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Large deviations for the Kardar–Parisi–Zhang equation from the Kadomtsev–Petviashvili equation

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Abstract. Recently, Quastel and Remenik (2019 (arXiv:1908.10353)) found a remarkable relation between some solutions of the finite time Kardar–Parisi–Zhang (KPZ) equation and the Kadomtsev–Petviashvili (KP) equation. Using this relation we obtain the large deviations at large time and at short time for the KPZ equation with droplet initial conditions, and at a short time with half-Brownian initial conditions. It is consistent with previous results and allows us to obtain sub-leading corrections, as well as results at intermediate time. In addition, we find that the appropriate generating function associated to the full Brownian initial condition also satisfies the KP equation. Finally, generating functions for some linear statistics of the Airy point process are also found to satisfy the KP property, and consequences are discussed.

Keywords: exact results, growth processes, kinetic roughening, Painlevé equations

J. Stat. Mech. (2020) 043201

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Contents

J. Stat. Mech. (2020) 043201

1. Introduction

The Kardar–Parisi–Zhang equation [2] in one dimension is a continuum model for the stochastic growth of the height field $h(x, t)$, $x \in \mathbb{R}$, as a function of time t, of an interface between two phases in a two-dimensional geometry. It reads

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \sqrt{2}\,\xi(x, t) \tag{1}$$

in the units chosen here, where $\xi(x, t)$ is a unit Gaussian space-time white noise. It maps to the equilibrium statistical mechanics problem of a directed polymer in a $d = 1 + 1$ random potential [3], of partition function $Z(x, t) = \mathrm{e}^{h(x,t)}$, which satisfies the stochastic heat equation (SHE)

$$\partial_t Z = \partial_x^2 Z + \sqrt{2}\,\xi(x,t)Z \tag{2}$$

defined here with the Ito prescription. Three initial conditions (IC) have been much studied: (i) the flat IC $Z(x,t=0)=1$, (ii) the droplet IC, $Z(x,t=0)=\delta(x)$, and (iii) the Brownian IC, $Z(x,t=0)=\mathrm{e}^{B_L(x)+w_L x}\theta(-x)+\mathrm{e}^{B_R(x)-w_R x}\theta(x)$ where $B_{L,R}(x)$ are two unit half-sided Brownian motions with $B_{L,R}(0)=0$. The case $w_{L,R}\to 0$ is of special interest as it corresponds to the stationary IC.

The KPZ equation is at the center of a vast universality class, the KPZ class, which, in one dimension (to which we restrict here), contains a number of solvable discrete models for e.g. growth [4–6], particle transport [7, 8], or polymers [9]. Exact solutions have also been obtained for the one-point cumulative distribution function (CDF) of the height at arbitrary time for the KPZ equation, for the three aforementioned special initial conditions [10–22]. These results exhibit universal convergence at large time, upon scaling h with $t^{1/3}$ and space x with $t^{2/3}$, to Tracy–Widom (TW)-type distributions [23], the precise type depending on the class of initial conditions, specifically the GOE-TW distribution for flat IC, the GUE-TW distribution for droplet IC and the Baik–Rains distribution [4] for stationary KPZ.

Recently, a very detailed characterization of the universal KPZ fixed point, which governs the infinite time limit of all models in the KPZ class, has been obtained from the large time asymptotics of the TASEP model, for essentially arbitrary deterministic initial condition [24, 25]. The single-time, multi-point CDF of the (properly scaled) height field $h(x,t)$ was expressed as a Fredholm determinant (FD) with a Airy-type kernel, quite complicated and non-explicit for general IC (formally constructed from a Brownian scattering operator) but which simplifies into explicit forms for a number of cases. In parallel, asymptotic results were also obtained for the TASEP and the KPZ class on a finite-size periodic ring [26–29].

More recently, from the general FD form, Quastel and Remenik showed [1] that the CDF of the (properly scaled) height field can be related to solutions of a well-known equation in the theory of integrable systems, the Kadomtsev–Petviashvili (KP) equation (for the one-point CDF) and the KP matrix equation for the multi-point CDF. This remarkable result, which holds for the KPZ fixed point, i.e. for the infinite time limit of the typical fluctuations of the height field, was termed 'completely unexpected'. The appearance of KP-like solitons in the description of the KPZ fixed point was also pointed out in [26].

Even more surprising, it was noted that the finite-time solution of the KPZ equation itself can be related (for arbitrary time) to the KP equation. This was obtained in [1] for the droplet and half-Brownian initial conditions. More precisely, let us define the following generating function for the KPZ height at one point x and time t, equivalently the Laplace transform of the probability distribution function (PDF) of $Z(x,t)$,

$$G(x,t,r) = \left\langle \exp(-\mathrm{e}^{h(x,t)+\frac{t}{12}-r}) \right\rangle = \left\langle \exp(-Z(x,t)\mathrm{e}^{\frac{t}{12}-r}) \right\rangle \tag{3}$$

where $\langle\cdots\rangle$ denotes the average w.r.t. the KPZ noise $\xi(x,t)$. It was shown a while back, using e.g. the replica Bethe ansatz, that this generating function can be calculated exactly for all t, as a Fredholm determinant (FD), for the droplet IC [10–14] and for the half-Brownian IC [17, 18] , that is (on $x>0$) $Z(x,t=0)=\mathrm{e}^{B(x)-wx}\theta(x)$. It was obtained in [1], using the FD expressions, that the following function $\phi(r,x,t)$ of three variables

$$\phi(x,t,r) = \partial_r^2 \log G(x,t,r) \tag{4}$$

obeys the KP equation

$$\partial_t \phi + \phi\, \partial_r \phi + \frac{1}{12} \partial_r^3 \phi + \partial_r^{-1} \partial_x^2 \phi = 0. \tag{5}$$

The question of the initial condition is somewhat subtle and is discussed below. Note that for any fixed x, t, $G(x,t,r)$ increases from 0 (for $r = -\infty$) to 1 (for $r = +\infty$), hence $\log G(x,t,r) \leqslant 0$ is increasing, i.e. $\partial_r \log G(x,t,r)$ is positive and $G(x,t,r)$ and its derivatives w.r.t. r vanish at $r = +\infty$.

The unexpected result (4) and (5) opens many questions, and we wish to address some of them here. We also wish to open the new toolbox that it provides. We first show that one can easily recover recent results about the large deviations for the KPZ equation, directly from the KP equation, and obtain a bit more. Large deviations mean rare fluctuations, away from the well-studied typical fluctuations $H \sim t^{1/3}$. There are two limits of interest, large time $t \gg 1$ and short time $t \ll 1$. It was shown that for droplet IC, the PDF of the (shifted) one point height $H = h(x,0) + \frac{t}{12}$, takes at large time, and in the scaling region $H \sim t$, the large deviation form [30]

$$P(H,t) \sim \exp(-t^2 \Phi_-(\frac{H}{t})). \tag{6}$$

This holds for the left large deviation tail $H/t < 0$ (similar results hold for the right tail, with a different rate, not addressed here). The exact rate function $\Phi_-(z)$ was obtained by *four* different, and non-trivial methods involving: (i) the WKB limit of a non-local Painleve equation [31] (ii) the free energy of a Coulomb gas perturbed at the edge [32] (iii) a variational formula using the stochastic Airy operator [33], (iv) a summation of the *short time* expansion using cumulants [34] (see related rigorous results for the tails in [35]). In [36] it was found how the four methods can be related and extended to treat a broader class of problems, involving linear statistics of the eigenvalues at the edge of the β-random matrix ensembles (described by the Airy_2 point process). Here we show that (6), together with the exact expression for $\Phi_-(z)$, arise quite naturally from the analysis of the KP equation, henceforth providing a fifth method. In addition, we extract the subleading corrections at large time.

At short time $t \ll 1$, it was shown that the large deviations occur in the regime $H \sim O(1)$ (while the typical fluctuations are $H \sim t^{1/4}$) and take, quite generically, the form

$$P(H,t) \sim \exp(-\frac{\Phi(H)}{\sqrt{t}}). \tag{7}$$

The rate function $\Phi(H)$ was (i) calculated from exact solutions for droplet IC, Brownian IC and for some half-space cases [37–41], (ii) related to solutions of (saddle point) differential equations using the weak noise theory, allowing us to extract the exact asymptotics of $\Phi(H)$ at large $|H|$ for a variety of IC [42–52]. Both methods were found to be consistent, and the results were also tested in very high precision numerical simulations [53–55]. The first method proceeds by first showing that at small time, for e.g. the droplet IC one has, for any $z > 0$

J. Stat. Mech. (2020) 043201

$$\big\langle \exp(-\frac{z e^H}{\sqrt{t}})\big\rangle \sim \exp(-\frac{\Psi(z)}{\sqrt{t}}) \quad , \quad \Psi(z) = \frac{-1}{\sqrt{4\pi}}\mathrm{Li}_{5/2}(-z) \tag{8}$$

with $H = h(0,t) + \frac{1}{2}\log t$. The rate function $\Phi(H)$ is then obtained via a (quite subtle) Legendre transform. In [34], we further calculated, for droplet IC, the subleading terms in the exponential in the rhs of (8), which takes the form of the series $\frac{1}{\sqrt{t}}\Psi(z) + \sum_{p\geqslant 1} t^{\frac{p-1}{2}}\Psi_p(z)$, up to a very high order, $O(t^3)$.

To address the small time large deviations, we first write the property (4) and (5) in terms of equations satisfied by the cumulants of $Z(x,t)$. Using the known expressions for the first lowest cumulants, we check that these equations are indeed satisfied for the droplet and the half-Brownian initial conditions (but not for the flat IC, which thus does not seem to be simply related to KP). One finds that, at short time, the nonlinear term in the KP equation enters only perturbatively. This allows us to determine iteratively the subleading terms in the rhs of (8) in terms of only the leading one, $\Psi(z)$. This procedure very efficiently recovers the systematic expansion obtained in [34], and allows us to go beyond. This in itself, provides a strong test of the KP property. However, we find that for the droplet IC the initial data problem is subtle, i.e. the leading term, $\Psi(z)$, remains undetermined. Specifying this large deviation rate function is equivalent to specifying the amplitudes of the $\sim 1/\sqrt{t}$ leading short time behavior for each cumulant of $Z(0,t)$. This thus appears as the initial data one must input in the KP equation. This subtlety is not so surprising, since the droplet IC, $Z(x,0) = \delta(x)$, needs some regularization, see below.

In section 2.5, we provide a bridge between the short time and the large time large deviations. This is achieved through a summation of cumulants, initiated in [34], and that we push here, thanks to the KP equation, to the next subdominant order. We show that it is equivalent to a semi-classical expansion, which takes the form of a perturbative expansion in the third derivative in the KP equation, whose leading order amounts to solve the Burgers equation.

The generating function (129) of the KPZ equation with droplet initial condition can be put in the form $G(x,t,r) = \hat{G}(t, r + \frac{x^2}{4t})$, where $\hat{G}(t,r)$ satisfies a reduced version of the KP equation (known as cylindrical KdV equation). We show that the KPZ equation with droplet IC is only one particular solution of a more general class of solutions, which encode for some linear statistics of the Airy$_2$ point process (denoted a_j)

$$\hat{G}(t,r) = \log \mathbb{E}_{\mathrm{Ai}}[\exp\big(\sum_{j=1}^{\infty} f(t^{1/3}a_j - r)\big)]. \tag{9}$$

This generating function was studied in [34, 36] (see definitions in equations (30)–(32) there), and $f(x)$ is a fairly general function, the special choice $f(x) = -\log(1+e^x)$ corresponds to the KPZ equation. We show here that for any $f(x)$ (where it exists), $\partial_r^2\hat{G}(t,r)$ satisfies the reduced KP equation, see remark 2. below, and the appendix E for the general analysis of a family of FD which satisfy the KP equation. This allows for a semi-classical expansion of the linear statistics of the Airy$_2$ point process using the KP equation, discussed in appendix C.

Next, we consider the half-Brownian initial condition. There, using the KP equation, we obtain in the short time limit $t \ll 1$, with $\tilde{x} = x/\sqrt{t}$ and $\tilde{w} = w\sqrt{t}$ being kept fixed

$$G(x,t,r) \simeq \left\langle \exp(-\frac{z e^{H}}{\sqrt{t}}) \right\rangle \sim \exp(-\frac{\Psi_{\tilde{w}}(\tilde{x} = \frac{x}{\sqrt{t}}, \sqrt{t} z))}{\sqrt{t}} \tag{10}$$

with $H = h(x,t) + \frac{1}{2}\log t$, together with an explicit expression for $\Psi_{\tilde{w}}(z,\tilde{x})$, see (158) and (160). Further taking the limit $\tilde{w} \to +\infty$ we finally obtain the result (8) for the droplet IC. Hence the half-Brownian solution to the KP equation, which is well defined at $t = 0$, can be used to regularize the droplet solution at small time.

Finally, comparing the equations that the cumulants of $Z(x,t)$ must obey so that KP holds, and the known expressions for these cumulants, e.g. from the replica Bethe ansatz, we identify the mechanism of solvability, see section 3 and appendix B . It then implies that any IC that has a 'decoupled' overlap with the Bethe eigenstates will similarly obey the KP equation. This is the case for the droplet and the half-Brownian IC, but since this is also the case for the full Brownian initial condition, we conclude that the full Brownian IC does also satisfy KP. More precisely, using an appropriately modified generating function $\tilde{G}$, $\phi = \partial_r^2 \tilde{G}$ must satisfy KP. We check explicitly this conjecture by comparing with the known small time large deviation rate function for the Brownian IC obtained in [38].

Note that the original theorem for the CDF of the KPZ fixed point obeying KP was shown *a priori* only for deterministic initial conditions. The fact that the KPZ equation with random IC (Brownian and half-Brownian) also obeys KP is thus a quite interesting development[1].

Note added. In a recent preprint [56], simultaneous with the first version of this work, Cafasso and Claeys obtain yet another derivation of the KPZ large deviation left tail using Riemann–Hilbert (RH) methods. These methods allow for a rigorous proof of the asymptotics. In remark 4 below we compare with our results. Their formula also applies to intermediate times. We show in section 2.6 that intermediate time results can also be obtained from the KP equation[2].

2. Droplet initial condition

2.1. Space-independent, reduced KP equation

We start with the droplet initial condition $Z(x, t = 0) = \delta(x)$. We first use the statistical tilt symmetry to eliminate the spatial variable x. For the droplet IC it is well known that $h(x,t) \equiv h(0,t) - \frac{x^2}{4t}$ where $\equiv$ means identity in law of the one point distributions. Hence one can write

$$G(x,t,r) = \left\langle \exp(-e^{h(0,t) - \frac{x^2}{4t} + \frac{t}{12} - r}) \right\rangle = \hat{G}(t, r + \frac{x^2}{4t}) \quad , \quad \phi(x,t,r) = \partial_r^2 G(x,t,r) = \psi(t, r + \frac{x^2}{4t}) \tag{11}$$

[1] Note that for the half-Brownian this was noted in the second version of the work [1].
[2] I thank Cafasso for discussions which led to these developments.

