© 2020 Chinese Physical Society and IOP Publishing Ltd Printed in China and the UK Communications in Theoretical Physics

Commun. Theor. Phys. 72 (2020) 055001 (12pp) https://doi.org/10.1088/1572-9494/ab7702

The basic principle of $m \times n$ resistor networks*

Zhi-Zhong Tan[1] **and Zhen Tan**[2]

[1] Department of Physics, Nantong University, Nantong, 226019, China
[2] School of Information Science and Technology, Nantong University, Nantong 226019, China

E-mail: tanz@ntu.edu.cn, tanzzh@163.com and zhzhtan@hotmail.com

Received 11 November 2019, revised 9 January 2020
Accepted for publication 13 January 2020
Published 2 April 2020

Abstract

The unified processing and research of multiple network models are implemented, and a new theoretical advance has been made, which sets up two new theorems on evaluating the exact electrical characteristics (potential and resistance) of the complex $m \times n$ resistor networks by the recursion-transform method with potential parameters, and applies to a variety of different types of lattice structure with arbitrary boundaries such as the nonregular $m \times n$ rectangular networks and the nonregular $m \times n$ cylindrical networks. Our research gives the analytical solutions of electrical characteristics of the complex networks (finite, semi-infinite and infinite), which has not been solved before. As applications of the theorems, a series of analytical solutions of potential and resistance of the complex resistor networks are discovered.

Keywords: complex network, RT-V method, electrical properties, boundary conditions, Laplace equation, mathematical physics

1. Introduction

Resistor network models are important in the field of physics and engineering since the issues of various disciplines can be studied by simulating resistor network, such as percolation and conduction [1], Nonlinear localized modes in two-dimensional electrical lattices [2], Electric circuit networks equivalent to chaotic quantum billiards [3], photonic crystal circuits [4], Manifesting the evolution of eigenstates from quantum billiards [5], topological properties of linear circuit lattices [6], three-dimensional printed meshes [7], topological insulator surface [8], the mean field theory [9, 10], lattice Green's functions [11–14], resistance distance [15], a recursion formula for resistance distances [16], and so on. In particular, two important equations of Poisson equation and Laplace equation [17, 18] can be simulated by resistor network model [19]. In addition, a real plane network of graphene exists in the real nature.

It is well known that calculating the equivalent resistance between two arbitrary lattice sites in a resistor network is always an important but difficult problem since it requires not only the circuit theory but also the innovative algebra. For example, when the boundary of resistor network is arbitrary, it is usually very difficult to obtain the exact potential and resistance of the complex networks with arbitrary boundaries. In fact, the boundary is like a wall or trap, which affects the solution of the problem. Therefore, the reality requires us to create new theories to accurately calculate the electrical characteristics (voltage and resistance) of the complex circuit network.

Let's review the research history of resistor networks. In 1845 Kirchhoff established the basic circuit theory (the node current law and the circuit voltage law). After 150 years, Cserti [20] calculated the two-point resistance of the infinite network by Green's function technique, which is mainly focused on infinite lattices, and some applications of Green's function technique were obtained in later literature [21, 22]. In 2004 Wu [23] formulated a different approach (call the Laplacian matrix method) and derived the explicit resistance in arbitrary finite and infinite lattices with normative boundary (such as free, periodic boundary etc) in terms of the eigenvalues and eigenvectors of the Laplacian matrix, which relies on two matrices along two vertical directions. Later, the Laplacian matrix analysis has also been applied to impedance networks [24], after some improvements, several new resistor network problems have been resolved [25–27]. However, the Laplacian approach cannot apply to the network with

* Project supported by a grant from the Natural Science Foundation of Jiangsu Province (No. BK20161278).

arbitrary boundary since it is impossible to give the explicit eigenvalues for the arbitrary matrix elements (associating arbitrary boundaries). But the boundary condition is important since it is real case occurring in real life.

In 2011 Tan pioneered a new technique for studying complex resistor networks [28], which now is called recursion-transform (RT) theory of resistor networks [19]. Tan's RT method depends on one matrix containing one directions, which is obviously different from the Laplacian method which depends on two matrices along two directions. With the development of the RT technique, many problems of non-regular network with zero resistor edges have been resolved [29–38]. In addition, the advantage of the RT method is that all resistance results are in a single summation differs from the Laplacian approach gave resistance results are in the form of a double summation. Recently, the RT method has been subdivided into two forms: one form is the matrix equation expressed by current parameters [31–37], which is simply called the RT-I method; another form is the matrix equation expressed by potential parameters [19, 38], which is simply called the RT-V method. Summarizing the previous applications of the RT (including RT-I and RT-V) method, it is not hard to see that the previous studies have not solved all the resistor networks, but only solved some personalized problems that depend on zero resistor boundary conditions, such as the globe network [29, 37] belongs to cylindrical network with two zero resistor boundaries, the cobweb network [19, 33] belongs to cylindrical network with one zero resistor boundary, the fan network [30, 38] belongs to nonregular rectangular network with one zero resistor boundary, and the hammock network [27, 36] belongs to nonregular rectangular network with two zero resistor boundaries. Obviously, how to study the complex network without zero resistor boundary by the RT method is a question.

This paper developed the RT theory to allow us to study arbitrary resistor networks without relying on zero resistor boundary, which can derive the electrical properties (potential and resistance) of the arbitrary $m \times n$ complex networks with complex boundaries. Here we build two new theorems lead to large problems to be resolved. Our study shows the universal RT method is very interesting and useful to solve the complex network. We focus on researching the electrical properties (potential and resistance) of figures 1 and 2 on two complex $m \times n$ resistor networks with two arbitrary boundaries by the advanced RT-V method, which have not been resolved before. It is worth emphasizing that the non-regular complex networks with two arbitrary boundaries are the multi-purpose network model because it can produce various geometrical structure as shown in figures 4 and 5. Thus a large number of problems of resistor networks will be resolved by this paper.

From the above analysis, professor Wu [23] was the first to give several accurate equivalent resistance formulas for the regular resistor networks by the Laplacian matrix method, for the sake of comparative study, here we introduce two main results of resistor networks from [23].

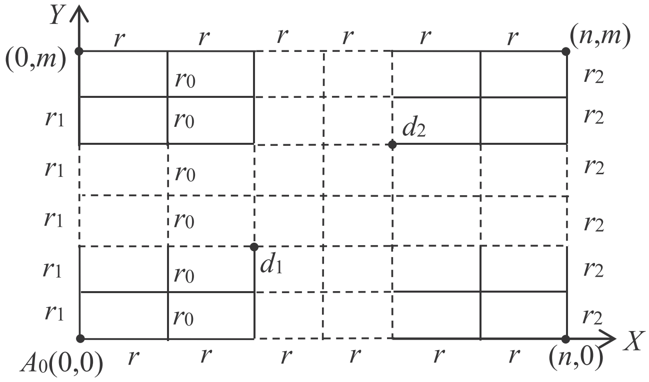

Figure 1. An arbitrary $m \times n$ resistor network with two arbitrary boundary resistors, where n and m are the maximum coordinate values of (n, m). Bonds in the horizontal and vertical directions are resistors r and r_0 except for two arbitrary boundary resistors of r_1 and r_2.

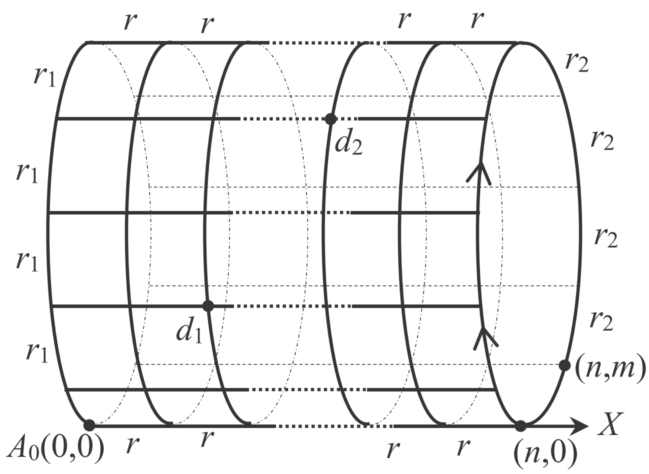

Figure 2. A nonregular cylindrical $m \times n$ resistor network, where n and m are the maximum coordinate value of (n, m), with the resistors r and r_0 in the respective horizontal and vertical (loop) directions except for two arbitrary boundary resistors of r_1 and r_2.

Case-1. Consider figure 1 with $r_1 = r_2 = r_0$ is a regular $m \times n$ rectangle network, where n and m are the maximum coordinate values of (n, m), resistors r and r_0 are bonded respectively in the horizontal and vertical directions. The resistance formula for figure 1 is

$$R_{m\times n}(d_1, d_2) = \frac{r}{m+1}|x_1 - x_2| + \frac{r_0}{n+1}|y_1 - y_2| + \frac{2}{(m+1)(n+1)} \times \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\left[C_{x_1,j}\cos\left(y_1+\frac{1}{2}\right)\theta_i - C_{x_2,j}\cos\left(y_2+\frac{1}{2}\right)\theta_i\right]^2}{r^{-1}(1-\cos\phi_j) + r_0^{-1}(1-\cos\theta_i)}, \tag{1}$$

where $C_{x_k,j} = \cos\left(x_k + \frac{1}{2}\right)\phi_j$, $\theta_i = i\pi/(m+1)$, $\phi_j = j\pi/(n+1)$ and $d_1(x_1, y_1)$ and $d_2(x_2, y_2)$ are arbitrary two nodes in the network.

