

On the Euler-alignment system with weakly singular communication weights

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Abstract

We study the pressureless Euler equations with nonlocal alignment interactions, which arises as a macroscopic representation of the Cucker–Smale model on animal flocks. For the Euler-alignment system with bounded interactions, a critical threshold phenomenon is proved in Tadmor and Tan (2014 *Phil. Trans. R. Soc. A* **372** 20130401), where global regularity depends on initial data. With strongly singular interactions, global regularity is obtained in Do *et al* (2018 *Arch. Ration. Mech. Anal.* **228** 1–37), for all initial data. We consider the remaining case when the interaction is weakly singular. We show a critical threshold, similar to the system with bounded interaction. However, different global behaviors may happen for critical initial data, which reveals the unique structure of the weakly singular alignment operator.

Keywords: Euler-alignment system, weakly singular interaction, critical threshold, blowup

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1. Introduction

We are interested in the Euler-alignment system, which takes the form

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad (1)$$

$$\partial_t u + u \cdot \nabla u = \int \psi(|x - y|)(u(y) - u(x))\rho(y)dy. \quad (2)$$

The system arises as a macroscopic representation of models characterizing collective behaviors, in particular alignment and flocking.

Here, ρ represents the density of the group, and u is the associated velocity. The term that appears on the right hand side of (2) is the *alignment force*. It was first proposed by Cucker and Smale in [9] in the microscopic agent-based model

$$\dot{x}_i = v_i, \quad m\dot{v}_i = \frac{1}{N} \sum_{j=1}^N \psi(|x_i - x_j|)(v_j - v_i). \quad (3)$$

$\psi : \mathbb{R}^+ \rightarrow \mathbb{R}$ is called the *communication weight*, measuring the strength of the alignment interaction. A natural assumption on ψ is that it is a decreasing function, as the strength of the interaction is weaker when the distance between agents becomes larger.

The alignment force in the Cucker–Smale system (3) tends to align the velocity of all agents as time becomes large. The corresponding flocking phenomenon has been proved in [12], under appropriate assumptions on the communication weight.

The Euler-alignment system (1) and (2) can be derived from the Cucker–Smale system (3), through a kinetic description, as a hydrodynamic limit. See [13] for a formal derivation, [6, 20] for discussions on the kinetic system, and [11, 15] for the rigorous passage to the limit.

1.1. Bounded interaction

The Euler-alignment system (1) and (2) with bounded Lipschitz interactions was first studied in [19], where a *critical threshold phenomenon* is proved: subcritical initial data lead to global smooth solutions, while supercritical initial data lead to finite-time singularity formations.

In a subsequent work [4], a sharp critical threshold condition is obtained in 1D, with the help of an important quantity

$$G(x, t) = \partial_x u(x, t) + \int \psi(|x - y|) \rho(y, t) dy. \quad (4)$$

One can easily obtain the dynamics of G , see [4], as follows

$$\partial_t G + \partial_x(Gu) = 0. \quad (5)$$

This together with the dynamics of ρ

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad (6)$$

can serve as an alternative representation of (1) and (2). The velocity field u can be recovered by the relation (4) and the conservation of momentum.

The following theorem describes the sharp critical threshold condition.

Theorem 1.1 ([4]). *Consider the 1D Euler-alignment system (5) and (6) with smooth initial data (ρ_0, G_0) , and a bounded Lipschitz interaction ψ .*

- *If $\inf_x G_0(x) \geq 0$, then there exists a globally regular solution.*
- *If $\inf_x G_0(x) < 0$, then the solution admits a finite time blowup.*

For the 2D Euler-alignment system, the threshold conditions are obtained in [19], and also in [14] with further improvement. However, neither result is sharp.

1.2. Strongly singular interaction

One family of influence functions has the form

$$\psi(r) = r^{-s}. \quad (7)$$

When $s > 0$, ψ is unbounded at $r = 0$. This corresponds to the case when the alignment interaction becomes very strong as the distance becomes smaller.

In the case when $s \geq n$, where n is the space dimension, $\psi(|x|)$ is not integrable at $x = 0$. It has been studied recently that the so-called *strongly singular interaction* has a regularization effect, which prevents the solution from finite-time singularity formations. In 1D, global regularity is obtained in [10] for $s \in (1, 2)$, and in [18] for $s \in [2, 3)$ through a different approach.

Theorem 1.2 ([10, 18]). *Consider the 1D Euler-alignment system (5) and (6) with smooth periodic initial data (ρ_0, G_0) , and a strongly singular interaction ψ . Suppose $\rho_0 > 0$. Then, there exists a globally regular solution to (5) and (6).*

Note that since ψ is not integrable, the quantity G in (4) is not well-defined. For ψ defined in (7), one can use an alternative quantity $G = \partial_x u - (-\Delta)^{(s-1)/2} \rho$. For general choices of ψ with the same singularity at $x = 0$, a similar global regularity result has been obtained in [16]. Very recently, global regularity has been proved in [1] for the critical interaction $\psi(r) = r^{-1}$.

The dynamics in 2D is much more complicated and far less understood. Global regularity has been obtained recently in [17] only for a small class of initial data.

1.3. Weakly singular interaction

We are interested in the Euler-alignment system (1) and (2) with weakly singular interactions. This corresponds to the case when $\psi(|x|)$ is integrable, namely $\psi(r)$ behaves like r^{-s} near the origin with $s \in (0, n)$.

In this case, the quantity G is well-defined as long as the solution (ρ, u) is smooth, since

$$\|G\|_{L^\infty} \leq \|u\|_{W^{1,\infty}} + \|\psi\|_{L^1} \|\rho\|_{L^\infty}.$$

So in 1D, one would expect a similar critical threshold phenomenon as theorem 1.1. However, the result is not always true.

