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Structured charged particles in external fields: Charge, current, action, hidden momentum

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Abstract – The charge and current densities of a structured particle are discussed and, for the case of small particles, expressed in terms of the charge and the electric and magnetic dipoles of the particle. The action for the motion of such particles in external electromagnetic fields is studied and the corresponding energy and momenta are investigated. It is seen that these contain the so-called hidden momenta, which do not vanish even when the particle is at rest.

Introduction. – A particle (molecule) is said to be structured if its configuration is not completely described by the configuration of a point particle, that is, if it has intrinsic properties as well. A well-known example is the electron. The electron has, in addition to its translational degrees of freedom, a spin degree of freedom, and the translation degrees of freedom and the spin enter in the evolution equations of each other. A charged molecule (ion) consisting of charged point particles each without any internal structure can be approximated by a point particle with internal structure, so long as it is moving in fields with characteristic lengths much larger than the size of the molecule. Such approximations are well studied in text books, [1] for example. Related to this is the subject of rewriting the field equations in matter in terms of the so-called free sources (of charge and current) and the auxiliary fields $\boldsymbol{D}$ and $\boldsymbol{H}$, as well as evaluating the force and power applied to matter in terms of free sources. Examples of works of this kind are those of Einstein and Laub [2,3].

The simplest approximation beyond the structureless point particle is when one incorporates the (electric and magnetic) dipoles of the molecule as well. Such a construction leads, among other things, to concepts like hidden momentum, a part of the momentum which is gauge invariant and arises only when there is an external electromagnetic field. One of its peculiarities is that it may be nonvanishing even when the molecule is at rest. The concept of hidden momentum has been discussed in many educational texts, among which [4–7]. But even as late as 2012, there have been claims that the Lorentz force law exerted on structured particles (to be more specific, magnetic dipoles) is not consistent with special relativity [8]. While it can be argued that the claim that the Lorentz force law is not compatible with special relativity cannot be mathematically correct ([9–12], for example), there has still been debate about the origin of the so-called hidden momentum, which should be taken into account in order to resolve apparent inconsistencies, [13,14] for example. It has also been argued that one should treat structures like dipoles intrinsically differently from monopoles, and write separate equations for the force (torque) acting on them, not something derived from a microscopic description of the dipoles and the Lorentz force acting on point charges [8].

The aim of this paper is to investigate the dipoles corresponding to a structured particle microscopically, *i.e.*, based on considering a particle as a collection of structureless point particles, and applying the field and force equations corresponding to point particles to them. The final results, however, would contain no trace of the structure except for those expressible in terms of the total charges and dipoles. So the results could be applicable to particles like the electron or proton (of course in regimes where classical treatment is plausible). The equations are, of course, Lorentz invariant. Defining the momentum of the molecule as the derivative of its Lagrangian with respect to its velocity (which is the usual definition of the momentum), it is explicitly shown that there are terms in the momentum of the particle which correspond to the hidden momentum. While the hidden momentum can be introduced as something which should be in the momentum in order to preserve the conservation of momentum, as

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in [12] for example, here it is derived at the particle level, without the usage of the energy and momentum of the electromagnetic field itself.

The scheme of the paper is the following. In the next section, some conventions are introduced. In the third section, the charge and current densities corresponding to a structured molecule are discussed. In the fourth section, the action corresponding to a structured particle moving in an external electromagnetic field is studied. In the fifth section an investigation of the energy and momentum of such a particle is presented. The last section is devoted to the concluding remarks.

Conventions. – A temporal component is characterized by a zero index, while spatial components are denoted by Latin indices. Greek indices denote both temporal (zero) or spatial components. Corresponding to the event $(t, \boldsymbol{x})$, the time t is denoted by x^0. The Minkowski line element $\mathrm{d}s$ is defined as

$$\begin{aligned} (\mathrm{d}s)^2 &= -c^2(\mathrm{d}t)^2 + \mathrm{d}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r}, \\ &=: \eta_{\alpha\beta}\mathrm{d}r^\alpha \mathrm{d}r^\beta, \end{aligned} \tag{1}$$

where c is the speed of light, and $\boldsymbol{r}$ is the position vector. Indices are lowered by η, and raised by its inverse,

$$\begin{aligned} \mathfrak{X}_\alpha &= \eta_{\alpha\beta}\mathfrak{X}^\beta, \\ \mathfrak{X}^\alpha &= \eta^{\beta\alpha}\mathfrak{X}_\beta. \end{aligned} \tag{2}$$