J. Stat. Mech. (2020) 043201

where we now denote

$$\hat{G}(t,r) = \langle \exp(-\mathrm{e}^{h(0,t)+\frac{t}{12}-r}) \rangle \quad , \quad \psi(t,r) = \partial_r^2 \log \hat{G}(t,r). \tag{12}$$

Here $\hat{G}$ is the standard generating function for the height at $x=0$. Injecting the form (11) into (5) one obtains, after some cancellations, a reduced KP equation

$$\partial_t \psi + \psi \partial_r \psi + \frac{1}{12}\partial_r^3 \psi + \frac{1}{2t}\psi = 0. \tag{13}$$

Note that this equation can be integrated once, with $\psi(t,r) = \partial_r \hat{\psi}(t,r)$, i.e. it can also be written for the function $\hat{\psi}(t,r) = \partial_r \log \hat{G}(t,r)$, as

$$\partial_t \hat{\psi} + \frac{1}{2}(\partial_r \hat{\psi})^2 + \frac{1}{12}\partial_r^3 \hat{\psi} + \frac{1}{2t}\hat{\psi} = 0 \tag{14}$$

where the integration constant must vanish since $\hat{\psi} = \partial_r \log \hat{G}$ vanishes at $r \to +\infty$.

Remark 1. Equation (13) is also called the cylindrical Korteweg-de Vries (KdV) equation (up to a rescaling of coefficients). Upon the change of variable (with $b > 0$)

$$\psi(t,r) = \frac{r}{2t} + \frac{B}{2t}u(y,\tau) \quad , \quad r = -\frac{y}{\tau} \quad , \quad t = \frac{b}{\tau^2} \tag{15}$$

it is transformed into the standard KdV equation[3]

$$\partial_\tau u + \frac{b}{6}\partial_y^3 u + B u \partial_y u = 0. \tag{16}$$

The canonical form is obtained for $b=6$ and $B=-6$, and the form $b=\frac{1}{2}$, $B=1$ arises in the description of the KPZ fixed point for flat IC [1]. Note that in the forthcoming paper [57], the KdV equation (16) is derived for the general case of $\hat{G}(t,r)$ defined as in (9) using the Riemann–Hilbert setting proposed in [56].

2.2. Checks and moment expansion

We can now perform a few checks. The function $\psi(t,r)$, from its definition (12), admits an expansion in powers of e^{-r} (e.g. for large positive r) whose coefficients are related to the cumulants $Z_n(t) = \langle Z(0,t)^n \rangle_c$ of the solution to the SHE, as

$$\psi(t,r) = \sum_{n \geqslant 1} \frac{(-1)^n n^2}{n!} Z_n(t) \mathrm{e}^{\frac{nt}{12} - nr} \tag{17}$$

using that $Z(x,t) = \mathrm{e}^{h(x,t)}$. Inserting into (13), we find that the reduced KP equation implies that the cumulants $Z_n(t)$, $n \geqslant 1$, must satisfy the following recursion relation

$$-\partial_t Z_n(t) + \frac{n^3 - n}{12} Z_n(t) - \frac{1}{2t} Z_n(t) = -\frac{(n-1)!}{2} \sum_{n_1+n_2=n, n_1,n_2 \geqslant 1} \frac{n_1^2 n_2^2}{n_1! n_2!} Z_{n_1}(t) Z_{n_2}(t). \tag{18}$$

[3] I thank Cafasso for pointing this out.

Before performing some simple checks that the known expressions for the $Z_n(t)$ do indeed satisfy this equation, we first ask whether this recursion determines the $Z_n(t)$. The answer is that this recursion determines iteratively the $Z_n(t)$, except that at each level $n \geqslant 1$ there is an unknown integration constant c_n, since the solution to the homogeneous part of (18) is $Z_n^{\rm hom}(t) = \frac{c_n}{\sqrt{t}} {\rm e}^{\frac{1}{12}(n^3-n)t}$. Let us examine the two lowest orders from (18)

$$-\partial_t Z_1(t) - \frac{1}{2t} Z_1(t) = 0 \tag{19}$$

$$-\partial_t Z_2(t) + \frac{1}{2} Z_2(t) - \frac{1}{2t} Z_2(t) = -\frac{1}{2} Z_1(t)^2. \tag{20}$$

The first equation gives $Z_1(t) = c_1/\sqrt{t}$. We known that, upon averaging the SHE (with the Ito convention), the first moment $\langle Z(x,t)\rangle$ satisfies the standard heat equation. Hence, for droplet IC one has $\langle Z(x,t)\rangle = \frac{1}{\sqrt{4\pi t}} {\rm e}^{-\frac{x^2}{4t}}$, i.e. the free diffusion kernel, and one must have $Z_1(t) = \langle Z(0,t)\rangle = \frac{1}{\sqrt{4\pi t}}$, which determines $c_1 = 1/\sqrt{4\pi}$. The general solution to the second equation is then

$$Z_2(t) = {\rm e}^{t/2}\left(\frac{c_2}{\sqrt{t}} + \frac{1}{4\sqrt{2\pi t}} {\rm Erf}\left(\sqrt{\frac{t}{2}}\right)\right). \tag{21}$$

On the other hand, the result of the Bethe ansatz calculation, i.e. equation (11) in [10], is

$$Z_2(t) = \frac{1}{4} \frac{1}{\sqrt{2\pi t}} {\rm e}^{t/2} \left(1 + {\rm Erf}\left(\sqrt{\frac{t}{2}}\right)\right). \tag{22}$$

The two formula (21) and (22) are indeed consistent, provided we choose the integration constant $c_2 = \frac{1}{4\sqrt{2\pi}}$. It is easy, but tedious, to check from the Bethe ansatz results, that a similar pattern holds for higher cumulants (see appendix B for related checks). Hence we conclude that the coefficients c_n play the role of the initial data. Equivalently, one can characterize the initial data by specifying the small time t limit as $Z_n(t) \simeq \frac{c'_n}{\sqrt{t}}$ (with, as we checked here $c'_n = c_n$ for $n = 1, 2$).

2.3. Short time large deviation expansion

It turns out that the coefficients c'_n are known, they were obtained in the large deviation analysis of the KPZ equation at short time in [37]. It was shown there that as $t \to 0$

$$\log \hat{G}(t,r) \simeq -\frac{1}{\sqrt{t}} \Psi(z = {\rm e}^{-r}) \quad , \quad \Psi(z) = \frac{-1}{\sqrt{4\pi}} {\rm Li}_{5/2}(-z). \tag{23}$$

Hence the short time behavior of $\psi(t,r)$ in (17) must be as $\sim \frac{1}{\sqrt{t}}$, uniformly in r, with

$$\psi(t,r) \simeq -\frac{1}{\sqrt{t}} \partial_r^2 \Psi(z = {\rm e}^{-r}) = \frac{1}{\sqrt{4\pi t}} {\rm Li}_{1/2}(-{\rm e}^{-r}) = -\frac{{\rm e}^{-r}}{\sqrt{4\pi t}} + \frac{{\rm e}^{-2r}}{2\sqrt{2\pi t}} + O({\rm e}^{-3r}). \tag{24}$$

This is consistent with the terms $n = 1, 2$ in (17) and the values $c'_1 = c_1 = 1/\sqrt{4\pi}$, $c'_2 = c_2 = \frac{1}{4\sqrt{2\pi}}$ obtained above (since at small t one can set the factor $\mathrm{e}^{\frac{nt}{12}}$ to unity to the leading order). The equation (24) determines all the c'_n, hence the initial data.

Let us backtrack one step and ask whether one can obtain the full small time expansion from the KP equation? In [34] it was shown that (23) was just the leading order of a systematic short-time expansion in powers of $\sqrt{t}$. Hence we will look for the form

$$\psi(t,r) = \frac{1}{\sqrt{t}}\,\partial_r\left(p_0(r) + \sum_{m\geqslant 1} p_m(r) t^{\frac{m}{2}}\right) \tag{25}$$

and insert it in the reduced KP equation (13). It gives a recursion (see appendix A) which can be solved in a hierarchical way

$$p_1(r) = -p'_0(r)^2 \tag{26}$$

$$p_2(r) = -p'_0(r)\,p'_1(r) - \frac{1}{12}p'''_0(r) = \partial_r(\frac{2}{3}p'_0(r)^3 - \frac{1}{12}p''_0(r)) \tag{27}$$

and so on (here and below we use indifferently the notations $\partial_r p(r) \equiv p'(r)$). We note that $p_0(r)$ is left undetermined, but all the $p_i(r)$ with $i \geqslant 1$ are obtained from $p_0(r)$. We will thus consider $p_0(r)$ as an 'initial condition' and set it equal to the known result

$$p'_0(r) = \frac{1}{\sqrt{4\pi}}\mathrm{Li}_{1/2}(-\mathrm{e}^{-r}). \tag{28}$$

We can now compare with the result from [34], obtained through a quite complicated calculation directly expanding the FD. The relevant formula there are (5), (30), (58), (61), (62). We check that

$$\psi(t,r) = \partial_r^2 q_{t,\beta=1}(\sigma) \quad , \quad \sigma = -\mathrm{e}^{-r} \quad , \quad \sigma\partial_\sigma = -\partial_r \tag{29}$$

with $q_{t,\beta=1}$ there equals $\log \hat{G}$ here, and the following functions were introduced there

$$\mathcal{L}_i(\sigma) = \frac{1}{\sqrt{4\pi}}\mathrm{Li}_{\frac{3}{2}-i}(\sigma) \quad , \quad \sigma\partial_\sigma \mathcal{L}_i = \mathcal{L}_{i+1}. \tag{30}$$

Let us check here the first two terms. One has $p'_0(r) = \mathcal{L}_1(-\mathrm{e}^{-r})$ hence (25) and (26) lead to

$$\psi(t,r) = \frac{1}{\sqrt{t}}\left(\mathcal{L}_1 - \sqrt{t}\partial_r\mathcal{L}_1^2 + t\partial_r^2(\frac{2}{3}\mathcal{L}_1^3 - \frac{1}{12}\partial_r\mathcal{L}_1) + O(t^{3/2})\right) \tag{31}$$

where $\mathcal{L}_i \equiv \mathcal{L}_i(-\mathrm{e}^{-r})$. Using that $-\partial_r\mathcal{L}_i = \mathcal{L}_{i+1}$ and comparing with (58) and (62) in [34] we see that it agrees. In the appendix A we have checked terms to a much higher order, we recover all terms displayed in [34] and show one order more, i.e. the term $O(t^{7/2})$. The present method is clearly much faster.

The conclusion of this subsection is that the KP equation allows us to recover easily the full systematic short time expansion for the KPZ generating function with droplet IC, obtained in [34] from the FD, provided it is given as an input, i.e. initial data, the leading term (28) for the large deviations which encodes the limits $c'_n = \lim_{t\to 0} t^{1/2} Z_n(t)$.

J. Stat. Mech. (2020) 043201

Remark 2. It is interesting to note that in [34] we have considered a more general problem, the evaluation of the following linear statistics over the Airy$_2$ point process a_i (see definitions and equations (30) and (32) there)

$$q_{t,\beta}(\sigma) = \log \mathbb{E}_{\mathrm{Ai}}\left[\exp\left(\beta\sum_{j=1}^{\infty} g(\sigma \mathrm{e}^{t^{1/3}a_j})\right)\right] = \log \mathrm{Det}[I - (1-\mathrm{e}^{\beta \hat{g}_{t,\sigma}})K_{\mathrm{Ai}}] \quad (32)$$

$\hat{g}_{t,\sigma}(a) = g(\sigma \mathrm{e}^{t^{1/3}a})$ (with $g(-e^x) = f(x)$ as defined in the introduction). It leads to exactly the same expansion of $q_{t,\beta}(\sigma)$ in terms of the $\mathcal{L}_i$, i.e. all coefficients being independent of the choice of the function $g(x)$, where now

$$\mathcal{L}_i \equiv \mathcal{L}_i(\sigma) = \frac{\beta}{\pi}(\sigma\partial_\sigma)^{i+1}\int_0^{+\infty} \mathrm{d}x\sqrt{x}g(\sigma\mathrm{e}^{-x}) = \beta(\sigma\partial_\sigma)^i \int_{-\infty}^{+\infty}\frac{\mathrm{d}p}{2\pi}g(\sigma\mathrm{e}^{-p^2}) \quad (33)$$

and the case of the KPZ equation is recovered for the choice $g(x) = g_{\mathrm{KPZ}}(x) = -\log(1-x)$. The important remark is thus that once the 'initial condition'

$$p_0'(r) = \mathcal{L}_1(-\mathrm{e}^{-r}) \quad (34)$$

is specified, then the hierarchy of equations for the $p_m(r)$ that we derived from the simplified KP equation, yields exactly the same result for the $p_m(r)$ as a function of the $\mathcal{L}_i(\sigma = -\mathrm{e}^{-r})$ as for the KPZ case. Hence it appears that for this more general problem $\phi(t,r) = \partial_r^2 q_{t,\beta}(\sigma = -\mathrm{e}^{-r})$ does satisfy also the KP equation, although with a different initial condition. This is shown here in appendix E using properties of FD. Note that, strictly, this holds only if the chosen function $g(x)$ is itself time-independent (hence, it does not apply *a priori* to the developments in section 5.2 in [34], although we do find an application below, see also appendix C).

Of course, considering (32) is a natural extension, once the conditions for a FD to lead to KP have been identified, i.e. some conditions were given in [1] (see also appendix E here). However the question of the initial condition was not discussed there, and it is enlightening to see how it works out on the small time expansion.

2.4. Large time large deviations

2.4.1. Leading order. We now study the large time limit, and search for a left tail large deviation form of the type (6), in the limit where both $-r, t \to +\infty$ with $z = r/t < 0$ fixed, that is

$$\log \hat{G}(t,r) = \log\langle \exp(-\mathrm{e}^{h(0,t)+\frac{t}{12}-r})\rangle = -t^2\Phi_-(z) + o(t^2) \quad , \quad r = zt \quad (35)$$

where one must have $\Phi_-(z) \geqslant 0$ and $\Phi_-'(z) \leqslant 0$ since $\log \hat{G}$ is a negative increasing function of r. Also one must have $\Phi_-(0) = 0$ since the regime of typical fluctuations, $r \sim t^{1/3}$, correspond to $z = 0$. It is useful to note that the generating function can also be written as

$$\log\langle \exp(-\mathrm{e}^{H-r})\rangle = \mathrm{Prob}(H + \gamma < r) \quad (36)$$

J. Stat. Mech. (2020) 043201

where γ is a unit Gumbel variable, i.e. of CDF $\mathrm{Prob}(\gamma < a) = \mathrm{e}^{-\mathrm{e}^{-a}}$, independent of H (here $H = h(0,t) + \frac{t}{12}$). Hence in the limit $t \to +\infty$ one has $\log\langle \exp(-\mathrm{e}^{H-zt})\rangle \simeq \log \mathrm{Prob}(\frac{H}{t} < z)$ and (35) is equivalent to (6).