Let us consider a special case when $G_0(x) \equiv 0$. Since G satisfies (5), it is easy to see that $G(x, t) = 0$ in all time. The dynamics of ρ can be written as

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad u(x, t) = - \int K'(x - y) \rho(y, t) dy, \quad K''(x) = \psi(x), \tag{8}$$

that is the *aggregation equation* with a convex potential K (as $\psi \geq 0$).

The global well-posedness of the aggregation equation has been well-studied. A sharp Osgood condition has been derived in [2, 3, 5], which distinguishes global regularity and finite time density concentration: the solution is globally regular if and only if

$$\int_0^1 \frac{1}{K'(r)} dr = \infty. \tag{9}$$

For weakly singular interaction $\psi \sim r^{-s}$ with $s \in (0, 1)$ near the origin, or more precisely,

$$\lambda r^{-s} \leq \psi(r) \leq \Lambda r^{-s}, \quad \Lambda \geq \lambda > 0, \quad s \in (0, 1), \tag{10}$$

uniformly in $r \in (0, 1]$, the Osgood condition (9) is violated, and hence the solution generates concentrations in finite time. The behavior is different from the bounded interaction case ($s = 0$), in which (9) holds.

In this paper, we study the global behavior of the Euler-alignment system with weakly singular interactions.

The following two theorems show similar behavior to the system with bounded interactions (theorem 1.1), for both supercritical and subcritical regions of initial data.

Theorem 1.3 (Supercritical threshold condition). *Consider the 1D Euler-alignment system (5) and (6) with smooth initial data (ρ_0, G_0) and weakly singular interaction ψ satisfying (10).*

If $\inf_x G_0(x) < 0$, then the solution admits a finite time blowup.

Theorem 1.4 (Subcritical threshold condition). *Consider the 1D Euler-alignment system (5) and (6) with smooth initial data (ρ_0, G_0) and weakly singular interaction ψ satisfying (10).*

If $\inf_x G_0(x) > 0$, then there exists a globally regular solution.

The theorems imply that different behaviors between systems with bounded and weakly singular interactions can only happen for critical initial data

$$\inf_x G_0(x) = 0.$$

The example above ($G_0(x) \equiv 0$) falls into this category. The following theorem describes a large set of critical initial data, with which the solution blows up in finite time.

Theorem 1.5 (Blowup for critical initial data). *Consider the 1D Euler-alignment system (5) and (6) with smooth initial data (ρ_0, G_0) and weakly singular interaction ψ satisfying (10).*

If $G_0(x) \geq 0$, and there exists an interval $I = [a, b]$ with $a < b$ such that for any $x \in I$, $G_0(x) = 0$ and $\rho_0(x) > 0$, then the solution admits a finite time blowup.

The theorem says, if G_0 reaches zero in any non-vacuum interval, then the solution will blow up in finite time. Very importantly, such initial data will lead to a global smooth solution if the communication weight is regular, due to theorem 1.1. The different long-time behaviors distinguish the two types of interactions, and reveal the unique property of the weakly singular interactions.

The rest of the paper is organized as follows. In section 2, we develop a local well-posedness theory of the 1D Euler-alignment system, as well as a Beale–Kato–Majda criterion that ensures the regularity. Sections 3 and 4 are devoted to proving theorem 1.3 and 1.4, respectively. A nonlinear maximum principle is introduced to take care of the weak singularity on the communication weight. The critical case is investigated in section 5. We introduce a new proof for the blowup of the aggregation equation. It utilizes local information and can be extended to the Euler-alignment system, proving theorem 1.5. Finally, in section 6, we make comments on the extension of our theory to higher dimensions.

2. Local well-posedness and blowup criterion

We start our discussion with a local well-posedness theory of our main system in 1D. Recall the 1D Euler-alignment system in (ρ, G) representation

$$\partial_t \rho + \partial_x(\rho u) = 0, \tag{11}$$

$$\partial_t G + \partial_x(Gu) = 0, \tag{12}$$

$$\partial_x u = G - \psi * \rho, \tag{13}$$

where $x \in \Omega$, $t \geq 0$, and $*$ stands for convolution in x variable. The spacial domain Ω can be either the whole real line \mathbb{R} , or the periodic domain \mathbb{T} .

Theorem 2.1 (Local well-posedness). *Consider the 1D Euler-alignment system (11)–(13) with smooth initial data with finite mass $(\rho_0, G_0) \in (H^s \cap L^1_+)(\Omega) \times H^s(\Omega)$, for $s > 1/2$. Suppose the communication weight is integrable:*

$$\psi \in L^1(\Omega). \tag{14}$$

Then, there exists a time $T > 0$ such that the solution

$$(\rho, G) \in \mathcal{C}([0, T]; (H^s \cap L^1_+)(\Omega)) \times \mathcal{C}([0, T]; H^s(\Omega)). \tag{15}$$

Moreover, given any finite time T , the solution stays smooth up to time T , in the sense of (15), if and only if

$$\int_0^T (\|\rho(\cdot, t)\|_{L^\infty} + \|G(\cdot, t)\|_{L^\infty}) dt < +\infty. \tag{16}$$

Proof. We first state an H^s -estimate on ρ

$$\frac{d}{dt} \|\rho(\cdot, t)\|_{H^s}^2 \lesssim [\|\rho\|_{L^\infty} + \|\partial_x u\|_{L^\infty}] (\|\rho\|_{H^s}^2 + \|\partial_x u\|_{H^s}^2). \tag{17}$$

The proof can be found, for instance, in [4, theorem appendix A.2]. We include a short sketch for the sake of completeness.