The proper time τ is related to the above through

$$\begin{aligned} \mathrm{d}\tau &= [-c^{-2}(\mathrm{d}s)^2]^{1/2}, \\ &= \gamma^{-1}\mathrm{d}t, \end{aligned} \tag{3}$$

where γ is the Lorentz factor

$$\gamma = \left(1 - c^{-2}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)^{-1/2}. \tag{4}$$

The velocity corresponding to the position $\boldsymbol{x}$ as a function of time is denoted by $\boldsymbol{v}$,

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}. \tag{5}$$

The zeroth component of such a velocity is denoted by v^0, and is equal to 1. The corresponding 4-velocity is denoted by u,

$$\begin{aligned} u^\alpha &= \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}, \\ &= \gamma v^\alpha. \end{aligned} \tag{6}$$

If $\mathfrak{X}$ is a function of t and $\boldsymbol{r}$, then $\dot{\mathfrak{X}}$ and $(\boldsymbol{\nabla}\mathfrak{X})$ mean derivatives of $\mathfrak{X}$ with respect to t and $\boldsymbol{r}$, respectively. If $\mathfrak{X}$ is a function of a single variable, then $(\mathsf{D}\mathfrak{X})$ is the derivative of $\mathfrak{X}$.

The momentum 4-vector p is defined with

$$p^0 = c^{-2}h, \tag{7}$$

where h is the energy, and the spatial components are equal to the spatial components of the momentum $\boldsymbol{p}$. The current density 4-vector J is defined with

$$J^0 = \rho, \tag{8}$$

where ρ is the charge density, and the spatial components are equal to the spatial components of the current density $\boldsymbol{J}$. The 4-potential A is defined with

$$A_0 = -\phi, \tag{9}$$

where ϕ is the scalar potential, and the spatial components are equal to the spatial components of the vector potential $\boldsymbol{A}$. The field strength tensor F is defined as an antisymmetric 4-tensor with

$$\begin{aligned} F_{i0} &= E_i, \\ F_{ij} &= \varepsilon_{ijk}B^k, \end{aligned} \tag{10}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic field strengths, respectively, and ε is the Levi-Civita tensor. The field strength is related to the 4-potential through

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{11}$$

SI convention is used, so that the dimension of the magnetic field is equal to that of the electric field divided by the dimension of speed.

Charge and current densities in terms of microscopic multipoles. – Consider a collection of point charges in particles. The charge of the point a is denoted by q_a. The position of the point a is denoted by $(\boldsymbol{x}+\boldsymbol{x}_a)$, where $\boldsymbol{x}$ and $\boldsymbol{x}_a$ denote the position of the particle and the position of the point a relative to that of the particle. One notes that the position of a particle is somehow arbitrary. It is somewhere not far from the positions of the points within that particle. The charge density corresponding to the particle is denoted by ρ, and one has

$$\rho(t, \boldsymbol{r}) = \sum_a q_a \delta[\boldsymbol{r} - \boldsymbol{x}(t) - \boldsymbol{x}_a(t)], \tag{12}$$

where the summation runs over the charges belonging to the particle. Similarly, the current density corresponding to the particle is denoted by $\boldsymbol{J}$, and one has

$$\boldsymbol{J}(t, \boldsymbol{r}) = \sum_a q_a[\boldsymbol{v}(t) + \boldsymbol{v}_a(t)]\delta[\boldsymbol{r} - \boldsymbol{x}(t) - \boldsymbol{x}_a(t)], \tag{13}$$

where $\boldsymbol{v}$ and $\boldsymbol{v}_a$ are the time derivatives of $\boldsymbol{x}$ and $\boldsymbol{x}_a$, respectively.

The Taylor expanding relation for a function $\mathfrak{A}$ reads

$$\mathfrak{A}(\boldsymbol{y}+\boldsymbol{z}) = [\exp(\boldsymbol{z}\cdot\boldsymbol{\nabla})]\mathfrak{A}(\boldsymbol{y}). \tag{14}$$

