

Quantum limited super-resolution of an unequal-brightness source pair in three dimensions

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Abstract

This paper extends our previous quantum Fisher information (QFI) based analysis of the problem of separating a pair of equal-brightness incoherent point sources in three dimensions to the case of a pair of sources that are unequally bright. When the pair's geometric center is perfectly known in advance, QFI with respect to the estimation of the three separation coordinates remains independent of the degree of brightness asymmetry. For the experimentally more relevant case of perfect prior knowledge of the pair's brightness centroid, however, such QFI becomes dependent on the pair separation vector in a way that is controlled by the degree of its brightness asymmetry. This study yields potentially useful insights into the analysis of a more general superresolution imaging problem involving extended incoherent sources with nontrivial brightness distributions.

Keywords: pair superresolution, quantum estimation, quantum metrology, quantum Fisher information, Cramer–Rao lower bound

(Some figures may appear in colour only in the online journal)

1. Introduction

In the photon counting limit, wavefront projections via spatial demultiplexing (SPADE), as proposed recently by Tsang *et al* [1, 2], can overcome the celebrated Rayleigh limit [3] for resolving a symmetrical pair of incoherent point sources at photon numbers that scale inverse-quadratically with separation even at the smallest separations. This scaling law, which was first derived in [1, 2] by applying quantum estimation theory to the problem of estimating the pair separation in one and two dimensions, seems to be entirely independent of the spatial dimensionality of the problem, as we showed in two papers [4, 5] via an analysis considerably more general than that of [6] for full three-dimensional (3D) source pair separation and localization using apertures of arbitrary shape and size. It amounts to a qualitatively more modest requirement than that entailed by intensity based detections of the source pair on an image sensor for which this scaling law, for small separations, is

no better than inverse quartic in separation. Extensions to thermal source states [7] and general quantum source states [8] have also been considered. A number of recent experiments have confirmed the achievability of the quantum estimation bound both for lateral [9–12] and axial [13] resolutions.

The dramatic difference in the scaling behaviors of the sensitivity of wavefront and intensity based methods opens a novel, highly sensitive approach of superresolution imaging of relatively faint extended sources for which image intensity based methods are likely to fail. More recently, SPADE has been analyzed as a potentially sensitive approach for optically acquiring an extended luminous object that is too faint and small to be resolved and imaged by ordinary intensity based methods. The method is essentially based on acquiring first, second, and higher-order intensity moments of an extended incoherent object [14–17] using a suitable projection basis. Successful acquisition of such moments can enable one to characterize the object in terms of its brightness attributes such as centroid, size, ellipticity, skewness, etc. These methods can be potentially generalized further to extended 3D sources.

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An alternative approach for imaging an extended incoherent source might regard it as a collection of point sources [18] with spatially varying intensity across it. The simplest example of such an extended object with a nonuniform brightness distribution is a pair of incoherent point sources of unequal intensity. We consider in this paper the problem of quantum limited superresolution of such a source pair. We will use the concepts of quantum Fisher information (QFI) and quantum Cramér Rao bound (QCRB) from quantum estimation theory put forth by Helstrom [19] to evaluate the fidelity of estimation of the separation vector for an arbitrary ratio of the intensities of the two point sources. The problem of calculating such a quantum limit on simultaneously estimating the geometric center, separation, and relative intensity of an unequally bright source pair using these concepts was first tackled by Řeháček, *et al* [20, 21], but their work was limited to one spatial dimension.

In the present paper, we will generalize the work by Řeháček's group, as well as our previous work [4] to an asymmetrically bright source pair in all three dimensions. Rather astoundingly, as we will show, the QFI whose independence from pair separation is the basis of such scaling remains independent of this separation even for an unequally bright source pair when its geometric center is well determined and fixed *a priori*. But for the experimentally more relevant scenario where only the brightness centroid, rather than the geometric center, can be reliably determined in advance, such independence obtains only for the equal-brightness case for which the two centroids coincide. We derive here the detailed dependence of QFI on the pair brightness ratio and separation for estimating the latter under these conditions.

Let us consider a pair of closely spaced incoherent point sources that have unequal average intensities I_{\pm} , so the quantum state of a photon emitted by such a source pair is described by the density operator (DO)

$$\hat{\rho} = p_+ |K_+\rangle \langle K_+| + p_- |K_-\rangle \langle K_-|, \quad p_{\pm} \stackrel{\text{def}}{=} \frac{I_{\pm}}{I_+ + I_-}, \quad (1)$$

in which p_{\pm} may be regarded as the probability of the photon being emitted by the source in the state $|K_{\pm}\rangle$, respectively. We consider only those photons that are captured and detected by the observing instrument, as only they can encode information about the source pair that is recoverable by means of measurements made on them. We further restrict attention to the classically ideal regime of photon counting without any detection noise under which the photons detected during a fixed observation time have a Poisson number distribution. In this regime, which applies to sources of small intensity fluctuations per mode such as ordinary thermal sources possessing a small degeneracy parameter, we may regard the photons as arriving in a statistically independent fashion at the instrument sensor. Consequently the Fisher information (FI) for the problem of N photocounts, whether classical or quantum, becomes simply N times the FI per photon. It is the quantum version of this FI per photon, namely QFI, that we set out to calculate. The diagonal elements of the inverse of the QFI matrix with respect to (w.r.t.) a set of n system

parameters, $\{\theta_1, \dots, \theta_n\}$, on which $\hat{\rho}$ depends determine the lowest possible variances for an unbiased estimation of these parameters, regardless of any specific quantum measurement, and depend only on the quantum state of the source photons.

2. Quantum limited estimation of source-pair separation for known geometric center and relative brightness

In the first half of the paper, we take the two point sources to be located at $\pm \mathbf{l}$, relative to their geometric center chosen to be the origin. Subsequently we shall refer their locations relative to the intensity centroid, a point that is likely to be easier to locate based on image intensity measurements on a pixelated sensor array when the sources are close together.

2.1. Photon wavefunction

As we have noted previously [4], the corresponding normalized wavefunctions take the following form over the exit pupil:

$$\langle \mathbf{u} | K_{\pm} \rangle = \exp(\pm i\phi) P(\mathbf{u}) \exp[\mp i\Psi(\mathbf{u}; \mathbf{l})], \quad (2)$$

in which $P(\mathbf{u})$ is a generally-complex pupil function obeying the normalization condition over the pupil plane

$$\int d^2u |P(\mathbf{u})|^2 = 1, \quad (3)$$

\mathbf{u} is the normalized pupil-plane position vector, which for a circular pupil is obtained by dividing the physical pupil-plane position vector by the radius of the pupil, and the phase function, $\Psi(\mathbf{u}; \mathbf{l})$, for a low numerical-aperture system [22] has the form

$$\Psi(\mathbf{u}; \mathbf{l}) = 2\pi \mathbf{u} \cdot \mathbf{l}_{\perp} + \pi u^2 l_z, \quad (4)$$

which is linear in the vector of 3D spatial-separation parameters, $\mathbf{l} \stackrel{\text{def}}{=} (\mathbf{l}_{\perp}, l_z)$. We adopt a convenient convention of writing 3D spatial vectors in terms of their 2D transverse projections, such as \mathbf{l}_{\perp} , and axial coordinates, such as l_z . The actual physical 3D separation coordinates are proportional to these parameters via two different, transverse and axial, diffraction scales, as [5]

$$\mathbf{R}_{\perp} = \frac{\lambda z_I}{R} \mathbf{l}_{\perp}; \quad Z = \frac{\lambda z_O^2}{R^2} l_z, \quad (5)$$

in which (\mathbf{R}_{\perp}, Z) is half the physical 3D vector separating the two sources, z_O, z_I are the average object-plane and image plane distances from the exit pupil of the imager, λ is the optical wavelength, and R , for a circular pupil, is the radius of the exit pupil. For a more general pupil shape, R could be regarded as a characteristic size of the pupil. For a thin-lens imager [22], the entrance and exit pupils coincide with the lens aperture.

