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Chebyshev centres, Jung constants, and their applications

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Abstract. The approximation of concrete function classes is the most common subject in the theory of approximations of functions. An important particular case of this is the problem of the Chebyshev centre and radius. As it turns out, this problem is not only a special case of the Kolmogorov width problem, but it is also related in a mysterious way to other important characteristics and results in the theory of functions and other more general branches of analysis and geometry. The aim of the present study is to give a survey of the current state of this problem and to discuss its possible applications.

Bibliography: 169 titles.

Keywords: Chebyshev centre, Chebyshev-centre map, Chebyshev net, Chebyshev point, Jung constant, fixed point theorem, normal structure coefficient.

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Chapter I

Chebyshev centre of a set

In this chapter we shall consider the problem of approximating a set by a class of sets. In this problem it is not only an evaluation of the approximation that is important but also a set that best approximates this class (an optimal set). In particular, for the class of singletons we arrive at the Chebyshev centre problem and the Chebyshev radius problem. The Chebyshev net problem appears if we consider the class consisting of nets of cardinality n (n-nets). For the class of all n-dimensional planes, the problem becomes the Kolmogorov width problem.

We shall be mostly concerned with the Chebyshev centre problem. In this problem, one searches for a point which best approximates (represents) a given set. Such a problem is naturally related to the practical situation when optimal estimates are required, for example, in settings when in a mathematical model a physical process is represented by some unknown point x in a space X. Suppose that from certain experiments or observations some estimates (possibly with errors) of a point x are known (for example, x is known to lie in a set M). The available information is usually insufficient for precise determination of the point x. In this setting, an element $\widehat{x}$ which best approximates (represents) the set M is called a *Chebyshev centre of the set* M. The Chebyshev centre problem is sometimes called the *problem of best simultaneous approximation*. It is clear that in this problem the set M is necessarily bounded.

1. Basic definitions and some results

Definition 1.1. For a non-empty bounded subset M of a metric space (X, ϱ), the quantity $\operatorname{diam} M = \sup_{x,y\in M} \varrho(x, y)$ is called the *diameter* of M, and

$$r_M := r(M) := \inf\bigl\{a \geqslant 0 \mid x \in X,\ M \subset B(x, a)\bigr\}$$

is called the *Chebyshev radius* of M. A point $x_0 \in X$ for which $M \subset B(x_0, r(M))$ is called a *Chebyshev centre* of M. (Here and below, $B(x, r)$ is the ball with centre x and radius r, and $\mathring{B}(x, r)$ is the open ball.)

Thus, a Chebyshev centre of a bounded subset of a normed space is the centre of a ball of smallest radius containing this set; in other words, a Chebyshev centre is a point in the space that 'best approximates' the entire set. The radius of this ball is the Chebyshev radius of this set.

In general, a Chebyshev centre is not unique. By $\mathrm{Z}(M)$ we denote the *set of all Chebyshev centres*[1] *of a bounded set* M. The (set-valued) operator

$$M \mapsto \mathrm{Z}(M) \tag{1.1}$$

is called the *Chebyshev-centre map.*

Example 1.1. The Chebyshev centre of an acute-angled triangle in the Euclidean plane $\mathbb{R}^2$ is unique and lies at the centre of the circumscribed circle. The Chebyshev centre of an obtuse-angled triangle in $\mathbb{R}^2$ lies at the middle of the largest edge.

Example 1.2. On the plane ℓ^∞_2 with max-norm, the set of Chebyshev centres of the set $M := \{(\lambda, 0) \mid |\lambda| \leqslant 1\}$ is the closed interval $\{(0, \mu) \mid |\mu| \leqslant 1\}$.

It is clear that the set $\mathrm{Z}(M)$ is bounded, closed, and convex (see Proposition 1.1 below); moreover, $\mathrm{Z}(M)$ has no interior points.

In some practical cases, the Chebyshev centre problem has to be solved under constraints on the centres of the balls under consideration (for example, the centres are frequently supposed to lie in a subspace, or in a more general setting, in a convex set). This leads to the relative Chebyshev centre problem.

Throughout, X is a real normed linear space.

Definition 1.2. Given a non-empty bounded subset M of X and a non-empty set $Y \subset X$, the quantity

$$r_Y(M) = \inf_{y\in Y} r(y, M),$$

where

$$r(x, M) := \inf\{r \geqslant 0 \mid M \subset B(x, r)\} = \sup_{y\in M} \|x - y\|,$$

is called the *relative Chebyshev radius* (of M with respect to Y).

The *set of relative Chebyshev centres* is defined by

$$\mathrm{Z}_Y(M) := \{y \in Y \mid r(y, M) = r_Y(M)\}. \tag{1.2}$$

[1] The use of the letter Z is traditional, and comes from the German *Zentrum* for *centre*.

Thus, the set of relative Chebyshev centres $\mathrm{Z}_Y(M)$ consists of the points y in Y such that any ball of smallest possible radius $r_Y(M)$ with centre at y contains M. For $Y = X$ we get the definitions of a Chebyshev centre and the Chebyshev radius. If $M = \{y\}$, then $r(x, M) = \|x - y\|$, $r_Y(M) = \rho(y, Y)$ is the distance from y to Y, and the relative Chebyshev-centre map $\mathrm{Z}_Y(M)$ is the metric projection on Y of the point y.

It is clear that the set of Chebyshev centres $\mathrm{Z}_Y(M)$ is closed in Y. Moreover,

$$r_Y(M) = r_Y(\overline{M}), \qquad \mathrm{Z}_Y(M) = \mathrm{Z}_Y(\overline{M}).$$

The *self Chebyshev radius of a set* M,

$$r_M(M) := r_M^{\mathrm{sc}} := \inf_{x\in M} r(x, M), \tag{1.3}$$

is a particular case of the relative Chebyshev radius.[2] The *set of self centres* of M is defined by

$$\mathrm{Z}_M^{\mathrm{sc}}(M) := \mathrm{Z}_M^{\mathrm{sc}} := \{x \in M \mid r(x, M) = r_M(M)\}. \tag{1.4}$$

The following inequalities are clear:

$$\begin{gathered} \operatorname{diam} M \leqslant 2r(M) \leqslant 2 \operatorname{diam} M, \\ |r(x, M) - r(y, M)| \leqslant \|x - y\| \leqslant r(x, M) + r(y, M) \end{gathered} \tag{1.5}$$

for any set $\varnothing \neq M \subset X$ and any $x, y \in X$. Next, if $c_M \in \mathrm{Z}(M)$ and $c_N \in \mathrm{Z}(N)$, then

$$\|c_M - c_N\| \leqslant r(M) + r(N) + d(M, N), \tag{1.6}$$

where $M, N \subset X$ and $d(M, N) := \sup_{x\in M} \rho(x, N)$ is the deviation of the set M from the set N. It is easily checked that these inequalities are sharp. Further, it can easily be verified that

$$r_M(M) \leqslant \operatorname{diam} M \leqslant 2\, r(M) \tag{1.7}$$

for any $M \subset X$.

In general, the sets $\mathrm{Z}(M)$ and $\mathrm{Z}_M(M)$ can be 'quite large': one can construct (see Example 3.1 below) a non-singleton set M such that $\operatorname{diam} \mathrm{Z}(M) = 2r(M)$ and $\operatorname{diam} \mathrm{Z}_M(M) = \operatorname{diam} M$.

The following properties follow from the continuity and convexity of the distance function $y \mapsto \|y - x\|$:

$$\begin{gathered} r(x, M) = r(x, \overline{M}) = r(x, \overline{\operatorname{conv}} M) = r(x, \operatorname{conv} M), \qquad (1.8) \\ r_Y(x, M) = r_Y(x, \overline{M}) = r_Y(x, \overline{\operatorname{conv}} M) = r_Y(x, \operatorname{conv} M), \\ \mathrm{Z}(M) = \mathrm{Z}(\overline{M}) = \mathrm{Z}(\overline{\operatorname{conv}}\, M) = \mathrm{Z}(\operatorname{conv} M), \\ \mathrm{Z}_Y(M) = \mathrm{Z}_Y(\overline{M}) = \mathrm{Z}_Y(\overline{\operatorname{conv}} M) = \mathrm{Z}_Y(\operatorname{conv} M), \\ \mathrm{Z}(M) = \mathrm{Z}(\operatorname{br}(M)), \quad \text{where} \quad \operatorname{br}(M) = \bigcap_{\substack{M\subset \Pi \\ \Pi \text{ is a closed span}}} \Pi; \qquad (1.9) \end{gathered}$$

[2]The acronym 'sc' refers to 'self centre'.

here a *span* is a set N such that $[\![x,y]\!]\subset N$ if $x,y\in N$, where $[\![x,y]\!]:=\{z\in X\mid \min\{\varphi(x),\varphi(y)\}\leqslant\varphi(z)\leqslant\max\{\varphi(x),\varphi(y)\}\ \forall\,\varphi\in\operatorname{ext}S^*\}$, and here and below, $\operatorname{ext}S^*$ is the set of extreme points of the unit sphere S^* of the dual space X^*.

A similar analysis shows that

$$\mathrm{Z}(M)=\mathrm{Z}(\mathrm{m}(M)),\quad\text{where}\quad \mathrm{m}(M)=\bigcap_{M\subset B(y,r)}B(y,r)$$

($\mathrm{m}(M)$ is the Banach–Mazur hull of a set M). In the (relative) Chebyshev centre problem one can thus assume without loss of generality that M is non-empty, closed, and convex.

It is easily seen that

$$M\subset\operatorname{conv}M\subset\overline{\operatorname{conv}}M\subset\operatorname{br}(M)\subset\mathrm{m}(M),\tag{1.10}$$

and in addition,

$$\mathrm{Z}(M)=\mathrm{Z}(\operatorname{conv}M)=\mathrm{Z}(\overline{\operatorname{conv}}M)=\mathrm{Z}(\operatorname{br}(M))=\mathrm{Z}(\mathrm{m}(M)).\tag{1.11}$$

For a subspace Y it follows that:

$$\mathrm{Z}_Y(M+y)=\mathrm{Z}_Y(M)+y,\qquad y\in Y;\tag{1.12}$$

$$\mathrm{Z}_Y(x,\alpha M)=|\alpha|\mathrm{Z}_Y(x,M);\tag{1.13}$$

$$\text{if}\quad x_0\in Y,\quad\text{then}\quad \{x_0\}=\mathrm{Z}_Y(M)\iff\{0\}=\mathrm{Z}_Y(M-x_0);\tag{1.14}$$

$$\text{if}\quad \alpha\geqslant 0,\quad\text{then}\quad \{0\}=\mathrm{Z}_Y(M)\iff\{0\}=\mathrm{Z}_Y(\alpha M).\tag{1.15}$$

It is clear that

$$\mathrm{Z}(M)=\bigcap_{x\in M}B(x,r(M))\tag{1.16}$$

and

$$\mathrm{Z}_V(M)=\bigcap_{x\in M}B(x,r_V(M))\cap V\tag{1.17}$$

for an arbitrary non-empty subset V of X. Taking into account that a closed ball is a closed bounded span (see [2], § 9,1) and that an intersection of spans is a span, we get that

$$\begin{gathered}\mathrm{Z}(M)\text{ is a closed bounded span,}\\ \mathrm{Z}_V(M)\text{ is the intersection of a closed bounded span with }V.\end{gathered}\tag{1.18}$$

From (1.16) it follows that

$$\mathrm{m}(\mathrm{Z}(M))=\mathrm{Z}(M).$$

The next result is an easy corollary of (1.17).

Proposition 1.1. *If Y is a convex set in X and $M\subset X$ is a non-empty bounded set, then the set $\mathrm{Z}_Y(M)$ is convex.*

The relation of Chebyshev centres to spans will be considered in more detail in § 2.

The width of a set (see the definition after Theorem 4.2) is a natural extension of the definition of the Chebyshev radius. Unfortunately, here we cannot discuss this question in more detail.

Another important and useful generalization of the definition of a Chebyshev centre is the concept of a *Chebyshev net of cardinality* n (a Chebyshev n-net, a best n-net) and the related concept of the entropy of a set.

Definition 1.3. Let $n\in\mathbb{N}$. By a *net of cardinality* n we mean any system $\mathrm{N}_n=\{y_1,\dots,y_n\}$ of n (not necessarily distinct) points in X, and the *covering radius* of a set $M\neq\varnothing$ by a net N_n is defined by

$$\mathrm{R}(M,\mathrm{N}_n):=\sup_{x\in M}\min_{1\leqslant i\leqslant n}\|x-y_i\|.$$

By a *Chebyshev* (or a *best*) *net of cardinality* n for a set M we mean a net $\mathrm{N}_n^*=\{y_1^*,\dots,y_n^*\}$ of cardinality n such that $\mathrm{R}(M,\mathrm{N}_n^*)=\inf\mathrm{R}(M,\mathrm{N}_n)$, where the infimum is taken over all possible nets of cardinality n in X. Thus, $R=\mathrm{R}(M,\mathrm{N}_n^*)$ is the radius of best covering of the set M by n balls of equal radius R.

Of course, the Chebyshev centre problem (that is, the problem of covering a set by a ball of smallest possible radius) is a particular case of the Chebyshev n-net problem.

The phrase 'best net of cardinality n' was used in an oral discussion of Kolmogorov's 1936 paper [100] on widths.

The Chebyshev n-net problem is *a fortiori* more involved than the Chebyshev centre problem even in the Euclidean setting — cf. the well-known spherical design problem (the problem of best distribution of a finite number of points on a sphere) [162].

In contrast to the Chebyshev centre problem, a Chebyshev n-net may not be unique even in a Euclidean space; moreover, some points of this net may lie outside the closed convex hull of the set under consideration. For some sets, best n-nets can be found analytically for small n (see [150]).

Note that if a sequence (M_n) of sets converges in the Hausdorff metric to a set M and if R_n is the best covering radius of M_n, $n=1,2,\dots$, then $\lim R_n=R$, where R is the radius of best covering of M.

Another extension of the definition of a Chebyshev centre is the concept of a Chebyshev point (see [28]).

Definition 1.4. Given a normed linear space X, consider a system of sets $\{G_\alpha\mid\alpha\in A\}$ for which the quantity $R(y):=\sup_{\alpha\in A}\rho(y,G_\alpha)$ is finite for any $y\in X$. A point $y^*\in X$ for which $R(y^*):=\inf_{y\in X}R(y)$ is called a *Chebyshev point* for this system of sets.

A Chebyshev centre of a set can be looked upon as a Chebyshev point of a system of singletons. Further, if $x^*\in X^*$, $x^*\neq 0$, $c\in\mathbb{R}$, and $H:=(x^*)^{-1}(c)=\{x\in X\mid x^*(x)=c\}$ is a closed hyperplane, then for any $x\in X$ the distance from x to H is given by the well-known formula $\rho(x,H)=|x^*(x)-c|/\|x^*\|$. This observation shows that the concept of a Chebyshev point of a system of sets is an extension of the concept of a Chebyshev point of an (inconsistent) system of linear equations (see [169]).

Some problems on the existence of a Chebyshev point of a system of hyperplanes of any cardinality in a finite-dimensional space and on the uniqueness of a Chebyshev point for a system of balls were solved by Belobrov [28], [31].

As another generalization of a Chebyshev centre, we mention the Steiner point. In a real Banach space $(X, \|\cdot\|)$, the *set of Steiner points* for any given n-tuple $\{x_1, \dots, x_n\}$, $n \geqslant 3$, of points in X is defined by

$$\operatorname{St}_n(x_1, \dots, x_n) = \biggl\{ s \in X \biggm| \sum_{k=1}^{n} \|x_k - s\| = \inf_{x \in X} \sum_{k=1}^{n} \|x_k - x\| \biggr\}.$$

Steiner points are also called Fermat points, Lamé points, or medians. The corresponding Steiner map $\operatorname{St}_n \colon X^n \to X$ of the space $X^n = \{(x_1, \dots, x_n) \mid x_k \in X\}$ with norm $\|(x_1, \dots, x_n)\|_n = \|x_1\| + \dots + \|x_n\|$ to X is, in general, set-valued, and its domain may not be the whole of X^n.

For a Hilbert space X and $n = 3$, a Steiner point $s(x_1, x_2, x_3)$ exists and is unique — it lies in the plane spanned by the points x_1, x_2, x_3 and either coincides with one of these points (if one of the angles in the triangle $x_1x_2x_3$ is at least 120°) or coincides with the Torricelli point (at which each side of the triangle subtends an angle of 120°); see [25].

Steiner points need not exist even for three-point subsets M_3 of a normed linear space. The first example of such a space X and a set M_3 was constructed by Garkavi [73] (for other examples, see [153], [20], [124], [38]). Veselý [153] proved that any non-reflexive Banach space X can be equivalently renormed so that some triple of points $M_3 \subset X$ has no Steiner point in the new norm. Kadets [91] proved this result using a different method. At the same time, in any Banach space X that is 1-complemented in its second dual (in particular, in any reflexive space, as well as in any space L^1), the set $\operatorname{St}_n(x_1, \dots, x_n)$ is non-empty for any n-tuple of points x_k and any natural number n. Bednov, Borodin, and Chesnokova [25] investigated the problem of the existence of Lipschitz selections of the Steiner map St_n (which associates with any n points of a Banach space X the set of their Steiner points) in dependence on the geometric properties of the unit sphere S of X, the dimension of X, and the number n.

Below in this chapter we shall be mainly concerned with the Chebyshev centre problem.

Klee [99] and, independently, Garkavi [71] proved that (some) Chebyshev centre of any bounded subset M of a space X lies in the closed convex hull of the subset if and only if X either is a Hilbert space or has dimension at most two. Garkavi showed that this result also holds if one considers only three-point sets. For any closed convex bounded subset of a Hilbert space, a Chebyshev centre exists, is unique, and lies in the subset (Theorem 4.4 below). Garkavi [70] showed that each bounded subset of a Banach space X has at most one Chebyshev centre if and only if X is *uniformly convex* (or *uniformly rotund*) *in every direction* (that is, for each $z \in X$ and any $\varepsilon > 0$, there exists a $\delta = \delta(z, \varepsilon) > 0$ such that if $\|x_1\| = \|x_2\| = 1$, $x_1 - x_2 = \lambda z$, and $\|x_1 + x_2\| > 2 - \delta$, then $|\lambda| < \varepsilon$). Belobrov [30] extended this condition by considering spaces in which the set of Chebyshev centres of any set has dimension at most $r<\infty$. Let us dwell on these results in more detail.

We first consider the space $\mathbb{R}^n$: the Chebyshev centre (of a non-empty bounded) subset of $\mathbb{R}^n$ exists, is unique, and lies in the convex hull of the subset; see Theorem 4.4 below.

Theorem 1.1. *The Chebyshev radius of a set $M\subset\mathbb{R}^n$ of diameter $\leqslant 2$ is bounded above by $\sqrt{2n/(n+1)}$. This estimate is attained if and only if the closure of M contains the vertices of a regular n-dimensional simplex with edge length* 2.

Proof. Without loss of generality it can be assumed that the set M is closed (see (1.8)). By Helly's classical theorem (see, for example, [128], § 1.10) if, for any $n+1$ points in M, the balls of radius $r=r(M)$ with centres at these points have non-empty intersection, then the balls of the same radius with centres at all the points of M also have non-empty intersection. It thus suffices to consider the case when M consists of at most $n+1$ points. Let us show that $r=r(M)\leqslant R:=\sqrt{2n/(n+1)}$. In $\mathbb{R}^n$, the (unique) Chebyshev centre y lies in the convex hull of M. In addition, we claim that the point y lies in the convex hull of the set $M_0:=\{x\in M\mid \|x-y\|=r\}=M\cap S(y,r)$. If this were not so, then it would be possible to strictly separate the point y and the set M_0 by a closed hyperplane. We drop the perpendicular $[y,y_0]$ to this hyperplane. The distance from a point $z\in(y,y_0]$ to the set M_0 is less than r. Moreover, the distance from y to any point in the set $M\setminus M_0$ is also less than r. Hence, the interval (y,y_0) has a point z at a distance less than r from M. But this contradicts the choice of r. By Carathéodory's theorem, there exist points $\{x_i\}_{i=0}^m\subset M_0$ such that $\|x_i-y\|=r$ $(m\leqslant n)$, and there exist numbers $\{\alpha_i\}_{i=0}^m\in\mathbb{R}_+$ such that $\sum_{i=0}^m\alpha_i=1$ and $\sum_{i=0}^m\alpha_ix_i=y$. Note that $a_{ij}=\|x_i-x_j\|\leqslant 2$. We can assume without loss of generality that

$$y=0=\sum_{i=0}^m\alpha_ix_i. \tag{1.19}$$

By the law of cosines,

$$a_{ij}^2=2r^2-2(x_i,x_j). \tag{1.20}$$

Hence, for each j

$$1-\alpha_j=\sum_{i\neq j}\alpha_i\geqslant\sum_{i=0}^m\frac{\alpha_ia_{ij}^2}{4}\overset{(1.20)}{=}\frac{r^2}{2}-\frac{1}{2}\biggl(\sum_{i=0}^m\alpha_ix_i,x_j\biggr)\overset{(1.19)}{=}\frac{r^2}{2}\,.$$

Summing these equalities over $j=0,\dots,m$ and using $\sum_{i=0}^m\alpha_i=1$, we obtain $m\geqslant\dfrac{(m+1)r^2}{2}$, and hence $r\leqslant\sqrt{\dfrac{2m}{m+1}}\leqslant\sqrt{\dfrac{2n}{n+1}}=R$.

The case $r=R$ implies that $m=n$ and $a_{ij}=2$, and so the points $\{x_i\}_{i=0}^m$ form the vertices of a regular n-dimensional simplex with edge length 2. $\square$

As a corollary, the Jung constant of the space $\mathbb{R}^n$ (see § 11 below) is

$$J(\mathbb{R}^n)=\sqrt{\frac{n}{2(n+1)}}.$$

2. Chebyshev centres and spans

A *segment* $[\![x,y]\!]$ in a normed linear space X is defined as (see [2])

$$\begin{aligned}[\![x,y]\!] :=&\bigl\{z \in X \mid \min\{\varphi(x),\varphi(y)\} \leqslant \varphi(z) \leqslant \max\{\varphi(x),\varphi(y)\}\ \forall\,\varphi \in \operatorname{ext} S^*\bigr\}\\ =&\bigl\{z \mid \varphi(z) \in [\varphi(x),\varphi(y)]\bigr\},\end{aligned} \tag{2.1}$$

where $\operatorname{ext} S^*$ is the set of extreme points of the unit sphere S^* of the dual space X^*.

In fact, if a subset $\mathscr{A} \subset \operatorname{ext} S^*$ is such that $\mathscr{A} \cup (-\mathscr{A}) = \operatorname{ext} S^*$, then

$$[\![x,y]\!] = \bigl\{z \in X \mid f(z) \in [f(x),f(y)]\ \forall\,f \in \mathscr{A}\bigr\}.$$

For example, for $X = C(Q)$ it is convenient to take the point evaluation functionals $(\varphi \mapsto \varphi(t))$ as $\mathscr{A}$. In this case, for any $\varphi,\psi \in C(Q)$

$$[\![\varphi,\psi]\!] = \bigl\{g \in C(Q) \mid g(t) \in [\varphi(t),\psi(t)]\ \forall t \in Q\bigr\}.$$

With any set $\mathscr{A} \subset \operatorname{ext} S^*$ for which $\mathscr{A} \cup (-\mathscr{A}) = \operatorname{ext} S^*$, one can associate the linear map $c\colon X \to c(X)$ that takes any $x \in X$ to the function $c(x)\colon \mathscr{A} \to \mathbb{R}$ defined by $x(x^*) = x^*(x)$ for $x^* \in \mathscr{A}$. Thus, $c(X)$ is the space of continuous functions on $\mathscr{A}$ which are the restrictions to $\mathscr{A}$ of the continuous linear functionals from X acting on X^*. We equip $c(X)$ with the uniform norm:

$$\|c(x)\| = \sup_{x^* \in \mathscr{A}} |x(x^*)|.$$

As a result, the map $c\colon X \to c(X)$ is an isometry. It will be convenient to identify a point x with its image $c(x)$ and a set $E \subset X$ with $c(E)$. Moreover, the ball $B(x,R)$ is identified with the set $b(x,R) := c(B(x,R))$. Since $B(x,R) := \{y \in X \mid |x^*(y-x)| \leqslant R \text{ for any } x^* \in \operatorname{ext} S^*\}$, we have

$$\begin{aligned}b(x,R) &= [\![x-R,\ x+R]\!]\\ &= \{y \in X \mid x(x^*) - R \leqslant y(x^*) \leqslant x(x^*) + R,\ \ x^* \in \operatorname{ext} S^*\}\\ &= \{y \in X \mid x(x^*) - R \leqslant y(x^*) \leqslant x(x^*) + R,\ \ x^* \in \mathscr{A}\}\\ &= \{c(y) \mid c(x) - R \leqslant c(y) \leqslant c(x) + R\}.\end{aligned}$$

The set $\mathrm{m}(E) := \bigcap_{E \subset B(x,R)} B(x,R)$ (the Banach–Mazur hull of the set E) is identified with the set

$$\mathrm{m}(c(E)) = \bigcap_{c(E) \subset b(x,R)} b(x,R).$$

Recall that a set E with $\varnothing \ne E \subset X$ is called a *span* (see [2]) if

$$[\![x,y]\!] \subset E \quad \text{for all } x,y \in E.$$

Any closed ball is a closed span.

Definition 2.1. Recall that a function $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is *lower semicontinuous* if it satisfies any one of the following equivalent conditions:

a) the set $\{x \in X \mid f(x) \leqslant \alpha\}$ is closed in X for any $\alpha \in \mathbb{R}$;

b) the set $\{x \in X \mid f(x) > \alpha\}$ is open in X for any $\alpha \in \mathbb{R}$;

c) the epigraph $\operatorname{epi} f$ of f is closed in $X \times \mathbb{R}$.

A map $F\colon X \to 2^Y$ is said to be *lower semicontinuous* at a point x_0 if, for any neighbourhood $O(y)$ of any $y \in F(x_0)$, there exists a neighbourhood $O(x_0)$ such that $F(x) \cap O(y) \neq \varnothing$ for any point $x \in O(x_0)$. As usual, F is lower semicontinuous on X if it is lower semicontinuous at any point $x_0 \in X$.