So, Taylor-expanding the Dirac delta in $\boldsymbol{x}_a$, ρ can be rewritten as

$$\begin{aligned}\rho &= \sum_a q_a[\exp(-\boldsymbol{x}_a\cdot\boldsymbol{\nabla})]\delta(\boldsymbol{r}-\boldsymbol{x}),\\ &= \left(\sum_a q_a\right)\delta(\boldsymbol{r}-\boldsymbol{x})\\ &\quad -\boldsymbol{\nabla}\cdot\left\{\sum_a q_a\boldsymbol{x}_a[\mathfrak{F}(\boldsymbol{x}_a\cdot\boldsymbol{\nabla})]\delta(\boldsymbol{r}-\boldsymbol{x})\right\}, \qquad (15)\end{aligned}$$

where the function $\mathfrak{F}$ is defined through

$$\mathfrak{F}(x) = \frac{1-\exp(-x)}{x}. \qquad (16)$$

The multiplier of the Dirac delta in the first term is clearly the total charge of the particle. Denoting this charge by q, and defining $\boldsymbol{\Pi}(t,\boldsymbol{r})$ through

$$\boldsymbol{\Pi} = \sum_a q_a\boldsymbol{x}_a[\mathfrak{F}(\boldsymbol{x}_a\cdot\boldsymbol{\nabla})]\delta(\boldsymbol{r}-\boldsymbol{x}), \qquad (17)$$

one arrives at

$$\rho = q\delta(\boldsymbol{r}-\boldsymbol{x}) - \boldsymbol{\nabla}\cdot\boldsymbol{\Pi}. \qquad (18)$$

$\boldsymbol{\Pi}$ is called the electric polarization of the particle. Expanding $\mathfrak{F}$, it is seen that $\boldsymbol{\Pi}$ contains the dipole density, the derivative of the quadruple density, and so on, corresponding to the particle. In chapter 6 of [1], a derivation is presented which gives the lowest-order terms (dipole and quadrupole). Neglecting higher multipoles is equivalent to Taylor-expanding $\mathfrak{F}$ and keeping only the lowest terms. Keeping only the zeroth term, one arrives at

$$\boldsymbol{\Pi} = \boldsymbol{\varpi}\delta(\boldsymbol{r}-\boldsymbol{x}), \qquad (19)$$

where $\boldsymbol{\varpi}$ is the electric dipole of the particle,

$$\boldsymbol{\varpi} = \sum_a q_a\boldsymbol{x}_a. \qquad (20)$$

Performing a similar analysis for the current density, one arrives at

$$\boldsymbol{J} = \sum_a q_a[(\boldsymbol{v}+\boldsymbol{v}_a)[1-(\boldsymbol{x}_a\cdot\boldsymbol{\nabla})\mathfrak{F}(\boldsymbol{x}_a\cdot\boldsymbol{\nabla})]\delta(\boldsymbol{r}-\boldsymbol{x}). \qquad (21)$$

Using

$$\begin{aligned}\dot{\boldsymbol{\Pi}} &= \sum_a q_a\{[\boldsymbol{v}_a-\boldsymbol{x}_a(\boldsymbol{v}\cdot\boldsymbol{\nabla})][\mathfrak{F}(\boldsymbol{x}_a\cdot\boldsymbol{\nabla})]\\ &\quad +\boldsymbol{x}_a(\boldsymbol{v}_a\cdot\boldsymbol{\nabla})[(\mathsf{D}\mathfrak{F})(\boldsymbol{x}_a\cdot\boldsymbol{\nabla})]\}\delta(\boldsymbol{r}-\boldsymbol{x}), \qquad (22)\\ \boldsymbol{\nabla}\cdot\Pi &= \sum_a q_a(\boldsymbol{x}_a\cdot\boldsymbol{\nabla})[\mathfrak{F}(\boldsymbol{x}_a\cdot\boldsymbol{\nabla})]\delta(\boldsymbol{r}-\boldsymbol{x}), \qquad (23)\\ (\boldsymbol{v}\cdot\boldsymbol{\nabla})\Pi &= \sum_a q_a\boldsymbol{x}_a(\boldsymbol{v}\cdot\boldsymbol{\nabla})[\mathfrak{F}(\boldsymbol{x}_a\cdot\boldsymbol{\nabla})]\delta(\boldsymbol{r}-\boldsymbol{x}), \qquad (24)\end{aligned}$$

one arrives after some algebra at

$$\begin{aligned}\boldsymbol{J} &= \boldsymbol{v}[q\delta(\boldsymbol{r}-\boldsymbol{x})-\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}]+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{\Pi}+\dot{\boldsymbol{\Pi}}\\ &\quad +\sum_a q_a\{\boldsymbol{v}_a[1-\mathfrak{F}(\boldsymbol{x}_a\cdot\boldsymbol{\nabla})-(\boldsymbol{x}_a\cdot\boldsymbol{\nabla})\mathfrak{F}(\boldsymbol{x}_a\cdot\boldsymbol{\nabla})]\\ &\quad -\boldsymbol{x}_a(\boldsymbol{v}_a\cdot\boldsymbol{\nabla})[(\mathsf{D}\mathfrak{F})(\boldsymbol{x}_a\cdot\boldsymbol{\nabla})]\}\delta(\boldsymbol{r}-\boldsymbol{x}). \qquad (25)\end{aligned}$$

