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Computational and numerical simulations for the nonlinear fractional Kolmogorov–Petrovskii–Piskunov (FKPP) equation

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Abstract

This research paper elucidates solitary, compacton, and peakon computational solutions, and numerical solutions of the nonlinear fractional Kolmogorov–Petrovskii–Piskunov (FKPP) equation that belongs to the class of reaction–diffusion equation. This equation describes the behavior of genetic models in the increase of microorganisms. Usually, it is used as a biological model to investigate the microbiological densities in bacteria cells as a result of diffusion mechanisms in terms of space-time. The present framework depends on applying the modified Khater method to the FKPP equation to extract the computational solutions then using these solutions to get necessary boundary conditions to implement the numerical B–spline schemes on the suggested equation. The reliability and accuracy of the computational method and solutions are verified by using numerical simulations. For more explanation of the obtained analytical solutions, some sketches are plotted in different types. Also, the comparison between the distinct types of obtained solutions is shown by calculating the absolute value of error.

Keywords: the nonlinear fractional kolmogorov–petrovskii–piskunov (FKPP) equation, fractional calculus, fractional nonlinear partial differential equation, modified khater method, septic b–spline scheme, computational & numerical solutions

(Some figures may appear in colour only in the online journal)

1. Introduction

Biomathematics is one of the newest exciting fields that use mathematical models to illustrate and investigate biological phenomena. The function and structure of the natural system components are studied in isolation by using modern experimental biology. Collecting the data from the biological experiment allows formulating the mathematical models of these phenomena. The dynamic of these models represents the system's components, structure, and processes of their interactions. These mathematical models are essential tools to simulate and analysis these biological phenomena to extend the biological theory that enables new hypotheses or experiments. Many information and quantitative answers are obtained by formulating these phenomena in mathematical equations such as the decline in sea turtle populations, predict the outcome accurately before action is taken, and so on [1, 2].

Recently, many Mathematicians and physicists are interested in studying the bacteria cell's properties where all animal life on Earth depends on it. They use theoretical analysis, mathematical models, and abstractions of the living organisms to analyze their structure, development, and behaviour. Bacteria poses a large domain of prokaryotic microorganisms. The general properties of it are few micrometers in length and ranging from spheres to rods and spirals. It is considered as the first life forms on the Earth and usually lives in the radioactive waste, the deep portions of Earth's crust, soil, water, radioactive waste, and acidic hot springs. Just 27% of bacteria cell can be grown in the laboratory where there are $[5 \times 10^{30}]$ bacteria on Earth, so you can imagine the

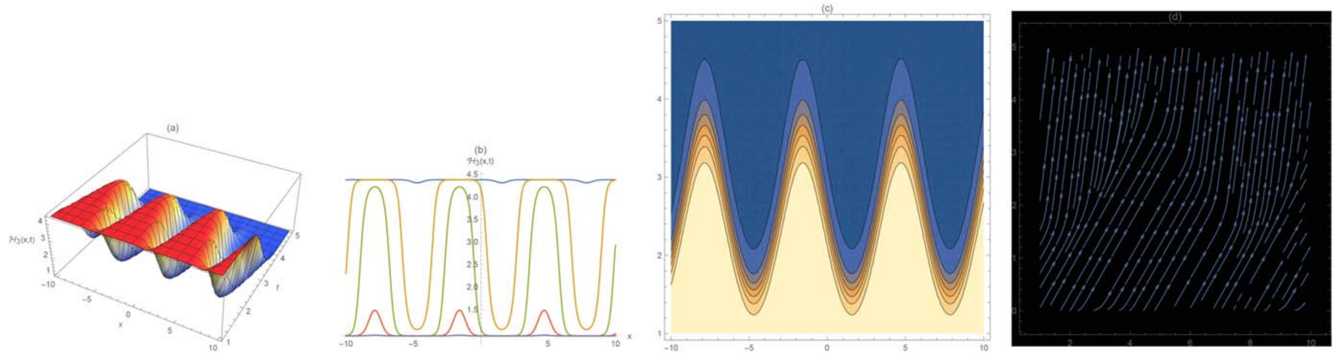

Figure 1. Compacton wave of (10) in three, two-dimensional, and contour plots when [$\vartheta = 0.5$, $\alpha = 3$, $\beta = 5$, $\delta = -2\sqrt{7}$, $k = 6$, $\rho = 4$, $\sigma = 2$, $\omega = -2\sqrt{21}$, $\varrho = 1$].

number of bacteria that can be grown in the laboratory. The bacteriology is the scientific name of studying the bacteria and its role in human life [3–5].

Fractional nonlinear evolution equations include systematic memory effects and non-local through time derivatives and fractional–order space that allows to formulate the phenomena across multiple time and space scales such as biological, chemical, engineering, viscoelastic materials, and physical models. These fractional models bases on the fractional derivatives can limit or capture salient features of complex phenomena that allow more investigation of these formulated mathematical models. Many kind of fractional derivatives are formulated to solve that kind of models such as Riemann–Liouville, Caputo, and conformable fractional derivatives [6–8].

This research paper applies the modified Khater and B–spline schemes to the nonlinear FKPP equation to find the computational and numerical solutions. It is a quasilinear parabolic one arising in the modeling of certain reaction–diffusion processes in the theory of combustion, mathematical biology, and other areas of natural sciences. This equation is given in the following formula:

$$\frac{\partial \mathcal{H}}{\partial t} - \Delta\, u(x, t) = F(\mathcal{H}(x, t)), \tag{1}$$

where $x = (x_1, x_2, \ldots, x_n)$ is a point of space $\mathfrak{R}^n$, $t \in \mathfrak{R}$, $n \geqslant 1$, $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2 + \ldots + \partial_n^2$ is the laplacian in $\mathfrak{R}^n$, and the right–hand side $f \in C^1([0, 1])$ is positive on the interval (0, 1) and satisfies the conditions

$$f(0) = f(1) = 0,\ E_0 := f'(0) > 0;$$

moreover, $f(\xi) \leqslant E_0\, \xi$ for $\xi \in (0, 1)$. Using the following wave transformation

$$\mathcal{H}(x, t) = \mathcal{M}(\eta(x, t)),\ \eta(x, t) := (x, \eta) - \omega\, t,\ \omega > 0,$$

coverts equation (1) that represents a stage of an uncompleted process at which a particular form of initial data has not had a noticeable effect on its steady–state behavior. substituting $E(\xi) = \xi\, f(\xi^\sigma)$ into equation (1) leads to a generalized Fisher equation that takes the following formula

$$\frac{d^2\,\mathcal{M}}{d\,\eta^2} + \omega\,\frac{d\,\mathcal{M}}{d\,\eta} + f_0\,\mathcal{M}(1 - \mathcal{M}^\sigma) = 0,\ \mathcal{M} \in (0, 1). \tag{2}$$

The nonlinear KPP equation (the so–called intermediate asymptotic regime) was derived by Kolmogorov, Petrovskii, and Piskunov in 1937 has the following formula [9, 10]:

$$\mathcal{H}_t - \mathcal{H}_{x\,x} + \rho\,\mathcal{H} + \delta\,\mathcal{H}^2 + \varrho\,\mathcal{H}^3 = 0, \tag{3}$$

where ρ, δ, $\varrho$are arbitrary constants and $\mathcal{H}(x, t)$ designates respectively, the state evolution over the spatial–temporal domain distinguished by the x, t coordinates. This equation is used to analyze the distinct physical, chemical, and biological models. Under specific conditions on the arbitrary constants, (3) contains different form of nonlinear evolution equations such as:

- When [$\rho = -1$, $\delta = 0$, $\varrho = 1$], equation (3) reduces to be the Newll–Whitehead equation [11, 12].
- When [$\rho = \mu$, $\delta = -(\mu + 1)$, $\varrho = 1$], equation (3) reduces to be the FitzHugh-Nagumo equation [13, 14].
- When [$\rho = -1$, $\delta = 1$, $\varrho = 0$, $(\mathcal{H}_t - \mathcal{H}_{x\,x} = \mathcal{H} - \mathcal{H}^2)$], equation (3) reduces to be the Fisher equation [15, 16].

While, the nonlinear FKPP equation is formulated to investigate the nonlocal property of the micro–morphogenesis which has a vital role in the elementary phenomena of contemporary microbiology[17–20].

$$D_t^\vartheta \mathcal{H} - \mathcal{H}_{x\,x} + \rho\,\mathcal{H} + \delta\,\mathcal{H}^2 + \varrho\,\mathcal{H}^3 = 0, \tag{4}$$

where $0 < \vartheta < 1$ and D is differential operator. Using the next definition of the conformable derivative properties $\left[\mathcal{H} = \mathcal{H}(x, t) = \mathcal{H}(\eth),\ \eth = k\,x + \frac{\omega\,t^\vartheta}{\vartheta}\right]$ on equation (4), leads to

$$\omega\,\mathcal{H}' - k\,\mathcal{H}'' + \rho\,\mathcal{H} + \delta\,\mathcal{H}^2 + \varrho\,\mathcal{H}^3 = 0. \tag{5}$$

using the homogeneous balance rule on equation (5), yields [$3\,N = N + 2 \ \rightarrow\ N = 1$].