Remark 2.1. It is well known that if $f(t_0)$ is finite, then a function $f(t)$ is lower semicontinuous at the point t_0 if and only if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $f(t_0) - \varepsilon < f(t)$ if $|t - t_0| < \delta$, $t \in [a, b]$.

Remark 2.2. Vasil'eva (see [152]) showed that a set $\Pi \subset C(Q)$, where Q is a compact Hausdorff space, is a non-empty closed span if and only if Π can be written as a *generalized segment*

$$[\![f_1, f_2]\!] = \bigl\{f \in C(Q) \mid f(t) \in [f_1(t), f_2(t)] \ \forall t \in Q\bigr\}, \tag{2.2}$$

where $f_1, f_2\colon Q \to \overline{\mathbb{R}}$, $f_1 \leqslant f_2$, f_1 is upper semicontinuous on Q, and f_2 is lower semicontinuous (in the definition of $[\![f_1, f_2]\!]$ the functions f_1 and f_2 need not lie in $C(Q)$). By the Katětov–Tong separation theorem for semicontinuous functions (see, for example, [68]), the set $[\![f_1, f_2]\!]$ is non-empty in $C(Q)$. Note that the generalized segment $[\![f_1, f_2]\!]$ is a singleton if and only if $f_1 = f_2$ (in this case f_1 and f_2 are continuous functions). Vasil'eva also showed that the metric projection onto a closed span has a continuous 1-Lipschitz selection if and only if the span is a segment $[\![f_1, f_2]\!]$ with $f_1, f_2 \in C(Q)$. For some properties of generalized segments see also [66].

Let $V \neq \varnothing$ and let $r := r_V(M)$ be the relative Chebyshev radius of a set M with respect to V. On $\mathscr{A}$ we set

$$m^*(\,\cdot\,) = \sup_{y \in M} y(\,\cdot\,) \quad \text{and} \quad m_*(\,\cdot\,) = \inf_{y \in M} y(\,\cdot\,). \tag{2.3}$$

Here y is regarded as $c(y)$ (that is, as a function on $\mathscr{A}$), and the functions $m^*(\,\cdot\,)$ and $m_*(\,\cdot\,)$ are lower and upper semicontinuous, respectively. Therefore, $\Pi := [\![m^* - r,\, m_* + r]\!]$ is a generalized segment.

Given $x^* \in \mathscr{A}$, consider the strip

$$\Pi_{x^*} := \bigl\{x \in X \mid m^*(x^*) - r \leqslant x(x^*) \leqslant m_*(x^*) + r\bigr\}.$$

It is easily seen that Π_{x^*} consists precisely of the points $x \in X$ such that

$$M \subset \{u \in X \mid |x^*(u - x)| \leqslant r\} =: \Pi_{x^*}(x).$$

Since $\bigcap\limits_{x^* \in \mathscr{A}} \Pi_{x^*}(x) = B(x, r)$, the following conditions are equivalent:

a) $v \in \mathrm{Z}_V(M)$;

b) $v \in V \cap \Pi_{x^*}$ for any $x^* \in \mathscr{A}$;

c) $v \in V \cap \Pi$, where $\Pi = \Pi_r := \bigcap_{x^* \in \mathscr{A}} \Pi_{x^*}$.

Thus,

$$\mathrm{Z}_V(M) = V \cap \Pi. \tag{2.4}$$

From (2.4) it follows, in particular, that *the set* $\mathrm{Z}_V(M)$ *of relative Chebyshev centres is convex for a convex set* $V \subset X$ (see Proposition 1.1). It is also clear that

$$\operatorname{diam} M,\ \operatorname{diam} \Pi \leqslant 2r(M).$$

The set of Chebyshev centres of a bounded set $\varnothing \neq M \subset X$ forms the closed segment

$$\Pi := \bigl\{x \in X \mid m^*(\,\cdot\,) - r \leqslant x(\,\cdot\,) \leqslant m_*(\,\cdot\,) + r\bigr\}, \tag{2.5}$$

where $r = r(M)$ is the Chebyshev radius of M.

Consider the functions

$$\overline{N}(\,\cdot\,) := \inf_{x \in X\colon m^* - r \leqslant x} x(\,\cdot\,) \quad \text{and} \quad \bar{n}(\,\cdot\,) := \sup_{x \in X\colon x \leqslant m_* + r} x(\,\cdot\,). \tag{2.6}$$

These functions are upper and lower semicontinuous, respectively. By construction, $m^*(\,\cdot\,) - r \leqslant \overline{N}(\,\cdot\,)$ and $\bar{n}(\,\cdot\,) \leqslant m_*(\,\cdot\,) + r$. Now from (2.5) and (2.6) we have the equality of generalized segments (cf. (2.2))

$$\Pi = [\![m^*(\,\cdot\,) - r,\, m_*(\,\cdot\,) + r]\!] = [\![\overline{N}(\,\cdot\,),\, \bar{n}(\,\cdot\,)]\!].$$

If X is a Banach space such that $c(X)$ contains the constants (for example, $X = C(Q)$), then replacing x in (2.6) by $y-r$ and $y+r$, we get that $\overline{N}(\,\cdot\,) = N(\,\cdot\,) - r$ and $\bar{n}(\,\cdot\,) = n(\,\cdot\,) + r$, respectively, where

$$N(\,\cdot\,) = \inf\{y(\,\cdot\,) \mid m^*(\,\cdot\,) \leqslant y(\,\cdot\,)\}, \tag{2.7}$$

$$n(\,\cdot\,) = \sup\{y(\,\cdot\,) \mid m_*(\,\cdot\,) \geqslant y(\,\cdot\,)\}. \tag{2.8}$$

Hence, if X is such that $c(X)$ contains the constants, then

$$\Pi = [\![N(\,\cdot\,) - r,\, n(\,\cdot\,) + r]\!]. \tag{2.9}$$

Let $\mathfrak{B}$ be the set of bars (see [2], § 8.4) of the form $\Pi = (x^*)^{-1}[a,b]$, where $-\infty \leqslant a \leqslant b \leqslant +\infty$.

Note that the 'bar hull'

$$\mathfrak{br}(M) := \bigcap \{\Pi \in \mathfrak{B} \mid \Pi \supset M\}$$

coincides with the generalized segment $[\![m_*(\,\cdot\,),\, m^*(\,\cdot\,)]\!]$.

We also note that in the space $C(Q)$

$$\mathfrak{br}(M) = \operatorname{br}(M) := \bigcap \{\Pi \mid \Pi \supset M,\ \Pi \text{ a closed span}\}.$$

Recall that the hull of the set $b(x,R)$ coincides with the generalized segment $[\![x(\,\cdot\,) - R,\, x(\,\cdot\,) + R]\!]$, and moreover,

$$\mathrm{m}(M) = \mathrm{m}(\operatorname{br}(M)) = \bigcap_{\operatorname{br}(M) \subset b(x,R)} b(x,R).$$

Hence, for all $x \in X$ and R such that $\operatorname{br}(M) \subset b(x,R)$ we have

$$m^*(\,\cdot\,) \leqslant x(\,\cdot\,) + R, \quad m_*(\,\cdot\,) \geqslant x(\,\cdot\,) - R,$$

and moreover, if X is a Banach space such that $c(X)$ contains the constants, then

$$[\![n(\,\cdot\,),\, N(\,\cdot\,)]\!] \subset b(x,R) \quad \text{provided that } \operatorname{br}(M) \subset b(x,R).$$

Therefore,

$$[\![n(\,\cdot\,),\, N(\,\cdot\,)]\!] \subset \mathrm{m}(\operatorname{br}(M)) = \mathrm{m}(M).$$

If Q is a topological space, then by $C(Q)$ we mean the space of continuous bounded functions on Q with the norm $\|f\| = \sup_{t\in Q}|f(t)|$. The following result was established by Tsar'kov [147].

Theorem 2.1. *Let $X = C(Q)$, where Q is a normal topological space, and let M be a non-empty bounded subset of X with a unique Chebyshev centre. Then*

$$\mathrm{m}(M) = B(z,r),$$

where z is the Chebyshev centre of M. Moreover, $z = (N(\,\cdot\,) + n(\,\cdot\,))/2$.

3. Chebyshev centre in the space $C(Q)$

In this section we recall and prove some results on the existence of Chebyshev centres in the space $C(Q)$. Many results here can be proved using the results in § 2 on representation of the set of Chebyshev centres $\mathrm{Z}(\,\cdot\,)$ as a closed span.

One of the most general results on the existence of Chebyshev centres was obtained independently by Amir [4] and Ka-Sing Lau (see also Theorem 6.8 and Remark 3.2 below).

Theorem 3.1 (Amir and Ka-Sing Lau). *Let Q be an arbitrary topological space and let X be a uniformly convex Banach space. Then any bounded set in the space $C(Q,X)$ admits a Chebyshev centre.*

Let M be a bounded subset of $C[a,b]$. Consider the functions

$$m_*(t) := \inf_{x\in M} x(t), \qquad m^*(t) := \sup_{x\in M} x(t),$$
$$n(t) := \varliminf_{\tau\to t} m_*(\tau), \qquad N(t) := \varlimsup_{\tau\to t} m^*(\tau).$$

The following new result is a direct consequence of the equality (2.9) in § 2. We note that in earlier results the characterization (3.1) of Chebyshev centres was proved under more stringent constraints on the space Q (paracompactness ([80], p. 186) and $\varkappa$-normality [167]).

Theorem 3.2 (Tsar'kov). *Each bounded subset M of the space $C(Q)$, where Q is a normal space, has a Chebyshev centre. The set $\mathrm{Z}(M)$ of all Chebyshev centres of the set M consists precisely of the continuous functions $y(\,\cdot\,)$ that satisfy the inequality*

$$N(t) - r \leqslant y(t) \leqslant n(t) + r \qquad \forall\, t\in Q, \tag{3.1}$$

where $r = r(M)$ is the Chebyshev radius of M.

Remark 3.1. An analogue of Theorem 3.2 for finite Chebyshev n-nets is unknown. Garkavi (see [70], § 2) showed that in the space $C[a,b]$, each *compact* set $M\subset C[a,b]$ admits a best net of cardinality n. Keener [96] gave sufficient conditions for the existence of a Chebyshev net of cardinality n. Garkavi proved that the problem of the existence of a Chebyshev net of cardinality n for *bounded* sets in the space c_0 has an affirmative solution. His arguments can also be carried over to $\ell^\infty(\Gamma)$-spaces. Amir and Mach (see [7]) proved that a best n-net for a *bounded* set may not exist: they showed that if a point ω of a compact Hausdorff space Q is a limit point for two disjoint sequences, then $C(Q)$ contains a bounded set for which no Chebyshev net of cardinality 2 exists (see also [72]).

Example 3.1. Consider the set M of all continuous functions y on $[-1,1]$ such that $0\leqslant y(t)\leqslant 1$ for $t>0$, $y(0)=0$, and $-1\leqslant y(t)\leqslant 0$ for $t<0$. It is clear that

$$m^*(t)=\begin{cases}0 & \text{for } t\leqslant 0,\\ 1 & \text{for } t>0\end{cases}\qquad\text{and}\qquad m_*(t)=\begin{cases}-1 & \text{for } t<0,\\ \ \ 0 & \text{for } t\geqslant 0.\end{cases}$$

Thus,

$$N(t)=\begin{cases}0 & \text{for } t<0,\\ 1 & \text{for } t\geqslant 0\end{cases}\qquad\text{and}\qquad n(t)=\begin{cases}-1 & \text{for } t\leqslant 0,\\ \ \ 0 & \text{for } t>0.\end{cases}$$

We have $2r=\sup_{t\in[a,b]}N(t)-n(t)=2=N(0)-n(0)$, which shows that the Chebyshev radius r_M of the set M is 1. The functions $n(t)+r_M$ and $N(t)-r_M$ coincide with $m^*(t)$ and $m_*(t)$, respectively, and now Theorem 3.2 gives us that the set of all Chebyshev centres of M coincides with M itself. Hence, the set M thus constructed provides an example of a non-trivial set coinciding with the set $\mathrm{Z}(M)$ of its Chebyshev centres.

For compact sets M, Theorem 3.2 assumes the following more manageable form.

Theorem 3.3. *Let M be a compact set in $C(Q)$, where Q is a topological space, and let $r=r(M)$ be the Chebyshev radius of M. Then the set $\mathrm{Z}(M)$ of all Chebyshev centres of M can be written as*

$$\mathrm{Z}(M)=\bigl\{y\in C(Q)\mid m^*(t)-r\leqslant y(t)\leqslant m_*(t)+r\bigr\}\neq\varnothing,$$

where $m^*(t)=\max_{x\in M}x(t)$, $m_*(t)=\min_{x\in M}x(t)$, and $r=\|m^*-m_*\|/2$. In particular,

$$\mathrm{Z}(M)=\mathrm{Z}(m^*,m_*):=\mathrm{Z}(\{m_*,m^*\}).$$

Corollary 3.1. *In the space $C(Q)$, where Q is a topological space, the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$ has a 1-Lipschitz selection on the class of non-empty compact sets $M\subset C(Q)$ with respect to the Hausdorff metric.*

As a 1-Lipschitz selection in Corollary 3.1 one can consider the map $x\mapsto(m^*(\,\cdot\,)+m_*(\,\cdot\,))/2$ (see § 2, and also [97]).

Remark 3.2. For vector-valued analogues of Theorems 3.2 and 3.3, see [160], [4], [167], and also Theorem 6.8 below. For example, Ward [160] established that if Q is a paracompact Hausdorff space and X is a finite-dimensional space, then in the space $C(Q,X)$ any bounded set has a Chebyshev centre. A similar result also holds in the space $C(Q, H)$, where Q is a normal space and H is a Hilbert space. Amir [4] (and independently Ka-Sing Lau) extended Ward's results to the case when X is a uniformly convex space, Q is an arbitrary topological space, and $C(Q, X)$ is the space of bounded continuous functions on Q with values in X (see Theorem 6.8). Zamyatin and Shishkin [167] extended this result by showing that a Chebyshev centre exists for any bounded subset of $C(Q, X)$, where Q is an arbitrary topological space and X is a KB-linear space of bounded elements. For other generalizations, see also [136], [130], and [49].

4. Existence of a Chebyshev centre in normed linear spaces

In this section, we formulate more results on the existence of Chebyshev centres in normed linear spaces (see, for example, [9], [8], Chap. 27 of [48], § 3 of [70], pp. 184–187 in [80], [101], [103], [154]–[156]).

Theorem 4.1 (Garkavi). *Suppose that the image of a space X under the canonical embedding in the second dual X^{**} is norm-1 complemented (that is, there exists a norm-1 projection of X^{**} onto X). Then any non-empty bounded subset of X has a Chebyshev centre.*

In particular, the hypotheses of Theorem 4.1 are satisfied for the L^1-spaces and for the dual spaces (it is well known that the first dual X^* is always 1-complemented in the third dual X^{***}).

Analogues of Theorem 4.1 and its corollary for dual spaces also hold for Chebyshev n-nets (see [70], § 2).

We mention another property of dual spaces related to the Chebyshev centre problem. A set $M \subset X$ can be regarded as a subset of the second dual X^{**}. However, the radius of a smallest ball for M in the space X^{**} can be smaller than $r(M) = \inf_{y \in X} \sup_{x \in M} \|x - y\|$. Nevertheless, if X is a dual space, then these quantities are equal for any $M \subset X$.

Remark 4.1. The condition of Theorem 4.1 is not necessary for each bounded subset of the space to have a Chebyshev centre. For example, Garkavi (see § 2 of [70]) showed that in the space c_0 any bounded set has a Chebyshev centre. However, it is well known that c_0 is not even complemented in the second dual: there is not even a bounded projection from ℓ^∞ onto c_0 (Phillips–Sobczyk theorem). The spaces c and $C[0, 1]$ are also not complemented in their second duals.

We mention the following simple result.

Theorem 4.2. *Suppose that any non-empty bounded set in a space X has a Chebyshev centre. Then in any 1-complemented affine subspace $L \subset X$ any bounded set has a Chebyshev centre.*

The *Kolmogorov width* of a non-empty bounded set M is defined by

$$d_n(M, X) = \inf_{L \in \operatorname{Aff}_n(X)} d(M, L),$$

where $\operatorname{Aff}_n(X)$ is the class of all affine subspaces of dimension $\leqslant n$ in the space X and

$$d(M, L) = \sup_{x \in M} \rho(x, L)$$

is the *deviation* of M from an affine subspace L. For $n=0$ the Kolmogorov width problem becomes the Chebyshev centre problem.

In the study of widths it is not only the values or order estimates for the widths (quantitative characteristics of the approximation) that are important, but also the concrete form of extreme subspaces. (A subspace $L_0 \in \operatorname{Aff}_n(X)$ is said to be *extreme* (or *best*) if $d(M, L_0) = d_n(M, X)$.)

Theorem 4.3 (Garkavi). *Assume that the image of a space X under the canonical embedding in the second dual X^{**} is 1-complemented. Then, for any non-empty bounded set $M \subset X$ and any $n \in \mathbb{N}$, there exist a Chebyshev net of cardinality n and an extreme subspace $L_0 \in \operatorname{Aff}_n(X)$. In particular, this is true for $X = Y^*$.*

Theorem 4.4 (Garkavi). *A necessary and sufficient condition for each bounded subset of a Banach space to have a Chebyshev centre lying in its closed convex hull is that the space either be a Hilbert space or have dimension at most two.*

Garkavi [71] gave the following characterization of Hilbert spaces in terms of the existence of a Chebyshev centre for three-point subsets.

Theorem 4.5 (Garkavi). *Let X be a Banach space of dimension at least three. If any three points of X admit a Chebyshev centre lying in their affine hull, then X is a Hilbert space.*

Theorem 4.6 (Garkavi). *Let X be a Banach space and let $\dim X \geqslant 3$. If any three points of the unit sphere of X can be covered by a ball of radius 1 with centre in their affine hull, then X is a Hilbert space.*

In [8] it was shown that if X is a Hilbert space, $Y \subset X$ is a closed convex set, and $K \subset X$ is a compact convex set, then $\mathrm{Z}_Y(K) \subset P_Y(K)$, where $P_Y(K) = \{y \in Y \mid \text{there exists an } x \in K \text{ such that } y \in P_Y x\}$. For a closed convex set K and $Y \subset X$ this conclusion ceases to hold. A corresponding example was constructed by Benítez (cf. [17]): let $X = \ell^2$, let (e_n) be the standard basis for ℓ^2, and let $Y = \operatorname{span} n^{-1} e_n$ and $K = \overline{\operatorname{conv}}\,\{n(n+1)^{-1} e_n \mid n \in \mathbb{N}\}$. Then

$$\mathrm{Z}_Y(K) = \{0\}, \quad 0 \notin P_Y K.$$

In the two-dimensional case Theorem 4.4 can be refined as follows.

Proposition 4.1. a) *If any Chebyshev centre of any two points in a Banach space X (of any dimension) lies on the line passing through these points, then X is a strictly convex (rotund) space.*

b) *A two-dimensional space X is strictly convex if and only if any Chebyshev centre of any bounded set $\varnothing \neq M \subset X$ lies in its convex hull.*

From the above theorems some characterizations of Hilbert spaces can be derived.

Theorem 4.7 (Garkavi). *A Banach space X of dimension $\geqslant 3$ is a Hilbert space if and only if*:

a) *each three points in X admit a Chebyshev centre*;

b) *for each three linearly independent points of the unit sphere S there exists a ball of radius < 1 containing these points.*

Given $\varepsilon > 0$ and $x^* \in S^*$, we set

$$A(x^*, \varepsilon) := \{x \in S \mid x^*(x) \geqslant 1 - \varepsilon\},$$

and we let $R(x^*, \varepsilon)$ be the Chebyshev radius of the set $A(x^*, \varepsilon)$. Consider the function

$$\mathrm{f}(\varepsilon) := \sup_{x^* \in S^*} R(x^*, \varepsilon).$$

The following result holds (see [71], and also [17]).

Theorem 4.8 (Garkavi). *A necessary and sufficient condition that X be a Hilbert space is that* $\mathrm{f}(\varepsilon)$ *tend strictly monotonically to zero as $\varepsilon \to 0$*

Among negative results, we mention the following ones.

Garkavi [69], [70] constructed an example of a Banach space which contains three points without a Chebyshev centre (see also Veselý [154]). The number three in this result is smallest possible, because any two points always admit a Chebyshev centre. For similar results for Chebyshev nets of cardinality n, see [70], § 2, and also Example 3.1 in [9]. Konyagin [101] showed that any non-reflexive Banach space X can be (equivalently) renormed so that the resulting space contains a three-point set without a Chebyshev centre.

The following question is natural. Assume that in a Banach space any finite (or any compact) set has a Chebyshev centre. Is it true that in such a space any *bounded* set has a Chebyshev centre? Veselý [155] answered this question in the negative for compact sets: he constructed a space of the form[3] $X = c_0(E)$ in which all compact sets admit Chebyshev centres, but which contains a bounded set without a Chebyshev centre. Earlier, Smith and Ward [142] presented an example of a proximinal hyperplane H in $C[0,1]$ and a bounded set $M \subset H$ without a relative Chebyshev centre in H. See also Rao [133].

The following example [156], which extends Garkavi's construction, gives an example of a set without a Chebyshev centre in a closed hyperplane of c_0.

Example 4.1. Let $f = (f^{(i)}) \in (c_0)^*$ be a functional with infinite support, and let $f^{(1)} = f^{(2)} = f^{(3)} = 1$ and $\sum_{i=1}^{\infty} f^{(i)} = 1$. Then the set M of three points

$$(-1, 1, 1, 0, 0, \dots), \quad (1, -1, 1, 0, 0, \dots), \quad (1, 1, -1, 0, 0, \dots)$$

lies in the hyperplane $f^{-1}(1)$ and has no Chebyshev centre there.

Veselý [156] also proved the following more general result.

Theorem 4.9. *Let $f = (f^{(i)}) \in (c_0)^*$ be a functional with infinite support and let $2\|f\|_\infty < \|f\|_1$. Then there exist a $\sigma \in \mathbb{R}$ and a three-point set $M = \{u, v, w\} \subset f^{-1}(\sigma)$ which has no Chebyshev centre in $f^{-1}(\sigma)$.*

Another result from [156] is worth pointing out (for a similar result for hyperplanes in $C(Q)$, see Zamyatin [166]).

[3] In his example $X = c_0(E)$ is the Banach space of null sequences in E with the norm $\|x\|_\infty = \max\{\|x(n)\| \mid n \in \mathbb{N}\}$, where E is a three-dimensional Banach space.

Theorem 4.10. *Let* $f = (f^{(i)}) \in (c_0)^*$ *and* $H = f^{-1}(0)$*. Then the following conditions are equivalent*:

a) *f has finite support or* $2\|f\|_\infty \geqslant \|f\|_1$;

b) *the hyperplane H is proximinal or is norm-*1 *complemented in* c_0;

c) *every non-empty bounded subset of H has a Chebyshev centre in H*;

d) *every finite subset of H has a Chebyshev centre in H*;

e) *every three-point subset of H has a Chebyshev centre in H.*

The next result follows from the Garkavi–Klee characterization of the spaces in which a Chebyshev centre of a set lies in its convex hull (see p. 781) and from Belobrov's [28] results on best net.

Theorem 4.11. *Let X be a Banach space. Then the following conditions are equivalent*:

a) *X is either a two-dimensional space or a Hilbert space*;

b) *there exists an $n \in \mathbb{N}$ such that for any non-empty set $M \subset X$ there exists a Chebyshev net of cardinality n lying in the convex hull of M*;

c) *for all $n \in \mathbb{N}$ and any non-empty set $M \subset X$ there exists a Chebyshev net of cardinality n lying in the convex hull of M.*

Quasi uniform convexity and existence of a Chebyshev centre. In this section we extend some of the above theorems on the existence of Chebyshev centres. Quasi uniformly convex spaces were introduced in 1973 in [44] (see also [157], [9], and [103]).

Definition 4.1. A Banach space X is said to be *quasi uniformly convex* ($X \in$ (QUC)) if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $x \in X$ there exists a $y \in B(0, \varepsilon)$ with $B(0, 1+\delta) \cap B(x, 1) \subset B(y, 1)$.

We note the following results.

Proposition 4.2 (Veselý [157]). *Let X be a Banach space. Then the following conditions are equivalent*:

a) $X \in$ (QUC);

b) *for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $x \in X$ and $\beta > 0$ there exists a $y \in B(0, \varepsilon)$ with $B(0, 1+\delta) \cap B(x, 1) \subset B(y, 1+\beta)$*;

c) *there exist sequences of positive numbers (ε_n) and (δ_n) such that $\delta_n \to 0$, $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, and for each $n \in \mathbb{N}$ and $x \in X$ there exists a $y_n \in B(0, \varepsilon_n)$ with $B(0, 1+\delta_n) \cap B(x, 1) \subset B(y_n, 1+\delta_n)$.*

Proposition 4.3 (see [9], [44]). *A Banach space is uniformly convex if and only if it is both quasi uniformly convex and strictly convex.*

Remark 4.2. a) The spaces ℓ^∞, c_0, c and $C[a, b]$ lie in the class (QUC) (see [44]).

b) If X is uniformly convex, then $C(Q, X) \in$ (QUC), where Q is a compact Hausdorff space (see [9]).

c) If $L^1(\mu)$ is infinite-dimensional, then $L^1(\mu) \notin$ (QUC) (see [9]).

In the following theorem, the existence of a Chebyshev centre was proved in [44], and its uniform continuity in the Hausdorff semimetric was proved in [9].

Theorem 4.12. *If $X \in$ (QUC) is a Banach space, then any non-empty bounded subset M of X has a Chebyshev centre. Moreover, the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$ is uniformly continuous in the Hausdorff semimetric on the class of non-empty subsets of X with uniformly bounded Chebyshev radii.*

For further results on the stability of the Chebyshev-centre map in quasi uniformly convex spaces, see § 6.1 below.

5. Uniqueness of the Chebyshev centre

The problem of uniqueness of a Chebyshev centre has been studied by Garkavi [70], M. Golomb, Laurent, and Pham-Dinh-Tuan [106], Rozema and Smith [137], Lambert and Milman [104], Smith and Ward [142], Amir and Ziegler [10], [11], Amir [5], Li and Watson [112], Laurent and Pai [105], and Peng and Li [125] (this list is by no means complete). Belobrov [30] considered spaces in which the set of Chebyshev centres of any set has dimension at most $r < \infty$.