Equation (35) implies that we must search for the following scaling form for $\psi(t,r)$ in that limit

$$\psi(t,r) = \partial_r^2 \log \hat{G}(t,r) = H_0(\frac{r}{t}) \quad , \quad H_0(z) = -\Phi_-''(z). \tag{37}$$

Substituting this form into the reduced equation (13), we find that $H_0(z)$ must satisfy

$$\frac{1}{2}H_0(z) - zH_0'(z) + H_0(z)H_0'(z) = 0 \tag{38}$$

where we can neglect the third derivative term $\frac{H_0'''(z)}{12t^2}$ in the large time limit. One can also write the integrated version inserting $\hat{\psi}(r,t) = -t\hat{\psi}_0(H/t) + o(t)$ into (14), with $\hat{\psi}_0(z) = -\Phi_-'(z)$ and $H_0(z) = \hat{\psi}_0'(z)$. The resulting equation is $\frac{3}{2}\hat{\psi}_0 - z\hat{\psi}_0' + \frac{1}{2}(\hat{\psi}_0')^2 = 0$, i.e. the integrated version of (38), with the integration constant automatically fixed from (14), a constraint which can be written $-3\Phi_-'(0) + \Phi_-''(0)^2 = 0$. Given that $\Phi_-'(z)$ must be negative it implies that $\Phi_-'(0) = \Phi_-''(0) = 0$.

The general solution of this equation with $H_0(0) = 0$ and which is real for any $z < 0$ reads

$$H_0(z) = \frac{1}{A\pi^2}(1 - \sqrt{1 - A\pi^2 z}) = \frac{z}{2} + \frac{A\pi^2}{8}z^2 + O\left(z^3\right) \tag{39}$$

where $A > 0$ is a constant, as yet undetermined. The factor π^2 has been introduced for convenience. The $+$ branch is also a solution but leads to $H(0) = 2/(A\pi^2)$ and can be discarded. Note that $H_0(z) = z/2$ is also a solution, corresponding to $A = 0$. Let us now integrate twice to obtain $\Phi_-(z)$, using that $\Phi_-(0) = \Phi_-'(0) = 0$, leading to

$$\Phi_-(z) = \frac{4}{15\pi^6 A^3}\left((1 - A\pi^2 z)^{5/2} - 1\right) - \frac{z^2}{2\pi^2 A} + \frac{2z}{3\pi^4 A^2} = -\frac{1}{12}z^3 + O(z^4). \tag{40}$$

This is precisely the known result from the four methods [31–34], if we choose $A = 1$. Note that the result at small $z < 0$, $\Phi_-(z) \simeq_{z\to 0} -\frac{1}{12}z^3$, is obtained independently of the choice for A. This result can also be obtained [30] by matching with the cubic tail of the Tracy Widom GUE distribution, from the regime of typical fluctuations $r \sim t^{1/3}$, assuming no intermediate regime (which seems to be indeed excluded by the present analysis).

Hence the KP equation provides a fifth and easy method to determine the large deviation function for the left tail. The only remaining question is thus how to determine the constant A. It can be set by the outer tail $\Phi_-(z) \simeq \frac{4}{15\pi}(-z)^{5/2}$ at large $z < 0$ which can be obtained from the simplest and first term in the cumulant expansion [32, 34, 36, 39], which, by Jensen's inequality, is also an exact upper bound for $\Phi_-(z)$. Equivalently, the constant A can be determined from the structure of the short time expansion (which is a cumulant expansion). This was used in [34] to obtain the full function $\Phi_-(z)$. We will implement that program below within the KP equation approach (in section 2.5), and show that indeed the constant $A = 1$ by relating the short time and large time behaviors.

J. Stat. Mech. (2020) 043201

2.4.2. Subleading orders. We can now search for a systematic large time expansion in the large deviation regime, of the form

$$\log \hat{G}(t,r) = \log\langle \exp(-\mathrm{e}^{h(0,t)+\frac{t}{12}-r})\rangle = -t^2\Phi_-(\frac{r}{t}) - t\,\Phi_1(\frac{r}{t}) - \Phi_2(\frac{r}{t}) + \ldots. \tag{41}$$

We thus insert in (13) the expansion

$$\psi(t,r) = \partial_r^2 \log \hat{G}(t,r) = \sum_{m\geqslant 0} t^{-m} H_m(r/t) \quad , \quad \hat{\psi}(t,r) = \partial_r \log \hat{G}(t,r) = \sum_{m\geqslant 0} t^{1-m}\hat{\psi}_m(r/t). \tag{42}$$

We find the following equation for $H_1(z)$

$$(H_0(z)-z)H_1'(z) + H_1(z)(H_0'(z)-\frac{1}{2}) = 0 \tag{43}$$

and the integrated version $2\hat{\psi}_1'(z-\hat{\psi}_0) = \hat{\psi}_1$ (which fixes $\hat{\psi}_1(0) = -\Phi_1'(0) = 0$). This leads to

$$H_1(z) = \frac{a_1}{\sqrt{1-A\pi^2 z}} \quad , \quad \hat{\psi}_1(z) = \frac{2a_1}{A\pi^2}(1-\sqrt{1-A\pi^2z^2}) \tag{44}$$

where a_1 is undetermined and A was discussed in the previous section. It leads to

$$\Phi_1(z) = \frac{4a_1}{3\pi^4A^2}(1-(1-\pi^2Az)^{3/2}) - \frac{2a_1 z}{\pi^2 A} = -\frac{a_1}{2}z^2 + O(z^2) = -\frac{4a_1(-z)^{3/2}}{3\pi\sqrt{A}} + O(z) \tag{45}$$

where we have used that $\Phi_1(0) = 0$. Indeed by matching to the typical regime we want $t\Phi_1(st^{-2/3})$ to be at most of order unity at large t, hence $\Phi_1(z)$ is at most $|z|^{3/2}$ which implies $\Phi_1(0) = \Phi_1'(0) = 0$.

However it turns out, as we see below from the examination of the cumulants in the short time expansion, that in fact $a_1 = 0$ for the KPZ equation. It can already be guessed from considering the subleading term in the small time expansion

$$\log \hat{G}(t,r) \simeq \frac{1}{\sqrt{4\pi t}}\mathrm{Li}_{5/2}(-\mathrm{e}^{-r}) = -\frac{1}{\sqrt{t}}(\frac{4}{15\pi}(-r)^{5/2} + \frac{\pi}{6}\sqrt{-r} + \ldots) \tag{46}$$

substituting $r = zt$, the first term is $O(t^2)$ and the second is $O(1)$, the $O(1/t)$ term proportional to $(-z)^{3/2}$ is missing. We have used for $x \to +\infty$ and $\nu > 3/2$, see [66]

$$\mathrm{Li}_\nu(-x) = -\frac{(\log x)^\nu}{\Gamma(\nu+1)} - \frac{\pi^2}{6}\frac{(\log x)^{\nu-2}}{\Gamma(\nu-1)} + \ldots. \tag{47}$$

The complete argument for $a_1 = 0$ is made below. Note however that for the more general problem defined in remark 2 above, this constant is not necessarily zero.

Let us consider the next correction. Setting $a_1 = 0$, i.e. $H_1(z) = 0$, the equation for $H_2(z)$ is

$$H_2(z)(H_0'(z)-\frac{3}{2}) - zH_2'(z) + H_0(z)H_2'(z) + \frac{1}{12}H_0'''(z) = 0 \tag{48}$$

and its integrated version $-\frac{1}{2}\hat{\psi}_2 + (\hat{\psi}_0' - z)\hat{\psi}_2' + \frac{1}{12}\hat{\psi}_0''' = 0$. We obtain the solution (from now on we set $A = 1$, its value for the KPZ equation, see above sections)

J. Stat. Mech. (2020) 043201

$$H_2(z) = \frac{96a_2\left(1-\pi^2 z\right)^{3/2} + \pi^4\left(2-3\sqrt{1-\pi^2 z}\right)}{96\left(\pi^2 z-1\right)^2\left(\sqrt{1-\pi^2 z}-1\right)^2} \quad ,$$

$$\hat{\psi}_2(z) = \frac{96a_2\left(\pi^2 z-1\right)+\pi^4}{48\pi^2\left(\pi^2 z-1\right)\left(\sqrt{1-\pi^2 z}-1\right)} = \frac{\frac{1}{24}-\frac{4a_2}{\pi^4}}{z} + O\left(z^0\right) \tag{49}$$

where a_2 is undetermined and we note that for general a_2 the function $\hat{\psi}_2(z) = -\Phi_2'(z)$ diverges at $z=0$. This leads to

$$\Phi_2(z) = \frac{192a_2\sqrt{1-\pi^2 z} - 2\left(\pi^4-96a_2\right)\log\left(\sqrt{1-\pi^2 z}-1\right) + \pi^4\log\left(1-\pi^2 z\right)}{48\pi^4} + b \tag{50}$$

where b is an (undetermined) integration constant. The constant a_2 is determined below from the cumulant expansion and is found to be $a_2 = \frac{\pi^4}{24}$. With this value of a_2 we find that at small $z \to 0^-$ one has

$$\Phi_2(z) \simeq \frac{1}{8}\log(-z) + \frac{4+\log\frac{\pi^6}{8}}{24} + b - \frac{7\pi^2 z}{96} + O\left(z^2\right). \tag{51}$$

One could try to match this result to the left tail asymptotics of the Tracy Widom GUE distribution, recalled in (114) below, which gives, naively, $\log F_2(s = zt^{2/3}) \simeq -\frac{t^2}{12}|z|^3 - \frac{1}{8}\log(|z|t^{2/3}) + C$. The first term is the correct leading small $|z|$ behavior of $\Phi_-(z)$, as discussed above, and the second term, i.e. the $-\frac{1}{8}\log|z|$ does agree indeed with (51) and (41). Furthermore, it is clear that b could also be time-dependent, and contain a $\log t$ term (which disappears in taking the derivative $\hat{\psi}(t,r)$). This makes the matching of the remaining $O(1)$ term delicate, i.e. the constant part of $\Phi_2(z)$ at small z, especially in view of the following remark.

Remark 3. Having determined the subdominant rate functions for the generating function in (41), we would like to translate it into large deviation rate functions for the probability for H. Tentatively, one would search for a formula of the type, at large t,

$$\log \mathrm{Prob}(H \leqslant zt) \simeq -t^2\Phi_-(z) - t\,\hat{\Phi}_1(z) - \hat{\Phi}_2(z) + \ldots. \tag{52}$$

While it is clear that the leading order involves the same function $\Phi_-(z)$ as in (41), determining the subleading orders is more delicate. First, there is a useful upper bound from the Markov inequality $\log \mathrm{Prob}(H \leqslant zt) = \log \mathrm{Prob}(\mathrm{e}^{-\mathrm{e}^{H-zt}} \geqslant \frac{1}{e}) \leqslant \log\langle\exp(-\mathrm{e}^{H-zt})\rangle + 1$. If the form (52) holds, it would imply $\hat{\Phi}_1(z) \geqslant \Phi_1(z) = 0$, and $\hat{\Phi}_2(z) > \Phi_2(z) + 1$. Second, if one follows [58] (section 3.1) one has the lower bound $\mathrm{Prob}(H \leqslant zt) \geqslant \langle\exp(-\mathrm{e}^{H-\tilde{z}t})\rangle$ $\mathrm{e}^{-\mathrm{e}^{t(z-\tilde{z})}}$, for any $\tilde{z}$. The best one can do is choose $\tilde{z} \simeq z - \frac{1}{t}\log(Ct^2)$ and obtain a bound of the type $\log \mathrm{Prob}(H \leqslant zt) \geqslant -t^2\Phi_-(\tilde{z}) - \hat{\Phi}_2(\tilde{z}) - a$ where $a > 0$. Inserting $-t^2\Phi_-(\tilde{z}) \simeq -t^2\Phi_-(z) + t\Phi_-'(z)\log(Ct^2)$. Hence it does not produce any useful lower bound on $\hat{\Phi}_1(z)$ (and even suggests a possible $O(t\log t)$ term).

Remark 4. In the very recent preprint [56] the following asymptotics is obtained by completely different methods

$$\log \hat{G}(t,r) = -t^2\Phi_-(\frac{r}{t}) - \frac{1}{6}\sqrt{1-\frac{\pi^2 r}{t}} + O(\log^2(|r|/t^{1/3})) + O(t^{1/3}) \tag{53}$$

holds uniformly in $M^{-1} < t < M^{2/3}r$ as $s = -r/t^{1/3} \to +\infty$. In the region studied here, both $-r, t \to +\infty$ with $r/t = z < 0$ fixed, this agrees with our leading order $O(t^2)$ result (the function $\Phi_-(z)$ is the same). It also agrees with our conclusion that our $O(t)$ result, i.e. the function $\Phi_1(z)$, vanishes. However it does not allow for a comparison of our $O(1)$ result, since corrections terms in (53) are larger. However formula (53) also holds for t fixed (e.g. of order $O(1)$) with $r \to -\infty$, which goes beyond the results of this section, but is discussed again in section 2.6.

2.5. Matching small time and large time: the cumulants

As we discovered above, inserting the expected scaling forms for the large time large deviation rate functions into the reduced KP equation, allows us to determine their form up to a few unknown constants (A, c_1, c_2 above). To determine those, we return to the small time expansion, and pursue the program started in [34]. There the following expansion in cumulants was considered[4] (see equations (129)–(132) there)

$$\log \hat{G}(t,r) = \sum_{n\geqslant 1} \frac{1}{n!}\kappa_n(t,r) \tag{54}$$

where $\log \hat{G}(t,r) = q_{t\beta}(\sigma = -e^{-r})$ there, for the choice $g(x) = g_{\mathrm{KPZ}}(x)$, $\beta = 1$, but it holds more generally, see remark 2. The cumulant $\kappa_n(t,r)$ re-groups all terms of degree n in the expansion studied above, i.e. it is by definition a homogeneous polynomial of total degree n in $p_0'(r) = \mathcal{L}_1$ and its derivatives $-\partial_r\mathcal{L}_i = \mathcal{L}_{i+1}$. Each cumulant has the following short time series expansion in t

$$\kappa_n(t,r) = t^{\frac{n}{2}-1}\kappa_n^0(r) + \sum_{p\geqslant 1} t^{\frac{n}{2}-1+p}\kappa_n^p(r). \tag{55}$$

The form of the leading term was found exactly in [34] (see below) and shown to sum up to produce the large time large deviation function $\Phi_-(z)$ studied above. Here we recover the result for $\kappa_n^0(r)$ from the KP equation, and obtain the next subleading term $\kappa_n^1(r)$.