This kind of model attracts the attention of the mathematicians and physics, where they can use them to discover more properties of them. In the context of the mathematical view of this survey, many computational, semi–analytical, and numerical schemes are derived to find distinct types of solutions such as The variational iteration method (VIM.), Adomian decomposition method, the generalized Kudryashov method, Riccati equation method, $\left(\frac{G'}{G}\right)$–expansion method, Khater method, the modified Khater method (modified auxiliary equation method) and so on [21–35].

The rest of this research paper is ordered in the following order: section 2 applies the modified Khater method [36–42], and B–spline schemes [43–47] to the nonlinear FKPP equation. Moreover, the comparison between the computational and numerical obtained solutions is explained, and some sketches are plotted to show more physical properties of this system. Section 3 discusses the obtained computational results and explain the comparison between them and that obtained in previous work. Moreover, it show the comparison between the obtained numerical results. Section 4 gives the conclusion of the whole research.

2. Computational and numerical solutions of the nonlinear FKPP equation

This section applies the computational and numerical schemes to the nonlinear FKPP equation to show more physical properties of the model in the optical illusions field by explaining the behavior of the Langmuir waves in an ionized plasma wave.

2.1. Solitary wave solutions

Implementation of the modified Khater method to the nonlinear FKPP equation, leads to derive the general form of solution of equation (5) in the following formula

$$\begin{aligned}\mathcal{H}(\eth) &= \textstyle\sum_{i=1}^{N} a_i K^{i\ \wp(\eth)} + \sum_{i=1}^{N} b_i K^{-i\ \wp(\eth)} + a_0\\ &= a_1 K^{\wp(\eth)} + a_0 + b_1 K^{-\wp(\eth)},\end{aligned} \tag{6}$$

where a_0, a_1, b_1 are arbitrary constants and $\wp(\eth)$ is the solution function of the following equation

$$\wp'(\eth) = \frac{\beta + \alpha K^{-\wp(\eth)} + \sigma K^{\wp(\eth)}}{\ln(K)}, \tag{7}$$

where K, α, β, σ are arbitrary constants. Additionally, N is the value of the balance between the highest order derivative term and nonlinear term. Substituting equation (6) along (7) into equation (5), yields a polynomial of $K^{\wp(\eth)}$. Gathering the coefficients with the same power of $K^{\wp(\eth)}$, leads to a system of algebraic equation. Solving this system by the Mathematica 11.2, yields

Family I

$$\left[b_1 \to 0,\ \omega \to -\frac{\sqrt{\frac{k\sigma^2}{\varrho}}\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)}}{\sigma}, a_1 \to \sqrt{\frac{2k\sigma^2}{\varrho}},\ a_0 \to \frac{\sigma\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)} + \beta\varrho\sqrt{\frac{k\sigma^2}{\varrho}}}{\sqrt{2}\,\sigma\varrho},\right.$$
$$\left.\delta \to -\sqrt{2}\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)},\ \text{where } \sigma \neq 0\right].$$

Thus, the solitary wave solutions of the nonlinear FKPP equation are in the following formulas:

When $\beta^2 - 4\alpha\sigma < 0 \& \sigma \neq 0$

$$\begin{aligned}\mathcal{H}_1(x, t) = \frac{1}{\sqrt{2}\,\sigma\varrho} &\times \left[\sigma\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)} + \varrho\sqrt{4\alpha\sigma - \beta^2}\sqrt{\frac{k\sigma^2}{\varrho}}\right.\\ &\left.\times \tan\left(\frac{1}{2}\sqrt{4\alpha\sigma - \beta^2}\left(kx - \frac{t^{\vartheta}\sqrt{\frac{k\sigma^2}{\varrho}}\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)}}{\sigma\vartheta}\right)\right)\right],\end{aligned} \tag{8}$$

$$\begin{aligned}\mathcal{H}_2(x, t) = \frac{1}{\sqrt{2}\,\sigma\varrho} &\times \left[\sigma\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)} + \varrho\sqrt{4\alpha\sigma - \beta^2}\sqrt{\frac{k\sigma^2}{\varrho}}\right.\\ &\left.\times \cot\left(\frac{1}{2}\sqrt{4\alpha\sigma - \beta^2}\left(kx - \frac{t^{\vartheta}\sqrt{\frac{k\sigma^2}{\varrho}}\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)}}{\sigma\vartheta}\right)\right)\right].\end{aligned} \tag{9}$$

When $\beta^2 - 4\alpha\sigma > 0 \& \sigma \neq 0$

$$\mathcal{H}_4(x, t) = \frac{1}{\sqrt{2}\,\sigma\varrho} \times \left[\sigma\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)} - \varrho\sqrt{\beta^2 - 4\alpha\sigma}\sqrt{\frac{k\sigma^2}{\varrho}} \times \tanh\left(\frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma}\left(kx - \frac{t^{\vartheta}\sqrt{\frac{k\sigma^2}{\varrho}}\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)}}{\sigma\vartheta}\right)\right)\right], \tag{10}$$

$$\mathcal{H}_5(x, t) = \frac{1}{\sqrt{2}\,\sigma\varrho} \times \left[\sigma\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)} - \varrho\sqrt{\beta^2 - 4\alpha\sigma}\sqrt{\frac{k\sigma^2}{\varrho}} \times \coth\left(\frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma}\left(kx - \frac{t^{\vartheta}\sqrt{\frac{k\sigma^2}{\varrho}}\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)}}{\sigma\vartheta}\right)\right)\right]. \tag{11}$$

When $\alpha\sigma > 0 \& \sigma \neq 0 \& \alpha \neq 0 \& \beta = 0$

$$\mathcal{H}_5(x, t) = \frac{\sqrt{\varrho(\rho - 2\alpha k\sigma)}}{\varrho} + \frac{\sqrt{2}\,\alpha\sqrt{\frac{k\sigma^2}{\varrho}}\tan\left(\sqrt{\alpha\sigma}\left(kx - \frac{t^{\vartheta}\sqrt{\frac{k\sigma^2}{\varrho}}\sqrt{\varrho(2\rho - 4\alpha k\sigma)}}{\sigma\vartheta}\right)\right)}{\sqrt{\alpha\sigma}}, \tag{12}$$

$$\mathcal{H}_6(x, t) = \frac{\sqrt{\varrho(\rho - 2\alpha k\sigma)}}{\varrho} - \frac{\sqrt{2}\,\alpha\sqrt{\frac{k\sigma^2}{\varrho}}\cot\left(\sqrt{\alpha\sigma}\left(kx - \frac{t^{\vartheta}\sqrt{\frac{k\sigma^2}{\varrho}}\sqrt{\varrho(2\rho - 4\alpha k\sigma)}}{\sigma\vartheta}\right)\right)}{\sqrt{\alpha\sigma}}. \tag{13}$$

When $\alpha\sigma < 0 \& \sigma \neq 0 \& \alpha \neq 0 \& \beta = 0$

$$\mathcal{H}_7(x, t) = \frac{\sqrt{\varrho(\rho - 2\alpha k\sigma)}}{\varrho} + \frac{\sqrt{2}\,\alpha\sqrt{\frac{k\sigma^2}{\varrho}}\tanh\left(\sqrt{-\alpha\sigma}\left(kx - \frac{t^{\vartheta}\sqrt{\frac{k\sigma^2}{\varrho}}\sqrt{\varrho(2\rho - 4\alpha k\sigma)}}{\sigma\vartheta}\right)\right)}{\sqrt{-\alpha\sigma}}, \tag{14}$$

$$\mathcal{H}_8(x, t) = \frac{\sqrt{\varrho(\rho - 2\alpha k\sigma)}}{\varrho} + \frac{\sqrt{2}\,\alpha\sqrt{\frac{k\sigma^2}{\varrho}}\coth\left(\sqrt{-\alpha\sigma}\left(kx - \frac{t^{\vartheta}\sqrt{\frac{k\sigma^2}{\varrho}}\sqrt{\varrho(2\rho - 4\alpha k\sigma)}}{\sigma\vartheta}\right)\right)}{\sqrt{-\alpha\sigma}}. \tag{15}$$