Case-2. Consider figure 2 with $r_1 = r_2 = r_0$ is a cylindrical $m \times n$ resistor network, where n and m are the

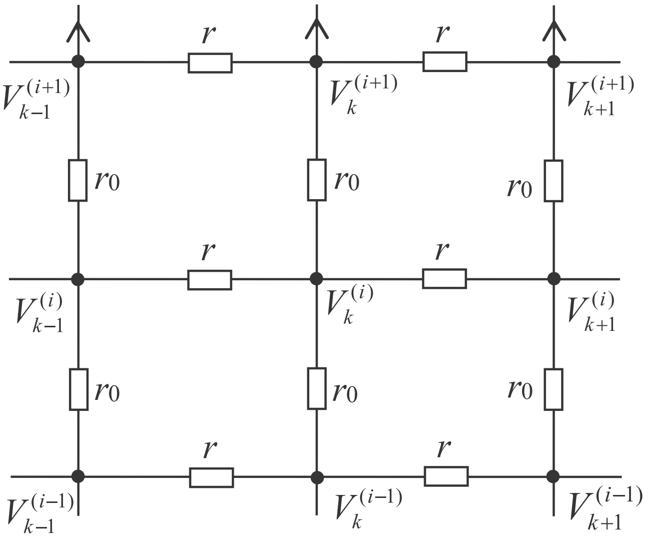

Figure 3. The resistor sub-network with the potential and resistor parameters.

maximum coordinate values of (n, m), resistors r and r_0 are bonded respectively in the horizontal and vertical directions. The resistance formula for figure 2 is

$$\begin{aligned} R_{m\times n}(d_1, d_2) &= \frac{r_0}{n+1}\left(|y_1 - y_2| - \frac{(y_1-y_2)^2}{m+1}\right) \\ &+ \frac{r}{m+1}|x_1 - x_2| + \frac{1}{(m+1)(n+1)} \\ &\times \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{C_{x_1,j}^2 + C_{x_2,j}^2 - 2C_{x_1,j}C_{x_2,j}\cos 2(y_2-y_1)\theta_i}{r_0^{-1}(1-\cos 2\theta_i) + r^{-1}(1-\cos\phi_j)}, \end{aligned} \tag{2}$$

where $C_{x_k,j} = \cos\left(x_k + \frac{1}{2}\right)\phi_j$, $\theta_i = i\pi/(m+1)$, $\phi_j = j\pi/(n+1)$.

The above results were found for the first time by Wu. Later [25–27] improved the Laplacian matrix method to make it applicable to regular cobweb and hammock networks. However, the improved Wu method still cannot resolve the resistor network with arbitrary boundary, such as the networks with two arbitrary boundaries of figures 1 and 2. In addition, the equivalent resistances in equations (1) and (2) are in the double summation but not in a single sum.

2. RT-V theory and Poisson equation

Consider two kinds of complex $m \times n$ resistor networks of figures 1 and 2, where n and m are the maximum coordinate values of (n, m). Assume A_0 (0, 0) is the origin of the rectangular coordinate system, and denoting nodes of the network by coordinate (x, y). Assume the electric current J goes from the input d_1 (x_1, y_1) to the output d_2 (x_2, y_2). Denote the nodal potential of the sub-network is shown in figure 3, and expressing the nodal potential at d (x, y) by $U_{m\times n}(x, y) = V_x^{(y)}$. We will study the complex resistor networks in four steps.

The first step, setting up discrete Poisson equation based on the sub-network of figure 3. By Kirchhoff law $(\sum r_i^{-1}V_k = 0)$ to set up the nodal potential equations along the vertical direction, we achieve a discrete static field equation (or call Poisson equation) for any network

$$(\Delta_x^2 + h\Delta_y^2)V_x^{(y)} = -rI_x^{(y)}\delta_{x,x_k}, \tag{3}$$

where $h = r/r_0$, and $I_x^{(y)} = J(\delta_{y,y_1} - \delta_{y,y_2})$ contains the input and output conditions of the current, $\Delta_x^2 V_x^{(y)} = V_{x+1}^{(y)} - 2V_x^{(y)} + V_{x-1}^{(y)}$ and $\Delta_y^2 V_x^{(y)} = V_x^{(y+1)} - 2V_x^{(y)} + V_x^{(y-1)}$ denote second order discrete equation, and when $x_k \neq x$, equation (3) reduces to the discrete Laplace equation $(\Delta_x^2 + h\Delta_y^2)V_x^{(y)} = 0$. For the arbitrary network together with the upper and lower boundary conditions, by equation (3) we are led to

$$\boldsymbol{V}_{k+1} = \boldsymbol{A}_{m+1}\boldsymbol{V}_k - \boldsymbol{V}_{k-1} - r\boldsymbol{I}_k\delta_{k,x}, \tag{4}$$

where $\boldsymbol{V}_k$ and $\boldsymbol{I}_x$ are respectively two column matrixes, and reads

$$\boldsymbol{V}_k = [V_k^{(0)}, V_k^{(1)}, V_k^{(2)}, \cdots, V_k^{(m)}]^{\mathrm{T}}, \tag{5}$$

$$\boldsymbol{I}_k = [I_k^{(0)}, I_k^{(1)}, I_k^{(2)}, \cdots, I_k^{(m)}]^{\mathrm{T}}, \tag{6}$$

and $\boldsymbol{A}_{m+1}$ is the matrix built along the vertical direction. For figures 1 and 2, the $\boldsymbol{A}_{m+1}$ is

$$\boldsymbol{A}_{m+1} = \begin{pmatrix} 2+h+bh & -h & 0 & \cdots & -bh \\ -h & 2+2h & -h & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -h & 2+2h & -h \\ -bh & \cdots & 0 & -h & 2+h+bh \end{pmatrix}, \tag{7}$$

where $b = r_0/r_3$, and r_3 is the resistor between $(x, 0)$ and (x, m) in figure 2, when $b = 0$ $(r_3 = \infty)$, the $\boldsymbol{A}_{m+1}$ belongs to figure 1; when $b = 1$ $(r_3 = r_0)$, the $\boldsymbol{A}_{m+1}$ belongs to figure 2. The purpose of introducing r_3 is to express two different resistor networks uniformly.

The second step, consider the boundary conditions of the left and right edges in the network of figures 1 and 2. Applying Kirchhoff's law $(\sum r_i^{-1}V_k = 0)$ to each of the left and right boundaries, we obtain two matrix equations of boundary conditions

$$h_1\boldsymbol{V}_1 = [\boldsymbol{A}_{m+1} - (2 - h_1)\boldsymbol{E}]\boldsymbol{V}_0, \tag{8}$$

$$h_2\boldsymbol{V}_{n-1} = [\boldsymbol{A}_{m+1} - (2 - h_2)\boldsymbol{E}]\boldsymbol{V}_n, \tag{9}$$

where $h_1 = r_1/r_0$, $h_2 = r_2/r_0$, $\boldsymbol{E}$ is the $(m+1)\times(m+1)$ identity matrix, matrix $\boldsymbol{A}_{m+1}$ is given by equation (7).

Equations (4)–(9) are all the equations we need to compute the node potential. However, it is impossible for us to get the solution of the above equations directly. Thanks to the RT theory of Tan that gave the matrix transform method [19, 31–33] and we create the new technique here. In the following we are going to give the transformation technology based on RT-V theory.

The third step, creating matrix transform. Firstly, we work out the eigenvalue t_i of matrix $\boldsymbol{A}_{m+1}$, which is given by solving determinantal equation of $\det|\boldsymbol{A}_{m+1} - t\boldsymbol{E}| = 0$ (just

$b=0$ and $b=1$), yields

$$t_i = 2(1+h) - 2h\cos\theta_i, \quad (i = 0, 1, \cdots, m), \tag{10}$$

where $\theta_i = (1+b)i\pi/(m+1)$, and $b=0$ for figure 1, $b=1$ for figure 2. Next to transform equations (4)–(9) by the following approaches

$$\boldsymbol{P}_{m+1}\boldsymbol{A}_{m+1} = \mathrm{diag}\{t_0, t_1, \cdots, t_m\}\boldsymbol{P}_{m+1}, \tag{11}$$

$$\boldsymbol{X}_k = \boldsymbol{P}_{m+1}\boldsymbol{V}_k \text{ or } \boldsymbol{V}_k = (\boldsymbol{P}_{m+1})^{-1}\boldsymbol{X}_k, \tag{12}$$

where $\boldsymbol{X}_k = [X_k^{(0)}, X_k^{(1)}, \cdots, X_k^{(m)}]^{\mathrm{T}}$. Assuming P_i is the row vectors of matrix $\boldsymbol{P}_{m+1}$, such as

$$P_i = [\zeta_{0,i}, \zeta_{1,i}, \zeta_{2,i}, \cdots, \zeta_{m,i}]. \tag{13}$$

Thus, we multiply equation (4) from the left-hand side by $\boldsymbol{P}_{m+1}$, we get

$$X_{k+1}^{(i)} = t_i X_k^{(i)} - X_{k-1}^{(i)} - rJ(\delta_{x_1,k}\zeta_{y_1,i} - \delta_{x_2,k}\zeta_{y_2,i}), \tag{14}$$

where equations (11) and (12) are used.

Similarly, applying $\boldsymbol{P}_{m+1}$ to equations (8) and (9), we are led to

$$h_1 X_1^{(i)} = (t_i + h_1 - 2)X_0^{(i)}, \tag{15}$$

$$h_2 X_{n-1}^{(i)} = (t_i + h_2 - 2)X_n^{(i)}. \tag{16}$$

The above equations (10)–(16) are all essential equations for evaluating the node potential.