Let us recall the iterative solution for the functions $p_m'(r)$ defined in (25) corresponding to the term of order $t^{\frac{m-1}{2}}$ in the small time expansion of $\psi(t,r) = \partial_r^2 \log \hat{G}(t,r)$. We see from e.g. (26) (see also appendix A) that the structure of the result for the p_m is a polynomial in p_0' and derivatives of the form (schematically, we only indicate the degree in p_0' and the total number of derivatives)

[4] In this paper we use the denomination 'cumulants' for two unrelated quantities. Above, we have studied the cumulants of $Z(x,t)$, i.e. $Z_n(x,t) = \langle Z(x,t)^n\rangle_c$. Here, we study an expansion of $\log \hat{G}$ in powers of $\mathcal{L}_1$, i.e. in powers of β, the parameter which appears in (32) (the nth cumulant is the term of order β^n in the expansion). Physically these arise from the expansion of the expectation value in (32) in the cumulants of the empirical measure of the Airy point process, equivalently, in the particular case $g(x) = g_{\mathrm{KPZ}}(x)$, in the KPZ noise. The precise definition of this cumulant expansion is given in [34, 39].

J. Stat. Mech. (2020) 043201

$$p'_m = (\partial_r)^m (p'_0)^{m+1} + (\partial_r)^{m+1} (p'_0)^{m-1} + \dots \tag{56}$$

The leading term corresponds to κ^0_{m+1}, the second one to κ^1_{m-1}, and so on. Below we will denote $p'_m = p^{0\prime}_m + p^{1\prime}_m + \dots$ the above decomposition.

We will thus introduce the following expansion for $\psi(t, r)$

$$\psi(t,r) = \psi_0(t,r) + \psi_1(t,r) + \dots \quad , \quad \psi_p(t,r) = \partial_r^2 \sum_{n\geqslant 1} \frac{\kappa^p_n(t,r)}{n!} \tag{57}$$

where $\psi_0(t, r)$ corresponds to κ^0_n, $\psi_1(t, r)$ corresponds to κ^1_n, and so on. This expansion corresponds to treating perturbatively the third derivative term in the reduced KP equation (13) and can be called *semi-classical expansion*. The leading term is thus obtained by neglecting the third derivative term in (13), i.e. solving

$$\partial_t \psi_0 + \psi_0 \partial_r \psi_0 + \frac{1}{2t}\psi_0 = 0. \tag{58}$$

This is precisely the equation which was solved in section 2.4.1, but only in the large t, large $r < 0$ limit, using the large deviation ansatz $\psi(t, r) \simeq H_0(r/t)$ with $H_0(z) = -\Phi''_-(z)$ leading to equation (38). Here we provide a more complete solution, valid at all time t and r. We will do it by two methods (i) a series expansion, allowing us to recover the result of [34] (ii) a mapping to Burgers equation, which is reminiscent of what was found in [36] and allows to calculate easily the next subleading term. Our result leads to a better understanding of how the small time and large time large deviations are related.

The first method is to search for a solution of (58) as a series expansion at small time of the form $\psi_0(r, t) = \sum_{m\geqslant 0} p^{0\prime}_m(r) t^{\frac{m-1}{2}}$. This leads to the simplified recursion for $m \geqslant 0$

$$\frac{m+1}{2} p^0_{m+1} + \frac{1}{2} \sum_{m_1+m_2=m, m_1\geqslant 0, m_2\geqslant 0} p^{0\prime}_{m_1} p^{0\prime}_{m_2} = 0 \tag{59}$$

and one obtains exactly the $p^{0\prime}_m$, $m \geqslant 0$ as a function of p'_0 as

$$p^{0\prime}_m(r) = \frac{2^m}{(m+1)!} (-1)^m (\partial_r)^m (p'_0(r))^{m+1} \tag{60}$$

using the identity for $m \geqslant 0$ for any function $f(r)$

$$\partial_r^m f^{m+2} = \frac{1}{2(m+1)} \sum_{m_1+m_2=m, m_1\geqslant 0, m_2\geqslant 0} \frac{(m+2)!}{(m_1+1)!(m_2+1)!} \partial_r^{m_1} f^{m_1+1} \partial_r^{m_2} f^{m_2+1} \tag{61}$$

which is easily checked with Mathematica. The result (60) is exactly equivalent to the formula (132) in [34], more precisely $p^{0\prime}_m(r)$ equals $\partial_r^2 \frac{\kappa^0_n(r)}{n!}$ with $n = m + 1$, using $\sigma = -\mathrm{e}^{-r}$, $\sigma\partial_\sigma = -\partial_r$ and $\mathcal{L}_1 = p'_0$. Obtaining the subleading term with that method is a bit tedious, so we now consider an equivalent, but more convenient method.

Let us first note that if we perform the following change of variable

$$\psi(t,r) = \frac{1}{\sqrt{t}} \tilde{\psi}(\sqrt{t}, r) \tag{62}$$

then the reduced KP equation (13) becomes, for the function $\tilde{\psi}(u,r)$

$$\partial_u \tilde{\psi}(u,r) + \partial_r(\tilde{\psi}(u,r)^2) + \frac{1}{6} u \partial_r^3 \tilde{\psi}(u,r) = 0 \quad , \quad u = \sqrt{t}. \tag{63}$$

We can now again perform the perturbative expansion in the cubic derivative term and write, for $p = 0, 1, ..$

$$\psi(t,r) = \sum_{p \geqslant 0} \psi_p(t,r) \quad , \quad \psi_p(t,r) = \frac{1}{\sqrt{t}} \tilde{\psi}_p(\sqrt{t}, r). \tag{64}$$

The leading term then obeys the Burgers equation

$$\partial_u \tilde{\psi}_0(u,r) + \partial_r(\tilde{\psi}_0(u,r)^2) = 0. \tag{65}$$

The general solution can be expressed as $F(\tilde{\psi}_0(u,r), r - 2u\tilde{\psi}_0(u,r)) = 0$ for some function $F(x,y)$. The solution of interest here has 'initial condition' $\tilde{\psi}_0(u=0,r) = p_0'(r)$. It is thus given by

$$\tilde{\psi}_0(u,r) = p_0'(r - 2u\tilde{\psi}_0(u,r)) \Rightarrow \psi_0(t,r) = \frac{1}{\sqrt{t}} p_0'(r - 2t\psi_0(t,r)). \tag{66}$$

By expanding at small t one can check that it produces the leading order of the cumulants as given above in (60).

The result (66) is quite general, see remark 2. In the present case of the KPZ equation, using (28), $\psi_0(t,r)$ is solution of

$$\psi_0(t,r) = \frac{1}{\sqrt{4\pi t}} \mathrm{Li}_{1/2}(-\mathrm{e}^{-r+2t\psi_0(t,r)}) \tag{67}$$

where we recall $\psi_0(t,r) = \partial_r^2 \sum_n t^{\frac{n}{2}-1} \frac{\kappa_n^0(r)}{n!}$ is the summation of the leading term in the small time expansion of each cumulant.

As was noticed in [34] for large negative $r \sim zt$, $z < 0$, the leading term in each cumulant $\kappa_n^0(r)$ allows us to obtain the large deviation function $\Phi_-(z)$. From the large negative r asymptotics of the polylogarithm [66] one has

$$p_0'(r) = \frac{1}{\sqrt{4\pi}} \mathrm{Li}_{1/2}(-\mathrm{e}^{-r}) = -\frac{1}{\pi}(-r)^{1/2} + \frac{\pi}{24}(-r)^{-3/2} + \ldots. \tag{68}$$

Inserting into (67) we obtain that the following limit exists, for $z < 0$

$$H_0(z) = \lim_{t \to +\infty} \psi_0(t, zt) \tag{69}$$

and $H_0(z)$ obeys the self-consistent equation

$$H_0(z) = -\frac{1}{\pi}\sqrt{-z + 2H_0(z)} \quad \Rightarrow \quad H_0(z) = \frac{1}{\pi^2}(1 \pm \sqrt{1 - \pi^2 z}). \tag{70}$$

The correct branch (actually reached in the large time limit) is $-$ and one recovers the result of the previous section for the large deviation function $H_0(z)$, with $A = 1$. This method thus relates the small and large time and allows to determine the missing constant $A = 1$ for the large time large deviation rate function.

Let us now briefly consider the general case of the linear statistics discussed in remark 2. For a general function $g(x)$ the leading term in the semi-classical expansion is given by (66), where the function $p_0'(r)$ is given by (34) and (33). This case is studied in detail in appendix C, where a connection with the section 5.2 in [34] and with the self-consistent equation in [36] is obtained. It shows that there are other large time large deviation solutions of the KP equation, of the form $\psi(t,r) \simeq t^{\alpha-1} H_0(r/t^\alpha)$, with continuously varying exponent α and function H_0, which corresponds to other functions $g(x)$ in remark 2 (and the 'monomial walls' in the Coulomb gas terminology of [36]). The case of the KPZ equation is recovered for $\alpha = 1$.

Let us now go back to the KPZ equation and study the subleading term. Inserting (64) into (63) we obtain the following equation for $\tilde{\psi}_1(u,r)$

$$\partial_u \tilde{\psi}_1(u,r) + 2\partial_r[\tilde{\psi}_1(u,r)\tilde{\psi}_0(u,r)] + \frac{1}{6} u \partial_r^3 \tilde{\psi}_0(u,r) = 0. \tag{71}$$

It is more convenient to write $\tilde{\psi}_1(u,r) = \partial_r \phi(u,r)$, integrate once w.r.t. r and obtain the equation for ϕ

$$\partial_u \phi(u,r) + 2\tilde{\psi}_0(u,r)\partial_r \phi(u,r) = -\frac{1}{6} u \partial_r^2 \tilde{\psi}_0(u,r) \tag{72}$$

which can be seen as a convection equation for a passive scalar $\phi(u,r)$ in the Burgers velocity field $2\tilde{\psi}_0(u,r)$. It is easy, and useful for later checks, to first extract the small u expansion of $\phi(u,r)$ from (72) and (66). One finds

$$\phi(u,r) = -\frac{1}{12} u^2 \partial_r^2 p_0'(r) + u^3 \partial_r [\frac{1}{6} {p_0}^{(3)}(r)\, p_0'(r) + \frac{1}{12} p_0''(r)^2] \tag{73}$$

$$+ u^4 \partial_r [-\frac{2}{3} {p_0}^{(3)}(r)\, p_0'(r)\, p_0''(r) - \frac{1}{6} {p_0}^{(4)}(r)\, p_0'(r)^2 - \frac{1}{9} p_0''(r)^3] + O(u^5). \tag{74}$$

To solve the equation (72) it is then natural to work in the variable $u, \tilde{\psi}_0$ instead of u, r. Indeed one has, from the solution (66) to Burgers equation

$$r = r(u, \tilde{\psi}_0) = 2u\tilde{\psi}_0 + g(\tilde{\psi}_0) \quad , \quad p_0'(g(a)) = a \tag{75}$$

where g is the inverse function of p_0'. Let us define

$$\tilde{\phi}(u, \psi_0) = \phi(u, r(u, \tilde{\psi}_0) = \phi(u, 2u\tilde{\psi}_0 + g(\tilde{\psi}_0)). \tag{76}$$

We now obtain an equation for this function. Taking a derivative w.r.t. u, one obtains, using the equation (72) for ϕ

$$\partial_u \tilde{\phi}(u, \tilde{\psi}_0) = \partial_u \phi(u, 2u\tilde{\psi}_0 + g(\tilde{\psi}_0)) + 2\tilde{\psi}_0\, \partial_r \phi(u, 2u\tilde{\psi}_0 + g(\tilde{\psi}_0)) = -\frac{1}{6} u \partial_r^2 \tilde{\psi}_0(u,r). \tag{77}$$

Let us evaluate the rhs One has from (75), by variation

$$\partial_r \tilde{\psi}_0 = \frac{1}{\partial_{\tilde{\psi}_0} r} = \frac{1}{2u + g'(\tilde{\psi}_0)}. \tag{78}$$

Taking a derivative ∂_r and using again (78), one finally obtains from (77)

$$\partial_u \tilde{\phi}(u,\tilde{\psi}_0) = \frac{1}{6} u \frac{g''(\tilde{\psi}_0)\partial_r \tilde{\psi}_0}{(2u+g'(\tilde{\psi}_0))^2} = \frac{1}{6}\frac{u g''(\tilde{\psi}_0)}{(2u+g'(\tilde{\psi}_0))^3}. \tag{79}$$

One can then easily integrate this equation w.r.t. u, with the constraint that ϕ must vanish at $u=0$ from (73). One obtains

$$\tilde{\phi}(u,\tilde{\psi}_0) = \frac{g''(\tilde{\psi}_0)u^2}{12 g'(\tilde{\psi}_0)(2u+g'(\tilde{\psi}_0))^2}. \tag{80}$$

From the definition of the inverse function $g(a)$ one can now use the relations

$$g'(a) = \frac{1}{p_0''(g(a))} \quad , \quad g''(a) = -g'(a)^2 \frac{p_0'''(g(a))}{p_0''(g(a))} \tag{81}$$

and we obtain our final result for the two first orders, summarized as follows

$$\psi(t,r) = \frac{1}{\sqrt{t}}[\tilde{\psi}_0(u,r) + \partial_r \phi(u,r) + \ldots] \quad , \quad u=\sqrt{t} \quad , \quad \tilde{\psi}_0(u,r) = p_0'(r - 2u\tilde{\psi}_0(u,r)) \tag{82}$$

$$\phi(u,r) = -\frac{u^2}{12}\frac{p_0'''(r-2u\tilde{\psi}_0(u,r))}{(1+2up_0''(r-2u\tilde{\psi}_0(u,r)))^2} \quad , \quad \tilde{\psi}_0(u,r) = \sum_{m\geqslant 0} u^m \frac{2^m}{(m+1)!}(-1)^m (\partial_r)^m (p_0'(r))^{m+1}. \tag{83}$$

It is easy to expand this result in powers of $u=\sqrt{t}$ and recover the result (73), which provides a check on our exact solution.

We now consider again the limit of large t, large negative r, with $z=r/t<0$ fixed. Up to higher order terms, $O(1/t^3)$, we only need the (semi-classical) expansion in the cubic derivative of the KP equation as $\psi(t,r) \simeq \psi_0(t,r) + \psi_1(t,r)$. We will now find that $\psi_0(t,r) \simeq H_0(z) + \frac{1}{t^2}H_2^{(2)}(z)$ and $\psi_1(t,r) \simeq \frac{1}{t^2}H_2^{(1)}(z)$. Hence (i) there are no correction of order $O(1/t)$, i.e. the function $H_1(z)$ is zero, as claimed in the previous section (ii) two pieces add up to give the total subleading rate function $H_2(z) = H_2^{(1)}(z) + H_2^{(2)}(z)$.