When $\beta = 0 \& \alpha = -\sigma$

$$\mathcal{H}_9(x, t) = \frac{\sqrt{\varrho(2\alpha^2 k + \rho)}}{\varrho} + \sqrt{2}\sqrt{\frac{\alpha^2 k}{\varrho}}\coth\left(\frac{t^{\vartheta}\sqrt{\frac{\alpha^2 k}{\varrho}}\sqrt{\varrho(4\alpha^2 k + 2\rho)}}{\vartheta} + \alpha kx\right). \tag{16}$$

When $\beta = \sigma = \kappa \& \alpha = 0$

$$\mathcal{H}_{10}(x, t) = \frac{\sqrt{\varrho(\kappa^2 k + 2\rho)} - \varrho\sqrt{\frac{\kappa^2 k}{\varrho}}\coth\left(\frac{1}{2}\left(\kappa kx - \frac{t^{\vartheta}\sqrt{\frac{\kappa^2 k}{\varrho}}\sqrt{\varrho(\kappa^2 k + 2\rho)}}{\vartheta}\right)\right)}{\sqrt{2}\,\varrho}. \tag{17}$$

When $\alpha = 0 \& \beta \neq 0 \& \sigma \neq 0$

$$\mathcal{H}_{11}(x, t) = \frac{\beta\sqrt{\frac{k\sigma^2}{\varrho}}\left(\frac{1}{\exp\left(\frac{\beta t^{\vartheta}\sqrt{\frac{k\sigma^2}{\varrho}}\sqrt{\varrho(\beta^2 k + 2\rho)}}{\sigma\vartheta} - \beta k x\right) - \frac{\sigma}{2}} + \frac{1}{\sigma}\right) + \frac{\sqrt{\varrho(\beta^2 k + 2\rho)}}{\varrho}}{\sqrt{2}}. \tag{18}$$

When $\beta = \alpha = 0 \& \sigma \neq 0$

$$\mathcal{H}_{12}(x, t) = \frac{2\sigma\vartheta}{2\sigma\sqrt{\rho\varrho}\,t^{\vartheta} - \sqrt{2}\,x\vartheta\varrho\sqrt{\frac{k\sigma^2}{\varrho}}} + \frac{\rho}{\sqrt{\rho\varrho}}. \tag{19}$$

When $\beta = 0 \& \alpha = \sigma$

$$\mathcal{H}_{13}(x, t) = \sqrt{2}\sqrt{\frac{\alpha^2 k}{\varrho}}\tan\left(C - \frac{t^{\vartheta}\sqrt{\frac{\alpha^2 k}{\varrho}}\sqrt{\varrho(2\rho - 4\alpha^2 k)}}{\vartheta} + \alpha k x\right) + \frac{\sqrt{\varrho(\rho - 2\alpha^2 k)}}{\varrho}. \tag{20}$$

When $\beta^2 - 4\alpha\sigma = 0$

$$\mathcal{H}_{14}(x, t) = \frac{\frac{\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)}}{\varrho} + \frac{\sqrt{\frac{k\sigma^2}{\varrho}}\left(-4\alpha\beta + \frac{\beta^3}{\sigma} + \frac{8\alpha\sigma\vartheta}{t^{\vartheta}\sqrt{\frac{k\sigma^2}{\varrho}}\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)} - k\sigma x\vartheta}\right)}{\beta^2}}{\sqrt{2}}. \tag{21}$$

Family II

$$\left[a_1 \to 0,\ \omega \to \frac{\sqrt{\frac{\alpha^2 k}{\varrho}}\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)}}{\alpha}, b_1 \to \sqrt{\frac{2\alpha^2 k}{\varrho}},\ a_0 \to \frac{\beta\varrho\sqrt{\frac{\alpha^2 k}{\varrho}} + \alpha\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)}}{\sqrt{2}\,\alpha\varrho},\right.$$

$$\left.\delta \to -\sqrt{2}\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)},\ \text{where } \alpha \neq 0\right].$$

Thus, the solitary wave solutions of the nonlinear FKPP equation are in the following formulas:

When $\beta^2 - 4\alpha\sigma < 0 \& \sigma \neq 0$

$$\mathcal{H}_{15}(x, t) = \frac{\sqrt{2}}{2} \times \left[\frac{\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)}}{\varrho} + \sqrt{\frac{\alpha^2 k}{\varrho}}\left(\frac{\beta}{\alpha} - \frac{4\sigma}{\beta - \sqrt{4\alpha\sigma - \beta^2}\tan\left(\frac{1}{2}\sqrt{4\alpha\sigma - \beta^2}\left(\frac{t^{\vartheta}\sqrt{\frac{\alpha^2 k}{\varrho}}\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)}}{\alpha\vartheta} + kx\right)\right)}\right)\right], \tag{22}$$

$$\mathcal{H}_{16}(x,t) = \frac{\sqrt{2}}{2} \times \left[\frac{\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)}}{\varrho} + \sqrt{\frac{\alpha^2 k}{\varrho}}\left(\frac{\beta}{\alpha} - \frac{4\sigma}{\beta - \sqrt{4\alpha\sigma - \beta^2}\cot\left(\frac{1}{2}\sqrt{4\alpha\sigma - \beta^2}\left(\frac{t^{\vartheta}\sqrt{\frac{\alpha^2 k}{\varrho}}\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)}}{\alpha\vartheta} + kx\right)\right)}\right)\right]. \tag{23}$$

When $\beta^2 - 4\alpha\sigma > 0\&\sigma \neq 0$

$$\mathcal{H}_{17}(x,t) = \frac{\sqrt{2}}{2} \times \left[\frac{\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)}}{\varrho} + \sqrt{\frac{\alpha^2 k}{\varrho}}\left(\frac{\beta}{\alpha} - \frac{4\sigma}{\beta + \sqrt{\beta^2 - 4\alpha\sigma}\tanh\left(\frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma}\left(\frac{t^{\vartheta}\sqrt{\frac{\alpha^2 k}{\varrho}}\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)}}{\alpha\vartheta} + kx\right)\right)}\right)\right], \tag{24}$$

$$\mathcal{H}_{18}(x,t) = \frac{\sqrt{2}}{2} \times \left[\frac{\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)}}{\varrho} + \sqrt{\frac{\alpha^2 k}{\varrho}}\left(\frac{\beta}{\alpha} - \frac{4\sigma}{\beta + \sqrt{\beta^2 - 4\alpha\sigma}\coth\left(\frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma}\left(\frac{t^{\vartheta}\sqrt{\frac{\alpha^2 k}{\varrho}}\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)}}{\alpha\vartheta} + kx\right)\right)}\right)\right]. \tag{25}$$

When $\alpha\sigma > 0\&\sigma \neq 0\&\alpha \neq 0\&\beta = 0$

$$\mathcal{H}_{19}(x,t) = \frac{\sqrt{\varrho(\rho - 2\alpha k\sigma)}}{\varrho} + \frac{\sqrt{2}\,\sigma\sqrt{\frac{\alpha^2 k}{\varrho}}\cot\left(\sqrt{\alpha\sigma}\left(\frac{t^{\vartheta}\sqrt{\frac{\alpha^2 k}{\varrho}}\sqrt{\varrho(2\rho - 4\alpha k\sigma)}}{\alpha\vartheta} + kx\right)\right)}{\sqrt{\alpha\sigma}}, \tag{26}$$

$$\mathcal{H}_{20}(x,t) = \frac{\sqrt{\varrho(\rho - 2\alpha k\sigma)}}{\varrho} + \frac{\sqrt{2}\,\sigma\sqrt{\frac{\alpha^2 k}{\varrho}}\tan\left(\sqrt{\alpha\sigma}\left(\frac{t^{\vartheta}\sqrt{\frac{\alpha^2 k}{\varrho}}\sqrt{\varrho(2\rho - 4\alpha k\sigma)}}{\alpha\vartheta} + kx\right)\right)}{\sqrt{\alpha\sigma}}. \tag{27}$$

When $\alpha\sigma < 0 \& \sigma \neq 0 \& \alpha \neq 0 \& \beta = 0$

$$\mathcal{H}_{21}(x, t) = \frac{\sqrt{\varrho(\rho - 2\alpha k\sigma)}}{\varrho} + \frac{\sqrt{2}\sqrt{-\alpha\sigma}\sqrt{\frac{\alpha^2 k}{\varrho}}\coth\left(\sqrt{-\alpha\sigma}\left(\frac{t^{\vartheta}\sqrt{\frac{\alpha^2 k}{\varrho}}\sqrt{\varrho(2\rho - 4\alpha k\sigma)}}{\alpha\vartheta} + kx\right)\right)}{\alpha}, \tag{28}$$

$$\mathcal{H}_{22}(x, t) = \frac{\sqrt{\varrho(\rho - 2\alpha k\sigma)}}{\varrho} + \frac{\sqrt{2}\sqrt{-\alpha\sigma}\sqrt{\frac{\alpha^2 k}{\varrho}}\tanh\left(\sqrt{-\alpha\sigma}\left(\frac{t^{\vartheta}\sqrt{\frac{\alpha^2 k}{\varrho}}\sqrt{\varrho(2\rho - 4\alpha k\sigma)}}{\alpha\vartheta} + kx\right)\right)}{\alpha}. \tag{29}$$

