IOP Publishing Physica Scripta

Phys. Scr. 95 (2020) 055304 (9pp) https://doi.org/10.1088/1402-4896/ab76f9

Exact solution for the Dirac oscillator in curved spacetime

M D de Oliveira[1,2,3] **and Alexandre G M Schmidt**[1,2]

[1] Instituto de Ciências Exatas, Universidade Federal Fluminense, 27213-145 Volta Redonda—RJ, Brazil
[2] Programa de Pós Graduação em Física Instituto de Física, Universidade Federal Fluminense, 24210-346 Niterói—RJ, Brazil

E-mail: dalpra.matheus@gmail.com and agmschmidt@gmail.com

Received 4 January 2020, revised 3 February 2020
Accepted for publication 17 February 2020
Published 2 March 2020

Abstract

In this paper we consider the Dirac equation in curved spacetime which has a line element given by $ds^2 = (1 + \alpha^2 U(r))^2(dt^2 - dr^2) - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$ with electromagnetic field $A_\mu = (V(r), cA_r(r), 0, 0)$. We calculate the spinorial wave function and the energy spectrum of the Dirac oscillator in the curved spacetime, and from this result, we are able to return the solution in the flat spacetime. We consider two forms of symmetry in the coupling of the spin $1/2$ particle with the electromagnetic field and curved spacetime: exact spin symmetry $V(r) = U(r)$ and pseudo-spin symmetry $V(r) = -U(r)$.

Keywords: curved spacetime, dirac equation, dirac-oscillator, spin symmetry, pseudo-spin symmetry

(Some figures may appear in colour only in the online journal)

1. Introduction

Oscillatory systems such as the harmonic oscillator arouse much interest by the many possibilities of application in various branches of physics. In quantum mechanics, one of the key and exactly solvable problems is the harmonic oscillator in the non-relativistic regime, where we have the well-known eigenenergies and eigenfunctions as in Thaller [1]. Gilbey and Goodman [2] studied the forced harmonic oscillator and, Pedrosa and Guedes [3] investigated the harmonic oscillator with mass and frequency depending on time. In view of the well-established theory of the harmonic oscillator in the non-relativistic regime, the need to investigate this system in the relativistic regime arose, and in 1967 Itô and Carriere [4] studied for the first time the Dirac oscillator. Later on, in the 1989 Moshinsky and Szczepaniak [5] solved Dirac equation for the relativistic harmonic oscillator using the minimum substitution method $\vec{p} \to \vec{p} - im\omega\beta\vec{r}$. Other studies have emerged as well, such as Bentez and Salas-Brito [6] that studied the hidden supersymmetry of a relativistic oscillator, and Villalba [7] who solved the two-dimensional oscillator. Nouicer [8] obtained an exact solution for the one-dimensional Dirac oscillator in the presence of minimal lengths. Boumali and Chetouani [9] that get exact solutions of the Dirac oscillator in Kemmer equation for spin-1 particles; and de Lima Rodriguez [10] that get a new representation for the Dirac oscillator based on the Clifford algebra. Based on this, interest arises in solving the Dirac oscillator in curved spacetime using exact spin and exact pseudo-spin symmetry approach [11], which have already been studied in the relativistic regime in flat spacetime by Ginocchio [12] and Lisboa and Fiolhais [13] respectively, as well as the way to obtain such oscillatory systems through geometry in curved spacetime.

The pseudo-spin symmetry is commonly related in nuclear physics to certain aspects of exotic and deformed nuclei [14, 15] and it is characterized by quasi-degeneracy between doublet single-particle states $(n, l, j = l + 1/2)$ and $(n - 1, l + 2, j = l + 3/2)$, where n, l and j are the radial, the orbital and the total angular momentum quantum numbers, respectively. In this symmetry the angular momentum is written as pseudo-angular momentum $\tilde{l} = l + 1$ and pseudo-spin $\tilde{s} = s = 1/2$ [16]. The spin and pseudo-spin symmetries are relationship between the scalar $U(r)$ and vectorial $V(r)$ potentials as follows: (i) spin symmetry $\Delta(r) = U(r) - V(r) = \text{constant}$, when $\Delta(r) = 0$ we have exact spin symmetry and (ii) pseudo-spin symmetry $\Sigma(r) = U(r) + V(r) =$

[3] Author to whom any correspondence should be addressed.

constant, when $\Sigma(r) = 0$ we have exact pseudo-spin symmetry. This approach has been applied in several works and we can mention: Ginocchio [12] investigated the possibility of the appearance of pseudospin symmetry in nuclei moving in a relativistic mean field; Guo and Xu [17] that investigated pseudo-spin symmetry in the relativistic harmonic oscillator; Xu and Zhu [18] that investigated the pseudo-spin symmetry and spin symmetry in the relativistic Woods-Saxon potential; Alhaidari [19] that generalized spin and pseudo-spin symmetry; Liang and Zhou [20] that approach the recent progress on the pseudo-spin and spin symmetry in various systems and potentials, including extensions of the pseudo-spin symmetry study from stable to exotic nuclei; and Gao and Zhang [21] that calculated an approximate analytical solution of the Dirac-Eckart problem with a Hulthén tensor interaction under the condition of the pseudo-spin symmetry.