5.1. Uniqueness of the Chebyshev centre of compact sets. We first consider the problem of uniqueness of a Chebyshev centre for compact sets.

Theorem 5.1 (Garkavi [69], [70]). *Each compact set M in a space X has at most one Chebyshev centre if and only if X is a strictly convex space.*

Amir and Ziegler [10] and later Amir [7] extended Theorem 5.1 as follows. First we need a definition.

Definition 5.1. A space X is said to be *strictly convex* (*rotund*) *in any direction in a convex subset* Y of it ($X \in$ (RED-Y)) if the unit sphere of X contains no non-degenerate interval parallel to an interval in Y. This is equivalent to the following condition:

$$\biggl[\ \|x\| = \|y\| = \biggl\|\frac{x+y}{2}\biggr\| = 1, \quad x-y \in Y-Y\biggr] \quad \Longrightarrow \quad x=y. \tag{5.1}$$

If Y is a subspace, then for brevity we say that X is strictly convex with respect to Y instead of saying that X is strictly convex in any direction in Y.

A space X is strictly convex (rotund) if and only if it is strictly convex with respect to $Y = X$. If X is strictly convex in any direction in a subspace Y, then clearly X is strictly convex in any direction in any subspace $Z \subset Y$. In this case, any subspace X_0 with $Y \subset X_0 \subset X$ is strictly convex with respect to any direction in Y, and in particular, Y is strictly convex. It is also clear that if X is strictly convex with respect to any 1-dimensional subspace, then X itself is strictly convex.

We recall the following definition.

Definition 5.2. Let $x \in X$ and $\varnothing \ne M \subset X$. A point $y_0 \in M$ is a *farthest point* in the set M from the point x if

$$\|x-y_0\| = \sup\{\|x-y\| \mid y \in M\} = r(x,M).$$

Theorem 5.2 (Amir [5]). *Let X be a normed linear space and let $Y \subset X$ be a convex set. Then the following conditions are equivalent*:

a) *X is strictly convex in any direction in Y (that is, the unit sphere S does not contain a non-trivial interval parallel to some interval in Y);*

b) $|\mathrm{Z}_Y(K)| \leqslant 1$ *for any compact set $K \subset X$;*

c) $|\mathrm{Z}_Y(K)| \leqslant 1$ *for any set $K \subset X$ such that any $y \in Y$ has a farthest point in K;*

d) $|\mathrm{Z}_Y(\{x, y\})| \leqslant 1$ *for any $x, y \in X$;*

e) *if $\|u\| = \|v\| = \|u + v\|/2$ and $u - v \in Y - Y$, then $u = v$;*

f) *any closed interval in Y is a Chebyshev set in X.*

Definition 5.3. A subspace Y of dimension n in a normed linear space X is called an *interpolating subspace* [14] if no non-trivial linear combination of n linearly independent extreme points of the dual ball B^* annihilates Y.

This definition is a natural extension of the definition of a Haar (Chebyshev) subspace.

Theorem 5.3 (Amir [5]). *Let Y be an interpolating subspace of a normed linear space X, and let $M \subset X$ be a compact set such that $r(M) < r_Y(M)$. Then the set $\mathrm{Z}_Y(M)$ is a singleton.*

Remark 5.1. The conclusion of Theorem 5.3 is not true for *bounded sets* (this was pointed out in [5], where the following counter-example was constructed disproving the corresponding erroneous assertion from [104]). Let $X = \{x \in C[-1, 1] \mid x(0) = (x(-1) + x(1))/2\}$ (with the Chebyshev norm). We set

$$y_0(t) = t, \quad Y := \operatorname{span} y_0, \quad M := \{x \in X \mid 0 \leqslant x(t) \leqslant 1 - |t|\}.$$

Since $\operatorname{ext} B^* = \{\pm e_t \mid 0 < |t| \leqslant 1\}$, where $e_t(x) := x(t)$, it follows that Y is an interpolating subspace, but

$$r(M) = r(2^{-1}, M) = 2^{-1} < 1 = r_Y(M) = r(\alpha y_0, M) \quad \text{for } |\alpha| \leqslant 1.$$

Let Q be a topological space and let L be a locally compact topological space. We denote by $C_0(L, X)$ the closed subspace of $C(L, X)$ consisting of all functions x that vanish at infinity (this means that for any $\varepsilon > 0$ the set $\{t \in L \mid |x(t)| \geqslant \varepsilon\}$ is compact).

We note the following simple fact.

Proposition 5.1. *The space $C_0(Q)$ is not strictly convex with respect to any subspace of dimension $\geqslant 2$.*

Proposition 5.2. *If μ is a measure, then there is no subspace of dimension $\geqslant 2$ in $L^1(\mu)$ with respect to which $L^1(\mu)$ is strictly convex. If the measure μ is atomless, then in $L^1(\mu)$ there is no subspace with respect to which $L^1(\mu)$ is strictly convex.*

Proof of Proposition 5.2. A trivial corollary of Phelps's characterization of finite-dimensional Chebyshev subspaces in the space $L^1(\Omega, \mu)$ (see [126]) is that $\operatorname{span} v$ is a Chebyshev subspace if and only if $\int_A v\, d\mu \neq \int_{\Omega\setminus A} v\, d\mu$ for any measurable set A. Let Y have dimension $\geqslant 2$, let v and w be two linearly independent points, and let A be a fixed set. By the intermediate value theorem, there exist numbers α and β with $\alpha^2 + \beta^2 = 1$ such that $\int_A (\alpha v + \beta w)\, d\mu = \int_{\Omega\setminus A} (\alpha v + \beta w)\, d\mu$. Hence, the 1-dimensional subspace spanned by the vector $z = \alpha v + \beta w$ is not a Chebyshev subspace, and therefore the space $L^1(\mu)$ is not strictly convex with respect to Y by the implication a) $\Rightarrow$ f) of Theorem 5.2.

If μ is atomless, then the required assertion follows from the following well-known result: there are no finite-dimensional Chebyshev subspaces in the space $L^1(\mu)$. $\square$

A similar analysis (see [10]) shows that the space $(C[a,b], \|\cdot\|_1)$ of continuous functions with the L^1-norm is not strictly convex with respect to any subspace Y of finite dimension $\geqslant 2$.

To conclude this section, we mention another result on uniqueness of relative Chebyshev centres. We need the following definition.

Definition 5.4. A space X is said to be *weakly uniformly convex* if the conditions $\|x_n\| \leqslant 1$, $\|y_n\| \leqslant 1$, $\|x_n + y_n\| \to 2$ imply that $x_n - y_n \xrightarrow{w} 0$. For the classical definition of a uniformly convex space, see, for example, [63].

Theorem 5.4 (Xiao and Zhu [163]). *Let X be a Banach space with* $\dim X \geqslant 2$, *and let $M \neq \varnothing$ be a convex weakly compact set in X. Assume that one of the following conditions is satisfied*:

a) *X is weakly uniformly convex*;

b) *X is locally uniformly convex and M is compact.*

Then $\mathrm{Z}_{\operatorname{conv} M}(M)$ *is a singleton which lies in M.*

5.2. Uniqueness of a Chebyshev centre of bounded sets. Following Garkavi, we say that a space X is *uniformly convex* (or *uniformly rotund*) *in every direction* ($X \in$ (URED)) if, for any $z \in X$ and any $\varepsilon > 0$, there exists a $\delta = \delta(z, \varepsilon) > 0$ such that if

$$\|x_1\| = \|x_2\| = 1, \quad x_1 - x_2 = \lambda z \quad \text{and} \quad \|x_1 + x_2\| > 2 - \delta,$$

then $|\lambda| \leqslant \varepsilon$.

Note that $X \in$ (URED) if and only if $x_n - y_n \to 0$ whenever $(x_n), (y_n) \subset S$, $\|(x_n + y_n)/2\| \to 1$, and $x_n - y_n \in \operatorname{span} v$ for some $v \in S$ and all $n \in \mathbb{N}$. Equivalently, $X \in$ (URED) if and only if, for any $z \in S$ and any bounded sequences $(x_n), (y_n)$ such that $2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \to 0$ and $x_n - y_n = \lambda_n z$ for some λ_n, we have $\lambda_n \to 0$ (see [63], p. 456).

The next theorem [69], [70] characterizes the spaces in which each bounded set has at most one Chebyshev centre.

Theorem 5.5 (Garkavi). *A necessary and sufficient condition that each bounded subset of a Banach space X have at most one Chebyshev centre is that X be uniformly convex in every direction* ($X \in$ (URED)).

Note that the condition $X \in$ (URED) is weaker than the condition that X is a uniformly convex space. Garkavi [69], [70] constructed an example of an (incomplete) normed space which is uniformly convex in every direction, but is not an (incomplete) uniformly convex space.

Let us give some extensions of Theorem 5.5.

Definition 5.5. Let Y be a convex set in X. A space X is *uniformly convex* (or *uniformly rotund*) *in every direction in* Y ($X \in (\text{URED-}Y)$) if for any $z \neq 0$ with $z \in Y - Y$ and any $\varepsilon > 0$ there exists a $\delta = \delta(z, \varepsilon) > 0$ such that

$$\left[\|x\| = \|y\| = 1,\ x - y = \lambda z,\ \left\|\frac{x+y}{2}\right\| > 1 - \delta\right] \quad \Longrightarrow \quad |\lambda| < \varepsilon. \tag{5.2}$$

Note that if $X \in (\text{URED-}Y)$, then X is strictly convex in every direction in Y. A number of properties of spaces that are uniformly convex in every direction can be found in [55].

The next result [10], [7] is a direct extension of Theorem 5.5.

Theorem 5.6. *Let Y be a convex subset of a Banach space X. Then the following conditions are equivalent*:

a) $|\mathrm{Z}_Y(M)| \leqslant 1$ *for any bounded set* $M \subset X$;

b) X *is uniformly convex in every direction in* Y ($X \in (\text{URED-}Y)$);

c) *if* $\|u_n\|, \|v_n\| \to 1$, $\|u_n + v_n\| \to 2$, *and if* $u_n - v_n = \lambda_n z \neq 0$ *for some* λ_n *and* $z \in Y - Y$, *then* $\lambda_n \to 0$.

We mention some further results (see also Theorem 6.26 below).

Theorem 5.7 (Amir [4], [9]). *A Banach space X is uniformly convex if and only if, for any non-empty bounded set $M \subset X$, the set $\mathrm{Z}(M)$ of Chebyshev centres is a singleton and the Chebyshev-centre map $M \mapsto \mathrm{Z}(M)$ is locally uniformly continuous.*

Definition 5.6. For a given subspace Y of a Banach space X, an equivalent definition of X as being *uniformly convex in every direction in* Y ($X \in (\text{URED-}Y)$) is that

$$\delta_Y(\varepsilon) := \inf\left\{1 - \frac{\|x+y\|}{2} \;\middle|\; \|x\| = \|y\| = 1,\ \|x - y\| \geqslant \varepsilon,\ x - y \in Y\right\} > 0 \tag{5.3}$$

for all $\varepsilon > 0$.

Theorem 5.8 (Amir [4], [9]). *Let Y be a subspace of a Banach space X. Then $X \in (\text{URED-}Y)$ if and only if the relative Chebyshev-centre map $M \mapsto \mathrm{Z}_Y(M)$ is single-valued and locally uniformly continuous on the class of bounded subsets of X.*

The next theorem is fairly simple. In it M is called a uniquely remotal set if every $x \in X$ has a unique farthest point in M.

Theorem 5.9. *Let X be strictly convex and let $Y \subset X$ be closed and convex. Then the set $\mathrm{Z}_Y(M)$ of Chebyshev centres is at most a singleton for any uniquely remotal set $M \subset X$.*

Remark 5.2. In fact, the conclusion of Theorem 5.9 holds if X is strictly convex (rotund) in every direction in Y ($X \in (\text{RED-}Y)$).

To conclude this section, we briefly discuss the problem of uniqueness of the absolute and relative Chebyshev centres in $C(Q)$.

The subsets of the space $C(Q)$ (Q a complete metric space) with unique Chebyshev centre were characterized by Smith and Ward [142] in terms of the behaviour of the functions $m_*(t) := \inf_{x\in M} x(t)$ and $m^*(t) := \sup_{x\in M} x(t)$. Their result was extended to not necessarily metrizable spaces Q by Zamyatin and Shishkin ([167], Theorem 9). In the more convenient language of the Banach–Mazur hull, the result on the uniqueness of a Chebyshev centre in $C(Q)$ (Q a normal space) is given in Theorem 2.1.

We recall (see Theorem 5.2) that the single-valuedness of the relative Chebyshev-centre map $\mathrm{Z}_Y(\,\cdot\,)$, where Y is a linear subspace, implies that the original space is strictly convex with respect to Y. As a corollary, in the space $C(Q)$ (Q a compact Hausdorff space), for any finite-dimensional subspace Y with $\dim Y \geqslant 2$ the relative Chebyshev-centre map $\mathrm{Z}_Y(\,\cdot\,)$ is not single-valued (see Proposition 5.1). Under the additional condition $r(M) < r_Y(M)$, the problem of uniqueness of a relative Chebyshev centre of M was investigated by Amir [5] for a certain special class of subspaces $Y \subset C(Q)$.

Definition 5.7. A subspace $Y = \operatorname{span}\{y_1, \dots, y_n\}$ of dimension n of a normed linear space X is said to be *strictly interpolating* [5] if no non-trivial linear combination of n linearly independent functionals in the w^*-closure $\overline{\operatorname{ext}}^{\,w^*} B^*$ annihilates Y.

Any one of the following conditions is equivalent to the strict interpolation condition [5]:

(i) $\det[f_i(y_j)] \neq 0$ for any linearly independent functionals $f_1, \dots, f_n \in \overline{\operatorname{ext}}\, B^*$;

(ii) for any linearly independent $f_1, \dots, f_n \in \overline{\operatorname{ext}}\, B^*$ and any scalars $c_1, \dots, c_n$, there exists a unique point $y \in Y$ such that $f_i(y) = c_i$ for each $i = 1, \dots, n$;

(iii) $X^* = Y^\perp \oplus \operatorname{span}\{f_1, \dots, f_n\}$ for any linearly independent $f_1, \dots, f_n \in \overline{\operatorname{ext}}\, B^*$.

It is easily seen that the interpolating subspaces of X are strictly interpolating if $\operatorname{ext} B^*$ is w^*-closed (this is so, for example, in the spaces $C(Q)$, where Q is a compact topological space, and in $L^1(\mu)$) or if $\operatorname{ext} B^* \cup \{0\}$ is w^*-closed (this condition is satisfied, for example, in the spaces $C_0(Q)$ with locally compact Q).

Theorem 5.10 (Amir [5]). *Let Y be a strictly interpolating subspace of a normed linear space X, and let $\varnothing \neq M \subset X$ be a bounded set such that $r(M) < r_Y(M)$. Then $\mathrm{Z}_Y(M)$ is a singleton.*

Theorem 5.11 (Amir [5]). *Let Y be a subspace of a normed space X such that for any $y_0 \in X$, $z \in X$, and $\varepsilon > 0$ there exists a $y_1 \in Y$ with $\inf\{f(y_1) \mid f \in \operatorname{ext} B^*,\ |f(z - y_0)| > \varepsilon\} > 0$. Let $\varnothing \neq M \subset X$ be a bounded set such that $r(M) < r_Y(M)$. Then $\mathrm{Z}_Y(M)$ is a singleton.*

Remark 5.3. An example of a space X and an infinite-dimensional subspace $Y \subset X$ satisfying the hypotheses of Theorem 5.11 was constructed in [5]:

$$X = \{x \in C[-1, 1] \mid x(0) = 0\},$$
$$Y = \{x \in X \mid x|_{[0,1]} \text{ is a polynomial of degree} \leqslant n\}.$$

6. Stability of the Chebyshev-centre map

The question of stability of the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$ in dependence on the properties of the space X and the set has been extensively studied. We mention Belobrov [27]–[29], Ward [159], Rozema and Smith [137], Bosznay [40], Mach

[114], [115], Szeptycki and Van Vleck [143], Prolla, Chiacchio, and Roversi [130], Amir, Mach, and Saatkamp [9], Amir and Mach [8], Beer and Pai [26], Chiacchio, Prolla, and Roversi [49], Tsar'kov [146], Al'brekht [1], Li and Lopez [111], Balashov and Polovinkin [128], Baronti and Papini [21], Alvoni and Papini [3], Balashov and Repovš [18], Balashov and G. E. Ivanov [85], G. E. Ivanov [84], Xiao and Zhu [163], Druzhinin [59], and Lalithambigai et al. [103]. (This list is by no means complete.)

6.1. Stability of the Chebyshev-centre map in arbitrary normed spaces. Let M and N be non-empty bounded subsets of a normed linear space X, let $z_M \in \mathrm{Z}(M)$ and $z_N \in \mathrm{Z}(N)$ be (some) Chebyshev centres of the sets M and N, and let $z'_M \in \mathrm{Z}^{\mathrm{sc}}_M$ and $z'_N \in \mathrm{Z}^{\mathrm{sc}}_N$ be (some) self Chebyshev centres of M and N (see (1.4)). Here and in what follows, $h(M,N)$ is the Hausdorff distance between sets M and N.

To begin with, we note that each of the numbers

$$h(M,N) \quad \text{and} \quad \|z_M - z_N\| \quad (\text{or } \|z'_M - z'_N\|)$$

can be larger, equal, or smaller than the other number (even if the Chebyshev centre is unique).

The following simple result holds (see, for example, [26], [21]).

Proposition 6.1. *Let M and N be non-empty closed bounded subsets of a normed linear space X, and let $x, y \in X$. Then*

$$|r(x,M) - r(y,M)| \leqslant \|x-y\|, \tag{6.1}$$

$$|r(x,M) - r(y,N)| \leqslant h(M,N) + \|x-y\|, \tag{6.2}$$

$$|r(M) - r(N)| \leqslant h(M,N). \tag{6.3}$$

Remark 6.1. *An analogue of (6.3) does not hold for self centres* $\mathrm{Z}^{\mathrm{sc}}(M)$ (see [21]). Indeed, let $X = C[0,1]$ and $0 < \varepsilon < 1/2$, and let

$$M := \bigl\{f \in X \mid 0 \leqslant f(t) \leqslant 1-\varepsilon \ \forall t \in [0,1],\ f(0) = 0\bigr\},$$
$$N := \bigl\{g \in X \mid 2^{-1} - \varepsilon \leqslant g(t) \leqslant 2^{-1} \ \forall t \in [0,1],\ g(0) = 2^{-1} - \varepsilon\bigr\}.$$

Then $r_M(M) = 1-\varepsilon$ and $r_N(N) = \varepsilon$, but $h(M,N) = 2^{-1} - \varepsilon < 1 - 2\varepsilon = r_M(M) - r_N(N)$.

For self centres, there is a weaker estimate (see [21]):

$$|r_M(M) - r_N(N)| \leqslant 2h(M,N). \tag{6.4}$$

Nevertheless, this estimate is sharp: it suffices to let $\varepsilon \to 0$ in the example in Remark 6.1.

We prove (6.4). Let $\varepsilon > 0$. Consider a point $b_\varepsilon \in N$ such that $r(b_\varepsilon, N) < r_N(N) + \varepsilon$ and choose $a_\varepsilon \in M$ such that $\|a_\varepsilon - b_\varepsilon\| < h(M,N) + \varepsilon$. Using (1.5) and (6.2), we find that

$$\begin{aligned} r(a_\varepsilon, M) &\leqslant r(b_\varepsilon, N) + h(M,N) + \|a_\varepsilon - b_\varepsilon\| < h(M,N) + r(b_\varepsilon, N) + h(M,N) + \varepsilon \\ &< 2h(M,N) + r_N(N) + 2\varepsilon, \end{aligned}$$

which gives $r_M(M) - r_N(N) \leqslant 2h(M,N)$. Now (6.4) follows by symmetry.

For self centres, the following analogue of the inequality (1.6) holds (see [21]).

Proposition 6.2. *If $z'_M \in \mathrm{Z}^{\mathrm{sc}}(M)$ and $z'_N \in \mathrm{Z}^{\mathrm{sc}}(N)$, then*

$$\|z'_M - z'_N\|^2 \leqslant \bigl(h(M,N) + r'_M\bigr)\bigl(h(M,N) + r'_N\bigr), \tag{6.5}$$

$$\|z'_M - z'_N\| \leqslant h(M,N) + \frac{r'_M + r'_N}{2}. \tag{6.6}$$

Proof. From (6.1) it follows that

$$\|z'_M - z'_N\| \leqslant r(z'_N, M) \leqslant r(z'_N, N) + h(M,N) = r'_N + h(M,N).$$

A similar argument shows that $\|z'_N - z'_M\| \leqslant r'_M + h(M,N)$. Now the required result follows by multiplying the inequalities obtained. □

Remark 6.2. The estimates in Proposition 6.2 are sharp: it suffices to consider the sets $M = \{(x,y) \mid 0 \leqslant x \leqslant 1,\ |y| \leqslant 1\}$ and $N = \{(x,y) \mid -1 \leqslant x \leqslant 0,\ |y| \leqslant 1\}$ in the space ℓ^∞_2.

6.2. Quasi uniform convexity and stability of the Chebyshev-centre map. Quasi uniformly convex (QUC) spaces were introduced above (see the end of § 4). We recall (see Theorem 4.12) that if X is a quasi uniformly convex Banach space ($X \in$ (QUC)), then any non-empty bounded set $M \subset X$ admits a Chebyshev centre, and the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$ is uniformly continuous on the class of non-empty subsets of X with uniformly bounded Chebyshev radii.

Below, $\mathscr{B}_H(X)$ is the semimetric space of all non-empty bounded subsets of a normed linear space X, equipped with the Hausdorff semimetric. We also require the following definitions from [157].

Definition 6.1. Given $A \in \mathscr{B}_H(X)$, we set

$$\mathrm{Z}^r(A) := \{x \in X \mid r(x,A) \leqslant r\} \qquad \forall r \geqslant r(A),$$

where $r(x,A) := \sup\{\|x-a\| \mid a \in A\}$.

The definitions of a Chebyshev centre and the Chebyshev radius have the following analogues for bounded nets in $\mathscr{B}_H(X)$.

Definition 6.2. Given a bounded decreasing (with respect to set inclusion) net $\mathscr{A} := (A_i)_{i\in I}$ in $\mathscr{B}_H(X)$ and $x \in X$, we define

$$\begin{aligned}
\varphi(\mathscr{A}, x) &:= \lim_{i\in I} r(x, A_i) = \inf_{i\in I} r(x, A_i),\\
r(\mathscr{A}) &:= \inf_{x\in X} \varphi(\mathscr{A}, x) = \inf_{x\in X} \inf_{i\in I} r(x, A_i),\\
\mathrm{Z}^r(\mathscr{A}) &:= \{x \in X \mid \varphi(\mathscr{A}, x) \leqslant r\} \qquad \forall r \geqslant r(\mathscr{A}),\\
\mathrm{Z}(\mathscr{A}) &:= \mathrm{Z}^{r(\mathscr{A})}(\mathscr{A}).
\end{aligned}$$

The non-negative number $r(\mathscr{A})$ is called the *asymptotic radius* of the net $\mathscr{A}$, and the (possibly empty) set $\mathrm{Z}(\mathscr{A})$ is known as the *set of asymptotic centres* of $\mathscr{A}$.

We also recall the classical notions of the asymptotic radius and the set of asymptotic centres of a sequence.

Definition 6.3. Given $x \in X$ and a bounded sequence (a_n) in X, we set

$$\begin{aligned}\rho((a_n), x) &:= \varlimsup_{n\to\infty} \|x - a_n\|, \quad r(a_n) := \inf_{x\in X} \rho((a_n), x),\\ \mathrm{Z}^r(a_n) &:= \{x \in X \mid \rho((a_n), x) \leqslant r\} \qquad \forall r \geqslant r(a_n),\\ \mathrm{Z}(a_n) &:= \mathrm{Z}^{r(a_n)}(a_n).\end{aligned}$$

The non-negative number $r(a_n)$ is called the *asymptotic radius* of a sequence (a_n), and the (possibly empty) set $\mathrm{Z}(a_n)$ is known as the *set of asymptotic centres* of a sequence (a_n).

Definitions 6.1 and 6.3 are particular cases of Definition 6.2.

Theorem 6.1 (Veselý [157]). *Let X be a Banach space. Then the following conditions are equivalent*:

a) $X \in (\mathrm{QUC})$;

b) *for any bounded net $\mathscr{A}$ in $\mathscr{B}_H(X)$ the set $\mathrm{Z}(\mathscr{A})$ is non-empty, and the map $r \mapsto \mathrm{Z}^r(\mathscr{A})$ is continuous on $[r(\mathscr{A}), \infty)$ and uniformly continuous on the class of nets with uniformly bounded asymptotic radii*;

c) *any set $A \in \mathscr{B}_H(X)$ admits a Chebyshev centre, and the map $r \mapsto \mathrm{Z}^r(A)$ is continuous on $[r(A), \infty)$ and uniformly continuous on the class of sets with uniformly bounded Chebyshev radii*;

d) *for any (some) $r_0 > 0$ the map $(A, r) \mapsto \mathrm{Z}^r(A)$ has values in $\mathscr{B}_H(X)$ and is uniformly continuous on the set*

$$\bigl\{(A, r) \in \mathscr{B}_H(X) \times (0, \infty) \mid r(A) \leqslant r \leqslant r_0\bigr\};$$

e) *for any (some) $r_0 > 0$ the map Z^{r_0} has values in $\mathscr{B}_H(X)$ and is uniformly continuous on the set*

$$\{A \in \mathscr{B}_H(X) \mid r(A) \leqslant r_0\};$$

f) *for any (some) $r_0 > 0$ the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$ has non-empty values and is uniformly continuous on the set*

$$\{A \in \mathscr{B}_H(X) \mid r(A) = r_0\}.$$

Any one of the conditions a)–f) *implies the following property*:

g) *any bounded sequence $(a_n) \subset X$ has an asymptotic centre, and the map $r \mapsto \mathrm{Z}^r(a_n)$ is continuous on $[r(a_n), \infty)$ and uniformly continuous on the class of sequences with uniformly bounded asymptotic radii.*

Moreover, if X is separable, then the conditions a)–g) *are equivalent.*

We mention some results regarding the space of bounded vector-valued continuous functions with values in a Banach space $X \in (\mathrm{QUC})$ (see [157]).

