

Global existence of weak solutions for Navier–Stokes–BGK system

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Abstract

In this paper, we study the global well-posedness of a coupled system of kinetic and fluid equations. More precisely, we establish the global existence of weak solutions for Navier–Stokes–BGK system consisting of the BGK model of Boltzmann equation and incompressible Navier–Stokes equations coupled through a drag forcing term. This is achieved by combining weak compactness of the particle interaction operator based on Dunford–Pettis theorem, strong compactness of macroscopic fields of the kinetic part relied on velocity averaging lemma and a high order moment estimate, and strong compactness of the fluid part by Aubin–Lions lemma.

Keywords: Vlasov equation, BGK model, incompressible Navier–Stokes equations, spray models, global existence of weak solutions

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1. Introduction

In the modeling of a fluid-particle system where the particles are dispersed in a fluid flow, it is often the case that the motions of the particles are described at the kinetic level and the fluid is described at the macroscopic level, with the acceleration of the particles caused by the surrounding fluid and the acceleration of the fluid caused by the immersed particles given by the drag force terms. When the inter-particle interactions are not negligible such as in the case of polydisperse multiphase flows, crossing jets, a collision type operator that captures the interactions between the particles must be included in the kinetic equations [8, 25, 31, 40]. The mathematical modeling for the interactions between particles and fluid is classified by

O'Rourke [40] according to the volume fraction of the gas. In the current work, we are interested in the dynamics of particles called *moderately thick sprays* where the volume fraction of the gas is negligible, but collisions between particles are taken into account. More precisely, we address the existence of weak solutions for a particle–fluid system in which the BGK model of Boltzmann equation and the incompressible Navier–Stokes equations are coupled through a drag force [5, 8, 25, 31]:

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f + \nabla_v \cdot ((u - v)f) &= \mathcal{M}(f) - f, \\ \partial_t u + u \cdot \nabla_x u + \nabla_x p - \mu \Delta_x u &= - \int_{\mathbb{R}^3} (u - v)f \, dv, \\ \nabla_x \cdot u &= 0, \end{aligned} \tag{1.1}$$

subject to initial data

$$(f(x, v, 0), u(x, 0)) =: (f_0(x, v), u_0(x)).$$

Here $f(x, v, t)$ denotes the number density function on the phase point $(x, v) \in \mathbb{T}^3 \times \mathbb{R}^3$ at time $t \in \mathbb{R}_+$, and $u(x, t)$ and $p(x, t)$ are the fluid velocity and the hydrostatic pressure on $x \in \mathbb{T}^3$ at time $t \in \mathbb{R}_+$, respectively. μ is the kinematic viscosity of the fluid. The first two terms in the kinetic equation in (1.1) represent the free transport of dispersed particles in a fluid. The third term is the drag force which explains the acceleration of the immersed particles driven by the surrounding fluid, which also appears as an external force in the fluid equations taking into account the acceleration of the fluid caused by the immersed particles.

The local Maxwellian $\mathcal{M}(f)$ is defined by

$$\mathcal{M}(f)(x, v, t) = \frac{\rho_f(x, t)}{\sqrt{(2\pi T_f(x, t))^3}} \exp\left(-\frac{|v - U_f(x, t)|^2}{2T_f(x, t)}\right),$$

where the macroscopic fields of f : ρ_f , U_f , and T_f are defined by

$$\begin{aligned} \rho_f(x, t) &:= \int_{\mathbb{R}^3} f(x, v, t) \, dv, \\ \rho_f(x, t)U_f(x, t) &:= \int_{\mathbb{R}^3} vf(x, v, t) \, dv, \\ 3\rho_f(x, t)T_f(x, t) &:= \int_{\mathbb{R}^3} |v - U_f(x, t)|^2 f(x, v, t) \, dv, \end{aligned}$$

respectively. These relations give the following cancellation properties:

$$\int_{\mathbb{R}^3} \{\mathcal{M}(f) - f\} (1, v, |v|^2) \, dv = 0.$$

Note that this provides the conservation of mass, momentum and energy for the uncoupled BGK model. However, in our coupled model (1.1), this only leads to conservation of mass due to the presence of drag force terms.

The most general model to describe the dynamics of rarefied particles suspended in a fluid is the Navier–Stokes–Boltzmann system coupled through the drag force term. Due to various technical difficulties, however, the global-in-time existence of solutions for such model is currently not available. In this paper, we consider the case in which the interactions between the particles are described by the nonlinear relaxation operator of the BGK model. This is meaningful in the following two senses.

First, the BGK model is one of the most widely used model equation of the Boltzmann equation in physics and engineering. This is due to the qualitatively reliable results produced by the BGK model at much lower computational cost compared to that of the Boltzmann equation.

Secondly, even though existence theories for particle-fluid systems are well studied nowadays, most of the results dealing with the interactions between the suspended particles consider the linear interaction operators. To the best knowledge of the authors, our result seems to be the first result to consider the particle-fluid model with a nonlinear collision operator for particle interactions.

History 1: Navier–Stokes–Vlasov system: Recently, the study on particle-fluid system is gathering a lot of attentions due to their applications, for example, in the study of sedimentation phenomena, fuel injector in engines, and compressibility of droplets of the spray, etc [3, 8, 40, 44, 49, 51]. Along with that applicative interest, the mathematical analysis for various modelling is also emphasized. In the case when the direct particle-particle interactions are absent, there are a number of literature on the global existence of solutions; weak solutions for Vlasov or Vlasov–Fokker–Planck equation coupled with homogeneous/inhomogeneous fluids are studied in [9, 15, 27, 34, 50, 54], strong solutions near a global Maxwellian for Vlasov–Fokker–Planck equation coupled with incompressible/compressible Euler system are obtained in [10, 12, 23]. We also refer to [13, 14] for the large-time behavior of solutions and finite-time blow-up phenomena in kinetic-fluid systems. Despite those fruitful developments on the existence theory, to the best knowledge of the authors, global existence of solutions for kinetic-fluid models where collisions between the particles are taken into account has not been studied so far. It is worth mentioning that the local-in-time smooth solutions for the Vlasov–Boltzmann/compressible Euler equations are studied in [31] and the global existence of weak solutions of Vlasov/incompressible Navier–Stokes equations with a linear particle interaction operator taking care of the breakup phenomena is established in [4, 53]. More recently, the existence of strong solutions to the inhomogeneous Navier–Stokes–BGK system is also discussed in [17]. In [1, 16], Vlasov/Navier–Stokes system with a nonlinear particle interaction operator describing an asymptotic velocity alignment behavior is considered and the global existence of weak solutions is obtained.

History 2: BGK model: In spite of its important role as a fundamental model connecting the particle level description and the fluid level description of gaseous systems, the applications of the Boltzmann equation at the physical or engineering level have often been limited by the high numerical cost involved in the numerical computations of the collision operator. This is especially so if one is interested in dealing with specific flow problems. Looking for a model equation that shares important features of the Boltzmann equation, and therefore, successfully mimics the dynamics of the Boltzmann equation, Bhatnagar *et al*, and independently Walender, introduced a relaxation model of the Boltzmann equation, which is called the BGK model. Since then, the BGK model has seen a wide range of applications in engineering and physics due to its reliable results at much lower computational cost compared to that of the Boltzmann equation.

The mathematical study of the BGK model can be traced back to [36] where Perthame established the existence of weak solutions. Perthame and Pulvirenti later studied the existence of unique mild solution in a weighted L^∞ space [38]. These works were fruitfully extended to several directions: gases in the presence of external forces [52], plasma [60, 61], solutions in L^p spaces [59] and gas mixture problems [28]. The existence of classical solutions and its exponential stabilization near equilibrium are studied in [55]. The results on the stationary problems in a slab can be found in [48]. The ellisoidal extension of the BGK model recently

drew much attention [2, 56–58]. BGK model also saw various applications in the study of various macroscopic limits [22, 29, 32, 33, 41–43]. We omit the survey on the numerical computations related to the BGK model, interested readers may refer to [19, 20, 24, 35, 39, 45, 46].

1.1. Main result

Before we define our solution concept and state the main result, we define norms, function spaces and notational conventions.

- We denote by C a generic, not necessarily identical, positive constant. It may depend on final time T , but not on x .
- For functions $f(x, v), g(x)$, $\|f\|_{L^p}$ and $\|g\|_{L^p}$ denote the usual $L^p(\mathbb{T}^3 \times \mathbb{R}^3)$ -norm and $L^p(\mathbb{T}^3)$ -norm, respectively.
- $\|f\|_{L^p_q}$ represents a weighted L^∞ -norm:

$$\|f\|_{L^p_q} := \operatorname{ess\,sup}_{x,v} (1 + |v|^q) f(x, v).$$

- For any nonnegative integer s , H^s denotes the s th order L^2 Sobolev space.
- $\mathcal{C}^s([0, T]; E)$ is the set of s -times continuously differentiable functions from an interval $[0, T] \subset \mathbb{R}$ into a Banach space E , and $L^p(0, T; E)$ is the set of the L^p functions from an interval $(0, T)$ to a Banach space E .

In order to state our main theorem on the global existence of weak solutions to the system (1.1), we also introduce functions spaces as follows:

$$\mathcal{H} := \{w \in L^2(\mathbb{T}^3) : \nabla_x \cdot w = 0\} \quad \text{and} \quad \mathcal{V} := \{w \in H^1(\mathbb{T}^3) : \nabla_x \cdot w = 0\}.$$

We then define a notion of weak solutions to the system (1.1).

Definition 1.1. We say that (f, u) is a weak solution to the system (1.1) if the following conditions are satisfied:

- (i) $f \in L^\infty(0, T; (L^1_+ \cap L^\infty)(\mathbb{T}^3 \times \mathbb{R}^3))$,
- (ii) $u \in L^\infty(0, T; \mathcal{H}) \cap L^2(0, T; \mathcal{V}) \cap \mathcal{C}^0([0, T]; \mathcal{V}')$,
- (iii) For all $\phi \in \mathcal{C}^1_c(\mathbb{T}^3 \times \mathbb{R}^3 \times [0, T])$ with $\phi(x, v, T) = 0$,

$$\begin{aligned} & - \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_0 \phi_0 \, dx dv - \int_0^T \int_{\mathbb{T}^3 \times \mathbb{R}^3} f (\partial_t \phi + v \cdot \nabla_x \phi + (u - v) \cdot \nabla_v \phi) \, dx dv dt \\ & = \int_0^T \int_{\mathbb{T}^3 \times \mathbb{R}^3} (\mathcal{M}(f) - f) \phi \, dx dv dt, \end{aligned}$$

- (iv) For all $\psi \in \mathcal{C}^1_c(\mathbb{T}^3 \times [0, T])$ with $\nabla_x \cdot \psi = 0$ for almost all t ,

$$\begin{aligned} & - \int_{\mathbb{T}^3} u_0 \cdot \psi_0 \, dx + \int_{\mathbb{T}^3} u \cdot \psi \, dx - \int_0^T \int_{\mathbb{T}^3} u \cdot \partial_t \psi \, dx dt + \int_0^T \int_{\mathbb{T}^3} (u \cdot \nabla_x) u \cdot \psi \, dx dt \\ & = -\mu \int_0^T \int_{\mathbb{T}^3} \nabla_x u : \nabla_x \psi \, dx dt - \int_0^T \int_{\mathbb{T}^3 \times \mathbb{R}^3} f(u - v) \cdot \psi \, dx dv dt. \end{aligned}$$

Remark 1.2. (1) L^1_+ means the set of non-negative L^1 functions. (2) The pressure p is not contained in the definition since it vanishes when it is tested on the divergence free vectors.

We are now ready to state our main result:

Theorem 1.3. *Let $T > 0$. Suppose that the initial data (f_0, u_0) satisfy*

$$[f_0 \in L^\infty(\mathbb{T}^3 \times \mathbb{R}^3), \int_{\mathbb{T}^3 \times \mathbb{R}^3} (1 + |v|^2 + |\ln f_0|) f_0 dx dv < \infty, \text{ and } u_0 \in L^2(\mathbb{T}^3)].$$

Then there exists at least one weak solution to the system (1.1) in the sense of definition 1.1 satisfying the following estimates:

(i) *Velocity distribution function is uniformly bounded:*

$$\|f\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3 \times (0, T))} \leq C \|f_0\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}.$$

(ii) *Total energy is uniformly bounded:*

$$\begin{aligned} & \frac{1}{2} \left(\int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^2 f dx dv + \int_{\mathbb{T}^3} |u|^2 dx \right) + \mu \int_0^t \int_{\mathbb{T}^3} |\nabla_x u|^2 dx ds \\ & + \int_0^t \int_{\mathbb{T}^3 \times \mathbb{R}^3} |u - v|^2 f dx dv ds \leq C \left(\int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^2 f_0 dx dv + \int_{\mathbb{T}^3} |u_0|^2 dx \right). \end{aligned}$$

(iii) *Entropy is uniformly bounded:*

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} f |\ln f| dx dv + \int_0^T \int_{\mathbb{T}^3 \times \mathbb{R}^3} \{ \mathcal{M}(f) - f \} \ln f dx dv dt \leq C_{f_0, T}$$

for almost every $t \in (0, T)$.

One of the key elements in the proof is the derivation of the third moment estimate that remains uniformly bounded with respect to the mollification parameter ε . To derive the weak compactness of the local Maxwellian, we first need to obtain the compactness of the macroscopic fields. For the compactness of the local density and bulk velocity, the second moment estimate combined with the velocity averaging lemma is enough to derive the desired result. However, we need a moment estimate strictly higher than 2 to derive the compactness of the local temperature (see [36]). In view of this, we observe that the third moment of the regularized distribution function f_ε can be controlled by the kinetic energy of the suspended particles and a fluid-particle type estimate (See section 3 for the definitions of f_ε and η_ε):

$$\int_0^T \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon |v|^3 dx dv dt \leq C \left(\|(\eta_\varepsilon \star u_\varepsilon - v) f_\varepsilon (1 + |v|)\|_{L^1} + \int_0^T \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon |v|^2 dx dv dt \right),$$

for some $C > 0$ independent of ε , which in turn is bounded by L^5 norm of the fluid velocity.