When $\beta = 0 \& \alpha = -\sigma$

$$\mathcal{H}_{23}(x, t) = \frac{\sqrt{\varrho(2\alpha^2 k + \rho)}}{\varrho} + \sqrt{2}\sqrt{\frac{\alpha^2 k}{\varrho}}\tanh\left(\frac{t^{\vartheta}\sqrt{\frac{\alpha^2 k}{\varrho}}\sqrt{\varrho(4\alpha^2 k + 2\rho)}}{\vartheta} + \alpha kx\right). \tag{30}$$

When $\beta = \frac{\alpha}{2} = \kappa \& \sigma = 0$

$$\mathcal{H}_{24}(x, t) = \frac{\sqrt{\frac{\kappa^2 k}{\varrho}}\left(\frac{4}{\exp\left(\frac{t^{\vartheta}\sqrt{\frac{\kappa^2 k}{\varrho}}\sqrt{\varrho(\kappa^2 k + 2\rho)}}{\vartheta} + \kappa kx\right) - 2} + 1\right) + \frac{\sqrt{\varrho(\kappa^2 k + 2\rho)}}{\varrho}}{\sqrt{2}}. \tag{31}$$

When $\beta = \sigma = 0 \& \alpha \neq 0$

$$\mathcal{H}_{25}(x, t) = \frac{2\alpha\vartheta}{\sqrt{2}x\vartheta\varrho\sqrt{\frac{\alpha^2 k}{\varrho}} + 2\alpha\sqrt{\rho\varrho}t^{\vartheta}} + \frac{\rho}{\sqrt{\rho\varrho}}. \tag{32}$$

When $\beta = 0 \& \alpha = \sigma$

$$\mathcal{H}_{26}(x, t) = \sqrt{2}\sqrt{\frac{\alpha^2 k}{\varrho}}\cot\left(C + \frac{t^{\vartheta}\sqrt{\frac{\alpha^2 k}{\varrho}}\sqrt{\varrho(2\rho - 4\alpha^2 k)}}{\vartheta} + \alpha kx\right) + \frac{\sqrt{\varrho(\rho - 2\alpha^2 k)}}{\varrho}. \tag{33}$$

When $\sigma = 0 \& \beta \neq 0 \& \alpha \neq 0$

$$\mathcal{H}_{27}(x, t) = \frac{\beta\sqrt{\frac{\alpha^2 k}{\varrho}}\left(\frac{1}{\alpha} - \frac{2}{\alpha - \beta\exp\left(\frac{\beta t^{\vartheta}\sqrt{\frac{\alpha^2 k}{\varrho}}\sqrt{\varrho(\beta^2 k + 2\rho)}}{\alpha\vartheta} + \beta kx\right)}\right) + \frac{\sqrt{\varrho(\beta^2 k + 2\rho)}}{\varrho}}{\sqrt{2}}. \tag{34}$$

When $\beta^2 - 4\alpha\sigma = 0$

$$\mathcal{H}_{28}(x, t) = \frac{1}{\sqrt{2}\varrho\left(\beta t^{\vartheta}\sqrt{\frac{\alpha^2 k}{\varrho}}\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)} + \alpha\vartheta(\beta kx + 2)\right)} \times [2\alpha\vartheta\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)} + \beta(-k)\varrho t^{\vartheta}(4\alpha\sigma - \beta^2)\sqrt{\frac{\alpha^2 k}{\varrho}} + 2\beta\varrho\sqrt{\frac{\alpha^2 k}{\varrho}}(\rho t^{\vartheta} + \vartheta) + \alpha\beta kx\vartheta\sqrt{\varrho(-4\alpha k\sigma + \beta^2 k + 2\rho)}]. \tag{35}$$

2.2. Numerical simulations

This section studies the numerical solutions of equation (5) by applying the B–spline techniques that are considered as the most accurate numerical tools to get this type of solutions. Using the computational solution of (5) with the following initial conditions $\left[\mathcal{H}_{exact} = \frac{1}{3}(-4)\tanh\left(\frac{\eth}{2}\right),\ \varrho = \frac{827}{6336},\ \alpha = 2,\ a_0 = 4,\ \beta = 3,\ \lambda = 5,\ \sigma = 1,\ \omega = -\frac{51}{2}\right]$, allows applying the following schemes, as follows:

2.2.1. Cubic-Spline. Based on the cubic B–spline, the suggested solution of the nonlinear KPP equation (5) is given by

$$\mathcal{H}(\eth) = \sum_{i=-1}^{n+1} \mho_i\, \aleph_i, \tag{36}$$

where $\mho_i$, $\aleph_i$ fulfill the next conditions:

$$L\ \mathcal{H}(\eth) = \varnothing(\eth_i,\ \mathcal{H}(\eth_i))\text{where } (i = 0,\ 1,\ldots,n)$$

and

$$\aleph_i(\eth) = \frac{1}{6\,h^3}\begin{cases} (\eth - \eth_{i-2})^3, & \eth \in [\eth_{i-2},\ \eth_{i-1}],\\ -3(\eth - \eth_{i-1})^3 + 3\,h\,(\eth - \eth_{i-1})^2 + 3\,h^2(\eth - \eth_{i-1}) + h^3, & \eth \in [\eth_{i-1},\ \eth_i],\\ -3(\eth_{i+1} - \eth)^3 + 3\,h\,(\eth_{i+1} - \eth)^2 + 3\,h^2\,(\eth_{i+1} - \eth) + h^3, & \eth \in [\eth_i,\ \eth_{i+1}],\\ (\eth_{i+2} - \eth)^3, & \eth \in [\eth_{i+1},\ \eth_{i+2}],\\ 0, & \text{otherwise}, \end{cases} \tag{37}$$

where L is a linear operator, $i \in [-2,\ n + 2]$. So that, the numerical formula of the solution is given as

$$\mathcal{H}_i(\eth) = \mho_{i-1} + 4\,\mho_i + \mho_{i+1}. \tag{38}$$

Substituting equation (38) into (5), leads to a system of equations. Solving this system of equations, gives the value of $\mho_i$. Replacing the values of $\mho_i$, $\aleph_i$ into equation (36), gives the following data that are shown in the next table 1

2.2.2. Quintic-spline. Based on the quintic B–spline, the suggested solution of the nonlinear KPP equation (5) is given by

$$\mathcal{H}(\eth) = \sum_{i=-1}^{n+1} \mho_i\, \aleph_i, \tag{39}$$

where $\mho_i$, $\aleph_i$ satisfies the following condition

$$L\ \mathcal{H}(\eth) = \varnothing(\eth_i,\ \mathcal{H}(\eth_i))\text{where } (i = 0,\ 1,\ldots,n)$$

and

$$\aleph_i(\eth) = \frac{1}{h^5}\begin{cases} (\eth - \eth_{i-3})^5, & \eth \in [\eth_{i-3},\ \eth_{i-2}],\\ (\eth - \eth_{i-3})^5 - 6(\eth - \eth_{i-2})^5, & \eth \in [\eth_{i-2},\ \eth_{i-1}],\\ (\eth - \eth_{i-3})^5 - 6(\eth - \eth_{i-2})^5 + 15(\eth - \eth_{i-1})^5, & \eth \in [\eth_{i-1},\ \eth_i],\\ (\eth_{i+3} - \eth)^5 - 6(\eth_{i+2} - \eth)^5 + 15(\eth_{i+1} - \eth)^5, & \eth \in [\eth_i,\ \eth_{i+1}],\\ (\eth_{i+3} - \eth)^5 - 6(\eth_{i+2} - \eth)^5, & \eth \in [\eth_{i+1},\ \eth_{i+2}],\\ (\eth_{i+3} - \eth)^5, & x \in [\eth_{i+2},\ \eth_{i+3}],\\ 0, & \text{otherwise}, \end{cases} \tag{40}$$

where L is a linear operator, $i \in [-2,\ n + 2]$. Hence, the numerical solution is formulated in the next form

$$v_i(\eth) = \mho_{i-2} + 26\,\mho_{i-1} + 66\,\mho_i + 26\,\mho_{i+1} + \mho_{i+2}. \tag{41}$$

