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Self-induction in two long parallel conductors connected to sinusoidal voltage source

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Abstract

The paper deals with the self-induction of two parallel infinitely long conductors of arbitrary cross section connected to an ideal sinusoidal voltage source. The conductors do not move. A quasi-stationary behavior is assumed, the displacement current is neglected and the permeability of the conductors and their surroundings equals the permeability of vacuum. The subject matter of the paper is the calculation of impedance, which in the theory of circuits with lumped elements can replace 1 m of conductors. The definition of inductance and the formulae for its calculation that form part of textbooks on physics and electromagnetism have been the same for such a long time that nobody doubts their correctness. Only the recently published method for the calculation of current density in long parallel conductors allows calculating accurately the equivalent impedance and self-inductance, and also evaluating critically the present knowledge.

Keywords: applied classical electromagnetism, induced currents, inductance, numerical simulation

1. Introduction

The paper is concerned with the self-induction of two parallel infinitely long conductors of arbitrary cross section, connected to an ideal sinusoidal voltage source with angular frequency ω. The conductors do not move. A quasi-stationary behavior is assumed, the displacement current is neglected, and the permeability of conductors and their surroundings equals the permeability of vacuum μ_0. In the chosen system of coordinates xyz the conductors are parallel to the axis z; their cross sections do not depend on z. Figure 1 gives an example of the cross sections of the conductors $\mathscr{A}_1$ and $\mathscr{A}_2$ in the plane xy; the cross sections are marked with the same symbols as the conductors. Conductor resistivity is a function of x and y, $\varrho = \varrho(x, y)$. On the assumption that the conductivity of the surroundings of conductors is zero, the two conductors form a current tube. According to [1], entry no. 121-11-30, electromagnetic induction is a phenomenon during which induced voltage or induced current is produced. According to [1], 121-11-31, self-induction in a current tube is caused by changes in electric current in the current tube.

The subject matter of the paper is the calculation of the impedance $\underline{Z}$, which in the theory of circuits with lumped elements can substitute 1 m of conductors when the above assumptions are satisfied. The segment of the two conductors between the planes $z = z_1$ and $z = z_2$, where $z_2 > z_1$, can be substituted by a circuit with lumped elements, see figure 2. The equivalent circuit is a series connection of an ideal source with voltage $V(z_1, t) - V(z_2, t)$, an ideal resistor with resistance R, and an ideal inductor on which there is a voltage $\mathrm{d}\Phi/\mathrm{d}t$ induced in the segment. The current I is given by the current density. The equivalent impedance $\underline{Z}$ is determined unambiguously by the current density in the conductors and by the voltage drop between the conductors $U(t) = [V(z_1, t) - V(z_2, t)]/(z_2 - z_1)$.

The considered segment of the two conductors in figure 2 can be replaced by the impedance

$$\underline{Z} = \frac{\underline{U}}{\underline{I}}, \qquad \underline{Z} = \Re(\underline{Z}) + \mathrm{j}\Im(\underline{Z}) = R_s + \mathrm{j}\omega L, \tag{1}$$

where the underlined symbols denote phasors or complex numbers. Using (1), L and R_s can be calculated for $\omega > 0$. For $\omega = 0$ it holds $R_s = R$. Admittedly, the value L for $\omega = 0$ has

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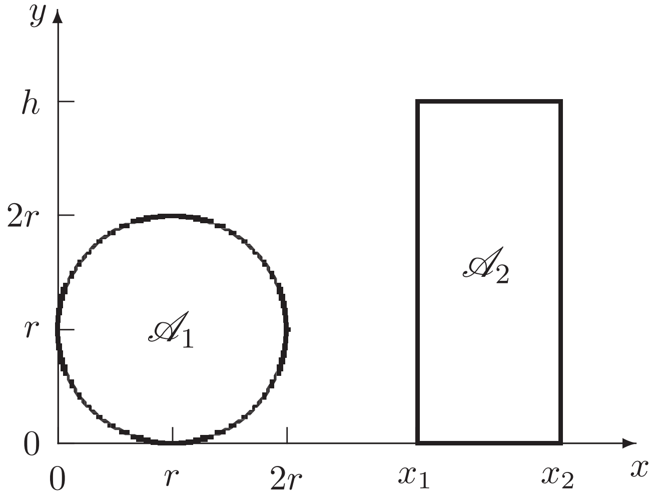

Figure 1. Cross sections of the conductors $\mathscr{A}_1$ and $\mathscr{A}_2$.

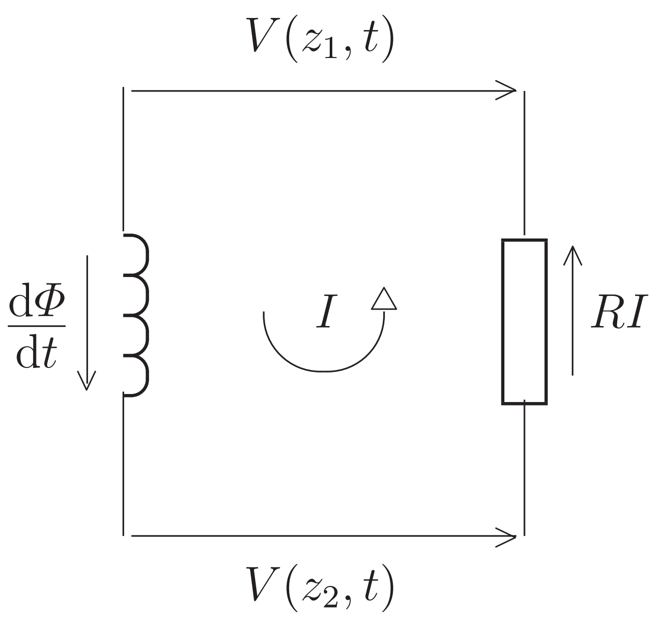

Figure 2. The equivalent circuit of the segment of conductors.

no meaning but it is equal to the limit of the values $\omega \to 0+$ calculated using (1). An accurate calculation of $\underline{Z}$ for an arbitrary shape of the cross sections of $\mathscr{A}_1$, $\mathscr{A}_2$ and for an arbitrary resistivity $\varrho(x, y)$ can be performed using the method recently published in [2].

