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Minimal embeddings of integrable processes in a Brownian motion

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Let $X = (X_s)_{s\geqslant 0}$ be a right-continuous random process with limits from the left. We say that X is embedded in a Brownian motion if, on some probability space with a filtration $\mathbb{F}$, there exist a standard $\mathbb{F}$-Brownian motion $B = (B_t)_{t\geqslant 0}$, $B_0 = 0$, and a time-change $(T_s)_{s\geqslant 0}$ (called an *embedding*), that is, a family of almost surely finite $\mathbb{F}$-stopping times T_s with right-continuous non-decreasing trajectories $s \rightsquigarrow T_s$ such that $\operatorname{Law}(X_s; s \geqslant 0) = \operatorname{Law}(B_{T_s}; s \geqslant 0)$. Recall that an almost surely finite $\mathbb{F}$-stopping time T is said to be *minimal* if the conditions that S is an $\mathbb{F}$-stopping time, $S \leqslant T$, and $\operatorname{Law}(B_S) = \operatorname{Law}(B_T)$ imply that $S = T$ almost surely. By a *minimal embedding* we mean a time-change $(T_s)_{s\geqslant 0}$ for which all $\mathbb{F}$-stopping times T_s are minimal.

It is known that a minimal embedding exists if X is a martingale (Theorem 11 in Monroe's paper [4]) or if X is a submartingale that is bounded above and has $\mathsf{E}X_0 \geqslant 0$ (Theorem 3.4 in [3]). The purpose of the present note is to characterize all integrable processes X (that is, processes satisfying $\mathsf{E}|X_s| < \infty$ for all $s \geqslant 0$) that are right-continuous with limits from the left and have a minimal embedding in a Brownian motion.

Theorem. *Let $X = (X_s)_{s\geqslant 0}$ be a right-continuous process and let $\mathsf{E}|X_s| < \infty$ for all $s \geqslant 0$. Then there is a minimal embedding of the process X in a Brownian motion if and only if X is a submartingale with $\mathsf{E}X_0 \geqslant 0$ or a supermartingale with $\mathsf{E}X_0 \leqslant 0$.*

The proof makes essential use of the minimality criteria for stopping times (see Theorem 3 in [4] and Theorem 5 in [1]). In particular, let $B = (B_t)_{t\geqslant 0}$ be an $\mathbb{F}$-Brownian motion and let T be an $\mathbb{F}$-stopping time, $\mathsf{P}(T < \infty) = 1$. Then the following conditions are equivalent: (i) $\mathsf{E}|B_T| < \infty$, $\mathsf{E}B_T \geqslant 0$, and T is mimimal; (ii) the family $\{B_S^+\}$ is uniformly integrable, where S is an arbitrary $\mathbb{F}$-stopping time such that $S \leqslant T$; (iii) $\mathsf{E}|B_T| < \infty$ and $\mathsf{E}B_{T\wedge H_c} \leqslant \mathsf{E}B_T$ for any $c > 0$, where $H_c = \inf\{t\colon B_t > c\}$ (cf. the condition (iv) in Theorem 5 of [1]).

The necessity follows from the implication (i) $\Rightarrow$ (ii) (note that (ii) is equivalent to saying that $(B_{t\wedge T})_{t\geqslant 0}$ is a closed submartingale). The proof of existence of a minimal embedding for the submartingale X follows the scheme of the proof of Theorem 11 in [4]. In the first step, we construct a minimal embedding for the submartingale $X = (X_n)_{n=0,1,\dots}$, $X_0 = 0$, with discrete time (with arbitrary B and $\mathbb{F} = (\mathscr{F}_t)$). Setting $T_0 = 0$, we assume that we have already constructed minimal $T_1, \dots, T_n$ such that $\mu = \operatorname{Law}(B_{T_1}, \dots, B_{T_n}) = \operatorname{Law}(X_1, \dots, X_n)$. Let $\nu_x(dy)$, $x \in \mathbb{R}^n$, be the regular conditional distribution $\mathsf{P}(X_{n+1} - X_n \in dy \mid (X_1, \dots, X_n) = x)$. Without loss of generality we assume that, for all x, the measure ν_x is integrable and has non-negative mean value.