Using the identity

$$x(\mathsf{D}\mathfrak{F})(x) = 1-\mathfrak{F}(x)-x\mathfrak{F}(x), \qquad (26)$$

one arrives at

$$\begin{aligned}\boldsymbol{J} &= \boldsymbol{v}[q\delta(\boldsymbol{r}-\boldsymbol{x})-\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}]+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{\Pi}+\dot{\boldsymbol{\Pi}}\\ &\quad +\sum_a q_a[\boldsymbol{v}_a(\boldsymbol{x}_a\cdot\boldsymbol{\nabla})-\boldsymbol{x}_a(\boldsymbol{v}_a\cdot\boldsymbol{\nabla})]\\ &\quad \times[(\mathsf{D}\mathfrak{F})(\boldsymbol{x}_a\cdot\boldsymbol{\nabla})]\delta(\boldsymbol{r}-\boldsymbol{x}). \qquad (27)\end{aligned}$$

Defining $\boldsymbol{M}$ and $\tilde{\boldsymbol{M}}$ (the magnetization and the intrinsic magnetization of the particle, respectively) through

$$\begin{aligned}\tilde{\boldsymbol{M}} &= \frac{1}{2}\sum_a q_a\boldsymbol{x}_a\times\boldsymbol{v}_a[-2(\mathsf{D}\mathfrak{F})(\boldsymbol{x}_a\cdot\boldsymbol{\nabla})]\delta(\boldsymbol{r}-\boldsymbol{x}), \qquad (28)\\ \boldsymbol{M} &:= \tilde{\boldsymbol{M}}+\boldsymbol{\Pi}\times\boldsymbol{v}, \qquad (29)\end{aligned}$$

one arrives at

$$\begin{aligned}\boldsymbol{J} &= \boldsymbol{v}[q\delta(\boldsymbol{r}-\boldsymbol{x})-\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}]+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{\Pi}+\dot{\boldsymbol{\Pi}}+\boldsymbol{\nabla}\times\tilde{\boldsymbol{M}}, \qquad (30)\\ \boldsymbol{J} &= \boldsymbol{v}\rho+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{\Pi}+\dot{\boldsymbol{\Pi}}+\boldsymbol{\nabla}\times\tilde{\boldsymbol{M}}, \qquad (31)\\ \boldsymbol{J} &= q\boldsymbol{v}\delta(\boldsymbol{r}-\boldsymbol{x})+\dot{\boldsymbol{\Pi}}+\boldsymbol{\nabla}\times\boldsymbol{M}. \qquad (32)\end{aligned}$$

Again, keeping only the zeroth term in the Taylor expansion of $(\mathsf{D}\mathfrak{F})$, one arrives at

$$\tilde{\boldsymbol{M}} = \tilde{\boldsymbol{\mu}}\delta(\boldsymbol{r}-\boldsymbol{x}), \qquad (33)$$

where $\tilde{\boldsymbol{\mu}}$ is the intrinsic magnetic moment of the particle,

$$\tilde{\boldsymbol{\mu}} = \frac{1}{2}\sum_a q_a\boldsymbol{x}_a\times\boldsymbol{v}_a. \qquad (34)$$

Also,

$$\boldsymbol{M} = \boldsymbol{\mu}\delta(\boldsymbol{r}-\boldsymbol{x}), \qquad (35)$$

where $\boldsymbol{\mu}$ is the magnetic moment of the particle,

$$\begin{aligned}\boldsymbol{\mu} &= \boldsymbol{\varpi}\times\boldsymbol{v}+\tilde{\boldsymbol{\mu}},\\ &= \sum_a q_a\boldsymbol{x}_a\times\left(\frac{1}{2}\boldsymbol{v}_a+\boldsymbol{v}\right). \qquad (36)\end{aligned}$$