The fourth step, solving the matrix equations (13)–(16) Selecting $\sum_{i=0}^{m} V_0^{(i)} = \frac{1}{2}(x_2 - x_1)rJ$ as the reference potential (notice: the potential is a relative reference value that you can artificially assume), by equations (14)–(16) we obtain after some algebra and reduction the solution

$$X_k^{(0)} = \frac{1}{\sqrt{2-b}}\left(\frac{x_1+x_2}{2} - x_\tau\right)rJ, \tag{17}$$

where $b=0$ for figure 1, $b=1$ for figure 2, and x_τ is defined in equation (28) below, and have

$$X_k^{(i)} = \frac{\beta_{k\vee x_1}^{(i)}\zeta_{y_1,i} - \beta_{k\vee x_2}^{(i)}\zeta_{y_2,i}}{(t_i - 2)G_n^{(i)}} rJ, \ (i \geqslant 1), \tag{18}$$

where $C_{k,i}$, $\beta_{k,s}^{(i)}$, $G_k^{(i)}$ are, respectively, defined in equations (19)–(25) below.

The RT-V theory. The above method of establishing recursive matrix equations with voltage parameters, implementing matrix transform and obtaining the solutions of matrix equations is called *RT-V* theory. The detailed content of the *RT-V* theory (recursion-transform theory with potential parameters) can be found by the above four steps in equation (3)–(18).

3. Two theorems of resistor networks

3.1. Several definitions

In order to facilitate and simplify the expression of the solutions of matrix equations, we define several variables of $C_{k,i}$ and λ_i, $\bar{\lambda}_i$ for later uses

$$C_{y_k,i} = \cos\left(y_k + \frac{1}{2}\right)\theta_i, \ C_{y_k-y} = \cos(y_k - y)\theta_i, \tag{19}$$

$$\begin{aligned}\lambda_i &= 1 + h - h\cos\theta_i + \sqrt{(1+h-h\cos\theta_i)^2 - 1}, \\ \bar{\lambda}_i &= 1 + h - h\cos\theta_i - \sqrt{(1+h-h\cos\theta_i)^2 - 1}.\end{aligned} \tag{20}$$

with

$$\theta_i = (1+b)i\pi/(m+1), \begin{cases} b=0 & \text{for figure 1,} \\ b=1 & \text{for figure 2} \end{cases}. \tag{21}$$

And define variables $F_k^{(i)}$, $\alpha_{s,x}^{(i)}$, $\beta_{k,s}^{(i)}$ and $G_k^{(i)}$ for later uses by

$$F_k^{(i)} = (\lambda_i^k - \bar{\lambda}_i^k)/(\lambda_i - \bar{\lambda}_i), \ \Delta F_k^{(i)} = F_{k+1}^{(i)} - F_k^{(i)}. \tag{22}$$

$$\alpha_{s,x}^{(i)} = \Delta F_x^{(i)} + (h_s - 1)\Delta F_{x-1}^{(i)}, \ h_s = r_s/r_0. \tag{23}$$

$$\beta_{x\vee x_s}^{(i)} = \begin{cases} \beta_{x,x_s}^{(i)} = \alpha_{1,x}^{(i)}\alpha_{2,n-x_s}^{(i)}, & \text{if } x \leqslant x_s \\ \beta_{x_s,x}^{(i)} = \alpha_{1,x_s}^{(i)}\alpha_{2,n-x}^{(i)}, & \text{if } x \geqslant x_s \end{cases}, \tag{24}$$

$$G_n^{(i)} = F_{n+1}^{(i)} + (h_1 + h_2 - 2)F_n^{(i)} + (h_2 - 1)(h_1 - 1)F_{n-1}^{(i)}. \tag{25}$$

The above definitions are applicable to the entire article. All of these definitions are meant to illustrate the following two fundamental theorems, and we always assume that the electric current J goes from the input $d_1(x_1, y_1)$ to the output $d_2(x_2, y_2)$ in our entire paper.

3.2. Two fundamental theorems

Theorem 1. *Consider the arbitrary $m \times n$ resistor networks of figures 1 and 2 whose maximum coordinate value is (n, m). Then the potential of node $d(x, y)$ in the $m \times n$ resistor network can be written as*

$$V_{m\times n}(x, y) = \frac{1}{(\zeta_{k,i}, \bar{\zeta}_{k,i})}\sum_{i=0}^{m} X_x^{(i)}\bar{\zeta}_{y,i}, \tag{26}$$

where $(\zeta_{k,i}, \bar{\zeta}_{k,i}) = \sum_{k=0}^{m}\zeta_{k,i}\bar{\zeta}_{k,i}$, and $\zeta_{y,i}$ is defined in equation (13), $\bar{\zeta}_{y,i}$ is the conjugate complex of $\zeta_{y,i}$, and $X_k^{(i)}$ is the solution of the matrix equation (14) together with the boundary condition equations. Formula (26) is a general formula which is suitable for any resistor network model.

In particular, when selecting $\sum_{i=0}^{m} V_0^{(i)} = \frac{1}{2}(x_2 - x_1)rJ$ as the reference potential, the potential of node $d(x, y)$ in the $m \times n$ resistor networks can be written as

$$V_{m\times n}(x, y) = \frac{\bar{x} - x_\tau}{m+1}rJ + \frac{2-b}{m+1}\sum_{i=1}^{m} X_x^{(i)}\bar{\zeta}_{y,i}, \tag{27}$$

where $\bar{x} = (x_1 + x_2)/2$, $\bar{\zeta}_{y,i}$ is the conjugate complex of $\zeta_{y,i}$ (there be $\bar{\zeta}_{y,i} = \zeta_{y,i}$ if $\zeta_{y,i}$ is just a real number), and x_τ is a piecewise function

$$\begin{aligned} x_\tau = \{x_1, 0 \leqslant x \leqslant x_1\} \cup \{x, x_1 \leqslant x \leqslant x_2\} \\ \cup \{x_2, x_2 \leqslant x \leqslant n\} \end{aligned} \tag{28}$$

and $X_k^{(i)}$ is given by (18) which is the solution of equations (13)–(16).

Theorem 2. *Consider the arbitrary $m \times n$ resistor networks of figures 1 and 2 whose maximum coordinate value is (n, m). Then the resistance between any two nodes $d_1(x_1, y_1)$ and $d_2(x_2, y_2)$ in the network is given by*

$$R_{m\times n}(d_1, d_2) = \frac{1}{(\zeta_{y,i}, \bar{\zeta}_{y,i})}\sum_{i=0}^{m}\frac{X_{x_1}^{(i)}\bar{\zeta}_{y_1,i} - X_{x_2}^{(i)}\bar{\zeta}_{y_2,i}}{J}, \tag{29}$$

where $X_k^{(i)}$ is the solution of the matrix equation (14) together with the boundary condition equations, Formula (29) is a general formula which is suitable for any resistor network model.

In particular, for the networks of figures 1 and 2, the resistance between two nodes $d_1(x_1, y_1)$ and d_2 (x_2, y_2) can be written as

$$R_{m\times n}(d_1, d_2) = \frac{|x_1 - x_2|}{m+1}r + \frac{2-b}{m+1}\sum_{i=1}^{m}\frac{X_{x_1}^{(i)}\bar{\zeta}_{y_1,i} - X_{x_2}^{(i)}\bar{\zeta}_{y_2,i}}{J}, \tag{30}$$

where $b = 0$ is the case of figure 1, and $b = 1$ is the case of figure 2, and $X_k^{(i)}$ is given by (18) which is the solution of equations (13)–(16).

The above two new theorems contain a wide variety of geometric structure of the network model, which can produce many new results of potential and resistance, we are going to prove the correctness of two theorems.

3.3. Proof of theorems

Consider the $m \times n$ resistor network with two arbitrary boundaries shown in figures 1 and 2, in the introduction, we have built the key equations (4)–(9) by the RT-V theory, and converted the equations to equations (14)–(16) and derived equations (17) and (18). Now we will work out the exact eigenvalues of matrix $\boldsymbol{A}_{m+1}$ in equation (7). Equation (10) can be derived by solving equation $\det|\boldsymbol{A}_{m+1} - t\boldsymbol{E}| = 0$, and then we need to consider two cases below.

One is for figure 1, substituting equation (10) into (11) with $b = 0$ in $\boldsymbol{A}_{m+1}$, we get the eigenvectors

$$\boldsymbol{P}_{m+1} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & \cdots & 1/\sqrt{2} \\ \cos(v_0\theta_1) & \cos(v_1\theta_1) & \cdots & \cos(v_m\theta_1) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(v_0\theta_m) & \cos(v_1\theta_m) & \cdots & \cos(v_m\theta_m) \end{pmatrix}, \tag{31}$$

where $v_k = k + 1/2$, and $\theta_i = i\pi/(m+1)$. By careful calculation, the inverse matrix can be easily obtained

$$\boldsymbol{P}_{m+1}^{-1} = \frac{2}{m+1}[\boldsymbol{P}_{m+1}]^{\mathrm{T}}, \tag{32}$$

where $[\ ]^{\mathrm{T}}$ denote matrix transpose.

