+\varepsilon}$ with $\beta \geqslant 1/4$ we have

$$\int_T^{T+H} |P(x)|^k \, dx \ll HT^{k/4+\varepsilon} + T^{1+\beta(k-2)+\varepsilon}. \tag{4.15}$$

4.2. The difference $P(x+U)-P(x)$. A precise formula for the quantity

$$\int_T^{T+H} [\Delta(x+U)-\Delta(x)]^2\,dx$$

was obtained in [29]. Repeating the arguments there and using the truncated Voronoi formula and (2.11), we obtain a precise formula for the quantity

$$Q \equiv Q(T,H,U) = \int_T^{T+H} [P(x+U)-P(x)]^2\,dx \tag{4.16}$$

under the conditions

$$\varphi(T) \ll U \ll T^{1/2}\varphi(T) \quad \text{and} \quad H \leqslant T \qquad (\varphi(T) = \overline{r}(T)\log T). \tag{4.17}$$

This formula has the form

$$Q = Q_0 + \Delta Q, \tag{4.18}$$

where

$$Q_0 = \frac{1}{2\pi^2} \sum_{n \leqslant T/(2U)} \frac{r^2(n)}{n^{3/2}} \int_T^{T+H} x^{1/2} |e^{i\pi U\sqrt{n/x}} - 1|^2\,dx \tag{4.19}$$

and

$$\Delta Q \ll U^{1/2} H \biggl(\log \frac{\sqrt{T}}{U}\biggr) \varphi(T) + T\varphi^2(T). \tag{4.20}$$

If $U \ll T^{1/2}$, then

$$Q_0 \asymp UH \log \frac{\sqrt{T}}{U}, \tag{4.21}$$

and thus for

$$UH \log \frac{\sqrt{T}}{U} \gg T\varphi^2(T), \qquad U \ll T^{1/2}, \tag{4.22}$$

we have the sharp two-sided estimate

$$Q \asymp HU \log \frac{\sqrt{T}}{U}, \quad \text{that is,} \quad C_1 HU \log \frac{\sqrt{T}}{U} \leqslant Q \leqslant C_2 HU \log \frac{\sqrt{T}}{U}. \tag{4.23}$$

In particular, this yields

$$P(x+U) - P(x) = \Omega\biggl(U \log \frac{\sqrt{T}}{U}\biggr)^{1/2}. \tag{4.24}$$

This was noted in [30], where it was shown that for $H = T$ it follows from (4.18) that

$$\begin{gathered} Q = TU\biggl(A_1 \log \frac{\sqrt{T}}{U} + A_2\biggr) + \Delta Q, \\ \Delta Q \ll U^2\sqrt{T} + U^{1/2} H \biggl(\log \frac{\sqrt{T}}{U}\biggr)^{1/2} + T\varphi^2(T). \end{gathered} \tag{4.25}$$

Apart from [29], some important results were obtained in [31]. It was shown there, for instance, that

$$\int_T^{2T} \max_{v\leqslant U}\bigl(E(t+v)-E(t)\bigr)^2\,dt \ll TU(\log T)^5 \qquad (U\ll T^{1/2}).$$

The estimate

$$\int_T^{2T} \max_{v\leqslant U}|\Delta(x+v)-\Delta(x)|^2\,dx \ll TU(\log T)^5 \qquad (U\ll T^{1/2}) \tag{4.26}$$

can be proved in a similar way (see [32]). Following [32], we deduce that

$$\int_T^{2T} \max_{v\leqslant U}|P(x+v)-P(x)|^2\,dx \ll TU(\log T)^3 \qquad (U\ll T^{1/2}). \tag{4.27}$$

The power of $\log T$ in (4.27) is lower than in (4.26) because

$$\sum_{n\leqslant x} r^2(n) \sim x\log x, \tag{4.28}$$

whereas

$$\sum_{n\leqslant x} d^2(n) \sim x(\log x)^3. \tag{4.29}$$

Following [32], assume now that

$$\int_T^{T+H} \max_{v\leqslant U}|P(x+v)-P(x)|^2\,dx \ll HUT^{\varepsilon} \tag{4.30}$$

with the conditions

$$T^{\varepsilon}\ll U\ll T^{1/2}, \qquad T^{1/2}\ll H\leqslant T, \quad \text{and} \quad HU\gg T^{1+\varepsilon}. \tag{4.31}$$

Then (see [32]) *if* (4.30) *holds for* $H\ll T^{1/2+\varepsilon}$, *then* $|P(T)|=O(T^{1/4+\varepsilon})$. In [32] the estimate (4.30) was proved for $H\gg T^{3/4}$.

