Evgenii Solomonovich Golod
(obituary)

The well-known Soviet and Russian algebraist Evgenii Solomonovich Golod passed away at the age of 82 on 5 July 2018 after a short serious illness.

Golod was born on 21 October 1935 in Moscow in a family of office workers. In 1953 he finished school with a gold medal in the city of Ivanovo and enrolled in the Faculty of Mechanics and Mathematics of Moscow State University, where he began attending I. R. Shafarevich's seminar during his first year and soon after became a student of Shafarevich.

After graduation he began Ph.D. studies in the same faculty, and on finishing in 1964 he defended his Ph.D. thesis, "On the homology of finite p-groups and local rings". Then for a number of years he taught at the Textile Institute in Moscow and at its branch in the nearby town of Pavlovskii Posad.

In 1961, on an invitation of A. G. Kurosh he started working in the Department of Higher Algebra of the Faculty of Mechanics and Mathematics at Moscow State University, first as an assistant (part-time) and after 1966 as a senior lecturer.

In 1999 Golod defended his D.Sc. thesis, "The Shafarevich complex and its applications", and from 2000 he worked in the Department of Higher Algebra as a professor, though it was in 2005 that he received the formal title of professor. In 2011 he was named a professor emeritus of Moscow University. The laconic and precise style with which he presented his research results and his academic material was perfected by his decades of work in the "Mathematics" section of the abstracts journal published by the All-Russian (formerly All-Union) Institute for Scientific and Technical Information, where he was the editor of one of the sections. Moreover, this work in the abstracts journal was invaluable from the viewpoint of keeping Soviet (and since the 1990s, Russian) mathematicians up-to-date about new results of foreign researchers.

Golod's research interests were concentrated around homological and commutative algebra. Altogether he published 23 research papers. Their total length is not great, but they contain quite a few beautiful and striking results, many of which are actively being used and developed to the present day.

He wrote his first research paper [1] as a third-year student in the Faculty of Mechanics and Mathematics. There he gave an affirmative answer to a question raised in a book by Cartan and Eilenberg,[1] on the existence of Noetherian projective but non-free modules over the integer group ring $\mathbb{Z}[G]$ of a finite group G. Let G be a cyclic group of order p, where p is a prime such that the pth cyclotomic field is multiclass. In this case, he constructed two non-free $\mathbb{Z}[G]$-modules M_1 and M_2 such that

$$M_1 \oplus M_2 \cong \mathbb{Z}[G] \oplus \mathbb{Z}[G],$$

that is, M_1 and M_2 are projective.

New striking theorems then followed: a proof that the cohomology algebra of a finite p-group is finitely generated [2] (this theorem was soon generalized to arbitrary finite groups by B. A. Venkov and L. Evens) and a description of the mathematical objects that have since come to be called Golod rings [3] — local and graded rings with maximal Betti numbers for given dimensions of the homology groups of the Koszul complex. If A is a commutative Noetherian local ring and a minimal system of generators of its maximal ideal $\mathfrak{m}$ consists of m elements, then the Poincaré function $F_A(t)$ of the ring A satisfies the inequality

$$F_A(t) \leqslant \frac{(1+t)^m}{1 - \sum c_i t^{i+1}},$$

where c_i is the dimension of the homology group $H_i(K)$ of the Koszul complex K. This inequality was discovered by J.-P. Serre, and thus Golod answered the question of when the inequality is sharp. He proved that it becomes an equality if and only if the homology algebra $H(K)$ of the Koszul complex has zero multiplication and all the Massey operations corresponding to simplexes are equal to zero.

The proof is based on the fact that the above inequality becomes an equality if and only if the Betti numbers of the ring A satisfy a certain recurrence relation, which can also be expressed in terms of the minimal projective resolution of the A-module $k = A/\mathfrak{m}$. This resolution is constructed explicitly, and it turns out that the construction can be carried out if and only if all the multiplications in $H(K)$ are zero, as well as all the Massey operations corresponding to simplexes, that is (using the modern terminology), the A_∞-algebra structure on the homology is trivial. These results became a basis for his Ph.D. thesis. They became widely known and lay at the basis of the notions of *Golod rings* in homological algebra and *Golod simplicial complexes* in toric topology.

It can be said without exaggeration that Golod's next paper [4] (written jointly with Shafarevich) made him known world-wide. It contains a negative solution of the famous class field tower problem. Already Hilbert conjectured (and proved in special cases) that for any algebraic number field K, its maximal Abelian unramified extension K_1 has finite degree over K, and the Galois group $G(K_1/K)$ is isomorphic to the ideal class group of the field K. Moreover, all the ideals of K become principal ideals of K_1. Subsequently these conjectures were proved, and the field K_1 became known as the Hilbert class field of the field K.