Thus, the term $\zeta_{k,i}$ appearing in equation (13) can be specifically rewritten as

$$\bar{\zeta}_{y,0} = \zeta_{y,0} = 1/\sqrt{2},\ (0 \leqslant y \leqslant m), \tag{33}$$

$$\bar{\zeta}_{y,i} = \zeta_{y,i} = \cos\left(y + \frac{1}{2}\right)\theta_i,\ (i \geqslant 1). \tag{34}$$

Another is for figure 2, substituting equation (10) into (11) with $b = 1$ in $\boldsymbol{A}_{m+1}$, the eigenvector is obtained after some algebra and derivation

$$\boldsymbol{P}_{m+1} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \exp(\mathrm{i}\theta_1) & \exp(\mathrm{i}2\theta_1) & \cdots & \exp(\mathrm{i}m\theta_1) \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & \exp(\mathrm{i}\theta_m) & \exp(\mathrm{i}2\theta_m) & \cdots & \exp(\mathrm{i}m\theta_m) \end{pmatrix}, \tag{35}$$

where $\mathrm{i}^2 = -1$, $\theta_i = 2i\pi/(m+1)$ $(i = 0, 1, 2,\cdots,m)$. According to strict calculations, the inverse matrix reads

$$\boldsymbol{P}_{m+1}^{-1} = \frac{1}{m+1}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \exp(-\mathrm{i}\theta_1) & \cdots & \exp(-\mathrm{i}\theta_m) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \exp(-\mathrm{i}m\theta_1) & \cdots & \exp(-\mathrm{i}m\theta_m) \end{pmatrix}. \tag{36}$$

Thus, the term $\zeta_{k,i}$ appearing in equation (13) can be specifically rewritten as

$$\zeta_{y,0} = \bar{\zeta}_{y,0} = 1,\ (0 \leqslant y \leqslant m), \tag{37}$$

$$\zeta_{y,i} = \exp(\mathrm{i}y\theta_i),\ \bar{\zeta}_{y,i} = \exp(-\mathrm{i}y\theta_i). \tag{38}$$

We find that equations (32) and (36) can be rewritten as a unified form below

$$\boldsymbol{P}_{m+1}^{-1} = \frac{1}{(\zeta_{k,i}, \bar{\zeta}_{k,i})}\begin{pmatrix} \bar{\zeta}_{0,0} & \bar{\zeta}_{0,1} & \cdots & \bar{\zeta}_{0,m} \\ \bar{\zeta}_{1,0} & \bar{\zeta}_{1,1} & \cdots & \bar{\zeta}_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\zeta}_{m,0} & \bar{\zeta}_{m,1} & \cdots & \bar{\zeta}_{m,m} \end{pmatrix}, \tag{39}$$

where $(\zeta_{k,i}, \bar{\zeta}_{k,i}) = \sum_{k=0}^{m}\zeta_{k,i}\bar{\zeta}_{k,i}$, and $\bar{\zeta}_{y,i}$ is the conjugate complex of $\zeta_{y,i}$. Equation (39) is an important innovation which is the key to our unified study of resistor networks.

Using equation (12), we have $\boldsymbol{V}_k = (\boldsymbol{P}_{m+1})^{-1}\boldsymbol{X}_k$, expanding this matrix equation, then we get

$$V_k^{(y)} = \frac{1}{(\zeta_{k,i}, \bar{\zeta}_{k,i})}\left(X_k^{(0)}\bar{\zeta}_{y,0} + \sum_{i=1}^{m}X_k^{(i)}\bar{\zeta}_{y,i}\right), \tag{40}$$

Equation (40) agrees with the formula (26) that we need to verify.

Further, we get $(\zeta_{k,i}, \bar{\zeta}_{k,i}) = (m+1)/(2-b)$ by comparing equation (39) with equations (32) and (36). And when selecting $\sum_{i=0}^{m}V_0^{(i)} = \frac{1}{2}(x_2 - x_1)rJ$, we have equation (17). Substituting equations (17), (33) and (37) into equation (40), then equation (27) can be verified immediately.

Next, we verify equations (29) and (30) by Ohm's Law, we have

$$R_{m\times n}(d_1, d_2) = \frac{1}{J}[V(x_1, y_1) - V(x_2, y_2)]. \tag{41}$$

Substituting equation (26) with $x = \{x_1, x_2\}$ and $y = \{y_1, y_2\}$ into equation (41), we therefore obtain equation (29).

Substituting equation (27) with $x = \{x_1, x_2\}$ and $y = \{y_1, y_2\}$ into equation (41), we immediately obtain equation (30). Thus, two theorems are verified.

In subsequent sections we consider applications of theorems to arbitrary lattices. In all applications, we stipulate all parameters in equations (18)–(39) apply to all resistor networks, and denote the resistors along the two principal directions by r and r_0 except for resistors on the left-right boundaries, and the input and output nodes of current are respectively at $d_1(x_1, y_1)$ and $d_2(x_2, y_2)$.

4. Electrical properties of complex rectangular network

4.1. Nodal potential of complex rectangular network

Consider the non-regular $m \times n$ resistor network shown in figure 1, where the maximum coordinate is (n, m), selecting $\sum_{i=0}^{m} V_0^{(i)} = \frac{1}{2}(x_2 - x_1)rJ$ as the reference potential, the potential of any node $d(x, y)$ in the finite and semi-infinite networks can be written as

$$\frac{U_{m\times n}(x, y)}{J} = \frac{\bar{x} - x_\tau}{m+1}r + \frac{r_0}{m+1} \times \sum_{i=1}^{m} \frac{\beta_{x_1\vee x}^{(i)} C_{y_1,i} - \beta_{x_2\vee x}^{(i)} C_{y_2,i}}{(1-\cos\theta_i)G_n^{(i)}} C_{y,i}, \tag{42}$$

$$\frac{U_{m\times\infty}(x, y)}{J} = \frac{\bar{x} - x_\tau}{m+1}r + \frac{r}{m+1} \times \sum_{i=1}^{m} \frac{\bar{\lambda}_i^{|x_1-x|} C_{y_1,i} - \bar{\lambda}_i^{|x_2-x|} C_{y_2,i}}{\sqrt{(1+h-h\cos\theta_i)^2 - 1}} C_{y,i}, \tag{43}$$

where $\theta_i = i\pi/(m+1)$, and $C_{k,i}$, $\beta_{k,s}^{(i)}$, $G_k^{(i)}$ are, respectively, defined in equations (19)–(25). For equation (43), there be $n \to \infty$, $x_1, x_2 \to \infty$ with finite $x_k - x$.

In particular, when $x_2 = x_1$ (means the input and output nodes of currents are at the same vertical axis), formulae (42) and (43) reduce to

$$\frac{U_{m\times n}(x, y)}{J} = \frac{r_0}{m+1}\sum_{i=1}^{m} \frac{(C_{y_1,i} - C_{y_2,i})C_{y,i}}{(1-\cos\theta_i)G_n^{(i)}} \beta_{x_1\vee x}^{(i)}, \tag{44}$$

$$\frac{U_{m\times\infty}(x, y)}{J} = \frac{r}{m+1}\sum_{i=1}^{m} \frac{\bar{\lambda}_i^{|x_1-x|}(C_{y_1,i} - C_{y_2,i})C_{y,i}}{\sqrt{(1+h-h\cos\theta_i)^2 - 1}}. \tag{45}$$

Proof of equation (42). For figure 1, substituting equation (34) with $y_k = \{y_1, y_2\}$ into (18), we achieve

$$X_k^{(i)} = \frac{\beta_{k\vee x_1}^{(i)} C_{y_1,i} - \beta_{k\vee x_2}^{(i)} C_{y_2,i}}{(t_i - 2)G_n^{(i)}} rJ, \ (1 \leqslant k \leqslant n). \tag{46}$$

Substituting equation (46) and (34) into (27) with $b = 0$, we therefore achieve equation (42).

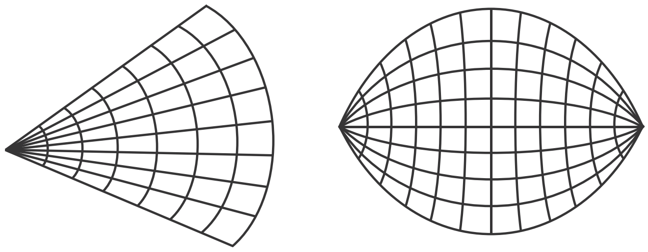

Figure 4. Two resistor network models. (a) is a Fan network with an arbitrary boundary resistor r_2; (b) is an arbitrary hammock network.

For proving equation (43), when $n \to \infty$, $x_1, x_2 \to \infty$ with finite $x_k - x$, it can be got a limit by using equations (20)–(25)

$$\lim_{\substack{n\to\infty \\ x\to\infty}} \frac{\beta_{x_k\vee x}^{(i)}}{G_n^{(i)}} = (t_i - 2)\frac{\bar{\lambda}_k^{|x_k - x|}}{\lambda_i - \bar{\lambda}_i}. \tag{47}$$

So, substituting equation (47) into (42) with $n \to \infty$, we therefore verified equation (43).

Formula (42) is a meaningful result because the network of figure 1 is very complex and has not been resolved before, which contains a lot of different network models since the different boundary resistors can produce different geometric structures. Here several special applications of formula (42) are given below.

Application 1. When $r_1 = r_2 = r_0$, figure 1 degrades into a regular $m \times n$ rectangular network, the potential of a node $d(x, y)$ in the network is

$$\frac{U_{(x,y)}}{J} = \frac{\bar{x} - x_\tau}{m+1}r + \frac{r_0}{m+1}\sum_{i=1}^{m} \frac{\beta_{x\vee x_1}^{(i)} C_{y_1,i} - \beta_{x\vee x_2}^{(i)} C_{y_2,i}}{(1-\cos\theta_i)F_{n+1}^{(i)}} C_{y,i}, \tag{48}$$

where $\beta_{x,x_s}^{(i)}$ reduces to $\beta_{x,x_s}^{(i)} = \Delta F_x^{(i)} \Delta F_{n-x_s}^{(i)}$.

