Indetermination versus incompatibility in dynamic systems with dry friction

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Abstract. The study of the dynamical behavior of a system becomes really complex when friction forces are introduced in calculus since they directly depend on the values of the normal reactions and on the motion of the system, but the motion, at its turn, depends upon the forces from the system. The analysis is more difficult when dry friction, characterized by inequalities, is present between the parts of the system. When inequalities are involved in dynamical equilibrium equations, theoretically simultaneous equilibrium states may occur. In scientific literature there are famous examples supporting this statement. The present paper presents the dynamical study of a simple system, namely a ball obliged to roll in a groove with flat walls.
It is proved experimentally that the motion of the ball is a planar one. The dynamical analysis of the system with planar motion leads to an undeterminate system of equations and therefore the spatial approach of the system is required. After writing the equations of the spatial motion of the ball, an incompatible system of equations is obtained.

1. Introduction

The dry friction is a complex phenomenon and presents numerous consequences in the behaviour of dynamical systems [1-7]. One of the most impressive phenomena happening in the systems with dry friction is the bifurcation occurrence [8], [9]. The base of bifurcation phenomenon is the strong dependency of the manner of evolution of a system on the initial conditions. Thus, when the behaviour of a dynamical system is described by a system of differential equations, a particular solution will be obtained by obliging the general solution to validate the initial conditions. Following this requirement, all the constants of integration from the expression of the general solution of differential equations can be found. By slightly varying the values of the parameters characteristic to the initial state of the system, two situations may occur: the behaviour of the system is quasi-identical to the one of the initial system and therefore the system is stable or, the behaviour is completely different from the one of the initial system and in this case, the system is instable.

The simplest dynamical system used in highlighting the bifurcation phenomenon is the plane pendulum with two degrees of freedom. When the double pendulum is launched from positions in which the two rods have a tilt in the vicinity of the vertical direction, the pendulum presents a stable behaviour. But when the two rods present a significant angle with the vertical, despite all the efforts to ensure rigorously identical conditions of launching, it will be noticed that the evolution of the system is completely different for two launches.

As emphasised at the beginning of the section, the dry friction is one of the most important reasons for the instable behaviour of a system [10], [11]. Actually, the fact that in the case of a dry friction contact, the value of the friction force is characterized by an inequality complicates the study of such systems. And when the rolling friction occurs the problem is more complex concerning the evolution of a dynamical system [12], [13].

The presence of pure rolling is defined as the motion between two bodies when a non-conforming contact exists between the bodies and therefore there is no relative motion between the contacting points. The above definition is valid for the assumption of smooth bodies. Actually, in the contact points between two real bodies a series of micro-contacts occur between the surface asperities and sliding friction comes about between these. In a recent work [14] it was highlighted the possibility that in a contact, sliding can exist on one direction and pure rolling can be present on a normal direction. All these concerns are supporting the statement that special carefulness must be considered when studying the evolution of a system with dry friction.

2. Experimental evidences regarding the plane-parallel motion of the ball

The present paper highlights another phenomenon taking place in dry friction dynamics. The system consists in a bearing ball obliged to move downward an inclined plane, the trajectory differing from the steepest slope. The trajectory of the centre of the ball is found due to the constraints: ball-inclined plane contact and ball-vertical support contact.

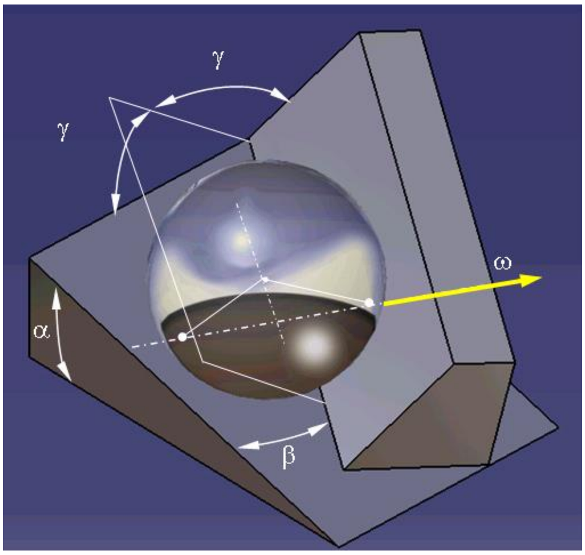

Figure 1. The geometrical characteristics of the system (ball and supporting planes)

The motion of a ball in a channel made according to the schematics from Figure 1 is presented in Figure 2, for the case when one of the walls of the groove is in the vertical plane. A rectangular mark is glued on the surface of the ball. The motion of the ball was filmed and by analyzing the frames of the film, it was noticed that for different moments the position of the mark with respect to the vertical wall is unchanged, fact that attests the existence of a rotation axis of fixed direction (characteristic to the plane-parallel motion).