For the existence of solutions to the fluid equations, a strong compactness is required to control the convection term. For this, we again need to have some uniform bounds for the local density and local moments together with the total energy estimates. This, combined with the smoothing effect from the viscosity enables us to use the Aubin–Lions lemma to have the strong compactness.

The outline of this paper is as follows: In section 2, we record several technical lemmas. In section 3, we set up a regularized approximate system for the Navier–Stokes–BGK model (1.1). Then, we prove the existence of the regularized model in section 4, and derive several key *a priori* estimates independent of the regularizing parameter in section 5. Section 6 is devoted to the proof of the main theorem.

2. Preliminaries: auxiliary lemmas

In this section, we record various technical lemmas that will be crucially used later. We first state the lower bound estimate of the local temperature, which is essential for the local Maxwellian to be well-defined.

Lemma 2.1 ([38, proposition 2.1]). *There exists a positive constant C_q , which depends only on q , such that*

$$\rho_f(x, t) \leq C_q \|f\|_{L_q^\infty} T_f^{3/2}(x, t) \quad (q > 3 \text{ or } q = 0).$$

We also need to control the growth of the local Maxwellian by that of the distribution functions:

Lemma 2.2 ([38, p 291]). *Suppose $\|f\|_{L_q^\infty} < \infty$ for $q > 5$. Then there exists a positive constant C_q , which depends only on q , such that*

$$\|\mathcal{M}(f)\|_{L_q^\infty} \leq C_q \|f\|_{L_q^\infty} \quad (q > 5 \text{ or } q = 0).$$

The next lemma says that, unlike the above estimate, the constant depends also on the final time and the lower bounds of macroscopic fields if we are to control the growth of derivatives either.

Lemma 2.3 ([56, proposition 4.1]). *Assume that f satisfies*

- (1) $\|f\|_{L_q^\infty} + \|\nabla_{x,v} f\|_{L_q^\infty} < C_1$,
- (2) $\rho_f + |U_f| + T_f < C_2$,
- (3) $\rho_f, T_f > C_3$,

for some constants $C_i > 0$ ($i = 1, 2, 3$). Then, we have

$$\|\mathcal{M}(f)\|_{L_q^\infty} + \|\nabla_{x,v} \mathcal{M}(f)\|_{L_q^\infty} \leq C_T \left\{ \|f\|_{L_q^\infty} + \|\nabla_{x,v} f\|_{L_q^\infty} \right\},$$

where $C_T > 0$ depends only on C_1, C_2, C_3 and the final time T .

The Lipschitz continuity of the local Maxwellian can be measured in the same weighted L_q^∞ space as follows:

Lemma 2.4 ([56, proposition 6.1]). *Assume f, g satisfy (h denotes either f or g)*

- (1) $\|h\|_{L_q^\infty} < C_1$,
- (2) $\rho_h + |U_h| + T_h < C_2$,
- (3) $\rho_h, T_h > C_3$,

for some constants $C_i > 0$ ($i = 1, 2, 3$). Then, we have

$$\|\mathcal{M}(f) - \mathcal{M}(g)\|_{L_q^\infty} \leq C_T \|f - g\|_{L_q^\infty},$$

where $C_T > 0$ depends only on C_1, C_2, C_3 and the final time T .

In the lemma below, we give an interpolation-type inequality for local moments of f . For this, we set ($k = 0, 1, 2, \dots$)

$$m_k f(x, t) := \int_{\mathbb{R}^3} |v|^k f(x, v, t) \, dv \quad \text{and} \quad M_k f(t) := \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^k f(x, v, t) \, dx dv.$$

Lemma 2.5 ([7, lemma 1]). *Let $\beta > 0$ and $g \in L^\infty_+(\mathbb{T}^3 \times \mathbb{R}^3 \times (0, T))$ with $m_\beta g(x, t) < \infty$ for almost every (x, t) . Then we have*

$$m_\alpha g(x, t) \leq \left(\frac{4\pi}{3} \|g(t)\|_{L^\infty} + 1 \right) (m_\beta g(x, t))^{\frac{\alpha+3}{\beta+3}} \quad \text{a.e. } (x, t),$$

for any $\alpha < \beta$.

We next state the velocity averaging lemma.

Lemma 2.6 ([9, lemma 3.2]). *For $1 \leq p < 5/4$, let $\{g^n\}_n$ be bounded in $L^p(\mathbb{T}^3 \times \mathbb{R}^3 \times (0, T))$. Suppose that f^n is bounded in $L^\infty(0, T; (L^1 \cap L^\infty)(\mathbb{T}^3 \times \mathbb{R}^3))$ and $|v|^2 f^n$ is bounded in $L^\infty(0, T; L^1(\mathbb{T}^3 \times \mathbb{R}^3))$. If f^n and g^n satisfy the equation*

$$\partial_t f^n + v \cdot \nabla_x f^n = \nabla_v^k g^n, \quad f^n|_{t=0} = f_0 \in L^p(\mathbb{T}^3 \times \mathbb{R}^3),$$

for a multi-index k . Then, for any $\psi(v)$, such that $|\psi(v)| \leq c|v|$ as $|v| \rightarrow \infty$, the sequence

$$\left\{ \int_{\mathbb{R}^3} f^n \psi(v) \, dv \right\}_n$$

is relatively compact in $L^p(\mathbb{T}^3 \times (0, T))$.

3. Global existence for a regularized system

In this section, we consider a regularized system of (1.1). As in [7], we regularize the fluid velocity in the drag forcing and convection terms, and apply a high-velocity cut-off to the drag force in the fluid part to relax some difficulties in the system (1.1). More precisely, let $\varepsilon > 0$ and η be a standard mollifier:

$$0 \leq \eta \in C_0^\infty(\mathbb{T}^3), \quad \text{supp}_x \eta \subseteq B(0, 1), \quad \int_{\mathbb{T}^3} \eta(x) \, dx = 1,$$

and we set a sequence of smooth mollifiers $\eta_\varepsilon(x) = (1/\varepsilon^3)\eta(x/\varepsilon)$. We also introduce a cut-off function $\gamma_\varepsilon \in C^\infty(\mathbb{R}^3)$:

$$\text{supp} \gamma_\varepsilon \subseteq B(0, 1/\varepsilon), \quad 0 \leq \gamma_\varepsilon \leq 1, \quad \gamma_\varepsilon = 1 \text{ on } B(0, 1/(2\varepsilon)), \quad \text{and} \quad \gamma_\varepsilon \rightarrow 1 \text{ as } \varepsilon \rightarrow 0.$$

Then the regularized system for the system (1.1) is defined as follows:

$$\begin{aligned} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \nabla_v \cdot ((\eta_\varepsilon \star u_\varepsilon - v) f_\varepsilon) &= \mathcal{M}(f_\varepsilon) - f_\varepsilon, \\ \partial_t u_\varepsilon + (\eta_\varepsilon \star u_\varepsilon) \cdot \nabla_x u_\varepsilon + \nabla_x p_\varepsilon - \mu \Delta_x u_\varepsilon &= - \int_{\mathbb{R}^3} \gamma_\varepsilon(v) (u_\varepsilon - v) f_\varepsilon \, dv, \quad (3.1) \\ \nabla_x \cdot u_\varepsilon &= 0, \end{aligned}$$

subject to regularized initial data:

$$(f_\varepsilon(x, v, 0), u_\varepsilon(x, 0)) =: (f_{0,\varepsilon}(x, v), u_{0,\varepsilon}(x)), \quad (x, v) \in \mathbb{T}^3 \times \mathbb{R}^3.$$

Here \star represents the convolution with respect to the spatial variable x . $u_{0,\varepsilon}$ is any C^∞ approximation of u_0 such that $u_{0,\varepsilon} \rightarrow u_0$ strongly in $L^2(\mathbb{T}^3)$ as $\varepsilon \rightarrow 0$, and $f_{0,\varepsilon}$ is defined by

$$f_{0,\varepsilon} = \eta \star \{f_0 1_{f_0 < 1/\varepsilon}\} + \varepsilon e^{-|v|^2}.$$

where 1_A denotes the characteristic function on A . Note that $f_{0,\varepsilon}$ satisfies $f_{0,\varepsilon} \rightarrow f_0$ strongly in $L^p(\mathbb{T}^3 \times \mathbb{R}^3)$ for all $p < \infty$ and weakly- \ast in $L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$, $M_2 f_{0,\varepsilon} \rightarrow M_2 f_0$ strongly in $L^\infty(\mathbb{T}^3)$ and uniformly bounded with respect to ε ,

In the following two sections, we prove the proposition below on the global-in-time existence of weak solutions and local-in-time uniform bound estimates of the regularized system (3.1).

Proposition 3.1.

- (1) For any $T > 0$ and $\varepsilon > 0$, there exists at least one weak solution $(f_\varepsilon, u_\varepsilon)$ of the regularized system (3.1) in the sense of definition 1.1.
- (2) Moreover, there exists a $T_\ast \in (0, T]$, which only depends on $T, \|u_0\|_{L^2} + M_2 f_0$, and $\|f_0\|_{L^\infty}$ such that

- Total energy estimate:

$$\sup_{0 \leq t \leq T_\ast} \left(\|u_\varepsilon(t)\|_{L^2}^2 + M_2 f_\varepsilon(t) + \int_0^t \|\nabla_x u_\varepsilon(s)\|_{L^2} ds \right) \leq C_1. \tag{3.2}$$

- Fluid-kinetic mixed estimate:

$$\|(\eta_\varepsilon \star u_\varepsilon - v)(1 + |v|)f_\varepsilon\|_{L^1} < C(f_0, u_0, T_\ast). \tag{3.3}$$

- Third moment and entropy estimate:

$$\|M_3 f_\varepsilon\|_{L^1(0, T_\ast)} + \sup_{0 \leq t \leq T_\ast} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon(t) |\ln f_\varepsilon(t)| dx dv \leq C(f_0, u_0, T_\ast).$$

Here, in particular, $C_1 > 0$ depends only on $T_\ast, T, \|u_0\|_{L^2} + M_2 f_0$, and $\|f_0\|_{L^\infty}$.

Since the proof is rather long, we divide the proof into two parts in sections 4 (Existence and Uniqueness) and 5 (Uniform-in- ε estimates) below.

4. Proof of proposition 3.1 (1): existence of $(f_\varepsilon, u_\varepsilon)$

We construct the solution $(f_\varepsilon, u_\varepsilon)$ to the regularized system (3.1) as a limit of the approximation sequence $(f_\varepsilon^n, u_\varepsilon^n)$ for the system (3.1) given by the following decoupled and linearized system:

$$\begin{aligned} \partial_t f_\varepsilon^{n+1} + v \cdot \nabla_x f_\varepsilon^{n+1} + \nabla_v \cdot ((\eta_\varepsilon \star u_\varepsilon^n - v)f_\varepsilon^{n+1}) &= \mathcal{M}(f_\varepsilon^n) - f_\varepsilon^{n+1}, \\ \partial_t u_\varepsilon^{n+1} + (\eta_\varepsilon \star u_\varepsilon^{n+1}) \cdot \nabla_x u_\varepsilon^{n+1} + \nabla_x p_\varepsilon^{n+1} - \mu \Delta_x u_\varepsilon^{n+1} &= - \int_{\mathbb{R}^3} \gamma_\varepsilon(v)(u_\varepsilon^n - v)f_\varepsilon^n dv, \\ \nabla_x \cdot u_\varepsilon^{n+1} &= 0, \end{aligned} \tag{4.1}$$

with the initial data and first iteration step:

$$(f_\varepsilon^n(x, v, t), u_\varepsilon^n(x, t))|_{t=0} = (f_{0,\varepsilon}(x, v), u_{0,\varepsilon}(x)) \quad \text{for all } n \geq 1,$$

and

$$(f_\varepsilon^0(x, v, t), u_\varepsilon^0(x, t)) = (f_{0,\varepsilon}(x, v), u_{0,\varepsilon}(x)), \quad (x, v, t) \in \mathbb{T}^3 \times \mathbb{R}^3 \times (0, T).$$

Before we consider (4.1), we consider the existence of characteristics:

Lemma 4.1. For $u \in L^\infty(0, T; L^2(\mathbb{T}^3))$ such that $\|u\|_{L^\infty(0,T;L^2)} < \infty$ and a fixed $\varepsilon > 0$, define the backward characteristic $Z_\varepsilon(s) := (X_\varepsilon(s), V_\varepsilon(s)) := (X_\varepsilon(s; t, x, v), V_\varepsilon(s; t, x, v))$ by

$$\begin{aligned} \frac{d}{ds} X_\varepsilon(s) &= V_\varepsilon(s), \\ \frac{d}{ds} V_\varepsilon(s) &= \eta_\varepsilon \star u(X_\varepsilon(s), s) - V_\varepsilon(s), \end{aligned} \tag{4.2}$$

with the terminal datum

$$X_\varepsilon(t) = x \quad \text{and} \quad V_\varepsilon(t) = v.$$

Then $Z_\varepsilon(s)$ is globally well-defined and satisfies

$$|Z_\varepsilon(s)| \leq C_{T,\varepsilon,u}(1 + |v|) \quad \text{and} \quad |\nabla_{x,v} Z_\varepsilon(s)| \leq C_{T,\varepsilon,u}, \tag{4.3}$$

for some positive constant $C_{T,\varepsilon,u} = C(T, \varepsilon, \|u\|_{L^\infty(0,T;L^2)})$.