The original contribution of this article is the presentation of the results of calculating current density in a pair of conductors $\mathscr{A}_1$ and $\mathscr{A}_2$, and the quantities L and R_s defined by relation (1). The knowledge of accurate values L and R_s for a pair of conductors of circular cross section allows a critical analysis of hitherto obtained formulae for their calculation.

2. Current density in conductors and equivalent impedance

This section is concerned with current density and equivalent impedance for the conductors $\mathscr{A}_1$ and $\mathscr{A}_2$, whose cross sections are illustrated in figure 1. The dimensions determining the cross sections are $r = 8$ mm, $x_1 = 2r$, $x_2 - x_1 = 10$ mm, $h = 24$ mm. The conductors are in contact in the straight line $x = 2r$, $y = r$, $z \in (-\infty, +\infty)$, but this contact is assumed not to be electrically conductive. The amplitude of steady sinusoidal voltage $U(t) = \sin(\omega t)$ is $\hat{U} = 1\ \mathrm{V{\cdot}m^{-1}}$. The temperature of the conductors is 300 K. On the conductor cross section the non-zero component of the current density $\boldsymbol{J}(x, y)$ is only $J_z(x, y) = J(x, y)$, where

$$J(x, y) = \hat{J}(x, y)\sin[\omega t + \varepsilon(x, y)].$$

The choice of the examined conductors $\mathscr{A}_1$ and $\mathscr{A}_2$ may seem artificial but it is not out of reality. The conductors have been chosen such that it can be seen at a glance that no method has so far been published (except [2]) that would allow the calculation of current density and equivalent impedance for the conductors mentioned. Moreover, all the potentials of the method published in [2] are far from having been fully exhausted, in particular as regards the shape of conductor cross sections, their number, their resistivity and the choice of $U(t)$ in transient or steady state.

Variant 1

Both conductors are of copper, their resistivity $\varrho = 1.725 \times 10^{-8}\ \Omega \cdot \mathrm{m}$ [3], the frequency of voltage source $f = \omega/(2\pi) = 50$ Hz.

Variant 2

Unlike in Variant 1, in this variant $f = 10^3$ Hz.

Variant 3

$f = 10^3$ Hz. Conductor $\mathscr{A}_1$ is made of an alloy composed of 75 mass% of Cu and 25 mass% of Al, its resistivity $\varrho_1 = 1.76 \times 10^{-7}\ \Omega \cdot \mathrm{m}$ [3], conductor $\mathscr{A}_2$ is of copper.

Variant 4

$f = 10^3$ Hz. Both conductors are of copper as in Variants 1 and 2, but their distance is not zero, $x_1 = 56$ mm.

Variant 5

The conductors are the same as in Variant 3, the source frequency $f = 10^6$ Hz.

Figures 3–6 illustrate the current density using the amplitude $\hat{J}$ and the initial phases ε for Variants 1–4. These figures give plots of the functions $\hat{J}(x, y)$ and $\varepsilon(x, y)$ for a constant y. The values y have been chosen such that the range of the values of $\hat{J}$ and ε over the whole cross sections of the conductors under examination should be evident. This is quite obvious in figure 3 for $\varepsilon(x, y)$ in the cross section $\mathscr{A}_2$. The cross section $\mathscr{A}_1$ is not symmetrical with the cross section $\mathscr{A}_2$ while the cross section $\mathscr{A}_1$ is symmetrical with respect to the straight line $y = r$ and the cross section $\mathscr{A}_2$ is symmetrical with respect to the straight line $y = h/2$. For two values of y that are symmetrical with respect to either of these straight lines the functions $\hat{J}(x, y)$ and $\varepsilon(x, y)$, do not differ by much as follows from figure 6, in which pairs of close curves can be seen. For this reason the values of y chosen for the function $\varepsilon(x, y)$ are lower than $h/2$ in figure 4.

With increasing frequency f the range of the values ε increases. As the values ε are modified so as to be in the interval $(-180°, 180°]$, the curves $\varepsilon(x, y)$ for some y in figure 4 are discontinuous. In Variant 5, see figure 7, $f = 10^6$ Hz and the discontinuities are so numerous that the

