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BRIEF COMMUNICATIONS

Dirichlet problem for the Yang–Mills equations

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This paper was motivated by Donaldson's paper [2] on the Yang–Mills equations on complex manifolds. He proved there the existence of a unique solution of the Dirichlet problem for Yang–Mills fields on a Hermitian vector bundle over a Kähler manifold with smooth boundary. One consequence of this result is a theorem on extending CR-structures on a Hermitian vector bundle over the boundary of a strictly pseudoconvex domain with smooth boundary in the interior of the domain, to a connection associated with a Hermitian Yang–Mills metric. In [2] this was proved in dimension 2. We show that Donaldson's result holds in each dimension (as was conjectured in [2]).

Let E be a holomorphic vector bundle over a complex manifold Z and let H be a fibrewise Hermitian metric on E. It is known that there exists a unique unitary connection on E that is compatible with the complex structure. Let F_H denote its curvature. Fix a local holomorphic trivialization of E by sections (s_j). In it H is given by a Hermitian matrix with entries $H_{ij} = (s_i, s_j)_H$. Then the connection has the matrix $H^{-1}\partial H$, and its curvature is given by

$$F_H = \overline{\partial}(H^{-1}\partial H) = H^{-1}(\overline{\partial}\partial H - \overline{\partial} H H^{-1}\partial H).$$

Let $\Omega^{p,q}(Z)$ denote the space of smooth (p,q)-forms on Z. Assume in addition that Z is endowed with a Kähler metric; let ω be its Kähler form. We define a contraction $\Lambda\colon \Omega^{1,1}(Z) \to \Omega^{0,0}(Z)$ by $\Lambda(\theta) = (\omega, \theta)$, where (ω,θ) denotes the (pointwise) scalar product on $\Omega^{1,1}(Z)$.

By definition the Hermitian Yang–Mills tensor of the metric H on E is $i\Lambda F_H$, and the corresponding Yang–Mills equation has the form

$$i\Lambda F_H = 0. \tag{1}$$

Its solutions are called Hermitian Yang–Mills metrics. Equation (1) is a non-linear partial differential equation of the second order with principal part equal to the

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Laplacian Δ. More precisely, on a Kähler manifold the Laplacian is given by $\Delta = 2\partial^*\partial = 2i\Lambda\overline{\partial}\partial$, so that we can write (1) as

$$Hi\Lambda\overline{\partial}(H^{-1}\partial H) = \frac{1}{2}\Delta H - i\Lambda\overline{\partial}HH^{-1}\partial H = 0.$$

In this note we consider the case when Z is the interior of a compact Kähler manifold $\overline{Z}$ with smooth boundary, and the Kähler form ω on Z extends to a smooth non-degenerate form on $\overline{Z}$. We also assume that the holomorphic bundle E extends smoothly to the boundary. It was proved in [2] that if f is a Hermitian metric on the restriction of E to ∂Z, then there exists a unique metric H on E such that $H = f$ over ∂Z and (1) holds on Z.

This can be seen as a solution of the Dirichlet problem for the Yang–Mills equation (1). As a consequence, we have a theorem on extension of CR-structures. (We refer the reader to [1] for a thorough presentation of the geometry of CR-structures.)

Let J denote the complex structure on $\overline{Z}$ which is smooth up to the boundary ∂Z. Let $V = T(\partial Z) \cap JT(\partial Z)$ be the complex tangent bundle over ∂Z and let $\overline{\partial}_b$ be the tangential Cauchy–Riemann operator induced on ∂Z by the structure J. A CR-structure on E over ∂Z is given by an operator

$$\overline{\partial}_b^E\colon \Gamma(E) \to \Gamma(E \otimes_{\mathbb{C}} V^{0,1})$$

such that $\overline{\partial}_b^E(fs) = \overline{\partial}_b(f)s + f\overline{\partial}_b^E(s)$, where s is a smooth section of E and f is a smooth function on ∂Z.

The following theorem is our main result here.

Theorem 1. *The CR-structure $\overline{\partial}_b^E$ over ∂Z extends to a connection on the vector bundle E over $\overline{Z}$ associated with a Hermitian Yang–Mills metric if and only if the operator $\overline{\partial}_b^E$ is holomorphically trivial, which means that there exists a trivialization of E in which $\overline{\partial}_b^E$ is a sum of several copies of $\overline{\partial}_b$.*

As in [2], Theorem 1 is a consequence of the fact that the Dirichlet problem for the Yang–Mills equation (1) is solvable. The proof of this fact in [2] was based on a version of the Oka–Grauert principle for manifolds with boundary that takes into account the smoothness up to the boundary. The indicated version was established in [2] for dimension 2, but here we use the proof in [3] of this version of the Oka–Grauert principle which is valid in any dimension (see also [4]).

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