The phase constant, ϕ , is conveniently chosen to make the inner product, $\Delta \stackrel{\text{def}}{=}} \langle K_- | K_+ \rangle$, real and positive, which implies two equivalent relations for Δ

$$\Delta = \langle K_- | K_+ \rangle = \langle K_+ | K_- \rangle. \quad (6)$$

In view of relations (2) and (4) for the wavefunction and Ψ , this inner product may be expressed as

$$\Delta \stackrel{\text{def}}{=} \langle K_+ | K_- \rangle = \exp(-2i\phi) \int d^2u |P(\mathbf{u})|^2 \times \exp(i4\pi \mathbf{1}_\perp \cdot \mathbf{u} + i2\pi l_\tau u^2). \quad (7)$$

Note that the phase constant, ϕ , being half of the complex phase of the integral on the RHS of equation (7), depends, in general, on the pair separation vector, \mathbf{l} . For the clear, unit-radius circular aperture, the pupil function $P(\mathbf{u})$ is simply $1/\sqrt{\pi}$ times the indicator function for the aperture, but our calculations make no reference to the actual form of the pupil function, which is subject only to the normalization condition (3).

2.2. Pair-Separation QFI

The QFI matrix, \mathbf{H} , is defined [19, 23–25] to have elements $H_{\mu\nu} \stackrel{\text{def}}{=} \text{Re Tr}(\hat{\rho} \hat{L}_\mu \hat{L}_\nu)$, where Re denotes the real part and \hat{L}_μ is the symmetric logarithmic derivative of the DO w.r.t. the μ th parameter. The QFI matrix elements may be readily shown [4] to have the form

$$H_{\mu\nu} = \text{Re } H_{\mu\nu}, \quad (8)$$

in which $H_{\mu\nu}$ may be expressed as

$$H_{\mu\nu} = \sum_{i=\pm} \frac{4}{e_i} \langle e_i | \partial_\mu \hat{\rho} \partial_\nu \hat{\rho} | e_i \rangle + \sum_{i=\pm} \sum_{j=\pm} \left[\frac{4e_i}{(e_i + e_j)^2} - \frac{4}{e_i} \right] \langle e_i | \partial_\mu \hat{\rho} | e_j \rangle \langle e_j | \partial_\nu \hat{\rho} | e_i \rangle. \quad (9)$$

Here e_\pm and $|e_\pm\rangle$ are the two non-vanishing eigenvalues and the corresponding eigenstates of DO (1), respectively, and $\partial_\mu \stackrel{\text{def}}{=} \partial/\partial\theta_\mu$ denotes the first order partial derivative w.r.t. the parameter θ_μ , of the quantity that immediately follows it.² By decomposing the double sum in equation (9) into its diagonal ($i = j$) and off-diagonal ($i \neq j$) terms and noting that (see [4], supplement)

$$\langle e_i | \partial_\mu \hat{\rho} | e_i \rangle = \partial_\mu e_i \quad (10)$$

and the DO trace norm relation

$$e_+ + e_- = 1, \quad (11)$$

we may express equation (9) as

$$H_{\mu\nu} = \sum_{i=\pm} \frac{4}{e_i} \langle e_i | \partial_\mu \hat{\rho} \partial_\nu \hat{\rho} | e_i \rangle - \frac{3}{e_+ e_-} \partial_\mu e_+ \partial_\nu e_+ - 4 \sum_{\substack{i \neq j \\ ij \in \{+, -\}}} \left(\frac{1}{e_i} - e_i \right) \langle e_i | \partial_\mu \hat{\rho} | e_j \rangle \langle e_j | \partial_\nu \hat{\rho} | e_i \rangle. \quad (12)$$

² We avoid the use of parentheses to ease notation except when the derivative is to be taken of multiple quantities which we enclose in parentheses.

The range space of $\hat{\rho}$ is spanned by the two states $|K_\pm\rangle$ in which we may find its eigenstates as the linear combinations

$$|e_\pm\rangle = \alpha_\pm |K_+\rangle + \beta_\pm |K_-\rangle. \quad (13)$$

The coefficients, α_\pm, β_\pm , as well as the eigenvalues e_\pm , may be determined by substituting expressions (13) and (1) into the eigen-relation

$$\hat{\rho} |e_\pm\rangle = e_\pm |e_\pm\rangle, \quad (14)$$

comparing coefficients of the two linearly independent states $|K_\pm\rangle$ on both sides of this equation, and requiring that the determinant of the underlying homogeneous linear system of equations in these coefficients vanish, we arrive, after some algebraic simplifications, at the following expressions for e_\pm and coefficients α_\pm, β_\pm :

$$e_\pm = \frac{1}{2}(1 \pm \delta e), \quad \delta e \stackrel{\text{def}}{=} \sqrt{\delta p^2 + \Delta^2(1 - \delta p^2)},$$

$$\alpha_\pm = \left[\frac{p_+(\delta e \pm \delta p)}{\delta e(1 \pm \delta e)} \right]^{1/2}, \quad \beta_\pm = \pm \left[\frac{p_-(\delta e \mp \delta p)}{\delta e(1 \pm \delta e)} \right]^{1/2}, \quad (15)$$

in which $\delta e = e_+ - e_-$ and $\delta p = p_+ - p_-$ are the differences between the two eigenvalues and the two state probabilities, respectively. Note that since $\Delta^2, \delta p^2 \leq 1$, it follows from equation (15) that $|\delta p| \leq \delta e \leq 1$. Equivalently, $e_+ \geq \max(p_+, p_-) \geq e_- \geq 0$. The unit normalization of the eigenstates, namely

$$\langle e_\pm | e_\pm \rangle = 1, \quad (16)$$

which in view of eigenstate expansion (13) and definition (7) of Δ is equivalent to the requirement,

$$\alpha_\pm^2 + \beta_\pm^2 + 2\alpha_\pm \beta_\pm \Delta = 1, \quad (17)$$

was used to fix the overall normalization of expressions (15) for the two pairs of coefficients.