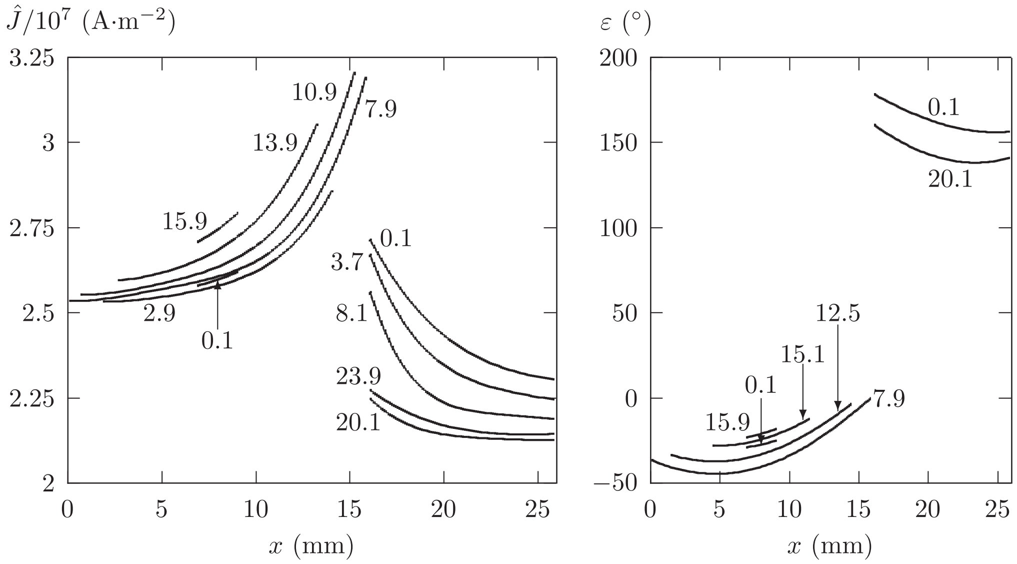

Figure 3. Variant 1: Dependence of the current density amplitude $\hat{J}$ and initial phase ε on x for a constant y in the conductors $\mathscr{A}_1$ (with circular cross section) and $\mathscr{A}_2$ (with rectangular cross section). The numbers at the curves are the values of y expressed in mm.

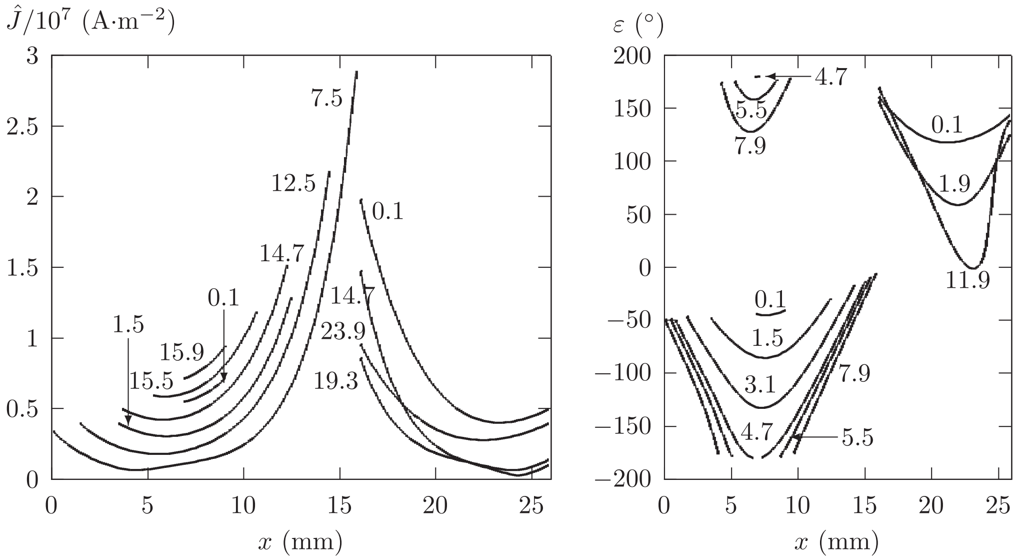

Figure 4. Variant 2: Dependence of the current density amplitude $\hat{J}$ and initial phase ε on x for a constant y in the conductors $\mathscr{A}_1$ and $\mathscr{A}_2$. The numbers at the curves are the values of y expressed in mm.

graph of function $\varepsilon(x, y)$ is not clearly arranged and therefore it is not shown. With increasing f the maximum and the minimum values of amplitude $\hat{J}$ decrease. These values are to a considerable extent affected by the conductor resistivity, as is obvious from the figures. It follows from the comparison of figures 4 and 6 and from [4] that with increasing conductor distance $x_1 - 2r$ the minimum and the maximum values of amplitude $\hat{J}$ decrease. The minimum value of $\hat{J}(x, y)$ and the number of phase $\varepsilon(x, y)$ discontinuities depend, in addition, on the dimensions of conductor cross sections. If, for example, in Variant 1 the values r, x_1, x_2, and h increase fivefold, then the graphs of $\hat{J}(x, y)$ and $\varepsilon(x, y)$ are at a glance the same as in figure 4 and therefore they are not given in the article.

The current density in the conductors determines the parameters of the conductor pair. The parameters are $Z = |\underline{Z}|$, R_s, L, $\hat{I}_1$ and β_1, where

$$I(t) = \hat{I}_1 \sin(\omega t + \beta_1)$$

is the current in the conductor $\mathscr{A}_1$. For the current in the conductor $\mathscr{A}_2$ it holds $\hat{I}_2 = \hat{I}_1$ and $\beta_2 = 180° + \beta_1$. The parameters of the examined conductor pair for Variants 1–5 are given in table 1.

3. Faraday's law, magnetic flux and self-inductance

By Faraday's electromagnetic induction law, in an arbitrary closed curve C time rate of change of magnetic flux induces