The theory of fermions in curved spacetime has already been studied by Parker [22] and in references [23–30]. Such studies in condensed matter and graphene bring possible applications of fermions in curved spacetime as interpretations of topological defects as Bakke [31] which makes a Kaluza-Klein description of the geometric phases in graphene, Stagmann and Szpak [32] that compare two different approaches to the electronic transport in deformed graphene: the condensed matter approach and the general relativistic approach in which classical trajectories of relativistic point particles moving in a curved surface; Iorio [33] studied the Hawking-Unruh phenomenon also in graphene; de Juan [34] studied dislocations and twisting in graphene and related systems; and, Minář [35] that show that a Dirac Hamiltonian in a curved background spacetime can be interpreted as a tight-binding Hamiltonian with non-unitary tunneling amplitudes. There are also experimental studies such as Louko [36] that examine the Unruh-DeWitt [37, 38] particle detector coupled to a scalar field in an arbitrary Hadamard state in four-dimensional curved spacetime. Boada [39] that used cold atoms to produce an artificial gravitational field, and fermions propagating in a curved spacetime could be simulated in such a system. We propose in this paper is a theoretical analysis of spin $1/2$ fermions in curved spacetime using spin and pseudo-spin symmetry. In this way we will use the Dirac equation with spherical symmetry in curved spacetime given by,

$$ds^2 = e^{2f(r)}dt^2 - e^{2g(r)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2, \quad (1)$$

where $f(r)$ and $g(r)$ are arbitrary functions of the radial coordinate r. Equation (1) has as special cases Schwarzschild [40], Reissner-Nordström [41] and anti de-Sitter [42] metrics. Because of spherical symmetry in (1), the angular component of the Dirac spinor will be given by the spherical harmonic functions in the same way as systems with the same symmetry in flat spacetime. So, what remains to be solved are the radial equations from the separation of the Dirac equation in curved spacetime. This is the main objective of our work.

The paper is organized as follows, in section 2 we present the Dirac equation in spherical coordinate in the curved spacetime for a general metric and symmetric spherical external electromagnetic field and solved the angular component of the wave function. In sections 3 and 4 we solve the Dirac oscillator system in curved spacetime with exact spin and exact pseudo-spin symmetry, respectively. Finally, in section 5 we have the conclusion.

2. Dirac equation in curved spacetime and exact solution of angular equation

In this section we will briefly review the Dirac equation in curved spacetime with external electromagnetic field, such as the the angular component of spinor. Let the metric on curved spacetime be (1) and consider the electromagnetic potential $A_\mu = (V(r), cA_r(r), 0, 0)$. The Dirac equation in atomic units ($m = e = \hbar = 1$ and $c = 1/\alpha$) is written as [26],

$$\begin{aligned} i\frac{1}{c}\frac{\partial}{\partial t}\Psi = \Bigg[& -ie^{f-g}\alpha^1\left(\partial_r + \frac{1}{r} + \frac{f_r'}{2} + iA_r\right) \\ & - i\alpha^2\frac{e^f}{r}\left(\partial_\theta + \frac{\cot(\theta)}{2}\right) - \frac{ie^f}{r\sin(\theta)}\alpha^3(\partial_\phi) \\ & + ce^f\beta + \frac{V(r)}{c}, \Bigg]\Psi, \end{aligned} \quad (2)$$

with $\alpha^i = \beta\gamma^i$. There are two ways of representing equation (2): (i) using the diagonal gauge, where $\gamma^i = (\gamma_d^1, \gamma_d^2, \gamma_d^3)$ are the canonical Dirac matrices with the spinor $\Psi = \Psi_d$, or (ii) using the Cartesian gauge with spinor $\Psi = \Psi_c$ and $\gamma^i = (\gamma_c^1, \gamma_c^2, \gamma_c^3)$, where

$$\begin{aligned} \gamma_c^1 &= (\gamma_d^1\cos\phi + \gamma_d^2\sin\phi)\sin\theta + \gamma_d^3\cos\theta, \\ \gamma_c^2 &= (\gamma_d^1\cos\phi + \gamma_d^2\sin\phi)\cos\theta - \gamma_d^3\sin\theta, \\ \gamma_c^3 &= -\gamma_d^1\sin\phi + \gamma_d^2\cos\phi, \end{aligned} \quad (3)$$

Henceforth we will write $\gamma_d^j = \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$, where σ^j are the Pauli matrices. These two representations are related via similarity transformation S, namely, $\gamma_c^i = S\gamma_d^iS^{-1}$ e $\Psi_c = S\Psi_d$. The matrix S is defined by,

$$\begin{aligned} S = & \exp\left(\frac{\phi}{2}\gamma^1\gamma^2\right)\exp\left(\frac{\theta}{2}\gamma^3\gamma^1\right) \\ & \times \frac{1}{2}(\mathbf{1}_4 + \gamma^2\gamma^1 + \gamma^1\gamma^3 + \gamma^3\gamma^2). \end{aligned} \quad (4)$$

Since we want to study spherically symmetric physical systems one have to use the Dirac equation in the Cartesian gauge, because this is the $SO(3)$ representation, which contains the symmetries we need to solve the angular equation of the systems in function of the spherical harmonics. However, in this gauge the Dirac equation is more complicated to work with, so to avoid this we will use Ψ_d because the spinor in the diagonal gauge is easier to work and with this we obtain the desired spinor in the Cartesian gauge through the relation $\Psi_c = S\Psi_d$. Using $H_d = i\partial/\partial_t$ in equation (2) we obtain Hamiltonian in the diagonal gauge given by,

$$\begin{aligned}\frac{H_d}{c} &= -i\alpha^1 e^{f-g}\left(\partial_r + \frac{1}{r} + \frac{f_r'}{2} + iA_r\right)\\ &\quad - i\frac{e^f}{r}\left\{\alpha^2\left(\partial_\theta + \frac{\cot\theta}{2}\right) + \frac{\alpha^3}{\sin\theta}\partial_\phi\right\}\\ &\quad + c\beta e^f + \frac{V(r)}{c}.\end{aligned} \tag{5}$$