One may easily evaluate the partial derivative of expression (15) for the eigenvalues w.r.t. a parameter θ_μ as

$$\partial_\mu e_\pm = \pm \frac{\Delta(1 - \delta p^2)}{2\delta e} \partial_\mu \Delta, \quad (18)$$

which allows the second term on the RHS of equation (12) to be evaluated in terms of a product of derivatives of the state-overlap function, Δ . We may thus simplify equation (12) as

$$H_{\mu\nu} = \sum_{i=\pm} \frac{4}{e_i} \langle e_i | \partial_\mu \hat{\rho} \partial_\nu \hat{\rho} | e_i \rangle - \frac{3\Delta^2(1 - \delta p^2)^2}{\delta e^2(1 - \delta e^2)} \partial_\mu \Delta \partial_\nu \Delta - 4 \sum_{\substack{i \neq j \\ ij \in \{+, -\}}} \left(\frac{1}{e_i} - e_i \right) \langle e_i | \partial_\mu \hat{\rho} | e_j \rangle \langle e_j | \partial_\nu \hat{\rho} | e_i \rangle. \quad (19)$$

By using expression (1) for DO in terms of the corresponding derivatives of the single-source emission states, $|K_{\pm}\rangle$, we may evaluate its derivatives and their bilinear products as

$$\begin{aligned} \partial_{\mu}\hat{\rho} &= p_{+}(\partial_{\mu}|K_{+}\rangle\langle K_{+}| + |K_{+}\rangle\partial_{\mu}\langle K_{+}|) \\ &+ p_{-}(\partial_{\mu}|K_{-}\rangle\langle K_{-}| + |K_{-}\rangle\partial_{\mu}\langle K_{-}|); \\ \partial_{\mu}\hat{\rho}\partial_{\nu}\hat{\rho} &= [p_{+}^2(\partial_{\mu}|K_{+}\rangle\langle K_{+}|\partial_{\nu}|K_{+}\rangle\langle K_{+}| + \partial_{\mu}|K_{+}\rangle\partial_{\nu}\langle K_{+}|) \\ &+ |K_{+}\rangle\partial_{\mu}\langle K_{+}|\partial_{\nu}|K_{+}\rangle\langle K_{+}| + |K_{+}\rangle\partial_{\mu}\langle K_{+}|K_{+}\rangle\partial_{\nu}\langle K_{+}|) \\ &+ p_{+}p_{-}(\partial_{\mu}|K_{+}\rangle\langle K_{+}|\partial_{\nu}|K_{-}\rangle\langle K_{-}| + \partial_{\mu}|K_{+}\rangle\Delta\partial_{\nu}\langle K_{-}| \\ &+ |K_{+}\rangle\partial_{\mu}\langle K_{+}|\partial_{\nu}|K_{-}\rangle\langle K_{-}| + |K_{+}\rangle\partial_{\mu}\langle K_{+}|K_{-}\rangle\partial_{\nu}\langle K_{-}|) \\ &+ \text{terms obtained by } + \leftrightarrow - \text{ interchange}]. \end{aligned} \quad (20)$$

Substituting expressions (20) and expansions (13) of the eigenstates $|e_{\pm}\rangle$ into equation (19), we can now straightforwardly calculate its sums, which involve the matrix elements $\langle e_{\pm}|\partial_{\mu}\hat{\rho}\partial_{\nu}\hat{\rho}|e_{\pm}\rangle$ and $\langle e_{+}|\partial_{\mu}\hat{\rho}|e_{-}\rangle$. This process is greatly simplified by noting from the form of the wavefunctions (2) that matrix elements like $\langle K_{\pm}|\partial_{\mu}|K_{\pm}\rangle$ are purely imaginary and those like $\partial_{\mu}\langle K_{\pm}|\partial_{\nu}|K_{\pm}\rangle$ purely real, since each partial derivative of a wavefunction w.r.t. a separation coordinate l_{μ} , $\mu = 1, 2, 3$, generates a purely imaginary factor. Further, since according to expression (2) the phases of $\langle u|K_{\pm}\rangle$ are equal but oppositely signed, a partial derivative of both sides of the equality, $\langle K_{\pm}|K_{\pm}\rangle = \Delta$, yields the simplified real expression, $\langle K_{\pm}|\partial_{\mu}|K_{\mp}\rangle = (1/2)\partial_{\mu}\Delta$, which is also needed for the calculation of QFI. We thus obtain the following final expressions for the real parts of the various matrix elements in terms of the eigenvector expansion coefficients, $\alpha_{\pm}, \beta_{\pm}$:

$$\begin{aligned} \text{Re}(\langle e_{+}|\partial_{\mu}\hat{\rho}|e_{-}\rangle\langle e_{-}|\partial_{\nu}\hat{\rho}|e_{+}\rangle) &= g_1\langle K_{+}|\partial_{\mu}|K_{+}\rangle \\ &\times \langle K_{+}|\partial_{\nu}|K_{+}\rangle + g_2\partial_{\mu}\Delta\partial_{\nu}\Delta; \\ \text{Re}\langle e_{+}|\partial_{\mu}\hat{\rho}\partial_{\nu}\hat{\rho}|e_{+}\rangle &= h_1^{(+)}\langle K_{+}|\partial_{\mu}|K_{+}\rangle\langle K_{+}|\partial_{\nu}|K_{+}\rangle \\ &+ h_2^{(+)}\partial_{\mu}\langle K_{+}|\partial_{\nu}|K_{+}\rangle + h_3^{(+)}\text{Re}\partial_{\mu}\langle K_{+}|\partial_{\nu}|K_{-}\rangle \\ &+ h_4^{(+)}\partial_{\mu}\Delta\partial_{\nu}\Delta; \\ \text{Re}\langle e_{-}|\partial_{\mu}\hat{\rho}\partial_{\nu}\hat{\rho}|e_{-}\rangle &= h_1^{(-)}\langle K_{+}|\partial_{\mu}|K_{+}\rangle\langle K_{+}|\partial_{\nu}|K_{+}\rangle \\ &+ h_2^{(-)}\partial_{\mu}\langle K_{+}|\partial_{\nu}|K_{+}\rangle + h_3^{(-)}\text{Re}\partial_{\mu}\langle K_{+}|\partial_{\nu}|K_{-}\rangle \\ &+ h_4^{(-)}\partial_{\mu}\Delta\partial_{\nu}\Delta, \end{aligned} \quad (21)$$

where

$$\begin{aligned} g_1 &= -(\alpha_{-}\beta_{+} - \alpha_{+}\beta_{-})^2\Delta^2; \\ g_2 &= \frac{1}{4}[(\alpha_{-}\beta_{+} + \alpha_{+}\beta_{-}) + 2\Delta(p_{+}\beta_{+}\beta_{-} + p_{-}\alpha_{+}\alpha_{-})]^2; \\ h_1^{(\pm)} &= p_{+}^2\alpha_{\pm}^2 + p_{-}^2\beta_{\pm}^2 + 2\alpha_{\pm}\beta_{\pm}\Delta(p_{+}^2 + p_{-}^2 + p_{+}p_{-}); \\ h_2^{(\pm)} &= p_{+}^2(\alpha_{\pm} + \beta_{\pm}\Delta)^2 + p_{-}^2(\alpha_{\pm}\Delta + \beta_{\pm})^2; \\ h_3^{(\pm)} &= 2p_{+}p_{-}(\alpha_{\pm} + \beta_{\pm}\Delta)(\alpha_{\pm}\Delta + \beta_{\pm}); \\ h_4^{(\pm)} &= p_{-}^2\alpha_{\pm}^2 + p_{+}^2\beta_{\pm}^2 + 2p_{+}p_{-}(1 + \alpha_{\pm}\beta_{\pm}\Delta). \end{aligned} \quad (22)$$

In view of expressions (19) and (21), we may express the QFI matrix element (8) in terms of four nontrivial matrix elements involving the single-source emission states, $|K_{\pm}\rangle$, as

$$\begin{aligned} H_{\mu\nu} &= 4\left[\frac{h_1^{(+)}}{e_{+}} + \frac{h_1^{(-)}}{e_{-}} - g_1\left(\frac{1}{e_{+}e_{-}} - 1\right)\right] \\ &\times \langle K_{+}|\partial_{\mu}|K_{+}\rangle\langle K_{+}|\partial_{\nu}|K_{+}\rangle \\ &+ 4\left(\frac{h_2^{(+)}}{e_{+}} + \frac{h_2^{(-)}}{e_{-}}\right)\partial_{\mu}\langle K_{+}|\partial_{\nu}|K_{+}\rangle \\ &+ 4\left(\frac{h_3^{(+)}}{e_{+}} + \frac{h_3^{(-)}}{e_{-}}\right)\text{Re}\partial_{\mu}\langle K_{+}|\partial_{\nu}|K_{-}\rangle \\ &+ 4\left[\frac{h_4^{(+)}}{e_{+}} + \frac{h_4^{(-)}}{e_{-}} - g_2\left(\frac{1}{e_{+}e_{-}} - 1\right)\right] \\ &- \frac{3\Delta^2(1 - \delta p^2)^2}{\delta e^2(1 - \delta e^2)}\partial_{\mu}\Delta\partial_{\nu}\Delta, \end{aligned} \quad (23)$$

in which we used the trace norm condition (11) to simplify the multipliers of g_1 and g_2 .

