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Unitarity of the time-evolution and observability of non-Hermitian Hamiltonians for time-dependent Dyson maps

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Abstract

We present an strategy for the derivation of a time-dependent Dyson map which ensures simultaneously the unitarity of the time evolution and the observability of the whole time-dependent non-Hermitian Hamiltonian or parts of it. The time-dependent Dyson map is derived through a constructed Schrödinger-like equation governed, in one case, by the non-Hermitian Hamiltonian itself or, in another case, by its parts. In the former case, when the whole non-Hermitian Hamiltonian is considered, our scheme ensures the time-independence of the metric operator despite the time-dependence of the Dyson map, a necessary condition for the observability of the non-Hermitian Hamiltonian. In the later case, however, when parts of the non-Hermitian Hamiltonian is considered, our method ensures the simultaneous time-dependence of the Dyson map and the metric operator. In this latter case what is ensured is the observability of the remaining part of the non-Hermitian Hamiltonian that was not chosen for the derivation of the Dyson map. Illustrative examples, for both cases, are derived from a driven non-Hermitian Harmonic oscillator.

Keywords: Pseudo-Hermiticity, time-dependent metric operator, time-dependent dyson map

(Some figures may appear in colour only in the online journal)

1. Introduction

Since the work by Bender and Boettcher [1] and further developments by Mostafazadeh [2], pseudo-Hermitian $\mathcal{PT}$-symmetric Hamiltonians have been extensively studied. While in the former reference it was suggested that Hamiltonians invariant under space-time reflection symmetry ($\mathcal{PT}$-symmetry) can have real spectra, in the latter the notion of pseudo-Hermiticity was introduced, establishing the grounds for treating non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians using time-independent metric operators [3]. Aiming to extend the scope of Hermitian quantum mechanics, the steps towards the deepening of our understanding of non-Hermitian systems has since been taken in virtually all fields of physics [3]. More recently, $\mathcal{PT}$-symmetry (PTS) and PTS breaking has been investigated in a variety of systems, such as waveguides [4], optical lattices [5] and optomechanics [6]. Moreover, a variety of phenomena such as disorder [7], localization [8], chaos [9] and solitons [10] have been investigated within $\mathcal{PT}$-symmetric systems and a linear response theory for a pseudo-Hermitian system-reservoir interaction [11] has been recently developed.

Despite the overall consensus on handling pseudo-Hermitian Hamiltonians through time-independent metric operators [3], controversies emerged regarding the generalization to time-dependent (TD) metric operators [12, 13]. Although it has been demonstrated that a TD metric operator can not ensure the unitarity of the time-evolution simultaneously with the observability of the Hamiltonian [12], some authors have disputed this claim [13], failing however to ensure the unitarity of time evolution by insisting on the observability of the Hamiltonian. A contribution has been recently presented in [14] for dealing with TD metric operators which, although in agreement with

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the theorem in [12], goes a step beyond; It has been demonstrated that a TD Dyson equation and a TD pseudo-Hermiticity relation (as first introduced in [14]) can be solved consistently at the cost of rendering the non-Hermitian Hamiltonian to be a nonobservable operator, *but showing that any other observable in the non-Hermitian system is derived in complete analogy with the time-independent scenario*. Non-trivial solutions to the TD Dyson equation and the proposed TD pseudo-Hermiticity relation have been presented, starting with a non-Hermitian linearly driven harmonic oscillator and a spin chain [14], and then going to the TD Swanson model [15] and a generalization of the TD Swanson model including the linear amplification term [16]. Many other interesting contributions to the subject of TD non-Hermitian Hamiltoinans are found in [17].

In the present contribution, however, we present a strategy to go even beyond [14], enabling us to account for the unitarity of the time-evolution simultaneously with the observability of a TD non-Hermitian Hamiltonian. To this end a Schrödinger-like equation is constructed from which we derive the TD Dyson map (*i*) from the TD pseudo-Hermitian Hamiltonian itself or (*ii*) from parts of it. In the former case *i*) our scheme remarkably ensures a time-independent metric operator despite the time-dependence of the Dyson map, a necessary condition for the observability of the whole pseudo-Hermitian Hamiltonian [12]. The distinction established between the time-dependence of the Dyson map and that of the metric operator is therefore central in that: *although we are in agreement with the main premise in* [12, 14]*, that a time-independent metric operator is needed for assuring the unitarity of the time evolution simultaneously with the observability of the pseudo-Hermitian Hamiltonian, here a TD Dyson map is considered, and this is an important point since for a TD non-Hermitian Hamiltonian, a time-independent Dyson map may result in severe restrictions on the time-dependent parameters of the Hamiltonian.* In the latter case *ii*) where only parts of the non-Hermitian Hamiltonian is considered for the derivation of the Dyson map, our method ensures the simultaneous time-dependence of the Dyson map and the metric operator, without conflicting with [12]. In fact, what is ensured in this case is the observability of the remaining part of the non-Hermitian Hamiltonian that was not chosen for the derivation of the Dyson map, and not the whole non-Hermitian Hamiltonian. Therefore, this latter case enabled us to expand the scenario presented by Mostafazadeh [12] for treating TD non-Hermitian Hamiltonian: If in the case *i*), where the observability of the non-Hermitian Hamiltonian is assured, we were in agreement with the Mostafazadeh theorem by assuring also a time-independent metric operator, in case *ii*) we expand the scenario presented by Mostafazadeh [12], advancing a new treatment for TD non-Hermitian Hamiltonians in which both the Dyson map and the metric are TD operators.

Our Schrödinger-like equation applies, however, to a more general scenario than the one for which it was constructed; apart from the TD non-Hermitian Hamiltonians, it also applies to time independent non-Hermitian Hamiltonians, in the latter case recovering exactly the standard procedure for handling non-Hermitian quantum mechanics as we show below. It also helps with unitary transformations within Hermitian quantum mechanics, providing us with the transformation operator from the Hamiltonian itself. As an illustration of our method we revisit the non-Hermitian linearly driven harmonic oscillator, deriving the TD Dyson map from the constructed Schrödinger-like equation and showing the time-independence of the associated metric operator. Finally, after deriving an eigenvalue equation for a TD non-Hermitian system, we analyze the $\mathcal{PT}$-symmetry breaking process.