The asymptotic equalities (4.23) and (4.24) allow us to conjecture that

$$P(x+U)-P(x)\ll U^{1/2}T^{\varepsilon} \quad \text{for } T<x<2T \quad (U\ll T^{1/2-\varepsilon}). \tag{4.32}$$

This is called *Jutila's conjecture* and is the strongest assertion made so far about the local behaviour of $P(x)$. Jutila's conjecture means that in the neighbourhood $|x-n|<U$ of n the quantity $P(x)$ behaves like a random walk starting from $P(n)$ at time n. *A solution of the circle problem is a consequence of* (4.32). Some estimates for $|P(x+U)-P(x)|$ weaker than (4.32) were obtained in [32]. In particular, it was proved there that

$$\begin{aligned} |P(x+U)-P(x)| &\ll_\varepsilon x^{1/4+\varepsilon}U^{1/4} \qquad (1\ll U\ll x^{3/5}),\\ |P(x+U)-P(x)| &\ll_\varepsilon x^{2/9+\varepsilon}U^{1/3} \qquad (1\ll U\ll x^{2/3}). \end{aligned} \tag{4.33}$$

In the conclusion of this subsection we present several results from [28]. Let $n \in [T-H, T+H]$ and $H \ll T$. Then

$$P(n) = \frac{1}{2H}\int_{n-H}^{n+H} (P(n) - P(x))\,dx + \Delta_H P(n), \tag{4.34}$$

$$\Delta_H P(n) \leqslant \begin{cases} C_1 \sqrt{\dfrac{n}{H}}, & H \leqslant \dfrac{1}{2\pi}\sqrt{n}, \\ C_2 \dfrac{n^{3/4}}{H}, & H \geqslant \dfrac{1}{2\pi}\sqrt{n}. \end{cases} \tag{4.35}$$

This is a simple consequence of Landau's formula (2.14). Thus,

$$\Delta_H P(n) \ll n^{1/4+\varepsilon} \quad \text{for } H \gg T^{1/2-\varepsilon} \tag{4.36}$$

and therefore $|P(n)| \ll Cn^{1/4+\varepsilon}$ if

$$\int_n^{n+H} |P(n) - P(x)|^2\,dx \ll Hn^{1/2+\varepsilon} \qquad (H \gg n^{1/2-\varepsilon}). \tag{4.37}$$

Assume that

$$|P(x) - P(n)| \leqslant B|P(n)|, \qquad B < 1, \tag{4.38}$$

if

$$|x - n| \leqslant H \quad \text{and} \quad H \leqslant n^{\alpha-\varepsilon}, \qquad 0 < \alpha \leqslant \frac{1}{2}. \tag{4.39}$$

Then it follows from (4.34) that

$$|P(n)| \leqslant n^{(1-\alpha)/2+\varepsilon}. \tag{4.40}$$

The same estimate holds if

$$|P(x) - P(n)| < n^{\beta+\varepsilon}H^{\gamma}, \qquad \beta + \alpha\gamma < \frac{1-\alpha}{2}. \tag{4.41}$$

Let n be a local maximum point of $|P(x)|$ and assume that $|P(n)| \geqslant Cn^{1/4}$. We call this a *broad* maximum if (4.38) holds for $|x-n| < n^{1/2-\varepsilon}$. It follows from (4.40) that *if a maximum is broad, then* $|P(x)| \leqslant O(x^{1/4+\varepsilon})$ *for* $|x-n| < n^{1/2-\varepsilon}$.

5. Distribution of values

It was shown in [20] that the function

$$F(t) = \frac{P(t^2)}{t^{1/2}} \tag{5.1}$$

has a distribution function with density $\rho(\xi)$. This means that for an arbitrary probability density function h on $[0,1]$ and any continuous function g we have

$$\lim_{T\to\infty} T^{-1}\int_0^T g(F(t))h\Big(\frac{t}{T}\Big)\,dt = \int_{-\infty}^{+\infty} g(\xi)\rho(\xi)\,d\xi. \tag{5.2}$$

The uniform distribution of $x = t^2$ corresponds to $h(s) = 2s$, and for each interval $I = [a, b]$ we have

$$\lim_{T\to\infty} T^{-1}\mu\left\{x \in [0,T],\ \frac{P(x)}{x^{1/4}} \in I\right\} = \int_a^b \rho(\xi)\,d\xi, \tag{5.3}$$

where $\mu\{A\}$ is the Lebesgue measure of a set A.