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By the strong Markov property, the process $\widetilde{B}_t = B_{T_n+t} - B_{T_n}$ is a Brownian motion relative to the filtration $\widetilde{\mathbb{F}} = (\mathscr{F}_{T_n+t})$; in particular, this process is independent of the σ-algebra $\mathscr{F}_{T_n}$. For each $x \in \mathbb{R}^n$, we construct the Skorokhod embedding for ν_x (by this we mean a minimal $\widetilde{\mathbb{F}}$-stopping time S_x such that $\operatorname{Law}(\widetilde{B}_{S_x}) = \nu_x$). Our purpose is to define T_{n+1} to be $T_n + S$, where $S = S_{(B_{T_1},\dots,B_{T_n})}$. For this purpose, S_x should depend 'measurably' on x. The required properties are satisfied, for example, in a variant of the Chacon–Walsh construction in Theorem 11 of [2] (in this construction, the sequence $\{v_n\}$ should be the same for all x). Then the conditional distribution $\widetilde{B}_S$ given $(B_{T_1},\dots,B_{T_n}) = x$ is ν_x. We claim that T_{n+1} is a minimal stopping time. For an arbitrary $c > 0$, we set $R = (T_n \vee H_c) \wedge T_{n+1}$ and $\widetilde{R} = R - T_n$. By (ii) and the definition of R, the condition (ii) is satisfied not only for $T = T_n$, but also for $T = R$. Hence $\mathsf{E}B_R \geqslant 0$. On the other hand, $\mathsf{E}\widetilde{B}_{S_x} \geqslant \mathsf{E}\widetilde{B}_{\widetilde{R}\wedge S_x}$ for any $x \in \mathbb{R}^n$ since S_x is minimal. Therefore, $\mathsf{E}\widetilde{B}_S = \int \mathsf{E}\widetilde{B}_{S_x}\,\mu(dx) \geqslant \int \mathsf{E}\widetilde{B}_{\widetilde{R}\wedge S_x}\,\mu(dx) = \mathsf{E}\widetilde{B}_{\widetilde{R}}$; that is, $\mathsf{E}B_{T_{n+1}} \geqslant \mathsf{E}B_R$.

The second step of the proof is to pass from discrete to continuous time. We give only the key points. Let $X^n_s = X_{2^{-n}\lfloor 2^n s\rfloor}$. Next, embeddings will be regarded as measures on the canonical space of trajectories corresponding to the pair of processes consisting of a Brownian motion and a time-change, the filtration being the smallest one with respect to which the first process is adapted and the second consists of stopping times. We thus get the embedding Q^n corresponding to the embedding for X^n constructed in the first step. The tightness condition, which guarantees the existence of a subsequence converging to some measure Q (then with respect to Q the canonical processes embed X), and the minimality of the embedding constructed are verified as in the proof of Theorem 11 in [4], in which the two-sided Doob inequality should be replaced by the one-sided inequality and the absolute values should be replaced by the positive parts (see (ii) and (iii)).

To conclude, we note that, in general, minimal embeddings no longer exist when we drop the condition that the random variables X_t be integrable. Consider the following example. Let $(X_n)_{n=0,1,\dots}$ be a Markov chain with set of states $\{0, \pm1, \pm2, \pm4, \dots, \pm2^n, \dots\}$ and transition probabilities $\mathsf{P}(X_1 = \pm1 \mid X_0 = 0) = 1/2$, $\mathsf{P}(X_1 = 0 \mid X_0 = 2^n) = \mathsf{P}(X_1 = 2^{n+1} \mid X_0 = 2^n) = 1/2$, and $\mathsf{P}(X_1 = 0 \mid X_0 = -2^n) = \mathsf{P}(X_1 = -2^{n+1} \mid X_0 = -2^n) = 1/2$, $n = 0, 1, \dots$. This chain has the invariant distribution $\mathsf{P}(X_0 = 0) = 1/3$ and $\mathsf{P}(X_0 = \pm 2^n) = 2^{-n}/6$, $n = 0, 1, \dots$. The process (X_n) is a generalized martingale (see [5]), that is, a discrete-time local martingale. However, $\operatorname{Law}(X_n)$ does not depend on n, and hence even the pair (X_0, X_1) cannot be realized by minimal stopping times.

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