DOI: https://doi.org/10.1070/RM9910

Linear systems with quadratic integral and complete integrability of the Schrödinger equation

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1. If a linear system

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n, \tag{1}$$

admits a quadratic first integral $f = (Bx, x)/2$, then it has the whole family of first integrals

$$f_m = \frac{1}{2}(B_m x, x), \qquad B_m = (A^*)^m B A^m, \tag{2}$$

where $m \in \mathbb{Z}_+$. If the operators A and B are non-singular, then n is even and (1) is a Hamiltonian system, with symplectic structure given by the antisymmetric operator BA^{-1} and with Hamiltonian f [1]. All the functions in (2) are pairwise in involution, and furthermore, if A has a simple spectrum, then precisely $n/2$ of the quadratic forms (2) are functionally independent [2]. (This is not so when the spectrum is multiple.)

These observations do not depend on the dimension of the phase space and thus can be carried over to the infinite-dimensional case. The case when linear systems acting in a Hilbert space are Hamiltonian was discussed in [3].

2. Using this trick, we can produce an infinite family of quadratic integrals for the wave equation in a domain $D \subset \mathbb{R}^p$:

$$u_{tt} = a^2 \Delta u, \qquad u\big|_{\partial D} = 0.$$

Theorem 1. *The wave equation admits the quadratic integrals*

$$\begin{aligned} f_0 &= \frac{1}{2}\int u_t^2\, d^p x + \frac{a^2}{2}\int \sum u_{x_i}^2\, d^p x, \\ f_1 &= \frac{a^2}{2}\int \sum u_{tx_i}^2\, d^p x + \frac{a^4}{2}\int (\Delta u)^2\, d^p x, \\ f_2 &= \frac{a^4}{2}\int (\Delta u_t)^2\, d^p x + \frac{a^6}{2}\int \sum (\Delta u_{x_i})^2\, d^p x, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{3}$$

The form f_0 is the total energy of the system. The integration is over D. In the general case the first integrals in (3) are independent, but the operators B_m involved in the definitions of these quadratic forms are unbounded (being operators of differentiation).

In some cases the operator A in the wave equation is invertible, and for $m < 0$ all the self-adjoint operators B_m are bounded in the metric on the phase space induced by the total energy. For instance, this is so for an elastic string with periodic

AMS 2010 *Mathematics Subject Classification.* Primary 37K05, 35L05, 35Q41.

boundary conditions and Cauchy data with mean value zero. In this case the wave equation should be regarded as a completely integrable infinite-dimensional Hamiltonian system.

3. Let v be a complete smooth vector field on a p-manifold $\Gamma = \{x\}$ whose phase flow preserves a measure $d\mu = \lambda(x)\, d^p x$ with smooth positive density λ: $\operatorname{div} \lambda v = 0$. Suppose that there is also another non-stationary invariant measure $\rho\, d\mu$. In this case Liouville's equation, a basic equation of statistical mechanics, leads to a linear evolution equation with respect to $\rho(t, x)$:

$$\frac{\partial \rho}{\partial t} + L\rho = 0, \tag{4}$$

where L is the operator of differentiation along the field v. In other words, ρ is a first integral for the dynamical system $\dot{x} = v(x)$ on Γ. It is known that for each solution of (4) we have

$$\int_\Gamma f(\rho)\, d\mu = \text{const},$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is an arbitrary measurable function such that $f(\rho)$ is μ-integrable.

Theorem 2. *Equation* (4) *admits the chain of first integrals*

$$\int_\Gamma f(L^{(m)}\rho)\, d\mu,$$

where the $m \geqslant 0$ are integers and $L^{(m)}$ is the mth power of L.

For a Hamiltonian system acting in $\mathbb{R}^{2n}$ the operator L is the Poisson bracket with the Hamiltonian function.

4. The evolution of a quantum system is described by the linear Schrödinger equation

$$i\,\frac{\partial \psi}{\partial t} = H\psi, \qquad H = H^*, \tag{5}$$

in the complex Hilbert space $\mathbb{H} = \{\psi\}$ (with Hermitian product $\langle\cdot,\cdot\rangle$). If F is another Hermitian operator which commutes with H, then $\langle F\psi, \psi\rangle$ is a first integral of (6). The Shrödinger equation is Hamiltonian with respect to the natural symplectic structure arising upon realification of (5).

Theorem 3. *Equation* (5) *admits the chain of first integrals in involution*

$$f_m = \langle H^m \psi, \psi\rangle, \qquad m \geqslant 0 \text{ an integer.} \tag{6}$$

If the Hamiltonian operator H is invertible, then for negative integers m the functions (6) are also integrals in involution. Furthermore, if H has a simple discrete spectrum, then the Hamiltonian system (5) can be regarded as completely integrable. In particular, when $\mathbb{H} = \mathbb{C}^n$, these conditions ensure that among the functions in (6) there is a complete system of functionally independent integrals in involution for the finite-dimensional quantum system.

The question of complete integrability of a quantum system has repeatedly been discussed from other standpoints (for instance, see [4]–[6]). In [6]–[8] the authors investigated conditions for the existence of differential operators commuting with the Hamiltonian operator. However, the fact that there are no additional 'non-trivial' symmetries does not affect the complete integrability of the Schrödinger equation in the case when the operator H has a simple discrete spectrum.

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Presented by D. V. Treschev
Accepted 10/SEP/19
Translated by N. KRUZHILIN