2. A Schrödinger-like equation for the evolution of the TD Dyson map

Starting with a brief review of the developments in [14], we consider a non-Hermitian TD Hamiltonian $H(t)$ associated with the Schrödinger equation $i\partial_t|\psi(t)\rangle = H(t)|\psi(t)\rangle$. A TD Dyson map $\eta(t)$ thus leads to the TD Dyson relation, i.e., the transformed Hamiltonian

$$h(t) = \eta(t)H(t)\eta^{-1}(t) + i[\partial_t\eta(t)]\eta^{-1}(t), \tag{1}$$

which generates the evolution of the equation $i\partial_t|\phi(t)\rangle = h(t)|\phi(t)\rangle$, where $|\phi(t)\rangle = \eta(t)|\psi(t)\rangle$. Due to the gauge-like term in equation (1) the non-Hermitian $H(t)$ and its Hermitian counterpart $h(t)$ are no longer related by means of a similarity transformation, resulting in that $H(t)$ is not a self-adjoint operator and, therefore, not observable. The Hermiticity of $h(t)$ leads however, as referred to in [14], to the *TD pseudo-Hermiticity relation*

$$H^\dagger(t)\rho(t) - \rho(t)H(t) = i\partial_t\rho(t), \qquad \rho(t) = \eta^\dagger(t)\eta(t), \tag{2}$$

which replaces the usual relation $H^\dagger\rho = \rho H$ for a time-independent metric. Assuming $\rho(t)$ to be a positive-definite TD metric operator, it is straightforward to verify that the generalized equation (2) leads to the expected relation between the TD probability densities in the Hermitian and non-Hermitian systems, given by

$$\langle\psi(t)|\tilde{\psi}(t)\rangle_{\rho(t)} = \langle\psi(t)|\rho(t)|\,\tilde{\psi}(t)\rangle = \langle\phi(t)|\tilde{\phi}(t)\rangle. \tag{3}$$

From the above observation one concludes, as in [14], that even for TD Dyson maps, any observable $o(t)$ in the Hermitian system possesses a counterpart $O(t)$ in the non-Hermitian one—except for the non-Hermitian Hamiltonian itself—given by

$$O(t) = \eta^{-1}(t)o(t)\eta(t), \tag{4}$$

in complete analogy for time-independent Dyson maps.

2.1. Case i): A time-independent metric from a TD Dyson map

Our strategy to restore a similarity transformation from equation (1) and, consequently, to restore the observability of $H(t)$—thus going beyond [14]—, is to impose the gauge-like term $i[\partial_t\eta(t)]\eta^{-1}(t)$ equal to to the operator $\eta(t)H(t)\eta^{-1}(t)$, thus leading to the Schrödinger-like equation

$$i\partial_t\eta(t) = \eta(t)H(t), \tag{5}$$

which enable us to compute the TD Dyson map from the non-Hermitian $H(t)$ itself. The equation (5) ensures the similarity transformation

$$h(t) = 2\eta(t)H(t)\eta^{-1}(t), \tag{6}$$

and by demanding $h(t)$ to be Hermitian, we derive the pseudo-Hermiticity relation

$$H^{\dagger}(t)\rho(t) = \rho(t)H(t), \tag{7}$$

where the factor 2 is obviously only a scale connecting the instantaneous eigenvalues of $h(t)$ and $H(t)$. From equations (5) and (7) we immediately verify the time-independence of the metric operator, $\partial_t \rho(t) = 0$, despite the time-dependence of the Dyson map. The relation (7) helps us to define the initial condition $\eta(t_0)$ for the exact solution of equation (5), given by

$$\eta(t) = \eta(t_0)T\exp\left[-i\int_{t_0}^{t} d\tau H(\tau)\right], \tag{8}$$

where T denotes the time ordering operator. Except for the initial condition $\eta(t_0)$, $\eta(t)$ is a determinist operator following from the non-Hermitian $H(t)$. In short, under the constructed Schrödinger-like equation (5) the TD Dyson equation (1) and pseudo-Hermiticity relation (2) reduce to their simplified forms in equations (6) and (7) which ensures the unitarity of the time evolution governed by $H(t)$ simultaneously with the observability of such a non-Hermitian Hamiltonian. Finally, with the unitarity of the time-evolution assuming the form of equation (3), the matrix elements of the observables in equation (4) becomes

$$\langle\psi(t)|\ O(t)|\tilde{\psi}(t)\rangle_{\rho(t)} = \langle\phi(t)|\ o(t)|\tilde{\phi}(t)\rangle. \tag{9}$$

In the appendix we present an alternative demonstration, using only equations (8) and (7), that $\rho(t) = \rho(t_0)$, with the initial value $\eta(t_0)$ following from the parameters defining $H(t)$.

2.2. Case ii): Simultaneous TD Dyson map and metric operator

If the Schrödinger-like equation (5) allowed us to bypass the constraints imposed by the Mostafazadeh's theorem [12]—in the sense that we restore the observability of a TD non-Hermitian Hamiltonian—by presenting a TD Dyson map associated with a time-independent metric operator, we now go even further presenting an approach that enable us to derive simultaneous TD Dyson map and metric operator. By describing the TD non-Hermitian hamiltonian as $H(t) = H_0(t) + V(t)$, the Schrödinger-like equation (5) may be substituted by

$$i\partial_t \eta(t) = -\eta(t)H_0(t), \tag{10}$$

or

$$i\partial_t \eta(t) = -\eta(t)V(t), \tag{11}$$

both relations leading respectively to the hamiltonians

$$h(t) = \eta(t)V(t)\eta^{-1}(t), \tag{12a}$$

$$h(t) = \eta(t)H_0(t)\eta^{-1}(t), \tag{12b}$$

the scale factor 2 appearing in equation (6) being now absent. From equations (12*a*) and (12*b*), where the evolution of the Dyson map is governed by $H_0(t)$ or $V(t)$, we obtain the pseudo-Hermiticity relations for $V(t)$ or $H_0(t)$, respectively:

$$V^{\dagger}(t)\rho(t) = \rho(t)V(t), \tag{13a}$$

$$H_0^{\dagger}(t)\rho(t) = \rho(t)H_0(t). \tag{13b}$$

Evidently, equations (11) and (13*b*) do not apply in the rather usual case where $H_0(t)$ is a diagonal Hermitian operator modelling the system of interest; likewise, equations (10) and (13*a*) do not apply in the case where $V(t)$ is a diagonal Hermitian operator acting over the system. The exact solution of equations (10) and (11) are given by

$$\eta(t) = \eta(t_0)T\exp\left[i\int_{t_0}^{t} d\tau H_0(\tau)\right], \tag{14a}$$

$$\eta(t) = \eta(t_0)T\exp\left[i\int_{t_0}^{t} d\tau V(\tau)\right]. \tag{14b}$$

The essential feature of this approach is that the metric, as well as the Dyson map, is a time-dependent operator, given by

$$\partial_t \rho(t) = i[\rho(t)H_0(t) - H_0^{\dagger}(t)\rho(t)],$$

or

$$\partial_t \rho(t) = i[\rho(t)V(t) - V^{\dagger}(t)\rho(t)],$$

when the Dyson map is derived from equation (10) or (11).

We observe that even in Hermitian quantum mechanics the observability of a time-dependent Hamiltonian is a sensitive problem, the Hamiltonian acting essentially as the generator of model dynamics. When considering, for example, the TD Hermitian $H(t) = H_0(t) + V(t)$, $H_0(t)$ representing the system of interest and $V(t)$ an external influence over it, the energy of the system is given by $U(t)H_0(t)U^{-1}(t)$, the term $V(t)$ contributing to the evolution operator $U(t) = T\exp\left[i\int_{t_0}^{t} d\tau H(\tau)\right]$ (see [18]).