Figure 2. The position of the stamp is unchanged for different instants, so, the rotation about an axis with constant direction (plane-parallel motion) is proved

Figure 3. The mark remains always in the vertical plane where it was placed at the initial moment

The hypothesis of plane parallel motion of the ball along the groove is underlined in Figures 3 and 4. In Figure 3 the ball moves along a groove made from two glass beans placed symmetrically with respect to the vertical direction. The mark from the surface of the ball, which was placed initially in the vertical plane of symmetry of the channel remains permanently in this plane and confirming the existence of an axis of rotation with constant direction, normal to the plane of symmetry, as seen in Figure 3. In Figure 4 one can observe the chalk powder particles arranged on the surface of the ball in the contact point. The two contact points ball-groove describe two identical circles placed in parallel planes, fact that confirms the existence of an axis of rotation with fixed direction (in this case, normal to the planes of the circles.

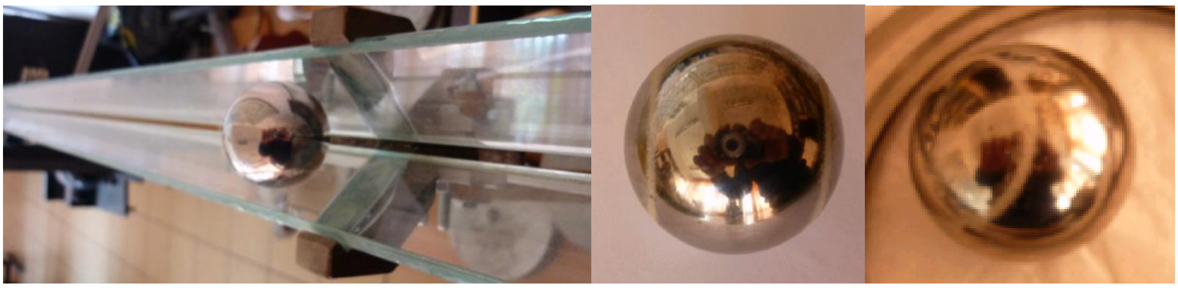

Figure 4. The chalk powder from the sides of the groove adheres to the ball under the shape of two circles, this showing that the contact points are immobile with respect to the instantaneous axis of rotation

3. Dynamical study of the motion of the ball

From experimental tests it is observed that the ball has plane-parallel motion but the dynamic study of the ball's motion is not possible under the above hypothesis since the number of scalar equations – obtained by the projection of the two fundamental theorems, the momentum theorem and the angular momentum theorem, is smaller than the number of unknowns. Then, the problem is *indeterminate.*

When the approach of the problem is made from spatial dynamics perspective, considering the same unknowns, there are obtained a number of equations greater than the number of unknowns, so the problem is over-constrained. In order to relax the problem, two more unknowns should be added, namely the components of the friction forces, normal to the direction of motion of the center of the ball.

In the case of *spatial approach of the problem*, the analysis of the number of unknowns shows that the law of motion must be found and under the assumption of pure rolling in both contacts, it is completely determined by the angle of rotation and by the components of the reactions in the contact points (five unknowns): the absolute value of the normal force, the components of the friction force on the direction of motion and on the normal to it, the spinning and rolling friction torques.

In the contact points C_1 and C_2 the following reactions occur, Figure 5:

- the *normal reactions* $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$, normal to the walls of the groove (channel);
- the *friction forces*, contained in the supporting planes, with the components $\boldsymbol{T}_1, \boldsymbol{T}_2$ normal to the instantaneous axis of rotation C_1C_2 and $\boldsymbol{T'}_1, \boldsymbol{T}_2$ which are normal to $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ respectively;
- the moments $\boldsymbol{M}_1, \boldsymbol{M}_2$ with the components $\boldsymbol{M}_{r1}, \boldsymbol{M}_{r2}$ (*rolling friction moments*) contained in the supporting planes and the components $\boldsymbol{M}_{s1}, \boldsymbol{M}_{s2}$ (*spinning moments*) normal to the supporting planes.