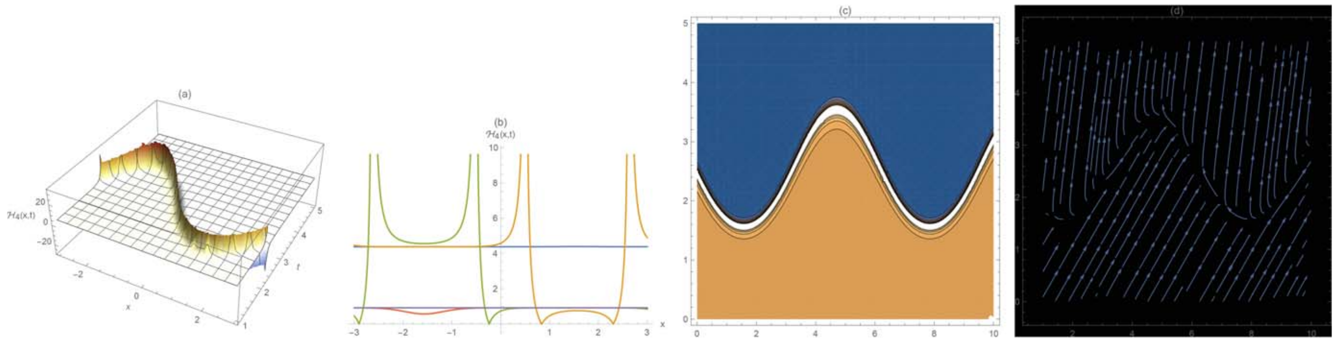

Figure 2. Peakon solitary wave of (11) in three, two-dimensional, and contour plots when $[\vartheta = 0.5,\ \alpha = 3,\ \beta = 5,\ \delta = -2\sqrt{7},\ k = 6,\ \rho = 4,\ \sigma = 2,\ \omega = -2\sqrt{21},\ \varrho = 1]$.

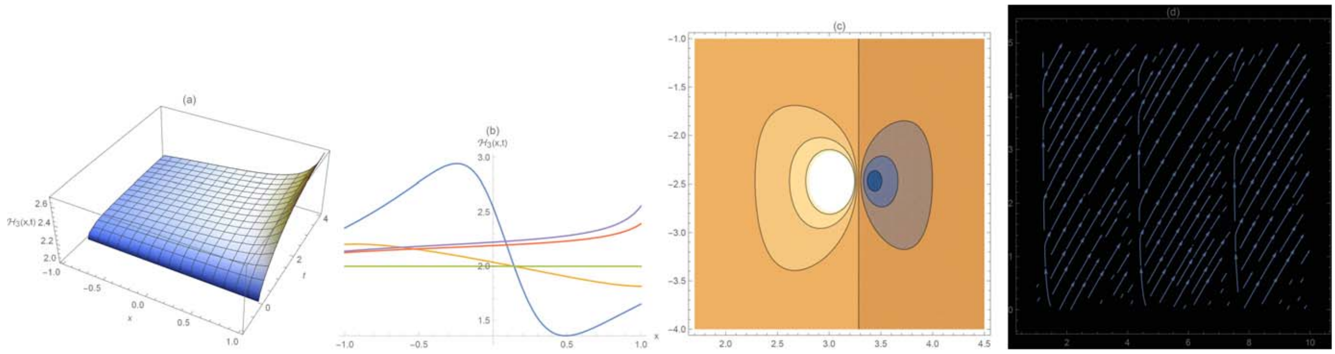

Figure 3. Solitary wave of (21) in three, two-dimensional, and contour plots when $\left[\beta = 5,\ \alpha = 1,\ \sigma = \frac{25}{4}\right]$.

Substituting equation (41) into equation (5), obtains a system of equations. Solving this system, leads to the value of $\mho_i$. Substituting the values of $\mho_i$, $\aleph_i$ into equation (39), obtains the following data in table 2

2.2.3. Septic-spline. Based on the septic B–spline, the suggested solution of the ordinary differential form of the nonlinear KPP equation (5) is given as follow

$$\mathcal{H}(\eth) = \sum_{i=-1}^{n+1} \mho_i\, \aleph_i, \tag{42}$$

where $\mho_i$, $\aleph_i$ satisfies the next conditions

$$L\ \mathcal{H}(\eth) = \varnothing(\eth_i,\ \mathcal{H}(x_i))\,\text{where}\ (i = 0,\ 1,\ldots,n)$$

and

$$\aleph_i(\eth) = \frac{1}{h^5}\begin{cases} (\eth - \eth_{i-4})^7, & \eth \in [\eth_{i-4},\ \eth_{i-3}],\\ (\eth - \eth_{i-4})^7 - 8(\eth - \eth_{i-3})^7, & \eth \in [\eth_{i-3},\ \eth_{i-2}],\\ (\eth - \eth_{i-4})^7 - 8(\eth - \eth_{i-3})^7 + 28\eth(\eth - \eth_{i-2})^7, & \eth \in [\eth_{i-2},\ \eth_{i-1}],\\ (\eth - \eth_{i-4})^7 - 8(\eth - \eth_{i-3})^7 + 28(\eth - \eth_{i-2})^7 + 56(\eth - \eth_{i-1})^7, & \eth \in [\eth_{i-1},\ \eth_i],\\ (\eth_{i+4} - \eth)^7 - 8(\eth_{i+3} - \eth)^7 + 28(\eth_{i+2} - \eth)^7 + 56(\eth_{i+1} - \eth)^7, & \eth \in [\eth_i,\ \eth_{i+1}],\\ (\eth_{i+4} - \eth)^7 - 8(\eth_{i+3} - \eth)^7 + 28(\eth_{i+2} - \eth)^7, & \eth \in [\eth_{i+1},\ \eth_{i+2}],\\ (\eth_{i+4} - \eth)^7 - 8(\eth_{i+3} - \eth)^7, & \eth \in [\eth_{i+2},\ \eth_{i+3}],\\ (\eth_{i+4} - \eth)^7, & \eth \in [\eth_{i+3},\ \eth_{i+4}],\\ 0, & \text{otherwise}, \end{cases} \tag{43}$$

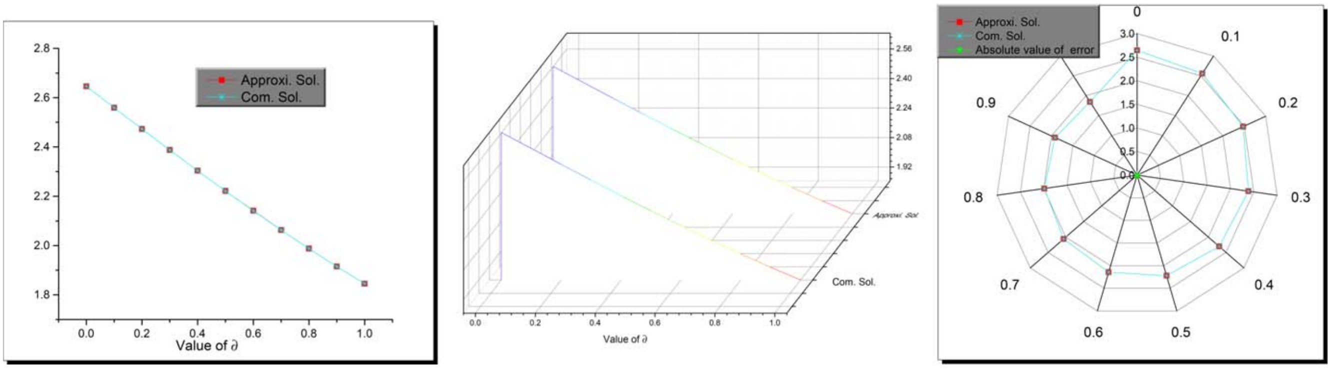

Figure 4. Exact, numerical, and absolute value of error by using cubic B–spline scheme on equation (5) according to the shown values in table 1.

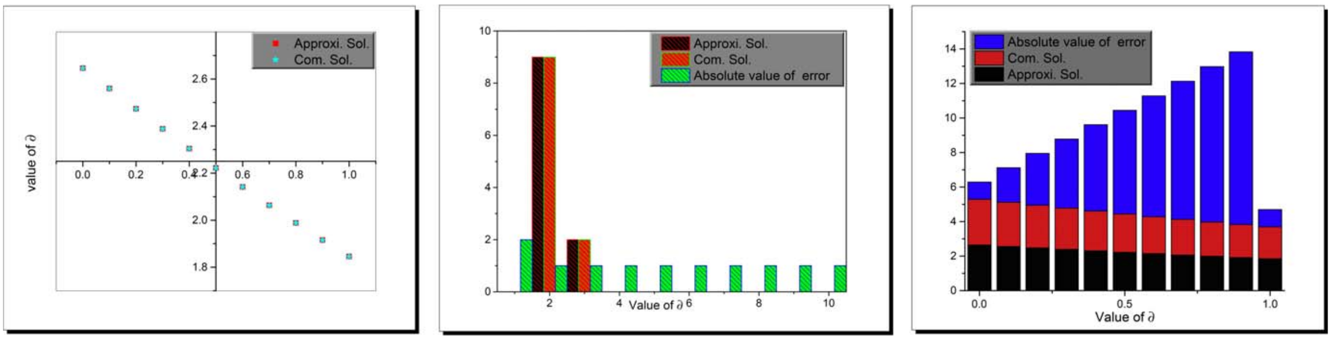

Figure 5. Exact, numerical, and absolute value of error by using quintic B–spline scheme on equation (5) according to the shown values in table 2.

where L is a linear operator, $i \in [-3, n+3]$. Thus, the approximate solution is given by

$$v_i(\eth) = \mho_{i-3} + 120\,\mho_{i-2} + 1191\,\mho_{i-1} + 2416\,\mho_i + 1191\,\mho_{i+1} + 120\,\mho_{i+2} + \mho_{i+3}. \tag{44}$$

Substituting equation (44) into equation (5), obtains a system of equations. Solving this system, gives the following data that are shown in the next table 3.