In particular, when $x_2 = x_1$, potential formula (48) reduces further to

$$\frac{U_{m\times n}(x, y)}{J} = \frac{r_0}{m+1}\sum_{i=1}^{m} \frac{(C_{y_1,i} - C_{y_2,i})C_{y,i}}{(1-\cos\theta_i)F_{n+1}^{(i)}} \beta_{x_1\vee x}^{(i)}. \tag{49}$$

Application 2. When $h_1 = 0$ ($r_1 = 0$), figure 1 degrades into a Fan network as shown in figure 4(a), where r and r_0 are the respective resistors along longitude (radius) and latitude (arc) directions, and the resistor element on the outer arc is r_2 (an arbitrary boundary resistor). The potential of a node $d(x, y)$ in the $m \times n$ Fan network can be written as

$$\frac{U(x, y)}{J} = \frac{\bar{x} - x_\tau}{m+1}r + \frac{2r}{m+1}\sum_{i=1}^{m} \frac{\beta_{x\vee x_1}^{(i)} C_{y_1,i} - \beta_{x\vee x_2}^{(i)} C_{y_2,i}}{\Delta F_n^{(i)} + (h_2 - 1)\Delta F_{n-1}^{(i)}} C_{y,i}, \tag{50}$$

where we redefine $\beta^{(i)}_{x\vee x_s}=F^{(i)}_x\alpha^{(i)}_{2,n-x_s}$(if $x\leqslant x_s$) and $\beta^{(i)}_{x\vee x_s}=F^{(i)}_{x_s}\alpha^{(i)}_{2,n-x}$(if $x\geqslant x_s$).

Please note that a non-regular Fan network (the outer arc resistor r_2 is arbitrary) is a scientific conundrum, which has not been solved before. Reference [19] has researched just the regular Fan network (the outer arc resistor is $r_2=r_0$), but our formula (50) with $r_2=r_0$ is different from the result in [19] because two results depends on the different matrices along the orthogonal direction.

Application 3. When $r_1=r_2=0$, figure 1 degrades into a hammock network as shown in figure 4(b), the potential of a node $d(x, y)$ in the $m\times n$ hammock network can be written as

$$\frac{U(x,y)}{J}=\frac{\bar{x}-x_\tau}{m+1}r+\frac{2r}{m+1}\sum_{i=1}^{m}\frac{\beta^{(i)}_{x\vee x_1}C_{y_1,i}-\beta^{(i)}_{x\vee x_2}C_{y_2,i}}{F^{(i)}_n}C_{y,i},\tag{51}$$

where we redefine $\beta^{(i)}_{x\vee x_s}=F^{(i)}_xF^{(i)}_{n-x_s}$(if $x\leqslant x_s$) and $\beta^{(i)}_{x\vee x_s}=F^{(i)}_{x_s}F^{(i)}_{n-x}$(if $x\geqslant x_s$).

In particular, when $d_1(0, y_1)$ and $d_2(n, y_2)$ are respectively at the left and right poles, the potential of equation (51) reduces to

$$\frac{U(x,y)}{J}=\frac{n-2x}{2(m+1)}r.\tag{52}$$

Please note that the hammock network has been solved by [36], but our formula (51) is different from the result in [36] because two results depends on the different matrices along the orthogonal direction.

Application 4. Assume figure 1 is a semi-infinite $\infty\times n$ network, and $m\to\infty$ but n, x and y are finite. Consider $d_1(0, y_1)$ is on the left edge, and $d_2(n, y_2)$ is on the right edge, when $r_1=r_2=r_0$, the potential of a node $d(x, y)$ in the semi-infinite $\infty\times n$ rectangular network is

$$\frac{U_{\infty\times n}(x,y)}{J}=\frac{r_0}{\pi}\int_0^{\pi}\frac{(\Delta F_{n-x}C_{y_1}-\Delta F_xC_{y_2})C_y}{(1-\cos\theta)F_{n+1}}\mathrm{d}\theta,\tag{53}$$

where $C_{y_k}=\cos\left(y_k+\frac{1}{2}\right)\theta$, $F_k=(\lambda^k-\bar{\lambda}^k)/(\lambda-\bar{\lambda})$ with $\lambda=1+h-h\cos\theta+\sqrt{(1+h-h\cos\theta)^2-1}$. Please note that these definitions apply to all such issues as appear below.

4.2. Resistance of complex rectangular network

Consider an $m\times n$ rectangular network with two arbitrary boundaries shown in figure 1, where the maximum coordinate is (n, m). Defining $\beta^{(i)}_{k,s}=\beta^{(i)}_{x_k,x_s}$, the resistance between two nodes $d_1(x_1, y_1)$ and $d_2(x_2, y_2)$ in the finite and semi-infinite networks are respectively

$$R_{m\times n}(d_1,d_2)=\frac{|x_2-x_1|}{m+1}r+\frac{r_0}{m+1}\sum_{i=1}^{m}\frac{\beta^{(i)}_{1,1}C^2_{y_1,i}-2\beta^{(i)}_{1,2}C_{y_1,i}C_{y_2,i}+\beta^{(i)}_{2,2}C^2_{y_2,i}}{(1-\cos\theta_i)G^{(i)}_n},\tag{54}$$

$$R_{m\times\infty}(d_1,d_2)=\frac{|x_2-x_1|}{m+1}r+\frac{r}{m+1}\sum_{i=1}^{m}\frac{C^2_{y_1,i}+C^2_{y_2,i}-2\bar{\lambda}_i^{|x_2-x_1|}C_{y_1,i}C_{y_2,i}}{\sqrt{(1+h-h\cos\theta_i)^2-1}},\tag{55}$$

where $\theta_i=i\pi/(m+1)$, and $C_{k,i}$, $G^{(i)}_k$ are, respectively, defined in equations (19)–(25) For equation (55), there be $n\to\infty$, $x_1, x_2\to\infty$ with finite x_1-x_2. Equation (55) can be derived by taking the limit $n\to\infty$ in equation (54).

Proof of equation (54). For figure 1, substituting equation (42) with $k=x_1$, x_2 into (41), then equation (54) is verified.

Equation (54) is an exact expression which still contains a variety of resistance results with all kinds of boundary conditions because the left and right boundaries are the arbitrary resistors. For clearly understanding formula (54), we set h_1 and h_2, m or n as special values, and give several special cases to understand its application and meaning.

Case 1. When $h_1=1$, the network of figure 1 degrades into a rectangular $m\times n$ network with an arbitrary right boundary, then formula (54) reduces to

$$R_{m\times n}(d_1,d_2)=\frac{|x_2-x_1|}{m+1}r+\frac{r_0}{m+1}\sum_{i=1}^{m}\frac{\beta^{(i)}_{1,1}C^2_{y_1,i}-2\beta^{(i)}_{1,2}C_{y_1,i}C_{y_2,i}+\beta^{(i)}_{2,2}C^2_{y_2,i}}{(1-\cos\theta_i)[F^{(i)}_{n+1}+(h_2-1)F^{(i)}_n]},\tag{56}$$

where $\beta^{(i)}_{k,s}$ reduces to $\beta^{(i)}_{k,s}=\Delta F^{(i)}_{x_k}[\Delta F^{(i)}_{n-x_s}+(h_2-1)\Delta F^{(i)}_{n-x_s-1}]$.

Case 2. When $h_2=h_1=1$, the network of figure 1 degrades into a normal $m\times n$ rectangular network, then formula (54) reduces to

$$R_{m\times n}(d_1,d_2)=\frac{|x_2-x_1|}{m+1}r+\frac{r_0}{m+1}\sum_{i=1}^{m}\frac{\beta^{(i)}_{1,1}C^2_{y_1,i}-2\beta^{(i)}_{1,2}C_{y_1,i}C_{y_2,i}+\beta^{(i)}_{2,2}C^2_{y_2,i}}{(1-\cos\theta_i)F^{(i)}_{n+1}},\tag{57}$$

where $\beta^{(i)}_{k,s}$ reduces to $\beta^{(i)}_{k,s}=\Delta F^{(i)}_{x_k}\Delta F^{(i)}_{n-x_s}$. This problem has been researched in [23], and gave equation (1) with a double sums. Clearly, our result (57) is different from equation (1). This also shows that the equivalent resistance can be expressed in different forms.

Case 3. When $h_1=0$, the network of figure 1 degrades into a non-regular $m\times n$ Fan network with an arbitrary boundary as shown in figure 4(a), by equation (54), we obtain the resistance of a Fan network

$$R_{m\times n}(d_1, d_2) = \frac{|x_2 - x_1|}{m+1}r + \frac{2r}{m+1}\sum_{i=1}^{m}\frac{\beta_{1,1}^{(i)}C_{y_1,i}^2 - 2\beta_{1,2}^{(i)}C_{y_1,i}C_{y_2,i} + \beta_{2,2}^{(i)}C_{y_2,i}^2}{\Delta F_n^{(i)} + (h_2 - 1)\Delta F_{n-1}^{(i)}}, \tag{58}$$

where $\beta_{k,s}^{(i)}$ is re-defined as $\beta_{k,s}^{(i)} = F_{x_k}^{(i)}[\Delta F_{n-x_s}^{(i)} + (h_2 - 1)\Delta F_{n-x_s-1}^{(i)}]$.

In particular, when $h_1 = 0$, $h_2 = 1$, the network of figure 1 degrades into a normal $m\times n$ Fan network, then formula (58) reduces to

$$R_{m\times n}(d_1, d_2) = \frac{|x_2 - x_1|}{m+1}r + \frac{2r}{m+1}\sum_{i=1}^{m}\frac{\beta_{1,1}^{(i)}C_{y_1,i}^2 - 2\beta_{1,2}^{(i)}C_{y_1,i}C_{y_2,i} + \beta_{2,2}^{(i)}C_{y_2,i}^2}{\Delta F_n^{(i)}}, \tag{59}$$

where $\beta_{k,s}^{(i)}$ is re-defined as $\beta_{k,s}^{(i)} = F_{x_k}^{(i)}\Delta F_{n-x_s}^{(i)}$. This case has been researched in [30], but the result is different from equation (59), however they are equivalent to each other. The reason is that they choice the different matrix along different direction. This also shows that the equivalent resistance can be expressed in different forms.