Using (5) on eigenvalue problem $(H_d/c)\Psi_d = (E/c)\Psi_d$ and applying a unitary transformation $D = (\mathbf{1}_4 + \gamma^2\gamma^1 + \gamma^1\gamma^3 + \gamma^3\gamma^2)/2$, we will have $D(H_d/c)D^\dagger D\Psi_d = (E/c)D\Psi_d$ and effecting the transformation we obtain,

$$\begin{aligned}\frac{E}{c}\Psi' &= \left[-i\alpha^3 e^{f-g}\left(\partial_r + \frac{1}{r} + \frac{f_r'}{2} + iA_r\right)\right.\\ &\quad - i\frac{e^f}{r}\left\{\alpha^1\left(\partial_\theta + \frac{\cot\theta}{2}\right) + \frac{\alpha^2}{\sin\theta}\partial_\phi\right\}\\ &\quad \left. + ce^f\beta + \frac{V(r)}{c}\right]\Psi',\end{aligned} \tag{6}$$

with $\Psi' = D\Psi_d$. So $\Psi_d = D^\dagger\Psi'$, thus $\Psi_c = \exp(\phi\gamma^1\gamma^2/2)\exp(\theta\gamma^3\gamma^1/2)\Psi'$. Then we can write the 4-component spinorial wave function as,

$$\Psi_c(r, \theta, \phi) = N\frac{e^{-f/2}}{r}\begin{pmatrix} R_1(r)\mathcal{Y}_l^{|m|j}(\theta, \phi)\\ iR_2(r)\mathcal{Y}_{\tilde{l}}^{|m|j}(\theta, \phi)\end{pmatrix}, \tag{7}$$

where l is the orbital angular momentum, $\tilde{l}$ is the pseudo-orbital angular momentum and $\mathcal{Y}_{j\pm 1/2}^{|m|j}$ are the spinor functions given by

$$\begin{aligned}&\mathcal{Y}_l^{j=l\pm 1/2, m}(\theta, \phi)\\ &= \frac{1}{\sqrt{2l+1}}\begin{pmatrix}\pm\sqrt{l \pm m + 1/2}\ Y_l^{m-1/2}(\theta, \phi)\\ \sqrt{l \mp m + 1/2}\ Y_l^{m+1/2}(\theta, \phi)\end{pmatrix},\\ &l = j \mp \frac{1}{2},\end{aligned} \tag{8}$$

with Y_l^m spherical harmonics [43], and $R_1(r)$ and $R_2(r)$ must satisfy,

$$\begin{pmatrix} e^f + \alpha^2 V(r) - \epsilon & -\alpha e^{f-g}\left[\frac{d}{dr} - A_r - \frac{\lambda}{r}e^f\right]\\ \alpha e^{f-g}\left[\frac{d}{dr} + A_r + \frac{\lambda}{r}e^f\right] & -e^f + \alpha^2 V(r) - \epsilon\end{pmatrix} \times \begin{pmatrix} R_1(r)\\ R_2(r)\end{pmatrix} = 0, \tag{9}$$

where $\epsilon = E/c^2$ and $\lambda = \pm(j + 1/2)$. Having determined the angular wave function of the system, we must now solve the radial differential equations, given by (9), for particle and anti-particle, namely $R_1(r)$ and $R_2(r)$ respectively. As in [26] here we will use $f = g$ in (1) for the purely theoretical interest of analyzing Dirac fermions in this type of metric. There are few studies devoted to this type of metric. On the other hand, there are works with other types of combinations of f and g such as: (i) Laguna and Celi [44] studied the case $f \neq 0$ and $g = 0$; (ii) Sabín [45] that proposed a quantum simulation using a system where $f = 0$ and $g \neq 0$; and (iii) Lyu [46] studied a system where $f = 1/g$. We have to use $e^f = 1 + \alpha^2 U(r)$ in order to eliminate the exponential for the sake of simplicity, because we want to analyze the system with exact spin and exact pseudo-spin symmetry between scalar and electrostatic potentials U and V, respectively, in curved spacetime. Thus, from (9) we have,

$$\begin{pmatrix} 1 + \alpha^2(V(r) + U(r)) - \epsilon & -\alpha\left[\frac{d}{dr} - \frac{\lambda}{r} - \frac{\lambda\alpha^2}{r}U(r) - A_r\right]\\ \alpha\left[\frac{d}{dr} + \frac{\lambda}{r} + \frac{\lambda\alpha^2}{r}U(r) + A_r\right] & -1 + \alpha^2(V(r) - U(r)) - \epsilon\end{pmatrix} \times \begin{pmatrix} R_1(r)\\ R_2(r)\end{pmatrix} = 0, \tag{10}$$

considering $\Sigma(r) = V(r) + U(r)$ and $\Delta(r) = V(r) - U(r)$ we have,

$$\begin{pmatrix} 1 + \alpha^2\Sigma(r) - \epsilon & -\alpha\left[\frac{d}{dr} - \frac{\lambda}{r} - \frac{\lambda}{r}V(r) - A_r\right]\\ \alpha\left[\frac{d}{dr} + \frac{\lambda}{r} + \frac{\lambda}{r}V(r) + A_r\right] & -1 + \alpha^2\Delta(r) - \epsilon\end{pmatrix} \times \begin{pmatrix} R_1(r)\\ R_2(r)\end{pmatrix} = 0. \tag{11}$$

Thus, from equation (11) we obtain,

$$\begin{cases}(1 + \alpha^2\Sigma(r) - \epsilon)R_1 = \alpha\left[\frac{d}{dr} - \frac{\lambda}{r} - \frac{\lambda}{r}V(r) - A_r\right]R_2\\ (-1 + \alpha^2\Delta(r) - \epsilon)R_2 = -\alpha\left[\frac{d}{dr} + \frac{\lambda}{r} + \frac{\lambda}{r}V(r) + A_r\right]R_1\end{cases}, \tag{12}$$

we have two forms for solve (12), the first begins with,

$$R_2 = \frac{\alpha}{1 - \alpha^2\Delta(r) + \epsilon}\left[\frac{d}{dr} + \frac{\lambda}{r}(1 + \alpha^2 U(r)) + A_r\right]R_1, \tag{13}$$

and R_1 must satisfy,

$$\begin{aligned}&\left\{\frac{d^2}{dr^2} + \frac{d}{dr}\left[\frac{\lambda}{r}(1 + \alpha^2 U) + A_r\right]\right.\\ &\quad + \alpha^2\frac{d\Delta(r)}{dr}\frac{[d/dr + \lambda(1 + \alpha^2 U)/r + A_r]}{1 + \epsilon - \alpha^2\Delta(r)}\\ &\quad - \left[\frac{\lambda}{r}(1 + \alpha^2 U) + A_r\right]^2\\ &\quad \left. - \frac{(1 - \epsilon + \alpha^2\Sigma(r))(1 + \epsilon - \alpha^2\Delta(r))}{\alpha^2}\right\}R_1 = 0.\end{aligned} \tag{14}$$