3. Result and discussion

This section is divided into two main parts. First, one is studying the computational solution and make a comparison between them and other obtained results in previous work. While the second part is making a comparison between the obtained numerical solutions in our paper to show and explain which one of them is more accurate than the other.

1. **Analytical solutions:**
   - **The obtained solutions**(8)–(35):

This method depends on auxiliary equation (7) that has a general solutions which is given by

$$\wp(\theta) = \frac{\log\left(\frac{\sqrt{4\alpha\sigma-\beta^2}\tan\left(\frac{1}{2}(c_1\log(K)\sqrt{4\alpha\sigma-\beta^2}+\theta\sqrt{4\alpha\sigma-\beta^2})\right)-\beta}{2\sigma}\right)}{\log(K)}, \tag{45}$$

where c_1 is arbitrary constant. Thus, all other solutions that are obtained and discussed in this paper are special forms of solutions which got by putting special conditions on equation (45). Moreover, we represent the equivalence between modified Khater method and some other recent methods.

- In [48] Hashemi *et al*, applied the simplest equation method to equation (5) and they obtained many kind of solutions. Comparison between our obtained solutions and that obtained in [48], yields equivalence between the following equation

$$u(x, t) = \frac{-1}{2\,\gamma}\left[\mu \;+\; \psi\,\tanh\left(\frac{\iota\,\sqrt{2\,\gamma}\,\psi\,x\,\alpha \;\mp\; 2\,\gamma\,\epsilon\,ln(\xi_0)\alpha - \mu\,\psi\,t^{\alpha}}{4\,\gamma\,\alpha}\right)\right] \tag{46}$$

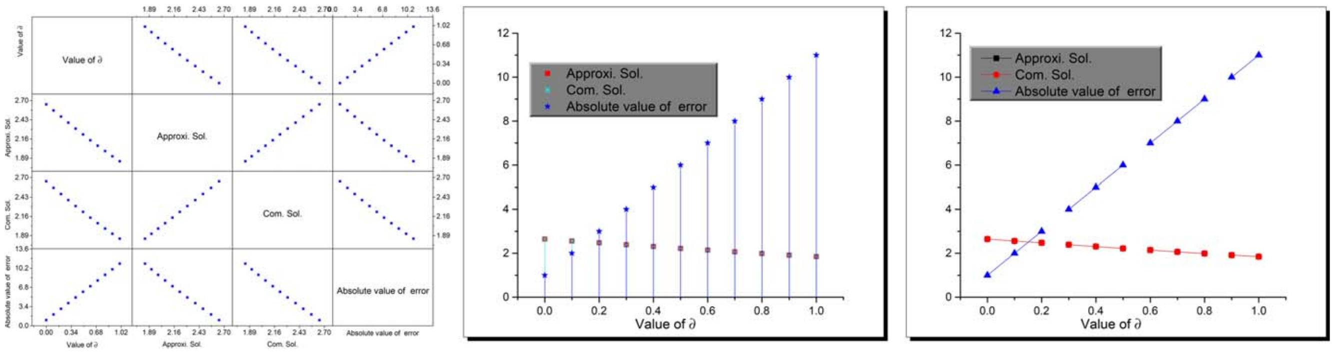

Figure 6. Exact, numerical, and absolute value of error by using septic B–spline scheme on equation (5) according to the shown values in table 3.

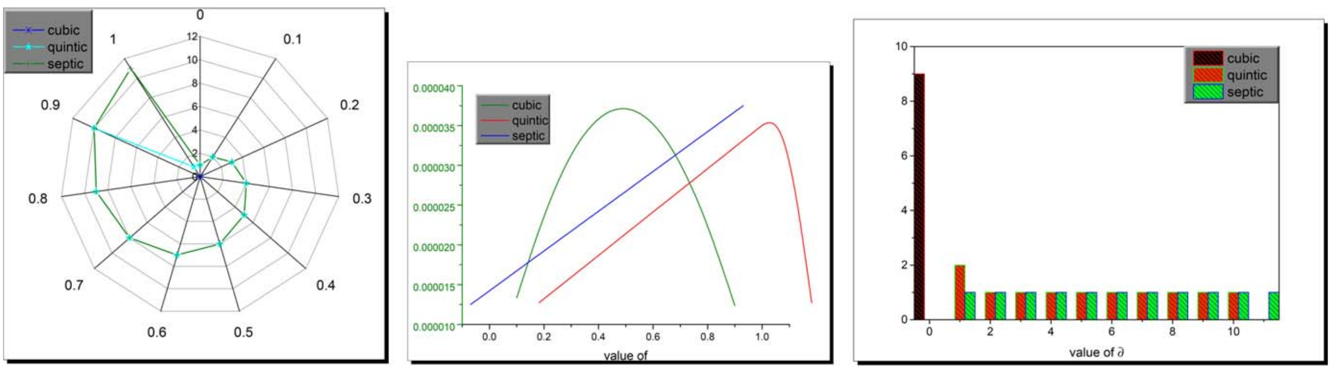

Figure 7. Comparison between the absolute value of error that obtained by each of B–spline schemes (cubic & quintic & septic).

and equation (14) when $\left[\frac{-\mu}{2\,\gamma}=\frac{\sqrt{\varrho(\rho-2\,\alpha\,k\,\sigma)}}{\varrho}, \frac{-\psi}{2\,\gamma}=\sqrt{\frac{2\,k\,\sigma^2}{-\varrho\,\alpha\,\sigma}}\,\alpha, \xi_0=0, \sqrt{\frac{\alpha\,\sigma}{2\,\gamma}}=\frac{\psi}{4\,\gamma\,k}, \frac{\mu\,\psi\,\vartheta}{4\,\gamma\,\alpha}=\sqrt{-\alpha\,\sigma\,k(2\,\rho-4\,\alpha\,k\,\sigma)}\right]$, whileall other obtained solutions in our paper are different from those obtained in [48]. That shows the superiority of our used computational method in this paper, which obtains many solutions and covers the solutions that can be obtained by other methods.

- The superiority of the modified Khater method is shown and discussed in detail in [36–42]. Now, we give the headline of this superiority as following:
  (a) **The $\left(\frac{G'}{G}\right)$- expansion method:**
  We can see that our new method is exactly same to the $\left(\frac{G'}{G}\right)$- expansion method when $\left[a^{f(\xi)}=\left(\frac{G'}{G}\right), \alpha=-\mu, \beta=-\lambda, \ \sigma=1\right]$. But, the $\left(\frac{G'}{G}\right)$-expansion method give only three kinds of solutions (hyperbolic, parabolic and rational) while our new method gives thirty different solutions.
  (b) **The $e^{(-i\,\phi\,\xi)}$- expansion method:**
  We can see how both equation are very closed however Khater method take the general form of exponential function of the $e^{(-i\,\phi\,\xi)}$- expansion method. But now, we will show how Khater method is more general than the $e^{(-i\,\phi\,\xi)}$- expansion method. Under specific condition which is $[a^{f(\xi)}=e^{(-i\,\phi(\xi))}, \alpha=-\mu, \sigma=-1, \beta=-\lambda]$ both method are similar but the $e^{(-i\,\phi\,\xi)}$- expansion method gives only five kind of solutions while our new method gives thirty different solutions.
  (c) **The extended tanh-function method:**
  Both methods are so closed to each other when $[a^{f(x)}=\phi, \alpha=b, \beta=0, \sigma=1]$. But the extended tanh-function method gives only three kind of solutions while our new method gives thirty different solutions.
  (d) **The Kudryashov and modified Kudryashov methods:**
  both methods are similar when $[a^{f(\xi)}=\phi, \alpha=0, \beta=-\sigma=-ln(a)]$. But the Kudryashov and modified Kudryashov methods give four solutions at most while our new method gives thirty different solutions and also Kudryashov method obtain only solitary wave solution and can not obtain elliptic solutions whilst Khater method obtain both of these solutions. That give another advantage of Khater's method about Kudryashov method.
  (e) **The improved $tan\left(\frac{\phi}{2}\right)$-expansion method:**
  We find both of methods are similar to each other when $\left[a^{f(\xi)}=tan\left(\frac{\phi}{2}\right), \alpha=(b+c),\right.$

Table 1. Exact, numerical, and absolute value of error by using cubic B–spline scheme on equation (5).