The most remarkable fact about the various coefficients in equation (23) is that when they are evaluated using expressions (15) for $\alpha_{\pm}, \beta_{\pm}$, the last two vanish identically and the first two evaluate to be equal to 4, independent of Δ, p_{\pm} , and the separation vector \mathbf{l} , i.e.

$$H_{\mu\nu} = 4(\langle K_{+}|\partial_{\mu}|K_{+}\rangle\langle K_{+}|\partial_{\nu}|K_{+}\rangle + \partial_{\mu}\langle K_{+}|\partial_{\nu}|K_{+}\rangle). \quad (24)$$

Indeed, this means that the QFI matrix elements become identical to those calculated in [4] for the problem of a pair of equally bright point sources. The independence of QFI on the relative brightness of the two sources and their separation supports our previous conjecture [4] that the determination of the separation of two point sources when their geometric center is known and fixed *a priori* (here at the coordinate origin) reduces fundamentally to a photon localization problem, independent of the nature, brightness, or locations of the point sources. This amounts, in effect, to the irrelevance of which source emits the photon. Rather, the only essential property of relevance to the estimation of pair separation w.r.t. a fixed geometric center is that the photon carrying the information about the sources was in fact emitted and observed by the measuring device.

Expression (24) for QFI, as we saw in [4], may be expressed in terms of derivatives of the phase function, Ψ , and phase constant, ϕ . In view of the wave function (2), we may write

$$\begin{aligned} \langle K_{+}|\partial_{\mu}|K_{+}\rangle &= i\langle\partial_{\mu}\phi - \partial_{\mu}\Psi\rangle; \\ \partial_{\mu}\langle K_{+}|\partial_{\nu}|K_{+}\rangle &= \langle(\partial_{\mu}\phi - \partial_{\mu}\Psi)(\partial_{\nu}\phi - \partial_{\nu}\Psi)\rangle, \end{aligned} \quad (25)$$

in which the angular brackets on the RHS denote pupil-plane averages of the quantity they enclose

$$\langle f(\mathbf{u}) \rangle \stackrel{\text{def}}{=} \int d^2u |P(\mathbf{u})|^2 f(\mathbf{u}), \quad (26)$$

for any function $f(\mathbf{u})$ of the pupil coordinates. When expressions (25) are substituted into the RHS of equation (24), we can see that the derivatives of the phase constant, namely $\partial_\mu \phi$, $\partial_\nu \phi$, which do not depend on \mathbf{u} and can thus be pulled out of such pupil averages, cancel out exactly, reducing QFI to the simple form

$$H_{\mu\nu} = 4(\langle \partial_\mu \Psi \partial_\nu \Psi \rangle - \langle \partial_\mu \Psi \rangle \langle \partial_\nu \Psi \rangle). \quad (27)$$

For a clear circular aperture, we may evaluate these averages further and thus obtain [4] the following diagonal form of the QFI matrix:

$$\mathbf{H} = \begin{pmatrix} 4\pi^2 & 0 & 0 \\ 0 & 4\pi^2 & 0 \\ 0 & 0 & \frac{\pi^2}{3} \end{pmatrix}. \quad (28)$$

3. Quantum limited pair separation for known intensity centroid and relative brightness

A more realistic and experimentally meaningful problem involving the separation of a closely spaced source pair is that in which the intensity centroid, rather than the geometric center, of the pair is known to be at a fixed non-random location. Such centroid can be well localized experimentally by finding the centroid of the photon counts recorded by a simple spatial image intensity based array sensor on which the pair is imaged. The intensity centroid, by its very definition, is closer to the brighter source, as for a bright star around which a faintly illuminated planet might be revolving. For the same single-photon DO of the previous section, given by equation (1), the intensity centroid of the pair of sources located at \mathbf{r}_\pm is at \mathbf{R} given by

$$\mathbf{R} = p_+ \mathbf{r}_+ + p_- \mathbf{r}_-, \quad (29)$$

which implies the following location vectors for the two sources relative to \vec{R} :

$$\boldsymbol{\rho}_\pm \stackrel{\text{def}}{=} \mathbf{r}_\pm - \mathbf{R} = \pm p_\mp \mathbf{r}, \quad (30)$$

in which

$$\mathbf{r} = \mathbf{r}_+ - \mathbf{r}_-, \quad (31)$$

is the separation vector of the two sources. Since the intensity centroid location, \mathbf{R} , can be assumed to be known well in advance, as we just noted, we may pick it to be the origin of coordinates, relative to which $\boldsymbol{\rho}_\pm$ represent the individual source location vectors that are simply proportional, via equation (30), to the pair separation vector, \mathbf{r} , which is to be estimated from source-pair photon measurements.

The single-photon wavefunctions for single-source emission, namely $\langle \mathbf{u} | K_\pm \rangle$, may be expressed in terms of these