Proof. The existence part is clear due to the regularization. For the estimate of (4.3), we rewrite (4.2) as

$$\begin{aligned} X_\varepsilon(s) &= x - \int_s^t V_\varepsilon(\tau) \, d\tau, \\ V_\varepsilon(s) &= e^{t-s} v - \int_s^t e^{\tau-s} (\eta_\varepsilon \star u)(X_\varepsilon(\tau), \tau) \, d\tau. \end{aligned} \tag{4.4}$$

A straightforward computation yields

$$|X_\varepsilon(s)| \leq |x| + \int_0^s |V_\varepsilon(\tau)| \, d\tau \leq C + \int_0^s |Z_\varepsilon(\tau)| \, d\tau$$

and

$$|V_\varepsilon(s)| \leq e^{t-s}|v| + \int_s^t e^{\tau-s} |(\eta_\varepsilon \star u)(X_\varepsilon(\tau), \tau)| \, d\tau \leq C_T|v| + \frac{C_T}{\varepsilon^{3/2}} \|\eta\|_{L^2} \|u\|_{L^2},$$

where we used

$$\|\eta_\varepsilon \star u\|_{L^\infty} \leq \|\eta_\varepsilon\|_{L^2} \|u\|_{L^2} \leq \frac{1}{\varepsilon^{3/2}} \|\eta\|_{L^2} \|u\|_{L^\infty(0,T;L^2)}.$$

Thus we obtain

$$|Z_\varepsilon(s)| \leq C_T|v| + C_{T,\varepsilon,u} + \int_0^s |Z_\varepsilon(\tau)| \, d\tau,$$

which gives

$$|Z_\varepsilon(s)| \leq C_{T,\varepsilon,u}(1 + |v|), \tag{4.5}$$

for some positive constant $C_{T,\varepsilon,u}$ depending on T, ε , and $\|u\|_{L^\infty(0,T;L^2)}$. Similarly, using

$$\|\nabla_x(\eta_\varepsilon \star u)\|_{L^\infty} \leq \|\nabla_x \eta_\varepsilon\|_{L^2} \|u\|_{L^2} \leq \frac{1}{\varepsilon^{5/2}} \|\nabla_x \eta\|_{L^2} \|u\|_{L^\infty(0,T;L^2)},$$

we get

$$\begin{aligned} |\nabla_{x,v} X_\varepsilon(s)| &\leq C + \int_0^s |\nabla_{x,v} V_\varepsilon(\tau)| \, d\tau, \\ |\nabla_{x,v} V_\varepsilon(s)| &\leq C_T + C_T \int_0^s |(\nabla_x \eta_\varepsilon) \star u(X_\varepsilon(\tau), \tau)| |\nabla_{x,v} Z_\varepsilon(\tau)| \, d\tau \\ &\leq C_T + \frac{C_T}{\varepsilon^{5/2}} \int_0^s \|\nabla_x \eta\|_{L^2} \|u\|_{L^2} |\nabla_{x,v} Z_\varepsilon(\tau)| \, d\tau. \end{aligned}$$

Thus we have

$$|\nabla_{x,v} Z_\varepsilon(s)| \leq C_T + C_{T,\varepsilon} \int_0^s |\nabla_{x,v} Z_\varepsilon(\tau)| \, d\tau,$$

which, from Gronwall’s inequality, yields

$$|\nabla_{x,v} Z_\varepsilon(s)| \leq C_{T,\varepsilon,u}.$$

Here, $C_{T,\varepsilon,u}$ is a positive constant depending on T, ε , and $\|u\|_{L^\infty(0,T;L^2)}$. □

We now state the results on existence and uniqueness of the regularized and decoupled system (4.1), and its uniform bound estimates in n in the proposition below.

Proposition 4.1. *Let $q > 5$. For any $T > 0$ and $n \in \mathbb{N}$, there exists a unique solution $(f_\varepsilon^n, u_\varepsilon^n)$ of the regularized and decoupled system (4.1) such that $f_\varepsilon^n \in L^\infty(0, T; L^q_q(\mathbb{T}^3 \times \mathbb{R}^3))$ and $u_\varepsilon^n \in (H^1(0, T; L^2(\mathbb{T}^3)) \cap L^2(0, T; H^1(\mathbb{T}^3)))$. Moreover, $(f_\varepsilon^n, u_\varepsilon^n)$ satisfies the following uniform-in- n estimates:*

- (i) $\|f_\varepsilon^n\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3 \times (0,T))} \leq C_1 \|f_{0,\varepsilon}\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}$,
- (ii) $\|u_\varepsilon^n\|_{L^\infty(0,T;L^2(\mathbb{T}^3)) \cap L^2(0,T;H^1(\mathbb{T}^3))} < C_{2,\varepsilon}$, $\|\partial_t u_\varepsilon^n\|_{L^2(\mathbb{T}^3 \times (0,T))} \leq C_{3,\varepsilon}$,
- (iii) $\|f_\varepsilon^n\|_{L^\infty(0,T;L^q_q(\mathbb{T}^3 \times \mathbb{R}^3))} + \|\nabla_{x,v} f_\varepsilon^n\|_{L^\infty(0,T;L^q_q(\mathbb{T}^3 \times \mathbb{R}^3))} \leq C_{4,\varepsilon}$,
- (iv) $\rho_{f_\varepsilon^n} + |U_{f_\varepsilon^n}| + T_{f_\varepsilon^n} < C_{5,\varepsilon}$, $\rho_{f_\varepsilon^n}, T_{f_\varepsilon^n} > C_{6,\varepsilon}$.

Here, $C_1 = C_1(T)$ depends only on T , whereas $C_{2,\varepsilon} = C_2(T, f_0, u_0, \varepsilon)$, $C_{3,\varepsilon} = C_3(T, f_0, u_0, \nabla u_0, \varepsilon)$ and $C_{i,\varepsilon} = C_i(T, f_0, \varepsilon)$ ($i = 4, 5, 6$).

Remark 4.2. The upper bound estimate of f_ε^n in $L^\infty(\mathbb{T}^3 \times \mathbb{R}^3 \times (0, T))$ does not depend on both ε and n .

Proof. We prove this proposition using induction. The case $n = 0$ is trivially satisfied. Assume that we have obtained $(f^n, u^n) \in L^\infty(\mathbb{T}^3 \times \mathbb{R}^3 \times (0, T)) \times L^\infty(0, T; L^2(\mathbb{T}^3))$ that satisfies all the statement of proposition 4.1.

- (1) **Existence and uniqueness of $(f_\varepsilon^{n+1}, u_\varepsilon^{n+1})$:** Under the assumption $(f^n, u^n) \in L^\infty(\mathbb{T}^3 \times \mathbb{R}^3 \times (0, T)) \times L^\infty(0, T; L^2(\mathbb{T}^3))$, (4.1)₁ can be seen as an inhomogeneous transport equation:

$$\partial_t f_\varepsilon^{n+1} + v \cdot \nabla_x f_\varepsilon^{n+1} + (\eta_\varepsilon \star u_\varepsilon^n - v) \cdot \nabla_v f_\varepsilon^{n+1} - 2f_\varepsilon^{n+1} = \mathcal{M}(f_\varepsilon^n). \tag{4.6}$$

Thus, in view of the uniform bound on $\mathcal{M}(f_\varepsilon^n)$ given by lemma 2.2, the existence follows straightforwardly once the well-posedness of the characteristic:

$$Z_\varepsilon^{n+1}(s) := (X_\varepsilon^{n+1}(s), V_\varepsilon^{n+1}(s)) := (X_\varepsilon^{n+1}(s; t, x, v), V_\varepsilon^{n+1}(s; t, x, v))$$

defined by

$$\begin{aligned} \frac{d}{ds} X_\varepsilon^{n+1}(s) &= V_\varepsilon^{n+1}(s), \quad 0 \leq s \leq T, \\ \frac{d}{ds} V_\varepsilon^{n+1}(s) &= \eta_\varepsilon \star u_\varepsilon^n(X_\varepsilon^{n+1}(s), s) - V_\varepsilon^{n+1}(s), \end{aligned} \tag{4.7}$$

with the terminal datum

$$X_\varepsilon^{n+1}(t) = x \quad \text{and} \quad V_\varepsilon^{n+1}(t) = v,$$

is verified, which is provided by lemma 4.1.

On the other hand the assumption $(f^n, u^n) \in L^\infty(\mathbb{T}^3 \times \mathbb{R}^3 \times (0, T)) \times L^\infty(0, T; L^2(\mathbb{T}^3))$ together with the high-velocity cut-off function $\gamma_\varepsilon(v)$ implies that the drag forcing term in the fluid part belongs to $L^2(\mathbb{T}^3 \times (0, T))$ at least. Thus, by a standard existence theory of incompressible Navier–Stokes equations with a mollified convection term [30], we can obtain the global-in-time existence and uniqueness of solution u_ε^{n+1} solving the fluid part in (4.1) with the regularity mentioned in proposition 4.1.

- (2) **Uniform bound estimates in n :** We now prove the uniform-in- n bounds in proposition 4.1.

- **Estimate of $\|f_\varepsilon^n(t)\|_{L^\infty}$:** Integrating (4.6) along the characteristic defined in (4.7), we get the mild form:

$$f_\varepsilon^{n+1}(x, v, t) = e^{2t} f_{0,\varepsilon}(Z_\varepsilon^{n+1}(0)) + \int_0^t e^{2(t-s)} \mathcal{M}(f_\varepsilon^n)(Z_\varepsilon^{n+1}(s), s) ds. \tag{4.8}$$

Then, lemma 2.2 gives

$$\begin{aligned} \|f_\varepsilon^{n+1}(t)\|_{L^\infty} &\leq \|f_{0,\varepsilon}\|_{L^\infty} e^{2T} + e^{2T} \int_0^t \|\mathcal{M}(f_\varepsilon^n)(s)\|_{L^\infty} ds \\ &\leq C_T \|f_{0,\varepsilon}\|_{L^\infty} + C_T \int_0^t \|f_\varepsilon^n(s)\|_{L^\infty} ds. \end{aligned}$$

Therefore, by Gronwall’s inequality, we have

$$\sup_{0 \leq t \leq T} \|f_\varepsilon^n(t)\|_{L^\infty} \leq C_T \|f_{0,\varepsilon}\|_{L^\infty} \quad \text{for } n \geq 1. \tag{4.9}$$

- **Estimate of $\|u_\varepsilon^n(t)\|_{L^\infty(0,T;L^2)}$ and $\|\partial_t u_\varepsilon^n(t)\|_{L^2(0,T;L^2)}$:** Multiplying (4.1) by u_ε^{n+1} and integrating it over \mathbb{T}^3 gives

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |u_\varepsilon^{n+1}|^2 dx + \mu \int_{\mathbb{T}^3} |\nabla u_\varepsilon^{n+1}|^2 dx = - \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\varepsilon(v)(u_\varepsilon^n - v) f_\varepsilon^n \cdot u_\varepsilon^{n+1} dx dv \tag{4.10}$$

due to

$$\int_{\mathbb{T}^3} \nabla_x p_\varepsilon u_\varepsilon^{n+1} dx = - \int_{\mathbb{T}^3} p_\varepsilon \nabla_x \cdot u_\varepsilon^{n+1} dx = 0,$$

and

$$\int_{\mathbb{T}^3} (\eta_\varepsilon \star u_\varepsilon^{n+1}) \cdot \nabla_x u_\varepsilon^{n+1} \cdot u_\varepsilon^{n+1} dx = 0.$$

On the other hand, the term on the right hand side of (4.10) can be estimated as

$$\begin{aligned} \left| \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\varepsilon(v)(u_\varepsilon^n - v) f_\varepsilon^n \cdot u_\varepsilon^{n+1} dx dv \right| &\leq C_\varepsilon \|f_\varepsilon^n\|_{L^\infty} (1 + \|u_\varepsilon^n\|_{L^2}^2 + \|u_\varepsilon^{n+1}\|_{L^2}^2) \\ &\leq C_{T,\varepsilon} (1 + \|u_\varepsilon^n\|_{L^2}^2 + \|u_\varepsilon^{n+1}\|_{L^2}^2), \end{aligned}$$

thanks to (4.9) and the cut-off function γ_ε . Thus we have

$$\frac{1}{2} \frac{d}{dt} \|u_\varepsilon^{n+1}\|_{L^2}^2 + \mu \|\nabla u_\varepsilon^{n+1}\|_{L^2}^2 \leq C_{T,\varepsilon} (1 + \|u_\varepsilon^n\|_{L^2}^2 + \|u_\varepsilon^{n+1}\|_{L^2}^2),$$

and this gives the uniform bound of u_ε^n in $L^\infty(0, T; L^2(\mathbb{T}^3)) \cap L^2(0, T; H^1(\mathbb{T}^3))$. Now we turn to the estimate of $\|\partial_t u_\varepsilon^n(t)\|_{L^2(0,T;L^2)}$. For this, we multiply (4.1) by $\partial_t u_\varepsilon^n(t)$, integrate over x , and use a similar argument as above to derive

$$\begin{aligned} \int_{\mathbb{T}^3} |\partial_t u_\varepsilon^{n+1}|^2 dx + \frac{\mu}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |\nabla_x u_\varepsilon^{n+1}|^2 dx &= - \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\varepsilon(v)(u_\varepsilon^n - v) f_\varepsilon^n \cdot \partial_t u_\varepsilon^{n+1} dx dv \\ &\leq C_\varepsilon \|\partial_t u_\varepsilon^{n+1}\|_{L^2} \\ &\leq C_\varepsilon + \frac{1}{2} \|\partial_t u_\varepsilon^{n+1}\|_{L^2}^2. \end{aligned}$$

Integrating the above inequality with respect to time, we obtain

$$\|\partial_t u_\varepsilon^{n+1}\|_{L^2(0,T;L^2)}^2 + \mu \|\nabla_x u_\varepsilon^{n+1}\|_{L^\infty(0,T;L^2)}^2 \leq C_\varepsilon T + \mu \|\nabla_x u_{0,\varepsilon}\|_{L^2}^2,$$

which gives $\|\partial_t u_\varepsilon^{n+1}\|_{L^2(0,T;L^2)} \leq C(\varepsilon)$.