In this case, spin symmetry, we have

$$\begin{cases} \lambda = \left(j + \frac{1}{2}\right) = l, & j = l - \frac{1}{2}, & \lambda > 0 \\ \lambda = -\left(j + \frac{1}{2}\right) = -(l+1), & j = l + \frac{1}{2}, & \lambda < 0 \end{cases}, \tag{15}$$

where l is the orbital angular momentum. The second approach to solve (12) is to write R_1 in terms of R_2, namely,

$$R_1 = \frac{\alpha}{1 + \alpha^2\Sigma(r) - \epsilon}\left[\frac{d}{dr} - \frac{\lambda}{r}(1 + \alpha^2 U(r)) - A_r\right]R_2, \tag{16}$$

then R_2 must satisfy,

$$\begin{aligned} &\left\{\frac{d^2}{dr^2} - \frac{d}{dr}\left[\frac{\lambda}{r}(1 + \alpha^2 U) + A_r\right]\right. \\ &+ \alpha^2\frac{d\Sigma(r)}{dr}\frac{[d/dr - \lambda(1 + \alpha^2 U)/r - A_r]}{1 - \epsilon + \alpha^2\Sigma(r)} \\ &- \left[\frac{\lambda}{r}(1 + \alpha^2 U) + A_r\right]^2 \\ &\left. - \frac{(1 - \epsilon + \alpha^2\Sigma(r))(1 + \epsilon - \alpha^2\Delta(r))}{\alpha^2}\right\}R_2 = 0, \end{aligned} \tag{17}$$

now, with pseudo-spin symmetry, we have

$$\begin{cases} \lambda = \left(j + \frac{1}{2}\right) = \tilde{l} + 1, & j = \tilde{l} + \frac{1}{2}, & \lambda > 0 \\ \lambda = -\left(j + \frac{1}{2}\right) = -\tilde{l}, & j = \tilde{l} - \frac{1}{2}, & \lambda < 0 \end{cases}, \tag{18}$$

where $\tilde{l}$ is the pseudo-orbital angular momentum. To solve the radial equations for $R_1(r)$ and $R_2(r)$ we need to define the functions $V(r)$, $U(r)$ and $A_r(r)$. In next sections we will solve the Dirac oscillator with exact spin and exact pseudo-spin symmetry.

3. Dirac oscillator with exact spin symmetry in curved spacetime

The first application we will analyze is the relativistic harmonic oscillator by minimum substitution $A_r = \omega\, r$ with exact spin symmetry. Thus we have $\Delta(r) = 0$, $\Sigma(r) = 2V(r)$, and we will consider the quadratic radial potential $V(r) = \mu r^2$, where μ and ω are real constants. As a result from (14) we obtain,

$$\begin{aligned} &\left[\frac{d^2}{dr^2} - \frac{\lambda(\lambda+1)}{r^2} - \{\beta^2 + 2\mu(1+\epsilon)\}r^2\right. \\ &\left. + \beta(1 - 2\lambda) + \frac{\epsilon^2 - 1}{\alpha^2}\right]R_1 = 0, \end{aligned} \tag{19}$$

where $\beta = \alpha^2\mu\lambda + \omega$ is coupling constant of the effective tensor potential. Replacing $x = \delta\, r^2$ with $\delta = \sqrt{\beta^2 + 2\mu(1+\epsilon)}$, $\lambda(\lambda+1) = l(l+1)$ and $R_1 = x^{-1/4}F(x)$ in (19) we obtain,

$$\begin{aligned} &\left[\frac{d^2}{dx^2} - \left\{\frac{l(l+1)}{4} - \frac{3}{16}\right\}\frac{1}{x^2}\right. \\ &\left. + \left\{\frac{\beta(1-2\lambda)}{4\delta} + \frac{\epsilon^2 - 1}{4\delta\alpha^2}\right\}\frac{1}{x} - \frac{1}{4}\right]F = 0. \end{aligned} \tag{20}$$

Such ordinary differential equation (20) has already been solved by Maghsoodi and Zarrinkamar [47] using the Nikiforov-Uvarov method [48], so we have

$$\left[\frac{d^2}{dx^2} + \frac{A}{x} + \frac{B}{x^2} + C\right]F(x) = 0, \tag{21}$$

where $F(x)$ is given by,

$$F(x) = (\sqrt{-C}x)^{1/2+\sqrt{1/4-B}}e^{-\sqrt{-C}x}L_n^{2\sqrt{1/4-B}}(2\sqrt{-C}x), \tag{22}$$

where L_n^k is associated Laguerre polynomial [43] and A, B and C must satisfy the constraint,

$$\left(2n + 1 + 2\sqrt{\frac{1}{4} - B}\right)\sqrt{-C} - A = 0. \tag{23}$$

Comparing (20) with (19) we identify $A = (\epsilon^2 - 1)/(4\delta\alpha^2)$, $B = -l(l+1)/4 + 3/16$ and $C = -1/4$, so from (22) we obtain,

$$F(x) = x^{(1+l+1/2)/2}\exp\left(-\frac{x}{2}\right)L_n^{l+1/2}(x), \tag{24}$$

using notation $R_1 = R_{nl}^1$ and $R_2 = R_{nl}^2$ from now on, we obtain,

$$R_{nl}^1(r) = (\sqrt{\delta}r)^{1+l}\exp\left(-\frac{\delta r^2}{2}\right)L_n^{l+1/2}(\delta r^2), \tag{25}$$

and using (13) we have,

$$\begin{aligned} R_{nl}^2(r) = \frac{\alpha}{1+\epsilon}&\left[\left\{\frac{\lambda - l}{r} + (\beta - \delta)r\right\}L_n^{l+1/2}(\delta r^2)\right. \\ &\left. + \frac{(2n + 2l + 1)}{r}L_n^{l-1/2}(\delta r^2)\right](\sqrt{\delta}r)^{1+l} \\ &\times \exp\left(-\frac{\delta r^2}{2}\right). \end{aligned} \tag{26}$$