Definition 6.4. Let Q be a topological space. As before, let $C(Q, X)$ be the space of all bounded continuous functions x on Q with values in X, equipped with the norm $\|x\| = \sup_{t\in Q} \|x(t)\|$.

The next result from [157] extends the assertion b) in Remark 4.2.

Theorem 6.2 (Veselý). *If $X \in (\mathrm{QUC})$ is a Banach space and Q is a topological space*, *then $C(Q,X) \in (\mathrm{QUC})$*.

Definition 6.5. Let Q be a topological space, $Q_0 \subset Q$ a closed subset, L a locally compact topological space, and Γ a non-empty set with the discrete topology. We consider the following spaces:

$C(Q,Q_0,X)$ is the closed subspace of $C(Q,X)$ consisting of all functions that vanish on Q_0;

$C_0(L,X)$ is the closed subspace of $C(L,X)$ consisting of all functions x that vanish at infinity (this means that the set $\{t \in L \mid |x(t)| \geqslant \varepsilon\}$ is compact for any $\varepsilon > 0$);

$\ell^\infty(\Gamma,X) := C(\Gamma,X)$ and $c_0(\Gamma,X) := C_0(\Gamma,X)$, where Γ is equipped with the discrete topology.

Theorem 6.3 (Veselý). *Let X be a Banach space. Then the following conditions are equivalent*:

a) $X \in (\mathrm{QUC})$;

b) $C(Q,X) \in (\mathrm{QUC})$ *for any topological space Q*;

c) $C(Q,Q_0,X) \in (\mathrm{QUC})$ *for any topological space Q and any closed subset $Q_0 \subset Q$*;

d) $C_0(L,X) \in (\mathrm{QUC})$ *for any locally compact space L*;

e) $c_0(\Gamma,X) \in (\mathrm{QUC})$ *for any Γ*;

f) $\ell^\infty(\Gamma,X) \in (\mathrm{QUC})$ *for any Γ*.

The next result follows from Theorems 6.1 and 6.3.

Theorem 6.4 (Veselý). *Let $X \in (\mathrm{QUC})$ be a Banach space and let $Y = C(Q,X)$* (*or Y is any other space in Theorem 6.3*). *Then the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$ is uniformly continuous on the class of sets $\{A \in \mathscr{B}_H(Y) \mid r(A) \leqslant r_0\}$ with any given $r_0 > 0$*.

To conclude this section, we mention one sufficient result in [157].

Theorem 6.5 (Veselý). *Let X be a finite-dimensional Banach space which is either polyhedral or two-dimensional. Then $X \in (\mathrm{QUC})$*, *and therefore $C(Q,X) \in (\mathrm{QUC})$*.

6.3. Stability of the Chebyshev-centre map in finite-dimensional polyhedral spaces. Let X_n be an n-dimensional polyhedral Banach space (that is, the unit ball of X_n is the convex hull of a finite number of points in X_n). Let $K \subset X_n$ be a convex polyhedral set (that is, K is the intersection of finitely many closed half-spaces in X_n).

In the polyhedral space X_n the metric projection onto a polyhedral set is globally Lipschitz continuous (Li [64]) and has a Lipschitz selection (Finzel and Li [65], Theorem 6.1). In this connection, we also mention the following results. Cline and, independently, V. I. Berdyshev showed that in ℓ^∞_n the metric projection onto a Chebyshev subspace is globally Lipschitz (uniformly continuous) on the entire space (see, for example, Theorem 2 in [35], [22], and § 5 in [64]). The next theorem extends these results.

Theorem 6.6 (Tsar'kov [147]). *Let $V \neq \varnothing$ be a polyhedral subset of ℓ^∞_n. Then the relative Chebyshev-centre map $\mathrm{Z}_V(\,\cdot\,)$ is a set-valued Lipschitz map on ℓ^∞_n*.

Proof. With each set $M \subset \ell_n^\infty$ we associate the pair of vectors $m_* = m_*(M) = (x_1,\dots,x_n)$, $m^* = m^*(M) = (y_1,\dots,y_n)$, where

$$x_i := \inf\{z_i \mid z = (z_1,\dots,z_n) \in M\}, \qquad y_i := \sup\{z_i \mid z = (z_1,\dots,z_n) \in M\}.$$

Consider the polyhedral set $U = \{(x,x) \mid x \in V\}$ in the space ℓ_{2n}^∞. According to Li (see [64]), the metric projection to any polyhedral set is Lipschitz continuous. As a corollary, the metric projection P_U is Lipschitz continuous. It follows that P_U is a Lipschitz map on the set of vectors $m(M) := (m_*(M), m^*(M))$. Moreover, any point $(x,x) \in U$ which is nearest to the vector $m(M)$ is characterized by the following property: $x \in V$ is a relative Chebyshev centre of the set M with respect to V. Thus, $\mathrm{Z}_V(\,\cdot\,)$ is a Lipschitz continuous map on the set of all bounded subsets of the space ℓ_n^∞. $\square$

The following result is a consequence of Theorem 6.6, because any finite-dimensional polyhedral space can be isometrically embedded in ℓ_N^∞ for some $N \in \mathbb{N}$, and since the Steiner-centre map is Lipschitz continuous on the class of bounded sets (see, for example, Theorem 2.1.2 in [128]). The Lipschitz selection required in Theorem 6.7 is the composition of the operator $\mathrm{Z}_V(\,\cdot\,)$ and the Steiner-centre map.

Theorem 6.7 (Tsar'kov [147]). *Let V be a non-empty polyhedral subset of a finite-dimensional polyhedral Banach space X. Then the relative Chebyshev-centre map*

$$M \mapsto \mathrm{Z}_V(M), \qquad \varnothing \neq M \subset X,$$

is globally Lipschitz continuous on X and admits a Lipschitz selection.

Theorem 6.7 implies Druzhinin's result on the existence of a Lipschitz selection of the Chebyshev-centre map in a finite-dimensional polyhedral space.

6.4. Stability of the Chebyshev-centre map in $C(Q)$-spaces. It is known that the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$ (which associates with a non-empty bounded set the set of its Chebyshev centres) is uniformly continuous in some class of spaces containing the uniformly convex spaces (see § 6.5) and spaces of type $C_0(Q)$ (in particular, the space $C(Q)$, where Q is a compact Hausdorff space); see [28], [4], [9], [7], [103], and also Theorem 5.7.

The next theorem (see [4], Theorem 2, and [9], Corollary 5.2) is one of the most general results on the existence and stability of the Chebyshev-centre map in the spaces $C(Q,X)$.

Theorem 6.8 (Amir and Ka-Sing Lau). *Let Q be an arbitrary topological space and X a uniformly convex Banach space. Then any bounded set in the space $C(Q,X)$ admits a Chebyshev centre, and the map $\mathrm{Z}(\,\cdot\,)$ is uniformly continuous on the class of non-empty bounded subsets with uniformly bounded diameters.*

Proof. Since X is a uniformly convex space, by definition there exists for any $\varepsilon > 0$ a $\delta(\varepsilon) > 0$ such that if $x, y \in S$, $x^* \in S^*$, $x^*(y) = 1$, and $x^*(x) > 1 - \delta(\varepsilon)$, then $\|x - y\| < \varepsilon$. Note that by Lemma 1 in [4] one can always assume that $\delta(\varepsilon) < \varepsilon/2$ and $\delta(\varepsilon/2) < \varepsilon/4$. Let $M \subset C(Q,X)$ be a non-empty bounded set. We can assume without loss of generality that $r(M) = 1$. For any $\varepsilon > 0$ we choose $f_0 \in C(Q,X)$ such that $r(f_0, M) \leqslant 1 + \delta(\varepsilon)$. We assert that there exists a function $f_1 \in C(Q,X)$

such that $r(f_1, M) \leqslant 1 + \delta(\varepsilon/2)$ and $\|f_1 - f_0\| \leqslant 2\varepsilon$. Indeed, consider an arbitrary function $g \in C(Q, X)$ such that $r(g, M) \leqslant 1 + \delta(\varepsilon/2)$, and define

$$\beta(t) = \begin{cases} 1, & \|g(t) - f_0(t)\| \leqslant 2\varepsilon, \\ \dfrac{2\varepsilon}{\|g(t) - f_0(t)\|}, & \|g(t) - f_0(t)\| > 2\varepsilon \end{cases}$$

and

$$f_1(t) = f_0(t) + \beta(t)(g(t) - f_0(t)).$$

It is clear that $f_1 \in C(Q, X)$ and $\|f_1 - f_0\| \leqslant 2\varepsilon$. Moreover, if $\|g - f_0\| > 2\varepsilon$, then $\|f_1 - f_0\| = 2\varepsilon$. Consider an arbitrary point $a \in M$. We need to show that $\|f_1(t) - a(t)\| \leqslant 1 + \delta(\varepsilon/2)$. This inequality is clear if $\beta(t) = 1$, since in this case $f_1(t) = g(t)$, and also if $\beta(t) < 1$ and $\|g(t) - a(t)\| \geqslant \|f_0(t) - a(t)\|$, since in this case $f_1(t) \in [f_0(t), g(t)]$. Thus, we can always assume that $1 + \delta(\varepsilon) \geqslant \|f_0(t) - a(t)\| > \|g(t) - a(t)\|$. Setting $u := f_0(t) - a(t)$ and $v := g(t) - a(t)$, we have $\|v\| \leqslant 1 + \delta(\varepsilon/2)$ and $1 + \delta(\varepsilon) \geqslant \|u\| > \|v\|$. Let us show that if we move a distance 2ε away from the point u in the direction v, then we fall in the ball $B(0, 1 + \delta(\varepsilon/2))$. Since $\delta(\varepsilon) < \varepsilon/2$ and $\delta(\varepsilon/2) < \varepsilon/4$, the required result holds for $\|v\| = 0$, and so it suffices to consider the case $\|v\| = 1 + \delta(\varepsilon/2)$.

In the two-dimensional space spanned by the points 0, u, and v, consider a point z lying on the sphere $S(0, \|v\|)$ on the same side as v of the line passing through 0 and u and such that the line uz supports the ball. We extend this line to a hyperplane $H := \psi^{-1}(1)$ that supports the sphere $S(0, \|v\|)$ in the space X. It is clear that $\|\psi\| = 1/\|v\|$. Let $\varphi := \|v\|\psi$, $x := u/\|u\|$, and $y := z/\|z\|$. Then $\|\varphi\| = \varphi(y) = 1 = \|y\| = \|x\|$ and $\varphi(x) = \|v\|/\|u\| \geqslant 1/\|u\| \geqslant 1/(1 + \delta(\varepsilon)) > 1 - \delta(\varepsilon)$. Since the space is uniformly convex, we then have $\|x - y\| < \varepsilon$ and $\|u - z\| < \varepsilon + \|u - x\| + \|z - y\| \leqslant \varepsilon + \delta(\varepsilon) + \delta(\varepsilon/2) < 2\varepsilon$, which proves the assertion, because the distance from u to the ball $B(0, \|v\|)$ in the direction v is smaller than the maximum of the distances in the direction x (which is at most $\delta(\varepsilon)$) and in the direction z (which is less than 2ε).

Arguing by induction, we can find an f_{n+1} such that $\|f_{n+1} - f_n\| \leqslant 2\varepsilon/2^n$ and $r(f_{n+1}, M) \leqslant 1 + \delta(\varepsilon/2^{n+1})$. The space $C(Q, X)$ is complete, and hence the Cauchy sequence (f_n) converges to some function f such that

$$\|f - f_0\| \leqslant 4\varepsilon \quad \text{and} \quad r(f, M) \leqslant \lim r(f_n, M) \leqslant 1.$$

This shows that $r(f, M) = 1$ and that f is a Chebyshev centre of the set M.

Thus, we have considered a point f_0 in the set of Chebyshev near-centres of the set M, and we have shown that there exists a Chebyshev centre f of M that lies sufficiently close to f_0. Hence, taking f_0 to be a Chebyshev centre of a set N which is close to M (in the Hausdorff metric), we get that f_0 (a Chebyshev centre of N) is close to some Chebyshev centre f of M. Therefore, the directed (one-sided) Hausdorff distance (and hence the Hausdorff distance) between the corresponding sets of Chebyshev centres of these close sets is small. And consequently, the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$ is uniformly continuous. $\square$

Amir [4] also showed that if $X = C(Q)$ and Y is a closed linear sublattice[4] of X, then any non-empty bounded set $M \subset X$ admits a Chebyshev centre and the Chebyshev-centre map $M \mapsto \mathrm{Z}_Y(M)$ is uniformly continuous on bounded subsets of the space of non-empty bounded sets in X (cf. Theorem 5.7).

The next result can be proved using Amir's arguments for the case of $C(Q,\mathbb{R})$.

Theorem 6.9. *Let M and N be non-empty bounded subsets of $C(Q)$, where Q is an arbitrary topological space. Then for any $\delta \geqslant 0$*

$$h(\mathrm{Z}^\delta(M), \mathrm{Z}^\delta(N)) \leqslant 2h(M,N).$$

Zamyatin and Kadec (see, for example, [167]) proved the result of Theorem 6.9 with $Q = [a,b]$ and $\delta = 0$.

Theorem 6.10 (Tsar'kov). *Let Q be a topological space. Then, for any non-empty closed span $Y \subset C(Q)$ and any non-empty bounded set $M \subset C(Q)$,*

$$\mathrm{Z}_Y(M) \neq \varnothing.$$

Moreover, for any $\delta \geqslant 0$ and arbitrary non-empty bounded sets $M, N \subset C(Q)$,

$$h(\mathrm{Z}_Y^\delta(M), \mathrm{Z}_Y^\delta(N)) \leqslant 2h(M,N).$$

Definition 6.6. Let V be a non-empty closed subset of a normed space X, and let $\mathscr{F}$ be a subfamily of the family of all non-empty closed bounded subsets of X. The triple $(X, V, \mathscr{F})$ is said to *have the property* (R_1) (see [123]) if the conditions $x \in V$, $M \in \mathscr{F}$, $r_1 > 0$, $r_2 > 0$, $V \cap \mathscr{O}_{r_1}(M) \neq \varnothing$ (where $\mathscr{O}_r(M) := \{x \in X \mid \rho(x,M) \leqslant r\}$), and $\rho(x,M) < r_1 + r_2$ imply that

$$V \cap B(x,r_1) \cap \mathscr{O}_{r_2}(M) \neq \varnothing.$$

According to Theorem 2.2 of [123], if V is a non-empty closed subset of a Banach space X, $\mathscr{F}$ is a family of non-empty closed bounded subsets of X, and the triple $(X, V, \mathscr{F})$ has the property (R_1), then

$$\mathrm{Z}_V(M) \neq \varnothing \quad \text{for any } M \in \mathscr{F}.$$

Pai and Nowroji [123] constructed several examples of triples $(X, V, \mathscr{F})$ with the property (R_1). In particular, they showed that if Q is a compact Hausdorff space, $X = C(Q,\mathbb{R})$, and V is a closed subalgebra of X, then the triple $(X, V, \mathscr{K}(X))$ (where $\mathscr{K}(X)$ is the family of compact subsets of X) has the property (R_1). The following result partially strengthens the Zamyatin–Kadec result on stability of the Chebyshev-centre map in $C[a,b]$.

Theorem 6.11 (Pai and Nowroji [123]). *Let V be a non-empty closed subset of a normed linear space X, and let the triple $(X, V, \mathscr{F})$ have the property (R_1). Then the Chebyshev-centre map $\mathrm{Z}_V(\,\cdot\,)$ is Lipschitz continuous in the Hausdorff metric:*

$$h(\mathrm{Z}_V(M), \mathrm{Z}_V(N)) \leqslant 2h(M,N) \tag{6.7}$$

for any non-empty closed bounded sets $M, N \subset X$. The constant 2 in (6.7) is best possible.

[4] A *linear sublattice* is a subspace L with the property that $f \in L \Rightarrow |f| \in L$.

For a similar result in $\ell^\infty(\Gamma)$, see [49].

The next result follows From Michael's selection theorem and Theorem 6.10.

Corollary 6.1. *In $C(Q)$, where Q is a topological space, the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$ has a continuous selection.*

By using Amir's results (see Theorem 6.8) it can be shown that in the space $C(Q,X)$, where Q is a topological space and X is a uniformly convex Banach space, the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$ has a continuous selection.

Remark 6.3. In $C(Q)$, where Q is an infinite compact Hausdorff space, the relative Chebyshev-centre map $\mathrm{Z}_V(\,\cdot\,)$, where V is a finite-dimensional Chebyshev subspace of dimension $\geqslant 2$, is not Lipschitz continuous (uniformly continuous). This result is a consequence of the fact that the restriction of the operator $\mathrm{Z}_V(\,\cdot\,)$ to singletons is the metric projection and since, according to a result of Cline, for any[5] finite-dimensional Chebyshev subspace V of dimension $\geqslant 2$, there exist functions $x,y\in C[-1,1]$ with $\|x\|=\|y\|=1$ such that $\|x-y\|<\varepsilon$ but $\|P_Vx-P_Vy\|\geqslant 1$. Cline's theorem also shows that the metric projection operator P is not uniformly continuous on the unit ball of the space $C(Q)$. In the particular case of the subspaces $\mathscr{P}_n$ generated by the algebraic polynomials of degree $\leqslant n$ with $n\geqslant 2$ in the space $C[a,b]$, this result dates back to a construction of S. N. Bernstein. For $n=1$ S. B. Stechkin showed that the metric projection operator to the subspace of linear functions in $C[a,b]$ is not uniformly continuous).

6.5. Stability of the Chebyshev-centre map in Hilbert and uniformly convex spaces. We first consider the Hilbert space setting. As mentioned above (see Theorem 4.4 or Lemmas 2.1.1 and 2.1.2 in [128]), for any closed convex bounded subset of a Hilbert space a Chebyshev centre exists, is unique, and lies in the subset. By z_M we denote the (unique) Chebyshev centre of a bounded subset M of a uniformly convex space or a Hilbert space, and we let $r_M=r(M)$ be its Chebyshev radius (see § 11).

The following result is due to Ward [159] and Alvoni and Papini [3].

Proposition 6.3. *Let H be a Hilbert space, and let $M=\{x_1,\dots,x_n\}\subset H$, $N=\{y_1,\dots,y_n\}\subset H$, $\|x_i\|\leqslant R$, $\|y_i\|\leqslant R$, and $\|x_i-y_i\|\leqslant h$, $i=1,\dots,n$. Then*

$$\|z_M-z_N\|\leqslant 2\sqrt{hR+h^2}. \tag{6.8}$$

Moreover, if $r_M\leqslant r_N$, then

$$\|z_M-z_N\|\leqslant h+\sqrt{5h^2+2hr_M+2hr_N}. \tag{6.9}$$

Consider now the case of arbitrary bounded sets. The following result holds (for compact sets see Theorem 1 in [143], and for the general case see [21], [3]).

Theorem 6.12. *Let M and N be non-empty bounded subsets of a Hilbert space, and let z_M and z_N be their Chebyshev centres. Then*

$$\|z_M-z_N\|\leqslant\sqrt{(h(M,N)+r_M+r_N)\,h(M,N)}. \tag{6.10}$$

[5]By Mairhuber's theorem the space $C(Q)$, where Q is a compact metric space, contains a Chebyshev subspace of any finite dimension $n=2,3,\dots$ if and only if Q is homeomorphic to an infinite closed subset of the closed unit interval $[0,1]$.

Remark 6.4. The estimate (6.10) is sharp: there exist pairs of sets (with the same Chebyshev radius) for which (6.10) becomes an equality. It is clear that equality in (6.10) is also attained for singletons, but (6.10) ceases to be true in spaces in which the Chebyshev-centre map is not single-valued.

The following strengthening of (6.10) was obtained in [3] with the use of a technique in [145].

Theorem 6.13. *Let M and N be non-empty bounded subsets of a Hilbert space, and let z_M and z_N be their Chebyshev centres. Suppose that $r_M \leqslant r_N$. Then*

$$\|z_M - z_N\|^2 \leqslant (r_M + h(M,N))^2 - r_N^2. \tag{6.11}$$

From (6.11) it follows that

$$\|z_M - z_N\| \leqslant \sqrt{2h(M,N)\,r_N + h^2(M,N)} \tag{6.12}$$

(see [3], Remark 3).

Remark 6.5. The estimate (6.11) can be seen as an estimate for the variation of the radius when the distance between the centres is known. For example (see [3], p. 431), if $\|z_M - z_N\| \geqslant h(M,N)$, then $r_N^2 \leqslant r_M^2 + 2h(M,N)\,r_M$, which implies that

$$r_M \geqslant \sqrt{h^2(M,N) - r_N^2} - h(M,N).$$

Some other estimates for the distance between Chebyshev centres in a Hilbert space can be found in [128], Theorem 2.1.1, [85], § 5, and [84], Lemma 3.5.3.

Remark 6.6. Examples showing that the estimate (6.10) cannot be significantly improved have been known for a long time (see, for example, [128]). Namely, for any $r > 0$ and any $\varepsilon \in (0, r)$ there exist closed convex sets $A_1, A_2 \subset B(0, r)$ such that

$$\varepsilon = \|c_1 - c_2\| \geqslant \sqrt{2rh(A_1, A_2)},$$

where c_i is the Chebyshev centre of the set A_i, $i = 1, 2$. Indeed, let $r > 0$ and $\varepsilon \in (0, r)$. On the Euclidean plane $\mathbb{R}^2$ (with Cartesian coordinates $(x^{(1)}, x^{(2)})$) consider two sets A_1 and A_2 defined as follows:

$$A_1 := \bigl\{x \in \mathbb{R}^2 \mid 0 \leqslant x^{(2)} \leqslant \varepsilon,\ (x^{(1)})^2 + (x^{(2)})^2 \leqslant r^2\bigr\}$$

and the set A_2 is symmetric to the set A_1 relative to the line $x^{(2)} = \varepsilon/2$. It is easily seen that the points $c_1 = \{(0, 0)\}$, $c_2 = \{(0, \varepsilon)\}$ are the Chebyshev centres of A_1 and A_2, respectively, and that the Hausdorff distance $h(A_1, A_2) = \sqrt{r^2 + \varepsilon^2} - r^2$ is at most $\varepsilon^2/(2r)$. As a result,

$$\|c_1 - c_2\| \geqslant \sqrt{2rh(A_1, A_2)}.$$

Theorem 6.13 shows that in a Hilbert space the Chebyshev-centre map is Hölder continuous with exponent 1/2 uniformly on sets of fixed diameter. Simple examples can be constructed to show that this map is not Lipschitz continuous.

We note the following result (see Proposition 4 in [3], and also Theorem 1 in [58]).

Proposition 6.4. *Let $M \subset H$ be a non-empty bounded subset of a Hilbert space and let z be its Chebyshev centre. Then*

$$r_M^2 + \|x - z\|^2 \leqslant r^2(x, M) \quad \text{for any } x \in X$$

(*here we recall that* $r(x, M) := \inf\{r \geqslant 0 \mid M \subset B(x, r)\}$).

For uniformly convex spaces we mention the following results.

Theorem 6.14 (see [3], [21]). *Let M and N be non-empty bounded subsets of a uniformly convex space X.*

1. *If $r_M \leqslant r_N$, then*

$$r_N \leqslant (r_M + h(M,N))\biggl(1 - \delta\biggl(\frac{\|z_M - z_N\|}{r_M + h(M,N)}\biggr)\biggr). \tag{6.13}$$

2. *The following inequality holds*:

$$\delta\biggl(\frac{\|z_M - z_N\|}{h(M,N) + \min\{r_M, r_N\}}\biggr) \leqslant 1 - \frac{\max\{r_M, r_N\}}{\min\{r_M, r_N\} + h(M,N)} \biggl(\leqslant \frac{h(M,N)}{\min\{r_M, r_N\} + h(M,N)}\biggr). \tag{6.14}$$

Here and below, $\delta(\varepsilon)$ is the modulus of convexity of a space (see [63]).

Another variant of (6.14) was established in Lemma 4 in [145].

For Hilbert spaces (for which $\delta(\varepsilon) = 1 - \sqrt{1 - \varepsilon^2/4}$, $0 \leqslant \varepsilon \leqslant 2$), Theorem 6.14 gives the estimate

$$\|z_M - z_N\| \leqslant 2\sqrt{(r_M + h(M,N))^2 - r_N^2},$$

which is weaker than (6.11). The following result [3] strengthens (6.13).

Theorem 6.15 (see [3]). *Let M and N be non-empty bounded subsets of a uniformly convex space X, and let $r_M \leqslant r_N$. Then*

$$r_N \leqslant (r_M + h(M,N))\biggl(1 - \delta\biggl(\frac{\|z_M - z_N\| + r_M - h(M,N) - r_N}{r_M + h(M,N)}\biggr)\biggr). \tag{6.15}$$

See [158] for analogues of the estimate in Theorem 6.14 and for its sharpness in p-uniformly convex spaces. We mention another result (see [21]).

Theorem 6.16. *Let X be a uniformly convex space and let M and N be bounded subsets of X. Let $c := \|z_M - z_N\|$, where z_M and z_N are the Chebyshev centres of the sets M and N, respectively, and define $\gamma := \max\{r_M + h, r_N + h\}$, where $h := h(M,N)$. Suppose that $0 \neq c \leqslant 2\gamma$. Then*

$$\gamma\,\delta\Bigl(\frac{c}{\gamma}\Bigr) \leqslant h \tag{6.16}$$

and

$$c \leqslant \gamma\,\varepsilon\Bigl(\frac{h}{\gamma}\Bigr). \tag{6.17}$$

In Theorem 6.16,

$$\varepsilon(\delta) := \sup\left\{\|x-y\| \;\middle|\; \|x\| \leqslant 1,\ \|y\| \leqslant 1,\ \frac{\|x+y\|}{2} \geqslant 1-\delta\right\}$$

is the inverse of the modulus of convexity $\delta(\varepsilon)$.

Remark 6.7. The estimates in Theorem 6.16 are quite rough in Hilbert spaces. For example, if $\|z_M - z_N\| \leqslant 2h$, then equality in (6.17) is impossible (see [21], Remark 4.2). It is unknown whether the estimates (6.16) and (6.17) are sharp in the class of uniformly convex spaces.

For L^q-spaces with $1 < q < \infty$, Prus and Smarzewski (see Theorem 4.1 in [132]) established the following result.