To evaluate ψ_1 we use the large negative r asymptotics.

$$p_0'(r) = -\frac{1}{\pi}(-r)^{1/2} \quad , \quad p_0''(r) = \frac{1}{2\pi}(-r)^{-1/2} \quad , \quad p_0'''(r) = \frac{1}{4\pi}(-r)^{-3/2}. \tag{84}$$

Inserting in the solution (82) we obtain

$$\psi_1(u,r) = -\frac{\sqrt{t}}{12}\frac{1}{4\pi}\partial_r\left[(-r+2tH_0)^{-3/2}\frac{1}{(1+\frac{1}{\pi}\sqrt{t}(-r-2tH_0)^{-1/2})^2}\right] \tag{85}$$

$$= \frac{1}{t^2}H_2^{(1)}(z) \quad , \quad H_2^{(1)}(z) = \frac{1}{48}\partial_z\left[\frac{1}{H_0(z)}\frac{1}{(\pi^2 H_0(z)-1)^2}\right] \tag{86}$$

where we have used that $-\pi H_0(z) = \sqrt{-z+2H_0(z)}$ and we recall that $H_0(z) = \frac{1}{\pi^2}(1-\sqrt{1-\pi^2 z})$.

Inserting now the large r expansion (68), and keeping the subdominant term in the equation for ψ_0, i.e. $\psi_0 = \frac{1}{\sqrt{t}}p_0'(r-2t\psi_0)$, we obtain $\psi_0 = H_0 + \frac{1}{t^2}H_2^{(2)}$ with

J. Stat. Mech. (2020) 043201

$$H_2^{(2)}(z) = \frac{1}{24H_0(z)^2(1-\pi^2 H_0(z))}. \tag{87}$$

Adding the two contributions we find

$$H_2(z) = \frac{1}{24H_0(z)^2(1-\pi^2 H_0(z))} + \frac{1}{48}\partial_z\left[\frac{1}{H_0(z)}\frac{1}{(\pi^2 H_0(z)-1)^2}\right] \quad , \quad H_0(z) = \frac{1}{\pi^2}(1-\sqrt{1-\pi^2 z}). \tag{88}$$

One can now check that this result is identical to the one obtained in (49) from the large time large deviation ansatz, provided one sets $a_2 = \frac{\pi^4}{24}$. In fact the second term in (88) corresponds to setting $a_2 = 0$, while the first is the one proportional to a_2 (i.e. the solution of the homogeneous part of the equation for H_2). Once again, the calculation of the cumulants from the short-time expansion allows us to fix the unknown constant a_2 in the large time large deviation subleading rate function.

2.6. A related expansion: intermediate times

One can consider a related expansion which allows a systematic study of the tail of $\log\hat{G}(t,r)$ and of its derivative $\hat{\psi}(t,r) = \partial_r \log\hat{G}(t,r)$ (which is solution of the integrated version (14) of the reduced KP equation) at large $r<0$ for any time t.

Consider again the small time series expansion (25). We now assume that the functions $p_m(r)$ have the following expansion $p_m(r) = \sum_{n\geqslant 0} p_{m,n}(-r)^{\frac{3-n}{2}}$ for $r\to-\infty$. This is certainly the case for the KPZ equation, and for some class of FD as in remark 2 (the more general class studied in appendix C can be studied by similar series expansions involving different exponents). Hence we look for a solution as a double series

$$\hat{\psi}(t,r) = \sum_{0\leqslant m\leqslant n} p_{m,n} t^{\frac{m-1}{2}}(-r)^{\frac{3-n}{2}} = \sum_{m\geqslant 0} p_m(r) t^{\frac{m-1}{2}} = \sum_{n\geqslant 0} q_n(t)(-r)^{\frac{3-n}{2}} \tag{89}$$

since, as we find below, $p_{m,n}=0$ for $m>n$. The coefficients $p_{m,n}$ encode information for several limits. (i) First, the leading small time behavior for $t\to 0$ at fixed $r<0$ (for $m=0$) is given by

$$\hat{\psi}(t,r) \simeq p_0(r) = \frac{1}{\sqrt{t}}\sum_{n\geqslant 0} p_{0,n}(-r)^{\frac{3-n}{2}} = -\frac{1}{\sqrt{4\pi}}\mathrm{Li}_{3/2}(-\mathrm{e}^{-r}) \tag{90}$$

where the last equality is valid only for the particular case of the KPZ equation, see (28). (ii) Second, the leading large time large deviations (35) for $r = zt < 0$ and $t\to+\infty$ (for $m=n$) is given by

$$\hat{\psi}(t,r) \simeq t\sum_{n\geqslant 0} p_{n,n}\left(-\frac{r}{t}\right)^{\frac{3-n}{2}} = -t\,\Phi'_-\left(z=\frac{r}{t}\right) \tag{91}$$

(iii) Finally, the series (89) contain the information about the large $r<0$ expansion at fixed time t, encoded in the functions $q_n(t)$, i.e.

$$\hat{\psi}(t,r) = \sum_{n\geqslant 0} q_n(t)(-r)^{\frac{3-n}{2}} \quad , \quad q_n(t) = \sum_{m=0}^{n} p_{m,n} t^{\frac{m-1}{2}}. \tag{92}$$

J. Stat. Mech. (2020) 043201

We will determine below some of these functions $q_n(t)$. To this aim one can note that they satisfy differential equations which can be solved recursively. We find it easier however to study instead the recursion for the $p_{m,n}$. Inserting in (14) we obtain recursion relations which show that all $p_{m>n,n} = 0$. The coefficients $p_{0,n}$ for $n \geqslant 0$ are arbitrary (i.e. determined as in the previous sections by the function $p_0(r)$). All $p_{m,n}$ with $m \geqslant 1$ and general $n \geqslant 1$ can be obtained from the set of $p_{0,n}$ as follows

$$p_{m,n} = \frac{-2}{m}\Big(\frac{(n-9)(n-7)(n-5)}{96} p_{m-2,n-6} + \sum_{n_1=0}^{n-1} \frac{(3-n_1)(4+n_1-n)}{8} \sum_{m_1=\max(0,m-n+n_1)}^{\min(n_1,m-1)} p_{m_1,n_1} p_{m-1-m_1,n-1-n_1}\Big) \tag{93}$$

where all $p_{m,n} = 0$ for either $m < 0$ or $n < 0$. This recursion is easily generated using Mathematica. One finds that $p_{1,1} = -\frac{9}{4}p_{0,0}^2$ and so on. If we suppress the first term on the rhs, which arises from the cubic derivative in the KP equation, one can check that one reproduces the expansion given in (60).

We now specialize to the KPZ equation, for which the $p_{0,n}$ are determined from (90). Let us recall the expansion, for $r \to -\infty$ (here we need only the formula for s a positive half-integer)

$$\mathrm{Li}_s(-\mathrm{e}^{-r}) = -2\sum_{k=0}^{+\infty}(1-2^{1-2k})\zeta(2k)\frac{(-r)^{s-2k}}{\Gamma(s+1-2k)}. \tag{94}$$

Hence $p_{0,n} = 0$ unless n is a multiple of 4 and

$$p_{n=4k} = \frac{1}{\sqrt{\pi}}\frac{(1-2^{1-\frac{n}{2}})\zeta(\frac{n}{2})}{\Gamma(\frac{5-n}{2})}. \tag{95}$$

The above recursion then leads to the following solutions for the lowest $q_n(t)$ up to $n = 12$

$$q_0(t) = \frac{2}{3\pi\sqrt{t}} \quad , \quad q_1(t) = -\frac{1}{\pi^2} \quad , \quad q_2(t) = \frac{\sqrt{t}}{\pi^3} \quad , \quad q_3(t) = -\frac{2t}{3\pi^4} \quad , \quad q_4(t) = \frac{t^{3/2}}{4\pi^5} + \frac{\pi}{12\sqrt{t}} \tag{96}$$

$$q_5(t) = \frac{1}{12} \quad , \quad q_6(t) = \frac{\sqrt{t}}{48\pi} - \frac{t^{5/2}}{24\pi^7} \quad , \quad q_7(t) = -\frac{t}{48\pi^2} \quad , \quad q_8(t) = \frac{t^{7/2}}{64\pi^9} + \frac{7\pi^3}{960\sqrt{t}} \tag{97}$$

$$q_9(t) = \frac{t^2}{48\pi^4} + \frac{5\pi^2}{144} \quad , \quad q_{10}(t) = -\frac{t^{9/2}}{128\pi^{11}} - \frac{t^{5/2}}{384\pi^5} + \frac{5\pi\sqrt{t}}{72} \quad , \quad q_{11}(t) = \frac{13t}{192} - \frac{t^3}{48\pi^6} \tag{98}$$

$$q_{12}(t) = \frac{7t^{11/2}}{1536\pi^{13}} + \frac{5t^{7/2}}{1536\pi^7} + \frac{259t^{3/2}}{9216\pi} + \frac{31\pi^5}{4608\sqrt{t}}. \tag{99}$$

One can verify that, keeping only the leading term at large t for each $q_n(t)$, i.e. $p_{n,n}t^{\frac{n-1}{2}}$, agrees, as it should according to (91), with the expansion of $-t\Phi'_-(z)$ in $z = r/t$ at large $z < 0$, from the solution (40) (with the correct value $A = 1$), a rather non-trivial check

$$t\hat{\psi}_0(r/t) = -t\,\Phi'_-(\frac{r}{t}) = \frac{2(-r)^{3/2}}{3\pi\sqrt{t}} - \frac{-r}{\pi^2} + \frac{\sqrt{-r}\sqrt{t}}{\pi^3} - \frac{2t}{3\pi^4} + \frac{\sqrt{-\frac{1}{r}}t^{3/2}}{4\pi^5} - \frac{\left(-\frac{1}{r}\right)^{3/2} t^{5/2}}{24\pi^7} \tag{100}$$

$$+\frac{\left(-\frac{1}{r}\right)^{5/2} t^{7/2}}{64\pi^9} - \frac{\left(-\frac{1}{r}\right)^{7/2} t^{9/2}}{128\pi^{11}} + \frac{7\left(-\frac{1}{r}\right)^{9/2} t^{11/2}}{1536\pi^{13}} + O((-\frac{1}{r})^{11/2}). \tag{101}$$

Furthermore, since $p_{n-2,n} = 0$ for all $n \geqslant 2$, we immediately see that the subleading function $H_1(z) = \hat{\psi}'_1(z)$ studied in section 2.4.2 is indeed zero, and that the above results are consistent with the series expansion of the next subleading function $H_2(z) = \hat{\psi}'_2(z)$ obtained in that section, i.e. one can check that

$$\hat{\psi}_2(z) = \sum_{n\geqslant 4} p_{n-4,n}(-z)^{\frac{3-n}{2}} \tag{102}$$

is indeed the function for $z < 0$ found in (49) with the correct value $a_2 = \frac{\pi^4}{24}$.

We can now integrate (92) over r to obtain the series expansion for large $r < 0$

$$\begin{aligned}\log \hat{G}(t,r) &= -\sum_{n\geqslant 0, n\neq 5} q_n(t)\frac{2}{5-n}(-r)^{\frac{5-n}{2}} - \frac{1}{12}\log(-r) + Q(t)\\ &= -\frac{4(-r)^{5/2}}{15\left(\pi\sqrt{t}\right)} + \frac{r^2}{2\pi^2} - \frac{2(-r)^{3/2}\sqrt{t}}{3\pi^3} - \frac{2rt}{3\pi^4} - \frac{\sqrt{-r}\,(3t^2+\pi^6)}{6\left(\pi^5\sqrt{t}\right)}\\ &\quad - \frac{1}{12}\log(-r) + Q(t) - \sum_{n\geqslant 6} q_n(t)\frac{2}{5-n}(-r)^{\frac{5-n}{2}}.\end{aligned} \tag{103}$$

Here $Q(t)$ is an undetermined integration constant of $O(1)$ in the large $r < 0$ expansion. Note that all terms with a positive power of r appearing in (103) are already contained in the function $-t^2\Phi_-(r/t)$, apart from the term $-\frac{\pi}{6}\sqrt{-r/t}$. This additional term is consistent with the one discussed in remark 4.

Remark 5. For the KPZ equation many of the $p_{m,n}$ vanish. Indeed they vanish if $m-n$ is not a multiple of 4. The series has the following structure

$$\begin{aligned}\hat{\psi}(t,r) = \sum_{0\leqslant q\leqslant k} [&p_{4q,4k}t^{-\frac{1}{2}+2q}(-r)^{\frac{3}{2}-2k} + p_{4q+1,4k+1}t^{2q}(-r)^{1-2k}\\ &+ p_{4q+2,4k+2}t^{\frac{1}{2}+2q}(-r)^{\frac{1}{2}-2k} + p_{4q+3,4k+3}t^{1+2q}(-r)^{-2k}].\end{aligned} \tag{104}$$

Hence it naturally splits in the sum of four functions, for which one can also obtain coupled series recursion relations.

2.7. Large time expansion, typical fluctuations

For completeness we now address briefly the regime of typical fluctuations in the large time limit. Not surprisingly, once the scaling form is introduced, it reproduces the KPZ fixed point result of [1]. However it allows in principle to study the finite time corrections.

J. Stat. Mech. (2020) 043201

In the large time and typical fluctations regime, we expect that the generating function (12) takes the form

$$\hat{G}(t,r) = P_0(r/t^{1/3}) + t^{-a}P_1(r/t^{1/3}) + \cdots \quad , \quad P_0(s) = \lim_{t\to+\infty} \mathrm{Prob}(\frac{h(0,t)+\frac{t}{12}}{t^{1/3}} < s) \tag{105}$$

where the last equality follows by construction of the generating function. At this stage we allow some freedom for the decay exponent a of the subleading corrections (see below). We thus look for a solution to the reduced KP equation (13) of the form

$$\psi(t,r) = \partial_r^2 \log \hat{G}(t,r) = t^{-2/3}(\psi_0(r/t^{1/3}) + t^{-a}\psi_1(r/t^{1/3}) + \cdots). \tag{106}$$

The function $\psi_0(s)$ must satisfy

$$12\psi_0\psi_0' - 4s\psi_0' - 2\psi_0 + \psi_0''' = 0. \tag{107}$$

Note that one can also consider the integrated version which, from (14) leads to

$$\psi(t,r) = \partial_r \log \hat{G}(t,r) = t^{-1/3}(\hat{\psi}_0(r/t^{1/3}) + t^{-a}\hat{\psi}_1(r/t^{1/3}) + \cdots) \tag{108}$$

$$6(\hat{\psi}_0')^2 - 4s\hat{\psi}_0' + 2\hat{\psi}_0 + \hat{\psi}_0''' = 0 \tag{109}$$

with $\psi_0 = \hat{\psi}_0'$.