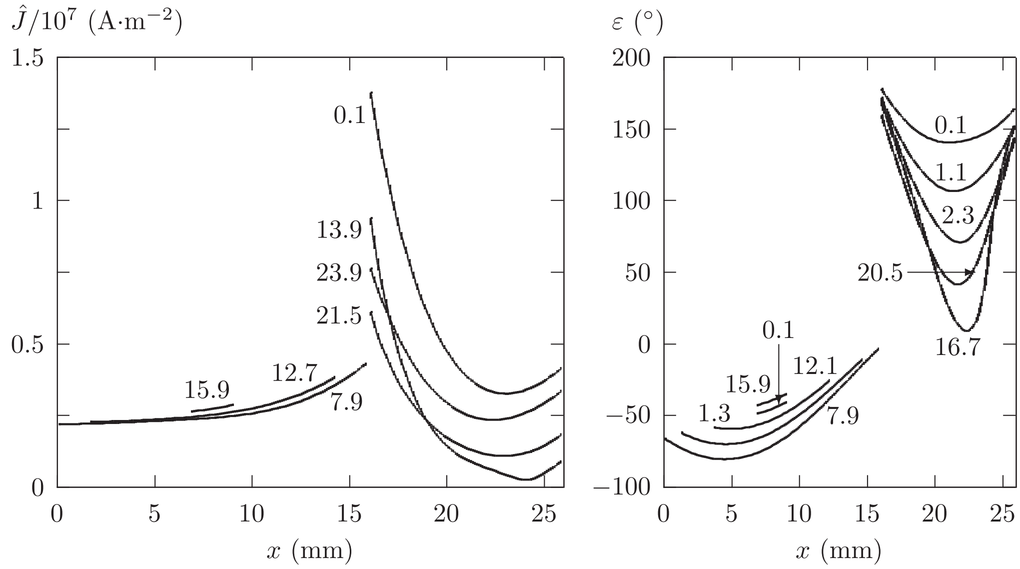

Figure 5. Variant 3: Dependence of the current density amplitude $\hat{J}$ and initial phase ε on x for a constant y in the conductors $\mathscr{A}_1$ and $\mathscr{A}_2$. The numbers at the curves are the values of y expressed in mm.

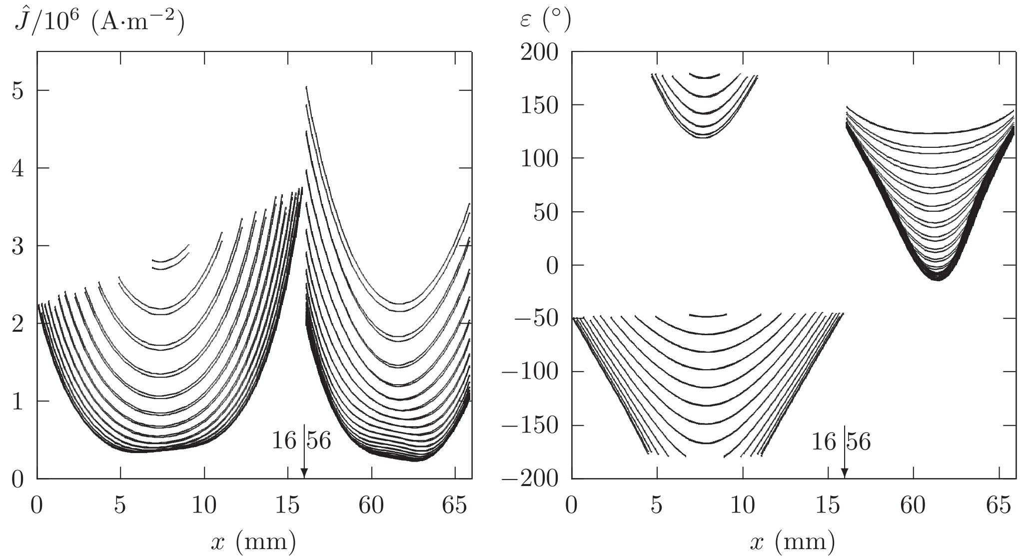

Figure 6. Variant 4: Dependence of the current density amplitude $\hat{J}$ and initial phase ε on x for a constant y in the conductors $\mathscr{A}_1$ and $\mathscr{A}_2$. The axis x at the point marked by the arrow ends in the value $x = 16$ mm and continues with the value $x = 56$ mm. Between these values is the gap between the conductors and the current density is zero.

the voltage

$$U_C = \frac{\mathrm{d}\Phi_C(t)}{\mathrm{d}t}, \tag{2}$$

where Φ_C is the flux of the vector $\boldsymbol{B}$ through a continuous oriented surface S_C bounded by the curve C. Φ_C does not depend on the shape of the surface S_C and therefore the term 'linked flux to C' is used in the following text. If the whole curve C lies in an electrically conductive medium, then U_C will produce a conductive electric current, i.e. induced current.

Current density in conductors can be considered as superposition of two current densities. The first density is produced by the voltage source; the second density is the density of induced current. Current is induced in each loop of a current filament [5] that is a closed curve C, and either $C \subset \mathscr{A}_1$ or $C \subset \mathscr{A}_2$, or the curve has two parts, $C = C_1 \cup C_2$, and it holds $C_1 \subset \mathscr{A}_1 \wedge C_2 \subset \mathscr{A}_2$. All such curves are taken into consideration in the calculation of the current density published in [2]. Magnetic fluxes linked to two different curves can have the same value but generally they are different. The quotient of the magnetic flux and the current $I(t)$

$$L_C = \frac{\Phi_C(t)}{\hat{I}_1 \sin(\omega t + \beta_1)}, \qquad t \in [0, 1/f),$$

takes all the values in the interval $(-\infty, +\infty)$ because the current goes twice through zero. With the assumed permeability μ_0 and for a constant current I (independent of t) the