We start with acting operator $\Lambda^s := (I - \Delta)^{s/2}$ on the continuity equation (11) and integrate by parts against $\Lambda^s \rho$. It yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\rho(\cdot, t)\|_{H^s}^2 &= - \int_{\Omega} [\Lambda^s \partial_x u] \rho \cdot \Lambda^s \rho dx + \frac{1}{2} \int_{\Omega} \partial_x u \cdot (\Lambda^s \rho)^2 dx \\ &\leq \|[\Lambda^s \partial_x u] \rho\|_{L^2} \|\rho\|_{H^s} + \frac{1}{2} \|\partial_x u\|_{L^\infty} \|\rho\|_{H^s}^2. \end{aligned}$$

The estimate (17) can be obtained by applying the following Kato–Ponce type commutator estimate

$$\|[\Lambda^s \partial_x u] \rho\|_{L^2} \lesssim \|\partial_x u\|_{L^\infty} \|\rho\|_{H^s} + \|\partial_x u\|_{H^s} \|\rho\|_{L^\infty}.$$

As G satisfies the same continuity equation as ρ , we have

$$\frac{d}{dt} \|G(\cdot, t)\|_{H^s}^2 \lesssim [\|G\|_{L^\infty} + \|\partial_x u\|_{L^\infty}] (\|G\|_{H^s}^2 + \|\partial_x u\|_{H^s}^2). \tag{18}$$

Putting these two estimate together, we obtain

$$\frac{d}{dt} (\|\rho(\cdot, t)\|_{H^s}^2 + \|G(\cdot, t)\|_{H^s}^2) \lesssim [\|\rho\|_{L^\infty} + \|G\|_{L^\infty} + \|\partial_x u\|_{L^\infty}] (\|\rho\|_{H^s}^2 + \|G\|_{H^s}^2 + \|\partial_x u\|_{H^s}^2). \tag{19}$$

From the relation (13), we can estimate $\partial_x u$ by ρ and G as follows. For a fixed time t ,

$$\begin{aligned} \|\partial_x u\|_{L^\infty} &\leq \|\psi * \rho\|_{L^\infty} + \|G\|_{L^\infty} \leq \|\psi\|_{L^1} \|\rho\|_{L^\infty} + \|G\|_{L^\infty}, \\ \|\partial_x u\|_{H^s} &\leq \|\psi * \rho\|_{H^s} + \|G\|_{H^s} \leq \|\psi\|_{L^1} \|\rho\|_{H^s} + \|G\|_{H^s}. \end{aligned}$$

Since $\|\psi\|_{L^1}$ is bounded, we now arrive at the estimate

$$\frac{d}{dt} (\|\rho(\cdot, t)\|_{H^s}^2 + \|G(\cdot, t)\|_{H^s}^2) \lesssim [\|\rho(\cdot, t)\|_{L^\infty} + \|G(\cdot, t)\|_{L^\infty}] (\|\rho(\cdot, t)\|_{H^s}^2 + \|G(\cdot, t)\|_{H^s}^2).$$

For $s > 1/2$, we have the embedding $H^s(\Omega) \subset L^\infty(\Omega)$. Therefore,

$$\|\rho(\cdot, t)\|_{L^\infty} + \|G(\cdot, t)\|_{L^\infty} \lesssim \|\rho(\cdot, t)\|_{H^s} + \|G(\cdot, t)\|_{H^s} \lesssim (\|\rho(\cdot, t)\|_{H^s}^2 + \|G(\cdot, t)\|_{H^s}^2)^{1/2}.$$

It implies

$$\frac{d}{dt} (\|\rho(\cdot, t)\|_{H^s}^2 + \|G(\cdot, t)\|_{H^s}^2) \lesssim (\|\rho(\cdot, t)\|_{H^s}^2 + \|G(\cdot, t)\|_{H^s}^2)^{3/2}.$$

Local well-posedness then follows directly from the Cauchy–Lipschitz theorem.

Moreover, applying Gronwall inequality to (19), we obtain

$$\|\rho(\cdot, T)\|_{H^s}^2 + \|G(\cdot, T)\|_{H^s}^2 \leq (\|\rho_0\|_{H^s}^2 + \|G_0\|_{H^s}^2) \exp \left[\int_0^T (\|\rho(\cdot, t)\|_{L^\infty} + \|G(\cdot, t)\|_{L^\infty}) dt \right].$$

Therefore, if condition (16) is satisfied, $\rho(\cdot, t), G(\cdot, t) \in H^s(\mathbb{R})$ for all $t \in [0, T]$.

On the other hand, if condition (16) is violated, namely

$$\int_0^T (\|\rho(\cdot, t)\|_{L^\infty} + \|G(\cdot, t)\|_{L^\infty}) dt = +\infty,$$

the same embedding $H^s(\Omega) \subset L^\infty(\Omega)$ would imply

$$\int_0^T (\|\rho(\cdot, t)\|_{H^s} + \|G(\cdot, t)\|_{H^s}) dt = +\infty.$$

Hence (15) cannot hold, and the solution cannot be smooth. This ends the proof of the theorem. □

We shall make several remarks regarding theorem 2.1.

Remark 2.1. A local well-posedness proof for 1D Euler-alignment system has been done in [4, theorem appendix A.1], with an additional assumption on ψ

$$\int_{\Omega} x\psi'(x)dx < +\infty.$$

Here, we relax the assumption by making use of the (ρ, G) formulation of the system.

If assumption (14) is violated, namely ψ is not integrable at the origin, then the behavior of the equation changes dramatically due to the strongly singular interaction. We refer to [10, 16] for discussions on local and global regularities under such setup.

Remark 2.2. Condition (16) is called the Beale–Kato–Majda (BKM) type criterion. It provides a sufficient and necessary condition under which the solution stays smooth. Condition (16) is equivalent to

$$\int_0^T \|\partial_x u(\cdot, t)\|_{L^\infty} dt < +\infty,$$

which is a standard sufficient condition to ensure the well-posedness of the characteristic paths for pressureless Euler dynamics. The equivalency is due to the following estimates

$$\begin{aligned} \int_0^T \|\partial_x u(\cdot, t)\|_{L^\infty} dt &\leq \int_0^T (\|\rho(\cdot, t)\|_{L^\infty} + \|\psi\|_{L^1} \|G(\cdot, t)\|_{L^\infty}) dt, \\ \|\rho(\cdot, t)\|_{L^\infty} + \|G(\cdot, t)\|_{L^\infty} &\leq (\|\rho_0\|_{L^\infty} + \|G_0\|_{L^\infty}) \exp \left[\int_0^t \|\partial_x u(\cdot, s)\|_{L^\infty} ds \right]. \end{aligned}$$

Remark 2.3. When $\Omega = \mathbb{R}$, assumption (14) can be further generalized to

$$\psi \in L^1(\mathbb{R}) + \text{const.}$$

This allows us to include more types of communication weight, for instance, $\psi \equiv 1$. We include a short proof for the sake of completeness.