A solution to the equation (107) is obtained from a solution $q(s)$ to the Painleve II equation as

$$\psi_0(s) = -q(s)^2 \quad , \quad q'' = sq + 2q^3. \tag{110}$$

This is verified by inserting into (107) leading to

$$12\psi_0\psi_0' - 4s\psi_0' - 2\psi_0 + \psi_0'''|_{\psi_0=-q^2} = -(2q\frac{\mathrm{d}}{\mathrm{d}s} + 6q')(q'' - sq - 2q^3) = 0. \tag{111}$$

This solution, together with $q(s) \sim -\mathrm{Ai}(s)$ for $s \to +\infty$, corresponds to the TW-GUE distribution $F_2(s)$

$$\partial_s^2 \log P_0(s) = -q(s)^2 \quad , \quad P_0(s) = \mathrm{e}^{-\int_s^{+\infty} \mathrm{d}u(u-s)q(u)^2} = F_2(s). \tag{112}$$

This is the standard analysis, also obtained from the KP equation satisfied by the KPZ fixed point in [1].

Let us recall the large negative s asymptotics for $q(s)$ and $F_2(s)$. From (110) one easily obtains (correcting a misprint in the last term in [59])

$$q(s) = \sqrt{\frac{-s}{2}}(1 + \frac{1}{8s^3} - \frac{73}{128s^6} + \frac{10\,657}{1024s^9} + O(|s|^{-12})) \quad , \quad \psi_0(s) = \frac{s}{2} + \frac{1}{8s^2} - \frac{9}{16s^5} + O(\frac{1}{|s|^8})\ldots \tag{113}$$

and, integrating twice,

$$\log F_2(s) = -\frac{1}{12}|s|^3 - \frac{1}{8}\log(|s|) + C + \frac{3}{2^6|s|^3} + O(\frac{1}{|s|^3}) \tag{114}$$

where obtaining the constant $C = \frac{1}{24}\log 2 + \zeta'(-1)$ requires more sophisticated methods [59].

Let us note that the right tail approximation of $F_2(s)$, i.e. the first order in the expansion of the FD in powers of the Airy kernel at large positive s, reads

$$F_2(s) = \mathrm{Det}[I - P_s K_{\mathrm{Ai}} P_s] \simeq 1 - \mathrm{Tr} P_s K_{\mathrm{Ai}} + O(\mathrm{e}^{-\frac{8}{3}s^{3/2}}) = 1 - \int_s^{+\infty} \mathrm{d}u \int_0^{+\infty} \mathrm{d}v \mathrm{Ai}(u+v)^2 + O(\mathrm{e}^{-\frac{8}{3}s^{3/2}}). \tag{115}$$

There is a corresponding approximation $\psi_0(s) = \psi_{00}(s) + \dots$ where one neglects the non-linear term in the KP equation, i.e. also in (107), leading to

$$-4s\psi_{00}' - 2\psi_{00} + \psi_{00}''' = 0 \tag{116}$$

which is solved as $\psi_0(s) = -q_0(s)^2$ with $q_0'' = sq_0$, solved as $q_0(s) = -\mathrm{Ai}(s)$, leading to (115). The approximation (115) corresponds to the large time limit of the contribution of the single string states in the Bethe ansatz. Hence the latter obeys the linear part of the KP equation, as discussed below (see also appendix B).

We write now the equation satisfied by the subleading term in (105). It is more convenient to give the result for the function $\hat{\psi}_1$ defined in (108) (with $\psi_1(s) = \hat{\psi}_1'(s)$). One finds that it must satisfy the following linear equation

$$2(1-6a)\hat{\psi}_1(s) - 4(s + 3q(s)^2)\hat{\psi}_1'(s) + \hat{\psi}_1'''(s) = 0. \tag{117}$$

At large s it implies that $\hat{\psi}_1(s) \sim (-s)^{6a-1}$. It is not so easy to solve this equation. However, the analysis of the Fredholm determinant was carried in [60] and it was found that (see appendix B there)

$$\hat{G}(t, r = st^{1/3}) \simeq F_2(s) + t^{-2/3}\frac{\pi^2}{6}F_2''(s) + O(t^{-4/3}). \tag{118}$$

This predicts that the exponent $a = 2/3$ and that a solution of (117) should be

$$\hat{\psi}_1(s) = \frac{\pi^2}{6}\partial_s \frac{F_2''(s)}{F_2(s)} = \frac{\pi^2}{6}\partial_s(\psi_0(s) + \hat{\psi}_0(s)^2) = \frac{\pi^2}{6}\left(\frac{|s|^3}{4} - \frac{7}{16} - \frac{27}{64|s|^3} - \frac{855}{128|s|^6} + O(|s|^{-9})\right). \tag{119}$$

Although we have not been able to show it directly, we carried a series expansion at large negative s, using (113) to a much higher order, which indeed indicates that this is the case.

Remark 6. We know from [10–13] that the finite time analog to (115) is (keeping only the first order in the expansion of the FD in traces of the finite time kernel)

$$\log \hat{G}(t,r) = \log \mathrm{Det}[I - \sigma_{t,r} K_{\mathrm{Ai}}] = -\mathrm{Tr}[\sigma_{t,r} K_{\mathrm{Ai}}] + \dots = -\int_{r/t^{1/3}}^{+\infty} \mathrm{d}v \int_{-\infty}^{+\infty} \mathrm{d}u \frac{\mathrm{Ai}(u+v)^2}{1 + \mathrm{e}^{-t^{1/3}u}} + \dots \tag{120}$$

$$\sigma_{t,r}(u,u') = \frac{1}{1 + \mathrm{e}^{r - t^{1/3}u}}\delta(u-u'). \tag{121}$$

It is easy to check that the (single trace) leading term

$$\hat{\psi}_1(t,r) = \partial_r \log(\hat{G}(t,r))|_{1 \text{ trace}} = t^{-1/3} \int_{-\infty}^{+\infty} \mathrm{d}u \frac{\mathrm{Ai}(u + \frac{r}{t^{1/3}})^2}{1 + \mathrm{e}^{-t^{1/3} u}} \tag{122}$$

is solution of the linear part of the integrated reduced KP equation (14)

$$\partial_t \hat{\psi}_1 + \frac{1}{12} \partial_r^3 \hat{\psi}_1 + \frac{1}{2t} \hat{\psi}_1 = 0. \tag{123}$$

Indeed one can write formally (upon expansion of the 'Fermi factor' in (122) and using the Airy propagator identity)

$$\hat{\psi}_1(t,r) = \sum_{n \geqslant 1} a_n \frac{1}{\sqrt{t}} \mathrm{e}^{-nr + \frac{n^3 t}{12}} \quad , \quad a_n = \frac{(-1)^n}{\sqrt{4\pi n}}. \tag{124}$$

It can also be written as

$$\hat{\psi}_1(t,r) = \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-t\partial_r^3} \mathrm{Li}_{1/2}(-\mathrm{e}^{-r}). \tag{125}$$

It is reminiscent but *different* from the exact expression of the first cumulant, see formula (111) in [34] (which is the same formula with $\mathrm{Li}_{3/2}$ instead of $\mathrm{Li}_{1/2}$). The first cumulant for $\log \hat{G}$ has the expression $\mathrm{Tr}[\log(1 - \sigma_{t,r}) K_{\mathrm{Ai}}]$ different indeed from $-\mathrm{Tr}[\sigma_{t,r} K_{\mathrm{Ai}}]$ above. It provides the correct small time limit $\hat{\psi}(r,t) \simeq p_0(r)/\sqrt{t}$ (see (25) and (28)) and does also satisfy the linearised version of KP (which can be checked by direct expansion as above, or see section 3.1). Hence, although $\hat{\psi}_1$ satisfies linear KP, it does not have the correct initial condition.

3. Other initial conditions: half-Brownian and Brownian

We now turn to the half-Brownian and Brownian initial conditions. We perform some checks from the known expressions for the cumulants of $Z(x,t)$, which hint at a general mechanism for the KP equation to hold. It agrees with the statement of [1] for the half-Brownian, and leads us to conjecture that the KP equation is also obeyed for the full Brownian. In the second part, we briefly study the small time large deviations for both cases, which confirms the conjecture.

It was stated in [1] that for the half-Brownian initial condition ($B(x)$ is a one-sided Brownian with $B(0) = 0$)

$$Z(x,0) = \mathrm{e}^{h(x,0)} = \mathrm{e}^{B(x) - wx} \theta(x) \tag{126}$$

the function $\phi(x,t,r) = \partial_r^2 \log G(x,t,r)$ associated to the same generating function G defined in (129), satisfies the KP equation, which we recall here

J. Stat. Mech. (2020) 043201

$$\partial_t \phi + \phi\, \partial_r \phi + \frac{1}{12} \partial_r^3 \phi + \partial_r^{-1} \partial_x^2 \phi = 0. \tag{127}$$

In addition we conjecture here that a similar property holds for the Brownian initial condition

$$Z(x,0) = \mathrm{e}^{h(x,0)} = \mathrm{e}^{B_R(x) - w_R x} \theta(x) + \mathrm{e}^{B_L(x) + w_L x} \theta(-x) \tag{128}$$

for the modified generating function

$$G(x,t,r) = \left\langle \exp(-\mathrm{e}^{h(x,t)+\chi+\frac{t}{12}-r}) \right\rangle = \left\langle \exp(-\tilde{Z}(x,t) \mathrm{e}^{\frac{t}{12}-r}) \right\rangle \tag{129}$$

where χ is a log-gamma random variable independent from $h(x,t)$, of PDF $P(\chi)\mathrm{d}\chi = \mathrm{e}^{-2w\chi - \mathrm{e}^{-\chi}} \mathrm{d}\chi / \Gamma(2w)$ and of exponential moments $\mathrm{e}^{n\chi} = \frac{\Gamma(w-n)}{\Gamma(w)}$, with $2w = w_R + w_L$. We claim that G also satisfies the KP equation (127).

We can again expand G in (129) in cumulants, now in the presence of space dependence. This leads to the series in e^{-r}

$$\phi(x,t,r) = \sum_{n \geqslant 1} \frac{(-1)^n n^2}{n!} Z_n(x,t) \mathrm{e}^{\frac{nt}{12} - nr} \quad , \quad Z_n(x,t) = \langle Z(x,t)^n \rangle_c. \tag{130}$$

In the case of the full Brownian, $Z(x,t)$ must be replaced by $\tilde{Z}(x,t) = \mathrm{e}^{h(x,t)+\chi}$ and the cumulants are averages over both h and χ. Inserting in the equation and identifying the terms e^{-nr} we obtain the recursion, for $n \geqslant 1$

$$-\partial_t Z_n(x,t) + \frac{n^3 - n}{12} Z_n(x,t) + \frac{1}{n} \partial_x^2 Z_n(x,t) = -\frac{1}{2}(n-1)! \sum_{n_1+n_2=n, n_1, n_2 \geqslant 1} \frac{n_1^2 n_2^2}{n_1! n_2!} Z_{n_1}(x,t) Z_{n_2}(x,t) \tag{131}$$

which, as in the previous section, allows us to determine the moments recursively from the first one, up to some undetermined solution of the homogeneous equation which enters at each level n. The latter reads

$$z_n(x,t) = \mathrm{e}^{\frac{n^3-n}{12} t} \int \frac{\mathrm{d}k}{2\pi} \hat{z}_n(k) \mathrm{e}^{-ntk^2 + \mathrm{i}nkx} \tag{132}$$

where $\hat{z}_1(k)$ is the Fourier transform of $Z_1(x,0)$ (which is usually specified by the initial data) and the $\hat{z}_n(k)$ are (up to a rescaling) the Fourier transforms of the $z_n(x,0)$ are *a priori* arbitrary (if we consider the general solution of KP). As a side remark, see appendix B for details and definitions, equation (132) corresponds to the evolution of the *moments* $\langle Z(x,t)^n \rangle_{1\ \mathrm{string}}$ retaining only the contribution of the eigenstates of the delta bose-gas Hamiltonian corresponding to a single string $n_s = 1$ with arbitrary momentum (which contains the ground state). The general solution of the KP equation is thus a functional of the set of functions $\hat{z}_j(k)$

$$Z_n(x,t) = F_n(x,t; \{\hat{z}_j(k)\}_{j=1,\ldots,n}) \tag{133}$$

and it is *a priori* far from obvious in general that this corresponds to the exact cumulants of the KPZ/SHE equation for *some* initial condition. Let us now make some more detailed analysis.

J. Stat. Mech. (2020) 043201

Let us start with $n = 1$, which reads

$$-\partial_t Z_1(x,t) + \partial_x^2 Z_1(x,t) = 0. \tag{134}$$

Hence $Z_1(x,t) = z_1(x,t)$ must satisfy the heat equation, which as we discussed above is a consequence of the Ito convention. Hence at this stage any initial condition of the SHE would work. Indeed we see that for the half-Brownian $Z_1(x,t) = \langle Z(x,t)\rangle$ given in (B.9) does satisfy (134). Although it does involve some additional averaging over the Brownian IC, the linearity of (134) guarantees that it works.

Let us write now the equation for the second cumulant $Z_2(x,t)$

$$\partial_t Z_2(x,t) - \frac{1}{2} Z_2(x,t) - \frac{1}{2}\partial_x^2 Z_2(x,t) = \frac{1}{2} Z_1(x,t)^2. \tag{135}$$

The general solution is

$$Z_2(x,t) = z_2(x,t) + \frac{1}{2}\int \mathrm{d}x' \int_0^t \mathrm{d}t' \frac{1}{\sqrt{2\pi(t-t')}} \mathrm{e}^{-\frac{(x-x')^2}{2(t-t')}} Z_1(x',t')^2 \mathrm{e}^{\frac{1}{2}(t-t')} \tag{136}$$

which is not illuminating. In appendix B the following form, suggested by the Bethe ansatz, is studied. Suppose that the generic integer moment can be written as

$$\begin{aligned}\langle Z(x,t)^n\rangle = n! \sum_{n_s=1}^{n} \frac{1}{n_s!} \sum_{(m_1,\dots,m_{n_s})_n} \prod_{j=1}^{n_s} \int \frac{\mathrm{d}k_j}{2\pi m_j} \mathrm{e}^{\mathrm{i}x\sum_{j=1}^{n_s} m_j k_j} \mathrm{e}^{t\sum_{j=1}^{n_s}[\frac{1}{12}(m_j^3 - m_j) - m_j k_j^2]} \\ \times \prod_{j=1}^{n_s} S_{k_j,m_j} \prod_{1\leqslant i<j\leqslant n_s} \frac{4(k_i-k_j)^2 + (m_i-m_j)^2}{4(k_i-k_j)^2 + (m_i+m_j)^2}\end{aligned} \tag{137}$$

which is the case for the half-Brownian with $S_{k,m} = \frac{\Gamma\left(w+\mathrm{i}k-\frac{m}{2}\right)}{\Gamma\left(w+\mathrm{i}k+\frac{m}{2}\right)}$, and for the Brownian (for the modified partition sum $\tilde{Z}(x,t)$) with $S_{k,m} = \frac{\Gamma\left(w_R+\mathrm{i}k-\frac{m}{2}\right)}{\Gamma\left(w_R+\mathrm{i}k+\frac{m}{2}\right)}\frac{\Gamma\left(w_L+\mathrm{i}k-\frac{m}{2}\right)}{\Gamma\left(w_L+\mathrm{i}k+\frac{m}{2}\right)}$, an information obtained from the replica Bethe ansatz solutions of [17, 20, 21]. Then, irrespective of the precise form of $S_{k,m}$ the equations (131) will be obeyed. This conjecture is verified in appendix B for $n = 2, 3$. A similar mechanism holds for the nested contour integral form (see Appendix).