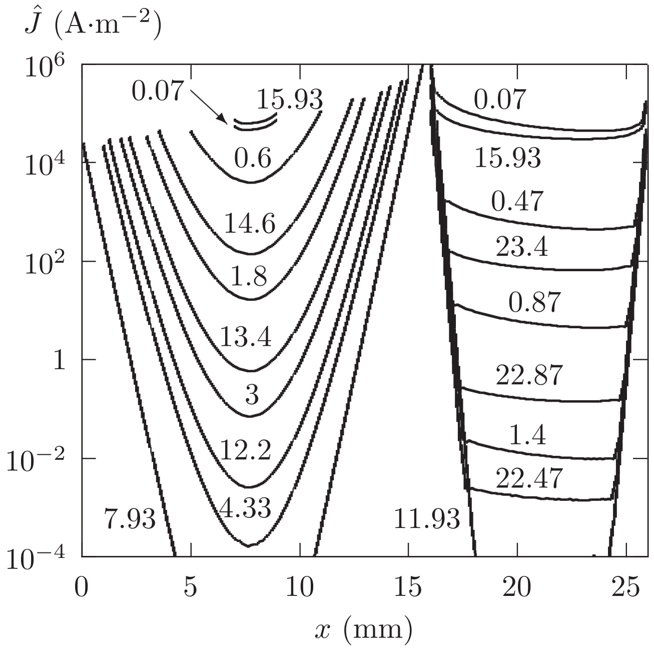

Figure 7. Variant 5: Dependence of the current density amplitude $\hat{J}$ on x for a constant y in the conductors $\mathscr{A}_1$ and $\mathscr{A}_2$. The numbers at the curves are the values of y expressed in mm.

magnetic field and the flux Φ_C are directly proportional to I and therefore L_C is constant for the given C.

Faraday's law is closely connected with the term 'self-inductance', which in textbooks on physics and electromagnetic field theory is defined for a loop formed by the current. This loop is called

- circuit carrying a steady current I [6];
- conductor, wire, current loop [7];
- thin loop, the condition of current-conductor thinness is fundamental [8];
- line current loop I flowing through contour C [9];
- circuit [10, 11];
- closed conducting circuit of one turn of wire [12];
- coil [13];
- loop of thin wire [14].

In the literature, the loop is usually replaced by a closed curve C (speaking strictly mathematically) with the direct current I and the self-inductance

$$L = \frac{\Phi_C}{I}, \qquad \Phi_C = \int_{S_C} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}_C, \tag{3}$$

where $\boldsymbol{B}$ is the vector of the magnetic field produced by the current I; the dependence of $\boldsymbol{B}$ on I is assumed to be linear.

A real circuit cannot be a curve because real conductors have non-zero cross sections. The assumption of dc current [15], 131-11-22, is a bit strange because a constant current I (independent on t) can produce only a constant (or undefined) magnetic flux Φ_C, for which it holds $\mathrm{d}\Phi_C/\mathrm{d}t = 0$. From an analysis of relation (3) it follows that to define self-inductance is problematic. Of fundamental significance in the study of the phenomenon of self-induction is the law (2), according to which the induced voltage is for the dc current I zero and thus there is nothing to be studied. However, in the quoted literature, and elsewhere, a constant I is assumed. As regards the cross section of the loop conductor, there are in essence three possibilities:

1. *The loop conductor is a current filament with zero cross section.*

Current in the conductor is constant and of finite magnitude. According to [5], the conductor is the current filament-1, and the integral in (3) is improper and divergent, as proved in [5]. The definition of L (3) does not make sense but this definition is still used to derive, for example, the Neumann formula and the relation between the inductance and the energy of magnetic field. The divergence of the integral in (3) is no secret. For example, in [14], p. 218, it is said: '... the flux linked to the curve C diverges, $\Phi \to \infty$. The inductance is also $L \to \infty$; there is no sense in introducing it'. Therefore it is assumed in [14] that the conductor cross section is small but not zero. This assumption, however, is not always satisfied in [14], maybe because it is irresistible due to its simplicity. The book [14] is often quoted because it was written only recently and in the list of references it has 104 items, most of which are textbooks on electromagnetism.

2. *The loop conductor cross section is not negligible.*

Let us consider two infinitely long conductors parallel to the axis z. Both conductors have a circular cross section of radius r. The axes of $\mathscr{A}_1$ and $\mathscr{A}_2$ go in the plane xy through the points (0, 0) and (c, 0), respectively; $c \geqslant 2r$. There is a constant current density J in the conductors. In this case self-inductance depends on the position of the curve to which the magnetic flux is linked. The self-inductance of conductors can be determined by calculating the magnetic flux Φ linked to the curve C_{12}, which is formed by line segments connecting the points (x_1, 0, z_1), (x_1, 0, z_2), (x_2, 0, z_2) and (x_2, 0, z_1), where $-r \leqslant x_1 \leqslant r$, $c - r \leqslant x_2 \leqslant c + r$, $z_2 > z_1$. According to (3)

$$\begin{aligned}\Phi(x_1, x_2) &= (z_2 - z_1)\int_{x_1}^{x_2} [B_{y1}(x) + B_{y2}(x)]\,\mathrm{d}x \\ &= \Phi_1(x_1) + \Phi_2(x_2) + \Phi_3(x_2) + \Phi_4(x_1),\end{aligned} \tag{4}$$

where B_{y1} and B_{y2} are y-components of magnetic field produced by the conductors $\mathscr{A}_1$ and $\mathscr{A}_2$, respectively. According to Ampere's circuital law [11]

$$\frac{\Phi_1(x_1)}{(z_2 - z_1)I} = \frac{1}{\pi r^2 J}\int_{x_1}^{r} B_{y1}(x)\mathrm{d}x = \frac{\mu_0}{4\pi}\left[1 - \left(\frac{x_1}{r}\right)^2\right],$$

$$\frac{\Phi_2(x_2)}{(z_2 - z_1)I} = \frac{1}{\pi r^2 J}\int_{r}^{x_2} B_{y1}(x)\mathrm{d}x = \frac{\mu_0}{2\pi}\ln\frac{x_2}{r},$$

$$\frac{\Phi_3(x_2)}{(z_2 - z_1)I} = \frac{1}{\pi r^2 J}\int_{c-r}^{x_2} B_{y2}(x)\mathrm{d}x = \frac{\mu_0}{4\pi}\left[1 - \left(\frac{c - x_2}{r}\right)^2\right],$$

$$\frac{\Phi_4(x_1)}{(z_2 - z_1)I} = \frac{1}{\pi r^2 J}\int_{x_1}^{c-r} B_{y2}(x)\mathrm{d}x = \frac{\mu_0}{2\pi}\ln\frac{c - x_1}{r},$$

The values of

$$L_{12} = \frac{\Phi(x_1, x_2)}{(z_2 - z_1)I} \tag{5}$$

for some x_1, x_2 and c are given in table 2.