Proof of remark 2.3. Let $\psi = \psi_0 + c$, where $\psi_0 \in L^1(\mathbb{R})$ and c is a constant. Define $G = \partial_x u + \psi_0 * \rho$. Then, the (ρ, G) representation of the 1D Euler-alignment system reads

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad \partial_t G + \partial_x(Gu) = -cm\partial_x u, \quad \partial_x u = G - \psi_0 * \rho,$$

where $m = \int_\Omega \rho_0(x) dx$ is the total mass which is preserved in time.

Due to the extra term in the dynamics of G , the H^s estimate on G (18) becomes

$$\begin{aligned} \frac{d}{dt} \|G(\cdot, t)\|_{H^s}^2 &\lesssim [\|G\|_{L^\infty} + \|\partial_x u\|_{L^\infty}] (\|G\|_{H^s}^2 + \|\partial_x u\|_{H^s}^2) + cm \|G\|_{H^s} \|\partial_x u\|_{H^s} \\ &\lesssim [1 + \|G\|_{L^\infty} + \|\partial_x u\|_{L^\infty}] (\|G\|_{H^s}^2 + \|\partial_x u\|_{H^s}^2). \end{aligned}$$

The rest of the proof stays the same as theorem 2.1. □

A natural question would be, whether the BKM criterion (16) can be further reduced to

$$\int_0^T \|G(\cdot, t)\|_{L^\infty} dt < +\infty. \tag{20}$$

In other word, whether boundedness of G implies boundedness of ρ . If so, the global regularity of the system becomes equivalent to the boundedness of G .

The following proposition shows that condition (20) indeed serves as a BKM criterion for the 1D Euler-alignment system, when the communication weight is bounded.

Proposition 2.1 [An enhanced BKM criterion for the system with bounded interactions] Consider the initial value problem of the 1D Euler-alignment system (11)–(13) with smooth initial data $(\rho_0, G_0) \in (H^s \cap L^1_+)(\Omega) \times H^s(\Omega)$, where $s > 1/2$. Suppose the communication weight is bounded and integrable:

$$\psi \in (L^1 \cap L^\infty)(\Omega) + \text{const.}$$

Suppose the criterion (20) is satisfied for time T . Then, the solution is smooth up to time T , namely

$$(\rho, G) \in \mathcal{C}([0, T]; (H^s \cap L^1_+)(\Omega)) \times \mathcal{C}([0, T]; H^s(\Omega)).$$

Proof. It suffices to prove that (20) implies (16).

Consider the characteristic path $X(t) := X(t; x_0)$ starting at $x_0 \in \Omega$

$$\frac{d}{dt}X(t; x_0) = u(X(t; x_0), t), \quad X(0; x_0) = x_0.$$

Since ψ is bounded, we can estimate

$$\|\partial_x u(\cdot, t)\|_{L^\infty} \leq \|G(\cdot, t)\|_{L^\infty} + m\|\psi\|_{L^\infty}. \tag{21}$$

Then, criterion (20) implies

$$\int_0^T \|\partial_x u(\cdot, t)\|_{L^\infty} \leq \int_0^T \|G(\cdot, t)\|_{L^\infty} dt + m\|\psi\|_{L^\infty} T < +\infty.$$

So the characteristic path $X(t; x_0)$ is well-defined for $t \in [0, T]$.

As ρ satisfies the continuity equation (11), we get

$$\frac{d}{dt}\rho(X(t), t) = -\partial_x u(X(t), t)\rho(X(t), t).$$

Then,

$$\rho(X(T), T) = \rho_0(x) \exp \left[-\int_0^T \partial_x u(X(t), t) dt \right] \leq \rho_0(x) \exp \left[\int_0^T \|\partial_x u(\cdot, t)\|_{L^\infty} dt \right] < +\infty.$$

Hence, the boundedness of G (20) does imply the boundedness of ρ (16). □

Using proposition 2.1, one can easily prove theorem 1.1, by showing criterion (20) is satisfied if and only if $\inf_x G_0(x) \geq 0$. We refer readers to [4] for details.

When the communication weight is weakly singular, proposition 2.1 might be false. In particular, the estimate (21) is no longer available. One alternative bound could be

$$\|\partial_x u(\cdot, t)\|_{L^\infty} \leq \|G(\cdot, t)\|_{L^\infty} + \|\rho(\cdot, t)\|_{L^\infty} \|\psi\|_{L^1}.$$

It implies an implicit bound

$$\|\rho(\cdot, t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} \exp \left[m\|\psi\|_{L^1} \int_0^t \|\rho(\cdot, s)\|_{L^\infty} ds + \int_0^t \|G(\cdot, t)\|_{L^\infty} ds \right],$$

which is not enough to obtain boundedness of ρ .

In fact, a counterexample such that proposition 2.1 fails for weakly singular interaction has been mentioned in the introduction, where $G_0(x) \equiv 0$. The corresponding aggregation system (8) is known to have a finite time loss of regularity as long as ψ is unbounded at the origin.

Therefore, the global regularity theory of the Euler-alignment system with bounded interaction cannot be directly extended to the case when the interaction is weakly singular.