Before presenting our illustrative examples, we summarize our developments: If Mostafazadeh's theorem [12] raised the issue of the lack of observability of TD non-Hermitian Hamiltonians with TD metric operators, and in [14] it was demonstrated that this issue does not affect other observables, in the present work we reestablish the observability of TD non-Hermitian Hamiltonians for a TD Dyson map. As follows from equation (4), the observables associated with the pseudo-Hermitian systems are generally composed of superpositions of canonically conjugated observables associated with the Hermitian ones. This poses an additional difficulty to the pseudo-Hermitian systems since they demand measurements of canonically conjugated variables [19, 20]. Aside from this additional difficulty, however, we want the observability of TD non-Hermitian Hamiltonians to be as much a sensitive problem as it currently is with TD Hermitian one, but nothing less.

3. Illustrative examples

3.1. Case i)

In order to illustrate our method for the case *i*) where the Dyson map is derived from the whole TD non-Hermitian Hamiltonian, we start from a Hamiltonian of the form

$$H(t) = H_0(t) + \kappa V(t), \qquad [H_0(t), H_0(t')] = 0, \quad (15)$$

where $H_0(t)$ stands for the usual free Hamiltonian and $\kappa V(t)$ stands for a non-Hermitian interaction with a real dimensionless strength κ, to be considered as a perturbation parameter. By acting de operator $\exp\left(i\int_{t_0}^{t} d\tau H_0(\tau)\right)$ on the right side of the Schrödinger-like equation (5), the TD Dyson map coming from equations (8) and (15) is thus given by

$$\eta(t) = \eta(t_0)\left[T\exp\left(-i\kappa\int_{t_0}^{t} d\tau\tilde{V}(\tau)\right)\right] \times \exp\left(-i\int_{t_0}^{t} d\tau H_0(\tau)\right), \quad (16)$$

where $\tilde{V}(t) = \exp\left(-i\int_{t_0}^{t} d\tau H_0(\tau)\right)V(t)\exp\left(i\int_{t_0}^{t} d\tau H_0(\tau)\right)$. For a TD harmonic oscillator under a TD non-Hermitian linear amplification, $H(t)$ is given by

$$H_0(t) = \omega(t)a^{\dagger}a, \qquad V(t) = \alpha(t)a + \beta(t)a^{\dagger}, \quad (17)$$

where we are assuming $\omega(t), \alpha(t), \beta(t) \in \mathbb{C}$. Evidently, $H(t)$ is not Hermitian when $\omega(t) \notin \mathbb{R}$ or $\alpha(t) \neq \beta^*(t)$, and it becomes $\mathcal{PT}$-symmetric when demanding $\omega(t)$ to be an even function in t or a generic function of *it*, simultaneously with demanding $\alpha(t)$, $\beta(t)$ to be odd functions in t or pure-imaginary generic functions of *it*.

In order to determine the Dyson map given by equation (16) we first consider the same ansatz as that in [14] for $\eta(t_0) = \exp[\gamma(t_0)a + \lambda(t_0)a^{\dagger}]$; we then compute the time-independent complex parameters γ and λ from the pseudo-Hermiticity relation (7) instead of the similarity transformation (6), avoiding the need for the perturbation expansion of the time-ordering operator in the TD Dyson map $\eta(t)$. The relation $H^{\dagger}(t) = \rho(t_0)H(t)\rho^{-1}(t_0)$, coming from equation (7), thus demands the functions $\omega(t)$ and $\alpha(t)\beta(t)$ to be real and $\gamma(t_0) + \lambda^*(t_0) = \kappa[\beta^*(t) - \alpha(t)]/\omega(t)$, such that $[\gamma^*(t_0) + \lambda(t_0)]\alpha(t) \in \mathbb{R}$. Without loss of generality we may assume $\lambda(t_0) = \gamma^*(t_0)$ such that $\gamma(t_0) = \kappa[\beta^*(t) - \alpha(t)]/2\omega(t)$ and $\gamma^*(t_0)\alpha(t) \in \mathbb{R}$. With the TD functions delimited in this way and guaranteeing the Hermiticity of $h(t)$, we then use the similarity transformation (6) to compute

$$h(t) = 2[\omega(t)a^{\dagger}a + u(t)a + u^*(t)a^{\dagger} + f(t)], \quad (18)$$

and $\tilde{V}(\tau) = \alpha(t)e^{i\chi(t)}a + \beta(t)e^{-i\chi(t)}a^{\dagger}$, with $\chi(t) = \int_{t_0}^{t}\omega(\tau)d\tau$.Evidently, the similarity transformation, and consequently the TD Dyson map, is as important to the problem as the pseudo-Hermiticity relation, and so the time-independent metric operator.Considering the perturbation parameter $\kappa \ll 1$, we have also verified in the appendix (up to first order of perturbation to avoid extending the already lengthy calculations), that $\rho(t) = \eta^{\dagger}(t_0)\eta(t_0)$, now without directly using equation (7), but using instead the restrictions imposed by this equation on the TD parameters of the Hamiltonian (17). Moreover, we compute the TD functions $u(t) = \omega(t)[\gamma(t_0) - i\kappa\tilde{\alpha}(t)] + \kappa\alpha(t)e^{i\chi(t)}$ and $f(t) = |u(t)|^2/\omega(t)$, where $\tilde{\alpha}(t) = \int_{t_0}^{t} d\tau\alpha(\tau)e^{i\chi(\tau)}$ and $\tilde{\beta}(t) = \int_{t_0}^{t} d\tau\beta(\tau)e^{-i\chi(\tau)}$.

Finally, we observe that the scale factor 2 in the Hamiltonian (18), is directly associated with the energy of the pumped harmonic oscillator, given by $2\omega(t)U(t)a^{\dagger}aU^{-1}(t)$, with $U(t) = T\exp\left[i\int_{t_0}^{t} d\tau h(\tau)\right]$.

3.1.1. Solutions of the Schrödinger equation for the pseudo-Hermitian Hamiltonian.

Using the Lewis and Riesenfeld invariants [21], as done in [18, 22], the basis state solutions of the Schrödinger equation governed by Hamiltonian $h(t)$ are given by the TD displaced number states

$$|\phi_m(t)\rangle = e^{i\Phi_m(t)}D[\theta(t)]|m\rangle; \qquad m = 0, 1, 2, \ldots, \quad (19)$$

where $\theta(t)$ follows from the equation $i\dot{\theta}(t) = 2\omega(t)\theta(t) + u^*(t)$, whereas the TD Lewis and Riesenfeld phases are given by

$$\Phi_m(t) = -\int_{t_0}^{t} d\tau\{m\varpi(\tau) + f(\tau) + \mathrm{Re}[u(\tau)\theta(\tau)]\}. \quad (20)$$