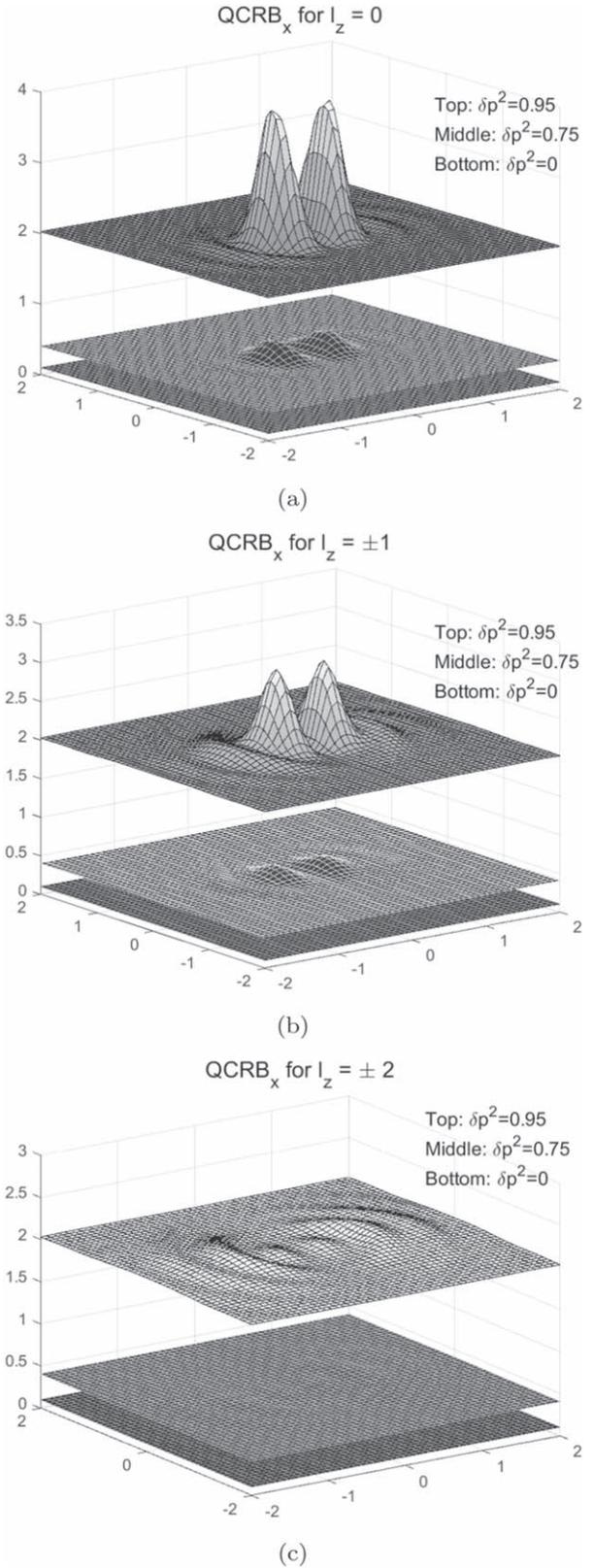


Figure 1. Surface plots of QCRB for l_z as a function of (l_x, l_y) for (a) $l_z = 0$, (b) $l_z = \pm 1$, (c) $l_z = \pm 2$, and for three different values of δp^2 , namely 0 (bottom surface), 0.45 (middle surface), and 0.95 (top surface).

relative source position vectors as

$$\langle \mathbf{u} | \mathbf{K}_\pm \rangle = \exp(\pm i p_\mp \phi) P(\mathbf{u}) \exp[\mp i p_\mp \Psi(\mathbf{u}; \mathbf{r})], \quad (32)$$

with the phase function $\Psi(\mathbf{u}; \mathbf{r})$ being the same as that defined in equation (4) with \mathbf{l} replaced by separation vector \mathbf{r} . The choice of the phase constant, ϕ , is such that the overlap integral between the two wavefunctions

$$\Delta = \langle K_+ | K_- \rangle = \exp(-i\phi) \int d^2u |P(\mathbf{u})|^2 \exp[i\Psi(\mathbf{u}; \mathbf{r})], \quad (33)$$

in which we used the probability normalization, $p_+ + p_- = 1$, becomes not only independent of p_\pm but also real and positive. For this to happen, ϕ must be chosen to be the phase of the complex integral on the RHS. In view of the form of the wave function (32), we also note the following important correspondences:

$$\begin{aligned} \langle K_- | \partial_\mu | K_- \rangle &= -\frac{p_-}{p_+} \langle K_+ | \partial_\mu | K_+ \rangle; \\ \langle K_+ | \partial_\mu | K_- \rangle &= \partial_\mu \langle K_- | K_+ \rangle = p_+ \partial_\mu \Delta; \\ \langle K_- | \partial_\mu | K_+ \rangle &= \partial_\mu \langle K_+ | K_- \rangle = p_- \partial_\mu \Delta. \end{aligned} \quad (34)$$

A straightforward implementation of these correspondences modifies expressions (21) and (22) into the form

$$\begin{aligned} \text{Re} \langle e_+ | \partial_\mu \hat{\rho} | e_- \rangle \langle e_- | \partial_\nu \hat{\rho} | e_+ \rangle &= \tilde{g}_1 \langle K_+ | \partial_\mu | K_+ \rangle \\ &\times \langle K_+ | \partial_\nu | K_+ \rangle + \tilde{g}_2 \partial_\mu \Delta \partial_\nu \Delta; \\ \text{Re} \langle e_+ | \partial_\mu \hat{\rho} \partial_\nu \hat{\rho} | e_+ \rangle &= \tilde{h}_1^{(+)} \langle K_+ | \partial_\mu | K_+ \rangle \langle K_+ | \partial_\nu | K_+ \rangle \\ &+ \tilde{h}_2^{(+)} \partial_\mu \langle K_+ | \partial_\nu | K_+ \rangle + \tilde{h}_3^{(+)} \text{Re} \partial_\mu \langle K_+ | \partial_\nu | K_- \rangle \\ &+ \tilde{h}_4^{(+)} \partial_\mu \Delta \partial_\nu \Delta; \\ \text{Re} \langle e_- | \partial_\mu \hat{\rho} \partial_\nu \hat{\rho} | e_- \rangle &= \tilde{h}_1^{(-)} \langle K_+ | \partial_\mu | K_+ \rangle \langle K_+ | \partial_\nu | K_+ \rangle \\ &+ \tilde{h}_2^{(-)} \partial_\mu \langle K_+ | \partial_\nu | K_+ \rangle + \tilde{h}_3^{(-)} \text{Re} \partial_\mu \langle K_+ | \partial_\nu | K_- \rangle \\ &+ \tilde{h}_4^{(-)} \partial_\mu \Delta \partial_\nu \Delta, \end{aligned} \quad (35)$$

where

$$\begin{aligned} \tilde{g}_1 &= -4p_+^2 (\alpha_- \beta_+ - \alpha_+ \beta_-)^2 \Delta^2; \\ \tilde{g}_2 &= 4p_+^2 p_-^2 [(\alpha_- \beta_+ + \alpha_+ \beta_-) + \Delta(\beta_+ \beta_- + \alpha_+ \alpha_-)]^2; \\ \tilde{h}_1^{(\pm)} &= p_+^2 (\alpha_\pm^2 + \beta_\pm^2 + 6\alpha_\pm \beta_\pm \Delta); \\ \tilde{h}_2^{(\pm)} &= p_+^2 [(\alpha_\pm + \beta_\pm \Delta)^2 + (\alpha_\pm \Delta + \beta_\pm)^2]; \\ \tilde{h}_3^{(\pm)} &= 2p_+ p_- (\alpha_\pm + \beta_\pm \Delta)(\alpha_\pm \Delta + \beta_\pm); \\ \tilde{h}_4^{(\pm)} &= p_+^2 p_-^2 [\alpha_\pm^2 + \beta_\pm^2 + 2(1 + \alpha_\pm \beta_\pm \Delta)]. \end{aligned} \quad (36)$$

These new expressions for the coefficients modify the final form of the QFI matrix elements presented earlier in

equation (23) to the value

$$\begin{aligned} \tilde{H}_{\mu\nu} &= 4 \left[\frac{\tilde{h}_1^{(+)}}{e_+} + \frac{\tilde{h}_1^{(-)}}{e_-} - \tilde{g}_1 \left(\frac{1}{e_+ e_-} - 1 \right) \right] \\ &\times \langle K_+ | \partial_\mu | K_+ \rangle \langle K_+ | \partial_\nu | K_+ \rangle \\ &+ 4 \left(\frac{\tilde{h}_2^{(+)}}{e_+} + \frac{\tilde{h}_2^{(-)}}{e_-} \right) \partial_\mu \langle K_+ | \partial_\nu | K_+ \rangle \\ &+ 4 \left(\frac{\tilde{h}_3^{(+)}}{e_+} + \frac{\tilde{h}_3^{(-)}}{e_-} \right) \text{Re} \partial_\mu \langle K_+ | \partial_\nu | K_- \rangle \\ &+ 4 \left[\frac{\tilde{h}_4^{(+)}}{e_+} + \frac{\tilde{h}_4^{(-)}}{e_-} - \tilde{g}_2 \left(\frac{1}{e_+ e_-} - 1 \right) \right. \\ &\left. - \frac{3\Delta^2(1 - \delta p^2)^2}{\delta e^2(1 - \delta e^2)} \right] \partial_\mu \Delta \partial_\nu \Delta. \end{aligned} \quad (37)$$

Since $\tilde{h}_3^{(\pm)} = h_3^{(\pm)} = 0$, the third term in expression (37) is identically zero. The coefficients of the other three terms no longer evaluate to forms that are independent of the pair separation vector \mathbf{r} or the relative intensities of the two sources, as measured by the probability difference, δp . Their dependence on \mathbf{r} arises through the involvement of the state overlap function, Δ , and its partial derivatives.