3. Supercritical threshold condition

3.1. Finite time blowup on G

In this section, we prove theorem 1.3: solution forms a singularity in finite time, for supercritical initial data

$$\inf_{x \in \Omega} G_0(x) < 0.$$

Under such configuration, there exists an $x_0 \in \Omega$ such that $G_0(x_0) < 0$. Denote $X(t)$ be the characteristic path starting at x_0

$$\frac{d}{dt} X(t) = u(X(t), t), \quad X(0) = x_0.$$

As long as the solution stays smooth, alongside $X(t)$, we have

$$\frac{d}{dt} G(X(t), t) = -\partial_x u(X(t), t) G(X(t), t).$$

This implies

$$G(X(t), t) = G_0(x_0) \exp \left[\int_0^t \partial_x u(X(s), s) ds \right] < 0.$$

Moreover, $\psi * \rho(\cdot, t) \geq 0$ for any $t \geq 0$. From (13), we get

$$\frac{d}{dt} G(X(t), t) = -G^2(X(t), t) + G(X(t), t) (\psi * \rho(\cdot, t))(X(t)) \leq -G^2(X(t), t).$$

Applying a classical comparison principle, we obtain

$$G(X(t), t) \leq \frac{1}{t + \frac{1}{G_0(x_0)}} \xrightarrow{t \rightarrow -\frac{1}{G_0(x_0)}} -\infty.$$

Therefore, there exists a finite time $T \leq -\frac{1}{G_0(x_0)}$, such that

$$\lim_{t \rightarrow T^-} G(X(t), t) = -\infty. \tag{22}$$

The BKM criterion (16) fails at time T , which leads to a loss of regularity.

Now, we discuss the behavior of the solution (ρ, u) , if the first blowup happens at time T , in the form (22).

Lemma 3.1. *Suppose there exists a finite time T such that the solution (ρ, u) stays smooth for $t \in [0, T)$, and G becomes singular at time T , and location $x = X(T; x_0)$. Then, the solution develops a shock at time T and location x , namely*

$$\lim_{t \rightarrow T} \partial_x u(X(t; x_0), t) = -\infty.$$

Moreover, if $\rho_0(x_0) > 0$, then the density concentrates at the shock location (called singular shock)

$$\lim_{t \rightarrow T} \rho(X(t; x_0), t) = +\infty.$$

Proof. From (13), we know $\partial_x u \leq G$. This together with (22) implies shock formation

$$\partial_x u(X(t; x_0), t) \leq G(X(t; x_0), t) \xrightarrow{t \rightarrow T} -\infty.$$

Define $F = G/\rho$. Then, F satisfies the local transport equation

$$\partial_t F + u \partial_x F = 0.$$

Since $\rho_0(x_0) > 0$, F_0 is bounded and smooth in a neighborhood of x_0 . Then, F is well-defined alongside the characteristic path $X(t; x_0)$, and

$$F(X(t; x_0), t) = F_0(x_0).$$

Therefore, we obtain a concentration of density

$$\rho(X(t; x_0), t) = \frac{\rho_0(x_0)}{G_0(x_0)} G(X(t; x_0), t) \xrightarrow{t \rightarrow T} +\infty. \quad \square$$

Lemma 3.1 does not rule out the possibility that blowup happens before G becomes singular. It says, if there are no other types of blowup, the solution has to form a singular shock at the time when G blows up.

Indeed, the BKM criterion (16) could fail if ρ becomes unbounded.

We now construct an example when ρ blows up before G . This would imply that the criterion (20) itself does not guarantee the regularity of the system. So proposition 2.1 is no longer true for the system with weakly singular interactions.

3.2. An example: ρ blows up before G

Take $\Omega = \mathbb{R}$. Let ρ_0 be a smooth function supported in $[0, 1]$.

Let η be a smooth function such that $\eta \geq 0$, $\max_x \eta(x) = 1$, and $\text{supp}(\eta) = [0, 1]$. Consider the following G_0

$$G_0(x) = -\epsilon \eta(x - L), \tag{23}$$

where $L > 0$ is a large number, and $\epsilon > 0$ is a small positive number to be chosen. As $\inf G_0(x) = -\epsilon < 0$, G_0 is a supercritical initial condition.

Note that $\text{supp}(G_0) = [L, L + 1]$. If L is large enough, $\text{supp}(\rho_0) \cap \text{supp}(G_0) = \emptyset$. Starting from any $x_0 \in \text{supp}(\rho_0)$, we have $G_0(x_0) = 0$ and consequently $G(X(t; x_0), t) = 0$. So, $\partial_x u = -\psi * \rho$ in the support of ρ , since $\partial_x u$ is locally depended on G . Therefore, the dynamics of ρ does not depend on G , and it is the same as the aggregation equation (8). Since ψ is singular, we know the density ρ concentrates at a finite time T_* , which is independent of L and ϵ .

The goal is to show G remains regular at time T_* . It suffices to prove that G is bounded from below at T_* . To this end, we shall obtain a lower bound estimate on G . Fix $x_0 \in \text{supp}(G_0)$. Then, along its characteristic path, we have

$$\frac{d}{dt} G(X(t; x_0), t) = -G(X(t; x_0), t) \partial_x u(X(t; x_0), t) = -G^2 + G \psi * \rho. \tag{24}$$

As $G(X(t; x_0), t) < 0$, we need an upper bound on $\psi * \rho$.