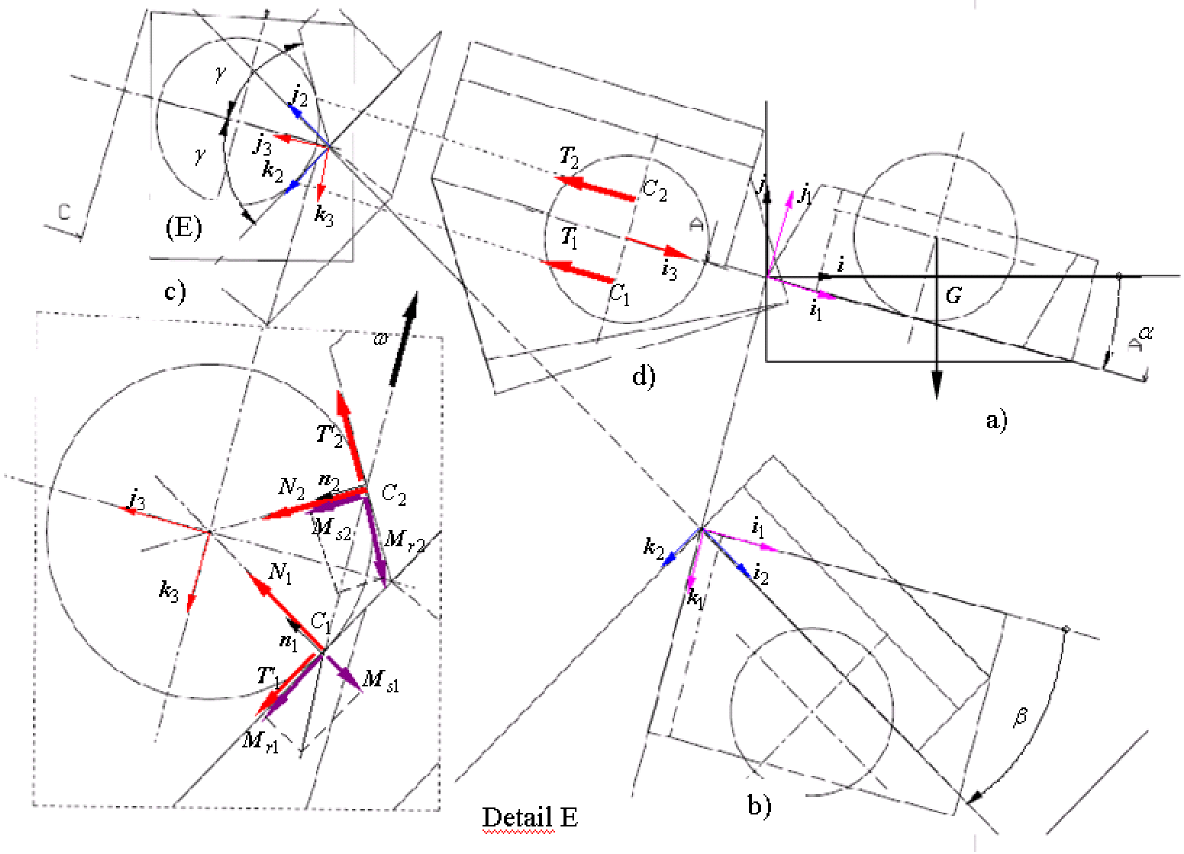

Figure 5. The forces and moments acting upon the ball in contact with the channel

An uncomplicated manner of applying the dynamics' theorems is the one which expresses the projections of the forces and moments in a coordinate system where most of these projections occur in natural size. The frame $\boldsymbol{i}_3\boldsymbol{j}_3\boldsymbol{k}_3$ is such a coordinate system, where the axis $\boldsymbol{i}_3$ is directed parallel to the common line of the supporting planes. The only force that doesn't occur in natural size in the $\boldsymbol{i}_3\boldsymbol{j}_3\boldsymbol{k}_3$ frame is the gravity force. In order to obtain the projections of this force on the axes of the system

$\boldsymbol{i}_3\boldsymbol{j}_3\boldsymbol{k}_3$, the methodology described by McCarthy is applied: the frame $\boldsymbol{i}\,\boldsymbol{j}\boldsymbol{k}$ is overlapped over the frame $\boldsymbol{i}_3\boldsymbol{j}_3\boldsymbol{k}_3$ performing a number of finite rotations about known axes.