When such small size approximations apply, one has the following expression for the charge and current density corresponding to the particle:

$$\begin{aligned}\rho &= q\delta(\boldsymbol{r}-\boldsymbol{x})-(\boldsymbol{\varpi}\cdot\boldsymbol{\nabla})\delta(\boldsymbol{r}-\boldsymbol{x}). \qquad (37)\\ \boldsymbol{J} &= q\boldsymbol{v}\delta(\boldsymbol{r}-\boldsymbol{x})+\dot{\boldsymbol{\varpi}}\delta(\boldsymbol{r}-\boldsymbol{x})-\boldsymbol{v}(\boldsymbol{\varpi}\cdot\boldsymbol{\nabla})\delta(\boldsymbol{r}-\boldsymbol{x})\\ &\quad -(\tilde{\boldsymbol{\mu}}\times\boldsymbol{\nabla})\delta(\boldsymbol{r}-\boldsymbol{x}). \qquad (38)\\ \boldsymbol{J} &= \boldsymbol{v}\rho+\dot{\boldsymbol{\varpi}}\delta(\boldsymbol{r}-\boldsymbol{x})-(\tilde{\boldsymbol{\mu}}\times\boldsymbol{\nabla})\delta(\boldsymbol{r}-\boldsymbol{x}). \qquad (39)\\ \boldsymbol{J} &= q\boldsymbol{v}\delta(\boldsymbol{r}-\boldsymbol{x})+\dot{\boldsymbol{\varpi}}\delta(\boldsymbol{r}-\boldsymbol{x})-\boldsymbol{\varpi}(\boldsymbol{v}\cdot\boldsymbol{\nabla})\delta(\boldsymbol{r}-\boldsymbol{x})\\ &\quad -(\boldsymbol{\mu}\times\boldsymbol{\nabla})\delta(\boldsymbol{r}-\boldsymbol{x}). \qquad (40)\end{aligned}$$

The electric polarization $\boldsymbol{\Pi}$ and the magnetization $\boldsymbol{M}$ can be combined in an antisymmetric tensor N, the polarization tensor

$$\begin{aligned}N^{i0} &= \Pi^i, \qquad (41)\\ N^{ij} &= \varepsilon^{ijk}M_k. \qquad (42)\end{aligned}$$

It is shown, in the appendix, that N actually transforms as a 4-tensor. $\boldsymbol{\varpi}$ and $\boldsymbol{\mu}$ could be combined in a tensor, in a similar manner. However, as shown in the appendix, it is that tensor times γ which transforms as a 4-tensor. The dipole tensor n is hence defined as

$$n^{i0} = \gamma\varpi^i, \tag{43}$$
$$n^{ij} = \gamma\varepsilon^{ijk}\mu_k. \tag{44}$$

The action. – The Lagrangian of a particle in an external electromagnetic field is

$$L_0 = K + \int \mathrm{d}V(\boldsymbol{J}\cdot\boldsymbol{A} - \rho\phi), \tag{45}$$

where L_0 is the Lagrangian, and K is the kinetic part of the action,

$$K = -mc^2\sqrt{1 - c^{-2}\boldsymbol{v}\cdot\boldsymbol{v}}, \tag{46}$$

where m is the mass of the particle. Using the expressions for the charge and current density of a particle, one arrives at

$$\begin{aligned} L_0 &= K + q(\boldsymbol{v}\cdot\boldsymbol{A} - \phi) \\ &\quad -\boldsymbol{\varpi}\cdot\boldsymbol{\nabla}\phi + \dot{\boldsymbol{\varpi}}\cdot\boldsymbol{A} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})(\boldsymbol{\varpi}\cdot\boldsymbol{A}) + \boldsymbol{\mu}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}), \\ &= K + q(\boldsymbol{v}\cdot\boldsymbol{A} - \phi) + \boldsymbol{\varpi}\cdot\boldsymbol{E} + \boldsymbol{\mu}\cdot\boldsymbol{B} + \frac{\mathrm{d}(\boldsymbol{\varpi}\cdot\boldsymbol{A})}{\mathrm{d}t}, \end{aligned} \tag{47}$$

where $(\mathrm{d}/\mathrm{d}t)$ is the complete (or comoving) time derivative,

$$\frac{\mathrm{d}}{\mathrm{d}t} := \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}. \tag{48}$$

Discarding the last term (which is a total time derivative) in the above Lagrangian, one arrives at an equivalent Lagrangian L

$$L = K + q(\boldsymbol{v}\cdot\boldsymbol{A} - \phi) + \boldsymbol{\varpi}\cdot\boldsymbol{E} + \boldsymbol{\mu}\cdot\boldsymbol{B}. \tag{49}$$

The first two terms are those of the Lagrangian of a point particle. The third and fourth terms correspond to the energies of the electric and magnetic dipoles in electric and magnetic fields. So essentially this is what one could expect from the beginning. The point is that here this Lagrangian has been obtained from first principles. This Lagrangian can be written in terms of 4-tensors,

$$L = \gamma^{-1}\left(-mc^2 + qu^\alpha A_\alpha + \frac{1}{2}n^{\alpha\beta}F_{\alpha\beta}\right). \tag{50}$$

The above Lagrangian looks gauge-invariant, apart from the second term. Applying the gauge transformation

$$A^\bullet_\alpha = A_\alpha + \partial_\alpha\chi, \tag{51}$$

the Lagrangian L is changed into $L^\bullet$, with

$$\begin{aligned} L^\bullet &= L + \gamma^{-1}qu^\alpha\partial_\alpha\chi, \\ &= L + \frac{\mathrm{d}(q\chi)}{\mathrm{d}t}. \end{aligned} \tag{52}$$

So the Lagrangian changes under the gauge transformation by a total time derivative, hence the gauge transformation is in fact a Noetherian symmetry of the system, as it should be.