In view of the eigenstate normalization condition (17), we can immediately simplify the expressions for the coefficients $\tilde{h}_1^{(\pm)}$, $\tilde{h}_2^{(\pm)}$, and $\tilde{h}_4^{(\pm)}$ given by equation (36) to the form

$$\begin{aligned} \tilde{h}_1^{(\pm)} &= p_\pm^2 (1 + 4\alpha_\pm \beta_\pm \Delta); \\ \tilde{h}_2^{(\pm)} &= p_\pm^2 (1 + \Delta)^2 (\alpha_\pm + \beta_\pm)^2; \\ \tilde{h}_4^{(\pm)} &= 3p_\pm^2 p_-^2. \end{aligned} \quad (38)$$

Using the fact that $\tilde{h}_3^{(\pm)} = 0$ and substituting the expressions for \tilde{g}_1 and \tilde{g}_2 , given by equation (36), and expressions (38) into equation (37), we may express the coefficients of the latter in terms of the eigenvector expansion coefficients, α_\pm and β_\pm , which we may evaluate by using their values given by equation (15). The final expression for the QFI matrix elements thus obtained is the following:

$$\begin{aligned} \tilde{H}_{\mu\nu} &= 4 \left(\frac{1 + \delta p}{1 - \delta p} \right) \left[\left(1 + \frac{\Delta^2 \delta p^2}{1 - \Delta^2} \right) \langle K_+ | \partial_\mu | K_+ \rangle \right. \\ &\times \langle K_+ | \partial_\nu | K_+ \rangle + \partial_\mu \langle K_+ | \partial_\nu | K_+ \rangle \\ &\left. - \delta p^2 (1 - \delta p^2) \partial_\mu \Delta \partial_\nu \Delta \right]. \end{aligned} \quad (39)$$

Note that, as expected, for equal-brightness sources for which $\delta p = 0$, expression (39) reduces to the simpler form (24) since the geometric and intensity centroids coincide in this case.

Expression (39) for QFI for estimating the 3D pair separation vector when the pair centroid is well localized may be expressed in terms of the derivatives of the phase function Ψ , as we did in the previous section. Note, however, that because of unequal coefficients of the first two terms in this expression, the derivatives of the phase constant ϕ no longer

drop out from the final result obtained in this way. In view of form (32) of the single-photon wavefunctions, the analogs of equations (25) are the following:

$$\begin{aligned} \langle K_+ | \partial_\mu | K_+ \rangle &= i p_\perp \langle \partial_\mu \phi - \partial_\mu \Psi \rangle; \\ \partial_\mu \langle K_+ | \partial_\nu | K_+ \rangle &= p_\perp^2 \langle (\partial_\mu \phi - \partial_\mu \Psi) (\partial_\nu \phi - \partial_\nu \Psi) \rangle, \end{aligned} \quad (40)$$

which when substituted into equation (39) yield the following expression for the QFI matrix elements in terms of such derivatives:

$$\begin{aligned} \tilde{H}_{\mu\nu} &= (1 - \delta p^2) (\langle \partial_\mu \Psi \partial_\nu \Psi \rangle - \langle \partial_\mu \Psi \rangle \langle \partial_\nu \Psi \rangle) \\ &\quad - \delta p^2 (1 - \delta p^2) \left[\frac{\Delta^2}{1 - \Delta^2} (\langle \partial_\mu \Psi \rangle - \partial_\mu \phi) \right. \\ &\quad \left. \times (\langle \partial_\nu \Psi \rangle - \partial_\nu \phi) + \partial_\mu \Delta \partial_\nu \Delta \right]. \end{aligned} \quad (41)$$

By differentiating the equality,

$$\Delta \exp(i\phi) = \langle \exp(i\Psi) \rangle, \quad (42)$$

which follows from equation (33), then dividing the result by $\Delta \exp(i\phi)$, and subsequently evaluating the real and imaginary part of the resulting expression, we may calculate $\partial_\mu \Delta$ and $\partial_\mu \phi$ as

$$\begin{aligned} \partial_\mu \Delta &= -\Delta \operatorname{Im} \left[\frac{\langle \partial_\mu \Psi \exp(i\Psi) \rangle}{\langle \exp(i\Psi) \rangle} \right]; \\ \partial_\mu \phi &= \operatorname{Re} \left[\frac{\langle \partial_\mu \Psi \exp(i\Psi) \rangle}{\langle \exp(i\Psi) \rangle} \right]. \end{aligned} \quad (43)$$

Use of these equalities and the equality, $\Delta = |\langle \exp(i\Psi) \rangle|$, in equation (41) turns it into an expression composed of pupil-plane averages of form (26) of quantities involving only the phase function Ψ , given by equation (4), and its derivatives.

4. Numerical results for a circular aperture

The results we have derived in the previous section apply to an arbitrarily-shaped aperture possessing any pupil function, $P(\mathbf{u})$, with the only constraint on the latter being the normalization condition (3). We illustrate our results with a numerical evaluation of equation (41) for the special case of a clear circular pupil of radius R , for which $\mathbf{u} = \boldsymbol{\rho}/R$ and $|P(\mathbf{u})|^2$ is simply $1/\pi$ for $|\mathbf{u}| < 1$ and 0 otherwise. For this pupil geometry, we see easily from its inversion symmetry the following pupil averages:

$$\langle \partial_x \Psi \rangle \sim \langle u_x \rangle = 0; \quad \langle \partial_y \Psi \rangle \sim \langle u_y \rangle = 0; \quad \langle \partial_z \Psi \rangle = \pi \langle u^2 \rangle = \frac{\pi}{2}. \quad (44)$$

Similar integrations over the pupil yield the following pupil averages:

$$\langle \partial_\mu \Psi \partial_\nu \Psi \rangle = \begin{cases} \pi^2, & \mu = \nu = x, y \\ \pi^2/3, & \mu = \nu = z \\ 0, & \text{otherwise.} \end{cases} \quad (45)$$

Results (44) and (45) imply the following diagonal form for the difference of averages on the first line of equation (41):

$$\langle \partial_\mu \Psi \partial_\nu \Psi \rangle - \langle \partial_\mu \Psi \rangle \langle \partial_\nu \Psi \rangle = \begin{cases} \pi^2 & \mu = \nu = x, y \\ \pi^2/12 & \mu = \nu = z \\ 0 & \text{otherwise} \end{cases},$$

which thus contribute the terms $(1 - \delta p^2)\pi^2(1, 1, 1/12)$ to the three diagonal elements of the QFI.