Value of ð	Approxi. Sol.	Com. Sol.	Absolute value of error
0	2.645 75	2.645 75	$4.440\,89 \times 10^{-16}$
0.1	2.559 23	2.559 22	1.33455×10^{-5}
0.2	2.473 15	2.473 12	$2.405\,74 \times 10^{-5}$
0.3	2.387 91	2.387 87	$3.180\,56 \times 10^{-5}$
0.4	2.303 92	2.303 89	$3.636\,91 \times 10^{-5}$
0.5	2.221 58	2.221 54	$3.765\,05 \times 10^{-5}$
0.6	2.141 22	2.141 18	0.000 035 681
0.7	2.063 16	2.063 13	$3.061\,66 \times 10^{-5}$
0.8	1.987 68	1.987 66	$2.272\,69 \times 10^{-5}$
0.9	1.915 01	1.915	$1.237\,59 \times 10^{-5}$
1	1.845 34	1.845 34	$2.220\,45 \times 10^{-16}$

Table 2. Exact, numerical, and absolute value of error by using quintic B–spline scheme on equation (5).

Value of ð	Approxi. Sol.	Com. Sol.	Absolute value of error
0	2.645 75	2.645 75	$4.440\,89 \times 10^{-16}$
0.1	2.559 22	2.559 22	$5.719\,78 \times 10^{-9}$
0.2	2.473 12	2.473 12	$1.288\,55 \times 10^{-8}$
0.3	2.387 87	2.387 87	$1.726\,11 \times 10^{-8}$
0.4	2.303 89	2.303 89	$1.982\,53 \times 10^{-8}$
0.5	2.221 54	2.221 54	$2.024\,79 \times 10^{-8}$
0.6	2.141 18	2.141 18	1.8733×10^{-8}
0.7	2.063 13	2.063 13	1.541×10^{-8}
0.8	1.987 66	1.987 66	$1.080\,92 \times 10^{-8}$
0.9	1.915	1.915	$4.418\,04 \times 10^{-9}$
1	1.845 34	1.845 34	$4.440\,89 \times 10^{-16}$

Table 3. Exact, numerical, and absolute value of error by using septic B–spline scheme on equation (5).

Value of ð	Approxi. Sol.	Com. Sol.	Absolute value of error
0	2.645 75	2.645 75	4.44089×10^{-16}
0.1	2.559 22	2.559 22	1.11866×10^{-11}
0.2	2.473 12	2.473 12	2.28364×10^{-11}
0.3	2.387 87	2.387 87	2.78315×10^{-11}
0.4	2.303 89	2.303 89	3.12879×10^{-11}
0.5	2.221 54	2.221 54	3.07567×10^{-11}
0.6	2.141 18	2.141 18	2.79203×10^{-11}
0.7	2.063 13	2.063 13	2.22311×10^{-11}
0.8	1.987 66	1.987 66	1.60598×10^{-11}
0.9	1.915	1.915	6.66134×10^{-12}
1	1.845 34	1.845 34	$2.220\,45 \times 10^{-16}$

$\beta = a$, $\sigma = (c - b)]$ but the improved $\tan\left(\frac{\phi}{2}\right)$-expansion method gives only Seventeen solutions while our new method gives thirty different solutions.

(f) **The novel $\left(\frac{G'}{G}\right)$- expansion method:**

The novel $\left(\frac{G'}{G}\right)$- expansion method is one of the methods which give many solutions like khater method but khater method more powerful, effective, felicitous and fabulous method than the novel $\left(\frac{G'}{G}\right)$- expansion method and both methods are similar to each other when $[a^{f(\xi)} = (d + \psi(\xi))$, $\alpha = \mu$, $\beta = \lambda$, $\sigma = (\nu - 1)]$.

(g) **The improved $\left(\frac{G'}{G}\right)$- expansion method:**

Both methods are similar to each other when

$$\left[a^{f(\xi)} = \frac{\left(\frac{G'}{G}\right)}{\delta + \left(\frac{G'}{G}\right)},\ \alpha = -\mu,\ \sigma = -1,\ \beta = 0\right].$$

But the improved $\left(\frac{G'}{G}\right)$- expansion method gives only three kind of solutions while our new method gives thirty different solutions.

- The relation between the used model and genetic models is given in detail in [19, 49]
- Representation of the obtained solutions We obtained some novel analytical solutions of the nonlinear FKPP equation and the physical meaning of them is given in the following form:

1. Equations (8), (9), (12), (13), (20), (22), (23), (26), (27) are trigonometric solutions.
2. Equations (10), (14), (24), (29), (30) are compacton hyperbolic solutions.
3. Equations (11), (15), (16), (16), (17), (25), (28) are Peakon hyperbolic solutions.
4. Equations (18), (19), (21), (31), (33), (34), (32) are rational solutions.

Now, we give the definition of some solutions for further information about them

(a) ***Compacton***:

In 1993, compacton waves were introduced by Rosenau and Hyman and were defined as solitons with finite wavelength or solitons free of exponential tails. Compacton wave is a solitary wave with the remarkable soliton property with compact spatial support where the nonlinear dispersion limits to a finite core and also, keeps its property after colliding with other compacton waves where it returns with the same coherent shape.

(b) ***Peakons***:

Peakon wave is a peaked solitary wave that retain their speed and shape after interacting in case of these solutions are smooth except for a peak at a corner of its crest. It has a finite jump in derivative of the solution $\mathcal{H}(x, t)$ where it is the point at which spatial derivative changes sign. In this text, you can see discontinuities in the x–derivative in peakons waves, and at the same time, there exist both one–sided derivatives differ only by a sign.

2. **Numerical solutions:**

The comparison between these types of solution depends on showing which one of the used schemes get the smallest value of the absolute value of error. To figure out these values, figure 7 shows the comparison between the obtained value of the absolute value of error in each used methods, explains the septic B–spline scheme is the more accurate than the other types of B–spline schemes for this model.

4. Conclusion

This article studied the computational and numerical solutions of the nonlinear FKPP equation by applying the modified Khater method, and B–spline schemes. The computational solutions are successfully obtained, and some of them are sketched to explain more physical properties of the model (figures 1–3). Moreover, the obtained computational solutions are used to find the approximate solutions by applying the B–spline schemes (cubic & quintic & septic). The comparison between the obtained distinct types of solutions is explained and investigated to show the absolute value of error between them and that is described in the shown (tables 1–3) and (figures 4–7). The performance of both computational and numerical schemes shows powerful, effective, and its ability for applying to many and various forms of nonlinear evolution equations.