It thus follows that $|\phi_m(t)\rangle = V(t, t_0)|m\rangle$, with the evolution operator $V(t, t_0) = \Upsilon(t)D[\theta(t)]R[\chi(t)]$, the rotation $R[\chi(t)] = \exp[-i2\chi(t)a^{\dagger}a]$, and the overall phase $\Upsilon(t) = \exp\left(-i\int_{t_0}^{t} d\tau\{f(\tau) + \mathrm{Re}[u(\tau)\theta(\tau)]\}\right)$. Consequently, $|\psi_m(t)\rangle = \eta^{-1}(t)|\phi_m(t)\rangle$ and for a generic superposition $|\phi(t)\rangle = \sum_m c_m|\phi_m(t)\rangle$, the generic solution of the Schrödinger equation for the pseudo-Hermitian $H(t)$ is given by

$$|\psi(t)\rangle = \eta^{-1}(t)|\phi(t)\rangle = \eta^{-1}(t)U(t, t_0)|\phi(t_0)\rangle, \quad (21)$$

with

$$\begin{aligned} U(t, t_0) &= V(t, t_0)V^{\dagger}(t_0, t_0) \\ &= \Upsilon(t)D[\theta(t)]R[\chi(t)](D[\theta(t_0)])^{-1}. \end{aligned} \quad (22)$$

3.1.2. Observables.

The observables associated with the pseudo-Hermitian $H(t)$, given by equation (4), are easily computed for the quadratures $x_\ell = [a^{\dagger} - (-1)^{\ell}a]/2(i)^{\ell-1}$, $\ell = 1, 2$, leading to the operators

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \cos[\chi(t)] & -\sin[\chi(t)] \\ \sin[\chi(t)] & \cos[\chi(t)] \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + i\begin{pmatrix} -\mathrm{Im}[\gamma(t_0)] + \kappa[\tilde{\alpha}(t) - \tilde{\beta}(t)]/2 \\ \mathrm{Re}\,\gamma(t_0) - i\kappa[\tilde{\alpha}(t) + \tilde{\beta}(t)]/2 \end{pmatrix} \quad (23)$$

where the first term on the *rhs* stands for the unperturbed diagonal Hamiltonian $H_0(t)$ whereas the second term stands for the perturbation correction. Regarding the Hamiltonian $H(t)$

itself, its matrix elements in Fock space states are given by

$$\langle\psi_m(t)|\ H(t)|\psi_n(t)\rangle_{\rho(t_0)}$$
$$= \langle\phi_m(t)|\ \frac{h(t)}{2}|\phi_n(t)\rangle = \langle m|\ V^{\dagger}(t,\, t_0)\frac{h(t)}{2}V(t,\, t_0)|n\rangle$$
$$= (\mathcal{A}(t)\delta_{mn} + \sqrt{n+1}\,\mathcal{B}(t)\delta_{m,n+1}$$
$$+ \sqrt{n}\,\mathcal{B}^{*}(t)\delta_{m,n-1})e^{i2\chi(m-n)} \tag{24}$$

where $\mathcal{A}(t) = \omega(t)[n + |\theta(t)|^2] + 2\,\mathrm{Re}\,[u(t)\theta(t)] + f(t)$ and $\mathcal{B}(t) = \omega(t)\theta(t) + u^{*}(t)$.

$\mathcal{PT}$-symmetry breaking. In spite of the time-dependence of the Hermitian Hamiltonian (18), we successfully derive an eigenvalue equation for this operator by defining, as in [22], the TD operators $b(t) = a + \xi^{*}(t)$ and $b^{\dagger}(t) = a^{\dagger} + \xi(t)$, associated with the relations $b^{\dagger}b|\zeta_m(t)\rangle = m|\zeta_m(t)\rangle$, $b|\zeta_m(t)\rangle = \sqrt{m}|\zeta_{m-1}(t)\rangle$, and $b^{\dagger}|\zeta_m(t)\rangle = \sqrt{m+1}|\zeta_{m+1}(t)\rangle$, where the wave vector $|\zeta_m(t)\rangle = D[-\xi^{*}(t)]|m\rangle$ stands for the displaced Fock states with $\xi(t) = u(t)/\omega(t)$. Now, up to sencond order of perturbation, in order to allow us to analyze the $\mathcal{PT}$-symmetry breaking, the operators $b(t)$and $b^{\dagger}(t)$ help us to rewrite equation (18)—with unchanged $u(t)$ but $f(t) = [|u(t)|^2 - \kappa^2\alpha(t)\beta(t)]/\omega(t)$—in the form $h(t) = 2\omega(t)b^{\dagger}b - 2\kappa^2\alpha(t)\beta(t)/\omega(t)$, thus leading to the TD eigenvalue equation

$$h(t)|\zeta_m(t)\rangle = \mathcal{E}_m(t)|\zeta_m(t)\rangle, \tag{25}$$

with $\mathcal{E}_m(t) = 2\omega(t)m - 2\kappa^2\alpha(t)\beta(t)/\omega(t)$. From equation (25) and the similarity transformation $h(t) = 2\eta(t)H(t)\eta^{-1}(t)$ we obtain (apart from an irrelevant factor 2)

$$H(t)[\eta^{-1}(t)|\zeta_m(t)\rangle] = \mathcal{E}_m(t)[\eta^{-1}(t)|\zeta_m(t)\rangle], \tag{26}$$

showing—as usual in the case of time-independent non-Hermitians Hamiltonians and Dyson maps—that the pseudo-Hermitian $H(t)$ and its Hermitian counterpart, are isospectral partners. From the eigenvalue equation (26) its is clear that the $\mathcal{PT}$-symmetry breaking occurs if $\omega(t)$ and/or $\alpha(t)\beta(t)$ cease to be real, resulting in the loss of the Hermiticity of $h(t)$ [3].

The eigenstates and the solutions of the Schrödinger equation for $h(t)$ are connected through the relation $|\phi_m(t)\rangle = \mathcal{U}(t,\, t_0)|\zeta_m(t)\rangle$ with $\mathcal{U}(t,\, t_0) = V(t,\, t_0)D^{\dagger}[-\xi^{*}(t)]$ and it is not difficult to find that the eigenstates of $h(t)$ are the solutions of the Schrödinger equation governed by the Hamiltonian $\mathcal{H}(t) = \mathcal{U}^{\dagger}(t,\, t_0)h(t)\mathcal{U}(t,\, t_0) + i\hbar\,\mathcal{U}^{\dagger}(t,\, t_0)\partial_t\mathcal{U}(t,\, t_0)$.