In order to calculate the eigenenergies we use (23) and one must solve the fourth-order equation,

$$\epsilon^4 + 2a\epsilon^2 - 2\mu b^2\epsilon + c = 0, \tag{27}$$

where $a = -1 + \alpha^2\beta(1 - 2\lambda)$, $b = 2\alpha^2(2n + \lambda + 3/2)$ and $c = a^2 - b^2(\beta^2 + 2\mu)$. So as to obtain the roots of (27) we use the so-called Ferrari's solution [49] which yields,

$$\epsilon = \left[-a(\pm)_1\frac{\mu b^2}{\sqrt{2m}}(\pm)_1(\pm)_2\sqrt{-m^2 - 2am(\pm)_1\mu b^2\sqrt{2m}}\right]^{1/2}, \tag{28}$$

where m is given by,

$$m = -\frac{2a}{3} + \left(-\frac{p}{2} + \sqrt{\frac{p^2}{4} + \frac{q^3}{27}}\right)^{1/3} + \left(-\frac{p}{2} - \sqrt{\frac{p^2}{4} + \frac{q^3}{27}}\right)^{1/3}, \tag{29}$$

with $p = -2a^3/27 + 2ac/3 - \mu^2 b^4/2$, $q = -a^2/3 - c$. However, the correct physical interpretation of the system imposes that when $\mu = 0$ the system takes the well-known flat spacetime [50] form. Therefore, $\epsilon(\mu = 0) = \sqrt{1 + 2\alpha^2\omega(2n + \lambda + l + 1)}$, so from (28) we note that only two eigenenergies satisfy such condition, thus

$$\epsilon_{nl}^{\pm} = \left[-a(\pm)\frac{\mu b^2}{\sqrt{2m}} + \sqrt{-m^2 - 2am(\pm)\mu b^2\sqrt{2m}}\right]^{1/2}. \tag{30}$$

It is important to note that for each chosen value of the parameters ω and μ we have a finite number for n in which the eigenenergies (30) are real, consequently we will have a finite number of square integrable spinors, and we have degeneracy between doublets states, e.g., $0p_{1/2}$ and $0p_{3/2}$ when the coupling constant β of the effective tensor potential vanishes. On the other hand, when β is not zero there is no degeneracy. Finally, we obtain the spinorial wave functions,

$$\Psi_c(r, \theta, \phi) = N_{nl}\frac{(1 + \alpha^2\mu r^2)^{-1/2}}{r}\begin{pmatrix} R_{nl}^1(r)\mathcal{Y}_l^{|m|j}(\theta, \phi) \\ iR_{nl}^2(r)\mathcal{Y}_{\bar{l}}^{|m|j}(\theta, \phi)\end{pmatrix}, \tag{31}$$

the normalization constant is calculated in appendix. The spinorial wave function in (31) can be written as $\Psi^{curved} = (g_{tt})^{-1/4}\Psi^{flat}$ where $g_{tt} = (1 + \alpha^2\mu r^2)^2$ is the first element of the metric given in (32) and Ψ^{flat} is the spinorial wave function in flat spacetime with effective mass $m(r) = 1 + \alpha^2\mu r^2$ and effective frequency δ.

In figure 1 we plot the normalized radial probability density $|\Psi_c|^2$ for some values of the parameter μ, for $\omega = 1$, $j = 1/2$, using ϵ^+ and $\lambda > 0$ in (15) with $\lambda = l = 1$.

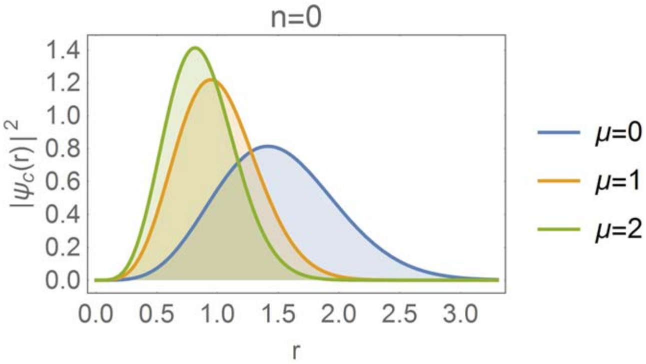

Figure 1. Plot of the probability density $|\Psi_c|^2$ in a curved space with line element given by (1) for $n = 0$. We plot the flat spacetime probability density ($\mu = 0$), and as the parameter μ increases we observe that the curves approach the origin. Since the problem can also be interpreted as relativistic oscillator on flat spacetime with position-dependent mass we observed that the probability density becomes higher in the region close to origin because of effective mass.

A physical interpretation of this problem in curved spacetime is that for $\mu \neq 0$ the particle behaves as it had an effective mass which depends on r, namely, $m(r) = 1 + \alpha^2\mu r^2$. This effect comes from the curvature of the space as well as from the interaction with the electromagnetic field. So, it is more likely to find the particle close to the origin as we can observe in figure 1. Conversely, when $\mu = 0$ we return to the well-known flat spacetime Dirac oscillator [50] without spin symmetry.

In summary, we calculated exactly the spinorial wave function, equation (31), and the eigenenergies, equation (30) of the relativistic harmonic oscillator in a curved spacetime whose line element is,

$$ds^2 = (1 + \alpha^2\mu r^2)^2(dt^2 - dr^2) - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{32}$$

where $\mu \geqslant 0$. The scalar curvature [40] of this problem is given by,

$$R = \frac{8\alpha^4\mu^2 r^2}{(1 + \alpha^2\mu r^2)^4} + \frac{2}{r^2}\left[1 - \frac{1}{(1 + \alpha^2\mu r^2)^2}\right], \tag{33}$$

so that in the limit of $r \to 0$ it implies that $R \to 4\mu\alpha^2$, i.e., the curvature is constant. On the other hand, when $r \to \infty$ it implies that $R \to 0$, so we obtain a vanishing curvature which means the spacetime is flat in this limit.