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[1]See H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, NY 1956.

For the field K_1, it is possible to consider its Hilbert class field K_2, and so on. Thus, we obtain a chain of fields

$$K \subseteq K_1 \subseteq K_2 \subseteq \cdots ,$$

which is called the class field tower. The class field tower problem consisted in the question as to whether the tower is always finite. This is equivalent to the statement that any algebraic number field K can be embedded into a one-class field, and the simplest examples apparently suggested that this was indeed the case.

A new approach to the problem was proposed in Shafarevich's paper "Extensions with given ramification points".[2] For a fixed prime p the so-called p-class field tower was considered there, that is, instead of the Hilbert class field K_1 of K, the Hilbert p-class field $K_1^{(p)}$ was considered, which is the maximal subfield of K_1 such that $K_1^{(p)}/K$ is a p-extension. If $K_2^{(p)}$ is the Hilbert p-class field of the field $K_1^{(p)}$ and, generally, $K_{i+1}^{(p)}$ is the Hilbert p-class field of the field $K_i^{(p)}$, then the chain of fields

$$K \subseteq K_1^{(p)} \subseteq \cdots \subseteq K_i^{(p)} \subseteq \cdots$$

is called the p-class field tower. For a negative solution of the class field tower problem it is sufficient to prove that the p-class field tower is infinite for some prime p. Let $K_\infty^{(p)} = \bigcup_{i=1}^\infty K_i^{(p)}$ be the maximal unramified p-extension of the field K, and let G be the Galois group of the extension $K_\infty^{(p)}/K$. Then G is a pro-p-group. Using class field theory, Shafarevich obtained in the indicated paper an estimate for the minimal number of generators $d = d(G)$ and the minimal number of relations $r = r(G)$, for G as a pro-p-group. In particular, he showed that the inequality $r - d \leqslant \rho + \delta$ always holds, where ρ is the rank of the group of units of the field K, while $\delta = 0$ if $\zeta_p \notin K$ and $\delta = 1$ otherwise. Here, ζ_p is a primitive root of unity of degree p. Thus if, for example, K is a cyclic extension of the field $\mathbb{Q}$ of degree p, then d is equal to the minimal number of generators of the class p-group of the field K, and d can be arbitrarily large if the discriminant of K contains sufficiently many different prime divisors, while $r - d \leqslant p - 1$. This follows from Dirichlet's unit theorem (if K is a real quadratic field, then $r - d \leqslant 2$).

By considering examples of various finite p-groups, Shafarevich conjectured that for any infinite sequence of finite p-groups $\{G_i\}$ such that $\lim_{i \to \infty} d(G_i) = \infty$, the condition

$$\lim_{i \to \infty} (r(G_i) - d(G_i)) = \infty$$

must also hold. Essentially, [4] was devoted to the proof of this conjecture. The paper also contained the proof of the famous Golod–Shafarevich inequality

$$r(G) > \left(\frac{d(G) - 1}{2} \right)^2 , \qquad (*)$$

which is valid for any finite p-group G.

The proof is based on a reformulation of the problem in terms of the group algebra A of G over the field of p elements, which is the quotient algebra of the algebra

[2] *Publ. Math. Inst. Hautes Études Sci.*, **18** (1963), 295–319.

of non-commutative polynomials in d variables. The cohomology of a certain complex M connected with A is considered, a complex that is a non-commutative analogue of the Koszul complex and is nowadays called the Shafarevich complex.

Golod obtained inequalities connecting the coefficients of certain Poincaré series that are defined in terms of the cohomology of the complex M and proved that these inequalities cannot hold for the finite-dimensional algebra A if $(*)$ does not hold.

The paper [4] presented a simplest example of a field that has an infinite p-class field tower (for $p = 2$). This is the imaginary quadratic field

$$k = \mathbb{Q}(\sqrt{-3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19}\,).$$

It is interesting to point out that even today, after more than half a century, the Golod–Shafarevich theorem remains the deepest result concerning unramified extensions of algebraic number fields.

In his paper [5], Golod used ideas and results from [4] to construct an example of an associative non-nilpotent nil-algebra with finitely many generators and an example of an infinite residually finite p-group with finitely many generators. The first of these examples gives a negative answer to Kurosh's question of whether an associative algebraic algebra, that is, an associative algebra in which every element is algebraic, is locally finite-dimensional, as well as to J. Levitzki's question of whether an associative nilring is locally nilpotent. The second example gives a negative solution of the general Burnside problem on periodic groups, namely, that there exist infinite finitely generated (and even residually finite) p-groups.