- **Estimate of $\|f_\varepsilon^n\|_{L^\infty(0,T;L^q_\eta)} + \|\nabla_{x,v} f_\varepsilon^n\|_{L^\infty(0,T;L^q_\eta)}$:** Let us take $C_{T,\varepsilon} > 0$ such that

$$\frac{e^{2T}}{\varepsilon^{3/2}} \int_0^T \|\eta\|_{L^2} \|u_\varepsilon^n\|_{L^2} d\tau \leq C_{T,\varepsilon}.$$

Note that the constant above $C_{T,\varepsilon}$ does not depend on n due to the uniform bound estimate of u_ε^n in the previous part. Then it follows from (4.4) that

$$|V_\varepsilon^{n+1}(t)| \geq |v| - \frac{e^{2T}}{\varepsilon^{3/2}} \int_0^T \|\eta\|_{L^2} \|u_\varepsilon^n\|_{L^2} d\tau \geq |v| - C_{T,\varepsilon},$$

that is,

$$1 + C_{T,\varepsilon} + |V_\varepsilon^{n+1}(t)| \geq 1 + |v| \quad \text{for } n \geq 1 \quad \text{and} \quad 0 \leq t \leq T.$$

Using the above estimate, we find

$$\begin{aligned} f_{0,\varepsilon}(Z_\varepsilon^{n+1}(0)) &= f_{0,\varepsilon}(Z_\varepsilon^{n+1}(0))(1 + C_{T,\varepsilon} + |V_\varepsilon^{n+1}(0)|)^q(1 + C_{T,\varepsilon} + |V_\varepsilon^{n+1}(0)|)^{-q} \\ &\leq C_{T,\varepsilon,q} \|f_{0,\varepsilon}\|_{L_q^\infty} (1 + |v|)^{-q}, \end{aligned}$$

for $0 < q < \infty$. Similarly, with the aid of lemma 2.2, we estimate

$$\begin{aligned} \mathcal{M}(f_\varepsilon^n)(Z_\varepsilon^{n+1}(\tau), \tau) &\leq \mathcal{M}(f_\varepsilon^n)(Z_\varepsilon^{n+1}(\tau), \tau)(1 + C_{T,\varepsilon} + |V_\varepsilon^{n+1}(\tau)|)^q(1 + C_{T,\varepsilon} + |V_\varepsilon^{n+1}(\tau)|)^{-q} \\ &\leq C_{T,\varepsilon,q} \|\mathcal{M}(f_\varepsilon^n)\|_{L_q^\infty} (1 + |v|)^{-q} \\ &\leq C_{T,\varepsilon,q} \|f_\varepsilon^n\|_{L_q^\infty} (1 + |v|)^{-q}. \end{aligned}$$

Combining all the above estimate, we have

$$|f_\varepsilon^{n+1}(x, v, t)| \leq C_{T,\varepsilon,q} \|f_{0,\varepsilon}\|_{L_q^\infty} (1 + |v|)^{-q} + C_{T,\varepsilon,q} \int_0^t \|f_\varepsilon^n(s)\|_{L_q^\infty} (1 + |v|)^{-q} ds.$$

This readily gives

$$\|f_\varepsilon^{n+1}(t)\|_{L_q^\infty} \leq C_{T,\varepsilon,q} \|f_{0,\varepsilon}\|_{L_q^\infty} + C_{T,\varepsilon,q} \int_0^t \|f_\varepsilon^n(s)\|_{L_q^\infty} ds. \tag{4.11}$$

We next estimate the first-order derivative for f_ε^{n+1} . Note that the estimate in lemma 4.1 is now uniform in n due to the uniform bound estimate of u_ε^n in $L^\infty(0, T; L^2(\mathbb{T}^3))$. This and using the similar argument as the above yield

$$\begin{aligned} |\nabla_{x,v} f_\varepsilon^{n+1}(x, v, t)| &\leq e^{2t} |\nabla_{x,v} f_{0,\varepsilon}(Z_\varepsilon^{n+1}(0))| |\nabla_{x,v} Z_\varepsilon^{n+1}(0)| \\ &+ \int_0^t e^{2(t-s)} |\nabla_{x,v} \mathcal{M}(f_\varepsilon^n)(Z_\varepsilon^{n+1}(s), s)| |\nabla_{x,v} Z_\varepsilon^{n+1}(s)| ds \\ &\leq C_{T,\varepsilon} \|\nabla_{x,v} f_{0,\varepsilon}\|_{L_q^\infty} (1 + |v|)^{-q} + C_{T,\varepsilon} \int_0^t \|\nabla_{x,v} \mathcal{M}(f_\varepsilon^n)\|_{L_q^\infty} (1 + |v|)^{-q} ds \\ &\leq C_{T,\varepsilon} \|\nabla_{x,v} f_{0,\varepsilon}\|_{L_q^\infty} (1 + |v|)^{-q} + C_{T,\varepsilon} \int_0^t \left(\|f_\varepsilon^n\|_{L_q^\infty} + \|\nabla_{x,v} f_\varepsilon^n\|_{L_q^\infty} \right) (1 + |v|)^{-q} ds. \end{aligned}$$

Hence we obtain

$$\|\nabla_{x,v} f_\varepsilon^{n+1}\|_{L_q^\infty} \leq C_{T,\varepsilon} \|\nabla_{x,v} f_{0,\varepsilon}\|_{L_q^\infty} + C_{T,\varepsilon} \int_0^t \left(\|f_\varepsilon^n\|_{L_q^\infty} + \|\nabla_{x,v} f_\varepsilon^n\|_{L_q^\infty} \right) ds. \tag{4.12}$$

Combining (4.11) and (4.12), we have

$$\begin{aligned} \|f_\varepsilon^{n+1}(t)\|_{L_q^\infty} + \|\nabla_{x,v} f_\varepsilon^{n+1}(t)\|_{L_q^\infty} &\leq C_{T,\varepsilon} \left(\|f_{0,\varepsilon}\|_{L_q^\infty} + \|\nabla_{x,v} f_{0,\varepsilon}\|_{L_q^\infty} \right) + C_{T,\varepsilon} \int_0^t \left(\|f_\varepsilon^n\|_{L_q^\infty} + \|\nabla_{x,v} f_\varepsilon^n\|_{L_q^\infty} \right) ds, \end{aligned}$$

which yields the desired result.

- **Estimates of macroscopic fields of f_ε^n :** We show that macroscopic fields of f satisfy $\rho_{f_\varepsilon^n} + |U_{f_\varepsilon^n}| + T_{f_\varepsilon^n} < C_{T,\varepsilon}$ and $\rho_{f_\varepsilon^n}, T_{f_\varepsilon^n} > C_{T,\varepsilon}$ for some positive constant $C_{T,\varepsilon}$. For this, we take into account the integration of (4.8) and recall how we regularized f_0 to see

$$\begin{aligned} \int_{\mathbb{R}^3} f_\varepsilon^n(x, v, t) \, dv &\geq e^{2t} \int_{\mathbb{R}^3} f_0(Z_\varepsilon^n(0)) \, dv \geq \int_{\mathbb{R}^3} \varepsilon e^{-|v_\varepsilon^n(0)|^2} \, dv \\ &\geq \int_{\mathbb{R}^3} \varepsilon e^{-C_{T,\varepsilon}(1+|v|)^2} \, dv \geq C_{T,\varepsilon}, \end{aligned} \tag{4.13}$$

where we used (4.5) together with the uniform estimate of u_ε^n . This gives the lower bound for $\rho_{f_\varepsilon^n}$. Then, the lower bound for $T_{f_\varepsilon^n}$ follows directly from lemma 2.1. The upper bounds can be obtained by exactly the same manner as in [38]. \square

4.1. Proof of proposition 3.1 (1)

We are now ready to prove the existence and uniqueness of $(f_\varepsilon, u_\varepsilon)$ stated in proposition 3.1. (1). We split the proof into five steps as follows.

Step A.- Cauchy estimate for f^n : It follows from (4.8) that

$$\begin{aligned} f_\varepsilon^{n+1}(x, v, t) - f_\varepsilon^n(x, v, t) &= \int_0^t e^{2(t-s)} (\mathcal{M}(f_\varepsilon^n)(Z_\varepsilon^{n+1}(s), s) - \mathcal{M}(f_\varepsilon^{n-1})(Z_\varepsilon^n(s), s)) \, ds \\ &= \int_0^t e^{2(t-s)} (\mathcal{M}(f_\varepsilon^n)(Z_\varepsilon^{n+1}(s), s) - \mathcal{M}(f_\varepsilon^n)(Z_\varepsilon^n(s), s)) \, ds \\ &\quad + \int_0^t e^{2(t-s)} (\mathcal{M}(f_\varepsilon^n)(Z_\varepsilon^n(s), s) - \mathcal{M}(f_\varepsilon^{n-1})(Z_\varepsilon^n(s), s)) \, ds \\ &=: I_1 + I_2, \end{aligned}$$

I_1 can be estimate as follows.

$$\begin{aligned} I_1 &= \int_0^t e^{2(t-s)} \nabla_{x,v} \mathcal{M}(f_\varepsilon^n) (\alpha Z_\varepsilon^{n+1}(s) + (1 - \alpha)Z_\varepsilon^n(s), s) \cdot (Z_\varepsilon^{n+1}(s) - Z_\varepsilon^n(s)) \, ds \\ &\leq C_{T,\varepsilon} \int_0^t \|\nabla_{x,v} \mathcal{M}(f_\varepsilon^n)\|_{L_q^\infty} |Z_\varepsilon^{n+1}(s) - Z_\varepsilon^n(s)| \, ds (1 + |v|)^{-q} \\ &\leq C_{T,\varepsilon} (1 + |v|)^{-q} \int_0^t (\|f_\varepsilon^n(s)\|_{L_q^\infty} + \|\nabla_{x,v} f_\varepsilon^n(s)\|_{L_q^\infty}) |Z_\varepsilon^{n+1}(s) - Z_\varepsilon^n(s)| \, ds. \end{aligned}$$

For I_2 , we define $\rho_\varepsilon^n, U_\varepsilon^n, T_\varepsilon^n$ to be the macroscopic fields constructed from f_ε^n , and recall from (4.11) and (4.13) that

$$\|f_\varepsilon^n\|_{L_q^\infty} \leq C_{T,\varepsilon} \quad \text{and} \quad \rho_\varepsilon^n > C_{T,\varepsilon}. \tag{4.14}$$

These estimates, together with lemma 2.1 gives the lower bound of T_ε^n independent of n :

$$T_\varepsilon^n \geq C_{T,\varepsilon}. \tag{4.15}$$

We can also derive from (4.14) the upper bounds for the regularized macroscopic fields:

$$\begin{aligned}
 \rho_\varepsilon^n &= \int_{\mathbb{R}^3} f_\varepsilon^n \, dv \leq C \|f_\varepsilon^n\|_{L^q_\nu} \leq C_{T,\varepsilon}, \\
 |U_\varepsilon^n| &= \frac{1}{\rho_\varepsilon^n} \left| \int_{\mathbb{R}^3} f_\varepsilon^n v \, dv \right| \leq \frac{C}{\rho_\varepsilon^n} \|f_\varepsilon^n\|_{L^q_\nu} \leq C_{T,\varepsilon}, \\
 T_\varepsilon^n &= \frac{1}{\rho_\varepsilon^n} \int_{\mathbb{R}^3} f_\varepsilon^n |v|^2 \, dv - \frac{1}{\rho_\varepsilon^n} \left| \int_{\mathbb{R}^3} f_\varepsilon^n v \, dv \right|^2 \leq \frac{C}{\rho_\varepsilon^n} \|f_\varepsilon^n\|_{L^q_\nu} + \frac{C}{\rho_\varepsilon^n} \|f_\varepsilon^n\|_{L^q_\nu}^2 \leq C_{T,\varepsilon}.
 \end{aligned}
 \tag{4.16}$$

Estimates (4.14), (4.15), (4.16) show that f_ε^n and its macroscopic fields $(\rho_\varepsilon^n, U_\varepsilon^n, T_\varepsilon^n)$ satisfy the assumptions of lemma 2.4. Therefore, we have from lemma 2.4:

$$I_2 \leq C_{T,\varepsilon} (1 + |v|)^{-q} \int_0^t \|(f_\varepsilon^n - f_\varepsilon^{n-1})(s)\|_{L^q_\nu} \, ds.
 \tag{4.17}$$

Here we used lemma 2.4 and the similar argument as in the proof of proposition 4.1. This yields

$$\|(f_\varepsilon^{n+1} - f_\varepsilon^n)(t)\|_{L^q_\nu} \leq C_{T,\varepsilon} \int_0^t \|(f_\varepsilon^n - f_\varepsilon^{n-1})(s)\|_{L^q_\nu} \, ds + C_{T,\varepsilon} \int_0^t |Z_\varepsilon^{n+1}(s) - Z_\varepsilon^n(s)| \, ds.$$

Step B.- Cauchy estimate for the characteristic Z_ε^{n+1} : We first find from (4.7) that

$$|X_\varepsilon^{n+1}(s) - X_\varepsilon^n(s)| \leq \int_s^t |V_\varepsilon^{n+1}(\tau) - V_\varepsilon^n(\tau)| \, d\tau.$$

We next estimate the characteristic for velocity as

$$\begin{aligned}
 |V_\varepsilon^{n+1}(s) - V_\varepsilon^n(s)| &\leq \int_s^t e^{\tau-s} |\eta_\varepsilon \star u_\varepsilon^n(X_\varepsilon^{n+1}(\tau), \tau) - \eta_\varepsilon \star u_\varepsilon^n(X_\varepsilon^n(\tau), \tau)| \, d\tau \\
 &\quad + \int_s^t e^{\tau-s} |\eta_\varepsilon \star u_\varepsilon^n(X_\varepsilon^n(\tau), \tau) - \eta_\varepsilon \star u_\varepsilon^{n-1}(X_\varepsilon^n(\tau), \tau)| \, d\tau \\
 &\leq C_{T,\varepsilon} \int_s^t \|\nabla \eta_\varepsilon\|_{L^2} \|u_\varepsilon^n\|_{L^2} |X_\varepsilon^{n+1}(\tau) - X_\varepsilon^n(\tau)| \, d\tau \\
 &\quad + C_T \int_0^T \|\eta_\varepsilon\|_{L^2} \|(u_\varepsilon^n - u_\varepsilon^{n-1})(\tau)\|_{L^2} \, d\tau \\
 &\leq C_{T,\varepsilon} \int_0^T |X_\varepsilon^{n+1}(\tau) - X_\varepsilon^n(\tau)| + \|(u_\varepsilon^n - u_\varepsilon^{n-1})(\tau)\|_{L^2} \, d\tau,
 \end{aligned}$$

where we used the uniform bound estimate of $\|u_\varepsilon^n\|_{L^\infty(0,T;L^2)}$ in n . Thus we have

$$|Z_\varepsilon^{n+1}(s) - Z_\varepsilon^n(s)| \leq C_{T,\varepsilon} \int_0^T |Z_\varepsilon^{n+1}(\tau) - Z_\varepsilon^n(\tau)| \, d\tau + C_{T,\varepsilon} \int_0^T \|(u_\varepsilon^n - u_\varepsilon^{n-1})(\tau)\|_{L^2} \, d\tau.$$

