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Selecting a dense weakly lacunary subsystem in a bounded orthonormal system

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Let $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ be an orthonormal system of functions (o.n.s.), on a probability space (X, μ). It is called a p-lacunary system ($p > 2$) or an S_p-system if for some constant K the following inequality holds for any polynomial $P = \sum_{k=1}^{N} a_k\varphi_k$:

$$\|P\|_{L_p} \leqslant K\|P\|_{L_2} \tag{1}$$

(see [1] and [2] for details). The following result was stated in [1] with a reference to Banach [3].

Theorem A. *If $p > 2$ and an o.n.s. $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ satisfies*

$$\|\varphi_k\|_{L_p} \leqslant C, \qquad k = 1, 2, \dots, \tag{2}$$

then there exists an infinite subset Λ of the natural numbers such that $\{\varphi_k\}_{k\in\Lambda}$ is an S_p-system.

Analogues of Theorem A for an o.n.s. whose elements are uniformly bounded in the norm of an Orlicz space L_Ψ, where the N-function $\Psi(t)$ increases slower than any power $|t|^p$ with $p > 2$ as $|t| \to \infty$, are due to Balykbaev [4], [5]. In particular, it was shown in [5] that the analogue of Theorem A holds for the Orlicz spaces L_{ψ_α} with

$$\psi_\alpha(t) = t^2\frac{\log^\alpha(e + |t|)}{\log^\alpha(e + 1/|t|)}, \qquad \alpha > 0. \tag{3}$$

The natural question of the maximum density of S_p-subsystems in a given o.n.s. (that is, of the density of the sequence Λ in Theorem A) proved to be quite complicated. Even for the trigonometric system it had remained open until the breakthrough paper [6], where Bourgain established the following result.

Theorem B. *If $p > 2$ and $\Phi = \{\varphi_k\}_{k=1}^{N}$ is an o.n.s. such that*

$$\|\varphi_k\|_{L_\infty} \leqslant M, \qquad k = 1, 2, \dots, N, \tag{4}$$

then there exists a set $\Lambda \subset \langle N\rangle$ such that $|\Lambda| \geqslant N^{2/p}$ and (1) holds for $K = K(M, p)$ for any polynomial $P = \sum_{k\in\Lambda} a_k\varphi_k$.

(Here and below, $\langle N\rangle := \{1, 2, \dots, N\}$ and $|\Lambda|$ is the cardinality of a finite set Λ. We also set $\Phi_\Lambda = \{\varphi_k\}_{k\in\Lambda}$.) Note that for quantitative results like Theorem B,

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some conditions in addition to the necessary condition (2) must be imposed on the original system Φ, such as (4) (see [6] for details).

Let $\Lambda \subset \langle N \rangle$ and let S_Λ be the operator acting by $S_\Lambda(\{a_k\}_{k\in\Lambda}) = \sum_{k\in\Lambda} a_k\varphi_k(x)$. It is clear that for the set Λ in Theorem B we have

$$\|S_\Lambda \colon l^\infty(\Lambda) \to L^p(X)\| \leqslant |\Lambda|^{1/2} \cdot K(M,p). \tag{5}$$

In this paper we establish analogues of (5) for Orlicz spaces L_{ψ_α} (see (3)) in the case of an arbitrary o.n.s. with uniformly bounded elements. Of course, in this case the guaranteed density of Λ is greater than in Theorem B. The proof of our results here is based on a modification of Bourgain's method in [6]. As in that paper, we average over all subsystems Φ with given cardinality. However, for the Orlicz space corresponding to the function (3) one cannot expect that a random subsystem with cardinality $\geqslant N/(\log N)^\beta$ (where β is an arbitrarily large constant) will be ψ_α-lacunary. Thus, it is natural to search for subsystems Φ_Λ with an analogue of the property (5), which is weaker than being p-lacunary.

Let $N \in \mathbb{N}$ and $\delta \in (0,1)$, and let $\{\xi_i(\omega)\}_{i=1}^N$ be a set of independent random variables (selectors) on a probability space (Ω,ν) such that $\xi_i(\omega) = 0$ or 1 and $\mathsf{E}\xi_i = \delta$ for $1 \leqslant i \leqslant N$. Also let $\Lambda(\omega,N) = \{i \in \langle N\rangle \colon \xi_i(\omega) = 1\}$ for $\omega \in \Omega$.

Theorem 1. *Fix $\alpha > 0$ and $\rho > 0$ and let $\Phi = \{\varphi_k\}_{k=1}^N$ be an arbitrary o.n.s. with the property (4). Then with probability greater than $1 - N^{-10}$ the random set $\Lambda = \Lambda(\omega,N)$ generated by the system of random variables $\{\xi_i(\omega)\}_{i=1}^N$ with $\mathsf{E}\xi_i = \delta = [\log(N+3)]^{-\rho}$ for $1 \leqslant i \leqslant N$ satisfies the inequality*

$$\|S_\Lambda \colon l^\infty(\Lambda) \to L_{\psi_\alpha}(X)\| \leqslant K(M,\alpha,\rho)|\Lambda|^{1/2}([\log(N+3)]^{\alpha/2-\rho/4} + 1). \tag{6}$$

Corollary 1. *For $\delta = [\log(N+3)]^{-2\alpha}$ the operator $S_\Lambda \cdot |\Lambda|^{-1/2}$ is bounded from $l^\infty(\Lambda)$ to $L_{\psi_\alpha}(X)$ with probability close to 1.*

For an o.n.s. $\Phi = \{\varphi_k\}_{k=1}^N$ consider the operator S^*_Φ taking the majorant of the partial sums, which acts on $\{a_k\}$ by the formula

$$S^*_\Phi(\{a_k\})(x) = \sup_{s=s(x)} \biggl|\sum_{k=1}^s a_k\varphi_k(x)\biggr|.$$

It is well known (see [2]) that a system Φ being lacunary is useful in the analysis of convergence almost everywhere of orthogonal series, because this property enables one to improve estimates for the norm of S^*_Φ. The following theorem is proved using results close to Theorem 1.

Theorem 2. *For $\rho > 4$ and an arbitrary o.n.s. $\Phi = \{\varphi_k\}_{k=1}^N$ with the property (4), there exists a set $\Lambda \subset \langle N\rangle$ with cardinality $|\Lambda| \geqslant N[\log(N+3)]^{-\rho}$ such that*

$$\|S^*_{\Phi_\Lambda} \colon l^\infty(\Lambda) \to L^2(X)\| \leqslant C(M,\rho)|\Lambda|^{1/2}. \tag{7}$$

Remark 1. For $\rho > 4$ the estimate (7) holds for most subsets of $\langle N\rangle$ with cardinality of order $N[\log(N+3)]^{-\rho}$.

Remark 2. For $\rho < 2$ the assertion of Theorem 2 does not hold.

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