Case 4. When $d_1(0, y_1)$ is on the left edge and $d_2(n, y_2)$ is on the right edge, formula (54) reduces to

$$R_{m\times n}(\{0, y_1\}, \{n, y_2\}) = \frac{nr}{m+1} + \frac{r_0}{m+1} \times \sum_{i=1}^{m}\frac{h_1\alpha_{2,n}^{(i)}C_{y_1,i}^2 - 2h_1h_2C_{y_1,i}C_{y_2,i} + h_2\alpha_{1,n}^{(i)}C_{y_2,i}^2}{(1-\cos\theta_i)G_n^{(i)}}, \tag{60}$$

where $\alpha_{k,n}^{(i)} = \Delta F_n^{(i)} + (h_k - 1)\Delta F_{n-1}^{(i)}$.

In particular, when $h_1 = h_2 = 1$, equation (60) reduces to

$$R_{m\times n}(\{0, y_1\}, \{n, y_2\}) = \frac{nr}{m+1} + \frac{r_0}{m+1}\sum_{i=1}^{m}\frac{(C_{y_1,i}^2 + C_{y_2,i}^2)\Delta F_n^{(i)} - 2C_{y_1,i}C_{y_2,i}}{(1-\cos\theta_i)F_{n+1}^{(i)}}. \tag{61}$$

Case 5. When $d_1(x_1, 0)$ is at the bottom edge and $d_2(x_2, m)$ is on the top edge, then formula (54) reduces to

$$R_{m\times n}(\{x_1, 0\}, \{x_2, m\}) = \frac{|x_2 - x_1|}{m+1}r + \frac{r_0}{m+1}\sum_{i=1}^{m}\frac{\beta_{1,1}^{(i)} - 2(-1)^i\beta_{1,2}^{(i)} + \beta_{2,2}^{(i)}}{2G_n^{(i)}}\cot^2\left(\frac{1}{2}\theta_i\right), \tag{62}$$

where $\beta_{k,s}^{(i)}$ and $G_n^{(i)}$ are defined in (24) and (25).

Case 6. When $d_1(0, 0)$ and $d_2(n, m)$ are two diagonal nodes, by (54) we have the resistance between two maximally separated nodes

$$R_{m\times n}(\{0, 0\}, \{n, m\}) = \frac{nr}{m+1} + \frac{r_0}{m+1}\sum_{i=1}^{m}\frac{h_1\alpha_{2,n}^{(i)} - 2(-1)^ih_1h_2 + h_2\alpha_{1,n}^{(i)}}{2G_n^{(i)}}\cot^2\left(\frac{1}{2}\theta_i\right), \tag{63}$$

where $\alpha_{k,n}^{(i)} = \Delta F_n^{(i)} + (h_k - 1)\Delta F_{n-1}^{(i)}$, $\theta_i = i\pi/(m+1)$.

In particular, when $r_1 = r_2 = r_0$, equation (63) reduces to

$$R_{m\times n}(\{0, 0\}, \{n, m\}) = \frac{n}{m+1}r + \frac{r_0}{m+1}\sum_{i=1}^{m}\left(\frac{\Delta F_n^{(i)} - (-1)^i}{F_{n+1}^{(i)}}\right)\cot^2\left(\frac{1}{2}\theta_i\right). \tag{64}$$

Please note that equation (64) is a desired equivalent resistance between two maximum separated nodes in an arbitrary $m\times n$ resistor network. This is an interesting result because it is simple and easy to research the asymptotic expansion for the maximum resistance. References [39, 40] studied the asymptotic expansion by making use of the result (1). Obviously, the concise equation (64) is more conducive to the study of the asymptotic expression of the maximum resistance.

From the above derivation, we find that formula (54) is a generalized result, which is applicable to many network problems and summarized a variety of complex network models since it contains six arbitrary elements (r_0, r, r_1, r_2, n, m).

5. Electrical properties of complex cylindrical network

5.1. Nodal potential of complex cylindrical network

Consider the non-regular $m\times n$ cylindrical network shown in figure 2, where the maximum coordinate is (n, m), selecting $\sum_{i=0}^{m}V_0^{(i)} = \frac{1}{2}(x_2 - x_1)rJ$ as the reference potential, defining $\theta_i = 2i\pi/(m+1)$, and $C_{y_k - y} = \cos(y_k - y)\theta_i$, the potential of any node $d(x, y)$ in the finite and sem-infinite networks can be written as

$$\frac{U_{m\times n}(x, y)}{J} = \frac{\bar{x} - x_\tau}{m+1}r + \frac{r_0}{2(m+1)}\sum_{i=1}^{m}\frac{\beta_{x_1\vee x}^{(i)}C_{y_1-y} - \beta_{x_2\vee x}^{(i)}C_{y_2-y}}{(1-\cos\theta_i)G_n^{(i)}}, \tag{65}$$

$$\frac{U_{m\times\infty}(x, y)}{J} = \frac{\bar{x} - x_\tau}{m+1}r + \frac{r}{2(m+1)}\sum_{i=1}^{m}\frac{\bar{\lambda}_i^{|x_1-x|}C_{y_1-y} - \bar{\lambda}_i^{|x_2-x|}C_{y_2-y}}{\sqrt{(1+h-h\cos\theta_i)^2 - 1}}, \tag{66}$$

where $\beta_{k,s}^{(i)}$, $G_k^{(i)}$ are, respectively, defined in equations (22)–(25). For equation (66), there be $n\to\infty$, x_1, $x_2\to\infty$ with finite $x_k - x$. Equation (66) can be derived by taking the limit $n\to\infty$ in equation (65).

In particular, when $x_2 = x_1$ (means the input and output nodes of currents are at the same vertical axis), formulae (65) and (66) reduce to

$$\frac{U_{m\times n}(x, y)}{J} = \frac{r_0}{2(m+1)}\sum_{i=1}^{m}\frac{C_{y_1-y} - C_{y_2-y}}{(1-\cos\theta_i)G_n^{(i)}}\beta_{x_1\vee x}^{(i)}, \tag{67}$$

$$\frac{U_{m\times\infty}(x, y)}{J} = \frac{r}{2(m+1)}\sum_{i=1}^{m}\frac{C_{y_1-y}-C_{y_2-y}}{\sqrt{(1+h-h\cos\theta_i)^2-1}}\bar{\lambda}_i^{|x_1-x|}. \tag{68}$$

Proof of equation (44). For figure 2, substituting equation (38) into equation (18), we achieve $(1\leqslant k\leqslant n)$

$$X_k^{(i)} = \frac{\beta_{k\vee x_1}^{(i)}\exp(\mathrm{i}y_1\theta_i)-\beta_{k\vee x_2}^{(i)}\exp(\mathrm{i}y_2\theta_i)}{(t_i-2)G_n^{(i)}}rJ. \tag{69}$$

The substitution of (69) into equation (27) yields

$$\frac{U(x, y)}{J} = \frac{\bar{x}-x_\tau}{m+1}r + \frac{r_0}{m+1}\times\sum_{i=1}^{m}\frac{\beta_{x_1\vee x}^{(i)}C_{y_1-y}-\beta_{x_2\vee x}^{(i)}C_{y_2-y}}{2(1-\cos\theta_i)G_n^{(i)}} + \mathrm{i}\frac{r_0}{m+1}\times\sum_{i=1}^{m}\frac{\beta_{x_1\vee x}^{(i)}\sin[(y_1-y)\theta_i]-\beta_{x_2\vee x}^{(i)}\sin[(y_2-y)\theta_i]}{2(1-\cos\theta_i)G_n^{(i)}}. \tag{70}$$

Because the elements r_k in the network is real number, the potential $U(x, y)$ must be real number. Thus, extracting the real part of equation (70) to produce equation (65).

Formula (65) is a meaningful result because the network of figure 2 is very complex and has not been resolved before, contains a lot of resistor network models, where each of the different boundary resistor represents a different network structure. So Formula (65) can create many interesting results. In the following applications we always assume that the current J goes from $d_1(x_1, y_1)$ to $d_2(x_2, y_2)$ except for special instructions.

Application 1. Consider an arbitrary $m\times n$ cylindrical network of figure 2 with $r_1 = r_2 = r_0$, by (65) we have the nodal potential

$$\frac{U(x, y)}{J} = \frac{\bar{x}-x_\tau}{m+1}r + \frac{r_0}{2(m+1)}\sum_{i=1}^{m}\frac{\beta_{x_1\vee x}^{(i)}C_{y_1-y}-\beta_{x_2\vee x}^{(i)}C_{y_2-y}}{(1-\cos\theta_i)F_{n+1}^{(i)}}, \tag{71}$$

where $\beta_{x,x_s}^{(i)}$ reduces to $\beta_{x,x_s}^{(i)} = \Delta F_x^{(i)}\Delta F_{n-x_s}^{(i)}$.