The other terms in equation (41), which only contribute for an unequal-brightness pair ($\delta p \neq 0$), may be evaluated, as equation (43) shows, by calculating the three pupil-plane averages

$$\begin{aligned} \Delta &= |\langle \exp(i\Psi) \rangle|; \\ \langle \partial_x \Psi \exp(i\Psi) \rangle &= 2\pi \langle u \cos \phi_u \exp(i2\pi \mathbf{u} \cdot \mathbf{l}_\perp) \exp(i\pi u^2 l_z) \rangle; \\ \langle \partial_y \Psi \exp(i\Psi) \rangle &= 2\pi \langle u \sin \phi_u \exp(i2\pi \mathbf{u} \cdot \mathbf{l}_\perp) \exp(i\pi u^2 l_z) \rangle; \\ \langle \partial_z \Psi \exp(i\Psi) \rangle &= \pi \langle u^2 \exp(i2\pi \mathbf{u} \cdot \mathbf{l}_\perp) \exp(i\pi u^2 l_z) \rangle. \end{aligned} \quad (46)$$

These averages, which are pupil-plane integrals over the circular aperture, may be simplified by use of the angular-integral identities

$$\begin{aligned} \oint d\phi_u \exp(i2\pi \mathbf{u} \cdot \mathbf{l}_\perp) &= 2\pi J_0(2\pi u l_\perp), \\ \oint d\phi_u \cos \phi_u \exp(i2\pi \mathbf{u} \cdot \mathbf{l}_\perp) &= i2\pi \cos \phi_l J_1(2\pi u l_\perp), \\ \oint d\phi_u \sin \phi_u \exp(i2\pi \mathbf{u} \cdot \mathbf{l}_\perp) &= i2\pi \sin \phi_l J_1(2\pi u l_\perp), \end{aligned} \quad (47)$$

in terms of Bessel functions, J_0 and J_1 , to the following radial integrals:

$$\begin{aligned} \langle \exp(i\Psi) \rangle &= 2 \int_0^1 du u J_0(2\pi u l_\perp) \exp(i\pi u^2 l_z), \\ \langle \partial_x \Psi \exp(i\Psi) \rangle &= 4i\pi \cos \phi_l \int_0^1 du u^2 J_1(2\pi u l_\perp) \exp(i\pi u^2 l_z), \\ \langle \partial_y \Psi \exp(i\Psi) \rangle &= 4i\pi \sin \phi_l \int_0^1 du u^2 J_1(2\pi u l_\perp) \exp(i\pi u^2 l_z), \\ \langle \partial_z \Psi \exp(i\Psi) \rangle &= 2\pi \int_0^1 du u^3 J_0(2\pi u l_\perp) \exp(i\pi u^2 l_z). \end{aligned} \quad (48)$$

We numerically evaluated these radial integrals in Matlab using its built-in *integral* code, from which by taking the real and imaginary parts of the ratios involved in equation (43) we were able to calculate all of the terms occurring on the second and third lines of equation (41). After evaluating all of the QFI matrix elements in this way, we were able to invert this 3×3 matrix to calculate the QCRBs as the diagonal elements of this inverse matrix. These QCRBs serve to provide the lowest possible bounds on the variance of unbiased estimation of the three separation coordinates of the pair per photon. Dividing the QCRB per photon for each coordinate by the total photon number involved in the measurement then generates the overall QCRB corresponding to the estimation of that coordinate.

In figures 1, we display, using a surface plot, QCRB per photon for the estimation of the x coordinate of the transverse separation as a function of (l_x, l_y) for three different values of

δp^2 , namely 0, 0.75, and 0.95, corresponding to the ratio of source brightnesses taking the values 1:1, 14:1, and 78:1, respectively. The three different sub-figures refer to l_z taking values 0 (in-focus), ± 1 , and ± 2 units, respectively. Figures 2, on the other hand, capture the numerically evaluated values of QCRB for the estimation of the axial separation coordinate, l_z , for the same conditions as in figures 1. Apart from the difference by a factor of 4 corresponding to a difference by a factor 2 in the definitions of the separation vector between [4] and the present section, the three QCRBs for the equally bright source pair, corresponding to $\delta p = 0$, are the same constants as those obtained in [4], namely $1/\pi^2$ for estimating each of the two transverse separations and $12/\pi^2$ for estimating the axial separation, independent of the actual value of the separation vector. These results are represented by the bottom flat surface in each sub-figure.

As the two source brightness values begin to differ from each other, QCRB with respect to the estimation of each of the transverse and axial separation coordinates shows significant variations at small transverse separations. These variations tend to be characterized by peaks and valleys over a considerable range of such short separations. They result from diffraction generated blurring of the wavefronts passing through a finite aperture, which is represented by the oscillatory behavior of Bessel functions of low order. When one source is considerably fainter than the other, then the two sources would be harder to separate from each other if the former were to lie close to a maximum of the transverse diffraction pattern of the latter. The same considerations apply to axial separations as well since there is a diffraction-generated oscillatory spreading in the axial dimension as well.

These oscillatory peaks and valleys are accentuated by increasing asymmetry of source brightness levels, particularly at short axial and lateral separations, as a comparison of the three surfaces in each figure shows. As the sources get well separated in their transverse dimensions, QCRB becomes asymptotically independent of separation as the sources then lie well outside the diffractive footprint of each source and can be well distinguished from each other. The asymptotic values of QCRB are, however, larger the larger the brightness asymmetry, since locating the fainter source—and thus estimating its separation from the brighter source—becomes noisier the larger such asymmetry.

The values of QCRB for the estimation of the second transverse separation coordinate, l_y , are identical to those in figures 1 when they are rotated by $\pi/2$ because of the full $l_x \leftrightarrow l_y$ symmetry of QCRB obtained by a $\pi/2$ rotation about the optical (z) axis, followed by a mirror reflection in one of the transverse axes. As such, we do not plot such QCRB values separately.

We also note that figures 1 and 2, corresponding to the estimation of the lateral and axial separation coordinates, respectively, differ in the asymptotic values of the two different QCRBs by a factor of 12 by which the two estimation variances differ from each other. This can be seen from the inverse of the diagonal QFI matrix corresponding to the first line of terms in equation (41). The second and third lines of