3.2. Case (ii)

Start again from Hamiltonian (17), we now consider the Dyson map (14*a*), which reduces to

$$\eta(t) = \eta(t_0)e^{i\chi(t)a^{\dagger}a}, \tag{27}$$

with $\chi(t) = \int_{t_0}^{t}\omega(\tau)d\tau$ and $\omega(t) \in \mathbb{C}$. Assuming $\eta(t_0) = \exp[\gamma(t_0)a^{\dagger}a]$, we then compute the time-independent complex parameter γ from the pseudo-Hermiticity relation (13*a*), leading us with the restriction $\varphi_{\beta} = -\varphi_{\alpha}$ and

$$\mathrm{Re}\,[\gamma(t_0)] = \mathrm{Im}\,[\chi(t)] - \ln\sqrt{\frac{|\beta(t)|}{|\alpha(t)|}}$$

With the TD functions delimited in this way and guaranteeing the Hermiticity of $h(t)$, we assume a real $\gamma(t_0)$ and use the similarity transformation (6) to compute

$$h(t) = \kappa[\alpha(t)e^{-i[\chi(t)-i\gamma(t_0)]}a + \alpha^{*}(t)e^{i[\chi^{*}(t)+i\gamma^{*}(t_0)]}a^{\dagger}]. \tag{28}$$

From equation (22) we verify that the evolution operator for the state vector governed by the Hamiltonian (28), i.e., $|\phi(t)\rangle = U(t,\, t_0)|\phi(t_0)\rangle$, is given by

$$U(t,\, t_0) = \Upsilon(t)D[\theta(t)]R[\chi(t)](D[\theta(t_0)])^{-1}, \tag{29}$$

with $i\dot{\theta}(t) = \kappa\alpha^{*}(t)e^{i[\chi^{*}(t)+i\gamma^{*}(t_0)]}/2$ and $\Upsilon(t) = \exp\left(-\frac{i\kappa}{2}\int_{t_0}^{t}d\tau\,\mathrm{Re}\,[\alpha(\tau)e^{-i[\chi(\tau)-i\gamma(t_0)]}\theta(\tau)]\right)$. Finaly, the observables associated with the pseudo-Hermitian $V(t)$, given by equation (4), are easily computed for the quadratures x_{ℓ}, leading to the operators

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} C_1(t) & C_2(t) \\ -C_2(t) & C_1(t) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{30}$$

where

$$C_1(t) = \cosh[\gamma(t_0)]\cos[\chi(t)] + i\sinh[\gamma(t_0)]\sin[\chi(t)],$$
$$C_2(t) = \cosh[\gamma(t_0)]\sin[\chi(t)] - i\sinh[\gamma(t_0)]\cos[\chi(t)].$$

We stress that schemes for simultaneous measurement os canonically conjugate varaibles has been discussed in the literature [19, 20].

4. On the generality of the Schrödinger-like equation

The Schrödinger-like equation (5) can be taken as a general procedure for the derivation of Dyson maps, even for time-independent non-Hermitian Hamiltonians. In fact, for time-independent H, all the expressions, from equation (1) to (9), remain valid except that the time ordering operator must be removed from equation (8), thus leading to $\eta(t) = \eta(t_0)\exp[-iH(t-t_0)]$ and, consequently, to a time-independent hermitian $h(t_0) = 2\eta(t)H\eta^{-1}(t) = 2\eta(t_0)H\eta^{-1}(t_0)$. We simply recover the time-independent scenario for non-Hermitian quantum mechanics. Even more generally, the strategy for the derivation of the Dyson map is not limited to the non-Hermitian quantum mechanics; it can be used when two Hermitian Hamiltonians are connected through a unitary transformation (instead of the non-unitary Dyson map) in the standard form $\tilde{h}(t) = \tilde{U}(t)\tilde{H}(t)\tilde{U}^{\dagger}(t) + i[\partial_t\tilde{U}(t)]\tilde{U}^{\dagger}(t)$. By defining the Schrödinger-like equation $i\partial_t\tilde{U}(t) = \tilde{U}(t)H(t)$, leading to the solution $\tilde{U}(t) = \tilde{U}(t_0)T\exp\left[-i\int_{t_0}^{t}d\tau\tilde{H}(\tau)\right]$, the relation between the Hamiltonians reduces to $\tilde{h}(t) = 2\tilde{U}(t)\tilde{H}(t)\tilde{U}^{\dagger}(t)$, thus simplifying the form of $\tilde{h}(t)$ by eliminating the need for a (not always easy to derive) Gauss decomposition for the time derivative of the operator $\tilde{U}(t)$.

Regarding the Schrödinger-like equation (10) in the case when H_0 happens to be a time-independent operator, we obtain the Dyson map $\eta(t) = \eta(t_0)\exp[iH_0(t-t_0)]$ and the pseudo-Hermiticity relation $h(t_0) = \eta(t_0)\exp[iH_0(t-t_0)]V(t)\exp[-iH_0(t-t_0)]\eta^{-1}(t_0)$. In the simplified case where

$\eta(t) = \exp[iH_0(t - t_0)]$, the pseudo-Hermiticity relation reduces to a kind of interaction picture in that $h(t_0) = \exp[iH_0(t - t_0)]V(t)\exp[-iH_0(t - t_0)]$, thus strengthening the general chacter of Schrödinger-like equations here introduced. For the Schrödinger-like equation (11) when V happens to be a time-independent operator, it follows that $\eta(t) = \eta(t_0)\exp[iV(t - t_0)]$ and $h(t_0) = \eta(t_0)\exp[iV(t - t_0)]H_0(t)\exp[-iV(t - t_0)]\eta^{-1}(t_0)$. When we choose the simplified form $\eta(t) = \exp[iV(t - t_0)]$ we then derive what appears to be an kind of inverted interaction representation, $h(t_0) = \exp[iV(t - t_0)]H_0(t)\exp[-iV(t - t_0)]$, in which instead of filtering the action of the free term H_0, we filter the interaction V.

We thus conclude that the Schrödinger-like equation can indeed be used as a general procedure for the derivation of TD Dyson maps with associated time-independent or even TD metric operators—i.e., within and outside Mostafazadeh́s premisses in [12]—, thus ensuring simultaneously the unitarity of the time evolution and the observability of a pseudo-Hermitian Hamiltonian or parts of it.

5. Conclusion

The main concern of the present contribution started with a theorem by Mostafazadeh [12] demonstrating that a TD metric operator can not ensure the unitarity of the time-evolution simultaneously with the observability of the Hamiltonian. As already stressed above, working in a scenario where TD metric operators are considered, in [14] it has been demonstrated that the TD Dyson equation and pseudo-Hermiticity relation can be solved consistently at the cost of rendering the non-Hermitian Hamiltonian to be a non-observable quantity in agreement with [12]. Therefore, in complete analogy to the time-independent scenario, where a time-independent Dyson map is used, it follows from [14] that any observable $o(t)$ in the Hermitian system possesses a counterpart $O(t)$ in the non-Hermitian system, given by equation (4), even though the Hamiltonian is not an observable.

Here we have presented two diferent schemes to ensure simultaneously the unitarity of the time evolution and the observability of a quasi-Hermitian Hamiltonian or parts of it.

i) Disconnecting the time-dependence of the Dyson map from that of the metric operator, we first construct a Schrödinger-like equation, governed by the non-Hermitian Hamiltonian itself, from which we derive a TD Dyson map which remarkably leads to a time-independent metric operator. Whereas the time-independence of the metric operator ensures the pseudo-Hermiticity relation and then the unitarity of the time evolution simultaneously with the observability of a pseudo-Hermitian Hamiltonian, the time-dependence of the Dyson map is an important demand since for a TD non-Hermitian Hamiltonian, a time-independent Dyson map is a rather restrictive choice. If this first scheme is within the premises of Mostafazadeh́s theorem, the second another scheme is outside these premises:

ii) By constructing a Schrödinger-like equation governed not by the non-Hermitian Hamiltonian itself but by parts of it, we them assure the time-dependence of both the Dyson map and the metric operator. In this way we ensure simultaneously the unitarity of the time evolution and the observability that remaining part of the non-Hermitian Hamiltonian that was not chosen for the derivation of the Dyson map.