In particular, when $x_2 = x_1$, potential formula (71) reduces further to

$$\frac{U(x, y)}{J} = \frac{r_0}{2(m+1)}\sum_{i=1}^{m}\frac{C_{y_1-y}-C_{y_2-y}}{(1-\cos\theta_i)F_{n+1}^{(i)}}\beta_{x_1\vee x}^{(i)}. \tag{72}$$

Application 2. Consider an $m\times n$ cylindrical network of figure 2. When $r_1 = 0$, figure 2 degrades into an $m\times n$ cobweb network as shown in figure 5(a), by (65) we have the nodal potential

$$\frac{U(x, y)}{J} = \frac{\bar{x}-x_\tau}{m+1}r + \frac{r}{m+1}\sum_{i=1}^{m}\frac{\beta_{x_1\vee x}^{(i)}C_{y_1-y}-\beta_{x_2\vee x}^{(i)}C_{y_2-y}}{\Delta F_n^{(i)}+(h_2-1)\Delta F_{n-1}^{(i)}}, \tag{73}$$

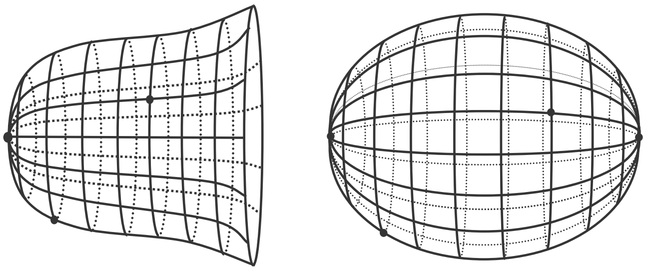

Figure 5. Two resistor network models. (a) is a cobweb network with an arbitrary boundary resistor r_2; (b) is an arbitrary globe network.

where $\beta_{x\vee x_s}^{(i)}$ is redefined as $\beta_{x\vee x_s}^{(i)} = F_x^{(i)}\alpha_{2,n-x_s}^{(i)}(\text{if } x\leqslant x_s)$ and $\beta_{x\vee x_s}^{(i)} = F_{x_s}^{(i)}\alpha_{2,n-x}^{(i)}(\text{if } x\geqslant x_s)$.

In particular, when $d_1(0, y_1)$ is at left edge, and $d_2(n, y_2)$ is at right edge. Equation (73) reduces to

$$\frac{U(x, y)}{J} = \frac{n-2x}{2(m+1)}r - \frac{r_2h}{m+1}\sum_{i=1}^{m}\frac{F_x^{(i)}\cos(y_2-y)\theta_i}{\Delta F_n^{(i)}+(h_2-1)\Delta F_{n-1}^{(i)}}. \tag{74}$$

Please note that the cobweb network with an arbitrary boundary has not been resolved before, the previous work only studied the normal cobweb network (the boundary resistor is $r_2 = r_0$) [19], equation (74) is an original result.

Application 3. Consider an arbitrary $m\times n$ globe network shown in figure 5(b). That is to say that figure 2 degrades into a globe network when $r_2 = r_1 = 0$, from (65) we have the nodal potential

$$\frac{U_{m\times n}(x, y)}{J} = \frac{\bar{x}-x_\tau}{m+1}r + \frac{r}{m+1}\sum_{i=1}^{m}\frac{\beta_{x_1\vee x}^{(i)}C_{y_1-y}-\beta_{x_2\vee x}^{(i)}C_{y_2-y}}{F_n^{(i)}}, \tag{75}$$

where we redefine $\beta_{x\vee x_s}^{(i)} = F_x^{(i)}F_{n-x_s}^{(i)}(\text{if } x\leqslant x_s)$ and $\beta_{x\vee x_s}^{(i)} = F_{x_s}^{(i)}F_{n-x}^{(i)}(\text{if } x\geqslant x_s)$.

In particular, when $d_1(0, y_1)$ is at left pole, and $d_2(n, y_2)$ is at right pole, equation (75) reduces to

$$\frac{U(x, y)}{J} = \frac{n-2x}{2(m+1)}r. \tag{76}$$

Formula (76) is very simple and very interesting because the potential distribution is only related to the x and has nothing to do with y, which shows the nodal potential is equal in the same latitude.

Application 4. Consider a non-regular $m\times n$ cylindrical network of figure 2. Assume $d_1(0, y_1)$ is on the left edge, and $d_2(n, y_2)$ is on the right edge. By (65) we have the nodal potential

$$\frac{U(x,y)}{J} = \frac{n-2x}{2(m+1)}r + \frac{r_0}{2(m+1)}\sum_{i=1}^{m}\frac{h_1\alpha_{2,n-x}^{(i)}C_{y_1-y} - h_2\alpha_{1,x}^{(i)}C_{y_2-y}}{(1-\cos\theta_i)G_n^{(i)}}, \tag{77}$$

where $\alpha_{k,x}^{(i)} = \Delta F_x^{(i)} + (h_k - 1)\Delta F_{x-1}^{(i)}$ is defined in equation (23), and $C_{y_k-y} = \cos(y_k - y)\theta_i$.

In particular, when $h_1 = h_2 = 1$, equation (77) reduces to

$$\frac{U(x,y)}{J} = \frac{n-2x}{2(m+1)}r + \frac{r_0}{2(m+1)}\sum_{i=1}^{m}\frac{\Delta F_{n-x}^{(i)}C_{y_1-y} - \Delta F_x^{(i)}C_{y_2-y}}{(1-\cos\theta_i)F_{n+1}^{(i)}}, \tag{78}$$

when $h_1 = h_2 = 1$, $y_2 = y_1$, equation (77) reduces to

$$\frac{U(x,y)}{J} = \frac{n-2x}{2(m+1)}r + \frac{r_0}{2(m+1)}\sum_{i=1}^{m}\frac{\Delta F_{n-x}^{(i)} - \Delta F_x^{(i)}}{(1-\cos\theta_i)F_{n+1}^{(i)}}C_{y_1-y}. \tag{79}$$

One know the potential function have important application value for solving the Laplace equation. In this paper, the analytical solutions of node potential functions under various conditions are given, which provides a new theory for practical application.

5.2. Resistance of complex m × n cylindrical network

Consider an $m \times n$ cylindrical network with two arbitrary boundaries shown in figure 2, where the maximum coordinate is (n, m). Defining $\beta_{k,s}^{(i)} = \beta_{x_k,x_s}^{(i)}$, the resistance between two nodes $d_1(x_1, y_1)$ and $d_2(x_2, y_2)$ in the finite and semi-infinite networks are respectively

$$R_{m\times n}(d_1,d_2) = \frac{|x_2-x_1|}{m+1}r + \frac{r_0}{2(m+1)}\sum_{i=1}^{m}\frac{\beta_{1,1}^{(i)} - 2\beta_{1,2}^{(i)}\cos(y\theta_i) + \beta_{2,2}^{(i)}}{(1-\cos\theta_i)G_n^{(i)}}, \tag{80}$$

$$R_{m\times\infty}(d_1,d_2) = \frac{|x_2-x_1|}{m+1}r + \frac{r}{m+1}\sum_{i=1}^{m}\frac{1-\bar{\lambda}_i^{|x_2-x_1|}\cos(y\theta_i)}{\sqrt{(1+h-h\cos\theta_i)^2-1}}, \tag{81}$$

where $\theta_i = 2i\pi/(m+1)$, $y = y_2 - y_1$, $\beta_{k,s}^{(i)}$ and $G_k^{(i)}$ are, respectively, defined in equations (24)–(25). For equation (81), there be $n \to \infty$, $x_1, x_2 \to \infty$, with finite m. Equation (81) can be derived by taking the limit $n \to \infty$ in equation (80).

Proof of equation (80). For figure 2, substituting equation (65) with $k = x_1, x_2$ into equation (41), we therefore achieve (80).

Formula (80) is an exact and exciting result because the network of figure 2 is very complex and has not been resolved before, and contains a lot of resistor network models, where each of the different boundary resistor represents a different network structure. In particular, when taking some specific value for r_1 and r_2, equation (80) gives rise to a series of special cases below.

Case 1. Consider a non-regular $m \times n$ cylindrical network of figure 2. When $r_1 = r_0$, the resistance of equation (80) reduces to

$$R_{m\times n}(d_1,d_2) = \frac{|x_2-x_1|}{m+1}r + \frac{r_0}{2(m+1)}\sum_{i=1}^{m}\frac{\beta_{1,1}^{(i)} - 2\beta_{1,2}^{(i)}\cos(y\theta_i) + \beta_{2,2}^{(i)}}{(1-\cos\theta_i)[F_{n+1}^{(i)} + (h_2-1)F_n^{(i)}]}, \tag{82}$$

where $\beta_{k,s}^{(i)}$ reduces to $\beta_{k,s}^{(i)} = \Delta F_{x_k}^{(i)}[\Delta F_{n-x_s}^{(i)} + (h_2-1)\Delta F_{n-x_s-1}^{(i)}]$.

Case 2. Consider a normal $m \times n$ cylindrical network of figure 2 with $r_2 = r_1 = r_0$, the resistance of equation (80) reduces to

$$R_{m\times n}(d_1,d_2) = \frac{|x_2-x_1|}{m+1}r + \frac{r_0}{2(m+1)}\sum_{i=1}^{m}\frac{\beta_{1,1}^{(i)} - 2\beta_{1,2}^{(i)}\cos(y\theta_i) + \beta_{2,2}^{(i)}}{(1-\cos\theta_i)F_{n+1}^{(i)}}, \tag{83}$$

where $y = y_2 - y_1$, $\beta_{k,s}^{(i)}$ reduces to $\beta_{k,s}^{(i)} = \Delta F_{x_k}^{(i)}\Delta F_{n-x_s}^{(i)}$. This problem has been researched in [23], and gave equation (2) with a double sums. Clearly, our result (83) is different from equation (2). This also shows that the equivalent resistance can be expressed in different forms.

Case 3. Consider a non-regular $m \times n$ cylindrical network of figure 2, when $h_1 = 0$, the left boundary collapses to a pole, the network of figure 2 degrades into a cobweb network with an arbitrary boundary resistor r_2 as shown in figure 5(a), we have the equivalent resistance

$$R_{m\times n}(d_1,d_2) = \frac{|x_2-x_1|}{m+1}r + \frac{r}{m+1}\sum_{i=1}^{m}\frac{\beta_{1,1}^{(i)} - 2\beta_{1,2}^{(i)}\cos(y\theta_i) + \beta_{2,2}^{(i)}}{\Delta F_n^{(i)} + (h_2-1)\Delta F_{n-1}^{(i)}}, \tag{84}$$

where $\beta_{k,s}^{(i)}$ is re-defined as $\beta_{k,s}^{(i)} = F_{x_k}^{(i)}[\Delta F_{n-x_s}^{(i)} + (h_2-1)\Delta F_{n-x_s-1}^{(i)}]$.