If the supports of $\rho(\cdot, t)$ and $G(\cdot, t)$ are well-separated, namely

$$\text{dist}(\text{supp}(\rho(\cdot, t)), \text{supp}(G(\cdot, t))) \geq 1, \tag{25}$$

for $t \in [0, T_*]$, then we have the estimate

$$\psi * \rho(X(t; x_0), t) = \int_{\text{supp}\rho(t)} \psi(X(t; x_0) - y, t) \rho(y, t) dy \leq \psi(1) \int_{\text{supp}\rho(t)} \rho(y, t) dy = \psi(1)m.$$

Let us denote the constant $C = \psi(1)m$. It is uniform in $x_0 \in \text{supp}(G_0)$ and $t \in [0, T_*]$. Apply the estimate to (24), we get

$$\frac{d}{dt} G(X(t; x_0), t) \geq -G^2 - CG.$$

An explicit calculation yields

$$G(X(t; x_0), t) \geq -\frac{C}{\frac{G_0(x_0)-c}{G_0(x_0)} e^{-Ct} - 1}.$$

So, if $G_0(x_0) \geq -C$, then

$$G(X(t; x_0), t) \geq -C, \quad \forall t \in \left[0, \frac{1}{C} \ln \left(\frac{C - G_0(x_0)}{-2G_0(x_0)} \right) \right].$$

Note that

$$\lim_{z \rightarrow 0^-} \left[\frac{1}{C} \ln \left(\frac{C - z}{-2z} \right) \right] = +\infty.$$

It means that if we pick ϵ small enough, $G(\cdot, t)$ can be bounded below by $-C$ for a sufficiently long time. In particular, we can choose ϵ small enough, e.g.

$$\epsilon = \frac{C}{2e^{CT_*} - 1},$$

so that G is bounded until $t = T_*$.

It remains to show that condition (25) holds at $t \in [0, T_*]$.

One important feature of the Euler-alignment system (1) and (2) is that the velocity is uniformly bounded in time

$$\|u(\cdot, t)\|_{L^\infty} \leq \|u_0\|_{L^\infty}.$$

Indeed, a maximum principle can be easily derived from (2) (see for instance [19]). Moreover, under additional assumptions, not only boundedness but also contraction on u can be proved, which reveals the so-called *flocking phenomenon*.

Take $x_1 \in \text{supp}(\rho_0)$ and $x_2 \in \text{supp}(G_0)$. Then,

$$\frac{d}{dt} (X(t; x_2) - X(t; x_1)) = u(X(t; x_2), t) - u(X(t; x_1), t) \geq -2\|u(\cdot, t)\|_{L^\infty} \geq -2\|u_0\|_{L^\infty}.$$

Hence,

$$X(t; x_2) - X(t; x_1) \geq (x_2 - x_1) - 2t\|u_0\|_{L^\infty} \geq (L - 1) - 2t\|u_0\|_{L^\infty}.$$

If we take L big enough (e.g. $L = 2 + 2T_*\|u_0\|_{L^\infty}$), then the distance will remain big at time T_* . Therefore, (25) holds for $t \in [0, T_*]$.

3.3. The BKM criterion

The example above states that ρ could blow up before G . On the other hand, G could blow up before ρ as well. Examples can be constructed similarly, by letting ϵ in (23) large.

Therefore, both terms in the BKM criterion (16) are necessary to ensure regularity. This is very different from the system with bounded interactions.

4. Subcritical threshold condition

In this section, we turn to study the Euler-alignment system with weakly singular interactions, for subcritical initial data

$$\inf_{x \in \Omega} G_0(x) > 0. \tag{26}$$

Since G satisfies the continuity equation (12), it is easy to show that positivity preserves in time, namely

$$G(x, t) > 0, \quad \forall x \in \Omega, t \geq 0.$$

Hence, the blowup (22) cannot happen. However, unlike the case with bounded interactions, the boundedness of G (criterion (20)) is not enough to ensure global regularity, as argued in section 3.3. In order to prove theorem 1.4, we need to obtain bounds on both G and ρ .

4.1. A global estimate on ρ

We start with an estimate on ρ . Along the characteristic path, we have

$$\frac{d}{dt} \rho(X(t), t) = -\rho(X(t), t) \partial_x u(X(t), t) = -\rho G + \rho \psi * \rho.$$

The first term on the right hand side is a good term that helps bring down the value of ρ alongside the characteristic path, while the second term is a bad term.

4.1.1. Step 1: An estimate on the good term. Let $q = \rho/G = 1/F$. Then, q satisfies the transport equation

$$\partial_t q + u \partial_x q = 0.$$

Since G_0 satisfies (26), q_0 is bounded and smooth. Clearly, we have

$$q(X(t; x), t) = q_0(x).$$

Therefore, we obtain a lower bound estimate on G

$$G(X(t; x), t) = \frac{\rho(X(t; x), t)}{q(X(t; x), t)} = \frac{\rho(X(t; x), t)}{q_0(x)} \geq \frac{\rho(X(t; x), t)}{\|q_0\|_{L^\infty}}.$$

It yields an estimate on the good term

$$-\rho G \leq -C_1 \rho^2, \tag{27}$$

where the constant $C_1 = 1/\|q_0\|_{L^\infty}$ is bounded and depend only on the initial data.

4.1.2. Step 2: An estimate on the bad term. To estimate the bad term, and to compare with the good term, we need a local bound on $\psi * \rho$.

A nonlinear maximum principle is introduced in [8] which offers a local bound, at the extrema of ρ , when ψ is strongly singular. Here, we state a lemma which serves as a nonlinear maximum principle for weakly singular interactions.

Lemma 4.1 (Nonlinear maximum principle). *Let ψ be a weakly singular communication weight satisfying condition (10). Consider a function $f \in L^1_+(\mathbb{R})$ and a point x_* such that $f(x_*) = \max f(x)$. Then, there exists a constant $C > 0$, depending on Λ, s and $\|f\|_{L^1}$, such that*

$$\psi * f(x_*) \leq Cf(x_*)^s. \tag{28}$$

Proof. First of all, since $\psi * f \leq (\Lambda|x|^{-s}) * f$, it suffices to prove (28) for $\psi(x) = |x|^{-s}$. For any $a > 0$, we compute

$$\begin{aligned} ((|x|^{-s}) * f)(x_*) &= \int_{|y| \leq a} f(x_* - y)|y|^{-s} dy + \int_{|y| > a} f(x_* - y)|y|^{-s} dy \\ &= f(x_*) \int_{|y| \leq a} |y|^{-s} dy - \int_{|y| \leq a} (f(x_*) - f(x_* - y))|y|^{-s} dy + \int_{|y| > a} f(x_* - y)|y|^{-s} dy \\ &\leq \frac{2a^{1-s}}{1-s} f(x_*) - a^{-s} \int_{|y| \leq a} (f(x_*) - f(x_* - y)) dy + a^{-s} \int_{|y| > a} f(x_* - y) dy \\ &= \frac{2a^{1-s}}{1-s} f(x_*) - 2a^{1-s} f(x_*) + a^{-s} \|f\|_{L^1} = \frac{2s}{1-s} a^{1-s} f(x_*) + a^{-s} \|f\|_{L^1}. \end{aligned}$$