Step C.- Cauchy estimate for the fluid velocity u^n : For notational simplicity, we set $w_\varepsilon^{n+1} := u_\varepsilon^{n+1} - u_\varepsilon^n$. Then it follows from (4.1)₂ that w_ε^{n+1} satisfies

$$\begin{aligned}
 \partial_t w_\varepsilon^{n+1} + (\eta_\varepsilon \star w_\varepsilon^{n+1}) \cdot \nabla_x u_\varepsilon^{n+1} + (\eta_\varepsilon \star u_\varepsilon^n) \cdot \nabla_x w_\varepsilon^{n+1} + \nabla_x (p_\varepsilon^{n+1} - p_\varepsilon^n) - \mu \Delta_x w_\varepsilon^{n+1} \\
 = - \int_{\mathbb{R}^3} \gamma_\varepsilon(v) w_\varepsilon^n f_\varepsilon^n \, dv - \int_{\mathbb{R}^3} \gamma_\varepsilon(v) (u_\varepsilon^{n-1} - v) (f_\varepsilon^n - f_\varepsilon^{n-1}) \, dv
 \end{aligned}
 \tag{4.18}$$

and $\nabla_x \cdot w_\varepsilon^{n+1} = 0$. Multiplying (4.18) by w_ε^{n+1} and integrating it over \mathbb{T}^3 gives

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|w_\varepsilon^{n+1}\|_{L^2}^2 + \mu \|\nabla_x w_\varepsilon^{n+1}\|_{L^2}^2 \\ &= - \int_{\mathbb{T}^3} (\eta_\varepsilon \star w_\varepsilon^{n+1}) \cdot \nabla_x u_\varepsilon^{n+1} \cdot w_\varepsilon^{n+1} \, dx - \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\varepsilon(v) w_\varepsilon^n \cdot w_\varepsilon^{n+1} f_\varepsilon^n \, dx dv \\ & \quad - \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\varepsilon(v) (u_\varepsilon^{n-1} - v) \cdot w_\varepsilon^{n+1} (f_\varepsilon^n - f_\varepsilon^{n-1}) \, dx dv \\ &=: J_1 + J_2 + J_3, \end{aligned}$$

thanks to

$$\int_{\mathbb{T}^3} (\eta_\varepsilon \star u_\varepsilon^n) \cdot \nabla_x w_\varepsilon^{n+1} \cdot w_\varepsilon^{n+1} \, dx = 0.$$

We then estimate $J_i (i = 1, 2, 3)$ as

$$\begin{aligned} J_1 &= \int_{\mathbb{T}^3} (\nabla_x \eta_\varepsilon \star w_\varepsilon^{n+1}) \cdot w_\varepsilon^{n+1} \cdot u_\varepsilon^{n+1} \, dx + \int_{\mathbb{T}^3} (\eta_\varepsilon \star w_\varepsilon^{n+1}) \cdot \nabla_x w_\varepsilon^{n+1} \cdot u_\varepsilon^{n+1} \, dx \\ &\leq C_\varepsilon \|u_\varepsilon^{n+1}\|_{L^2} \|w_\varepsilon^{n+1}\|_{L^2}^2 + C_\varepsilon \|u_\varepsilon^{n+1}\|_{L^2} \|w_\varepsilon^{n+1}\|_{L^2} \|\nabla_x w_\varepsilon^{n+1}\|_{L^2}, \\ J_2 &\leq C_\varepsilon \|f_\varepsilon^n\|_{L^\infty} \|w_\varepsilon^n\|_{L^2} \|w_\varepsilon^{n+1}\|_{L^2}, \\ J_3 &\leq C_\varepsilon (1 + \|u_\varepsilon^{n-1}\|_{L^2}) \|w_\varepsilon^{n+1}\|_{L^2} \|f_\varepsilon^n - f_\varepsilon^{n-1}\|_{L^\infty}. \end{aligned}$$

Hence, together with the uniform bound estimate of $(f_\varepsilon^n, u_\varepsilon^n)$ in n and Young’s inequality, we get

$$\begin{aligned} J_1 + J_2 + J_3 &\leq C_\varepsilon \|w_\varepsilon^{n+1}\|_{L^2}^2 + C_\varepsilon \|w_\varepsilon^{n+1}\|_{L^2} \|\nabla_x w_\varepsilon^{n+1}\|_{L^2} \\ &\quad + C_\varepsilon \|w_\varepsilon^n\|_{L^2} \|w_\varepsilon^{n+1}\|_{L^2} + C_\varepsilon \|w_\varepsilon^{n+1}\|_{L^2} \|f_\varepsilon^n - f_\varepsilon^{n-1}\|_{L^\infty} \\ &\leq C_\varepsilon (\|w_\varepsilon^n\|_{L^2}^2 + \|w_\varepsilon^{n+1}\|_{L^2}^2 + \|f_\varepsilon^n - f_\varepsilon^{n-1}\|_{L^\infty}^2), \end{aligned}$$

so that

$$\frac{d}{dt} \|w_\varepsilon^{n+1}\|_{L^2}^2 + \mu \|\nabla_x w_\varepsilon^{n+1}\|_{L^2}^2 \leq C_\varepsilon (\|w_\varepsilon^n\|_{L^2}^2 + \|w_\varepsilon^{n+1}\|_{L^2}^2 + \|f_\varepsilon^n - f_\varepsilon^{n-1}\|_{L^\infty}^2).$$

Step D.- Cauchy estimate for $(f_\varepsilon^n, u_\varepsilon^n, Z_\varepsilon^n)_{n \in \mathbb{N}}$: Combining the estimates in previous steps, we have for all $0 \leq t < T$

$$\begin{aligned} & \|f_\varepsilon^{n+1}(t) - f_\varepsilon^n(t)\|_{L_q^\infty} + \|Z_\varepsilon^{n+1}(t) - Z_\varepsilon^n(t)\|_{L^\infty} + \|u_\varepsilon^{n+1}(t) - u_\varepsilon^n(t)\|_{L^2} \\ & \leq C_{T,\varepsilon} \int_0^t \|f_\varepsilon^n(\tau) - f_\varepsilon^{n-1}(\tau)\|_{L_q^\infty} + \|Z_\varepsilon^{n+1}(\tau) - Z_\varepsilon^n(\tau)\|_{L^\infty} + \|u_\varepsilon^n(\tau) - u_\varepsilon^{n-1}(\tau)\|_{L^2} \, d\tau, \end{aligned}$$

from which we can conclude that $(f_\varepsilon^n, u_\varepsilon^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^\infty(0, T; L_q^\infty(\mathbb{T}^3 \times \mathbb{R}^3)) \times L^\infty(0, T; L^2(\mathbb{T}^3))$. Therefore, for a fixed $\varepsilon > 0$, there exist limiting functions $f_\varepsilon, u_\varepsilon, Z_\varepsilon$ such that

$$\sup_{0 \leq t \leq T} \left(\|f_\varepsilon^n(t) - f_\varepsilon(t)\|_{L_q^\infty} + \|Z_\varepsilon^n(t) - Z_\varepsilon(t)\|_{L^\infty} + \|u_\varepsilon^n(t) - u_\varepsilon(t)\|_{L^2} \right) \rightarrow 0 \tag{4.19}$$

as $n \rightarrow \infty$.

Step E.- $(f_\varepsilon, u_\varepsilon, Z_\varepsilon)_{n \in \mathbb{N}}$ solve the regularized system (3.1): Now we will show that

$$\|\mathcal{M}(f_\varepsilon^n) - \mathcal{M}(f_\varepsilon)\|_{L^q_\infty} \rightarrow 0,$$

which, combined with the standard argument as in [4], leads to the conclusion that $(f_\varepsilon, Z_\varepsilon, u_\varepsilon)$ solve the regularized system (3.1).

For this, we note from (4.19) the assumption $q > 5$ that, for $\phi(v) = 1, v, |v|^2$

$$\begin{aligned} \left| \int_{\mathbb{R}^3} f_\varepsilon^n \phi(v) \, dv - \int_{\mathbb{R}^3} f_\varepsilon \phi(v) \, dv \right| &\leq \int_{\mathbb{R}^3} |f_\varepsilon^n - f_\varepsilon| |\phi(v)| \, dv \\ &\leq \|f_\varepsilon^n - f_\varepsilon\|_{L^q_\infty} \int_{\mathbb{R}^3} \frac{|\phi(v)|}{(1 + |v|)^q} \, dv \\ &\leq C \|f_\varepsilon^n - f_\varepsilon\|_{L^q_\infty} \rightarrow 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \rho_\varepsilon^n &\rightarrow \rho_\varepsilon, \\ \rho_\varepsilon^n U_\varepsilon^n &\rightarrow \rho_\varepsilon U_\varepsilon, \\ \rho_\varepsilon^n |U_\varepsilon^n|^2 + 3\rho_\varepsilon^n T_\varepsilon^n &\rightarrow \rho_\varepsilon |U_\varepsilon|^2 + 3\rho_\varepsilon T_\varepsilon \end{aligned}$$

uniformly in x and t . Here $(\rho_\varepsilon, U_\varepsilon, T_\varepsilon)$ represent the macroscopic fields constructed from f_ε . Then, since we have $\rho_\varepsilon^n > C_{T,\varepsilon}$ from (4.13), this yields

$$\rho_\varepsilon^n \rightarrow \rho_\varepsilon, U_\varepsilon^n \rightarrow U_\varepsilon, T_\varepsilon^n \rightarrow T_\varepsilon \quad \text{uniformly in } x, t.$$

Now, recall that we proved in **Step A** that f_ε^n and its macroscopic fields $(\rho_\varepsilon^n, U_\varepsilon^n, T_\varepsilon^n)$ satisfy the assumptions of lemma 2.4. Thus, the convergence of f_ε^n in $\|\cdot\|_{L^q_\infty}$ and the uniform convergence of $(\rho_\varepsilon^n, U_\varepsilon^n, T_\varepsilon^n)$ to $(\rho_\varepsilon, U_\varepsilon, T_\varepsilon)$ imply that f_ε and $(\rho_\varepsilon, U_\varepsilon, T_\varepsilon)$ also satisfy the assumptions of lemma 2.4. Therefore, we conclude from lemma 2.4 and (4.19) that

$$\|\mathcal{M}(f_\varepsilon^n) - \mathcal{M}(f_\varepsilon)\|_{L^q_\infty} \leq C_{T,\varepsilon} \|f_\varepsilon^n - f_\varepsilon\|_{L^q_\infty} \rightarrow 0.$$

This completes the proof.

5. Proof of proposition 3.1. (2): uniform-in- ε estimates on $(f_\varepsilon, u_\varepsilon)$

In this section, we establish several uniform-in- ε estimates for $(f_\varepsilon, u_\varepsilon)$ given in proposition 3.1. (2). For notational simplicity, we drop the subscript f in $\rho_{f_\varepsilon}, U_{f_\varepsilon}$, and T_{f_ε} when there is no confusion, i.e. we denote by $\rho_\varepsilon := \rho_{f_\varepsilon}, U_\varepsilon := U_{f_\varepsilon}$, and $T_\varepsilon := T_{f_\varepsilon}$.

- **Uniform bounds of the total energy:** A straightforward computation yields from (3.1)₁ that

$$\frac{d}{dt} M_2 f_\varepsilon + 2M_2 f_\varepsilon \leq 2 \int_{\mathbb{T}^3} |\eta_\varepsilon \star u_\varepsilon(x, t)| m_1 f_\varepsilon \, dx.$$

This, together with lemma 2.5; $m_1 f_\varepsilon \leq C(m_2 f_\varepsilon)^{4/5}$, Minkowski's inequality; $\|\eta_\varepsilon \star u_\varepsilon(t)\|_{L^5} \leq C\|u_\varepsilon(t)\|_{L^5}$, the uniform bound estimate of $\|f_\varepsilon\|_{L^\infty}$ in proposition 4.1, (see also Remark 4.2), and Hölder inequality gives

$$\frac{d}{dt} M_2 f_\varepsilon + 2M_2 f_\varepsilon \leq \|\eta_\varepsilon \star u_\varepsilon(t)\|_{L^5} \|m_1 f_\varepsilon(t)\|_{L^{5/4}} \leq C\|u_\varepsilon(t)\|_{L^5} (M_2 f_\varepsilon)^{4/5}. \quad (5.1)$$

Applying Gronwall’s inequality, we obtain

$$M_2 f_\varepsilon(t) \leq C \left((M_2 f_{0,\varepsilon})^{1/5} + \int_0^t \|u_\varepsilon(s)\|_{L^5} ds \right)^5 \leq C \left(1 + \int_0^t \|u_\varepsilon(s)\|_{L^5} ds \right)^5, \tag{5.2}$$

due to $M_2 f_{0,\varepsilon} \leq CM_2 f_0$, where $C > 0$ is independent of ε . We next turn to the uniform estimate of the fluid velocity. For this, we multiply (3.1)₂ by u_ε , integrate over x to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_\varepsilon\|_{L^2}^2 + \mu \|\nabla_x u_\varepsilon\|_{L^2}^2 &= - \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon u_\varepsilon \cdot (u_\varepsilon - v) \gamma_\varepsilon(v) dx dv \\ &= - \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon |u_\varepsilon|^2 \gamma_\varepsilon(v) dx dv + \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon u_\varepsilon \cdot v \gamma_\varepsilon(v) dx dv \\ &\leq \int_{\mathbb{T}^3} |u_\varepsilon| m_1 f_\varepsilon dx. \end{aligned}$$