We have shown that our Schrödinger-like equations applies for the derivation of a TD Dyson map either from a TD or a time-independent non-Hermitian Hamiltonian, in the latter case recovering exactly the standard procedure for time-independent non-Hermitian quantum mechanics. We have, in addition, presented illustrative examples starting from a non-Hermitian Hamiltonian describing a harmonic oscillator with a TD frequency under a TD linear amplification process. This Hamiltonian has been solved using the Lewis and Riesenfed TD invariants, in a similar fashion to what has been done in [14], but now on a framework where the pseudo-Hermitian Hamiltonian or pseudo-Hermitian component of the Hamiltonian is also an observable quantity. We also succeeded in achieving a TD eigenvalue equation for our pseudo-Hermitian Hamiltonian, which has helped us to analyze the $\mathcal{PT}$-symmetry breaking process.

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Appendix

A.1. Proof that $\rho(t) = \eta^\dagger(t)\eta(t) = \eta^\dagger(t_0)\eta(t_0)$ from the pseudo-Hermiticity relation $H^\dagger(t)\rho(t_0) = \rho(t_0)H(t)$, with $\rho(t_0) = \eta^\dagger(t_0)\eta(t_0)$

i) We first present a formal proof, starting from the pseudo-Hermiticity relation $H^\dagger(t)\rho(t_0) = \rho(t_0)H(t)$ which implies that $F[iH^\dagger(t)]\rho(t_0) = \rho(t_0)F[iH^\dagger(t)]$, and consequently:

$$\begin{aligned}\eta^\dagger(t)\eta(t) &= T\left\{\exp\left[i\int_{t_0}^{t} d\tau H^\dagger(\tau)\right]\right\}\rho(t_0)T\\ &\quad\times\exp\left[-i\int_{t_0}^{t} d\tau H(\tau)\right]\\ &= \rho(t_0)T\left\{\exp\left[i\int_{t_0}^{t} d\tau H(\tau)\right]\right\}T\\ &\quad\times\exp\left[-i\int_{t_0}^{t} d\tau H(\tau)\right] = \rho(t_0).\end{aligned}$$

ii) Next, for a more explicit proof, we expand the time-ordering operator to obtain

$$\begin{aligned}\eta^{\dagger}(t)\eta(t) = \Bigg[&1 + i\int_{t_0}^{t} dt_1 H^{\dagger}(t_1) + \sum_{\ell=2}^{\infty}(i)^{\ell}\\ &\times \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{\ell-1}} dt_{\ell} H^{\dagger}(t_1)\cdots H^{\dagger}(t_{\ell})\Bigg]\rho(t_0)\\ &\times\Bigg[1 - i\int_{t_0}^{t} dt_1 H(t_1) + \sum_{\ell=2}^{\infty}(-i)^{\ell}\\ &\times \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{\ell-1}} dt_{\ell} H(t_{\ell})\cdots H(t_1)\Bigg].\end{aligned}$$

Using the pseudo-Hermiticity relation $H^{\dagger}(t)\rho(t_0) = \rho(t_0)H(t)$, we rewrite the metric operator in the form

$$\begin{aligned}\eta^{\dagger}(t)\eta(t) = \rho(t_0)\Bigg[&1 + i\int_{t_0}^{t} dt_1 H(t_1)\\ &+ \sum_{\ell=2}^{\infty}(i)^{\ell}\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{\ell-1}} dt_{\ell} H(t_1)\cdots H(t_{\ell})\Bigg]\\ \times\Bigg[&1 - i\int_{t_0}^{t} dt_1 H(t_1)\\ &+ \sum_{\ell=2}^{\infty}(-i)^{\ell}\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{\ell-1}} dt_{\ell} H(t_{\ell})\cdots H(t_1)\Bigg].\end{aligned}$$

In order to prove that the above metric operator equals $\rho(t_0)$ (i.e, that the product of the terms in both brackets gives us the indentity), we must show that, with the exception of zero order term, all terms associated with all other orders are equal to zero separately. To this end we start by considering the term associated with the generic odd nth-order, given by

$$\begin{aligned}&\left[(-i)^n\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-1}} dt_n H(t_n)\cdots H(t_1)\right]\\ &+\left[i\int_{t_0}^{t} dt_1 H(t_1)\right]\Bigg[(-i)^{n-1}\int_{t_0}^{t} dt_1\\ &\cdots\int_{t_0}^{t_{n-2}} dt_{n-1} H(t_{n-1})\cdots H(t_1)\Bigg]\\ &+\left[(i)^2\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2 H(t_1)H(t_2)\right]\\ &\times\left[(-i)^{n-2}\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-3}} dt_{n-2} H(t_{n-2})\cdots H(t_1)\right]\\ &\vdots\\ &+\left[(i)^{n-2}\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-3}} dt_{n-2} H(t_1)\cdots H(t_{n-2})\right]\\ &\times\left[(-i)^2\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2 H(t_2)H(t_1)\right]\\ &+\left[(i)^{n-1}\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-2}} dt_{n-1} H(t_1)\cdots H(t_{n-1})\right]\\ &\times\left[-i\int_{t_0}^{t} dt_1 H(t_1)\right]\\ &+\left[(i)^n\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-1}} dt_n H(t_1)\cdots H(t_n)\right].\end{aligned}$$

Now, as our first step, we relabel the mute indices coming from the terms within the second brackets, to obtain, in each term of the sum, the time-ordered product $H(t_1)\cdots H(t_n)$. We thus end up with

$$\begin{aligned}= (i)^n\Bigg\{&(-1)^n\int_{t_0}^{t_2} dt_1\int_{t_0}^{t_3} dt_2\int_{t_0}^{t_4} dt_3\\ &\cdots\int_{t_0}^{t_{n-1}} dt_{n-2}\int_{t_0}^{t_n} dt_{n-1}\int_{t_0}^{t} dt_n\\ &+(-1)^{n-1}\int_{t_0}^{t} dt_1\int_{t_0}^{t_3} dt_2\int_{t_0}^{t_4} dt_3\\ &\cdots\int_{t_0}^{t_{n-1}} dt_{n-2}\int_{t_0}^{t_n} dt_{n-1}\int_{t_0}^{t} dt_n\\ &+(-1)^{n-2}\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\int_{t_0}^{t_4} dt_3\\ &\cdots\int_{t_0}^{t_{n-1}} dt_{n-2}\int_{t_0}^{t_n} dt_{n-1}\int_{t_0}^{t} dt_n\\ &\vdots\\ &+(-1)^{2}\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\int_{t_0}^{t_2} dt_3\\ &\cdots\int_{t_0}^{t_{n-3}} dt_{n-2}\int_{t_0}^{t_n} dt_{n-1}\int_{t_0}^{t} dt_n\\ &+(-1)^{1}\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\int_{t_0}^{t_2} dt_3\\ &\cdots\int_{t_0}^{t_{n-3}} dt_{n-2}\int_{t_0}^{t_{n-2}} dt_{n-1}\int_{t_0}^{t} dt_n\\ &+(-1)^{0}\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\int_{t_0}^{t_2} dt_3\cdots\int_{t_0}^{t_{n-3}} dt_{n-2}\\ &\times\int_{t_0}^{t_{n-2}} dt_{n-1}\int_{t_0}^{t_{n-1}} dt_n\Bigg\}H(t_1)\cdots H(t_n).\end{aligned}$$