3. *The diameter of the loop conductor cross section converges to zero.*

Table 1. Parameters of the conductors in Variants 1–5. R_1 and R_2 are the resistances of conductors $\mathscr{A}_1$ and $\mathscr{A}_2$, respectively.

Variant	R_1 $\mu\Omega\cdot m^{-1}$	R_2 $\mu\Omega\cdot m^{-1}$	Z $\mu\Omega\cdot m^{-1}$	R_s $\mu\Omega\cdot m^{-1}$	L $nH\cdot m^{-1}$	$\hat{I}_1$ A	β_1 °
1	85.794	71.875	189.82	164.42	301.94	5268.1	−29.98
2	85.794	71.875	1282.6	604.35	180.06	779.65	−61.89
3	875.35	71.875	2011.4	1272.8	247.88	497.17	−50.74
4	85.794	71.875	4816.8	365.59	764.41	207.61	−85.65
5	875.35	71.875	288 849	103 470	42.921	3.4620	−69.01

Table 2. The values of L_{12} in 10^{-7}H·m^{-1}, for some x_1, x_2 and c.

x_1	x_2	$c = 2r$	$c = 10r$
r	$c - r$	0	$4 \ln 9$
0	c	$2 + 4 \ln 2$	$2 + 4 \ln 10$
$-r$	$c + r$	$4 \ln 3$	$4 \ln 11$

According to (3), (4) and (5), $L_{12} \to +\infty$ for $r \to 0+$ or for $c \to +\infty$ if r is constant.

It follows from the three above possibilities that the definition (3) does not allow an accurate calculation of self-inductance. The first possibility holds quite generally for an arbitrary curve C and does not make sense. The other two possibilities were demonstrated on a pair of parallel cylindrical conductors. Basically the same result could also be derived for a pair of conductors of other than circular cross section, but the result could not be expressed by simple formulae. Of fundamental significance in the definition (3) is the curve C, which in a real conductor can be chosen in infinitely many ways and it is *a priori* not known which the correct one is.

A drawback of the definition (3) and formula (5) is that the resistivity of the real loop conductor is not taken into consideration. By table 1, self-inductance in the current tube greatly depends on conductor resistance, which shows in particular when the conductor cross section is small. Table 3 gives the parameters of the conductors in Variant 2 for several values of the number κ, if the conductor cross sections are determined by κr, κx_1, κx_2 and κh. With the value of κ decreasing, the value L first increases until it reaches the value $L_0 = 305.76$ nH·m^{-1}, which is the value of inductance for $f = 0$. This inductance value is preserved even when κ continues decreasing. This means that the imaginary component of impedance $\Im(\underline{Z})$ also does not change while $\Re(\underline{Z})$ increases. The consequence is that the induced voltage and the induced current are negligible. The current density in the conductors is practically constant and the initial phase β_1 of the current is equal to zero. The values in table 3 were determined using the formula (1), in which the phasor $\underline{I}$ was calculated using the current density obtained by a method published in [2]. Further calculations have confirmed that the described dependence of parameters on κ is similar even for other pairs than $\mathscr{A}_1$, $\mathscr{A}_2$.

Another drawback of the definition (3) and formula (5) is that self-inductance does not depend on the frequency of current I. It follows from figures 3–7 and, in particular, table 1 that the dependence of I and L on f is not negligible for higher frequencies. What is considered to be high depends on the required parameter accuracy, on the size and shape of the cross sections of the conductors, and on their mutual position.

4. Definition of energetic self-inductance

Using the relation (3), which in the literature is considered to be also valid for conductors of a small cross section, relation is derived among L, the current I flowing through the loop C, and the energy W_m of the magnetic field produced by the current I

$$W_m = \frac{1}{2} L I^2. \tag{6}$$

In [14], p. 297, it is said: 'relation (6) for the energy of the magnetic field of a thin loop will by way of definition be extended to massive loops. The self-inductance of a massive loop is then defined by the equation'

$$L_{ener} = \frac{2\, W_m}{I^2}. \tag{7}$$

(7) is the definition of energetic self-inductance. Relation (3) does not make sense, as argued above, and it is thus inadmissible to derive anything using (3), even if relations (6) and (7) are valid. The validity of relation (6) can be derived in a correct way. If the current $I(t) = \hat{I} \sin \omega t$ flows through an inductor with the inductance L, the voltage on the inductor is

$$U_L(t) = \omega L \hat{I} \sin(\omega t + \pi/2)$$

and the instantaneous power is

$$Q_L(t) = U_L(t) I(t) = \omega L\, I_{eff}^2 \sin(2\omega t), \qquad I_{eff}^2 = \hat{I}^2/2.$$

The energy of the inductor magnetic field is

$$W_m(t) = \int_0^t Q_L(x) dx = \frac{1}{2} L\, I_{eff}^2\, [1 - \cos(2\omega t)].$$

Therefore (6) and (7) hold for the constant current.

By [1], 121-11-64, the energy W_m in a volume V in a linear medium is given by the integral

$$W_m = \frac{1}{2} \int_V \boldsymbol{H} \cdot \boldsymbol{B}\, dV. \tag{8}$$

The volume V in the integral (8) denotes part of the space in which there is a non-zero magnetic field produced by the current I, i.e. the whole space. According to the literature, L_{ener} can be assigned to any part of the space $V_1 \subset V$; it

Table 3. Parameters of the conductors in Variant 2 and their dependence on κ. R is the resistance of a segment of both conductors.