The momentum and the energy. – The momentum is the derivative of the Lagrangian with respect to the velocity. Writing the Lagrangian (50) as

$$L = \gamma^{-1}\tilde{L}, \tag{53}$$

one arrives at

$$p_j = -c^{-2}\gamma v_j\tilde{L} + \tilde{p}_j, \tag{54}$$

where

$$\tilde{p}_j = \gamma^{-1}\frac{\partial\tilde{L}}{\partial v^j}. \tag{55}$$

Also,

$$\begin{aligned} p_0 &= L - v^j p_j, \\ &= \gamma\tilde{L} + \tilde{p}_0, \end{aligned} \tag{56}$$

where

$$\tilde{p}_0 = -\gamma^{-1}v^j\frac{\partial\tilde{L}}{\partial v^j}. \tag{57}$$

So,

$$p_\alpha = -c^{-2}u_\alpha\tilde{L} + \tilde{p}_\alpha, \tag{58}$$
$$v^\alpha\tilde{p}_\alpha = 0. \tag{59}$$

The velocity dependence of $\tilde{L}$, apart from u, comes from the tensor n, as the dipoles are functions of the rest-frame dipoles as well as the velocity. The change of n in terms of the velocity change is a special case of the change of a tensor-valued degree of freedom of the particle in terms of the velocity change. To obtain this dependence, one considers the tensor as a function of the *rest-frame tensor* and the velocity of the particle. The change of the tensor as a result of the velocity change is the change of the tensor when the change of the rest-frame tensor is zero. This is given by the Fermi-Walker transport: the no-change condition for the rest-frame tensor is the condition that the tensor is Fermi-Walker–transported, [15] for example.

For a vector W, one has

$$\Delta_{\mathrm{FW}}W = \Delta W + \frac{u(W\cdot\Delta u) - (u\cdot W)\Delta u}{u\cdot u}, \tag{60}$$

where the Fermi-Walker transport corresponds to the vanishing of the left-hand side. So the change of the internal vector W with the rest-frame vector kept fixed is the following:

$$\Delta W = -\frac{u(W\cdot\Delta u) - (u\cdot W)\Delta u}{u\cdot u}. \tag{61}$$

The change of the 4-velocity u can be expressed in terms of the change of $\boldsymbol{v}$,

$$\Delta u^\nu = \gamma(\delta^\nu_j + c^{-2}u_j u^\nu)\Delta v^j. \tag{62}$$

One then arrives at

$$\Delta W^{\alpha} = c^{-2}\gamma(W_j u^{\alpha} - \delta^{\alpha}_j u_{\nu} W^{\nu})\Delta v^j. \tag{63}$$

Similarly, for the tensor n,

$$\begin{aligned}\Delta n^{\alpha\beta} &= c^{-2}\gamma(n_j{}^{\beta} u^{\alpha} - \delta^{\alpha}_j u_{\nu} n^{\nu\beta} \\ &\quad + n^{\alpha}_j u^{\beta} - \delta^{\beta}_j u_{\nu} n^{\alpha\nu})\Delta v^j. \end{aligned} \tag{64}$$

The coefficient of (Δv^j) in the right-hand side is in fact the partial derivative of $(\Delta n^{\alpha\beta})$ with respect to (Δv^j). So,

$$\begin{aligned}\tilde{p}_{\sigma} &= c^{-2}[(n_{\sigma}{}^{\beta} F_{\alpha\beta} - n_{\alpha}{}^{\beta} F_{\sigma\beta})u^{\alpha} \\ &\quad + q(u_{\sigma} u^{\alpha} A_{\alpha} - u_{\alpha} u^{\alpha} A_{\sigma})]. \end{aligned} \tag{65}$$

$$p_{\sigma} = \left[m - c^{-2}\left(q u^{\alpha} A_{\alpha} + \frac{1}{2} n^{\alpha\beta} F_{\alpha\beta}\right)\right] u_{\sigma} + \tilde{p}_{\sigma}. \tag{66}$$

This is the complete form the momentum of a structured particle in external fields. Although parts of this momentum (in addition to the momentum of a point particle) has been already discussed, to my knowledge the complete form has not been presented yet.