4. Exact pseudo-spin symmetry on the Dirac oscillator into curved spacetime

In this section let us turn to the pseudo-spin symmetry. Consider the relativistic harmonic oscillator again by minimum substitution $A_r = \omega\ r$, but now with $\Sigma(r) = 0$, $\Delta(r) = 2V(r)$. Let the external potential be the quadratic radial potential $V(r) = -U(r) = \mu r^2$, where μ and ω are real constants. As a result of (17) we obtain,

$$\left[\frac{d^2}{dr^2} - \frac{\lambda(\lambda-1)}{r^2} - \{\beta^2 - 2\mu(1-\epsilon)\}r^2 - \beta(1+2\lambda) + \frac{\epsilon^2-1}{\alpha^2}\right]R_2 = 0, \quad (34)$$

where $\beta = \omega - \alpha^2\mu\lambda$ is coupling constant of the effective tensor potential. Replacing $x = \gamma r^2$ with $\gamma = \sqrt{\beta^2 - 2\mu(1-\epsilon)}$, $\lambda(\lambda-1) = \tilde{l}(\tilde{l}+1)$ and $R_2 = x^{-1/4}G(x)$ in (34) we obtain,

$$\left[\frac{d^2}{dx^2} - \left\{\frac{\tilde{l}(\tilde{l}+1)}{4} - \frac{3}{16}\right\}\frac{1}{x^2} + \left\{\frac{\epsilon^2-1}{4\delta\alpha^2} - \frac{\beta(1+2\lambda)}{4\delta}\right\}\frac{1}{x} - \frac{1}{4}\right]G = 0. \quad (35)$$

As in the previous case we solve (35) using (22). One easily identifies $A = (\epsilon^2-1)/(4\delta\alpha^2) - \beta(1+2\lambda)/(4\gamma)$, $B = -\tilde{l}(\tilde{l}+1)/4 + 3/16$ and $C = -1/4$, so we obtain,

$$G(x) = x^{(\tilde{l}+1/2)/2}\exp\left(-\frac{x}{2}\right)L_n^{\tilde{l}-1/2}(x), \quad (36)$$

using notation $R_2 = R^2_{n\tilde{l}}$ and $R_1 = R^1_{n\tilde{l}}$ from now on, we obtain,

$$R^2_{n\tilde{l}}(r) = (\sqrt{\gamma}r)^{\tilde{l}}\exp\left(-\frac{\gamma r^2}{2}\right)L_n^{\tilde{l}-1/2}(\gamma r^2), \quad (37)$$

and using (16) we write $R^1_{n\tilde{l}}$,

$$R^1_{n\tilde{l}}(r) = \frac{\alpha}{1-\epsilon}\left[\left\{\frac{\tilde{l}-\lambda}{r} + (\gamma-\beta)r\right\}L_n^{\tilde{l}-1/2}(\gamma r^2) - 2\gamma r L_n^{\tilde{l}+1/2}(\gamma r^2)\right](\sqrt{\gamma}r)^{\tilde{l}}\exp\left(-\frac{\gamma r^2}{2}\right). \quad (38)$$

Again, using (23) we obtain same fourth order algebraic equation given in (27) for the eigenenergies, with $a = -1 - \alpha^2\beta(1+2\lambda)$, $b = 2\alpha^2(2n+\tilde{l}+3/2)$ and $c = a^2 - b^2(\beta^2 - 2\mu)$. Therefore the eigenenergies will be given by the same fourth-degree polynomial of the previous case, but the values of a, b and c are different, so the eigenenergies satisfying the condition $\epsilon(\mu=0) = \sqrt{1 + 2\alpha^2\omega(2n+\lambda+\tilde{l}+2)}$ will be

$$\epsilon^{\pm}_{n\tilde{l}} = \left[-a(\pm)\frac{\mu b^2}{\sqrt{2m}} + \sqrt{-m^2 - 2am(\pm)\mu b^2\sqrt{2m}}\right]^{1/2}, \quad (39)$$

where m is given by (29). Again for each chosen value of the parameters ω and μ we have a finite number for n in which the eigenenergies in (39) are real, consequently we will have a finite number of square integrable spinors. Here we have degeneracy between doublet states, for instance $1s_{1/2}$ and $0d_{3/2}$, when the coupling constant β of the effective tensor potential vanishes. Conversely, when β is not zero we have no degeneracy. Finally, we write the spinorial wave function,

$$\Psi_c(r,\theta,\phi) = N_{n\tilde{l}}\frac{(1-\alpha^2\mu r^2)^{-1/2}}{r}\begin{pmatrix} R^1_{n\tilde{l}}(r)\mathcal{Y}_l^{|m|j}(\theta,\phi) \\ iR^2_{n\tilde{l}}(r)\mathcal{Y}_{\tilde{l}}^{|m|j}(\theta,\phi)\end{pmatrix}, \quad (40)$$

with $\tilde{l} = l+1$. As in the previous case the spinorial wave function in (40) can be written as $\Psi^{curved} = (g_{tt})^{-1/4}\Psi^{flat}$ where $g_{tt} = (1-\alpha^2\mu r^2)^2$ is the first element of the metric given in (41) and Ψ^{flat} is the spinorial wave function in flat spacetime with effective mass $m(r) = 1 - \alpha^2\mu r^2$ and effective frequency γ. In figure 2 we plot some of these probability densities for $\mu \neq 0$ and $\mu = 0$, when $\omega = 1$ and $\lambda < 0$ in (18) with $\lambda = -\tilde{l} = -1$. In the present case the particle effective is $m(r) = 1 - \alpha^2\mu r^2$, and the coupling with both curved spacetime and electromagnetic field does probability density confining itself close to the origin, as we can observe in figure 2. In the special case of $\mu = 0$ we obtain the Dirac oscillator in flat spacetime without pseudo-spin symmetry.