κ	R $\Omega \cdot m^{-1}$	R_s $\Omega \cdot m^{-1}$	L $nH \cdot m^{-1}$	$\hat{I}_1$ A	β_1 $^\circ$
10	1.577×10^{-6}	1.407×10^{-4}	51.83	2819	−66.64
1	1.577×10^{-4}	6.044×10^{-4}	180.1	779.6	−61.89
0.5	6.308×10^{-4}	1.043×10^{-3}	258.7	517.7	−57.31
0.1	1.577×10^{-2}	1.580×10^{-2}	305.6	62.83	−6.93
0.01	1.577	1.577	305.8	0.6341	−0.07
0.005	6.308	6.308	305.8	0.1585	−0.02
0.001	157.7	157.7	305.8	6.341×10^{-3}	0.00

Table 4. Dependence of inductance on the conductor distance c for $r = 10$ mm. The values L, L_{ener} and L_{lit} hold for $f = 0$ while the values L_{kHz} and L_{MHz} hold for $f = 1$ kHz and $f = 1$ MHz, respecitvely.

c mm	L $\mu H \cdot m^{-1}$	L_{ener} $\mu H \cdot m^{-1}$	L_{lit} $\mu H \cdot m^{-1}$	L_{kHz} $\mu H \cdot m^{-1}$	L_{MHz} $\mu H \cdot m^{-1}$
20	0.377	0.377	0.1	0.205	0.035
50	0.744	0.744	0.655	0.672	0.628
100	1.021	1.021	0.979	0.959	0.918
200	1.298	1.298	1.278	1.239	1.199
500	1.666	1.664	1.657	1.606	1.566
1000	1.942	1.941	1.938	1.883	1.843

suffices to replace V by V_1 in the integral (8). Such replacement does not constitute any problem from the mathematical viewpoint, in contrast to the physical viewpoint.

For two infinitely long cylindrical conductors parallel with the axis z, a formula is derived in [14] for the calculation of the energetic self-inductance of 1 m of conductors

$$L_{\mathrm{lit}} = \frac{\mu_0}{\pi}\left[0.25 + \ln\frac{c-r}{r}\right], \tag{9}$$

where r is the radius of the conductor cross section; the conductor axes go in the plane xy through the points (0, 0) and (c, 0) and there is a constant current density in the conductors. According to [14], (9) holds for $c \gg r$. It follows from table 4 that for low frequencies the quantity L_{lit} is usable for only $c \gg r$; another drawback of this quantity is that it does not depend on f.

In the literature, L_{lit} is usually given as the energetic definition of inductance but from a comparison of the third and the fourth columns of table 4 it follows that the relation $L_{\mathrm{lit}} \doteq L_{\mathrm{ener}}$ holds for only $c \gg r$, as is also given in [14]. This is because when deriving the formula (9) the magnetic field energy was used only partially. The conductors are infinitely long and inductance is determined for 1 m of conductors and therefore the relation (8) has the form

$$W_{\mathrm{m}} = (z_2 - z_1)\int_{xy} \boldsymbol{H}\cdot\boldsymbol{B}\,\mathrm{d}x\,\mathrm{d}y, \tag{10}$$

where $z_2 - z_1 = 1$ m. When deriving (9), the plane xy in the integral (10) was divided into two parts. One part is the cross sections of the two conductors, the other part is the remainder of the plane xy. In the calculation of the magnetic field in each of the conductors the magnetic field produced by the other conductor is neglected, which is only possible for $c \gg r$. The integral over the other part of the plane xy is replaced by the value of the quotient Φ/I, Φ is the magnetic field flux through the rectangle $[z_1, z_2] \times [r, c - r]$ in the plane xy. Such replacement is evidently incorrect; in the case of $c = 2r$ it holds $\Phi = 0$, while the energy of the magnetic field in the other part of the plane xy is obviously non-zero, as is evident from a comparison of L and L_{lit} on the first line of table 4.

5. Characterization of self-induction in a loop

The subject of the present article is self-induction in a pair of long conductors and therefore it would be useful to propose a quantity that characterizes self-induction quantitatively. The first to come to mind is the self-inductance L. The value L is the highest for $f = 0$, then the self-induction phenomenon does not arise and therefore the value L is no suitable as a characteristic of self-induction. Another quantity that comes into consideration is the inductive reactance ωL or the induced voltage amplitude $\omega L\hat{I}$ on the inductor. With increasing f, the values L and $\hat{I}$ decrease while the values ω and ωL increase. With increasing ω the value $\omega L\hat{I}$ converges to $\hat{U}$. In view of the fact that $\hat{I}$ is directly proportional to $\hat{U}$, the quantity that characterizes self-induction appears to be the quotient of induced voltage and exciting voltage amplitudes

$$\mathscr{L} = \frac{\omega L\hat{I}}{\hat{U}}, \qquad \mathscr{L} \in [0, 1]. \tag{11}$$

table 5 gives the values of $\mathscr{L}$ in dependence on f and on the distance c of the conductor axes. The pattern of the dependence of $\mathscr{L}$ on f and on c is shown in detail in figure 8.

6. Real component of equivalent impedance

Self-induction in the current tube considered is characterized not only by the inductance L or the characteristic $\mathscr{L}$ but also by the real component $\Re(\underline{Z}) = R_s$ of the equivalent impedance $\underline{Z}$. The value of R_s depends on the resistivity of the current tube and on the frequency of the voltage source, as follows from the analysis in [4], and is determined by the relation (1). However, using the relation (1) is conditional on the knowledge of the current density in the conductors.