Then, by using the argument in (5.1) and (5.2), we can bound the last term as

$$\begin{aligned} \int_{\mathbb{T}^3} |u_\varepsilon| m_1 f_\varepsilon dx &\leq \|u_\varepsilon\|_{L^5} \|m_1 f_\varepsilon\|_{L^{5/4}} \\ &\leq C \|u_\varepsilon(t)\|_{L^5} (M_2 f_\varepsilon)^{4/5} \\ &\leq C \|u_\varepsilon\|_{L^5} \left(1 + \int_0^t \|u_\varepsilon(s)\|_{L^5} ds \right)^4 \\ &\leq C \|u_\varepsilon\|_{H^1} \left(1 + \left(\int_0^t \|u_\varepsilon(s)\|_{H^1}^2 ds \right)^{1/2} \right)^4, \end{aligned} \tag{5.3}$$

where we used the Sobolev embedding $L^5(\mathbb{T}^3) \hookrightarrow H^1(\mathbb{T}^3)$ in the last line. We then use the Young’s inequality to proceed

$$\begin{aligned} \int_{\mathbb{T}^3} |u_\varepsilon| m_1 f_\varepsilon dx &\leq \frac{\mu}{2} \|u_\varepsilon\|_{H^1}^2 + C \left(1 + \left(\int_0^t \|u_\varepsilon(s)\|_{H^1}^2 ds \right)^{1/2} \right)^8 \\ &\leq C + C \|u_\varepsilon\|_{L^2}^2 + \frac{\mu}{2} \|\nabla_x u_\varepsilon\|_{L^2}^2 + C \left(\int_0^t \|u_\varepsilon(s)\|_{L^2}^2 ds \right)^4 + C \left(\int_0^t \|\nabla_x u_\varepsilon(s)\|_{L^2}^2 ds \right)^4 \\ &\leq C + C \|u_\varepsilon\|_{L^2}^2 + \frac{\mu}{2} \|\nabla_x u_\varepsilon\|_{L^2}^2 + C \int_0^t \|u_\varepsilon(s)\|_{L^2}^8 ds + C \left(\int_0^t \|\nabla_x u_\varepsilon(s)\|_{L^2}^2 ds \right)^4. \end{aligned}$$

In the last line, we used Hölder inequality:

$$\int_0^t \|u_\varepsilon(s)\|_{L^2}^2 ds \leq C_T \left(\int_0^t \|u_\varepsilon(s)\|_{L^2}^8 ds \right)^{1/4}.$$

Therefore, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_\varepsilon\|_{L^2}^2 + \mu \|\nabla_x u_\varepsilon\|_{L^2}^2 &\leq C + C \|u_\varepsilon\|_{L^2}^2 + \frac{\mu}{2} \|\nabla_x u_\varepsilon\|_{L^2}^2 + C \int_0^t \|u_\varepsilon(s)\|_{L^2}^8 ds + C \left(\int_0^t \|\nabla_x u_\varepsilon(s)\|_{L^2}^2 ds \right)^4. \end{aligned}$$

Integrating the above inequality over the time interval $[0, t]$, we find

$$\begin{aligned} \|u_\varepsilon\|_{L^2}^2 + \mu \int_0^t \|\nabla_x u_\varepsilon(s)\|_{L^2}^2 ds &\leq \|u_{0,\varepsilon}\|_{L^2}^2 + C + C \int_0^t \|u_\varepsilon(s)\|_{L^2}^2 ds + C \int_0^t \|u_\varepsilon(s)\|_{L^2}^8 ds \\ &\quad + C \int_0^t \left(\int_0^s \|\nabla_x u_\varepsilon(\tau)\|_{L^2}^2 d\tau \right)^4 ds. \end{aligned}$$

We then apply the Gronwall's inequality to obtain that there exists a $0 < T_* \leq T$ such that

$$\|u_\varepsilon(t)\|_{L^2}^2 + \int_0^t \|\nabla_x u_\varepsilon(s)\|_{L^2}^2 ds \leq C \quad \text{for } 0 \leq t \leq T_*, \tag{5.4}$$

due to $\|u_{0,\varepsilon}\|_{L^2} \leq C\|u_0\|_{L^2}$, where $C > 0$ is independent of ε . We also combine (5.2) and (5.4) to have

$$M_2 f_\varepsilon(t) \leq C \quad \text{for } 0 \leq t \leq T_*, \tag{5.5}$$

where $C > 0$ is independent of ε .

• **Uniform bound of $\|(\eta_\varepsilon \star u_\varepsilon - v)(1 + |v|)f_\varepsilon\|_{L^1}$:** We divide the integral as

$$\begin{aligned} &\|(\eta_\varepsilon \star u_\varepsilon - v)(1 + |v|)f_\varepsilon\|_{L^1} \\ &= \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} |(\eta_\varepsilon \star u_\varepsilon - v)| f_\varepsilon dx dv dt + \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} |(\eta_\varepsilon \star u_\varepsilon - v)| |v| f_\varepsilon dx dv dt \\ &=: I_1 + I_2, \end{aligned}$$

and estimate I_1 and I_2 separately. For the estimate of I_1 , we first note that

$$\begin{aligned} I_1 &\leq \int_0^{T_*} \int_{\mathbb{T}^3} |\eta_\varepsilon \star u_\varepsilon| \rho_\varepsilon dx dt + \int_0^{T_*} \int_{\mathbb{T}^3} m_1 f_\varepsilon dx dt \\ &\leq \int_0^{T_*} \|u_\varepsilon(t)\|_{L^{5/2}} \|\rho_\varepsilon(t)\|_{L^{5/3}} dt + C \int_0^{T_*} \|m_1 f_\varepsilon\|_{L^{5/4}} dt, \end{aligned}$$

where the second term on the right hand side of the above inequality can be uniformly bounded as

$$\begin{aligned} \|m_1 f_\varepsilon\|_{L^{5/4}} &\leq \left(1 + \int_0^t \|u_\varepsilon(s)\|_{L^5} ds \right)^4 \\ &\leq C \left(1 + \left(\int_0^t \|u_\varepsilon(s)\|_{H^1}^2 ds \right)^{1/2} \right)^4 \\ &\leq C \quad \text{for } t \in (0, T_*), \end{aligned}$$

by using the same argument as in the estimate of the total energy. For the first term, we use lemma 2.5; $m_0 f_\varepsilon \leq C(m_1 f_\varepsilon)^{3/4}$ to get $\|\rho_\varepsilon\|_{L^{5/3}} \leq C\|m_1 f_\varepsilon\|_{L^{5/4}}^{3/4}$, where $C > 0$ is independent of ε . A similar argument as in the previous estimate then yields

$$\begin{aligned}
 & \int_0^{T_*} \|u_\varepsilon(t)\|_{L^{5/2}} \|\rho_\varepsilon(t)\|_{L^{5/3}} dt \\
 & \leq C \int_0^{T_*} \|u_\varepsilon(t)\|_{L^{5/2}} \|m_1 f_\varepsilon(t)\|_{L^{5/4}}^{3/4} dt \\
 & \leq C \int_0^{T_*} \|u_\varepsilon(t)\|_{H^1} \left(1 + \int_0^t \|u_\varepsilon(s)\|_{H^1} ds\right)^3 dt \\
 & \leq C \int_0^{T_*} \|u_\varepsilon(t)\|_{H^1} dt + C \int_0^{T_*} \|u_\varepsilon(t)\|_{H^1} \left(\int_0^t \|u_\varepsilon(s)\|_{H^1}^2 ds\right)^{3/2} dt \\
 & \leq C \int_0^{T_*} \|u_\varepsilon(t)\|_{H^1} dt \\
 & \leq C \left(\int_0^{T_*} \|u_\varepsilon(t)\|_{H^1}^2 dt\right)^{1/2} \\
 & \leq C,
 \end{aligned}$$

for $0 \leq t \leq T_*$, where $C > 0$ is independent of ε due to (5.4). For I_2 , we decompose similarly as

$$I_2 \leq \int_0^{T_*} \int_{\mathbb{T}^3} |\eta_\varepsilon \star u_\varepsilon| m_1 f_\varepsilon dx dt + \int_0^{T_*} M_2 f_\varepsilon dt.$$

The uniform boundedness of the second term on the right hand side is obtained in (5.5). The computation for the first term is treated in (5.3). This concludes the desired result.

- **Uniform bound of third moment:** We adopt the argument from [6, 37], unlike in [6, 37], we show that the third moment is controlled by the kinetic-fluid mixed estimate due to the presence of the drag force term. We multiply (3.1) by

$$\Phi(x, v) = \frac{(1 + |v|^2)^{1/2} x \cdot v}{(1 + |x|^2)^{1/2}}$$

and integrate on $\mathbb{T}^3 \times \mathbb{R}^3 \times [0, T_*]$ to get

$$\begin{aligned}
 - \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} v \cdot \nabla_x f_\varepsilon \Phi dx dv dt &= \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \partial_t f_\varepsilon \Phi dx dv dt \\
 &+ \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \nabla_v \cdot \{(\eta_\varepsilon \star u_\varepsilon - v) f_\varepsilon\} \Phi dx dv dt \\
 &- \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \{\mathcal{M}(f_\varepsilon) - f_\varepsilon\} \Phi dx dv dt.
 \end{aligned}$$

We denote the left hand side by L and the three terms on the right hand side by $R_i (i = 1, 2, 3)$.

- ◊ **The estimate of L :** By divergence theorem, we have

$$\begin{aligned}
 L &= \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon v \cdot \nabla_x \Phi \, dx dv dt \\
 &= \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon \{v(1 + |v|^2)^{1/2}\} \cdot \nabla_x \left\{ \frac{x \cdot v}{(1 + |x|^2)^{1/2}} \right\} dx dv dt \\
 &= \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon \{v(1 + |v|^2)^{1/2}\} \cdot \left\{ \frac{v}{(1 + |x|^2)^{1/2}} + \frac{-x(x \cdot v)}{(1 + |x|^2)^{3/2}} \right\} dx dv dt \\
 &= \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon (1 + |v|^2)^{1/2} \cdot \left\{ \frac{|v|^2}{(1 + |x|^2)^{1/2}} + \frac{-(x \cdot v)^2}{(1 + |x|^2)^{3/2}} \right\} dx dv dt \\
 &= \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon \frac{|v|^2(1 + |v|^2)^{1/2}}{(1 + |x|^2)^{1/2}} \left\{ 1 - \frac{(x \cdot v)^2}{(1 + |x|^2)|v|^2} \right\} dx dv dt.
 \end{aligned}$$

On the other hand, we observe

$$1 - \frac{(x \cdot v)^2}{(1 + |x|^2)|v|^2} \geq 1 - \frac{|x|^2|v|^2}{(1 + |x|^2)|v|^2} = \frac{1}{1 + |x|^2} \geq 1/4,$$

and

$$\frac{|v|^2(1 + |v|^2)^{1/2}}{(1 + |x|^2)^{1/2}} \geq \frac{1}{2}|v|^3,$$

for $(x, v) \in \mathbb{T}^3 \times \mathbb{R}^3$. This yields

$$L \geq \frac{1}{8} \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^3 f_\varepsilon \, dx dv dt.$$

◇ **The estimate of R_1 :** Since Φ does not depend on t , we can integrate in time as

$$\begin{aligned}
 R_1 &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} (f_\varepsilon(T_*) - f_\varepsilon(0)) \Phi \, dx dv \\
 &\leq \int_{\mathbb{T}^3 \times \mathbb{R}^3} (f_\varepsilon(T_*) - f_\varepsilon(0)) (1 + |v|^2) \, dx dv \\
 &\leq C,
 \end{aligned}$$

where we used $\Phi(x, v) \leq (1 + |v|^2)$ for $(x, v) \in \mathbb{T}^3 \times \mathbb{R}^3$ and (3.2), and the constant $C > 0$ is independent of ε .

◇ **The estimate of R_2 :** Using divergence theorem, we estimate

$$\begin{aligned}
 R_2 &= \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \nabla_v \cdot \{(\eta_\varepsilon \star u_\varepsilon - v) f_\varepsilon\} \frac{(1 + |v|^2)^{1/2} x \cdot v}{(1 + |x|^2)^{1/2}} \, dx dv dt \\
 &= - \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \{(\eta_\varepsilon \star u_\varepsilon - v) f_\varepsilon\} \cdot \nabla_v \left\{ \frac{(1 + |v|^2)^{1/2} x \cdot v}{(1 + |x|^2)^{1/2}} \right\} \, dx dv dt \\
 &= - \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \{(\eta_\varepsilon \star u_\varepsilon - v) f_\varepsilon\} (1 + |v|^2)^{1/2} \left\{ \frac{v\{x \cdot v\} + x(1 + |v|^2)}{(1 + |v|^2)(1 + |x|^2)^{1/2}} \right\} \, dx dv dt.
 \end{aligned}$$

Note that

$$\left| \frac{v\{x \cdot v\} + x(1 + |v|^2)}{(1 + |v|^2)(1 + |x|^2)^{1/2}} \right| \leq 2 \quad \text{for } (x, v) \in \mathbb{T}^3 \times \mathbb{R}^3,$$

which gives

$$|R_2| \leq 2 \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} |\eta_\varepsilon \star u_\varepsilon - v| f_\varepsilon (1 + |v|) \, dx dv dt = 2 \|(\eta_\varepsilon \star u_\varepsilon - v)(1 + |v|) f_\varepsilon\|_{L^1} \leq C,$$

where we used (3.3).