In summary, we exactly calculated the spinorial wave function, equation (40), and eigenenergies, equation (39), of the relativistic harmonic oscillator on curved spacetime whose metric is

$$ds^2 = (1-\alpha^2\mu r^2)^2(dt^2 - dr^2) - r^2d\theta^2 - r^2\sin^2\theta d\phi^2, \quad (41)$$

where $\mu \geqslant 0$. The scalar curvature [40] of this problem is given by,

$$R = \frac{8\alpha^4\mu^2r^2}{(1-\alpha^2\mu r^2)^4} + \frac{2}{r^2}\left[1 - \frac{1}{(1-\alpha^2\mu r^2)^2}\right], \quad (42)$$

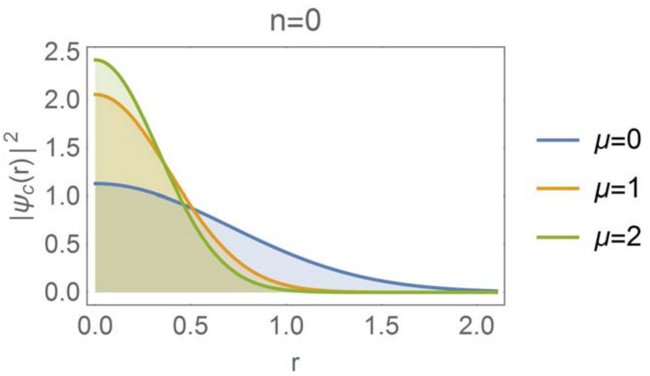

Figure 2. Plot of the probability density $|\Psi_c|^2$ in a curved space with line element given by (1) for $n = 0$. We plot the flat spacetime probability density ($\mu = 0$), and as the parameter μ increases we observe that the curves approach the origin. Since the problem again can also be interpreted as relativistic oscillator on flat spacetime with position-dependent mass we observed that, the probability density becomes higher in the region where the particle gets more massive.

so in the limit of $r \to 0$ we have $R \to -4\mu\alpha^2$, i.e., the curvature is negative and constant. When $r \to \infty$ it implies $R \to 0$, so we obtain a vanishing curvature, thus the spacetime is flat in this limit.

5. Conclusion

We studied the Dirac oscillator in this work by the method of minimum substitution with external electromagnetic field given by $A_\mu = (\mu r^2, c\omega r, 0, 0)$ in a curved spacetime with $ds^2 = (1 + \alpha^2 U(r))(dt^2 - dr^2) - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$. We apply two different metrics: in the first case $U(r) = \mu r^2$ and in the second case $U(r) = -\mu r^2$, so that with these couplings two symmetries appeared: exact spin and exact pseudo-spin. We solved the Dirac equation for two cases so that for the exact spin symmetry comes up an effective mass $m(r) = 1 + \alpha^2\mu r^2$ and effective frequency δ, and for exact pseudo-spin symmetry the effective mass and frequency is $m(r) = 1 - \alpha^2\mu r^2$ and γ, respectively. In both cases, the probability densities are confined close to the origin for $\mu \neq 0$. Another interesting fact is that we were able to write the spinors in curved spacetime as the product of the first element of the metric with the spinor in flat spacetime with effective mass and frequency as follows: $\Psi^{curved} = (g_{tt})^{-1/4}\Psi^{flat}$. The metric element g_{tt} is often related to the Fermi velocity, which is proportional to $(g_{tt})^{1/2}$ and also in optical metric where the refraction index of light is equal to $(g_{tt})^{1/2}$. In both cases with spin and pseudo-spin symmetry, an effective tensor potential emerged that broke the degenerescence between the double states, thus breaking both symmetries. In the limit of $\beta \to 0$, the degeneracy of doubled states returned, as for example, in states $0p_{1/2}$ and $0p_{3/2}$ for spin symmetry and $1s_{1/2}$ and $0d_{3/2}$ for pseudo-spin symmetry.

Acknowledgments

The authors gratefully acknowledge CNPq, CAPES and FAPERJ for continued partial financial support. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior—Brasil (CAPES)—Finance Code 001.

Appendix

A.1. Calculation of the normalization constants

Let us calculate explicitly the normalization constants for the spinor wave functions we studied in the previous sections. The normalization condition for each system is given by,

$$\int_0^\infty \int_0^\pi \int_0^{2\pi} |\Psi_c(r, \theta, \phi)|^2 r^2 (1 + \alpha^2 U(r))^2 \times \sin\theta \, dr d\theta d\phi = 1. \tag{43}$$

Since the integrals over θ and ϕ are already normalized we impose,

$$N^2 \int_0^\infty (|R_1|^2 + |R_2|^2)(1 + \alpha^2 U(r)) dr = 1, \tag{44}$$

where N must be calculated. For the Dirac oscillator with exact spin symmetry in section (III), the normalization constant is,

$$\begin{aligned}
N_{nl} = \Bigg[& I_n\left(\delta, l+1, l+\frac{1}{2}, l+\frac{1}{2}\right) \\
& + \frac{\mu\alpha^2}{\delta} I_n\left(\delta, l+2, l+\frac{1}{2}, l+\frac{1}{2}\right) \\
& + \frac{\alpha^2(\beta-\delta)^2}{\delta(1+\epsilon)^2}\Bigg\{ I_n\left(\delta, l+2, l+\frac{1}{2}, l+\frac{1}{2}\right) \\
& + \frac{\mu\alpha^2}{\delta} I_n\left(\delta, l+3, l+\frac{1}{2}, l+\frac{1}{2}\right)\Bigg\} \\
& + \frac{2\alpha^2(\beta-\delta)(2n+2l+1)}{(1+\epsilon)^2} \\
& \times \Bigg\{ I_n\left(\delta, l+1, l+\frac{1}{2}, l-\frac{1}{2}\right) \\
& + \frac{\mu\alpha^2}{\delta} I_n\left(\delta, l+2, l+\frac{1}{2}, l-\frac{1}{2}\right)\Bigg\} \\
& + \frac{\alpha^2(2n+2l+1)^2\delta}{(1+\epsilon^2)}\Bigg\{ I_n\left(\delta, l, l-\frac{1}{2}, l-\frac{1}{2}\right) \\
& + \frac{\mu\alpha^2}{\delta} I_n\left(\delta, l+1, l-\frac{1}{2}, l-\frac{1}{2}\right)\Bigg\} \\
& + \frac{\alpha^2}{(1-\epsilon)^2}(\lambda - l)^2\delta\Bigg\{ I_n\left(\delta, l, l+\frac{1}{2}, l+\frac{1}{2}\right) \\
& + \frac{\mu\alpha^2}{\delta} I_n\left(\delta, l+1, l+\frac{1}{2}, l+\frac{1}{2}\right)\Bigg\} \\
& + \frac{2\alpha^2}{(1+\epsilon)^2}(\lambda - l)(\beta-\delta)\Bigg\{ I_n\left(\delta, l+1, l+\frac{1}{2}, l+\frac{1}{2}\right) \\
& + \frac{\mu\alpha^2}{\delta} I_n\left(\delta, l+2, l+\frac{1}{2}, l+\frac{1}{2}\right)\Bigg\} \\
& + \frac{2\alpha^2}{(1+\epsilon)^2}(\lambda - l)(2n+2l+1)\delta \\
& \times \Bigg\{ I_n\left(\delta, l, l+\frac{1}{2}, l-\frac{1}{2}\right) \\
& + \frac{\mu\alpha^2}{\delta} I_n\left(\delta, l+1, l+\frac{1}{2}, l-\frac{1}{2}\right)\Bigg\}\Bigg]^{-1/2},
\end{aligned}$$