Hence we expect that the KP equation will be obeyed for any IC such that the overlap factorizes. Let us now examine the question of the initial condition, and the short time large deviations.

3.1. Half-Brownian

For the half-Brownian IC, the initial data for the moments read

$$\langle Z(x,0)^n\rangle = \langle \mathrm{e}^{nB(x)}\rangle_B \mathrm{e}^{-nwx}\theta(x) = \mathrm{e}^{n(\frac{n}{2}-w)x}\theta(x) \tag{138}$$

and one can also write explicitly the initial condition for G (which is discontinuous at $x = 0$)

$$G(x,0,r) = \langle \mathrm{e}^{-\theta(x)\mathrm{e}^{B(x)}\mathrm{e}^{-wx-r}}\rangle = \int_{-\infty}^{+\infty} \frac{db}{\sqrt{2\pi x}} \mathrm{e}^{-\frac{b^2}{2x} - \mathrm{e}^{b-wx-r}}\theta(x) + \theta(-x). \tag{139}$$

J. Stat. Mech. (2020) 043201

We now study the time evolution at short time $t \ll 1$. The regime of interest will correspond to small x, large w and xw fixed, so we will need the initial condition in that region. Let us write explicitly the initial condition for the first three cumulants and their expansion at small x, large w and xw fixed

$$Z_1(x,t=0) = \mathrm{e}^{(\frac{1}{2}-w)x}\theta(x) \simeq \mathrm{e}^{-wx}\theta(x) \tag{140}$$

$$Z_2(x,t=0) = (\mathrm{e}^{2(1-w)x} - \mathrm{e}^{2(\frac{1}{2}-w)x})\theta(x) \simeq x\mathrm{e}^{-2wx}\theta(x) \tag{141}$$

$$Z_3(x,t=0) = (e^x-1)^2\,(e^x+2)\,\mathrm{e}^{\left(\frac{3}{2}-3w\right)x}\theta(x) \simeq 3x^2\mathrm{e}^{-3wx}\theta(x). \tag{142}$$

It is easy to guess that the general formula is

$$Z_n(x,t=0) \simeq n^{n-2}x^{n-1}\mathrm{e}^{-nwx}\theta(x). \tag{143}$$

We will need the definition and series expansion of the standard branch $W_0(z)$ of the Lambert function [61], $W(z)$

$$W(z)\mathrm{e}^{W(z)} = z \quad , \quad W(z) = W_0(z) = -\sum_{n\geqslant 1}\frac{(-1)^n}{n!}n^{n-1}z^n. \tag{144}$$

From the moments we can thus write the initial condition for G, in that regime (for small x, large w, with wx fixed) as

$$\log G(x,t=0,r) = \sum_{n\geqslant 1}\frac{(-1)^n}{n!}Z_n(x,t=0)\mathrm{e}^{-nr} \simeq \frac{1}{x}\sum_{n\geqslant 1}\frac{(-1)^n}{n!}n^{n-2}\mathrm{e}^{-n(r-\log(x)+wx)}\theta(x) \tag{145}$$

and its derivative,

$$\hat{\phi}(x,t=0,r) = \partial_r\log G(x,t=0,r) = \frac{1}{x}W_0(\mathrm{e}^{-(r-\log x+wx)})\theta(x). \tag{146}$$

For fixed r and small x with wx fixed one has $\partial_r\log G(x,t=0,r) \simeq \mathrm{e}^{-r-wx}\theta(x)$ as for a half-wedge, however there is a fluctuation region, with fixed $r-\log x+wx$ where $\partial_r\log G(x,t=0,r) \sim 1/x$ is large.

To study the small time large deviations of small t, small x and large w with wx fixed we define

$$\tilde{x} = \frac{x}{\sqrt{t}} \quad , \quad \tilde{w} = w\sqrt{t}. \tag{147}$$

By analogy with the study of the Brownian IC in [38], we expect in the case of the half-Brownian, the large deviation form at $\tilde{x}=0$

$$\log G(0,t,r) \simeq -\frac{1}{\sqrt{t}}\Psi(t^{1/2}\mathrm{e}^{-r}). \tag{148}$$

At finite $\tilde{x}$ we thus expect the following form (which we obtain explicitly below)

$$\log G(x,t,r) \simeq -\frac{1}{\sqrt{t}}\hat{\Psi}(\tilde{x},r-\frac{1}{2}\log t) \quad , \quad \phi(x,t,r) = \partial_r^2\log G(x,t,r) \simeq -\frac{1}{\sqrt{t}}\psi(\tilde{x},r-\frac{1}{2}\log t) \tag{149}$$

J. Stat. Mech. (2020) 043201

with $\psi = \partial_r^2 \hat{\Psi}$. Inserting into the KP equation (5) gives to leading order $O(t^{-3/2})$

$$\partial_r \psi + \partial_r^2 \psi + x\partial_r \partial_x \psi - 2\partial_x^2 \psi = 0. \tag{150}$$

Note that for the droplet IC the same equation degenerates and does not allow us to determine the function. Since (150) is a linear equation, an alternate method, which we now use, is to study the linear part of the recursion (131) for the cumulants

$$[\partial_t - \frac{n^3 - n}{12} - \frac{1}{n}\partial_x^2] Z_n(x,t) = 0. \tag{151}$$

At short time one can neglect the term $(n^3 - n)/12$, and the solution can be written explicitly from the knowledge of the initial condition (143), as

$$Z_n(x,t) \simeq n^{n-2} \int_0^{+\infty} \frac{\mathrm{d}y}{\sqrt{4\pi t/n}} \mathrm{e}^{-\frac{n(x-y)^2}{4t}} y^{n-1} \mathrm{e}^{-nwy}. \tag{152}$$

Other, more explicit, expressions for these cumulants are given in appendix D.

Before evaluating this expression, let us first show that at large w one recovers exactly the droplet result. The large w limit is obtained setting $y \to y/w$ which leads to

$$Z_n(x,t) \simeq \frac{1}{\sqrt{4\pi t}} n^{n-\frac{3}{2}} w^{-n} \mathrm{e}^{-\frac{nx^2}{4t}} \int_0^{+\infty} \mathrm{d}y y^{n-1} \mathrm{e}^{-ny} = \frac{1}{\sqrt{4\pi t}} n^{-\frac{3}{2}} \Gamma(n) w^{-n} \mathrm{e}^{-\frac{nx^2}{4t}} \tag{153}$$

and to

$$\log G(x,t,r) \simeq \frac{1}{\sqrt{4\pi t}} \sum_{n\geqslant 1} (-1)^n n^{-5/2} (\frac{1}{w} \mathrm{e}^{-r-\frac{x^2}{4t}})^n = \frac{1}{\sqrt{4\pi t}} \mathrm{Li}_{\frac{5}{2}}(-\frac{1}{w} \mathrm{e}^{-r-\log w-\frac{x^2}{4t}}). \tag{154}$$

Apart from a shift, this is exactly the result obtained in [37]. The shift is easy to understand. Indeed, since at large w, $\mathrm{e}^{-wx+B(x)}\theta(x) \to \frac{1}{w}\delta(x)$, we expect $Z_{hb}(x,t) \to \frac{1}{w} Z_d(x,t)$ (the index hb refers to half-Brownian and d to droplet), that is $\langle \exp(-\mathrm{e}^{h_{hb}(x,t)-r}) \rangle \to \langle \exp(-\mathrm{e}^{h_d(0,t)-r-\log w-\frac{x^2}{4t}}) \rangle = \langle \exp(-\mathrm{e}^{h_d(x,t)-r-\log w}) \rangle$. Hence using the half-Brownian IC as a regularisation to run the KP equation, one can indeed calculate the leading term $p_0'(r) = \mathcal{L}_1$ in the short time expansion for the droplet IC (which was missing from a direct approach).

Let us return to the solution (152) for the cumulants and perform the summation (one can neglect the term $t/12$ in the exponential at small time)

$$\log G(x,t,r) \simeq \sum_{n\geqslant 1} \frac{(-1)^n}{n!} Z_n(x,t) \mathrm{e}^{-nr} \simeq \frac{1}{\sqrt{4\pi t}} \int_0^{+\infty} \frac{\mathrm{d}y}{y} \sum_{n\geqslant 1} \frac{(-1)^n}{n!} n^{n-\frac{3}{2}} [y \mathrm{e}^{-r-\frac{(x-y)^2}{4t}-wy}]^n \tag{155}$$

$$= \frac{1}{\sqrt{4\pi t}} \int_0^{+\infty} \frac{\mathrm{d}y}{y} \sum_{n\geqslant 1} \frac{(-1)^n}{n!} n^{n-\frac{3}{2}} [y \mathrm{e}^{-r+\frac{1}{2}\log t-\frac{(\tilde{x}-y)^2}{4}-\tilde{w}y}]^n \tag{156}$$

where in the second expression we have rescaled $y \to \sqrt{t}y$. We use now the following integral representation for the series

$$\sum_{n\geqslant 1} \frac{(-1)^n}{n!} n^{n-\frac{3}{2}} z^n = -\int_0^{+\infty} \frac{\mathrm{d}u}{\sqrt{\pi u}} \sum_{n\geqslant 1} \frac{(-1)^{n-1}}{n!} n^{n-1} (z\mathrm{e}^{-u})^n = -\int_0^{+\infty} \frac{\mathrm{d}u}{\sqrt{\pi u}} W_0(z\mathrm{e}^{-u}) \tag{157}$$

J. Stat. Mech. (2020) 043201

using the formula (144) for the Lambert function W_0. Hence we find our final result for the small time large deviation function from the Brownian IC

$$\log G(x,t,r) \simeq -\frac{1}{\sqrt{t}} \hat{\Psi}_{\tilde{w}}(\tilde{x}, r - \frac{1}{2}\log t) \quad , \quad \tilde{x} = x/\sqrt{t} \quad , \quad \tilde{w} = w\sqrt{t}$$
$$\hat{\Psi}_{\tilde{w}}(\tilde{x}, r) = \frac{1}{\sqrt{4\pi}} \int_0^{+\infty} \frac{\mathrm{d}y}{y} \int_0^{+\infty} \frac{\mathrm{d}u}{\sqrt{\pi u}} W_0(y\mathrm{e}^{-r-u-\frac{(\tilde{x}-y)^2}{4}-\tilde{w}y}) \tag{158}$$

which gives equation (10) in the introduction, with $\Psi(\tilde{x}, z = \mathrm{e}^{-r}) = \hat{\Psi}(\tilde{x}, r)$.

This result can be put in an equivalent form footnote[5]. Going back to the series (156) we can rescale $y \to y/n$, expand the square in the exponential, and use the representation $\mathrm{e}^{-\frac{y^2}{4n}} = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}k \mathrm{e}^{\mathrm{i}ky - nk^2}$ to put it in the form

$$\log G(x,t,r) \simeq \frac{1}{\sqrt{t}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} \sum_{n \geqslant 1} \frac{(-1)^n}{n n!} (\mathrm{e}^{-r-\frac{\tilde{x}^2}{4}+\frac{1}{2}\log t - k^2})^n \int_0^{+\infty} \frac{\mathrm{d}y}{y} y^n \mathrm{e}^{-y(\tilde{w}-\frac{\tilde{x}}{2}-\mathrm{i}k)}. \tag{159}$$

When $\tilde{w} - \frac{\tilde{x}}{2} > 0$ one can perform the integral over y, and one recognizes the series expansion of the function $\mathrm{Li}_2(s) = \sum_{n\geqslant 1} \frac{s^n}{n^2}$, leading to

$$\log G(x,t,r) \simeq \frac{1}{\sqrt{t}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} \mathrm{Li}_2(-\frac{\mathrm{e}^{-r-\frac{\tilde{x}^2}{4}+\frac{1}{2}\log t}\mathrm{e}^{-k^2}}{\tilde{w} - \frac{\tilde{x}}{2} - \mathrm{i}k}). \tag{160}$$

This form of the large deviation function is similar to the generic form which is obtained for other solvable cases[6].

3.2. Brownian

We now consider briefly the full Brownian IC. The initial condition is

$$\tilde{Z}(x,0) = \mathrm{e}^{\chi}(\mathrm{e}^{B_R(x)}\mathrm{e}^{-w_R x}\theta(x) + \mathrm{e}^{B_L(x)}\mathrm{e}^{w_L x}\theta(-x)) \tag{161}$$

and its moments are given by (we denote $2w = w_L + w_R$)

$$\langle \tilde{Z}(x,0)^n \rangle = \frac{\Gamma(2w-n)}{\Gamma(2w)} \left(\mathrm{e}^{n(\frac{n}{2}-w_R)x}\theta(x) + \mathrm{e}^{n(\frac{n}{2}+w_L)x}\theta(-x) \right). \tag{162}$$

The corresponding initial condition for G can be written as

$$G(x,0,r) = \langle \mathrm{e}^{-\tilde{Z}(x,0)\mathrm{e}^{-r}} \rangle = \int_{-\infty}^{+\infty} \mathrm{d}\chi P(\chi) \int_{-\infty}^{+\infty} \frac{db}{\sqrt{2\pi|x|}} \mathrm{e}^{-\frac{b^2}{2|x|}} (\mathrm{e}^{-\mathrm{e}^{b-w_R x+\chi-r}}\theta(x) + \mathrm{e}^{-\mathrm{e}^{b+w_L x+\chi-r}}\theta(-x)). \tag{163}$$

Let us specify to the case $w_L = w_R = w$. As for the half-Brownian, we want to study the limit of small x, w large with wx fixed. We would like to apply the same method as for the half-Brownian, i.e. evolve the cumulants with the linear part of the KP equation as

$$\tilde{Z}_n(x,t) \simeq \int_{-\infty}^{+\infty} \frac{\mathrm{d}y}{\sqrt{4\pi t/n}} \mathrm{e}^{-\frac{n(x-y)^2}{4t}} \tilde{Z}_n(y, t=0). \tag{164}$$

[5] I thank A Krajenbrink for help in this transformation.
[6] see e.g. table 7.1 in [41].