Our second step is to add the first and the second lines as well as the $(n-1)$th and nth one, to obtain

$$\begin{aligned}= (i)^n\Bigg\{&(-1)^{n-1}\int_{t_2}^{t} dt_1\int_{t_0}^{t_3} dt_2\int_{t_0}^{t_4} dt_3\\ &\cdots\int_{t_0}^{t_{n-1}} dt_{n-2}\int_{t_0}^{t_n} dt_{n-1}\int_{t_0}^{t} dt_n\\ &+(-1)^{n-2}\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\int_{t_0}^{t_4} dt_3\\ &\cdots\int_{t_0}^{t_{n-1}} dt_{n-2}\int_{t_0}^{t_n} dt_{n-1}\int_{t_0}^{t} dt_n\\ &+(-1)^{n-3}\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\int_{t_0}^{t_2} dt_3\\ &\cdots\int_{t_0}^{t_{n-1}} dt_{n-2}\int_{t_0}^{t_n} dt_{n-1}\int_{t_0}^{t} dt_n\\ &\vdots\\ &+(-1)^{3}\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\int_{t_0}^{t_2} dt_3\int_{t_0}^{t_3} dt_4\\ &\cdots\int_{t_0}^{t_{n-1}} dt_{n-2}\int_{t_0}^{t_n} dt_{n-1}\int_{t_0}^{t} dt_n\\ &+(-1)^{2}\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\int_{t_0}^{t_2} dt_3\int_{t_0}^{t_3} dt_4\\ &\cdots\int_{t_0}^{t_{n-3}} dt_{n-2}\int_{t_0}^{t_n} dt_{n-1}\int_{t_0}^{t} dt_n\\ &+(-1)^{1}\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\int_{t_0}^{t_2} dt_3\int_{t_0}^{t_3} dt_4\\ \cdots&\int_{t_0}^{t_{n-3}} dt_{n-2}\int_{t_0}^{t_{n-2}} dt_{n-1}\int_{t_{n-1}}^{t} dt_n\Bigg\}\\ &\times H(t_1)\cdots H(t_n).\end{aligned}$$

Next, as our third step we redefine the region of integration in the variables t_1 and t_2 (t_n and t_{n-1}) from the second to the $(n-2)$th [the first to the $(n-3)$th] line, using the relations

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 f(t_1, t_2) = \int_{t_0}^{t} dt_2 \int_{t_2}^{t} dt_1 f(t_1, t_2),$$
$$\int_{t_0}^{t_n} dt_{n-1} \int_{t_0}^{t} dt_n g(t_{n-1}, t_n) = \int_{t_0}^{t} dt_{n-1} \int_{t_{n-1}}^{t} dt_n g(t_{n-1}, t_n).$$

We are thus able to factorize the integrals in the variables t_1 and t_n to obtain the simplified expression

$$\begin{aligned} =(i)^n \int_{t_2}^{t} dt_1 \int_{t_{n-1}}^{t} dt_n \Big\{ & (-1)^{n-1} \int_{t_0}^{t_3} dt_2 \int_{t_0}^{t_4} dt_3 \cdots \int_{t_0}^{t_{n-1}} dt_{n-2} \int_{t_0}^{t} dt_{n-1} \\ & + (-1)^{n-2} \int_{t_0}^{t} dt_2 \int_{t_0}^{t_4} dt_3 \cdots \int_{t_0}^{t_{n-1}} dt_{n-2} \int_{t_0}^{t} dt_{n-1} \\ & \vdots \\ & + (-1)^2 \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_3 \cdots \int_{t_0}^{t_{n-3}} dt_{n-2} \int_{t_0}^{t} dt_{n-1} \\ & + (-1)^1 \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_3 \cdots \int_{t_0}^{t_{n-3}} dt_{n-2} \int_{t_0}^{t_{n-2}} dt_{n-1} \Big\} H(t_1) \cdots H(t_n). \end{aligned}$$

Now, the second and the third steps must be performed $(n-3)/2$ times, leading to the final expression

$$\begin{aligned} = (i)^n & \int_{t_2}^{t} dt_1 \int_{t_3}^{t} dt_2 \cdots \int_{t_{(n+1)/2}}^{t} dt_{(n-1)/2} \\ & \times \int_{t_{(n+1)/2}}^{t} dt_{(n+3)/2} \cdots \int_{t_{n-2}}^{t} dt_{n-1} \int_{t_{n-1}}^{t} dt_n \\ & \times \left[(-1)^{(n+3)/2} \int_{t_0}^{t} dt_{(n+1)/2} + (-1)^{(n+1)/2} \int_{t_0}^{t} dt_{(n+1)/2} \right] H(t_1) \cdots H(t_n), \end{aligned}$$

which clearly equals zero when we factorize the sign $(-1)^{(n+1)/2}$.

A.2. Verifying that $\rho(t) = \eta^{\dagger}(t)\eta(t) = \eta^{\dagger}(t_0)\eta(t_0)$ for our illustrative example

Starting with $\tilde{V}(\tau) = \alpha(t) e^{i\chi(t)} a + \beta(t) e^{-i\chi(t)} a^{\dagger}$, we obtain, up to first order of perturbation, that

$$T \exp\left(-i\kappa \int_{t_0}^{t} d\tau \tilde{V}(\tau)\right) \simeq e^{-i\kappa[\tilde{\alpha}(t)a + \tilde{\beta}(t)a^{\dagger}]} \tag{31}$$

thus leading to the metric operator

$$\begin{aligned} \rho(t) = \eta^{\dagger}(t)\eta(t) = & e^{i\chi(t)a^{\dagger}a} e^{i\kappa[\tilde{\alpha}^*(t)a^{\dagger} + \tilde{\beta}^*(t)a]} e^{\gamma^*(t_0)a^{\dagger} + \lambda^*(t_0)a} \\ & \times e^{\gamma(t_0)a + \lambda(t_0)a^{\dagger}} e^{-i\kappa(\tilde{\alpha}(t)a + \tilde{\beta}(t)a^{\dagger})} e^{-i\chi(t)a^{\dagger}a}. \end{aligned} \tag{32}$$