and for the Dirac oscillator with exact pseudo-spin symmetry in section (IV), the normalization constant is,

$$
\begin{aligned}
N_{n\tilde{l}} = \Bigg[& I_n\left(\gamma, \tilde{l}, \tilde{l} - \frac{1}{2}, \tilde{l} - \frac{1}{2}\right) \\
& - \frac{\mu\alpha^2}{\gamma} I_n\left(\gamma, \tilde{l} + 1, \tilde{l} - \frac{1}{2}, \tilde{l} - \frac{1}{2}\right) \\
& + \frac{\alpha^2(\beta - \gamma)^2}{\gamma(1 - \epsilon)^2} \left\{ I_n\left(\gamma, \tilde{l} + 1, \tilde{l} - \frac{1}{2}, \tilde{l} - \frac{1}{2}\right) \right. \\
& \left. - \frac{\mu\alpha^2}{\gamma} I_n\left(\gamma, \tilde{l} + 2, \tilde{l} - \frac{1}{2}, \tilde{l} - \frac{1}{2}\right) \right\} \\
& + \frac{4\alpha^2\gamma}{(1 - \epsilon)^2} \left\{ I_n\left(\gamma, \tilde{l} + 1, \tilde{l} + \frac{1}{2}, l + \frac{1}{2}\right) \right. \\
& \left. - \frac{\mu\alpha^2}{\gamma} I_n\left(\gamma, \tilde{l} + 2, \tilde{l} + \frac{1}{2}, \tilde{l} + \frac{1}{2}\right) \right\} \\
& - \frac{4\alpha^2(\gamma - \beta)}{(1 - \epsilon)^2} \left\{ I_n\left(\gamma, \tilde{l} + 1, \tilde{l} - \frac{1}{2}, \tilde{l} + \frac{1}{2}\right) \right. \\
& \left. - \frac{\mu\alpha^2}{\gamma} I_n\left(\gamma, \tilde{l} + 2, \tilde{l} - \frac{1}{2}, \tilde{l} + \frac{1}{2}\right) \right\} \\
& + \frac{\alpha^2(\tilde{l} - \lambda)^2}{(1 - \epsilon)^2} \gamma \left\{ I_n\left(\gamma, \tilde{l} - 1, \tilde{l} - \frac{1}{2}, \tilde{l} - \frac{1}{2}\right) \right. \\
& \left. - \frac{\mu\alpha^2}{\gamma} I_n\left(\gamma, \tilde{l}, \tilde{l} - \frac{1}{2}, \tilde{l} - \frac{1}{2}\right) \right\} \\
& - \frac{4\alpha^2\gamma}{(1 - \epsilon)^2} (\tilde{l} - \lambda) \left\{ I_n\left(\gamma, \tilde{l}, \tilde{l} - \frac{1}{2}, l + \frac{1}{2}\right) \right. \\
& \left. - \frac{\mu\alpha^2}{\gamma} I_n\left(\gamma, \tilde{l} + 1, \tilde{l} - \frac{1}{2}, \tilde{l} + \frac{1}{2}\right) \right\} \\
& + \frac{2\alpha^2(\gamma - \beta)(\tilde{l} - \lambda)}{(1 - \epsilon)^2} \left\{ I_n\left(\gamma, \tilde{l}, \tilde{l} - \frac{1}{2}, \tilde{l} - \frac{1}{2}\right) \right. \\
& \left. \left. - \frac{\mu\alpha^2}{\gamma} I_n\left(\gamma, \tilde{l} + 1, \tilde{l} - \frac{1}{2}, \tilde{l} - \frac{1}{2}\right) \right\} \right]^{-1/2} .
\end{aligned}
$$

In both cases I_n is defined as,

$$
I_n(s, a, b, c) = \int_0^\infty (sr^2)^a \exp(sr^2) L_n^b(sr^2) L_n^c(sr^2) dr, \tag{45}
$$

taking $x = sr^2$ and using the associated Laguerre polynomial identity [51],

$$
L_n^k(x) = \sum_{i=0}^{n} (-1)^i \binom{n+k}{n-i} \frac{x^i}{i!} \tag{46}
$$

we have,

$$
\begin{aligned}
I_n(s, a, b, c) = & \frac{s^{-1/2}}{2} \sum_{i,j=0}^{n} \frac{(-1)^{i+j}}{i!j!} \\
& \times \binom{n+b}{n-i} \binom{n+c}{n-j} \int_0^\infty x^{a+i+j-1/2} e^{-x} dx
\end{aligned} \tag{47}
$$

perusal of reference [51] yields,

$$
\begin{aligned}
I_n(s, a, b, c) = & \frac{s^{-1/2}}{2} \sum_{i,j=0}^{n} \frac{(-1)^{i+j}}{i!j!} \\
& \times \binom{n+b}{n-i} \binom{n+c}{n-j} \left(a + i + j - \frac{1}{2}\right)!
\end{aligned} \tag{48}
$$

ORCID iDs

Alexandre G M Schmidt https://orcid.org/0000-0001-8336-2135

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