It was shown in [20] that $\rho(\,\cdot\,)$ can be continued analytically to an entire function that decreases more rapidly than any power function on the real axis. This result was refined in [33]. Consider the quantities

$$D(u) = \int_{-\infty}^{u} \rho(\xi)\,d\xi \quad \text{and} \quad \widetilde{D}(u) = \begin{cases} D(u), & u < 0, \\ 1 - D(u), & u > 0. \end{cases} \tag{5.4}$$

It was shown in [33] that

$$\widetilde{D}(u) \ll \exp(-|u|^{4-\varepsilon}) \quad \forall\, \varepsilon > 0. \tag{5.5}$$

This was improved in [34], where it was shown that

$$\begin{gathered} \exp\biggl(-C_2\frac{|u|^4}{(\log u)^\alpha}\biggr) \ll \widetilde{D}(u) \ll \exp\biggl(-C_1\frac{|u|^4}{(\log u)^\alpha}\biggr), \\ \alpha = 3(2^{1/3} - 1). \end{gathered} \tag{5.6}$$

It was observed in [26] that in (5.3) the interval of change of x can be taken to be short and

$$\lim_{T\to\infty} \frac{1}{Y(T)}\,\mu\left\{x \in [T, T+Y(T)],\ \frac{P(x)}{x^{1/4}} \in I\right\} = \int_a^b \rho(\xi)\,d\xi \tag{5.7}$$

for all $Y(T) \gg T^{\varepsilon}$.

6. Change of sign and the behaviour of $P(x)$ on a long interval $[T, 2T]$

It was shown in [8] that for any $\tau \gg 1$ each of the two inequalities

$$\pm\, P(x) > C_1 x^{1/4} \tag{6.1}$$

is solvable on the interval $\tau < x < \tau + C_2\sqrt{\tau}$. (Here and below the assertion is that such constants C_1 and C_2 exist.) This also follows from later, more general results in [31], [35], and [36]. In particular, it was proved in [31] that if $|f(x)| < C_1 x^{1/4}$, then $P(x) + f(x)$ changes sign on the interval $[T, T + C_2\sqrt{T}\,]$. Thus, any interval $[T, T + C_2\sqrt{T}\,]$ contains points $x_1, x_2 \in [T, T + C_2\sqrt{T}\,]$ such that

$$P(x_1) > C_1 x_1^{1/4}, \quad P(x_2) < -C_1 x_2^{1/4} \qquad (x_i \in [T, 2T]). \tag{6.2}$$

The main result on the behaviour of $P(x)$ on an interval $[T, 2T]$ follows from [31] (see also [32]). In fact, the quantity $E(T)$ was considered in [31], and $\Delta(x)$ was considered in [32]. Using the methods from these papers, we can show that there exists a $\delta_0 > 0$ (which can be explicitly found) such that for any $\delta \leqslant \delta_0$ the interval $[T, 2T]$ contains disjoint subintervals

$$U_\alpha^\pm = [x_\alpha^\pm, x_\alpha^\pm + C_1(\delta)T^{1/2}(\log T)^{-3}] \tag{6.3}$$

such that

$$P(x) > \delta x^{1/4} \quad (x \in U_\alpha^+) \quad \text{and} \quad P(x) < -\delta x^{1/4} \quad (x \in U_\alpha^-). \tag{6.4}$$

Furthermore,

$$\mu\{V^\pm\} \geqslant C_2(\delta)T, \quad \text{where } V^\pm = \bigcup_\alpha U_\alpha^\pm, \tag{6.5}$$

and if $x, x+v \in U_\alpha^\pm$, then

$$\frac{1}{2}|P(x)| \leqslant |P(x+v)| \leqslant \frac{3}{2}|P(x)|. \tag{6.6}$$

Using (4.40), *we get that*

$$|P(x)| \leqslant Cx^{1/4+\varepsilon}, \qquad x \in V = V^+ \cup V^-, \tag{6.7}$$

and under the assumption that the maximum points of $|P(x)|$ *lie in* V, *we have* $P(x) = O(T^{1/4+\varepsilon})$ *for* $x \in [T, 2T]$.

Note that we cannot conclude from this result that the intervals $U_\alpha^\pm$ alternate on $[T, 2T]$, and the existence of large narrow maxima of $|P(x)|$ in the complement of V cannot be ruled out.