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References

[1] Figueiredo I N, Leal C, Romanazzi G and Engquist B 2019 Biomathematical model for simulating abnormal orifice patterns in colonic crypts *Math. Biosci.* **315** 108221
[2] Hawkins-Daarud A, Johnston S K and Swanson K R 2019 Quantifying uncertainty and robustness in a biomathematical model-based patient-specific response metric for glioblastoma *JCO Clinical Cancer Informatics* **3** 1–8
[3] Dudhagara P R and Patel A 2018 Enhance the bacterial nitrate reductase production using mathematical and statistical model to formulate the affordable silver nanoparticle for the production of nanofinished fabric *Biomath Communications Supplement* **5** 1
[4] Leclerc Q J, Lindsay J A and Knight G M 2019 Mathematical modelling to study the horizontal transfer of antimicrobial resistance genes in bacteria: current state of the field and recommendations *Journal of the Royal Society Interface* **16** 20190260
[5] Romero J, Ibargüen E and Esteva L 2018 Modeling mathematically the influence of plasmids on the propagation of bacterial resistance to antibiotics *Biomath Communications Supplement* **5** 1
[6] Atangana A and Gómez-Aguilar J F 2018 Numerical approximation of Riemann-Liouville definition of fractional derivative: from riemann-liouville to atangana-baleanu *Numerical Methods for Partial Differential equations* **34** 1502–23
[7] Ezz-Eldien S S and El-Kalaawy A A 2018 Numerical simulation and convergence analysis of fractional optimization problems with right-sided Caputo fractional derivative *J. Comput. Nonlinear Dyn.* **13** 011010
[8] Martinez L, Rosales J J, Carreño C A and Lozano J M 2018 Electrical circuits described by fractional conformable derivative *Int. J. Circuit Theory Appl.* **46** 1091–100
[9] Morgado G, Nowakowski B and Lemarchand A 2019 Fisher-Kolmogorov-Petrovskii-Piskunov wave front as a sensor of perturbed diffusion in concentrated systems *Phys. Rev.* E **99** 022205
[10] Pikulin S V 2018 Traveling-wave solutions of the Kolmogorov-Petrovskii-Piskunov equation *Comput. Math. Math. Phys.* **58** 230–7
[11] Prakash A, Goyal M and Gupta S 2019 Fractional variational iteration method for solving time-fractional Newell-Whitehead-Segel equation *Nonlinear Engineering* **8** 164–71
[12] Valls C 2019 On the global dynamics of the Newell-Whitehead system *J. Nonlinear Math. Phys.* **26** 569–78
[13] Namjoo M and Zibaei S 2018 Numerical solutions of Fitzhugh-Nagumo equation by exact finite-difference and NSFD schemes *Comput. Appl. Math.* **37** 1395–411
[14] Schouten-Straatman W M and Hupkes H J 2019 Nonlinear stability of pulse solutions for the discrete FitzHugh-Nagumo equation with infinite-range interactions *Discrete & Continuous Dynamical Systems-A* **39** 5017–83
[15] Jiao Y, Wang T, Shi X and Liu W 2018 Mixed Jacobi-Fourier spectral method for Fisher equation *Mathematical Modelling and Analysis* **23** 240–61
[16] Khan A T and Naher H 2018 Solitons and periodic solutions of the fisher equation with nonlinear ordinary differential equation as auxiliary equation *American Journal of Applied Mathematics and Statistics* **6** 244–52
[17] Hasheim M S *et al* 2019 Symmetry properties and exact solutions of the time fractional Kolmogorov-Petrovskii-Piskunov equation *Revista Mexicana de Física* **65** 529–35
[18] Qin C-Y, Tian S-F, Wang X-B, Zou L and Zhang T-T 2018 Lie symmetry analysis, conservation laws and analytic solutions of the time fractional Kolmogorov-Petrovskii-Piskunov equation *Chin. J. Phys.* **56** 1734–42
[19] Veeresha P, Prakasha D G and D Baleanu 2019 An efficient numerical technique for the nonlinear fractional Kolmogorov-Petrovskii-Piskunov equation *Mathematics* **7** 265
[20] Zhou X, Shan W, Niu Z, Xiao P and Wang Y 2019 Lie symmetry analysis, explicit solutions and conservation laws for the time fractional Kolmogorov-Petrovskii-Piskunov equation *Waves Random Complex Medium* **00** 1–16
[21] Arioli G and Koch H 2019 Traveling wave solutions for the FPU chain: a constructive approach arXiv:1903.01299
[22] Ballo M T, Urman N, Lavy-Shahaf G, Grewal J, Bomzon Z and Toms S 2019 Correlation of tumor treating fields dosimetry to survival outcomes in newly diagnosed glioblastoma: a large-scale numerical simulation-based analysis of data from the phase 3 ef-14 randomized trial *International Journal of Radiation Oncology* Biology* Physics* **104** 5
[23] Cai P, Nie W, Chen D, Yang S and Liu Z 2019 Effect of air flowrate on pollutant dispersion pattern of coal dust particles at fully mechanized mining face based on numerical simulation *Fuel* **239** 623–35
[24] Elghobashi S 2019 Direct numerical simulation of turbulent flows laden with droplets or bubbles *Annual Review of Fluid Mechanics* **51** 217–44
[25] Gao F, Yang X-J and Ju Y 2019 Exact traveling-wave solutions for one-dimensional modified Korteweg-de Vries equation defined on cantor sets *Fractals* **27** 1940010
[26] Goodarzi H, Akbari O A, Sarafraz M M, Karchegani M M, Safaei M R and Shabani G A S 2019 Numerical simulation of natural convection heat transfer of nanofluid with Cu, MWCNT, and Al_2o_3 nanoparticles in a cavity with different

aspect ratios *Journal of Thermal Science and Engineering Applications* **11** 1061020

[27] Gui G, Liu Y and Luo T 2019 Model equations and traveling wave solutions for shallow-water waves with the Coriolis effect *Journal of Nonlinear Science* **29** 993–1039

[28] Han M, Zhang L, Wang Y and Khalique C M 2019 The effects of the singular lines on the traveling wave solutions of modified dispersive water wave equations *Nonlinear Anal. Real World Appl.* **47** 236–50

[29] Hayashi M 2019 Long-period limit of exact periodic traveling wave solutions for the derivative nonlinear Schrödinger equation *Annales de l'Institut Henri Poincaré C, Analyse non linéaire* (Amsterdam: Elsevier)

[30] Hsu C-H and Lin J-J 2019 Stability of traveling wave solutions for a spatially discrete SIS epidemic model *Zeitschrift für angewandte Mathematik und Physik* **70** 62

[31] Kumar S, Zhou Q and Liu W 2019 Invariant traveling wave solutions of parity-time-symmetric mixed linear-nonlinear optical lattices with three types of nonlinearity *Laser Phys.* **29** 045401

[32] Nasiri H, Jamalabadi M Y A, Sadeghi R, Safaei M R, Nguyen T K and Shadloo M S 2019 A smoothed particle hydrodynamics approach for numerical simulation of nano-fluid flows *J. Therm. Anal. Calorim.* **135** 1733–41

[33] Ninomiya H 2019 Entire solutions and traveling wave solutions of the Allen-Cahn-Nagumo equation *Discrete & Continuous Dynamical Systems-A* **39** 2001–19

[34] Rajabzadeh B, Najafabadi M H and Karimipour A 2019 Effects of new methods of porosity arrangement on forced convection in a variable BDPM using numerical simulation *Int. J. Therm. Sci.* **146** 106004

[35] Zhu W, Xia Y, Zhang B and Bai Y 2019 Exact traveling wave solutions and bifurcations of the time-fractional differential equations with applications *Int. J. Bifurcation Chaos* **29** 1950041

[36] Attia R A M, Lu D and Khater M M A 2019 Chaos and relativistic energy-momentum of the nonlinear time fractional Duffing equation *Mathematical and Computational Applications* **24** 10

[37] Khater M, Attia R and Lu D 2019 Modified auxiliary equation method versus three nonlinear fractional biological models in present explicit wave solutions *Mathematical and Computational Applications* **24** 1

[38] Khater M, Attia R A M and Lu D 2019 Explicit lump solitary wave of certain interesting (3+ 1)-dimensional waves in physics via some recent traveling wave methods *Entropy* **21** 397

[39] Khater M M A, Lu D and Attia R A M 2019 Dispersive long wave of nonlinear fractional Wu-Zhang system via a modified auxiliary equation method *AIP Adv.* **9** 025003

[40] Khater M M A, Lu D and Attia R A M 2019 Lump soliton wave solutions for the $(2+1)$-dimensional Konopelchenko-Dubrovsky equation and KdV equation *Mod. Phys. Lett.* B **33** 18

[41] Osman M S, Lu D and Khater M M A 2019 A study of optical wave propagation in the nonautonomous Schrödinger-Hirota equation with power-law nonlinearity *Results in Physics* **13** 102157

[42] Rezazadeh H, Korkmaz A, Khater M M A, Eslami M, Lu D and Attia R A M 2019 New exact traveling wave solutions of biological population model via the extended rational sinh-cosh method and the modified Khater method *Mod. Phys. Lett.* B **33** 28

[43] Edwards M C, Meyer R and Christensen N 2019 Bayesian nonparametric spectral density estimation using B-spline priors *Stat. Comput.* **29** 67–78

[44] Gavriil K, Schiftner A and Pottmann H 2019 Optimizing B-spline surfaces for developability and paneling architectural freeform surfaces *Comput.-Aided Des.* **111** 29–43

[45] Goss A, Schmidt M, Erdogan E, Görres B and Seitz F 2019 High-resolution vertical total electron content maps based on multi-scale B-spline representations *Ann. Geophys.* **37** 4

[46] Khater M M A, Attia R A M and Lu D 2019 Numerical solutions of nonlinear fractional Wu-Zhang system for water surface versus three approximate schemes *Journal of Ocean Engineering and Science* (https://doi.org/10.1016/j.joes.2019.03.002)

[47] Penner A 2019 Odf using a 5-point b-spline *Fitting Splines to a Parametric Function* (Berlin: Springer) pp 37–42

[48] Hasheim M S, Bayram M *et al* 2019 Symmetry properties and exact solutions of the time fractional Kolmogorov-Petrovskii-Piskunov equation *Revista Mexicana de Física* **65** 529–35

[49] Tikhomirov V M 1991 A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem *In Selected works of AN Kolmogorov* (Berlin: Springer) pp 242–70