Take $a = \|f\|_{L^1} / (2f(x_*))$, we obtain

$$((|x|^{-s}) * f)(x_*) \leq \left(\frac{2-s}{1-s} 2^s \|f\|_{L^1}^{1-s} \right) f(x_*)^s.$$

□

We now apply lemma 4.1 with $f = \rho(\cdot, t)$. Fix any time t , and let x_* be the location where maximum of $\rho(\cdot, t)$ is attained. Then,

$$\rho(x_*, t) \psi * \rho(x_*, t) \leq C_2 \rho(x_*, t)^{1+s}, \tag{29}$$

where the constant $C_2 = C_2(\Lambda, s, m) > 0$.

4.1.3. Step 3: a uniform upper bound on ρ . Combining the two estimates (27) and (29), we obtain that if x_* is a point such that $\rho(x_*, t) = \max_x \rho(x, t)$, then

$$\partial_t \rho(x_*, t) \leq -C_1 \rho(x_*, t)^2 + C_2 \rho(x_*, t)^{1+s}.$$

So, when ρ is large enough such that $\rho \geq (C_2/C_1)^{1/(1-s)}$, then $\partial_t \rho(x_*, t) \leq 0$. Therefore, we obtain an *a priori* bound on the density

$$\|\rho(\cdot, t)\|_{L^\infty} \leq \max \left\{ \|\rho_0\|_{L^\infty}, (C_2/C_1)^{1/(1-s)} \right\} =: C_\rho, \quad \forall t \geq 0.$$

Note that the bound $C_\rho = C_\rho(\Lambda, s, m, \|\rho_0\|_{L^\infty}, \|q_0\|_{L^\infty})$ is independent of time. Therefore, $\|\rho(\cdot, t)\|_{L^\infty}$ is uniformly bounded.

4.2. An estimate on G

We are left to bound G , which is not hard to obtain given the *a priori* estimate on ρ .

Along the characteristic path, G satisfies (24). Recall

$$\frac{d}{dt} G(X(t), t) = -G^2 + G \psi * \rho.$$

The uniform bound on ρ implies a bound on $\psi * \rho$

$$\|\psi * \rho\|_{L^\infty} \leq \|\psi\|_{L^1} \|\rho\|_{L^\infty} \leq \|\psi\|_{L^1} C_\rho, \quad \forall t \geq 0.$$

Therefore, we obtain

$$\frac{d}{dt} G(X(t), t) \leq -G(G - \|\psi\|_{L^1} C_\rho).$$

So, G cannot grow along the characteristic path if $G \geq \|\psi\|_{L^1} C_\rho$. It yields a uniform bound on G

$$\|G(\cdot, t)\|_{L^\infty} \leq \{\|G_0\|_{L^\infty}, \|\psi\|_{L^1} C_\rho\}.$$

5. The critical case

This section is devoted to discussing the critical case, when the initial condition satisfies

$$\min_{x \in \mathbb{R}} G_0(x) = 0. \tag{30}$$

Theorems 1.3 and 1.4 show that the behavior of the Euler-alignment system with weakly singular interactions is the same as the system with bounded interactions, in both subcritical and supercritical regimes. Therefore, different behaviors can only happen in the critical scenario.

One special critical initial condition is $G_0(x) \equiv 0$. The system reduces to the aggregation equation (8). As we have argued in the introduction, the weakly singular interaction will drive the solution towards a finite time blowup. Therefore, theorem 1.1 no longer holds.

A natural question arises: what happens for general critical initial data?

Recall the dynamics of the density ρ along the characteristic path

$$\frac{d}{dt} \rho(X(t), t) = -\rho G + \rho \psi * \rho.$$

If $G_0(x_0) = 0$, then from (24) we have $G(X(t), t) = 0$. Therefore, the good term $-\rho G$ turns off near $X(t)$, and the local behavior of dynamics becomes the same as the aggregation equation.

To capture such behavior, we shall first provide an alternative proof to the blowup of the aggregation equation. Unlike [2], the proof traces local information along the characteristic paths. The idea is partly inspired by [7].

5.1. A 'local' proof for the blowup of the aggregation equation

Let us consider the aggregation equation in the form

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad \partial_x u = -\psi * \rho.$$

Without loss of generality, we assume that ρ_0 is strictly positive in some interval $I = [a, b]$, namely

$$\rho_0(x) \geq c > 0, \quad \forall x \in [a, b]. \tag{31}$$

Let $r(t) = X(t; b) - X(t; a)$. Then,

$$\begin{aligned} \frac{d}{dt}r(t) &= u(X(t; b), t) - u(X(t; a), t) = \int_{X(t;a)}^{X(t;b)} \partial_x u(y, t) dy \\ &= - \int_{X(t;a)}^{X(t;b)} (\psi * \rho)(y, t) dy = - \int_{X(t;a)}^{X(t;b)} \int_{-\infty}^{\infty} \psi(y - z) \rho(z, t) dz dy \\ &\leq - \int_{X(t;a)}^{X(t;b)} \int_{X(t;a)}^{X(t;b)} \psi(y - z) \rho(z, t) dz dy. \end{aligned}$$

For $y, z \in [X(t; a), X(t; b)]$, we have $|y - z| \leq 2r(t)$. By weakly singular condition (10), this implies $\psi(y - z) \geq \lambda(2r(t))^{-s}$. Therefore, we obtain

$$\frac{d}{dt}r(t) \leq -\lambda(2r(t))^{-s} r(t) \int_{X(t;a)}^{X(t;b)} \rho(z, t) dz. \tag{32}$$

The following lemma shows a local conservation of mass along characteristic paths.