Theorem 6.17. *Suppose that for some $q \geqslant 2$ there exists a positive constant C such that*

$$\delta_X(\varepsilon) \geqslant C\,\varepsilon^q \qquad (0 < \varepsilon \leqslant 2).$$

Let $M \subset X$ and $\mathrm{Z}_M = \{x\}$ (or $\mathrm{Z}_M^{\mathrm{sc}} = \{x\}$). Then

$$r(x, M) \leqslant r(y, M) - k\|x-y\|^q \quad \text{for any } y \in M,$$

where the constant k depends only on q and C.

We also mention the following Hilbert space criterion from [3].

Theorem 6.18. *Let X be a Banach space. Then the norm on X is Euclidean if and only if the inequality*

$$\|z_M - z_N\|^2 \leqslant (r_M + h(M,N))^2 - r_N^2 \quad \text{for } r_M \leqslant r_N \tag{6.18}$$

is satisfied for any sets $M, N \subset X$ that have Chebyshev centres.

6.6. Stability of the self Chebyshev-centre map. The stability problem for the self Chebyshev-centre map was first examined by Borwein and Keener [39].

Given a non-empty bounded set M, we denote by z'_M and $r'(M)$ the self Chebyshev centre and self Chebyshev radius of M, respectively (see (1.4), (1.3)).

Let $\mathscr{M}$ be the class of non-empty closed convex bounded sets with the property that $\mathring{B}(z'_M, r'_M) \cap \mathring{B}(z'_N, r'_N) = \varnothing$ for any non-equal $M, N \in \mathscr{M}$ with $\mathrm{Z}_M^{\mathrm{sc}} = \{z'_M\}$ and $\mathrm{Z}_N^{\mathrm{sc}} = \{z'_N\}$ (here, as before, $\mathrm{Z}_M^{\mathrm{sc}} = \{z'_M\}$ is the set of self Chebyshev centres of a set M; see (1.4)).

Theorem 6.19 (Borwein and Keener [39]). *Let X be a Hilbert space and let $M, N \in \mathscr{M}$. Then*

$$\|z'_M - z'_N\| \leqslant \frac{1+\sqrt{5}}{2}\, h(M,N).$$

Theorem 6.20 (Borwein and Keener [39]). *Let X be a strictly convex space and let $M, N \in \mathscr{M}$. Then*

$$\|z'_M - z'_N\| \leqslant c\, h(M,N), \quad \text{where } c \in \left[\frac{1+\sqrt{5}}{2}, 2\right].$$

6.7. Upper semicontinuity of the Chebyshev-centre map and the Chebyshev near-centre map. Recall that a map $F\colon X \to 2^Y$ is *upper semicontinuous* at a point x_0 if, for any open set $V \subset Y$ with $F(x_0) \subset V$, there exists a neighbourhood $\mathscr{O}(x_0)$ such that $F(x) \subset V$ for any $x \in \mathscr{O}(x_0)$. A map F is *upper semicontinuous* on X if it is upper semicontinuous at any point $x_0 \in X$.

Theorem 6.21 (Belobrov [28]). *Let X be an Efimov–Stechkin space,*[6] *and let the sequence of sets $M_n \subset X$ converge to a compact set M in the Hausdorff metric. Then for any $\varepsilon > 0$*

$$\mathrm{Z}_{M_n} \subset \mathscr{O}_\varepsilon(\mathrm{Z}_M)$$

for n starting from some $N \in \mathbb{N}$.

For extensions of Theorem 6.21, see Theorem 6.23 below.

Remark 6.8. Belobrov [28] also showed that *in an Efimov–Stechkin space the set of Chebyshev centres of a compact set is compact.*

Remark 6.9. The relation $\mathrm{Z}_M \subset \mathscr{O}_\varepsilon(\mathrm{Z}_{M_n})$, which is the reverse of the inclusion in Theorem 6.21, does not hold in general even in finite-dimensional Banach spaces. In this regard, consider the following example (see [28] and [103], Example 3.2). Let a_1 and a_2 be opposite points on a circle O in $\mathbb{R}^3$. By h_1 and h_2 we denote closed intervals of equal length having midpoints a_1 and a_2 and being perpendicular to the plane of the circle O. Let B be the convex hull of O and the intervals h_1 and h_2. Consider the sets $M_n = \{x_n', x_n''\}$, where x_n' and x_n'' are opposite points on the circle O but such that $x_n' \to a_1$ and $x_n'' \to a_2$ in the norm of the space with unit ball B, $x_n' \neq a_1$, $x_n'' \neq a_2$, $n = 1, 2, \dots$. Each set M_n, $n = 1, 2, \dots$, has a unique Chebyshev centre (the centre of the circle O). On the other hand, any point on the diameter of B parallel to the interval h_1 (or h_2) is a Chebyshev centre of M.

Definition 6.7. Given $\delta > 0$, a bounded set $M \subset X$, and a non-empty closed subset V of X, we define the *set of relative Chebyshev δ-centres* of M by

$$\mathrm{Z}_V^\delta(M) := \bigl\{y \in V \mid r(y, M) \leqslant r_V(M) + \delta\bigr\}. \tag{6.19}$$

Definition 6.8. Let $V \subset X$ be a non-empty closed convex set, and let $\mathscr{F}'(X)$ be a family of closed bounded subsets of X with non-empty sets of relative Chebyshev centres: $\mathrm{Z}_V(M) \neq \varnothing$ for any $M \in \mathscr{F}'$. Following [115] and [103], we say that a pair $(V, \mathscr{F}')$ has the *property* (P_1) if for any $M \in \mathscr{F}$ and any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\mathrm{Z}_V^\delta(M) \subset \mathrm{Z}_V(M) + \varepsilon B.$$

The property (P_1) was introduced by Mach [115] and further studied in [103].

Remark 6.10. Mach [115] found some pairs $(V, \mathscr{F})$ with the property (P_1). For example, a pair $(V, \mathscr{F})$ (where $\mathscr{F}$ is the class of closed bounded subsets of X) has the property (P_1) in the following cases:

1) X is a Banach space and $V \subset X$ is a finite-dimensional subspace;

[6] A Banach space X is an *Efimov–Stechkin space* if, for any $x_n \in S$ and $f \in S^*$ such that $f(x_n) \to 1$, the sequence (x_n) has a convergent subsequence (see [2]). Efimov–Stechkin spaces are also called reflexive Kadec–Klee spaces.

2) $X=\ell^1$ and V is a w^*-closed convex subset of X;

3) X is a uniformly convex space and V is a closed bounded convex subset of X;

4) X is a Lindenstrauss space (that is, a space predual to $L^1(\mu)$), V is an M-ideal in X, and $\mathscr{F}$ is the class of non-empty compact sets in X.

It was shown in Theorem 2.5 of [103] that if[7] $X\in(\mathrm{CLUR})$, V is a closed convex bounded subset of X, and $\mathscr{F}=\mathscr{K}(X)$ is the class of non-empty compact sets in X, then the pair $(V,\mathscr{F})$ has the property (P_1).

Remark 6.11. Note that the property (P_1) does not in general imply the continuity of the operator $\mathrm{Z}_V(\,\cdot\,)$. Finite-dimensional subspaces with metric projection which is not lower semicontinuous can be constructed in many (even finite-dimensional) spaces. Let V be such a subspace and let $\mathscr{F}_p$ be the class of singletons in X. According to Remark 6.10, the pair $(V,\mathscr{F}_p)$ has the property (P_1), but the (relative) Chebyshev-centre map

$$\{x\}\mapsto \mathrm{Z}_V(\{x\})=P_V(x)$$

is not continuous.

Mach (see [115], Theorem 5) proved the following result.

Theorem 6.22 (Mach). *Let X be a Banach space and let $(V,\mathscr{F})$ be a pair with the property* (P_1). *Then the map*

$$M\longmapsto \mathrm{Z}_V(M),\qquad M\in\mathscr{F},$$

is upper semicontinuous in the Hausdorff metric.

The next result (see [103], Theorem 2.3) extends Theorem 6.21.

Theorem 6.23. *The following conditions on a Banach space X are equivalent*:

a) *X is an Efimov–Stechkin space*;

b) *if V is a non-empty closed convex subset of X and $\mathscr{F}=\mathscr{K}(X)$ is the class of non-empty compact sets in X, then the pair $(V,\mathscr{F})$ has the property* (P_1).

The relative Chebyshev δ-centre map Z_V^δ (see (6.19)) can be shown to have stronger continuity properties without assumptions on the space structure or on the properties of the pair $(V,\mathscr{F})$. The following result holds [147].

Theorem 6.24 (Tsar'kov). *If $V\neq\varnothing$ is a closed convex subset of a Banach space X and $\delta>0$, then the Chebyshev $\delta r(M)$-centre map*

$$M\mapsto \mathrm{Z}_V^{\delta\, r(M)}(M)$$

is Lipschitz continuous on the class of non-empty bounded subsets of X.

Corollary 6.2. *Let X be a finite-dimensional Banach space, let $V\neq\varnothing$ be a closed convex bounded subset of X, and let $\delta>0$. Then the Chebyshev $\delta r(M)$-centre map*

$$M\mapsto \mathrm{Z}_V^{\delta\, r(M)}(M)$$

has a continuous Lipschitz selection.

---

[7] $X\in(\mathrm{CLUR})$ if the conditions $x,x_n\in S(X)$ and $\|x+x_n\|/2\to 1$ imply that (x_k) has a convergent subsequence.

The following property (P_2) (see [115], [103]), which is stronger than (P_1), gives a sufficient condition for the continuity of the Chebyshev-centre map.

Definition 6.9. Let $V \subset X$ be a non-empty closed convex set, and let $\mathscr{F}(X)$ be a family of closed bounded subsets of X with non-empty sets of relative Chebyshev centres: $\mathrm{Z}_V(M) \neq \varnothing$ for any $M \in \mathscr{F}$. A pair $(V, \mathscr{F})$ is said to have the *property* (P_2) if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\mathrm{Z}_V^{\delta}(M) \subset \mathrm{Z}_V(M) + \varepsilon B \tag{6.20}$$

for any $M \in \mathscr{F}$.

Theorem 6.25 (see [115], [103]). *Let V be a closed bounded subset of X, and let $\mathscr{F}$ be a family of non-empty closed bounded subsets of X. Suppose that the pair $(V, \mathscr{F})$ has the property* (P_2). *Then the Chebyshev-centre map*

$$M \mapsto \mathrm{Z}_V(M)$$

is uniformly continuous in the Hausdorff metric on the family $\mathscr{F}$.

The next two results (see [103]) extend Theorems 5.7 and 5.8.

Theorem 6.26. *The following conditions on a Banach space X are equivalent*:

a) *X is uniformly convex*;

b) *for any subspace $V \subset X$, any number $\alpha > 0$, and the family $\mathscr{F}$ of non-empty closed bounded sets $M \subset X$ with $r_V(M) \leqslant \alpha$, the pair $(V, \mathscr{F})$ has the property* (P_2), *and the relative Chebyshev-centre map $\mathrm{Z}_V(M)$ is single-valued for any $M \in \mathscr{F}$*;

c) *for any subspace $V \subset X$ the relative Chebyshev-centre map $M \mapsto \mathrm{Z}_V(M)$ is single-valued and uniformly continuous on the class of non-empty bounded subsets of the space X*;

d) *the Chebyshev-centre map $M \mapsto \mathrm{Z}(M)$ is single-valued and uniformly continuous on the class of non-empty bounded subsets of X.*

Theorem 6.27. *Let V be a subspace of a Banach space X. Then the following conditions are equivalent*:

a) *X is uniformly convex with respect to V (see* (5.3)*)*;

b) *for any $\alpha > 0$ the pair $(V, \mathscr{F})$ has the property* (P_2), *where $\mathscr{F}$ consists of all non-empty closed bounded subsets $M \subset X$ with $r_V(M) \leqslant \alpha$, and the relative Chebyshev-centre map $\mathrm{Z}_V(M)$ is single-valued for any $M \in \mathscr{F}$.*

The next theorem characterizes the uniformly convex Banach spaces in terms of the uniform approximative stability of the Chebyshev near-centre map on the class of sets of Chebyshev radius 1.

Theorem 6.28 (see [103]). *The following conditions on a Banach space X are equivalent*:

a) *X is uniformly convex*;

b) *for any $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\mathrm{Z}_V^{\delta}(M) \subset \mathrm{Z}_V(M) + \varepsilon B$$

for any closed subspace $V \subset X$ and any closed bounded set $M \subset X$ with $r_V(M) = 1$.

Let us briefly discuss the problem of convergence of Chebyshev nets. The following result holds.

Theorem 6.29 (Belobrov). *Suppose that a sequence $(M_n)_{n=1}^{\infty}$ of non-empty convex subsets of a Hilbert space H converges in the Hausdorff metric to a compact set $M \subset H$. Then there exists a sequence $(S_n^*)_{n=1}^{\infty}$ of best N-nets for the sets M_n which contains a subsequence converging in the Hausdorff metric to some best N-net for the set M.*

Knowing that in a Euclidean space there is no continuous selection from the set of Chebyshev 2-nets, many who study the properties of Chebyshev nets were satisfied with this state of affairs and did not consider the problem of the existence of continuous selections from the set of Chebyshev nets. Druzhinin [60] investigated this problem for various spaces and showed that any selection from the set of Chebyshev n-nets for $n \geqslant 2$ is discontinuous in any non-strictly convex Banach space. In addition, he proved the absence of a Lipschitz selection in an arbitrary Banach space of dimension $\geqslant 2$ that has a smooth exposed point on the unit sphere.

6.8. Lipschitz selection of the Chebyshev-centre map. As before, by $\mathscr{B}_H(X)$ we denote the class of non-empty bounded subsets of a space X, equipped with the Hausdorff semimetric. A map $\varphi\colon \mathscr{B}_H(X) \to X$ is called a selection of the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$ if $\varphi(M) \in \mathrm{Z}(M)$ for any $M \in \mathscr{B}_H(X)$. We first give the following simple result.

Theorem 6.30. *In $X = \ell^\infty(\Gamma)$ the Chebyshev radius $r(M)$ of any non-empty bounded set $M \subset X$ is equal to half the diameter of M. Moreover, the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$ admits a 1-Lipschitz selection.*

Proof. Let $M \in \mathscr{B}_H(X)$. For any $t \in \Gamma$ we set

$$m(t) := \frac{1}{2}\Bigl(\inf_{x\in M} x(t) + \sup_{x\in M} x(t)\Bigr).$$

It is easily checked that the point m is a Chebyshev centre of M, and

$$r(M) = \frac{1}{2}\operatorname{diam} M, \qquad \operatorname{diam} M = \sup_{t\in\Gamma}\Bigl(\sup_{x\in M} x(t) - \inf_{x\in M} x(t)\Bigr).$$

The map $\varphi(M) := m$ is the required 1-Lipschitz selection. $\square$

The problem of the existence of a Lipschitz selection of the Chebyshev-centre map has been investigated by Amir ([7], § 6.4), Amir, Mach and Saatkamp ([9], § 4), Pai and Nowroji [123], Druzhinin [59], and others.

We say that the Chebyshev-centre map Z admits a *Lipschitz selection* with constant θ if there exists a single-valued operator T which associates with any bounded set some (single) Chebyshev centre of this set and is such that

$$\|T(M) - T(N)\| \leqslant \theta \cdot h(M,N)$$

for some $\theta > 0$ and any non-empty bounded sets M and N, where, as before, $h(M,N)$ is the Hausdorff distance.

It was shown above (see Remark 6.6) that in a Hilbert space the (single-valued) Chebyshev-centre map is not Lipschitz continuous.

Recall that a point s of the unit sphere S of a normed space X is called a *smooth point* if the support hyperplane to S at this point is unique; a point s is an *exposed point* of the sphere S (or of the unit ball B) if there exists a hyperplane H supporting B at s such that $H \cap B = \{s\}$.

Theorem 6.31 (Druzhinin). *If the unit sphere of a Banach space X has a smooth exposed point, then the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$ does not have a Lipschitz selection.*

Theorem 6.32 (Druzhinin). *If X is a finite-dimensional Banach space, then the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$ admits a Lipschitz selection if and only if X is polyhedral.*

For the existence of a Lipschitz selection in polyhedral spaces, see also § 6.3.

The problem of existence of a Lipschitz selection of the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$ in $C(Q)$, where Q is a compact Hausdorff space, has not been solved in the general case (cf. Corollary 6.1). However, if the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$ is restricted to compact sets, then the existence of a Lipschitz selection of the map $\mathrm{Z}(\,\cdot\,)$ is fairly clear. Druzhinin [59] obtained the following partial answer in the problem of the existence of a Lipschitz selection of the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$.

Theorem 6.33. *Let Q be a compact Hausdorff space with finitely many limit points. Then in the space $C(Q)$ the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$ has a Lipschitz selection.*

6.9. Discontinuity of the Chebyshev-centre map. For infinite-dimensional L^1-spaces, the Chebyshev-centre map $\mathrm{Z}(\,\cdot\,)$ (see (1.1)) is not lower semicontinuous even on the class of two-point sets (as a consequence, $\mathrm{Z}(\,\cdot\,)$ does not have a continuous selection) [7].

It is well known that in a finite-dimensional space X the metric projection onto a finite-dimensional subspace need not be continuous. Let V be such a subspace. According to Remark 6.11, the (relative) Chebyshev-centre map

$$\{x\} \mapsto \mathrm{Z}_V(\{x\}) = P_V(x)$$

is not continuous; here we associate with a singleton $\{x\}$ the set of relative (with respect to V) Chebyshev centres. A similar example of a discontinuous single-valued relative Chebyshev-centre map $\mathrm{Z}_V\colon X \to V$ can be constructed in any space X containing a Chebyshev subspace V with discontinuous metric projection (and, in particular, in ℓ^1; see § 2 in [115]).

Note that in a finite-dimensional space the Chebyshev-centre map is always upper semicontinuous (see Theorem 6.21). One can easily construct an example of a three-dimensional space in which the Chebyshev-centre map is not lower semicontinuous (see [9], Example 2.5). Let B be the (symmetric) convex hull of the circle $\{e_1 \cos t + e_2 \sin t \mid 0 \leqslant t \leqslant 2\pi\}$ and the closed interval $\{se_1 + e_3 \mid -1 \leqslant s \leqslant 1\}$, and let $\|\cdot\|$ be the norm generated by the body B. In other words,

$$\|(x^{(1)}, x^{(2)}, x^{(3)})\| = \begin{cases} \sqrt{(x^{(2)})^2 + (|x^{(1)}| - |x^{(3)}|)^2} + |x^{(3)}|, & |x^{(1)}| \geqslant |x^{(3)}|, \\ |x^{(2)}| + |x^{(3)}|, & |x^{(1)}| \leqslant |x^{(3)}|. \end{cases}$$

We have $\mathrm{Z}(\{-e_1, e_1\}) = [-e_3, e_3]$ and $\mathrm{Z}(\{-e_1 - \eta e_2, e_1 + \eta e_2\}) = \{0\}$ for $\eta \neq 0$, so the map $M \mapsto \mathrm{Z}(M)$ is not lower semicontinuous, and thus not continuous (see also Remark 6.9).

7. Characterization of a Chebyshev centre, a decomposition theorem

We first mention one simple result for Hilbert spaces (see, for example, [8], Corollary 2.5, [53], Lemma 4, [122], Lemma 2, and [3], Lemma 0).

Theorem 7.1. *Let M be a bounded subset of a Hilbert space. Then a point z is a Chebyshev centre of M if and only if*

$$z \in \bigcap_{\varepsilon>0} \overline{\operatorname{conv}}\,\{y \in M \mid \|y-z\| \geqslant r(M) - \varepsilon\}.$$

Remark 7.1. A similar characterization also holds in an arbitrary two-dimensional strictly convex space.

We also mention two more characterizations of the relative Chebyshev centre in a Hilbert space [17].

Theorem 7.2 (Balaganskii). *Let X be a Hilbert space, let $Y \subset X$ be a non-empty closed convex set, let $K \subset X$ be a non-empty closed convex bounded set, and let $y \in Y$ and $r := r(y, K)$. Then*

1) $\{y\} = \mathrm{Z}_Y(K) \iff y \in P_Y(\overline{\operatorname{conv}}\,(K \setminus B(y,t)))\ \forall\, 0 < t < r$;

2) $\{y\} = \mathrm{Z}_Y(K) \iff y \in P_Y(\overline{\operatorname{conv}}\,(K \cap S(y,r)))$.

The following result [127] describes the Chebyshev centres and the Chebyshev radii in finite-dimensional Banach spaces.

Theorem 7.3 (Pichugov). *Let M be a closed convex subset of a finite-dimensional Banach space of dimension n, and let $r(M) = r$. Then a point y is a Chebyshev centre of M ($y \in \mathrm{Z}(M)$) if and only if there exists a natural number $N \leqslant n+1$ such that*:

a) *there are points x_i in M, $i = 1, \dots, N$, such that $\|x_i - y\| = r$;*

b) *there are functionals f_i in $(X_n)^*$, $i = 1, \dots, N$, such that*

$$(f_i, x_i - y) = \|x_i - y\|, \quad \|f_i\| = 1;$$

c) *there exist numbers α_i, $i = 1, \dots, n$, with $\sum_{i=1}^n \alpha_i = 1$ and $\alpha_i \geqslant 0$ such that $\sum_{i=1}^N \alpha_i f_i = 0$.*

Proof. The convex function $F(x) = \max\{\|x - z\| \mid z \in M\}$ has a minimum at a point y if and only if $0 \in \partial F(y)$. Now the required assertion follows from the finite-dimensional decomposition theorem (see, for example, [128]). $\square$

8. Chebyshev centres which are not farthest points

The paper [139] considers the problem of when a Chebyshev centre of a bounded subset of a Banach space can be a farthest point of the subset. Following [139], we say that a set M is *non-trivial* if $|M| \geqslant 2$.

We first recall the definition of a farthest point of a set.

Let $x \in X$ and $\varnothing \neq M \subset X$. A point $y_0 \in M$ is called a *farthest* point of M from the point x if

$$\|x - y_0\| = \sup\{\|x - y\| \mid y \in M\} = r(x, M);$$

that is, $\|x - y_0\| \geqslant \|x - y\|$ for any point $y \in M$. The set of all farthest points in M from a point x (the metric antiprojection, the max-projection) is denoted by $\mathrm{F}(x, M)$:

$$\mathrm{F}(x, M) = \{y \in M \mid \|x - y\| = r(x, M)\}.$$

By $\operatorname{Far} M$ we denote the set of all points in M on which the supremum in the definition of $r(x, M)$ is attained at some $x \in X$; that is,

$$\operatorname{Far} M = \bigcup_{x \in X} \mathrm{F}(x, M).$$

Baronti and Papini (see Proposition 6.4 above) showed that if $M \subset H$ is a non-empty bounded subset of a Hilbert space and z_M is a Chebyshev centre of M, then

$$r_M^2 + \|x - z_M\|^2 \leqslant r^2(x, M) \quad \text{for any } x \in X.$$

Hence, in a Hilbert space

$$r(x, M) > \|x - z_M\|,$$

which implies that $z_M \notin \operatorname{Far} M$, where z_M is the unique Chebyshev centre of a non-trivial bounded subset M of H.

Remark 8.1. Clearly, if X is not strictly convex, that is, the unit sphere of X contains a non-trivial interval M, then all points in M lie at distance 1 from the origin, and hence $M = \operatorname{Far} M$. In particular, the Chebyshev centre $(u + v)/2$ of M belongs to $\operatorname{Far} M$.

This observation led Sain to the following question: in a strictly convex Banach space can a Chebyshev centre of a bounded non-trivial set be a farthest point of this set from a point (cf. Remark 8.1)? The answer to this question turns out to be positive.

Definition 8.1. Following [139], we say that M is a CCF-*set* if there is a Chebyshev centre for M lying in $\operatorname{Far} M$.[8] Correspondingly, M is an NCCF-*set* if $M \notin (\mathrm{CCF})$.

A space X is said to lie in the class (CCF) if it contains a non-trivial CCF-set; X lies in (NCCF) if $X \notin (\mathrm{CCF})$, that is, any non-trivial subset of X is an NCCF-set.

From Remark 8.1 it follows that if $X \in (\mathrm{NCCF})$, then X is strictly convex. Theorem 8.2 gives the converse: each two-dimensional strictly convex Banach space lies in the class (NCCF). This ceases to be true if the dimension of the space is greater than two.

Sain, Kadets, Paul, and Ray [139] characterized the NCCF-spaces as follows. In the next theorem, $r_{t,z}$ denotes the Chebyshev radius of the set $A_{t,z} := B \cap B(z, t)$.

[8]'CCF' is an acronym for 'a Chebyshev centre lies in $\operatorname{Far} M$'.

Theorem 8.1. *Let X be a Banach space. Then the following three conditions are equivalent*:

a) $X \in$ (NCCF);

b) *the inequality* $r_{t,z} < t$ *holds for any* $z \in S$ *and* $t \in (0,1]$;

c) *for any* $\varepsilon \in (0,1]$ *there exists a* $t_0 \in (0,\varepsilon)$ *such that* $r_{t,z} < t$ *for all* $z \in S$ *and* $t \in (0,t_0]$.

The next result [139] shows that in the two-dimensional setting the class (NCCF) coincides with the class of strictly convex Banach spaces. This is not true for dimension $\geqslant 3$.

Theorem 8.2. *A two-dimensional Banach space X is strictly convex if and only if any non-trivial bounded set $M \subset X$ which contains some Chebyshev centre of M is an* NCCF*-set.*

Example 8.1 (see [139]). On the space $X = (\mathbb{R}^n, \|\cdot\|)$, $n \geqslant 3$, consider the norm

$$\|(x^{(1)}, x^{(2)}, \dots, x^{(n)})\| = \sum_{i=1}^{n} |x^{(i)}| + \frac{1}{2}\sqrt{\sum_{i=1}^{n} |x^{(i)}|^2}$$

(such a norm is strictly convex). By Theorem 5.1, a Chebyshev centre of any bounded set is unique. Let $\theta := (0,0,\dots,0)$ and let $\{e_1,\dots,e_n\}$ be the standard basis for $\mathbb{R}^n$. Consider the set $M := \{\theta, e_1, e_2, \dots, e_n\}$, and define $z = (1,1,\dots,1) \in \mathbb{R}^n$. Then $\mathrm{Z}(M) = \{\theta\}$, and θ is a farthest point in M. Using this example, it can be shown (see [139]) that there exists a finite-dimensional strictly convex normed space X containing a non-trivial compact *convex* CCF-subset.

