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Derivations on Murray–von Neumann algebras

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For a given algebra $\mathscr{A}$, a linear operator $D\colon \mathscr{A}\to\mathscr{A}$ is called a *derivation* if D satisfies the Leibniz rule, that is, $D(xy)=D(x)y+xD(y)$ for all $x,y\in\mathscr{A}$. Each element $a\in\mathscr{A}$ implements a derivation $\operatorname{ad}(a)$ on $\mathscr{A}$ defined by $\operatorname{ad}(a)(x)=[a,x]=ax-xa$, $x\in\mathscr{A}$. Such derivations are said to be *inner*.

Let H be a Hilbert space, let $B(H)$ be the $*$-algebra of all bounded linear operators on H, and let $\mathscr{M}$ be a von Neumann algebra, that is, a weakly closed unital $*$-subalgebra of $B(H)$ (for details see [11]).

A densely defined closed linear operator $x\colon \operatorname{dom}(x)\to H$ (here the domain $\operatorname{dom}(x)$ of x is a linear subspace of H) is said to be *affiliated* with $\mathscr{M}$ if $yx\subset xy$ for all y in the commutant $\mathscr{M}'$ of the algebra $\mathscr{M}$.

Denote the set of all projections in $\mathscr{M}$ by $P(\mathscr{M})$. Recall that two projections $e,f\in P(\mathscr{M})$ are said to be *equivalent* if there exists an element $u\in\mathscr{M}$ such that $u^*u=e$ and $uu^*=f$. A projection $p\in\mathscr{M}$ is said to be *finite* if the conditions $q\leqslant p$ and $q\sim p$ imply that $q=p$. A linear operator x affiliated with $\mathscr{M}$ is said to be *measurable* with respect to $\mathscr{M}$ if $\chi_{(\lambda,+\infty)}(|x|)$ is a finite projection for some $\lambda>0$. (Here $\chi_{(\lambda,+\infty)}(|x|)$ is the spectral projection of $|x|$ corresponding to the interval $(\lambda,+\infty)$). We denote the set of all measurable operators by $S(\mathscr{M})$.

Let $x,y\in S(\mathscr{M})$. It is well known that $x+y$ and xy are densely-defined and preclosed operators. Moreover, the closures of $x+y$, xy, and x^* are also in $S(\mathscr{M})$. The closures of $x+y$ and xy are called the *strong sum* and *strong product*, respectively. When equipped with these operations, $S(\mathscr{M})$ becomes a unital $*$-algebra over $\mathbb{C}$ (see [12] and [15]). It is clear that $\mathscr{M}$ is a $*$-subalgebra of $S(\mathscr{M})$. In the case when $\mathscr{M}$ is a finite von Neumann algebra, $S(\mathscr{M})$ is referred to as the *Murray-von Neumann algebra* associated with $\mathscr{M}$ [9].

The hypothesis that all derivations of the algebra $S(\mathscr{M})$ associated with a von Neumann algebra $\mathscr{M}$ of type II are inner was first conjectured by Ayupov (see [2] and [3]). As Kadison and Liu noted in [10], pp. 210–211 (see also [9], p. 2090), for type II_1 algebras "the complete cohomological result would say that each derivation of $S(\mathscr{M})$ is inner.... The authors *strongly* feel that this is true; but it is still open". In this paper we announce the complete solution of this cohomological problem for type II_1 von Neumann algebras $\mathscr{M}$.

Theorem 1. *Let $\mathscr{M}$ be a type* II_1 *von Neumann algebra, and let $S(\mathscr{M})$ be the Murray-von Neumann algebra of all operators affiliated with $\mathscr{M}$. Then any derivation of $S(\mathscr{M})$ is inner.*

In fact, we prove that any derivation of $S(\mathscr{M})$ is continuous in the topology of convergence in measure on $S(\mathscr{M})$, and then we use known results from [4], [5], and [7] giving us that any derivation of $S(\mathscr{M})$ which is continuous in this topology is necessarily inner.

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When $\mathscr{M}$ is an arbitrary von Neumann algebra, Sankaran [14] and Yeadon [16] introduced the algebra $LS(\mathscr{M})$ of locally measurable operators affiliated with $\mathscr{M}$, with the operations of strong sum and strong multiplication. An operator x affiliated with $\mathscr{M}$ is said to be locally measurable (with respect to $\mathscr{M}$) if there is a sequence $\{z_n\}_{n=0}^{\infty} \subset Z(\mathscr{M})$ of projections in the centre $Z(\mathscr{M})$ of $\mathscr{M}$ such that $z_n \uparrow \mathbf{1}$, $z_n(H) \subset \operatorname{dom}(x)$, and $xz_n \in S(\mathscr{M})$ for all $n \geqslant 0$.

Using Theorem 1 and results from [1], [6], and [7], we obtain a necessary and sufficient condition for the existence of a non-inner derivation of the algebras $S(\mathscr{M})$ and $LS(\mathscr{M})$. This result provides a complete answer to the problem posed by Ayupov in [2] and an adaptation of the celebrated Kadison–Sakai theorem [8], [13] to algebras of unbounded operators.

Corollary 2. *Let $\mathscr{M}$ be an arbitrary von Neumann algebra. Then the following assertions are equivalent*:

(a) *any derivation of $LS(\mathscr{M})$ (of $S(\mathscr{M})$) is inner*;

(b) *a type I_{fin} direct summand of $\mathscr{M}$ is atomic.*

In other words, the algebra $S(\mathscr{M})$ (or $LS(\mathscr{M})$) admits non-inner derivations if and only if the type I_{fin} direct summand of $\mathscr{M}$ is non-trivial and non-atomic.

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