The final result is

$$p_{\sigma} = \mathfrak{m} u_{\sigma} + q A_{\sigma} + \mathfrak{p}_{\sigma}, \tag{67}$$

where

$$\mathfrak{m} = m - c^{-2}\left(\frac{1}{2} n^{\alpha\beta} F_{\alpha\beta}\right), \tag{68}$$

$$\mathfrak{p}_{\sigma} = c^{-2}(n_{\sigma}{}^{\beta} F_{\alpha\beta} - n_{\alpha}{}^{\beta} F_{\sigma\beta})u^{\alpha}. \tag{69}$$

$\mathfrak{m}$ is essentially an effective mass, [16]: $(\mathfrak{m}c^2)$ is the rest-frame energy. (qA) is the usual gauge contribution to the momentum. The quantity $(p - qA)$, which is equal to $(\mathfrak{m}u + \mathfrak{p})$, is the gauge-invariant part of the momentum, just as $(p - qA)$ is the gauge-invariant part of the momentum for point particles. $\mathfrak{p}$ is a nontrivial contribution to the momentum, arising from the structure of the particle. Decomposing these in terms of 3-quantities,

$$\mathfrak{m} = m - c^{-2}\gamma(\boldsymbol{\varpi}\cdot\boldsymbol{E} + \boldsymbol{\mu}\cdot\boldsymbol{B}), \tag{70}$$

$$\mathfrak{p}_0 = -\gamma^2 \boldsymbol{v}\cdot(-\boldsymbol{\varpi}\times\boldsymbol{B} + c^{-2}\boldsymbol{\mu}\times\boldsymbol{E}), \tag{71}$$

$$\begin{aligned}\boldsymbol{\mathfrak{p}} &= \gamma^2(-\boldsymbol{\varpi}\times\boldsymbol{B} + c^{-2}\boldsymbol{\mu}\times\boldsymbol{E}) \\ &\quad + c^{-2}\gamma^2 \boldsymbol{v}\times(\boldsymbol{\varpi}\times\boldsymbol{E} + \boldsymbol{\mu}\times\boldsymbol{B}). \end{aligned} \tag{72}$$

The rest-frame quantities are

$$\mathfrak{m}_{\mathrm{rf}} = m - c^{-2}(\boldsymbol{\varpi}\cdot\boldsymbol{E} + \boldsymbol{\mu}\cdot\boldsymbol{B}), \tag{73}$$

$$\mathfrak{p}_{0\mathrm{rf}} = 0, \tag{74}$$

$$\boldsymbol{\mathfrak{p}}_{\mathrm{rf}} = -\boldsymbol{\varpi}\times\boldsymbol{B} + c^{-2}\boldsymbol{\mu}\times\boldsymbol{E}. \tag{75}$$

It is seen that a particle at rest does have a non-vanishing gauge-invariant momentum.

Concluding remarks. – A particle was studied, which contains a collection of point particles. The charge and current densities of the particle were investigated and expressed in terms of the total charge and the electric and magnetic dipoles. The corresponding action was presented and from which expressions were deduced for the energy and momentum of the particle. It was shown that these contain the so-called hidden momentum. This part in the momentum comes from the internal structure of the particle, and is a result of the dependence of that structure (the dipoles) on the velocity of the particle. It was seen that to obtain that part of the energy and momentum, no special treatment of the structure of the particle is needed: there is no need to consider forces other than the usual Lorentz force to arrive at that.

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Appendix: the transformation properties of the polarizations and the dipoles. – J transforms as a 4-vector. One way to see that, is to note that J is the sum of terms like

$$U^{\alpha}(t,\boldsymbol{r}) = Q v^{\alpha}(t)\delta[\boldsymbol{r} - \boldsymbol{x}(t)], \tag{A.1}$$

or

$$U^{\alpha}(t,\boldsymbol{r}) = Q u^{\alpha}(t)\frac{\delta[\boldsymbol{r} - \boldsymbol{x}(t)]}{\gamma(t)}, \tag{A.2}$$

where $\boldsymbol{x}$ is some position and Q is some Lorentz scalar. Now consider the Lorentz transformation corresponding to the velocity $\boldsymbol{V}$. The corresponding Lorentz factor is denoted by Γ. One has

$$\delta[\boldsymbol{r}' - \boldsymbol{x}'(t')] = \Gamma[1 + c^{-2}\boldsymbol{V}\cdot\boldsymbol{v}(t)]\delta[\boldsymbol{r} - \boldsymbol{x}(t)], \tag{A.3}$$