◇ **The estimate of R_3 :** A straightforward computation gives

$$\begin{aligned} R_3 &\leq \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \{\mathcal{M}(f_\varepsilon) + f_\varepsilon\} (1 + |v|^2) \, dx dv dt \\ &= 2 \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon (1 + |v|^2) \, dx dv dt \\ &\leq C_{f_0, u_0, T_*}. \end{aligned}$$

Combining all these estimates, we obtain

$$\int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon |v|^3 \, dx dv dt \leq C_{f_0, u_0, T_*}.$$

● **Uniform bound of entropy:** Multiply (3.1)₁ by $\ln f_\varepsilon$ and integrate with respect to x and v to get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon \ln f_\varepsilon \, dx dv + \int_{\mathbb{T}^3 \times \mathbb{R}^3} (v \cdot \nabla_x f_\varepsilon) \ln f_\varepsilon \, dx dv + \int_{\mathbb{T}^3 \times \mathbb{R}^3} \nabla_v \cdot ((\eta_\varepsilon \star u_\varepsilon - v) f_\varepsilon) \ln f_\varepsilon \, dx dv \\ = \int_{\mathbb{T}^3 \times \mathbb{R}^3} (\mathcal{M}(f_\varepsilon) - f_\varepsilon) \ln f_\varepsilon \, dx dv. \end{aligned}$$

The second term on the left hand side vanishes due to the divergence theorem. Using divergence theorem and integration by parts, we can estimate the third term on the left hand side as

$$\begin{aligned} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \nabla_v \cdot ((\eta_\varepsilon \star u_\varepsilon - v) f_\varepsilon) \ln f_\varepsilon \, dx dv &= - \int_{\mathbb{T}^3 \times \mathbb{R}^3} (\eta_\varepsilon \star u_\varepsilon - v) \nabla_v f_\varepsilon \, dx dv \\ &= -3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon \, dx dv. \end{aligned}$$

Since the local Maxwellian shares the same moments up to second order with f_ε , we get

$$\begin{aligned} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \{\mathcal{M}(f_\varepsilon) - f_\varepsilon\} \ln \mathcal{M}(f_\varepsilon) \, dx dv &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} \{\mathcal{M}(f_\varepsilon) - f_\varepsilon\} \left\{ \ln \frac{\rho_\varepsilon}{\sqrt{(2\pi T_\varepsilon)^3}} - \frac{|v - U_\varepsilon|^2}{2T_\varepsilon} \right\} \, dx dv \\ &= 0, \end{aligned}$$

which immediately gives

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} \{\mathcal{M}(f_\varepsilon) - f_\varepsilon\} \ln f_\varepsilon \, dx dv = - \int_{\mathbb{T}^3 \times \mathbb{R}^3} \{\mathcal{M}(f_\varepsilon) - f_\varepsilon\} \{\ln \mathcal{M}(f_\varepsilon) - \ln f_\varepsilon\} \, dx dv \leq 0.$$

Thus, we obtain

$$\frac{d}{dt} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon \ln f_\varepsilon \, dx dv - 3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon \, dx dv = - \int_{\mathbb{T}^3 \times \mathbb{R}^3} \{\mathcal{M}(f_\varepsilon) - f_\varepsilon\} (\ln \mathcal{M}(f_\varepsilon) - \ln f_\varepsilon) \, dx dv.$$

Integrating in time, we get

$$\begin{aligned} & \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon(t) \ln f_\varepsilon(t) \, dx dv + \int_0^t \int_{\mathbb{T}^3 \times \mathbb{R}^3} \{\mathcal{M}(f_\varepsilon) - f_\varepsilon\} (\ln \mathcal{M}(f_\varepsilon) - \ln f_\varepsilon) \, dx dv ds \\ & \leq \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_{0,\varepsilon} \ln f_{0,\varepsilon} \, dx dv + 3M_0 f_0 T \quad \text{for } t \in (0, T). \end{aligned}$$

Then, it is standard to show that (see for example, [11, 21])

$$\sup_{0 \leq t \leq T} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon(t) |\ln f_\varepsilon(t)| \, dx dv \leq C(f_0, T).$$

This completes the proof.

6. Global existence of weak solutions

6.1. Weak compactness of f_ε and $\mathcal{M}(f_\varepsilon)$

In this part, we use the uniform estimates in ε obtained in the previous subsection to derive compactness of $(f_\varepsilon, u_\varepsilon)$ and the relaxation operators.

We have derived in the previous section that there exists a constant C , independent of ε such that

$$\int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} (1 + |v|^3 + |\ln f_\varepsilon|) f_\varepsilon \, dx dv dt \leq C.$$

Dunford–Pettis theorem then implies that f_ε , $f_\varepsilon v$ and $f_\varepsilon |v|^2$ are weakly compact in $L^1(\mathbb{T}^3 \times \mathbb{R}^3 \times (0, T_*))$. To derive the weak compactness of $\mathcal{M}(f_\varepsilon)$, we compute for $R > 1$

$$\begin{aligned} \mathcal{M}(f) - f &= \{\mathcal{M}(f) - f\} 1_{\mathcal{M}(f) < Rf} + \{\mathcal{M}(f) - f\} 1_{\mathcal{M}(f) \geq Rf} \\ &\leq (R - 1)f 1_{\mathcal{M}(f) < Rf} + \frac{1}{\ln R} (\mathcal{M}(f) - f) (\ln \mathcal{M}(f) - \ln f) 1_{\mathcal{M}(f) \geq Rf}, \end{aligned}$$

so that

$$\mathcal{M}(f) \leq Rf + \frac{1}{\ln R} (\mathcal{M}(f) - f) (\ln \mathcal{M}(f) - \ln f).$$

Now, we take an arbitrary measurable set $B_{x,v} \subseteq \mathbb{T}^3 \times \mathbb{R}^3$ and integrate over $B_{x,v} \times [0, T_*]$ to get

$$\begin{aligned} & \int_0^{T_*} \int_{B_{x,v}} \mathcal{M}(f) \, dx dv dt \\ & \leq R \int_0^{T_*} \int_{B_{x,v}} f \, dx dv dt + \frac{1}{\ln R} \int_0^{T_*} \int_{B_{x,v}} (\mathcal{M}(f) - f) (\ln \mathcal{M}(f) - \ln f) \, dx dv dt \\ & \leq R \int_0^{T_*} \int_{B_{x,v}} f \, dx dv dt + \frac{1}{\ln R} \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} (\mathcal{M}(f) - f) (\ln \mathcal{M}(f) - \ln f) \, dx dv dt \\ & \leq R \int_0^{T_*} \int_{B_{x,v}} f \, dx dv dt + \frac{1}{\ln R} \left(\int_{\mathbb{T}^3 \times \mathbb{R}^3} f_0 \ln f_0 \, dx dv + 3M_0 f_0 T \right) \\ & \leq R \int_0^{T_*} \int_{B_{x,v}} f \, dx dv dt + \frac{1}{\ln R} \left(\int_{\mathbb{T}^3 \times \mathbb{R}^3} f_0 |\ln f_0| \, dx dv + C_{f_0, T} \right). \end{aligned}$$

Then, Dunford–Pettis theorem again gives the weak compactness of $\mathcal{M}(f)$ in $L^1(\mathbb{T}^3 \times \mathbb{R}^3 \times (0, T_*))$.

6.2. Strong compactness of $\rho_\varepsilon, U_\varepsilon$ and T_ε

From the argument in the previous section, we see that there exists $f \in L^1(\mathbb{T}^3 \times \mathbb{R}^3 \times (0, T_*))$ such that $f_\varepsilon, f_\varepsilon v, f_\varepsilon |v|^2$ converge to $f, f v, f |v|^2$ weakly in $L^1(\mathbb{T}^3 \times \mathbb{R}^3 \times (0, T_*))$ respectively, which also implies

$$\rho_\varepsilon = \int_{\mathbb{R}^3} f_\varepsilon \, dv \rightharpoonup \int_{\mathbb{R}^3} f \, dv = \rho, \quad \rho_\varepsilon U_\varepsilon = \int_{\mathbb{R}^3} v f_\varepsilon \, dv \rightharpoonup \int_{\mathbb{R}^3} v f \, dv = \rho U,$$

and

$$3\rho_\varepsilon T_\varepsilon + \rho_\varepsilon |U_\varepsilon|^2 = \int_{\mathbb{R}^3} f_\varepsilon |v|^2 \, dv \rightharpoonup \int_{\mathbb{R}^3} f |v|^2 \, dv = 3\rho T + \rho |U|^2$$

in $L^1(\mathbb{T}^3 \times (0, T_*))$. Thanks to the velocity averaging lemma [26], the above convergence actually is strong, which gives the almost everywhere convergence of the macroscopic fields:

$$\rho_\varepsilon \rightarrow \rho \text{ a.e on } \mathbb{T}^3 \times [0, T_*], \quad U_\varepsilon \rightarrow U \text{ a.e on } E, \quad \text{and} \quad T_\varepsilon \rightarrow T \text{ a.e on } E, \tag{6.1}$$

where

$$E = \{(x, t) \in \mathbb{T}^3 \times (0, T_*) \mid \rho(x, t) \neq 0\}.$$

Next, we need to show that $\mathcal{M}(f_\varepsilon)$ converges weakly in L^1 to $\mathcal{M}(f)$.

6.3. $\mathcal{M}(f_\varepsilon)$ converges to $\mathcal{M}(f)$ in $L^1(\mathbb{T}^3 \times \mathbb{R}^3 \times (0, T_*))$

Since (6.1) implies

$$\mathcal{M}(\rho_\varepsilon, U_\varepsilon, T_\varepsilon)\varphi \rightarrow \mathcal{M}(\rho, U, T)\varphi \text{ a.e on } E \times \mathbb{R}^3$$

for any non-negative L^∞ function φ , we have from Fatou’s lemma that

$$\int_{E \times \mathbb{R}^3} \mathcal{M}(\rho, U, T)\varphi \, dx \, dv \, dt \leq \liminf_{\varepsilon \rightarrow 0} \int_{E \times \mathbb{R}^3} \mathcal{M}(\rho_\varepsilon, U_\varepsilon, T_\varepsilon)\varphi \, dx \, dv \, dt.$$

On the other hand, from the weak L^1 compactness of $\mathcal{M}(f_\varepsilon)$, we can find a L^1 function M such that

$$\lim_{\varepsilon \rightarrow 0} \int_{E \times \mathbb{R}^3} \mathcal{M}(\rho_\varepsilon, U_\varepsilon, T_\varepsilon)\varphi \, dx \, dv \, dt = \int_{E \times \mathbb{R}^3} M\varphi \, dx \, dv \, dt.$$

Thus we obtain

$$\int_{E \times \mathbb{R}^3} \mathcal{M}(\rho, U, T)\varphi \, dx \, dv \, dt \leq \int_{E \times \mathbb{R}^3} M\varphi \, dx \, dv \, dt,$$

for all $\varphi \in L^\infty(\mathbb{T}^3 \times \mathbb{R}^3 \times (0, T_*))$, from which we can conclude that

$$\mathcal{M}(\rho, U, T) \leq M \tag{6.2}$$

almost everywhere on $E \times \mathbb{R}^3$. Now, taking $\varphi = 1$, we find

$$\begin{aligned} \int_{E \times \mathbb{R}^3} M \, dx \, dv \, dt &= \lim_{\varepsilon \rightarrow 0} \int_{E \times \mathbb{R}^3} \mathcal{M}(\rho_\varepsilon, U_\varepsilon, T_\varepsilon) \, dx \, dv \, dt \\ &= \lim_{\varepsilon \rightarrow 0} \int_E \rho_\varepsilon \, dx \, dt \\ &= \int_E \rho \, dx \, dt \\ &= \int_{E \times \mathbb{R}^3} \mathcal{M}(\rho, U, T) \, dx \, dv \, dt. \end{aligned}$$

This, together with (6.2) implies $\mathcal{M}(\rho, U, T) = M$ almost everywhere on E . On the other hand, we observe

$$\begin{aligned} \left| \lim_{\varepsilon \rightarrow 0} \int_{E^c \times \mathbb{R}^3} \mathcal{M}_\varepsilon \varphi \, dx \, dv \, dt \right| &\leq \lim_{\varepsilon \rightarrow 0} \int_{E^c \times \mathbb{R}^3} \mathcal{M}_\varepsilon |\varphi| \, dx \, dv \, dt \\ &\leq \lim_{\varepsilon \rightarrow 0} \|\varphi\|_{L^\infty} \int_{E^c \times \mathbb{R}^3} \mathcal{M}_\varepsilon \, dx \, dv \, dt \\ &= \lim_{\varepsilon \rightarrow 0} \|\varphi\|_{L^\infty} \int_{E^c} \rho_\varepsilon \, dx \, dt \\ &= \|\varphi\|_{L^\infty} \int_{E^c} \rho \, dx \, dt \\ &= 0. \end{aligned}$$

Hence we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_{E^c \times \mathbb{R}^3} M \, dx \, dv \, dt = 0.$$

In conclusion, we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \mathcal{M}(f_\varepsilon) \varphi \, dx \, dv \, dt &= \lim_{\varepsilon \rightarrow 0} \int_{E \times \mathbb{R}^3} \mathcal{M}(f_\varepsilon) \varphi \, dx \, dv \, dt + \lim_{\varepsilon \rightarrow 0} \int_{E^c \times \mathbb{R}^3} \mathcal{M}(f_\varepsilon) \varphi \, dx \, dv \, dt \\ &= \int_{E \times \mathbb{R}^3} \mathcal{M}(f) \varphi \, dx \, dv \, dt + 0 \\ &= \int_0^{T_*} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \mathcal{M}(f) \varphi \, dx \, dv \, dt. \end{aligned}$$

This provides the desired result.

6.4. Compactness of u_ε in $L^2(0, T_*; L^2(\mathbb{T}^3))$

In this subsection, we show that u_ε is compact in $L^2(0, T_*; L^2(\mathbb{T}^3))$. For this, we are going to show that $\partial_t u_\varepsilon$ is uniformly bounded in $L^{3/2}(0, T_*; \mathcal{V}')$ so that we can employ the Aubin–Lions lemma that guarantees the strong compactness [18, 47].