Definition 8.2. A set M is said to be *centreable* if $r(M) = (1/2)\operatorname{diam} M$.

A similar example with a *centreable* set can be constructed in any infinite-dimensional strictly convex space (see [139]). Theorem 8.3 below implies that such an example cannot be constructed in the finite-dimensional setting.

Theorem 8.3 (see [139]). *Let X be a uniformly convex Banach space, and let M be a non-trivial bounded centreable subset of X containing a Chebyshev centre z_M of it. Then M is an* NCCF*-set.*

Remark 8.2. The uniform convexity condition in Theorem 8.3 cannot be replaced by the strict convexity condition.

Definition 8.3 (see [139]). Let $x \in X$. A sequence $(a_n) \subset M$ is said to be *maximizing* (*minimizing*) if

$$\|x - a_n\| \to r(x, M) \quad (\text{respectively, } \|x - a_n\| \to \rho(x, M)).$$

A point $x \in X$ is a point of MAX-*approximative compactness* (a point of MIN-*approximative compactness*) of a set $\varnothing \neq M \subset X$ if any maximizing (minimizing) sequence in M contains a subsequence converging to a point in M. A set M is said to be MAX-*approximatively compact* (MIN-*approximatively compact*) if any point $x \in X$ is a point of MAX-*approximative compactness* (MIN-*approximative compactness*) for the set M. In these terms, MIN-approximative compactness is the classical approximative compactness (see [2]).

Theorem 8.4 (see [139]). *Let X be a strictly convex Banach space, and let $M \subset X$ be a non-trivial bounded centreable* MAX*-approximatively compact set containing a Chebyshev centre z_M of it. Then M is an* NCCF*-set.*

Remark 8.3. The MAX-approximative compactness assumption in Theorem 8.4 cannot be dropped.

The following characterization of strictly convex spaces follows from Theorem 8.4 and the fact that a closed interval is a centreable MAX-approximatively compact set.

Theorem 8.5 (see [139]). *A Banach space X is strictly convex if and only if any non-trivial bounded centreable* MAX*-approximatively compact subset M of X containing some Chebyshev centre of it is an* NCCF*-set.*

A space L_p with $p \neq 2$ of dimension > 2 lies in the class (CCF). The spaces L_1 and L_∞ are not strictly convex, and hence by the above they lie in (CCF).

The next result is auxiliary. Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, and $e_3 = (0,0,1)$.

Proposition 8.1 (see [139]). *For the set $A_0 = \{e_1, e_2, e_3\} \subset \ell_3^p$, the set of its Chebyshev centres is the point $x_p = (s_p, s_p, s_p)$, where $s_p = 1/(1 + 2^{1/(p-1)})$.*

We set $A_p = \{e_1, e_2, e_3, x_p\} \subset \ell_3^p$ (where x_p is the point in the statement of Proposition 8.1).

Proposition 8.2 (see [139]). *For $p \in (1,2) \cup (2,\infty)$ the set A_p is a* CCF*-set. As a consequence, $\ell_3^p \in$* (CCF).

Theorem 8.6 (see [139]). *Let (Ω, Σ, μ) be a finite or σ-finite measure space containing a disjoint triple $\{\Delta_i\}_{i=1}^3 \subset \Sigma$ of subsets of finite positive measure. Then*

$$L_p = L_p(\Omega, \Sigma, \mu) \in (\mathrm{CCF}) \quad \text{for any } p \in (1,2) \cup (2,\infty).$$

For further results in this direction, see [138], [121], and [23].

9. Smooth and continuous selections of the Chebyshev near-centre map

Let $V \subset X$ be a non-empty convex set, and let $\varnothing \neq M \subset X$, be a bounded set. Consider the following *sets of relative Chebyshev near-centres*:

$$\mathrm{Z}_V^\varepsilon(M) := \{y \in V \mid r(y, M) \leqslant r(M) + \varepsilon\},$$
$$\mathring{\mathrm{Z}}_V^\varepsilon(M) := \{y \in V \mid r(y, M) < r(M) + \varepsilon\}$$

(the quantities $r(y, M)$ and $r(M)$ were defined in § 1). In the proof of the next result, $\mathrm{Z}^\varepsilon(M) := \mathrm{Z}_X^\varepsilon(M)$ and $\mathring{\mathrm{Z}}^\varepsilon(M) := \mathring{\mathrm{Z}}_X^\varepsilon(M)$ $(M \subset X)$.

Theorem 9.1. *The map $M \mapsto \mathrm{Z}^\varepsilon(M)$ admits a continuous selection for any $\varepsilon > 0$.*

Proof. Let $\delta(y) := \varepsilon + r(M) - r(y, M)$. Clearly, $\delta(y) > 0$ for $y \in \mathring{\mathrm{Z}}^\varepsilon(M)$. We assert that the map $M \mapsto \mathring{\mathrm{Z}}^\varepsilon(M)$ is lower semicontinuous. Indeed, if $M_n \to M$ and $y \in \mathring{\mathrm{Z}}^\varepsilon(M)$, then there exists an $N \in \mathbb{N}$ such that $y \in \mathring{\mathrm{Z}}^\varepsilon(M_n)$ for any $n \geqslant N$. This shows that the map $M \mapsto \mathring{\mathrm{Z}}^\varepsilon(M)$ is lower semicontinuous. The sets $\mathring{\mathrm{Z}}^\varepsilon(M)$ and $\mathrm{Z}^\varepsilon(M)$ are convex, and hence by Michael's classical selection theorem (see, for example, § 2.9 of [128]) the map $M \mapsto \mathring{\mathrm{Z}}^\varepsilon(M)$ has a continuous selection, being lower semicontinuous on the metric space of bounded sets with the Hausdorff metric. Since $\mathring{\mathrm{Z}}^\varepsilon(M)$ is a body, the map $M \mapsto \mathrm{Z}^\varepsilon(M)$ also has a continuous selection. $\square$

Remark 9.1. If V is a non-empty closed convex set, then the map $M \mapsto \mathrm{Z}_V^{\varepsilon}(M)$ is lower semicontinuous and admits a continuous selection for any $\varepsilon > 0$ (see also Theorem 6.24 above).

The next result shows that the set of Chebyshev near-centres has a Lipschitz selection in the space $C(Q)$.

Theorem 9.2. *Let $X = C(Q)$, where Q is a compact metric space. Then there exists a 2-Lipschitz map $\varphi\colon \mathscr{B}_H(X) \to X$ such that*

$$M \subset B(\varphi(M), \operatorname{diam} M) \quad \textit{for any } M \in \mathscr{B}_H(X).$$

Proof. It is known that $C(Q)$ can be isometrically embedded in the space $\ell^\infty(Q)$. By Theorem 6.30, there is a 1-Lipschitz selection ψ of the Chebyshev-centre map in the space $Y = \ell^\infty(Q)$. According to the Lindenstrauss–Kalton theorem (see [92], Theorem 3.5), there is a 2-Lipschitz retraction π from $\ell^\infty(Q)$ onto $C(Q)$, where Q is a compact metric space. It is easily seen that $\varphi = \pi \circ \psi$ is the required map. □

Tsar'kov [146] investigated the existence of smooth selections of the Chebyshev near-centre map. Let X be a Banach space. For an arbitrary body $M \subset X$ consider the quantity

$$q(M) := \sup\{a \geqslant 0 \mid B(x,a) \subset Mr\}.$$

Given $\tau > 1$, we denote by $\mathfrak{N}(X,\tau)$ the metric space of all closed convex bodies $M \subset X$ such that

$$\operatorname{diam} M = 1 \quad \text{and} \quad \frac{r(M)}{q(M)} \leqslant \tau.$$

The space $\mathfrak{N}(X,\tau) \subset \mathscr{H}(X)$ is equipped with the standard Hausdorff metric.

For metric spaces (X_1,ρ_1) and (X_2,ρ_2) and a map $\varphi\colon X_1 \to X_2$, the modulus of continuity is defined by

$$\omega(\varphi,\delta) := \sup\bigl\{\rho_2(\varphi(x),\varphi(y)) \mid \rho_1(x,y) \leqslant \delta\bigr\}, \qquad \delta \geqslant 0.$$

Given $\gamma \in [0,1]$, we let UC^γ denote the class of all maps $\varphi\colon \mathfrak{N}(X,\tau) \to X$ such that $\omega(\varphi,\delta) = o(\delta^\gamma)$ for $\delta \to 0+$.

Theorems 9.3–9.5 are due to Tsar'kov [146]. The following result shows that better smoothness for a selection of the Chebyshev near-centre map cannot be achieved if Chebyshev near-centres are considered near the corresponding sets.

Theorem 9.3. *Let X be an infinite-dimensional Banach space. Then for any $\tau > 1$ and $\varepsilon \in (1/\tau, 1)$ the class $\mathrm{UC}^{1/2}$ does not contain a map $\varphi\colon \mathfrak{N}(X,\tau) \to X$ such that*

$$\rho(\varphi(M), M) < r(M)\,(1-\varepsilon) \qquad \forall M \in \mathfrak{N}(X,\tau).$$

We recall (see [63], p. 51) that a Banach space X has *type p* if there exists a constant C such that for all $x_1,\dots,x_n \in X$

$$\frac{1}{2^n}\sum_{\varepsilon_i=\pm1}\biggl\|\sum_{i=1}^n \varepsilon_i x_i\biggr\| \leqslant C\biggl(\sum_{i=1}^n \|x_i\|^p\biggr)^{1/p}.$$

Theorem 9.4. *Let X be an infinite-dimensional Banach space of type $p>1$. Then for any $\tau>1$ the class* $\mathrm{UC}^{1/2}$ *does not contain a map $\varphi\colon \mathfrak{N}(X,\tau)\to X$ such that*

$$\sup\bigl\{\rho(\varphi(M),M)\mid M\in\mathfrak{N}(X,\tau)\bigr\}<\infty.$$

The next result establishes a link between smoothness of selections of the Chebyshev near-centre map and smoothness of operators of non-linear projection onto a subspace.

Theorem 9.5. *Let X be an infinite-dimensional Banach space. Suppose that there exist numbers $\tau>1$ and $\gamma\geqslant 1/2$ and a map $\varphi\in\mathrm{UC}^{\gamma}$ such that*

$$\sup\bigl\{\rho(\varphi(M),M)\mid M\in\mathfrak{N}(X,\tau)\bigr\}<\infty.$$

Then there exists a subspace $L\subset X$ such that no uniformly continuous projection onto L exists in any uniform neighbourhood of L.

The *Steiner centre* of a compact convex set $M\subset\mathbb{R}^n$ is defined as

$$\mathrm{s}(M)=\frac{1}{v_n}\int_{S(0,1)} s(p,M)\,dp, \tag{9.1}$$

where v_n denotes the volume of the unit ball in $\mathbb{R}^n$ (see § 2.1 of [128])), $s(p,M):=\sup\{\langle p,x\rangle\mid x\in M\}$, and $p\in X^*$ is a support function of the set M.

The Steiner-centre map $\mathrm{s}(\,\cdot\,)$ is Lipschitz continuous in the standard Hausdorff metric as a function of compact convex sets in $\mathbb{R}^n$, namely:

$$\|\mathrm{s}(A_1)-\mathrm{s}(A_2)\|\leqslant L_n h(A_1,A_2), \tag{9.2}$$

for any compact convex sets A_1 and A_2 in $\mathbb{R}^n$, where

$$L_n:=\frac{2}{\sqrt{\pi}}\,\frac{\Gamma(n/2+1)}{\Gamma((n+1)/2)}\,,$$

and the Lipschitz constant L_n in (9.2) is best possible (see § 2.1 of [128]). We note that L_n is bounded from above by $\sqrt{n}$ and behaves roughly like $\sqrt{n}$ with increasing n. Hence, as the dimension n of the space increases, the Steiner-centre map cannot be extended as a Lipschitz selection to any infinite-dimensional space (or even to a Hilbert space). We also note that on the class of compact convex subsets of $\mathbb{R}^n$ one can consider different metrics (not equivalent to the Hausdorff distance), in which the Steiner-centre map (as a function of compact convex sets) provides a Lipschitz selection with Lipschitz constant 1 (see Theorem 2.1.3 in [128]).

10. Algorithms and applied problems connected with Chebyshev centres

The problem of approximation of geometrically complicated shapes M by more convenient sets (balls of radius $r(M)$ in the Chebyshev centre problem or by sets of balls of fixed covering radius in the best n-net problem) is a classical problem in computational geometry [129], and it is interesting both from the theoretical point of view and in relation to multiple applications to problems of cellular [168] and space communication [75], logistics [43], construction of reachability sets for control systems [79], and also optimization problems (see [88]) and approximation of optimal packings [151]. For applications of Chebyshev centres to problems of optimal recovery of linear operators, see [144], [13], [48], [56]. As a recent application of the Chebyshev centre machinery, we mention the construction of space-filling designs for computer experiments based on an extension of Lloyd's clustering algorithm [131]. A Chebyshev centre can also be naturally regarded as a centre of an information set in control problems with uncertain disturbances and errors in the state information.

The problem of constructing an approximation of an information set (which characterizes the uncertainty in the evaluation of the state vector from observations) as a system of linear inequalities was considered in [148]. This choice of the class of approximating regions and the method of approximation is superior to the vertex representation in terms of memory space and is better then approximation by ellipsoids in terms of accuracy. To be able to control an object, one has to know a point estimate rather than a guaranteed estimate, which leads to the problem of finding a point estimate from an information set. If a guaranteed state estimate (that is, an information set) is known, then as a point estimate one can choose a point in this information set. For a minimax filtration problem, we again choose one point (a Chebyshev centre of the information set) from the whole set of points.

The problem of constructing a Chebyshev centre of a given finite point set $\{a_1,\dots,a_N\}$ in a finite-dimensional space is a classical optimization problem, for which many algorithms are available (for a historical survey see [62], and for more references see [165], and also [41], [140], [82], [150], and [93]). The earliest such algorithms had been known long before the maturation of the theory of Chebyshev centres.

Exact algorithms for constructing Chebyshev centres of finite sets are based on various linear programming methods (see, for instance, [41], [74], [161]); approximate algorithms can deliver $(1+\varepsilon_k)$-approximations of a Chebyshev centre as $\varepsilon_k \to 0$. Exact algorithms depend exponentially on the dimension n of the space, and hence can be used only for small dimensions (say, for $n \lesssim 20$). Approximate algorithms can work for large n to produce an $(1+\varepsilon)$-approximation of a Chebyshev centre with $O(Nn/\varepsilon)$ arithmetic operations (see [165], [51]). An efficient algorithm for finding a conditional Chebyshev centre in the space ℓ^1 was constructed in [45].

The problem of finding a Chebyshev centre for a set M is an NP-hard problem, except for a few simple cases (when M is finite and the metric is Euclidean, when M is polyhedral in ℓ^∞_n, or when $M \subset \mathbb{C}$ is given by two ellipsoids [24]; see [164]).

The problem of approximating objects by subsets of a finite point set was considered in [108], [109] in the Euclidean plane setting, in [149] for a sphere in a Euclidean space, and in [94], [95] for a plane with inhomogeneous metric.[9]

An algorithm constructing best nets of cardinality N in general metric spaces is given in [93]. In the Euclidean plane setting, the algorithm for finding best N-nets

[9]The inhomogeneous metric is defined as follows: $\rho_f(a,b) := \int_{\Gamma\in\Gamma(a,b)} (f(x,y))^{-1}\,d\Gamma$, where $0 < f(x,y) < c$ is a piecewise-continuous function, and $\Gamma(a,b)$ is the class of continuous paths between points a and b. If $f \equiv 1$, then the metric $\rho_f(\,\cdot\,,\,\cdot\,)$ is Euclidean. Such metrics appear in transport and infrastructure logistics problems, for example, in optimal placement of objects within a fixed number of logistics centres.

involves a partition of a given set into subsets with the subsequent step of finding their Chebyshev centres (see [93]); for small N this method is optimal from the viewpoint of computational operations. For $n = 2$ and $n = 3$, numerical algorithms for constructing best n-nets were developed in [107], [108].

Little is known about the construction of the set of Chebyshev centres in the non-Euclidean case (although, of course, general optimization arguments can also be applied to this setting).

We give an algorithm of Botkin and Turova-Botkina [41] for finding the Chebyshev centre of a finite set $A := \{a_1, \dots, a_m\} \subset \mathbb{R}^n$. We set

$$E(x) := \Bigl\{a \in A \Bigm| \|x - a\| = \max_{w \in A} \|x - w\|\Bigr\}.$$

Let $\widehat{x}$ be the (unique) Chebyshev centre of $Z(A)$ in $\mathbb{R}^n$.

Step 0. Choose an initial point $x_0 \in \operatorname{conv} A$ and set $k = 0$.

Step 1. Find $E(x_k)$. If $E(x_k) = A$, then stop (with the answer $\widehat{x} = x_k$), else go to Step 2.

Step 2. Let y_k be the nearest point in $\operatorname{conv} E(x_k)$ to x_k. If $y_k = x_k$, then stop (with the answer $\widehat{x} = x_k$), else go to Step 3.

Step 3. Evaluate

$$\alpha_k := \min_{i \in I_k^-} \frac{\|a_i - x_k\|^2 - d^2_{\max}(x_k)}{2(y_k - x_k, a_i - y_k)}\,,$$

where $I_k^- := \{i \mid a_i \in A \setminus E(x_k),\ (y_k - x_k, a_i - y_k) < 0\}$, and $d_{\max}(x) := \max_{a \in A} \|x - a\|$ (we formally set $\alpha_k = +\infty$ if $I_k^- = \varnothing$). If $\alpha_k \geqslant 1$, then stop (with the answer $\widehat{x} = y_k$), else go to Step 4.

Step 4. Let $x_{k+1} := x_k + \alpha_k(y_k - x_k)$, $k := k + 1$. Go to Step 1.

This algorithm includes the non-trivial operation of finding a nearest point in a polyhedron, and for this the recursive algorithm in [140] can be applied. Note (see [41]) that the point y_k in Step 2 is the Chebyshev centre of the set $E(x_k)$. A similar algorithm for finding the relative Chebyshev centre in $\mathbb{R}^n$ can be found in [82].

One can also mention several algorithms for finding a Chebyshev point (see Definition 1.4) for a system of sets or hyperplanes. In finite-dimensional spaces such algorithms were designed by Zukhovitskij [169]; the existence of a Chebyshev point in an arbitrary Banach space for a finite number of sets was established by Garkavi.

Chebyshev points proved useful in constructing iteration processes for solving convex programming problems, for solving inconsistent systems of linear equations, in the study of dynamical systems, and also in optimal control problems and in problems of optimal quality and reliability provision for complex systems (see Chap. 5 of [169], [102], [42], [32]).

Chapter II

The Jung constant

11. Definition of the Jung constant

The *Jung constant* of a normed linear space X is defined by

$$J(X)=\sup\biggl\{\frac{r(M)}{\operatorname{diam} M}\biggm| M\subset X,\ \operatorname{diam} M<+\infty\biggr\},$$

where, as above, $r(M)=\inf\{a\geqslant 0\mid M\subset B(x,a)\}$ is the *Chebyshev radius* of a non-empty bounded set $M\subset X$ and $\operatorname{diam} M$ is the diameter of M.

In other words, the Jung constant of a Banach space is the radius of a smallest ball that can cover any set of diameter 1. The inverse of the Jung constant is sometimes called the normal structure coefficient of a space. The term 'Jung constant' was introduced by Grünbaum in 1959.

Jung constants play an important role in the geometry of Banach spaces. One should also mention the relation between Jung constants and Jackson inequalities, in which the best approximation of a function by finite-dimensional subspaces is estimated in terms of its modulus of continuity.

We first note that for any normed space X

$$\frac{1}{2}\leqslant J(X)\leqslant 1$$

(both bounds can be attained).

The first Jung constant was found in 1901 for a finite-dimensional Euclidean space by Jung himself [90]. Bohnenblust [37] (see also [110], Chap. 2, § 11, and [6]) showed that $J(X_n)\leqslant n/(n+1)$ for any n-dimensional Banach space X_n and proved that this inequality is sharp for any n. Leichtweiß (see [110], Chap. 2, § 11) proved that in an n-dimensional space the equality $r(M)/\operatorname{diam} M=n/(n+1)$ holds if M is the vertex set of an n-dimensional simplex Σ and B (the unit ball) is the difference body $\Sigma+(-1)\Sigma$ of Σ.

It is easily shown that $J(\ell^\infty_n)=1/2$. For an infinite-dimensional Hilbert space H, Routledge [135] (and later V. I. Berdyshev [34]) established that $J(H)=1/\sqrt{2}$. Further substantial advances in the study of Jung constants were made by Dol'nikov [57], V. I. Ivanov and Pichugov [127], [83], [86], Manokhin [116], and Ball [19]. The Jung constant for L^p, $1\leqslant p<\infty$, was found by Pichugov [127], [87] and independently by Ball [19]:

$$J(L^p)=J(\ell^p)=2^{-1/r},\quad \text{where}\ \ r:=\max\biggl\{p,\frac{p}{p-1}\biggr\},\ \ 1\leqslant p<\infty$$

(for $p=1$ it is assumed that $p/(p-1)=\infty$). For $0<p<1$, Pichugov (see [127], § 3) showed that $J(L^p)=1$. For rearrangement-invariant spaces, some estimates of the Jung constant were given by Semenov and Franchetti [141]. For example, they established that if X is a rearrangement-invariant space and $X\neq L^\infty$, then $J(X)\geqslant 1/\sqrt{2}$. (Recall that a Banach space X of Lebesgue-measurable functions on $[0,1]$ is said to be rearrangement-invariant or symmetric if 1) $|x(t)|\leqslant|y(t)|$ for all $t \in [0,1]$ and $y \in X$ implies that $x \in X$ and $\|x\| \leqslant \|y\|$; 2) if $x(t)$ and $y(t)$ are equimeasurable and if $y \in X$ then $x \in X$ and $\|x\| = \|y\|$.) Sharp estimates of the Jung constant for Banach lattices were found in [12]. Some estimates of the Jung constant in Orlicz spaces can be found in [134].

12. The measure of non-convexity of a space and the Jung constant

Following Gulevich [78], we define

$$G(X) = \sup\{\eta(A) \mid A \subset X,\ \operatorname{diam} A = 1\},$$

where $\eta(A) = \eta_X(A)$ is the *measure of non-convexity* (or the EL-measure of non-convexity[10]) of a bounded subset A of X (see [61]), which is defined by

$$\eta(A) = \sup_{x \in \overline{\operatorname{conv}} A} \inf_{y \in A} \|x - y\|.$$

It is clear that

$$\eta(A) = \inf\biggl\{r > 0 \biggm| \operatorname{conv} A \subset \bigcup_{a \in A} B(a, r)\biggr\} = h(A, \operatorname{conv} A),$$

where $h(A, C)$ is the Hausdorff distance between sets A and C.

The measure of non-convexity $\eta(A)$ is defined for any bounded set A.

The following simple properties of the measure of non-convexity $\eta(A)$ are immediate (see, for example, [118]):

(i) $\eta(A) = 0$ if and only if the set $\overline{A}$ is convex;
(ii) $\eta(\alpha A) = |\alpha| \eta(A)$, $\alpha \in \mathbb{R}$;
(iii) $\eta(A + C) \leqslant \eta(A) + \eta(C)$;
(iv) $|\eta(A) - \eta(C)| \leqslant \eta(A + (-C))$;
(v) $\eta(\overline{A}) = \eta(A)$;
(vi) $\eta(A) \leqslant \operatorname{diam} A$, where $\operatorname{diam} A$ is the diameter of A;
(vii) $|\eta(A) - \eta(C)| \leqslant 2h(A, C)$.

The measure of non-convexity is non-monotone in the sense that the inclusion $A \subset C$ does not imply that $\eta(A) \leqslant \eta(C)$.

The following results (Theorems 12.1–12.4) are due to Gulevich [78].

Theorem 12.1. *For any Banach space X,*

$$G(X) = J(X).$$

Theorem 12.2. *For any Banach space X,*

$$J(X) = \sup\{J(L) \mid L \text{ is a finite-dimensional subspace of } X\}.$$

Theorem 12.3. *For any Banach space X,*

$$J(X) = J(X^{**}).$$

[10] EL stands for Eisenfeld and Lakshmikantham, who in [61] introduced the measure of non-convexity $\eta(A)$ by analogy with the Kuratowski measure of non-compactness.

In particular, $J(X)=\sup\{r(A)\mid A\subset X \text{ is a finite set, } \operatorname{diam}A=1\}$ (see [78]). For the definition of the modulus of convexity $\delta(\varepsilon)$ of a space in the next theorem and below, see, for example, [128], § 2.7.

Theorem 12.4. *Let X be a Banach space and let $\delta(\varepsilon)$ be the modulus of convexity of X. Then $J(X)\leqslant t_0$, where t_0 is the root of the equation $t+2\delta(2t-1)=1$ on the interval* $[1/2,1]$.

Proof of Theorem 12.4. Consider an arbitrary set $A\subset X$ with $\operatorname{diam}A=1$. We can assume without loss of generality that $0\in A$. The function $t\mapsto t+2\delta(2t-1)$ is strictly increasing and continuous on the interval $[1/2,1]$, and hence the equation $t+2\delta(2t-1)=1$ has a unique root $t_0\in[1/2,1]$. Assume on the contrary that $\eta(A)>t_0$. Then there exist a number t_1 ($t_0<t_1<\eta(A)$) and a point $b\in\operatorname{conv}A$ such that $A\cap B(b,t_1)=\varnothing$. Therefore, for any $a\in A$

$$\biggl\|a-\frac{b}{\|b\|}\biggr\|\geqslant\biggl|\,\biggl\|b-\frac{b}{\|b\|}\biggr\|-\|a-b\|\,\biggr|=\bigl|\|a-b\|+\|b\|-1\bigr|\geqslant 2t_1-1,$$

because $\|a-b\|\geqslant t_1$ and $1\geqslant\|b\|\geqslant t_1$. Hence, $\bigl\|a+b/\|b\|\bigr\|\leqslant 2(1-\delta(2t_1-1))$. Let $\varphi\in X^*$ be such that $\varphi(b)=\|b\|$ and $\|\varphi\|=1$. For any $a\in A$ we have

$$\begin{aligned}\varphi(a)=\varphi\biggl(a+\frac{b}{\|b\|}\biggr)-\varphi\biggl(\frac{b}{\|b\|}\biggr)&\leqslant\biggl\|a+\frac{b}{\|b\|}\biggr\|-1\\&\leqslant 1-2\delta(2t_1-1)\leqslant 1-2\delta(2t_0-1)=t_0,\end{aligned}$$

which gives $\varphi(b)\leqslant t_0$. But $\varphi(b)=\|b\|>t_0$. This contradiction shows that $\eta(A)\leqslant t_0$. By Theorem 12.1, $J(X)\leqslant t_0$. $\square$

Definition 12.1. Following [118], we say that the measure of non-convexity $\eta_X(\,\cdot\,)$ has the *Cantor property* if a nested sequence $(A_n)_{n=1}^\infty$ of non-empty closed bounded (not necessarily convex) subsets of X has non-empty intersection provided that $\eta(A_n)\to 0$. The measure of non-convexity $\eta_X(\,\cdot\,)$ has the *Cantor property in a set* $M\subset X$ if any nested sequence $(A_n)_{n=1}^\infty$ of non-empty closed bounded subsets of M has non-empty intersection provided that $\eta(A_n)\to 0$.