Table 5. Dependence of $\mathscr{L}$ on frequency f and on distance c of conductor axes for conductors of rectangular (rec) cross section 10 × 32 mm^2 and cylindrical conductors (cyl) of 20 mm in diameter.

f (Hz)	rec $c = 2$ cm	cyl $c = 2$ cm	cyl $c = 10$ cm	cyl $c = 1$ m
0.01	2.108×10^{-4}	2.175×10^{-4}	5.888×10^{-4}	1.120×10^{-3}
0.1	2.108×10^{-3}	2.175×10^{-3}	5.888×10^{-3}	1.120×10^{-2}
1	2.107×10^{-2}	2.175×10^{-2}	5.878×10^{-2}	0.1113
10	0.2059	0.2114	0.5069	0.7456
100	0.8771	0.8276	0.9825	0.9952
10^3	0.9860	0.9054	0.9990	0.9997
10^5	0.9999	0.9203	1.0000	1.0000

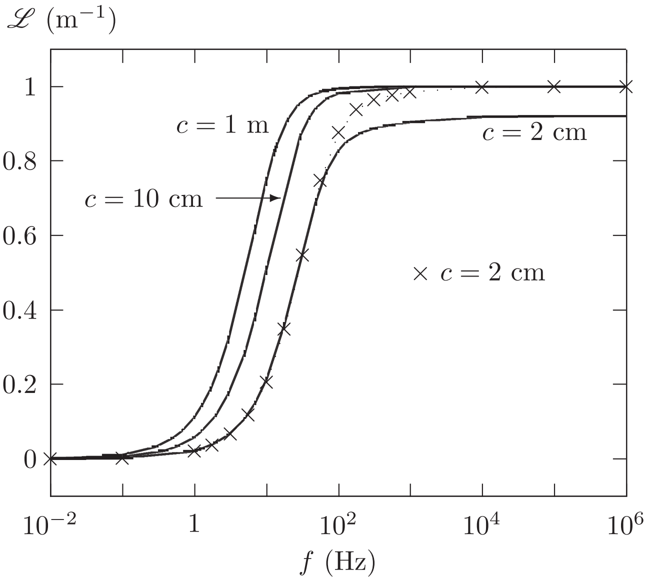

Figure 8. Dependence of $\mathscr{L}$ on the frequency f and on the distance c of conductor axes for conductors of rectangular cross section 10 × 32 mm^2 (×) and cylindrical conductors of 20 mm in diameter (lines).

In [16, 17] two methods were proposed and compared for the calculation of the real component R_s of impedance and the inductance L of a pair of long copper strip conductors, whose conductivity 56 MS·m^{-1} is constant over the conductor cross section. In [18], the formulae are given for the calculation of R_s and L but, unlike in [16, 17], the formulae have been extended to include the dependence of R_s and L on the dimensions of conductor cross sections and on their distance. The results in [16–18] and in the papers quoted therein were not obtained using accurate current density values. It is assumed that for low values of f the current density in the conductor cross section is constant while for higher f the current flows through a thin layer near the conductor surface. The assumption of constant current density is acceptable in strip conductors for lower frequencies, as given in [19], section 4.3. The values of the real component of impedance and of inductance given in [16–18] are erroneous because in their calculation via (1) inaccurate values of current were used. Current in the conductors had been calculated using inaccurate values of current density.

Another method for the calculation of R_s was published in [20–22]. The results given in [20–22] are often quoted and used, for example [23], and they have led to the introduction of the term skin effect. In [20–22], Art. 689, Maxwell describes the method for the calculation of current density in a long solitary conductor of circular cross section. Although he uses the word circuit several times in the text, he only considers one conductor and does not concern himself with the existence of such arrangement. A solitary conductor is not a suitable model for the calculation of self-inductance, as proved in [24–26], because it is a conductor connected to the ideal current source [27]. In [20–22], the calculation is performed of the resistance of a conductor segment as a quotient of the voltage on this segment and the current flowing through the conductor, with the assumption that the voltage on a conductor segment of finite length is of finite magnitude. This is impossible in the case of conductor connected to the ideal current source and therefore the calculated value of resistance cannot be correct either. Thomson prepared the publication of the book and he added an extensive note to Art. 689 in [22], in which he applies Maxwell's method to the calculation of R_s for the sinusoidal dependence of current on time. This calculation is not correct either.

7. Conclusion

Two long parallel conductors and a source of sinusoidal voltage form a long loop. The mathematical model of a loop segment characterized by one or more relations between integral quantities is a circuit element [15], 131-11-03. These integral quantities for the loop under examination are the source voltage $U(t)$ and the loop current $I(t)$. The method for the calculation of current density [2] allows the calculation of $I(t)$ in the loop for an arbitrary (not only sinusoidal) $U(t)$. The relation between the sinusoidal voltage $U(t)$ and the current $I(t)$ is given by the impedance (1), which replaces 1 m of loop. The imaginary component of equivalent impedance determines the loop inductance L. A quantity $\mathscr{L}$ was proposed (see (11)) whose value aptly characterizes self-induction in a loop.

From the analysis of the existing definitions of L and of the methods for calculating its value it follows that they are inaccurate. They start from the magnetic flux linked to a loop, but we have shown that magnetic flux cannot uniquely be assigned to a pair of conductors of finite cross section and furthermore it is not even given in [1]. The inductance L, the same as the real component R_s of impedance (1), depends, among other things, on the frequency of the voltage source and on the resistivity of the loop conductors. In the literature, except [4, 19, 24], this dependence is not respected for L and it is not correct for R_s either.

Equivalent impedance (1) completely describes self-induction in the considered long loop, which is the simplest possible arrangement of a pair of conductors. For a more complex arrangement of conductors, for example for conductors of finite length, no method for the calculation of current density in conductors, which is indispensable for the calculation of equivalent impedance, is currently available.

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