It follows from the weak formulation for the fluid part that for all $\psi \in \mathcal{C}^1(\mathbb{T}^3 \times [0, T_*])$ with $\nabla_x \cdot \psi = 0$ for almost everywhere t

$$\begin{aligned} \int_0^t \int_{\mathbb{T}^3} \partial_t u_\varepsilon \cdot \psi \, dx ds &= - \int_0^t \int_{\mathbb{T}^3} ((\eta_\varepsilon \star u_\varepsilon) \cdot \nabla_x) u_\varepsilon \cdot \psi \, dx ds - \mu \int_0^t \int_{\mathbb{T}^3} \nabla_x u_\varepsilon : \nabla_x \psi \, dx ds \\ &\quad - \int_0^t \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon(u_\varepsilon - v) \gamma_\varepsilon(v) \cdot \psi \, dx dv ds \\ &=: J_1 + J_2 + J_3. \end{aligned}$$

Using the integration by parts together with the divergence free condition, we get

$$J_1 = \int_0^t \int_{\mathbb{T}^3} ((\eta_\varepsilon \star u_\varepsilon) \cdot \nabla_x) \psi \cdot u_\varepsilon \, dx ds.$$

By Hölder inequality, we have

$$|J_1| \leq \int_0^t \|\nabla_x \psi\|_{L^2} \| |\eta_\varepsilon \star u_\varepsilon| |u_\varepsilon| \|_{L^2} \, ds. \tag{6.3}$$

Then, by Hölder inequality again,

$$\begin{aligned} \| |\eta_\varepsilon \star u_\varepsilon| |u_\varepsilon| \|_{L^2} &= \left(\int_{\mathbb{R}^3} |\eta_\varepsilon \star u_\varepsilon|^2 |u_\varepsilon|^2 \, dx \right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}^3} |\eta_\varepsilon \star u_\varepsilon|^4 \, dx \right)^{1/4} \left(\int_{\mathbb{R}^3} |u_\varepsilon|^4 \, dx \right)^{1/4} \\ &\leq \|u_\varepsilon\|_{L^4}^2 \end{aligned}$$

and Minkowski integral inequality,

$$\left(\int_{\mathbb{R}^3} |\eta_\varepsilon \star u_\varepsilon|^4 \, dx \right)^{1/4} \leq \|u_\varepsilon\|_{L^4},$$

we obtain from (6.3) that

$$|J_1| \leq \int_0^t \|\nabla_x \psi\|_{L^2} \|u_\varepsilon\|_{L^4}^2 \, ds \leq \|\nabla_x \psi\|_{L^3(0, T_*; L^2)} \|u_\varepsilon\|_{L^3(0, T_*; L^4)}^2, \tag{6.4}$$

where $\|u_\varepsilon\|_{L^3(0, T_*; L^4)}$ is uniformly bounded in ε due to the uniform boundedness of u_ε in $L^\infty(0, T_*; L^2(\mathbb{T}^3)) \cap L^2(0, T_*; H^1(\mathbb{T}^3))$ and the Sobolev embedding:

$$L^\infty(0, T_*; L^2(\mathbb{T}^3)) \cap L^2(0, T_*; H^1(\mathbb{T}^3)) \hookrightarrow L^3(0, T_*; L^4(\mathbb{T}^3)).$$

Thus we obtain

$$\psi \mapsto - \int_0^t \int_{\mathbb{T}^3} ((\eta_\varepsilon \star u_\varepsilon) \cdot \nabla_x) u_\varepsilon \cdot \psi \, dx ds$$

is bounded in $L^{3/2}(0, T_*; \mathcal{V}')$. The estimate of J_2 can be easily done as

$$|J_2| \leq \mu \int_0^t \|\nabla_x u_\varepsilon\|_{L^2} \|\nabla_x \psi\|_{L^2} \, ds \leq \|\nabla_x \psi\|_{L^3(0, T_*; L^2)} \|\nabla_x u_\varepsilon\|_{L^{3/2}(0, T_*; L^2)}.$$

Thus it gives the same result as the above. Finally, we estimate J_3 as

$$\begin{aligned}
 |J_3| &\leq \int_0^t \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon (|u_\varepsilon| + |v|) |\psi| \, dx dv ds \\
 &\leq \int_0^t (\|u_\varepsilon\|_{L^6} \|\psi\|_{L^6} \|\rho_\varepsilon\|_{L^{3/2}} + \|\psi\|_{L^5} \|m_1 f_\varepsilon\|_{L^{5/4}}) \, ds \\
 &\leq \|u_\varepsilon\|_{L^2(0, T_*; L^6)} \|\psi\|_{L^2(0, T_*; L^6)} \|\rho_\varepsilon\|_{L^\infty(0, T_*; L^{3/2})} + \|\psi\|_{L^2(0, T_*; L^5)} \|m_1 f_\varepsilon\|_{L^2(0, T_*; L^{5/4})}.
 \end{aligned}$$

On the other hand, it follows from lemma 2.5; $m_0 f \leq C_{\|f_\varepsilon\|_{L^\infty}} (m_2 f)^{3/5}$ and Hölder inequality that

$$\begin{aligned}
 \|\rho_{f_\varepsilon}\|_{L^{3/2}} &\leq C \left(\int_{\mathbb{T}^3} (m_2 f_\varepsilon)^{9/10} \, dx \right)^{2/3} \\
 &\leq C \left(\int_{\mathbb{T}^3} \left\{ (m_2 f_\varepsilon)^{9/10} \right\}^{10/9} \, dx \right)^{9/10} \left(\int_{\mathbb{T}^3} 1^{10} \, dx \right)^{1/10} \\
 &\leq C (M_2 f_\varepsilon)^{3/5}.
 \end{aligned}$$

Thus we get the uniform boundedness of $\|\rho_{f_\varepsilon}\|_{L^\infty(0, T_*; L^{3/2})}$ in ε . Similarly, we find

$$\|m_1 f_\varepsilon\|_{L^{5/4}} \leq C (M_2 f_\varepsilon)^{4/5},$$

i.e. $m_1 f_\varepsilon$ is uniformly bounded in $L^2(0, T_*; L^{5/4}(\mathbb{T}^3))$. Combined with the uniform boundedness of $\|u_\varepsilon\|_{L^2(0, T_*; L^6)}$ in ε , this yields

$$|J_3| \leq C \|\psi\|_{L^2(0, T_*; L^6)} \leq C \|\psi\|_{L^2(0, T_*; H^1)}.$$

Thus we obtain that $\partial_t u_\varepsilon$ is uniformly bounded in $L^{3/2}(0, T_*; \mathcal{V}')$. Then, by Aubin–Lions lemma, we have the following strong convergences of u_ε :

$$u_\varepsilon \rightarrow u \text{ in } L^2(0, T_*; L^2(\mathbb{T}^3)), \quad u_\varepsilon \rightarrow u \text{ in } \mathcal{C}([0, T_*]; \mathcal{V}'),$$

as $\varepsilon \rightarrow 0$. These convergence together with the weak convergences allow us to pass to the limit to conclude the existence of weak solutions.

In order to extend that local-in-time weak solutions to the global ones, we give the following energy estimate showing the total energy of the system (1.1) is not increasing. Then, by using the same strategy based on the continuity argument as in [7, section 3.6], we have the global-in-time existence of weak solutions and complete the proof of theorem 1.3. Even though the proof of following lemma is almost same with [7, lemma 2], for the completeness and the readers’ convenience, we provide its details.

Lemma 6.1. *Let (f, u) be the solutions to the system (1.1) obtained above. Then we have the following total energy estimate*

$$\begin{aligned}
 \frac{1}{2} M_2 f(t) + \frac{1}{2} \|u(t)\|_{L^2}^2 + \mu \int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds \\
 + \int_0^t \int_{\mathbb{T}^3 \times \mathbb{R}^3} f |u - v|^2 \, dx dv ds \leq \frac{1}{2} M_2 f_0 + \frac{1}{2} \|u_0\|_{L^2}^2
 \end{aligned}$$

for almost every $t \in [0, T_*]$.

Proof. A straightforward computation yields

$$\begin{aligned} & \frac{1}{2}M_2f_\varepsilon(t) + \frac{1}{2}\|u_\varepsilon(t)\|_{L^2}^2 + \mu \int_0^t \|\nabla u_\varepsilon(s)\|_{L^2}^2 ds \\ & + \int_0^t \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon |u_\varepsilon - v|^2 dx dv ds = \frac{1}{2}M_2f_0^\varepsilon + \frac{1}{2}\|u_0^\varepsilon\|_{L^2}^2 + R_\varepsilon(t), \end{aligned}$$

where the remnant $R_\varepsilon(t)$ is given by

$$\begin{aligned} R_\varepsilon(t) &= \int_0^t \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon |u_\varepsilon|^2 (1 - \gamma_\varepsilon(v)) dx dv ds - \int_0^t \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon u_\varepsilon \cdot v (1 - \gamma_\varepsilon(v)) dx dv ds \\ &+ \int_0^t \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon (u_\varepsilon - \eta_\varepsilon \star u_\varepsilon) \cdot v dx dv ds \\ &=: R_\varepsilon^1 + R_\varepsilon^2 + R_\varepsilon^3. \end{aligned}$$

We now show $R_\varepsilon(t) \rightarrow 0$ as $\varepsilon \rightarrow 0$ uniformly in $t \in [0, T_*]$.

- **Estimate of $R_\varepsilon^1(t)$:** Set $h_\varepsilon(x, t) := \int_{\mathbb{R}^3} f_\varepsilon(x, v, t) (1 - \gamma_\varepsilon(v)) dv$. Then we use lemma 2.5 to obtain

$$\begin{aligned} |R_\varepsilon^1(t)| &\leq \int_0^t \int_{\mathbb{T}^3} |u_\varepsilon|^2 |h_\varepsilon| dx ds \leq \int_0^t \|u_\varepsilon\|_{L^6}^2 \|m_0 h_\varepsilon\|_{L^{3/2}} ds \\ &\leq C \int_0^t \|u_\varepsilon\|_{H^1}^2 |M_{3/2} h_\varepsilon|^{3/2} ds \\ &\leq C \|M_{3/2} h_\varepsilon\|_{L^\infty(0, T_*; L^{3/2})}^{3/2} \|u_\varepsilon\|_{L^2(0, T_*; H^1)}^2. \end{aligned}$$

On the other hand, we find

$$\begin{aligned} |M_{3/2} h_\varepsilon(t)| &\leq \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^{3/2} f_\varepsilon (1 - \gamma_\varepsilon) dx dv \\ &\leq \int_{\mathbb{T}^3 \times \{v: |v| \geq \frac{1}{2\varepsilon}\}} |v|^{3/2} f_\varepsilon dx dv \\ &\leq \sqrt{2\varepsilon} \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^2 f_\varepsilon dx dv \leq C\sqrt{\varepsilon}. \end{aligned}$$

Thus we have

$$|R_\varepsilon^1(t)| \leq C\sqrt{\varepsilon} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

- **Estimate of $R_\varepsilon^2(t)$:** Taking a similar argument as the above, we get

$$\begin{aligned} |R_\varepsilon^2(t)| &\leq \int_0^t \int_{\mathbb{T}^3} |u_\varepsilon| |m_1 h_\varepsilon| dx ds \leq \int_0^t \|u_\varepsilon\|_{L^6} \|m_1 h_\varepsilon\|_{L^{6/5}} ds \\ &\leq \int_0^t \|u_\varepsilon\|_{H^1} |M_{9/5} h_\varepsilon|^{5/6} ds \\ &\leq \sqrt{T} \|u_\varepsilon\|_{L^2(0, T_*; H^1)} \|M_{9/5} h_\varepsilon\|_{L^\infty(0, T_*; L^{5/3})}^{5/6} \\ &\leq C\varepsilon^{1/5} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \end{aligned}$$

where we used

$$|M_{9/5}h_\varepsilon(t)| \leq \int_{\mathbb{T}^3 \times \{v: |v| \geq \frac{1}{2\varepsilon}\}} |v|^{9/5} f_\varepsilon \, dx dv \leq (2\varepsilon)^{1/5} M_2 f_\varepsilon(t) \leq C\varepsilon^{1/5}.$$

• **Estimate of $R_\varepsilon^3(t)$:** We again divide it into two terms $R_{\varepsilon,\delta}^{3,i}$, $i = 1, 2$ as follows.

$$\begin{aligned} R_\varepsilon^3(t) &= \int_0^t \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon(u_\varepsilon - \eta_\varepsilon \star u_\varepsilon) \cdot v(1 - \gamma_\delta(v)) \, dx dv ds \\ &\quad + \int_0^t \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon(u_\varepsilon - \eta_\varepsilon \star u_\varepsilon) \cdot v \gamma_\delta(v) \, dx dv ds \\ &=: R_{\varepsilon,\delta}^{3,1}(t) + R_{\varepsilon,\delta}^{3,2}(t), \end{aligned}$$

for any $\delta > 0$. First, we easily find that $|R_{\varepsilon,\delta}^{3,1}(t)| \leq C\delta^{1/5} \rightarrow 0$ as $\delta \rightarrow 0$ uniformly in ε using the same argument as the above. For the estimate $R_{\varepsilon,\delta}^{3,2}$, we use the uniform bound estimate of f_ε in $L^\infty(\mathbb{T}^3 \times \mathbb{R}^3 \times (0, T_*))$ to get

$$|R_{\varepsilon,\delta}^{3,2}(t)| \leq \int_0^t \int_{\mathbb{T}^3 \times \{v: |v| \leq \frac{1}{\delta}\}} |u_\varepsilon - \eta_\varepsilon \star u_\varepsilon| |f_\varepsilon| |v| \, dx dv ds \leq C_\delta \|f_\varepsilon\|_{L^\infty} \|u_\varepsilon - \eta_\varepsilon \star u_\varepsilon\|_{L^1(0, T_*; L^1)}.$$

Then since $u_\varepsilon \rightarrow u$ in $L^2(0, T_*; L^2_{\text{loc}}(\mathbb{T}^3))$ we obtain

$$|R_{\varepsilon,\delta}^{3,2}(t)| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Thus we first let $\varepsilon \rightarrow 0$ to have $|R_\varepsilon^1(t)| + |R_\varepsilon^2(t)| + |R_\varepsilon^3(t)| \leq C\delta^{1/5}$ for all $\delta > 0$, and then let $\delta \rightarrow 0$ to have $R_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ uniformly in $t \in [0, T_*]$. We next use the weak- \star convergence of f_ε in $L^\infty(\mathbb{T}^3 \times \mathbb{R}^3 \times (0, T_*))$ to get

$$M_2 f(t) \leq \liminf_{\varepsilon \rightarrow 0} M_2 f_\varepsilon(t) \quad \text{for almost every } t \in [0, T_*].$$

Using that idea together with the strong convergence of u_ε in $L^2(\mathbb{T}^3 \times (0, T_*))$, we can also deal with the terms $\int_0^t \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon |u_\varepsilon - v|^2 \, dx dv ds$, $\|u_\varepsilon\|_{L^2(\mathbb{T}^3)}$, and $\int_0^t \|\nabla_x u_\varepsilon(s)\|_{L^2}^2 \, ds$. This completes the proof. □

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