Remark 12.1. It is well known that any nested sequence of non-empty closed convex bounded subsets of a reflexive Banach space has non-empty intersection. Gulevich [78] (see Theorems 12.5 and 12.6 below) extended this result to the case of not necessarily convex sets. On the other hand, the following simple example shows that $\eta_X(\,\cdot\,)$ does not have the Cantor property in any non-reflexive space X. Indeed, in any non-reflexive space X one can easily construct a nested sequence of non-empty closed convex bounded subsets with empty intersection: it suffices to take a functional $f\in X^*$ which does not attain its norm on the unit ball B and to consider the sets $\{x\in B\mid f(x)\geqslant 1-1/n\}$, $n\in\mathbb{N}$.

Theorem 12.5 (Gulevich). *Let $(A_n)_{n=1}^\infty$ be a nested sequence of non-empty closed bounded* (*not necessarily convex*) *subsets of a reflexive Banach space. If $\eta(A_n)\to 0$ as $n\to\infty$, then $\bigcap_{n=1}^\infty A_n$ is a non-empty closed convex set.*

Theorem 12.6 (Gulevich). *For a Banach space X the following conditions are equivalent*:

a) *X is reflexive*;

b) *the measure of non-convexity $\eta_X(\,\cdot\,)$ has the Cantor property*; *that is*, *any nested sequence $(A_n)_{n=1}^\infty$ of non-empty closed bounded subsets of X has non-empty intersection provided that $\eta(A_n)\to 0$.*

Theorem 12.7 (Marrero [118]). *Let X be a Banach space*, *F a non-empty closed subset of X*, *and $x_0\in X\setminus F$. Let $d=\rho(x_0,F)$ and*

$$F_n[x_0,F]:=F\cap\biggl[x_0+\biggl(d+\frac{1}{n}\biggr)B\biggr],\qquad n\in\mathbb{N}.$$

Then the following conditions are equivalent:

a) *X is reflexive* (*respectively*, *strictly convex*);

b) *if $F\subset X$ is a non-empty closed set and $\eta(F_n[x_0,F])\to 0$ for any $x_0\notin F$*, *then F is a set of existence* (*a set of uniqueness*).

The next theorem characterizes the weakly compact subsets of a Banach space in terms of the Cantor property of the measure of non-convexity of the space.

Theorem 12.8 (Marrero [119]). *Let X be a Banach space*, *let $\eta_X(\,\cdot\,)$ be the measure of non-convexity*, *and let M be a non-empty weakly closed bounded subset of X. Then the following conditions are equivalent*:

a) *M is weakly compact*;

b) *the measure of non-convexity $\eta_X(\,\cdot\,)$ has the Cantor property in the set $\overline{\operatorname{conv}}M$.*

13. The Jung constant and fixed points of condensing and non-expansive maps

Recall that the *self Chebyshev radius* of a bounded set $M\ne\varnothing$ is defined by (see § 1)

$$r_M(M):=\inf_{y\in M}\sup_{x\in M}\|x-y\|.$$

A classical result due to Klee and Garkavi [71] (see Theorem 1 in [99]) asserts that in a normed linear space X the equality $r_M(M)=r(M)$ holds for each convex bounded subset $M\subset X$ if and only if X either is a Euclidean space or $\dim X=2$.

This remark justifies the consideration of (in addition to the Jung constant $J(X)$ and the relative Jung constant $J_s(X)$ of a space X; see § 17 below) the *relative Jung constant for convex sets*[11]

$$J_{\rm cv}(X):=\sup\biggl\{\frac{r_M(M)}{\operatorname{diam}M}\biggm| M\subset X\text{ is convex},\ 0<\operatorname{diam}M<\infty\biggr\}.$$

It is clear that $J_{\rm cv}(X)\geqslant J(X)$.

The corresponding definitions of a Banach space with normal structure and the coefficient of normal structure $\mathrm{N}(X)$ of a space are closely related to the Jung constant of a space.

[11] Here 'cv' is an acronym for 'convex'.

Let X be a normed linear space and let $\varnothing \neq M \subset X$ be a bounded set. We define

$$r^{\mathrm{cv}}(M) := r_{\operatorname{conv} M} M := \inf_{y \in \operatorname{conv} M} \sup_{x \in M} \|x-y\| = \inf_{y \in \operatorname{conv} M} r(y, M)$$

and

$$\mathrm{Z}^{\mathrm{cv}}(M) := \{y \in \operatorname{conv} M \mid r(y,M) = r_{\operatorname{conv} M} M\},$$

where, as before, $r(x,M) := \inf\{r \geqslant 0 \mid M \subset B(x,r)\} = \sup_{y \in M} \|x-y\|$. If M is convex, then we write $r_M(M) := r^{\mathrm{cv}}(M)$ and

$$\mathrm{Z}^{\mathrm{cv}}(M) = \mathrm{Z}_M(M) := \mathrm{Z}_M^{\mathrm{sc}}(M) := \{x \in M \mid r(x,M) = r_M(M)\}.$$

We mention several results on the constant $J_{\mathrm{cv}}(X)$. It is known (see § 2 of [6]) that if X is a reflexive space, then

$$J_{\mathrm{cv}}(X) = \sup\{r^{\mathrm{cv}}(M) \mid M \subset X \text{ is finite, } \operatorname{diam} M = 1\}.$$

It is also known (see [6]) that if X is non-reflexive, then $J_{\mathrm{cv}}(X) = 1$. Lim (see [113], [6]) showed that

$$\begin{aligned} J_{\mathrm{cv}}(X) &= \sup\,\{r^{\mathrm{cv}}(M) \mid M \subset X \text{ is separable and } \operatorname{diam} M = 1\} \\ &= \max\{J_{\mathrm{cv}}(Y) \mid Y \text{ is a separable subspace of } X\}. \end{aligned}$$

Amir [6] proved that if $J_{\mathrm{cv}}(X) < 1$, then

$$\begin{aligned} J_{\mathrm{cv}}(X) &= \sup\{r^{\mathrm{cv}}(M) \mid M \subset X \text{ is finite and } \operatorname{diam} M = 1\} \\ &= \sup\{J_{\mathrm{cv}}(Y) \mid Y \text{ is a finite-dimensional subspace of } X\}. \end{aligned}$$

Remark 13.1 (see [6]). A similar estimate for the (absolute) Jung constant $J(X)$ in terms of the Jung constants of subspaces does not hold. For example, any space Y is a subspace of some space $X = \ell^\infty(\Gamma)$, hence $J(X) = 1/2$, but $J(Y) \in [1/2, 1]$, and in particular, there exists a Y such that $J(Y) > J(X) = 1/2$. A lower estimate for $J(X)$ in terms of the Chebyshev radii of finite subsets also does not hold. It is known that $J(c_0) = 1$ (for a proof it suffices to consider the set $M = \{(-1)^n e_n \mid n \in \mathbb{N}\}$). However, for any finite set $M = \{x^{(1)}, \dots, x^{(1)}\} \subset c_0$ the point

$$\bar{x} := \frac{1}{2}\Bigl(\max_{1 \leqslant i \leqslant n} x^{(i)} - \min_{1 \leqslant i \leqslant n} x^{(i)}\Bigr)$$

lies in c_0, which gives $r(\bar{x}, M) = (1/2) \operatorname{diam} M$.

We consider another example from [6]. Let Γ be an uncountable set and let $X = \{x \in \ell^\infty(\Gamma) \mid \text{the set } \{\gamma \in \Gamma \mid x(\gamma) \neq 0\} \text{ is at most countable}\}$. Thus, X is a closed subspace of $\ell^\infty(\Gamma)$ and therefore is a Banach space. Any separable subset of X lies in a subspace of $\ell^\infty(\Gamma_0)$, where $\Gamma_0 \subset \Gamma$ is countable and $\ell^\infty(\Gamma_0)$ is a subspace of X isometric to ℓ^∞. Recall that $J(\ell^\infty) = 1/2$. On the other hand, $\Gamma = \Gamma_0 \sqcup \Gamma_1$, where Γ_1 is uncountable. Let $M_i := \{x \in X \mid 0 \leqslant x \leqslant \chi_{\Gamma_i}(x)\}$, $i = 0, 1$, where $\chi_A(\,\cdot\,)$ is the indicator function of a set A, and let $M = M_0 \cup (-M_1)$. It is easily seen that $\operatorname{diam} M = 1$, but $r(M) = 1$. Hence $J(X) = 1$.

Amir [6] found some estimates for $J(\,\cdot\,)$ and $J_{\mathrm{cv}}(\,\cdot\,)$ in ℓ^p-spaces. For example,

$$J_{\mathrm{cv}}(\ell^p_n) \geqslant \frac{2^{-1/p}}{n}\bigl[(n-1)+(n-1)^p\bigr]^{1/p};$$
$$J(\ell^p) \geqslant 2^{1/p-1}, \qquad J_{\mathrm{cv}}(\ell^p) \geqslant \max\{2^{1/p-1}, 2^{-1/p}\}.$$

We also note the following results. If X is infinite-dimensional, then $J_{\mathrm{cv}}(X) \geqslant 1/\sqrt{2}$, and if $\dim X \leqslant n$, then $J_{\mathrm{cv}}(X) \leqslant n/(n+1)$. To formulate another estimate of $J_{\mathrm{cv}}(\,\cdot\,)$, we need the definition of the modulus of n-convexity of a space X, as given by F. Sullivan:

$$\delta^{(n)}_X(\varepsilon) := \inf\biggl\{1-\frac{1}{n+1}\biggl\|\sum_{i=0}^n x_i\biggr\| \biggm| x_i\in B(0,1),\ i=0,\dots,n, \\ \mathrm{vol}_n(x_0,\dots,x_n)\geqslant\varepsilon\biggr\},$$

where $\mathrm{vol}_n(x_1,\dots,x_n)$ is the n-dimensional volume of the convex hull of a finite set $x_0,\dots,x_n$; that is,

$$\mathrm{vol}_n(x_0,\dots,x_n) = \sup\left\{\begin{vmatrix} 1 & \dots & 1\\ f_1(x_0) & \dots & f_1(x_n)\\ \dots & \dots & \dots \\ f_n(x_0) & \dots & f_n(x_n)\end{vmatrix} : f_i\in S^*,\ i=1,\dots,n\right\}.$$

It is clear that $\delta^{(1)}_X(\varepsilon)=\delta_X(\varepsilon)$.

Amir showed that $J_{\mathrm{cv}}(X)\leqslant \min_\varepsilon \max\bigl\{1-\delta^{(2)}_X(\varepsilon),\, 2\varepsilon/3+1/2\bigr\}$. For $X=\ell^2$, Amir's estimate gives $J_{\mathrm{cv}}(\ell^2)\leqslant 0.805$ (this estimate is better than Bynum's estimate $J_{\mathrm{cv}}(X)\leqslant 2(1-\delta_X(1))$, which implies that $J_{\mathrm{cv}}(\ell^2)\leqslant\sqrt{3}/2$). Note that the exact value of $J_{\mathrm{cv}}(\ell^2)$ is $\sqrt{2}/2$. More results on estimates of $J_{\mathrm{cv}}(\,\cdot\,)$ for ℓ^p-spaces with $p>2$ can be found in [113] and [6].

Some fixed-point results for non-expansive maps can be formulated in terms of the normal structure of Banach spaces and the coefficient $\mathrm{N}(X)$ of normal structure of a space ($\mathrm{N}(X)=1/J_{\mathrm{cv}}(X)$).

Definition 13.1. A non-empty closed convex subset M of a Banach space X is said to be *diametral* if $\operatorname{diam} M = r_M(M)$. A Banach space X has *normal structure* (*weak normal structure*) if any non-empty closed bounded (respectively, weakly compact) convex diametral subset of X is a singleton.

Remark 13.2. The spaces ℓ^p and $L^p(\Omega)$ with $1<p<\infty$ have normal structure [15]. The space c_0 does not have normal structure. Indeed, consider the set $M := \overline{\mathrm{conv}}\{e_n \mid n\in\mathbb{N}\}$, where (e_n) is the standard basis. We have $\operatorname{diam} M=1$ and $r^{\mathrm{cv}}(M)=1$, because $\lim\|x-e_n\|\geqslant 1$ for any $x\in c_0$. Since (e_n) is weakly null, the set M is weakly compact. Hence, c_0 also does not have weak normal structure. A similar example shows that the space ℓ^1 does not have normal structure. Theorem 3.3 of [15] shows that the space ℓ^1 (as well as any Banach space with the Schur property) has weak normal structure.

A map $f\colon X\to X$ is said to be *non-expansive* if $\|f(x)-f(y)\|\leqslant\|x-y\|$ for all $x,y\in X$.

Theorem 13.1 (Kirk; see [15]). *Let X be a Banach space with weak normal structure, let C be a weakly compact convex subset of X, and let $f\colon C \to C$ be a non-expansive map. Then f has a fixed point.*

Proof. Let $\mathscr{B}$ be the family of all non-empty weakly compact convex subsets of C which are invariant under f. If these sets are partially ordered by inclusion, then it is easy to check that any net in $\mathscr{B}$ has a maximal element. Hence, by Zorn's lemma the family $\mathscr{B}$ has a minimal element K. We have $f(K) \subset K$, and hence $\overline{\operatorname{conv}}(f(K)) \subset K$. Thus, $\overline{\operatorname{conv}}(f(K))$ is a weakly compact subset of K which is invariant under f. The minimality of K implies that $K = \overline{\operatorname{conv}}(f(K))$. Since K is weakly compact and convex, the set $\mathrm{Z}_K(K)$ is non-empty, being the intersection of the nested sequence of convex weakly compact sets $\mathrm{Z}^{\varepsilon}_M(M) := \{y \in K \mid r(y,K) \leqslant r_K(K)+\varepsilon\}$. Let $x \in \mathrm{Z}_K(K)$; that is, $r(x,K) = r_K(K)$. For any $y \in K$ we have $\|f(x)-f(y)\| \leqslant \|y-x\| \leqslant r_K(K)$. Therefore, $f(K)$ is contained in the closed ball $B(f(x), r_K(K))$, which implies that $\overline{\operatorname{conv}}(f(K)) = K \subset B(f(x), r_K(K))$. Consequently, $r(f(x),K) \leqslant r_K(K)$, which gives $f(x) \in \mathrm{Z}_K(K)$. Thus, $\mathrm{Z}_K(K)$ is a non-empty convex weakly compact subset of K which is invariant under f. From the minimality of K we have $\mathrm{Z}_K(K) = K$. Since X has weak normal structure, $\operatorname{diam} K = 0$; that is, K is a fixed point of f. $\square$

Theorem 13.1 can be used in establishing the existence of periodic solutions of differential equations [15].

Another important characteristic of a space is the *normal structure coefficient*

$$\mathrm{N}(X) := \inf\biggl\{\frac{\operatorname{diam} M}{r_M(M)} \biggm| M \subset X\biggr\}, \tag{13.1}$$

where the infimum is taken over all bounded *convex* subsets $M \subset X$ with $\operatorname{diam} M > 0$. It is clear that $1 \leqslant \mathrm{N}(X) \leqslant 2$. Unlike the definition of the constant $1/J(X)$, in (13.1) we consider $r_M(M)$ rather than $r(M)$, and the set M is assumed to be convex. Then $\mathrm{N}(X) = 1/J_{\mathrm{cv}}(X)$. For basic properties of the constant $\mathrm{N}(X)$ see the above results on $J_{\mathrm{cv}}(X)$, and also [67].

From the definition it is clear that if $\mathrm{N}(X) > 1$, then X has normal structure (Example 5 in [15] shows that the converse implication does not hold). Spaces X with the property $\mathrm{N}(X) > 1$ are called *spaces with uniformly normal structure.*

Bynum (see Theorem 2.2 in [15]) showed that if $\delta_X(\,\cdot\,)$ is the modulus of convexity of a Banach space, then $\mathrm{N}(X) \geqslant 1/(1-\delta_X(1))$. This estimate is not sharp. For example, for $X = \ell^2$ we have $\mathrm{N}(X) = \sqrt{2}$, and $\delta_{\ell^2}(\varepsilon) = 1-\sqrt{1-\varepsilon^2/4}$, which gives the lower estimate $2/\sqrt{3}$. It is also known that if $\delta(3/2) > 1/4$, then $\mathrm{N}(X) > 1$. Prus showed that $\mathrm{N}(X) \geqslant \alpha - (\alpha^2-4)^{1/2}$, where $\alpha := \inf\{\varepsilon/2 + 2 - \delta(\varepsilon) \mid 1 \leqslant \varepsilon \leqslant 3/2\}$. We remark that $\mathrm{N}(L^p(\Omega)) = \min\{2^{1-1/p}, 2^{1/p}\}$ for infinite-dimensional $L^p(\Omega,\Sigma,\mu)$-spaces, where μ is a σ-finite measure and $1 \leqslant p < \infty$. According to Theorem 2.6 of [15], if X is a Banach space and $\mathrm{N}(X) > 1$, then X is reflexive. But if X is reflexive, then the infimum in (13.1) can be taken over the convex hulls of finite subsets of X. We also note the general estimate $\mathrm{N}(X) \leqslant \sqrt{2}$, where X is any infinite-dimensional Banach space.

For non-expansive maps we mention the following result, which is a corollary to Theorem 13.1: *if $N(X) > 1$, then any non-expansive map of a non-empty closed convex bounded subset of X has a fixed point.*

For further details see [15] and the references cited there.

The definitions and results that follow below in this section are due to Marrero [118], [119].

Definition 13.2. For a non-empty closed bounded subset Y of a Banach space X, we say that a map $f\colon Y\to Y$ has the *property* (C) if $\lim_{n\to\infty}\eta(Y_n)=0$, where (Y_n) is the nested sequence of non-empty closed bounded subsets of Y defined by

$$Y_1=\overline{f(Y)},\quad Y_{n+1}=\overline{f(Y_n)},\quad n\in\mathbb{N} \tag{13.2}$$

(as before, $\eta(\,\cdot\,)$ is the measure of non-convexity).

Definition 13.3. Let Y be a non-empty bounded subset of a Banach space X. A map $f\colon Y\to Y$ is said to be δ-*condensing* if

$$\operatorname{diam} f(M)<\operatorname{diam} M$$

for any set $M\subset Y$ such that $f(M)\subset M$ and $\operatorname{diam} M>0$.

A map $f\colon Y\to Y$ is said to be η-*condensing* if

$$\eta\bigl(f(M)\bigr)<\eta(M)$$

for any set $M\subset Y$ such that $f(M)\subset M$ and $\eta(M)>0$.

The following two results extend Ćirić's fixed-point theorem [50] to the case of not necessarily convex sets.

Theorem 13.2. *Let $Y\ne\varnothing$ be a closed bounded subset of a Banach space X such that $\overline{\operatorname{conv}}\,Y$ is weakly compact. Suppose that $f\colon Y\to Y$ is a continuous η-condensing map with the property* (C). *Then f has a fixed point.*

Theorem 13.3. *Let $Y\ne\varnothing$ be a closed bounded subset of a reflexive space X, and let $f\colon Y\to Y$ be a δ- or η-condensing map with the property* (C). *Then f has a fixed point.*

Definition 13.4. Let $M\subset X$ be a non-empty bounded set. A point $x\in M$ is said to be *diametral* if $\sup\{\|x-y\|\mid y\in M\}=\operatorname{diam} M$. A convex set M is said to have *normal structure* if, for any $N\subset M$ containing more than one point, there exists a point $x\in N$ which is not a diametral point of N (see also Definition 13.1). Sets with normal structure were introduced by Brodskii and Milman. For more details see [76], Chap. 4.

The next result (see [118], [119]) extends Kirk's fixed-point theorem in [98].

Theorem 13.4. *Let $Y\ne\varnothing$ be a closed bounded subset of a reflexive space X or a non-empty weakly compact subset of a Banach space X. Suppose that Y has normal structure and that $f\colon Y\to Y$ is a non-expansive map with the property* (C). *Then f has a fixed point.*

14. On an approximate solution of the equation $f(x) = x$

In this section, X is a real Banach space and $\varnothing \neq M \subset X$. Let $\mathfrak{N}(X,M)$ (respectively, $\mathfrak{N}(M,M)$) be some class of maps from X (from M) to M. We set

$$\beta_0 = \beta_0(\mathfrak{N}, X, M) := \sup_{f\in\mathfrak{N}(X,M)} \inf_{x\in X} \|f(x) - x\|, \tag{14.1}$$

and

$$\beta = \beta(\mathfrak{N}, X, M) \ := \sup_{f\in\mathfrak{N}(M,M)} \inf_{x\in M} \|f(x) - x\|. \tag{14.2}$$

From the definition it follows that $\beta_0 \leqslant \beta$. Babenko, Konyagin, and Tsar'kov [16] found estimates of β and β_0 for classes of continuous (respectively, Lipschitz) maps.

I. Let $\mathfrak{N}(X,M) = C(X,M)$ ($\mathfrak{N}(M,M) = C(M,M)$). The Schauder fixed-point theorem gives us that $\beta_0 = 0$ (respectively, $\beta = 0$) if M is compact (respectively, if M is convex and compact). According to a result of Dugundji (see, for example, [77], p. 108), if $\dim X = \infty$, then for $M := \{x \in X \mid \|x\| \leqslant 1\}$ there exists an $f \in C(M,M)$ such that $f(x) \neq x$ for any $x \in X$ (this implies that the unit sphere of an infinite-dimensional space is contractible to a point). Klee (see [77]) extended this result to the case of convex sets in metrizable locally convex spaces. The following theorem can be proved by using Dugundji's fixed-point theorem.

Theorem 14.1 (see [16]). *Let X be an infinite-dimensional Banach space and let $M := \{x \in X \mid \|x\| \leqslant 1\}$. Then for any $\varepsilon > 0$ there exists an $f \in C(X,M)$ such that*

$$\|f(x) - x\| > 1 - \varepsilon \qquad \forall x \in X$$

(*that is*, $\beta_0(M) = \beta(M) = 1$).

We set

$$\alpha_0 = \alpha_0(X,M) := \inf_{T\subset X} \sup_{y\in M} \rho(y,T),$$

where $T \subset X$ is a finite set of points, $\rho(y,T) := \min_{x\in T} \|y - x\|$, and α_0 is the radius of best covering of the set M by balls (the self radius of best covering of M by balls is

$$\alpha = \alpha(X,M) := \inf_{T\subset M} \sup_{y\in M} \rho(y,T)\Bigr).$$

It is clear that $\alpha/2 \leqslant \alpha_0 \leqslant \alpha$. Note that if $M = B \subset X$ is the unit ball of an infinite-dimensional space, then $\alpha(B) = \alpha_0(B) = 1$, and hence by Theorem 14.1, $\beta_0(B) = \beta(B) = \alpha_0(B) = \alpha(B)$, which shows that the estimate in the next theorem is sharp.

Theorem 14.2 (see [16]). *Let X be a Banach space and let $M \subset X$ be a non-empty convex set. Then $\beta_0 \leqslant \alpha_0$ and $\beta \leqslant \alpha$.*

In the next theorem we estimate β and β_0 from below. Let $\mathscr{F}(X)$ be the class of all non-empty closed subsets of a Banach space X.

Theorem 14.3 (see [16]). *Let X be an infinite-dimensional Banach space and let $M \in \mathscr{F}(X)$ be convex. Then:*

a) *if M is separable, then $\alpha_0/2 \leqslant \beta_0$, $\alpha/3 \leqslant \beta$, $\beta_0 \leqslant \alpha_0$, and $\beta \leqslant \alpha$ (the estimates are attained in the space of bounded sequences);*

b) *if M is not separable, then $\alpha_0/4 \leqslant \beta_0$ and $\alpha/6 \leqslant \beta$.*

Sharper estimates can be obtained in the Hilbert space setting (the estimates are attained on some subset M of a Hilbert space.

Theorem 14.4 (see [16]). *Let $M \in \mathscr{F}(H)$ be convex. Then*

$$\frac{\sqrt{2}}{2}\,\alpha_0 = \frac{\sqrt{2}}{2}\,\alpha \leqslant \beta \leqslant \beta_0.$$

II. Let $\mathfrak{N}(X,M)$ and $\mathfrak{N}(M,M)$ be the classes of Lipschitz maps with fixed Lipschitz constant $k \geqslant 0$. It is known that if $k < 1$, then $\beta = \beta_0 = 0$ by the Banach contraction principle. Below we assume that $k \geqslant 1$.

Theorem 14.5 (see [16]). *Let X be a Banach space and let $M \in \mathscr{F}(X)$ be a convex set. Then*

$$\beta_0 \leqslant \frac{k-1}{k}\,\alpha_0 \quad \textit{and} \quad \beta \leqslant \frac{k-1}{k}\,\alpha$$

(the estimate is attained in $X = \ell^\infty$).