a) $\boldsymbol{i}\,\boldsymbol{j}\boldsymbol{k}$ over $\boldsymbol{i}_1\,\boldsymbol{j}_1\boldsymbol{k}_1$, rotation about z with ($-\alpha$) angle:

$$\boldsymbol{R}_z = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

b) $\boldsymbol{i}_1\,\boldsymbol{j}_1\boldsymbol{k}_1$ over $\boldsymbol{i}_2\boldsymbol{j}_2k_2$, rotation about y with ($-\beta$) angle:

$$\boldsymbol{R}_y = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{2}$$

c) $\boldsymbol{i}_2\boldsymbol{j}_2\boldsymbol{k}_2$ over $\boldsymbol{i}_3\boldsymbol{j}_3\boldsymbol{k}_3$ Rotation about x with ($\pi/2-\gamma$) angle:

$$\boldsymbol{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \tag{3}$$

The $\boldsymbol{j}$ versor of the direction of gravity force $\boldsymbol{G}$ is obtained by computing the product of the three matrices in the specified succession and is expressed by the projections on the axes of the coordinate frame $\boldsymbol{i}_3\boldsymbol{j}_3\boldsymbol{k}_3$:

$$\boldsymbol{j} = -\boldsymbol{i}_3 \sin\alpha\cos\beta + \boldsymbol{j}_3(\cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma) + \boldsymbol{k}_3(-\cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma) \tag{4}$$

The theorem of momentum is written, in vector format:

$$M\boldsymbol{a} = \boldsymbol{G} + \boldsymbol{N}_1 + \boldsymbol{N}_2 + \boldsymbol{T}_1 + \boldsymbol{T}_2 + \boldsymbol{T}'_1 + \boldsymbol{T}'_2 \tag{5}$$

or by using the projections:

$$\begin{bmatrix} Ma \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -Mg\sin\alpha\cos\beta - T_1 - T_2 \\ Mg\cos\alpha\sin\gamma + Mg\sin\alpha\sin\beta\cos\gamma + N_1\sin\gamma + N_2\sin\gamma + T'_1\cos\gamma + T'_2\cos\gamma \\ -Mg\cos\alpha\cos\gamma + Mg\sin\alpha\sin\beta\sin\gamma - N_1\cos\gamma + N_2\cos\gamma + T'_1\sin\gamma - T'_2\sin\gamma \end{bmatrix} \begin{matrix} (6a) \\ (6b) \\ (6c) \end{matrix}$$

The moment of momentum theorem is applied:

$$\boldsymbol{J}_G\varepsilon = \overline{OC_1} \times (\boldsymbol{T}_1 + \boldsymbol{T}'_1) + \overline{OC_2} \times (\boldsymbol{T}_2 + \boldsymbol{T}'_2) + \boldsymbol{M}_{r1} + \boldsymbol{M}_{p1} + \boldsymbol{M}_{r2} + \boldsymbol{M}_{p2} \tag{7}$$

or expressed by the projections:

$$\begin{bmatrix} 0 \\ 0 \\ -J_z\varepsilon \end{bmatrix} = \begin{bmatrix} -RT'_1 + RT'_2 \\ -RT_1\cos\gamma + RT_2\cos\gamma + M_{r1}\cos\gamma - M_{p1}\sin\gamma - M_{r2}\cos\gamma + M_{p2}\sin\gamma \\ -RT_1\sin\gamma - RT_2\sin\gamma + M_{r1}\sin\gamma + M_{p1}\cos\gamma + M_{r2}\sin\gamma + M_{p2}\cos\gamma \end{bmatrix} \begin{matrix} (8a) \\ (8b) \\ (8c) \end{matrix}$$

The pure rolling condition requires a linear dependence between ε and a, under the form:

$$a = \varepsilon r. \tag{9}$$

where r is the distance from the centre of the ball to the straight line C_1C_2. The equations $1\div 9$ give a system of seven equations with eight unknowns, $a, \varepsilon, N_1, N_2, T_1, T_2, T'_1, T'_2$ $M_{r1}, M_{r2}, M_{p1}, M_{p2}$. The equation (8a) shows that:

$$T'_1 = T'_2 = T \tag{10}$$

have the same value. All the other unknowns will be expressed as function of the *T* parameter. For two different values of the T parameter, the motion of the system and the reactions will be the same. Now, writing the equations $1 \div 9$ for the two values of the T parameter, a system of 14 equations with 8 unknowns is obtained, which is generally an incompatible system.

4. Conclusions

The dynamical systems where dry friction occurs are difficult to model and contradictory results may come out, even for actually simple systems.

The motion of a ball in free falling on an inclined straight groove is analyzed.

The motion is a plan parallel one, experimentally shown, but the theorems of dynamics don't allow finding the reactions from the two contact points.

Considering spatial dynamics approach, two new reaction components must be considered and the system becomes indeterminate, all kinematical parameters and reactions depending on the same parameter. For two different values of the parameter, despite the fact that the motion state and the values of the reaction forces are always the same, an incompatible system is obtained and the aimed solutions cannot be found.

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