7. $P(x)$ and the universality conjecture

The universality conjecture, when it is valid, enables one to use results on $P(x)$ in a much more general framework. Let $M[g]$ be a closed Riemannian 2-manifold. With the Riemannian metric g we can canonically associate the Laplace operator $\Delta \equiv \Delta[g]$. This operator is negative definite and has a purely discrete spectrum $\{\lambda_i\}$ $(\Delta\varphi_i + \lambda_i\varphi_i = 0,\ 0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots)$. If $N(M[g])(x)$ is the distribution function for the eigenvalues, namely,

$$N(M[g])(x) = \#\{i \mid \lambda_i \leqslant x\}, \tag{7.1}$$

then by Weyl's formula

$$\begin{gathered} N(M[g])(x) = \frac{|M|}{4\pi}x + \Delta N(M[g])(x), \\ \Delta N([M[g])(x) = O(x^{1/2}). \end{gathered} \tag{7.2}$$

Here $|M|$ is the area of M with respect to the measure induced by g. Let M be the torus $\mathbb{T}^2 = [-\pi, \pi]^2$ with the flat metric g_0. Then $\lambda_i = n = l^2 + m^2$ $(l, m \in \mathbb{Z})$, $r(n)$ is the multiplicity of the eigenvalue n, and

$$\Delta N(M[g])(x) = P(x). \tag{7.3}$$

The theory of quantum chaos studies the statistical properties of the sequence $\{\lambda_i\}$ (see [2], [3], [37]).

The universality conjecture was formulated in the framework of this theory. It states that for generic metrics g the behaviour of $\Delta N(M[g])(x)$ on short intervals $(T, T + CT^{1/2})$ depends only on whether or not the geodesic flow $\Gamma_t[g]$ of the metric g is integrable [37]. If it is, then the normalized eigenvalues $\tilde{\lambda}_i = \dfrac{4\pi}{|M|}\lambda_i$ have a Poisson distribution, but if it is ergodic (non-integrable) then the $\tilde{\lambda}_i$ are distributed in accordance with the Gaussian Orthogonal Ensemble (GOE) [38].

It particular, the asymptotic equalities

$$T^{-1}\int_T^{2T} \bigl(\Delta N(M[g])(x+U) - \Delta N(M[g])(x)\bigr)^2\,dx \sim \begin{cases} U, & \Gamma_t[g] \text{ is integrable,} \\ \log U, & \Gamma_t[g] \text{ is ergodic} \end{cases} \quad (T \to \infty,\ U \ll \sqrt{T}\,), \tag{7.4}$$

follow from the universality conjecture (see [2] and [3]). *If a flat torus is a typical representative of $M[g]$ in the 'integrable class' of the universality conjecture, then the quantity $\Delta N(M[g])(x)$ must behave like $P(x)$ on short intervals.* In particular, the equality (4.25) must hold:

$$T^{-1}\int_T^{2T} \bigl(\Delta N(M[g])(x+U) - \Delta N(M[g])(x)\bigr)^2\,dx \sim AU\log\frac{\sqrt{T}}{U} + A_2U \quad (T \to \infty,\ U \ll \sqrt{T}\,). \tag{7.5}$$

For integrable geodesic flows the quantity $\Delta N(M[g])(x)$ has been investigated in two cases:

1) the metrics of surfaces of revolution in $\mathbb{R}^3$ [39];
2) Liouville metrics on $\mathbb{T}^2$ [40], [41].

In both cases it has been proved that

$$\Delta N(M[g])(x) = O(x^{1/3}). \tag{7.6}$$

(Here we mean generic metrics: for instance, there exist Liouville metrics on $\mathbb{T}^2$ satisfying certain Diophantine conditions such that $\Delta N(M[g])(x) = \Omega(x^{1/2-\varepsilon})$ for these metrics; see [42].)

It was shown in [39]–[41] that in these two cases the quantities $\Delta N(M[g])(t^2) \times t^{-1/2}$ have distribution functions with the same properties as we indicated in § 5 in the case of $P(t^2)t^{-1/2}$.

In conclusion we add a few words about the non-integrable case. On the one hand, for a generic metric there is a conjecture that

$$\Delta N(M[g])(x) = O(x^{\varepsilon}) \tag{7.7}$$

(see [40]). On the other hand, there are no examples where it has been shown that the power can be reduced, that is, where an estimate

$$\Delta N(M[g])(x) = O(x^{\beta+\varepsilon}), \qquad \beta < \frac{1}{2}\,, \tag{7.8}$$

holds. In particular, this is not known for Riemann surfaces $M = \Gamma\backslash H$ (where Γ is a generic strictly hyperbolic group and H is the upper half-plane with the Poincaré metric). We note that Selberg (see [43]) indicated an infinite series of groups Γ such that

$$\Delta N(M[g])(x) = \Omega\biggl(\frac{x^{1/4}}{\log x}\biggr) \qquad (M = \Gamma\backslash H). \tag{7.9}$$

However, these groups do not correspond to generic points in the Teichmüller space.

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