Proof of Theorem 14.5. We fix an arbitrary $\varepsilon > 0$ and choose a finite set of points $T \subset X$ ($T \subset M$) such that

$$\rho(x,T) \leqslant \alpha_0 + \varepsilon \quad (\text{respectively } \leqslant \alpha + \varepsilon) \qquad \forall\, x \in M.$$

Let $f \in \mathfrak{N}(X,M)$ ($f \in \mathfrak{N}(M,M)$) and let L be the convex hull of T. We define the map $\varphi\colon L \to M$ as follows: with each point $x \in L$ we associate a point $y \in X$ ($y \in M$) such that $x + (k+\varepsilon)^{-1}(f(y) - x) = y$ (such a point exists and is unique, because $\psi(z) = x + (k+\varepsilon)^{-1}(f(z) - x)$ is a contraction of M). It is easily checked that $\varphi \in C(L,M)$ and $\chi := P_L \circ f \circ \varphi$ is an upper semicontinuous map from L to 2^L with compact convex values. Hence, by Kakutani's fixed-point theorem there exists a point $x_0 \in L$ such that $x_0 \in \chi(x_0)$. Therefore, $\|(f\circ\varphi)(x_0)\| \leqslant \alpha_0 + \varepsilon$ (respectively $\leqslant \alpha + \varepsilon$). By construction

$$\begin{gathered}(f\circ\varphi)(x_0) - x_0 = (k+\varepsilon)(\varphi(x_0) - x_0),\\ (f\circ\varphi)(x_0) - \varphi(x_0) = (k+\varepsilon-1)(\varphi(x_0) - x_0),\end{gathered}$$

and hence

$$\begin{aligned}\|(f\circ\varphi)(x_0) - \varphi(x_0)\| &\leqslant \frac{k+\varepsilon-1}{k+\varepsilon}\,\|(f\circ\varphi)(x_0) - x_0\| \\ &\leqslant \frac{k+\varepsilon-1}{k+\varepsilon}(\alpha_0+\varepsilon) \quad \biggl(\text{respectively, } \leqslant \frac{k+\varepsilon-1}{k+\varepsilon}(\alpha+\varepsilon)\biggr).\end{aligned}$$

The theorem follows by letting $\varepsilon \to 0$. $\square$

One more result is worth noting.

Theorem 14.6 (see [16]). *There exists a* $C > 0$ *such that*

$$C\,\frac{k-1}{k}\,\alpha \leqslant \beta$$

for any infinite-dimensional Banach space X *and any convex set* $M \in \mathscr{F}(X)$.

A similar estimate also holds in a Hilbert space for α_0 and β_0.

15. The Jung constant of the space ℓ_n^1

The general inequality for the Jung constant in an n-dimensional space implies that

$$J(\ell_n^1) \leqslant \frac{n}{n+1}.$$

It can be shown [87] that this becomes an equality if and only if n is a Hadamard dimension (the definition is given below).

By a *Hadamard matrix* A of order n we mean a square $n \times n$ matrix with all entries ± 1 and with pairwise orthogonal rows; that is, $AA^{\mathrm{T}} = nE$, where E is the identity matrix and A^{T} is the transpose of A. It is known that if a Hadamard matrix of order n exists, then n either is a multiple of 4 or equals 1 or 2 (for more details, see [81]). There is a conjecture that this condition is also sufficient for the existence of a Hadamard matrix. In particular, Hadamard matrices have been constructed for any $n = 2^k$ and for all $n = 4k < 668$ (see, for example, [120], [52]).

Theorem 15.1 (Dol'nikov [57]). *The inequality*

$$J(\ell_n^1) \leqslant \frac{n}{n+1} \tag{15.1}$$

(*which holds for any* n*-dimensional space*) *becomes an equality if and only if there exists a Hadamard matrix of order* $n+1$.

The proof of Dol'nikov's theorem depends essentially on the classical existence problem for Hadamard matrices.

We also mention some results on the Jung constant for ℓ_n^1 (see [57], [117]).

Theorem 15.2. 1. $J(\ell_{2m}^1) \geqslant (2m-1)/(2m)$.

2. *If there exists a Hadamard matrix of order* $2m$, *then* $J(\ell_{2m}^1) = \dfrac{2m-1}{2m}$.
3. *Assume that there exists a Hadamard matrix of order* $n+1$. *Then*

$$J(\ell_{n+1}^1) \geqslant \frac{n}{n+1}, \qquad J(\ell_{n+2}^1) \geqslant \frac{n}{n+1}, \qquad J(\ell_{n+3}^1) \geqslant \frac{n}{n+1}.$$

V. I. Ivanov's conjecture is that if there exists a Hadamard matrix of order $n+1$, then $J(\ell_{n+1}^1) = J(\ell_{n+2}^1) = J(\ell_{n+3}^1) = n/(n+1)$.

From a result of Dol'nikov (Theorem 2 of [57]) it follows that $J(\ell_4^1) = 3/4$. At present, the value of the Jung constant for the space ℓ_5^1 is not known.

16. On the relation between the Jung constant and the Jackson constant

Stechkin and V. I. Berdyshev (see [34], and also [87], Chap. 4, § 5) established a link between the Jung constant (a geometric characteristic of a space) and the sharp constant in Jackson's theorem on best approximation of functions by constants (an approximative characteristic of a space).

Namely, let X be the Banach space of integrable 2π-periodic functions $f\colon \mathbb{R} \to Y$ (Y is a normed linear space) with a norm satisfying the conditions

$$\|f(\,\cdot\,+u)\|_X = \|f(\,\cdot\,)\|_X, \qquad |u| \leqslant \pi. \tag{16.1}$$

We assume that X contains the constant functions and that the curve $M_f := \{f(\,\cdot\,+t) \mid |t| \leqslant \pi\}$ is compact for any function $f \in X$. In particular, the above properties are satisfied by the uniform norm on the space of continuous 2π-periodic functions.

Let $E_0(f) = E_0(f)_X := \inf\{\|f(\,\cdot\,) - c\|_X \mid c \in Y\}$ be the best approximation of a function $f(\,\cdot\,)$ by constants $c \in Y$, and let $\omega(f, \delta, X) := \sup_{|t|\leqslant\delta} \|f(\,\cdot\,+t) - f(\,\cdot\,)\|_X$ be the (first) *modulus of continuity* of the function $f(\,\cdot\,)$. The Jackson constant is defined as

$$K(X) := \sup_{f\in X} \frac{E_0(f)_X}{\omega(f, \pi, X)}.$$

We also define the Jung constant for the class of compact sets:

$$J^*(X) := \sup \frac{r(M)}{\operatorname{diam} M},$$

where the supremum is taken over all *compact* sets $M \subset X$. It is clear that $J^*(X) \leqslant J(X)$. In the next theorem, the case $Y = \mathbb{R}$ was considered by Stechkin [34].

Theorem 16.1 (Stechkin). *Let X be a Banach space of 2π-periodic integrable functions $f\colon \mathbb{R} \to Y$ with a norm satisfying the conditions* (16.1), *and suppose that X contains the constant functions. Then*

$$K(X) \leqslant J^*(X).$$

Proof. Let $f \in X$, $f \neq 0$. By (16.1), for a function $\varphi \in X$

$$\|f(\tau + t) - \varphi(\tau)\| = \|f(\tau - u + t) - \varphi(\tau - u)\| \overset{u=t}{=} \|f(\tau) - \varphi(\tau - t)\|.$$

Then

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} \|f(\tau+t) - \varphi(\tau)\|\,dt &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \|f(\tau) - \varphi(\tau-t)\|\,dt \\
&\geqslant \frac{1}{2\pi}\left\|\int_{-\pi}^{\pi} [f(\tau) - \varphi(\tau-t)]\,dt\right\| = \left\|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\,dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\tau-t)\,dt\right\| \\
&= \left\|f(\tau) - \frac{1}{2\pi}\int_{\tau-\pi}^{\tau+\pi} \varphi(\tau)\,d\tau\right\| = \left\|f(\tau) - \frac{1}{2\pi}\int_{\pi}^{\pi} \varphi(\tau)\,d\tau\right\| \geqslant E_0(f),
\end{aligned}$$

and therefore,

$$r(M_f)_X := \sup_{|t|\leqslant\pi} \|f(\tau+t)-\varphi(\tau)\| \geqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \|f(\tau+t)-\varphi(\tau)\|\,dt \geqslant E_0(f),$$

where $M_f := \{f(\,\cdot\,+t) \mid |t|\leqslant\pi\}$. On the other hand,

$$\operatorname{diam} M_f = \sup_{t_1,t_2} \|f(\tau+t_1)-f(\tau+t_2)\| = \sup_{|t|\leqslant\pi} \|f(\tau+t)-f(\tau)\| = \omega(f,\pi,X),$$

and since M_f is compact, we have $\dfrac{E_0(f)}{\omega(f,\pi,X)} \leqslant \dfrac{r(M_f)}{\operatorname{diam} M_f} \leqslant J^*(X)$. $\square$

The next result follows from Theorem 16.1 and a result due to V.I. Berdyshev [33].

Corollary 16.1. *In the space $L^p[-\pi,\pi]$ with $1\leqslant p<\infty$,*

$$\max\{2^{(1-p)/p}, 2^{-1/p}\} \leqslant K(L^p) \leqslant J^*(L^p) \leqslant J(L^p) \tag{16.2}$$

(*and for $p=1,2$ all the inequalities in* (16.2) *become equalities*).

Note that (see [34])

$$J^*(C[-\pi,\pi]) = K(C[-\pi,\pi]) = \frac{1}{2}\,, \tag{16.3}$$

but

$$J(C[-\pi,\pi]) = 1. \tag{16.4}$$

To prove (16.4) it suffices to consider the following example presented by Stechkin (see [34]). We let

$$f_k(t) = \begin{cases} 1, & 0\leqslant t\leqslant \dfrac{\pi}{2}-\dfrac{1}{k}, \\ -1, & \dfrac{\pi}{2}+\dfrac{1}{k}\leqslant t\leqslant\pi, \\ k\Bigl(\dfrac{\pi}{2}-t\Bigr), & \dfrac{\pi}{2}-\dfrac{1}{k}<t<\dfrac{\pi}{2}+\dfrac{1}{k}, \end{cases} \tag{16.5}$$

and put $M=(f_k(t))_{k=1}^{\infty}$. The set $M\subset C[-\pi,\pi]$ consists of continuous 2π-periodic functions such that $f_k(t)=f_k(-t)$ and $f_k(t+2\pi)=f_k(t)$ for $k=1,2,\dots$. It is easily seen that $r(M)=\operatorname{diam} M=1$, and hence $J(C[-\pi,\pi])=1$. For the compact set M, the function $f_0(t) := \bigl[\max_{f\in M} f(t)+\min_{f\in M} f(t)\bigr]/2$ lies in $C[-\pi,\pi]$ and $r(M)=\max_{f\in M}\|f_0(t)-f(t)\| = (1/2)\operatorname{diam} M$ (see Theorem 3.3), which proves the equality $J^*(C[-\pi,\pi])=1/2$ in (16.3).

It is easily verified that the Jung constant of the space of bounded functions is $1/2$. In general, the Jung constant $J(X)$ is equal to $1/2$ on the class of so-called P_1-spaces (see [54]). In particular, $J(L^\infty)=J(\ell^\infty)=1/2$. It is also easy to see that the Jung constants of a space and some subspace of it can be different (for example, $J(c_0)=1$).

For more on the relation between the Jung constant and the Jackson constant in L^p-spaces, see [87].

17. The relative Jung constant

For a non-empty bounded subset M of X, its *self Chebyshev radius* is defined by $r_M(M) := \inf_{y\in M}\sup_{x\in M}\|x-y\|$ (see (1.3)). The *relative Jung constant* of a space X is defined by

$$J_s(X) := \sup\biggl\{\frac{r_M(M)}{\operatorname{diam} M}\biggm| M\subset X,\ 0<\operatorname{diam} M<\infty\biggr\}.$$

Note that if X is non-reflexive, then $J_s(X)=1$ (see [6]).

Pichugov (see, for example, [87]) showed that

$$J_s(\ell_p^n)\leqslant \frac{1}{\sqrt[q']{2}}\sqrt[q]{\frac{n}{n+1}},\qquad q=\min(p,p'),\quad 1\leqslant p<\infty;$$

this becomes an equality for $p<2$ and for n such that there exists a Hadamard matrix of order $n+1$ (p' and q' are determined by $1/p+1/p'=1$ and $1/q+1/q'=1$, respectively).

S. V. Berdyshev [36] found the relative Jung constant of the space ℓ_n^∞, $n\in\mathbb{N}$ (there he also described the *extremal subsets* of ℓ_n^∞, that is, the sets M with $J_s(\ell_n^\infty)=r_M(M)/\operatorname{diam} M$).

Theorem 17.1 (S. V. Berdyshev). *The following equality holds*:

$$J_s(\ell_n^\infty)=\frac{n-1}{n},\qquad n\geqslant 2\,.$$

Unlike the case of the Jung constant $J(X)$, Theorem 17.1 gives the precise value of the relative Jung constant $J_s(\ell_n^\infty)$ *for all* n.

We set

$$\gamma(X)=\sup\frac{r_M(M)}{r(M)},$$

where the supremum is taken over all non-empty closed convex subsets of X. The constant γ was introduced by Arestov (see [13]) in the study of a problem involving operator recovery. It is clear that for all sets $M\subset X$

$$r(M)\leqslant r_M(M)\leqslant \operatorname{diam} M\leqslant 2r(M).$$

Hence, in view of (1.5) we get that $\max\{1,J_s(X)\}\leqslant\gamma(X)\leqslant 2J_s(X)$.

Klee [99] and, independently, Garkavi [71] (see Theorem 4.4 above) showed that (some[12]) Chebyshev centre of any bounded subset of a space X lies in the convex hull of this set if and only if X is a Hilbert space or if $\dim X\leqslant 2$. In [36] it was noted that this result implies that either of these two conditions is equivalent to saying that $\gamma(X)=1$ for the space X.

Corollary 17.1. $\gamma(\ell_n^\infty)=2(n-1)/n$.

This result is a corollary of Theorem 17.1 (see [36]).

[12]In a Hilbert space the Chebyshev centre of a non-empty bounded set is unique (see § 5).

18. The Jung constant of a pair of spaces

Let X_1 and X_2 be normed linear spaces with $X_2 \subset X_1$, and let $M \subset X_2$ be a bounded set. The Jung constant of $M \subset X_2 \subset X_1$ for the pair of spaces (X_1, X_2) is defined by

$$J(M, X_1, X_2) = \frac{r(M)_{X_1}}{\operatorname{diam}(M)_{X_2}}$$

(see [87]), where $r(M)_{X_1}$ is the Chebyshev radius of M in X_1, and $\operatorname{diam}(M)_{X_2}$ is the diameter of M in the space X_2. The relative Jung constant of M for a pair of spaces (X_1, X_2) is defined by

$$J_s(M, X_1, X_2) = \frac{r_M(M)_{X_1}}{\operatorname{diam}(M)_{X_2}}\,.$$

By the Jung constant of a pair of spaces (X_1, X_2) we mean

$$J(X_1, X_2) = \sup_{M\subset X_2} J(M, X_1, X_2) \tag{18.1}$$

(see [87]), and by the relative Jung constant of a pair of spaces we mean

$$J_s(X_1, X_2) = \sup_{M\subset X_2} J_s(M, X_1, X_2). \tag{18.2}$$

It can easily be checked that

$$r(M)_{X_1} = r(\operatorname{conv} M)_{X_1} = r(\overline{\operatorname{conv}} M)_{X_1},$$
$$\operatorname{diam}(M)_{X_2} = \operatorname{diam}(\operatorname{conv} M)_{X_2} = \operatorname{diam}(\overline{\operatorname{conv}} M)_{X_2}.$$

Indeed, if $x_1, \dots, x_n \in M$, $y_1, \dots, y_m \in M$, $\alpha_1 + \dots + \alpha_n = 1$, $\alpha_i \geqslant 0$, and $\beta_1 + \dots + \beta_m = 1$, $\beta_j \geqslant 0$, then

$$\begin{aligned}\biggl\|\sum_{i=1}^{n} \alpha_i x_i - \sum_{j=1}^{m} \beta_j y_j\biggr\|_{X_2} &= \biggl\|\sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i\beta_j (x_i - y_j)\biggr\|_{X_2} \\ &\leqslant \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} \beta_j \max_{i,j} \|x_i - y_j\| \leqslant \operatorname{diam}(M)_{X_2}.\end{aligned}$$

Hence, the set M in (18.1) and (18.2) can be assumed to be closed and convex.

Consider the following class of spaces:

$$\ell^p_{nk} = \bigl\{x = (x^{(1)}, \dots, x^{(n)}) \mid x^{(i)} \in \mathbb{R}^k,\ \ i = 1, \dots, n\bigr\},$$
$$\|x\|_p = \biggl(\frac{1}{n}\sum_{i=1}^{n} |x^{(i)}|^p\biggr)^{1/p} = \biggl(\frac{1}{n}\sum_{i=1}^{n}\biggl(\sum_{s=1}^{k} |x^{(i)}_s|^2\biggr)^{p/2}\biggr)^{1/p}, \qquad 1 \leqslant p < \infty,$$
$$\|x\|_\infty = \max_i |x^{(i)}| = \max_i \biggl(\sum_{i=1}^{k} |x^{(i)}_s|\biggr)^{1/2}, \qquad p = \infty.$$

Below, q' is the conjugate exponent of q.

Theorem 18.1 (see [87], Chap. 4). *Let* $1 \leqslant p, q \leqslant \infty$, $n, k \in \mathbb{N}$. *Then*:

$$J(\ell^p_{nk}, \ell^q_{nk}) \leqslant \frac{1}{2^{1/q'}} \left(\frac{nk}{nk+1}\right)^{1/q} \quad \textit{if} \;\; 1 \leqslant p \leqslant q \leqslant 2;$$

$$J(\ell^p_{nk}, \ell^q_{nk}) \leqslant \frac{n^{1/q-1/p}}{2^{1/q}} \left(\frac{nk}{nk+1}\right)^{1/q'} \quad \textit{if} \;\; 2 \leqslant q \leqslant p < \infty;$$

$$J(\ell^1_n, \ell^q_n) \leqslant \frac{1}{2^{1/q'}} \left(\frac{n}{n+1}\right)^{1/q} \quad \textit{if} \;\; 1 \leqslant q \leqslant \infty, \quad k = 1;$$

$$J(\ell^p_n, \ell^\infty_n) = \frac{1}{2} \quad \textit{if} \;\; 1 \leqslant p \leqslant \infty, \quad k = 1;$$

$$J(\ell^\infty_n, \ell^q_n) \leqslant \frac{n^{1/q}}{2} \quad \textit{if} \;\; 1 \leqslant q \leqslant \infty, \quad k = 1.$$

Theorem 18.2 (see [87], Chap. 4). *Let* $1 \leqslant p, q < \infty$, $n, k \in \mathbb{N}$. *Then*:

$$J_s(\ell^p_{nk}, \ell^q_{nk}) \leqslant \frac{1}{2^{1/q'}} \left(\frac{nk}{nk+1}\right)^{1/q} \quad \textit{if} \;\; 1 \leqslant p \leqslant q \leqslant 2;$$

$$J_s(\ell^p_{nk}, \ell^q_{nk}) \leqslant \frac{1}{2^{1/q}} \left(\frac{nk}{nk+1}\right)^{1/q'} \quad \textit{if} \;\; 2 < q = p < \infty;$$

$$J_s(\ell^p_{nk}, \ell^4_{nk}) \leqslant \left(\frac{k}{4(k+1)}\right)^{1/4} \left(\frac{nk}{nk+1}\right)^{1/4} \quad \textit{if} \;\; 1 \leqslant p \leqslant \frac{8(k+1)}{3k+4};$$

$$J_s(\ell^p_{nk}, \ell^4_{nk}) \leqslant \frac{(p-2)^{1/2-1/p}(k(4-p))^{1/p-1/4}}{(2k+4)^{1/p-1/4}\,p^{1/4}} \left(\frac{nk}{nk+1}\right)^{1/4}$$
$$\textit{if} \;\; \frac{8(k+1)}{3k+4} \leqslant p \leqslant 4;$$

$$J_s(\ell^p_{nk}, \ell^q_{nk}) \leqslant \frac{1}{2^{1/q'}} \left(\frac{n}{n+1}\right)^{1/q} \quad \textit{if} \quad \begin{array}{ll} 3/2 \leqslant p \leqslant 2, & q = 3, \\ 1 \leqslant p \leqslant q/2, & 2 < q \leqslant 7/2, \\ 1 \leqslant p \leqslant 2, & q > 7/2; \end{array}$$

$$J_s(\ell^p_{nk}, \ell^q_{nk}) \leqslant \frac{1}{2^{1/q'}} \left(\frac{n}{n+1}\right)^{1/q} \quad \textit{if} \quad \begin{array}{ll} q/2 \leqslant p \leqslant 2q/3, & 2 < q \leqslant 3, \\ q/2 \leqslant p \leqslant 4q/7, & 3 < q \leqslant 7/2, \\ 2 \leqslant p \leqslant 4q/7, & 7/2 < q \leqslant 4, \\ 2 \leqslant p \leqslant q/2, & q > 4. \end{array}$$

19. Some remarks on intersections of convex sets. Connection with the Jung constant

Any closed bounded subset of a finite-dimensional Banach space is compact. Hence, the intersection of any nested sequence of closed bounded subsets of a finite-dimensional Banach space is non-empty. The converse also holds (which is an easy consequence of the characterization of finite-dimensional spaces in terms of compactness of the unit ball): if the intersection of each sequence of nested closed bounded sets in a Banach space is non-empty, then this space is finite-dimensional.

Questions connected with intersections of nested families of closed bounded subsets of infinite-dimensional Banach spaces have been considered by Vakhania, Kartsivadze, Chelidze, Papini, Jachymski, and others (see [47], [89] and the references given there). An example of a nested sequence of balls with empty intersection can be constructed in a complete linear metric space. However, according to Vakhania and Kartsivadze, in a (real or complex) Banach space the intersection of any nested sequence of *closed balls* is always non-empty (even in the case when the radii do not tend to zero).

By an *admissible sequence of sets* (A_n) we shall mean a sequence of nested closed bounded sets $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$ in a normed linear space X.

In order to study intersections of sets more general than balls, we recall a definition from [47], where with any bounded subset of a Banach space one associates a certain number characterizing (in a sense) the deviation of this set from a ball.

Definition 19.1. Let M be a bounded subset of a Banach space, and let

$$R_x(M) := \sup_{y\in M} \|x-y\| \quad (=: r(x,M)) \quad \text{and} \quad r_x(M) := \inf_{y\in X\setminus M} \|x-y\|.$$

We exclude the trivial case when the set M is empty or is a singleton, and we consider the quantity

$$\chi(M) := \sup_{x\in M} \frac{r_x(M)}{R_x(M)}.$$

It is known (see [46]) that if (A_n) is an admissible sequence of sets in a Banach space and if $\varlimsup \chi(A_n) > 1/2$, then the sequence (A_n) has non-empty intersection. On the other hand, there exists a Banach space X (for example, $X=c$) such that for any $\chi < 1/2$ there exists an admissible sequence (A_n) of closed convex bounded subsets of X such that $\chi(A_n)=\chi$ for $n=1,2,\dots$ and $\bigcap_{n=1}^\infty A_n = \varnothing$ (see [47]). In the particular case when the space is reflexive and all the closed bounded sets A_n in the nested sequence are convex, the intersection $\bigcap_{n=1}^\infty A_n$ is always non-empty regardless of the behaviour of the sequence $\chi(A_n)$. It is also known (see [46]) that in the spaces ℓ^p with $1\leqslant p<\infty$ the intersection of an admissible sequence (A_n) is non-empty if $\varlimsup \chi(A_n) \geqslant 1/(1+2^{1/p})$.

Definition 19.2. A number $\alpha\in\mathbb{R}_+$ is called a *critical value* [46] for a Banach space X (notation: $\alpha = \operatorname{cv}(X)$) if:

a) any admissible sequence (A_n) has non-empty intersection if $\varlimsup \chi(A_n) > \alpha$;

b) for any $\varepsilon>0$ there exists an admissible sequence (A_n) such that $\varlimsup \chi(A_n) > \alpha-\varepsilon$ and $\bigcap_{n=1}^\infty A_n=\varnothing$.

The values of $\operatorname{cv}(X)$ are known for some spaces. In particular, $\operatorname{cv}(X)\in[1/3,1/2]$ and $\operatorname{cv}(\ell^p) = 1/(1+2^{1/p})$ for $1\leqslant p<\infty$ (so that for any α in the interval $(1/3,1/2)$ there exists a reflexive space for which α is a critical value). An example of a reflexive space X with critical value $1/2$ was constructed in [47] (X is a subspace of ℓ^1). It is unknown whether there exists a reflexive space with critical value $1/3$.

A set A is said to be *non-trivial* if it contains at least two points. For an admissible sequence consisting of non-trivial sets (A_n), we define (see [47])

$$\Delta(A_n) := \varlimsup_{n\to\infty} \frac{r'(A_n)}{\operatorname{diam} A_n}, \quad \text{where} \ \ r'(A_n) := \sup_{x\in X} r_x(A_n) = \sup_{x\in A_n} r_x(A_n).$$

Definition 19.3. A number $\alpha \in \mathbb{R}_+$ is said to be CV-*critical* for X (notation: $\alpha = \mathrm{CV}(X)$) if:

a) any admissible sequence of non-trivial sets (A_n) has non-empty intersection if $\Delta(A_n) > \alpha$;

b) for any $\varepsilon > 0$ there exists an admissible sequence (A_n) of non-trivial sets such that $\Delta(A_n) > \alpha - \varepsilon$ and $\bigcap_{n=1}^{\infty} A_n = \varnothing$.

Chelidze and Papini [47] showed that for any Banach space X

$$\mathrm{CV}(X) = \frac{\mathrm{cv}(X)}{1+\mathrm{cv}(X)}\,,$$

and thus, $\mathrm{CV}(X) \in [1/4, 1/3]$.

Following [47], for any closed bounded set A we define

$$\overline{\chi}(A) = \frac{r'(A)}{r(A)}\,, \quad \text{where} \ \ r'(A) := \sup\{r_x(A) \mid x \in A\}.$$

It is clear that $\chi(A) \leqslant \overline{\chi}(A) \leqslant 1$ for any A. Moreover, the intersection of an admissible sequence (A_n) is non-empty if $\varlimsup \overline{\chi}(A_n) > 1/2$.

In connection with these questions, we mention a result on the Jung constant (see [47]).

Theorem 19.1 (Chelidze and Papini). *The intersection of an admissible sequence (A_n) is non-empty if*

$$\varlimsup \overline{\chi}(A_n) > \frac{J(X)}{4}\,.$$

Consequently, $\mathrm{CV}(X) \leqslant J(X)/2$ *and* $\mathrm{cv}(X) \leqslant J(X)/(2-J(X))$.

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