The solution by Golod of several well-known Burnside-type problems became a sensation in the mid-1960s and made the notion of a Golod group familiar to every algebraist. To the present day, generalizations and developments of this construction are leading to the solution of ever more problems in ring theory and group theory.

Golod's talk [6] at the 1966 International Congress of Mathematicians in Moscow was also devoted to a method for constructing similar counterexamples to certain problems of Burnside type.

A solution of the Burnside problem in the case of bounded exponent was obtained by S. I. Adian and P. S. Novikov. They showed the existence, for any sufficiently large odd positive integer n, of an infinite 2-generated group of exponent n. The case of a sufficiently large even exponent n was settled by S. V. Ivanov and I. G. Lysënok. The boundedness of the orders of finite groups of exponent n with a fixed number of generators was proved by A. I. Kostrikin for prime n, and then by E. I. Zel'manov for an arbitrary n. A. Lubotzky discovered the presence of the Golod–Shafarevich inequality in the topology of hyperbolic 3-manifolds, which fact helped him to solve Serre's problem on arithmetic lattices in $\mathrm{SL}_2(\mathbb{C})$. Kurosh's problem for PI-algebras was solved in the affirmative by Levitzki and I. Kaplansky.

Most of Golod's papers from the 1970s and 80s were connected with commutative algebra, especially with one of the important invariants of local and graded rings, namely, with the homology of the Koszul complex. These papers included a characterization (jointly with his student L. L. Avramov in [7]) of the Gorenstein local rings as rings for which the homology algebra of the Koszul complex of the maximal

ideal is a Frobenius algebra (subsequently, Golod obtained one of the generalizations of this theorem in [14]), and also a calculation of the Koszul homology algebra and the Poincaré series of almost complete intersections of embedding dimension three [8].

One of his most cited papers [9] (along with the papers on the Golod–Shafarevich theorem and Burnside-type problems) appeared in 1984. It contains a number of results about the relative Gorenstein dimension: this far-reaching generalization of the projective dimension of modules over a local ring was introduced by H.-B. Foxby. In particular, in [9] Golod obtained a formula connecting the relative Gorenstein dimensions of a module over a ring and over a quotient ring. One of his last published papers [17] was also connected with the Gorenstein dimension; in particular, there the modules of zero Gorenstein dimension over Stanley–Reisner algebras were classified. In [10] he established a number of isomorphisms between the Koszul cohomology and local cohomology.

At the same time, in the 1980s and 90s Golod returned to the study of the chain complex that appeared in the first proof of the Golod–Shafarevich theorem (he called it the Shafarevich complex, though it is now also called the *Golod–Shafarevich complex*). This is a non-commutative analogue of the Koszul complex, obtained from a (differential graded) associative algebra by adjoining new free variables on which the differential takes values in the original algebra. Golod investigated both the homology algebras of the Shafarevich complex of a free algebra [12], and the behaviour of this complex and its commutative analogue in a more general situation that arises in the construction of the differential graded model (that is, the cofibrant resolution) of an associative algebra and, respectively, the Tate resolution in the commutative case. In particular, he obtained new homological descriptions of non-commutative complete intersections in the sense of D. Anick, algebras of homological dimension two [13]. He proposed a remarkable homological interpretation of one of the main tools of computer algebra, namely, commutative and non-commutative Gröbner bases and their generalizations: it turned out that the first homology of the Koszul and Shafarevich complexes contains information about the process of constructing bases by the standard critical pairs algorithm [11].

Golod's papers from the 2000s are also connected with the theoretical foundations of the theory of standard bases in their most general form — for filtered modules over rings. These papers established interesting connections between the Diamond Lemma and distributive lattices of ideals. Among the most beautiful results is the characterization of commutative arithmetic rings as rings over which B. Buchberger's criterion for Gröbner bases of modules is valid, as well as the theorem that an extension of a commutative ring preserves all the Gröbner bases precisely in the case when the extension is flat. His other recent results, apart from the aforementioned paper on modules of zero Gorenstein dimension, are also connected with ring theory: he obtained formulae for the annihilators of modules over commutative arithmetic rings [15] and (jointly with A. A. Tuganbaev) over right-invariant rings [16].

Evgenii Golod was characteristically remarkably modest, considerate of others, and demanding of himself. He always treated his students with sincere interest and benevolence, generously sharing his ideas and helping them write their first papers.

Four of his students are now doctors of the physical and mathematical sciences: L. L. Avramov, V. M. Galkin, L. V. Kuz'min, and D. I. Piontkovskii.

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V. A. Artamonov, *V. M. Buchstaber*, *È. B. Vinberg*, *L. V. Kuz'min*,
V. S. Kulikov, *V. N. Latyshev*, *A. V. Mikhalev*, *A. Yu. Ol'shanskii*,
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