In particular, when $h_1 = 0$, $h_2 = 1$, the network of figure 5(a) degrades into a regular cobweb network, the resistance of equation (84) reduces to

$$R_{m\times n}(d_1,d_2) = \frac{|x_2-x_1|}{m+1}r + \frac{r}{m+1}\sum_{i=1}^{m}\frac{\beta_{1,1}^{(i)} - 2\beta_{1,2}^{(i)}\cos(y\theta_i) + \beta_{2,2}^{(i)}}{\Delta F_n^{(i)}} \tag{85}$$

where $\beta_{k,s}^{(i)}$ is re-defined as $\beta_{k,s}^{(i)} = F_{x_k}^{(i)}\Delta F_{n-x_s}^{(i)}$.

Please note that Case 2 has been researched in [30], but the result is different from equation (85), however they are equivalent to each other. The reason is that they choice the different matrix along different direction, where [30] set up matrix along the longitude, but this paper set up matrix along the latitude.

Case 4. When $h_1 = h_2 = 0$, the left and right boundary collapse respectively to two poles, the network of figure 2 degrades into an $m \times n$ globe network as shown in figure 5(b), we have

$$R_{m\times n}(d_1, d_2) = \frac{|x_2 - x_1|}{m+1}r + \frac{r}{m+1}\sum_{i=1}^{m}\frac{\beta_{1,1}^{(i)} - 2\beta_{1,2}^{(i)}\cos(y\theta_i) + \beta_{2,2}^{(i)}}{F_n^{(i)}}, \tag{86}$$

where $\beta_{k,s}^{(i)}$ is re-defined as $\beta_{k,s}^{(i)} = F_{x_k}^{(i)}F_{n-x_s}^{(i)}$.

Please note that Case 4 has been researched in [29], but the result is different from equation (86), however they are equivalent to each other. The reason is that they choice the different matrix along different axes. This also shows that the equivalent resistance can be expressed in different forms.

Case 5. Consider a non-regular $m \times n$ cylindrical network of figure 2, when both $d_1(x_1, 0)$ and $d_2(x_2, 0)$ are on the same horizontal axis, we have

$$R_{m\times n}(\{x_1, 0\}, \{x_2, 0\}) = \frac{|x_2 - x_1|}{m+1}r + \frac{r_0}{2(m+1)}\sum_{i=1}^{m}\frac{\beta_{1,1}^{(i)} - 2\beta_{1,2}^{(i)} + \beta_{2,2}^{(i)}}{(1-\cos\theta_i)G_n^{(i)}}. \tag{87}$$

Case 6. Consider a non-regular $m \times n$ cylindrical network of figure 2, when $d_1(0, 0)$ is on the left edge and $d_2(n, y)$ is on the right edge, the resistance between two edges is

$$R_{m\times n}(\{0, 0\}, \{n, y\}) = \frac{n}{m+1}r + \frac{r_0}{2(m+1)}\sum_{i=1}^{m}\frac{h_1\alpha_{2,n}^{(i)} + h_2\alpha_{1,n}^{(i)} - 2h_1h_2\cos(y\theta_i)}{(1-\cos\theta_i)G_n^{(i)}}. \tag{88}$$

In particular, when $h_1 = h_2 = 1$, equation (88) reduces to

$$R_{m\times n}(\{0, 0\}, \{n, y\}) = \frac{n}{m+1}r + \frac{r_0}{m+1}\sum_{i=1}^{m}\frac{\Delta F_n^{(i)} - \cos(y\theta_i)}{(1-\cos\theta_i)F_{n+1}^{(i)}}. \tag{89}$$

And when $h_1 = 0$, equation (88) reduces to

$$R_{m\times n}(\{0, 0\}, \{n, y\}) = \frac{n}{m+1}r + \frac{rh_2}{m+1}\sum_{i=1}^{m}\frac{F_n^{(i)}}{\Delta F_n^{(i)} + (h_2 - 1)\Delta F_{n-1}^{(i)}}. \tag{90}$$

From the above results we know formula (80) is a general results which, contains many results in a variety of lattice structures, can produce many new resistance formulae.

6. Conclusion and comment

This paper developed the RT-V theory (RT theory with potential parameters) and reveals the basic principle of electrical characteristics of complex resistor networks for the first time, such as two theorems of theorems 1 and 2 are proposed, and the explicit electrical characteristics (potential and resistance) formulae of the complex networks are given, which contains the results of finite and infinite networks. As applications of two theorems, the analytical solutions of the electrical characteristics (potential function and equivalent resistance) in the complex $m \times n$ resistor networks with arbitrary boundaries are given, and many interesting results of the various types of resistor networks are produced.

It must be emphasized that the previous theories (Mainly refers Green's function technique and Laplacian matrix method) cannot solve resistor networks with complex boundaries, because the Green's function technique is usually used to solve infinite network problems, and the Laplacian matrix method depends on the solution of two eigenvalues which relies on two matrices along two orthogonal directions. Using Tan's RT-V method to study resistor networks just relies on one matrix along one vertical directions, which avoids the confusion of another matrix with arbitrary elements that cannot be solved explicitly, and also gives concise results in a single summation, such as the all equations given by this paper.

In addition, resistance formulae (54), (55), (80) and (81) *et al* can be extended to impedance networks since the grid elements r_k can be either resistors or impedances in figures 1 and 2. For example, assume $r = Z_L = R + j\omega L$, $r_0 = Z_C = -j/\omega C$, then we can therefore study the arbitrary $m \times n$ *RLC* network if we do a plural analysis [24, 34] to the resistance results obtained in this paper.

References

[1] Kirkpatrick S 1973 *Rev. Mod. Phys.* **45** 574
[2] English L Q *et al* 2013 *Phys. Rev.* E **88** 022912
[3] Bulgakov E N, Maksimov D N and Sadreev A F 2005 *Phys. Rev.* E **71** 046205
[4] McGurn A R 2000 *Phys. Rev.* B **61** 13235
[5] Tuan P H *et al* 2015 *Phys. Rev.* E **92** 062906
[6] Albert V V, Glazman L I and Jiang L 2015 *Phys. Rev. Lett.* **114** 173902
[7] Melnikov A V, Shuba M and Lambin P 2018 *Phys. Rev.* E **97** 043307
[8] Lupke F *et al* 2017 *Nat. Commun.* **8** 15704
[9] Barabási A L, Albert R and Jeong H 1999 *Physica* A **272** 173–87
[10] Caracciolo R *et al* 1998 *Ann. Phys.* **262** 105–31
[11] Joyce G S 2017 *J. Phys. A: Math. Theor* **50** 425001
[12] Komnik A and Heinze S 2017 *Phys. Rev.* B **96** 155103
[13] Koutschan C 2013 *J. Phys. A: Math. Theor* **46** 125005
[14] Yasi J A and Trinkle D R 2012 *Phys. Rev.* E **85** 066706
[15] Klein D J and Randi M 1993 *J. Math. Chem.* **12** 8195
[16] Yang Y J and Klein D J 2013 *Discrete Appl. Math.* **161** 2702–15
[17] Lai M C and Wang W-C 2002 *Numer. Methods Partial Differ. Equ.* **18** 56–68
[18] Borges L and Daripa P 2001 *J. Comput. Phys.* **169** 151–92
[19] Tan Z Z 2017 *Chin. Phys.* B **26** 090503
[20] Cserti J 2000 *Am. J. Phys.* **68** 896
[21] Asad J H 2013 *J. Stat. Phys.* **150** 1177–82
[22] Owaidat M Q and Asad J H 2016 *Eur. Phys. J. Plus* **131** 309
[23] Wu F Y 2004 *J. Phys. A: Math. Gen.* **37** 6653

[24] Tzeng W J and Wu F Y 2006 *J. Phys. A: Math. Gen.* **39** 8579
[25] Izmailian N S, Kenna R and Wu F Y 2014 *J. Phys. A: Math. Theor.* **47** 035003
[26] Izmailian N S and Kenna R 2014 *J. Stat. Mech.* 1742–5468
[27] Essam J W, Izmailian N S, Kenna R and Tan Z Z 2015 *R. Soc. Open Sci.* **2** 140420
[28] Tan Z Z 2011 *Resistance Network Model* (Xi'an: Xidian Univ. Press) (in Chinese)
[29] Tan Z Z, Essam J W and Wu F Y 2014 *Phys. Rev.* E **90** 012130
[30] Essam J W, Tan Z Z and Wu F Y 2014 *Phys. Rev.* E **90** 032130
[31] Tan Z Z 2015 *Chin. Phys.* B **24** 020503
[32] Tan Z Z 2015 *Phys. Rev.* E **91** 052122
[33] Tan Z Z 2015 *Sci. Rep.* **5** 11266
[34] Tan Z Z 2016 *Chin. Phys.* B **25** 050504
[35] Tan Z Z 2017 *Commun. Theor. Phys.* **67** 280–8
[36] Tan Z, Tan Z Z and Zhou L 2018 *Commun. Theor. Phys.* **69** 610–616
[37] Tan Z and Tan Z Z 2018 *Sci. Rep.* **8** 9937
[38] Tan Z, Tan Z Z and Chen J-X 2018 *Sci. Rep.* **8** 5798
[39] Essam J W and Wu F Y 2009 *J. Phys. A: Math. Theor.* **42** 025205
[40] Izmailian N S and Huang M-C 2010 *Phys. Rev.* E **82** 011125