By inserting the identity operator $e^{-i\chi(t)a^{\dagger}a} e^{i\chi(t)a^{\dagger}a}$ after the second, the third, and the fourth exponentials in equation (32), and using the relations

$$e^{i\chi(t)a^{\dagger}a} a e^{-i\chi(t)a^{\dagger}a} = e^{-i\chi(t)} a, \tag{33a}$$

$$e^{i\chi(t)a^{\dagger}a} a^{\dagger} e^{-i\chi(t)a^{\dagger}a} = e^{i\chi(t)} a^{\dagger}, \tag{33b}$$

we are able to rewrite the metric in the form

$$\begin{aligned} \rho(t) = & \exp\{i\kappa[\tilde{\alpha}^*(t) e^{i\chi(t)} a^{\dagger} + \tilde{\beta}^*(t) e^{-i\chi(t)} a]\} \\ & \times \exp[\gamma^*(t_0) e^{i\chi(t)} a^{\dagger} + \lambda^*(t_0) e^{-i\chi(t)} a] \\ & \times \exp[\gamma(t_0) e^{-i\chi(t)} a + \lambda(t_0) e^{i\chi(t)} a^{\dagger}] \\ & \times \exp\{-i\kappa[\tilde{\alpha}(t) e^{-i\chi(t)} a + \tilde{\beta}(t) e^{i\chi(t)} a^{\dagger}]\}. \end{aligned} \tag{34}$$

Next, we insert the identity operators

$$\begin{aligned} & \exp[\gamma^*(t_0) a^{\dagger} e^{i\chi(t)} + \lambda^*(t_0) a e^{-i\chi(t)}] \\ & \times \exp[-\gamma^*(t_0) a^{\dagger} e^{i\chi(t)} - \lambda^*(t_0) a e^{-i\chi(t)}], \end{aligned} \tag{35}$$

$$\begin{aligned} & \exp[-\gamma(t_0) e^{-i\chi(t)} a - \lambda(t_0) e^{i\chi(t)} a^{\dagger}] \\ & \times \exp[\gamma(t_0) e^{-i\chi(t)} a + \lambda(t_0) e^{i\chi(t)} a^{\dagger}] \end{aligned} \tag{36}$$

before the first and after the last exponentials, respectively, and use the relations

$$e^{xa + ya^{\dagger}} a e^{-xa - ya^{\dagger}} = a - y, \tag{37a}$$

$$e^{xa + ya^{\dagger}} a^{\dagger} e^{-xa - ya^{\dagger}} = a^{\dagger} + x, \tag{37b}$$

with x and y being c-numbers, to obtain

$$\begin{aligned} \rho(t) = & \exp[\gamma^*(t_0) e^{i\chi(t)} a^{\dagger} + \lambda^*(t_0) e^{-i\chi(t)} a] \\ & \times \exp\{i\kappa[\tilde{\alpha}^*(t) e^{i\chi(t)} a^{\dagger} + \tilde{\beta}^*(t) e^{-i\chi(t)} a]\} \\ & \times \exp\{-i\kappa[\tilde{\alpha}(t) e^{-i\chi(t)} a + \tilde{\beta}(t) e^{i\chi(t)} a^{\dagger}]\} \\ & \times \exp[\gamma(t_0) e^{-i\chi(t)} a + \lambda(t_0) e^{i\chi(t)} a^{\dagger}] \\ & \times \exp\{i\kappa[\tilde{\alpha}(t)\lambda(t_0) - \tilde{\alpha}^*(t)\lambda^*(t_0) \\ & + \tilde{\beta}^*(t)\gamma^*(t_0) - \tilde{\beta}(t)\gamma(t_0)]\}. \end{aligned} \tag{38}$$

We then apply the special case of the Baker-Hausdorff theorem

$$e^X e^Y = e^{X+Y} e^{\frac{1}{2}[X,Y]}, \tag{39}$$

with X and Y being operators, to merge together the second and third exponentials in equation (38), leading us with the expression

$$\begin{aligned} \rho(t) = & e^{\gamma^*(t_0) e^{i\chi(t)} a^{\dagger} + \lambda^*(t_0) e^{-i\chi(t)} a} \exp\{i\kappa[(\tilde{\alpha}^*(t) - \tilde{\beta}(t)) \\ & \times e^{i\chi(t)} a^{\dagger} + (\tilde{\beta}^*(t) - \tilde{\alpha}(t)) e^{-i\chi(t)} a]\} \\ & \times e^{\gamma(t_0) e^{-i\chi(t)} a + \lambda(t_0) e^{i\chi(t)} a^{\dagger}} \exp\{i\kappa[\tilde{\alpha}(t)\lambda(t_0) \\ & - \tilde{\alpha}^*(t)\lambda^*(t_0) + \tilde{\beta}^*(t)\gamma^*(t_0) - \tilde{\beta}(t)\gamma(t_0)]\}. \end{aligned} \tag{40}$$

Remembering the relation $\kappa[\beta(t) - \alpha^*(t)]/\omega(t) = \gamma^*(t_0) + \lambda(t_0)$, we may use the result

$$\begin{aligned} \kappa[\tilde{\beta}(t) - \tilde{\alpha}^*(t)] = & \int_{t_0}^{t} \left(\kappa \frac{\beta(\tau) - \alpha^*(\tau)}{\omega(\tau)}\right) \omega(\tau) e^{-i\chi(\tau)} d\tau \\ = & i[\gamma^*(t_0) + \lambda(t_0)](e^{-i\chi(t)} - 1) \end{aligned} \tag{41}$$

to further rewrite the metric in the form

$$\rho(t) = e^{\gamma^*(t_0)e^{i\chi(t)}a^\dagger + \lambda^*(t_0)e^{-i\chi(t)}a} \exp\{[\gamma^*(t_0) + \lambda(t_0)] \times (1 - e^{i\chi(t)})a^\dagger + [\gamma(t_0) + \lambda^*(t_0)](1 - e^{-i\chi(t)})a\} \times e^{\gamma(t_0)e^{-i\chi(t)}a + \lambda(t_0)e^{i\chi(t)}a^\dagger} \exp\{i\kappa[\tilde{\alpha}(t)\lambda(t_0) - \tilde{\alpha}^*(t)\lambda^*(t_0) + \tilde{\beta}^*(t)\gamma^*(t_0) - \tilde{\beta}(t)\gamma(t_0)]\}. \tag{42}$$

Using again equations (39) and (41), the former to merge together the first two exponentials in equation (42), we obtain after some algebra

$$\rho(t) = \exp\{[\gamma^*(t_0) + \lambda(t_0)]a^\dagger + [\gamma(t_0) + \lambda^*(t_0)]a\} \times \exp\left\{\frac{1}{2}[|\lambda(t_0)|^2 - |\gamma(t_0)|^2]\right\} \tag{43}$$

Using equation (39), now to break the exponential up into two parts, it follows that

$$\rho(t) = e^{\gamma^*(t_0)a^\dagger + \lambda^*(t_0)a} e^{\lambda(t_0)a^\dagger + \gamma(t_0)a} = \rho(t_0).$$

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