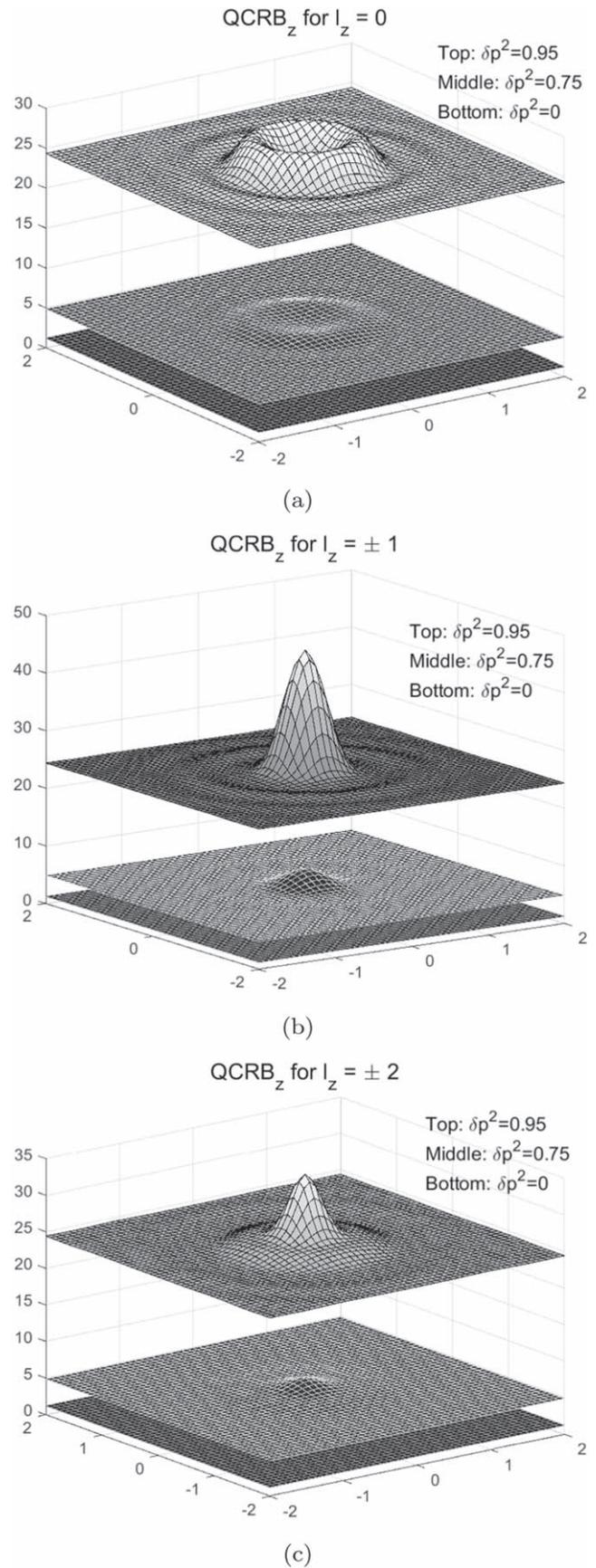


Figure 2. Surface plots of QCRB for l_z for the same parameter values as in figure 1.

terms in that equation tend to vanish in the asymptotic limit of large separations.

5. Concluding remarks

This paper has computed the quantum lower bounds on the variances for estimating the three components of the vector separation of a pair of incoherent point sources of unequal brightness. For the case of fixed intensity centroid of the source pair, the QFI matrix with respect to these three components and its inverse, whose diagonal elements furnish the said lower bounds, are both diagonal and independent of the specific values of the separation parameters only when the sources are equally bright. On the other hand, when the source pair's geometric center is well determined *a priori*, then QFI becomes independent of the separation coordinates as well as the brightness ratio for the pair, reducing to the previously obtained simple form [4] for an equally bright pair of sources. The latter result supports the conjecture that for fixed geometric center of the pair, a photon bringing the information about the location of either source to the imager can be interrogated with a fidelity that is independent of which source emitted it, since the two, by the very definition of the geometric center, have equal but oppositely signed vector separation from it. The experimental usefulness of this result is somewhat limited, however, since the more asymmetric the brightnesses of the two sources, the more difficult it would be to locate their geometric center precisely. The intensity centroid, by contrast, can be more readily and precisely located within the standard imaging protocol by computing the centroid of the photon counts observed on an array imaging sensor.

In [4] we demonstrated, by means of a numerical simulation, the use of low-order Zernike modes to saturate QCRB for the case of a symmetrical pair in the limit of small separations. We expect the Zernike projection basis to saturate QCRB regardless of the pair's brightness asymmetry, since the dependence of the phase of the photon wavefunction on separation, as seen from equation (32), is formally invariant under a change of such asymmetry represented by a change of p_{\pm} away from $1/2$.

Future work will enlarge the scope of the problem to include the quantum bound on the fidelity for the estimation of the brightness ratio for the source pair and for separating more than two unequally bright point sources in close vicinity of one another with a known brightness centroid. Such studies could be important as a first step in calculating the quantum limits on 3D superresolution imaging of extended continuous sources.

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References

- [1] Tsang M, Nair R and Lu X-M 2016 Quantum theory of superresolution for two incoherent optical point sources *Phys. Rev. X* **6** 031033
- [2] Ang S, Nair R and Tsang M 2017 Quantum limit for two-dimensional resolution of two incoherent optical point sources *Phys. Rev. A* **95** 063847
- [3] Rayleigh L 1879 Investigations in optics, with special reference to the spectroscope *Phil. Mag.* **8** 261–74
- [4] Yu Z and Prasad S 2018 Quantum limited superresolution of an incoherent source pair in three dimensions *Phys. Rev. Lett.* **121** 180504
- [5] Prasad S and Yu Z 2019 Quantum-limited superlocalization and superresolution of a source pair in three dimensions *Phys. Rev. A* **99** 022116
- [6] Napoli C, Piano S, Leach R, Adesso G and Tufarelli T 2019 Towards superresolution surface metrology: Quantum estimation of angular and axial separations *Phys. Rev. Lett.* **122** 140505
- [7] Nair R and Tsang M 2016 Far-field superresolution of thermal electromagnetic sources at the quantum limit *Phys. Rev. Lett.* **117** 190801
- [8] Lupo C and Pirandola S 2016 Ultimate precision bound of quantum and subwavelength imaging *Phys. Rev. Lett.* **117** 190802
- [9] Pařr M, Stoklasa B, Hradil Z, Sánchez-Soto L and Řeháček J 2016 Achieving the ultimate optical resolution *Optica* **10** 1144–7
- [10] Yang F, Tashchilina A, Moiseev E, Simon C and Lvovsky A 2016 Far-field linear optical superresolution via heterodyne detection in a higher-order local oscillator mode *Optica* **3** 1148–52
- [11] Tham W K, Ferretti H and Steinberg A M 2017 Beating Rayleigh's curse by imaging using phase information *Phys. Rev. Lett.* **118** 070801
- [12] Parniak M, Borowka S, Boroszko K, Wasilewski W, Bonaszek K and Demkowicz-Dobrzanski R 2018 Beating the Rayleigh limit using two-photon interference *Phys. Rev. Lett.* **121** 250503
- [13] Zhou Y, Hassett J, Rafsanjani S, Mirhosseni M, Vamivikas A, Jordan A, Shi Z and Boyd R 2019 Quantum-limited estimation of the axial separation of two incoherent point sources *Optica* **6** 534–41
- [14] Tsang M 2017 Subdiffraction incoherent optical imaging via spatial-mode demultiplexing *New J. Phys.* **19** 023054
- [15] Chrostowski A, Demkowicz-Dobrzanski R, Jazyrna M and Banaszek K 2017 On super-resolution imaging as a multi-parameter estimation problem *Int. J. Quantum Inf.* **15** 1740005
- [16] Tsang M 2019 Quantum limit to subdiffraction incoherent optical imaging *Phys. Rev. A* **99** 012305
- [17] Zhou S and Jiang L 2019 A modern description of Rayleigh's criterion *Phys. Rev. A* **99** 013808
- [18] Dutton Z, Kerviche R, Ashok A and Guha S 2019 Attaining the quantum limit of superresolution in imaging an object's length via predetection spatial-mode sorting *Phys. Rev. A* **99** 033847

- [19] Helstrom C 1976 *Quantum Detection and Estimation Theory* 123 (New York: Academic)
- [20] Řeháček J, Hradil Z, Stoklasa B, Pař M, Grover J, Krzic A and Sánchez-Soto L L 2017 Multiparameter quantum metrology of incoherent point sources: Towards realistic superresolution *Phys. Rev. A* **96** 062107
- [21] Řeháček J, Hradil Z, Koutný D, Grover J, Krzic A and Sánchez-Soto L L 2018 Optimal measurements for quantum spatial superresolution *Phys. Rev. A* **98** 012103
- [22] Goodman J 1996 *Introduction to Fourier Optics* 2nd edn (New York: McGraw Hill) section 5.3
- [23] Helstrom C 1968 The minimum variance of estimates in quantum signal detection *IEEE Trans. Inf. Theory* **IT-14** 234–42
- [24] Helstrom C 1970 Estimation of object parameters by a quantum-limited optical system *J. Opt. Soc. Am.* **60** 233–42
- [25] Braunstein S and Caves C 1994 Statistical distance and the geometry of quantum states *Phys. Rev. Lett.* **72** 3439–43