Lemma 5.1 (Conservation of mass). *Let ρ be a strong solution of the continuity equation*

$$\partial_t \rho + \partial_x(\rho u) = 0.$$

Let $X(t; x_1), X(t; x_2)$ be two characteristic paths starting at x_1 and x_2 , respectively. Then,

$$\int_{X(t;x_1)}^{X(t;x_2)} \rho(x, t) dx = \int_{x_1}^{x_2} \rho_0(x) dx, \quad \forall t \geq 0. \tag{33}$$

Namely, the mass in the interval $[X(t; x_1), X(t; x_2)]$ is conserved in time.

Proof. Compute

$$\begin{aligned} &\frac{d}{dt} \int_{X(t;x_1)}^{X(t;x_2)} \rho(x, t) dx \\ &= \rho(X(t; x_2), t) \frac{d}{dt} X(t; x_2) - \rho(X(t; x_1), t) \frac{d}{dt} X(t; x_1) + \int_{X(t;x_1)}^{X(t;x_2)} \partial_t \rho(x, t) dx \\ &= \rho(X(t; x_2), t) u(X(t; x_2), t) - \rho(X(t; x_1), t) u(X(t; x_1), t) + \int_{X(t;x_1)}^{X(t;x_2)} \partial_t \rho(x, t) dx \\ &= \int_{X(t;x_1)}^{X(t;x_2)} \partial_x (\rho(x, t) u(x, t)) dx + \int_{X(t;x_1)}^{X(t;x_2)} \partial_t \rho(x, t) dx = 0. \end{aligned}$$

This directly implies the conservation of mass (33). □

Applying lemma 5.1 to (32) with $x_1 = a$ and $x_2 = b$, and using the lower bound assumption (31), we obtain

$$\frac{d}{dt}r(t) \leq -2^{-s} \lambda(r(t))^{1-s} \int_a^b \rho_0(z) dz \leq -2^{-s} c(b - a) \lambda(r(t))^{1-s}.$$

For $s \in (0, 1)$, it is easy to show that $r(t)$ reaches zero in finite time. Indeed, a standard comparison principle yields

$$r(t) \leq [(b - a)^s - 2^{-s}c(b - a)\lambda st]^{1/s},$$

where the right hand side touches zero at

$$T_* = \frac{2^s}{c(b - a)^{1-s}\lambda s} < \infty.$$

Then, $r(t)$ should reach zero no later than T_* .

The quantity $r(t) = 0$ means that two characteristic paths run into each other. It indicates a shock formation with $\partial_x u(x, t) \rightarrow -\infty$. Therefore, the solution loses regularity in finite time.

5.2. Finite-time blowup for a class of critical initial data

Now, let us consider the Euler-alignment system (11)–(13) with critical initial data (30).

Suppose there exists an interval $I = [a, b]$ such that ρ_0 is strictly positive, and G_0 is zero, namely

$$\forall x \in [a, b], \quad \rho_0(x) \geq c > 0, \quad G_0(x) = 0. \tag{34}$$

Then, from (24) we obtain

$$G(x, t) = 0, \quad \forall t \geq 0, \quad x \in [X(t; a), X(t; b)].$$

Therefore, the dynamics of ρ between the two characteristic paths $X(t; a)$ and $X(t; b)$ should be the same as the dynamics of the corresponding aggregation equation, as long as the solution stays smooth. The blowup estimates for the aggregation equation in section 5.1 can be directly applied to the Euler-alignment system. Therefore, the same type of blowup as the aggregation equation happens in finite time. This ends the proof of theorem 1.5.

Remark 5.1. The condition (34) contains a large family of critical initial data, under which the global behaviors of the Euler-alignment system is different between the bounded and weakly singular interactions. The condition is sharp in the following sense.

Consider the following critical initial data (ρ_0, G_0) :

$$\exists q_0 \in L^\infty, \quad \text{such that} \quad \rho_0(x) = q_0(x)G_0(x). \tag{35}$$

Then, the arguments in section 4 can be easily extended, allowing $G_0(x) = 0$. Hence, the solution exists globally in time.

Note that condition (35) implies that $G_0(x) = 0$ only occurs at $\rho_0(x) = 0$. which is almost the opposite of condition (34). Therefore, condition (34) is a sharp condition so that the global behaviors of systems with bounded and weakly singular interactions are different from each other.

Rare exceptions could happen. For instance, $G_0(x) = 0$ only at a single point x_0 , with $\rho_0(x_0) > 0$. It satisfies neither (34) nor (35). In this case, a more subtle ‘local’ proof is required for the corresponding aggregation system in order to obtain a finite time blowup. This will be left for further investigations.

6. Extensions to higher dimensions

The global behaviors of the Euler-alignment system (1) and (2) are much less understood in higher dimensions. With bounded interactions, the system was first studied in [19] in two dimensions. Threshold conditions on initial data were obtained, but the result was not sharp.

The G quantity can be defined as $G = \nabla \cdot u + \psi * \rho$. However, it does not satisfy the continuity equation anymore. The dynamics of G reads

$$\partial_t G + \nabla \cdot (Gu) = \text{tr}(\nabla u^{\otimes 2}) - (\nabla \cdot u)^2.$$

The right hand side is called the *spectral gap*, which is generally non-zero in two or higher dimensions.

The system in (ρ, G) formulation in 2D has been studied in [14], where improved threshold conditions are obtained. However, the result is still far from being sharp, due to the lack of control in the spectral gap.

For the Euler-alignment system with weakly singular interactions, our arguments on local well-posedness (section 2) as well as both supercritical and subcritical threshold conditions (sections 3 and 4) can be extended to higher dimensions, using similar techniques to handle the spectral gap. However, we are not able to distinguish the behaviors between the systems with bounded and weakly singular interactions (section 5) until we get a sharp threshold condition.

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