where r' and $[t', \boldsymbol{x}'(t')]$ are the Lorentz transformeds of r and $[t, \boldsymbol{x}(t)]$, respectively. One also has

$$\frac{\gamma'(t')}{\gamma(t)} = \Gamma[1 + c^{-2}\boldsymbol{V}\cdot\boldsymbol{v}(t)], \tag{A.4}$$

where γ and γ' are the Lorentz factors corresponding to $\boldsymbol{v}$ and $\boldsymbol{v}'$, respectively, and $\boldsymbol{v}'$ is the Lorentz transformed of $\boldsymbol{v}$,

$$\boldsymbol{v}'(t') = \frac{\gamma}{\gamma'}\{\Gamma[\boldsymbol{V} + \boldsymbol{v}_{\parallel}(t)] + \boldsymbol{v}_{\perp}(t)\}. \tag{A.5}$$

So,

$$\frac{\delta[\boldsymbol{r}' - \boldsymbol{x}'(t')]}{\gamma'(t')} = \frac{\delta[\boldsymbol{r} - \boldsymbol{x}(t)]}{\gamma(t)}. \tag{A.6}$$

This, combined with the facts that u transforms as a 4-vector and Q is a Lorentz scalar, shows that U transforms as a 4-vector. As a result, J transforms as a 4-vector.

The current and charge densities corresponding to a structured particle could be decomposed into those corresponding to a point particle and a so-called polarization current and charge density, J_{p},

$$J(t,\boldsymbol{r}) = q v(t)\delta[\boldsymbol{r} - \boldsymbol{x}(t)] + J_{\mathrm{p}}(t,\boldsymbol{r}), \tag{A.7}$$

where

$$J_{\mathrm{p}}^{0} = -\boldsymbol{\nabla} \cdot \boldsymbol{\Pi}, \tag{A.8}$$

$$\boldsymbol{J}_{\mathrm{p}} = \dot{\boldsymbol{\Pi}} + \boldsymbol{\nabla} \times \boldsymbol{M}. \tag{A.9}$$

The point particle current is of the form (A.1), so it transforms as a 4-vector. Hence the polarization current should transform as a 4-vector as well. In terms of the polarization tensor N, one has

$$J_{\mathrm{p}}^{\alpha} = \partial_{\beta} N^{\alpha\beta}. \tag{A.10}$$

So the polarization tensor transforms as a 4-tensor.

Finally, taking into account only the first nonzero terms of the polarization tensor corresponding to a small particle, one has

$$N^{\alpha\beta}(t, \boldsymbol{r}) = n^{\alpha\beta}(t) \frac{\delta[\boldsymbol{r} - \boldsymbol{x}(t)]}{\gamma(t)}, \tag{A.11}$$

and as the second factor on the right-hand side transforms as a Lorentz scalar, n transforms as a 4-tensor.

REFERENCES

[1] Jackson J. D., *Classical Electrodynamics*, 3rd edition (John Wiley & Sons) 1999.
[2] Einstein A. and Laub J., *Ann. Phys. (Leipzig)*, **331** (1908) 532.
[3] Einstein A. and Laub J., *Ann. Phys. (Leipzig)*, **331** (1908) 541.
[4] Bedford D. and Krumm P., *Am. J. Phys.*, **54** (1986) 1036.
[5] Namias V., *Am. J. Phys.*, **57** (1989) 171.
[6] Vaidman L., *Am. J. Phys.*, **58** (1990) 978.
[7] Babson D., *Am. J. Phys.*, **77** (2009) 826.
[8] Mansuripur M., *Phys. Rev. Lett.*, **108** (2012) 193901.
[9] Vanzella D. A. T., *Phys. Rev. Lett.*, **110** (2013) 089401.
[10] Barnett S. M., *Phys. Rev. Lett.*, **110** (2013) 089402.
[11] Saldanha P. L., *Phys. Rev. Lett.*, **110** (2013) 089403.
[12] Khorrami M., *Phys. Rev. Lett.*, **110** (2013) 089404.
[13] Oliveira-Filho J. S. and Saldanha P. L., *Phys. Rev. A*, **92** (2015) 052107.
[14] Cameron R. P., *Phys. Rev. A*, **97** (2018) 042125.
[15] Rindler W., *Relativity: Special, General, and Cosmological*, 2nd edition (Oxford University Press) 2006.
[16] Khorrami M., *Int. J. Mod. Phys. A*, **34** (2019) 1950043.