However the initial condition for $\tilde{Z}_n(x, t = 0)$ in the regime of interest is now more complicated, e.g.

$$\tilde{Z}_1(x, t = 0) \simeq \frac{1}{2w} e^{-w|x|} \quad , \quad \tilde{Z}_2(x, t = 0) \simeq \frac{1}{8w^3}(1 + 2w|x|)e^{-2w|x|} \tag{165}$$

$$\tilde{Z}_3(x, t = 0) \simeq \frac{1}{8w^5}(1 + 3w|x|(1 + w|x|))e^{-3w|x|} \quad , \quad \tilde{Z}_4(0, t = 0) \simeq \frac{15}{64w^5} \tag{166}$$

and so on, with more and more complicated polynomials. Hence we have performed only a few checks on the following conjectured formula

$$\log G(x, t) = \sum_{n \geqslant 1} \frac{(-1)^n}{n!} \tilde{Z}_n(x, t) e^{-nr}|_{x=\sqrt{t}\tilde{x}, w=\tilde{w}/\sqrt{t}}|_{\tilde{x}=0} \simeq -\frac{1}{\sqrt{t}} \Psi(te^{-r}) \tag{167}$$

$$\Psi(z) = \frac{1}{\pi} \int_0^{+\infty} dy (1 + \frac{1}{y + \tilde{w}^2}) \sqrt{y} \log(1 + \frac{ze^{-y}}{y + \tilde{w}^2}) = -\int_{-\infty}^{+\infty} \frac{dk}{2\pi} \mathrm{Li}_2(-z \frac{e^{-k^2}}{k^2 + \tilde{w}^2}) \tag{168}$$

where the second line is the exact result for the Brownian IC obtained from the FD in [38] (the equivalent last expression was obtained in [34] and note that all known solvable IC for KPZ in full space and half-space can be put in similar forms [41]). Inserting the $\tilde{Z}_n(x, t)$ obtained from (164) and the initial condition (165) we have verified by series expansion in e^{-r} that it holds for $n = 1, 2, 3$. Although much remains to be done, this provides a nice check that the full Brownian IC indeed satisfies the KP equation, as claimed here.

4. Conclusion

In conclusion we have studied some of the consequences of the property recently discovered in [1], that the generating function of the droplet and half-Brownian IC solutions of the KPZ equation satisfy the KP equation. We have also studied the mechanism for this property to hold on the cumulants $Z_n(x,t)$, which led to the conclusion that the modified generating function for the full Brownian IC (or any IC with a 'decoupled' overlap) should also satisfy this property.

The main consequences of the KP property studied here concerns the large deviations, both at short time and at large time. For the droplet IC, we have found that the question of which initial condition should be used for the KP equation is intimately related to the small time large deviations. In the case of the droplet IC, the KP equation simplifies into a reduced KP equation. We have shown how to recover, from this reduced KP equation, and in a rather effortless way, the full systematic short time expansion obtained previously in [34]. On the other hand, substituting the large time large deviation form in the reduced KP equation provides a (rather simple) fifth method to obtain the rate function $\Phi_-(z)$, up to a single undetermined parameter. We showed how this parameter can be determined using the so-called cumulant summation of the short time expansion. This method, which allows us to relate the short time and the large time large deviation regimes, was studied within the KP setting. It takes the form of a semi-classical expansion where one treats the third derivative term in the KP equation as a perturbation. It can nicely be solved in terms of Burgers equation. This allowed us also to obtain in addition the first subdominant corrections, not obtained in [34]. We have shown that not only the KPZ problem, but a variant of a more general problem of linear statistics of the Airy point process, defined and studied in [34] and [36], do obey the KP property. We have re-obtained some results for these linear statistics, by a completely different method. The ensuing connections between the KP equation, the Coulomb gas, the non-local Painleve equation, and the stochastic Airy operator (the connections between the latter three were unveiled in [36]) remain to be investigated deeper.

Note two interesting consequences. The first arises from the connection [60, 62] of the droplet solution of the KPZ equation to N non interacting fermions in a 1D harmonic trap at finite temperature T, with Hamiltonian $H = \frac{p^2}{2} + \frac{x^2}{2}$, in the limit $N, T \to +\infty$ at fixed $b = N^{1/3}/T$. Denoting $x_{\max}(T)$ the position of the rightmost fermion, and $\xi = \sqrt{2}N^{1/6}(x_{\max}(T) - \sqrt{2N})$ the centered and scaled position, then $\partial_r^2 \log \mathrm{Prob}(b\xi < r)$ satisfies the reduced KP equation (13) with $t = b^3$. A second consequence arises from the connection between the KPZ equation and the non-relativistic limit of the $\mathrm{D} = 1 + 1$ sine Gordon field theory [14]. It implies that, in that limit, the two time correlation function of the field $\mathrm{e}^{a\varphi(0,t)}$ (at time zero and t, φ being the sine Gordon field) identifies with the generating function $\hat{G}(t, r)$ studied in the present work, with the correspondence $\mathrm{e}^{\frac{t}{12} - r} = (2\sinh\frac{a}{2})^2\mathrm{e}^{-Mc_\ell^2 t}$. Hence, in that limit the two time correlation function obeys a differential equation related to KP via this change of variable.

For the half-Brownian and Brownian IC, the study is technically more involved since one should handle space and time, and many questions remain. However, we have obtained the small time large deviation space-time rate function for the half-Brownian IC. We have also checked that the formal solution at short time in the case of the Brownian IC does agree with known results for the large deviation rate function at the origin, from [38].

Although the present study does not explain the deep reason of why the KP equation appears in some finite time solutions to the KPZ equation (which is related to why they can be expressed as a specific form of Fredholm determinants) it does show that this property provides a new interesting angle to study properties of these solutions.

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Appendix A. Short time expansion

We give here more details on the short time expansion for droplet IC. It is also valid for more general linear statistics problems see remark 2.

Inserting the series (25) into (13) and integrating once over r (as in (14)) we obtain the recursion, for $m \geqslant 0$

$$\frac{m+1}{2} p_{m+1} + \frac{1}{12}\theta_{m\geqslant 1} p'''_{m-1} + \frac{1}{2}\sum_{m_1+m_2=m, m_1\geqslant 0, m_2\geqslant 0} p'_{m_1} p'_{m_2} = 0. \tag{A.1}$$

Performing the recursion with Mathematica we find that all p_m are total derivatives of the form $p_m = \partial_r P_m(p'_0, p''_0, ..)$ where P_m are polynomials. Using $-\partial_r \mathcal{L}_i = \mathcal{L}_{i+1}$ and $p'_0(r) = \mathcal{L}_1$, we find exactly all seven terms (up to $O(t^3)$) given in the lengthy equation (61) in [34]. It is easy to quickly obtain the next terms, and we will show here only the next one $O(t^{7/2})$

$$\begin{aligned}
p_8(r) = \partial_r \Big[& \frac{2}{315}\mathcal{L}_7\mathcal{L}_1^8 + \frac{32}{63}\mathcal{L}_4^2\mathcal{L}_1^7 + \frac{16}{21}\mathcal{L}_3\mathcal{L}_5\mathcal{L}_1^7 + \frac{32}{105}\mathcal{L}_2\mathcal{L}_6\mathcal{L}_1^7 + \frac{16}{3}\mathcal{L}_3^3\mathcal{L}_1^6 + \frac{64}{3}\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4\mathcal{L}_1^6 + \frac{16}{3}\mathcal{L}_2^2\mathcal{L}_5\mathcal{L}_1^6 + \frac{1}{135}\mathcal{L}_8\mathcal{L}_1^6 \\
& + 96\mathcal{L}_2^2\mathcal{L}_3^2\mathcal{L}_1^5 + \frac{128}{3}\mathcal{L}_2^3\mathcal{L}_4\mathcal{L}_1^5 + \frac{10}{9}\mathcal{L}_4\mathcal{L}_5\mathcal{L}_1^5 + \frac{32}{45}\mathcal{L}_3\mathcal{L}_6\mathcal{L}_1^5 + \frac{4}{15}\mathcal{L}_2\mathcal{L}_7\mathcal{L}_1^5 + \frac{28}{3}\mathcal{L}_2\mathcal{L}_4^2\mathcal{L}_1^4 + 160\mathcal{L}_2^4\mathcal{L}_3\mathcal{L}_1^4 + \frac{118}{9}\mathcal{L}_3^2\mathcal{L}_4\mathcal{L}_1^4 \\
& + \frac{44}{3}\mathcal{L}_2\mathcal{L}_3\mathcal{L}_5\mathcal{L}_1^4 + \frac{10}{3}\mathcal{L}_2^2\mathcal{L}_6\mathcal{L}_1^4 + \frac{1}{432}\mathcal{L}_9\mathcal{L}_1^4 + \frac{128}{3}\mathcal{L}_2^6\mathcal{L}_1^3 + \frac{128}{3}\mathcal{L}_2\mathcal{L}_3^3\mathcal{L}_1^3 + \frac{17}{90}\mathcal{L}_5^2\mathcal{L}_1^3 + \frac{280}{3}\mathcal{L}_2^2\mathcal{L}_3\mathcal{L}_4\mathcal{L}_1^3 + \frac{160}{9}\mathcal{L}_2^3\mathcal{L}_5\mathcal{L}_1^3 \\
& + \frac{83}{270}\mathcal{L}_4\mathcal{L}_6\mathcal{L}_1^3 + \frac{89}{540}\mathcal{L}_3\mathcal{L}_7\mathcal{L}_1^3 + \frac{1}{18}\mathcal{L}_2\mathcal{L}_8\mathcal{L}_1^3 + \frac{320}{3}\mathcal{L}_2^3\mathcal{L}_3^2\mathcal{L}_1^2 + \frac{109}{36}\mathcal{L}_3\mathcal{L}_4^2\mathcal{L}_1^2 + 40\mathcal{L}_2^4\mathcal{L}_4\mathcal{L}_1^2 + \frac{7}{3}\mathcal{L}_3^2\mathcal{L}_5\mathcal{L}_1^2 + \frac{17}{5}\mathcal{L}_2\mathcal{L}_4\mathcal{L}_5\mathcal{L}_1^2 \\
& + \frac{32}{15}\mathcal{L}_2\mathcal{L}_3\mathcal{L}_6\mathcal{L}_1^2 + \frac{5}{12}\mathcal{L}_2^2\mathcal{L}_7\mathcal{L}_1^2 + \frac{\mathcal{L}_{10}\mathcal{L}_1^2}{5184} + \frac{14}{9}\mathcal{L}_3^4\mathcal{L}_1 + \frac{29}{6}\mathcal{L}_2^2\mathcal{L}_4^2\mathcal{L}_1 + 32\mathcal{L}_2^5\mathcal{L}_3\mathcal{L}_1 + \frac{40}{3}\mathcal{L}_2\mathcal{L}_3^2\mathcal{L}_4\mathcal{L}_1 + \frac{22}{3}\mathcal{L}_2^2\mathcal{L}_3\mathcal{L}_5\mathcal{L}_1 \\
& + \frac{10}{9}\mathcal{L}_2^3\mathcal{L}_6\mathcal{L}_1 + \frac{37\mathcal{L}_5\mathcal{L}_6\mathcal{L}_1}{1512} + \frac{103\mathcal{L}_4\mathcal{L}_7\mathcal{L}_1}{6048} + \frac{17\mathcal{L}_3\mathcal{L}_8\mathcal{L}_1}{2160} + \frac{1}{432}\mathcal{L}_2\mathcal{L}_9\mathcal{L}_1 + \frac{16\mathcal{L}_2^7}{21} + \frac{14}{3}\mathcal{L}_2^2\mathcal{L}_3^3 + \frac{583\mathcal{L}_4^3}{18\,144} + \frac{607\mathcal{L}_2\mathcal{L}_5^2}{15\,120} \\
& + \frac{58}{9}\mathcal{L}_2^3\mathcal{L}_3\mathcal{L}_4 + \frac{5}{6}\mathcal{L}_2^4\mathcal{L}_5 + \frac{1121\mathcal{L}_3\mathcal{L}_4\mathcal{L}_5}{7560} + \frac{17}{360}\mathcal{L}_3^2\mathcal{L}_6 + \frac{503\mathcal{L}_2\mathcal{L}_4\mathcal{L}_6}{7560} + \frac{77\mathcal{L}_2\mathcal{L}_3\mathcal{L}_7}{2160} + \frac{5}{864}\mathcal{L}_2^2\mathcal{L}_8 + \frac{\mathcal{L}_{11}}{497\,664}\Big]
\end{aligned}$$

where we recall that $p'_m(r)$ is also the term of order $t^{\frac{m-1}{2}}$ in $q_{t,\beta=1}(\sigma)$ as defined in equation (61) in [34].

Appendix B. Checks on cumulants from the Bethe ansatz for the droplet, half-Brownian and Brownian IC

Let us recall briefly that the moments of the solution of the SHE can be obtained as a sum over the eigenstates of the delta Bose gas (Lieb Liniger) Hamiltonian [63] $H_n = -\sum_{\alpha=1}^n \partial_{x_\alpha}^2 - 2\bar{c}\sum_{1\leqslant\alpha<\beta\leqslant n}\delta(x_\alpha - x_\beta)$, as follows

$$\langle Z(x,t)^n\rangle = \sum_\mu \Psi_\mu(x,\ldots,x)\frac{\mathrm{e}^{-tE_\mu}}{||\mu||^2}\langle\Psi_\mu|\Phi_0\rangle = \sum_\mu \Psi_\mu^*(x,\ldots,x)\frac{\mathrm{e}^{-tE_\mu}}{||\mu||^2}\langle\Phi_0|\Psi_\mu\rangle \tag{B.1}$$

in quantum mechanical notations, where Ψ_μ denote the eigenfunctions (and $||\mu||$ their norm) and E_μ the eigenvalues of H_n. Note that $\bar{c} = 1$ in our units for the study of the SHE (corresponding to attractive interactions), but the case $\bar{c} = -c < 0$ is also of interest in the context of repulsive bosons [64][7]. The eigenfunctions [63] are parameterized by a set of rapidities $\mu \equiv \{\lambda_1, ..\lambda_n\}$, are totally symmetric in the x_α, and in the sector $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$, take the (un-normalized) form of a sum over permutations P

[7] For an application of the repulsive case to Brownian coincidences, see [64].

$$\Psi_\mu(x_1,..x_n)=\sum_{P\in S_n}A_P\prod_{j=1}^{n}\mathrm{e}^{\mathrm{i}\sum_{\alpha=1}^{n}\lambda_{P_\alpha}x_\alpha}\ ,\quad A_P=\prod_{1\leqslant\alpha<\beta\leqslant n}\Big(1+\frac{\mathrm{i}}{\lambda_{P_\beta}-\lambda_{P_\alpha}}\Big).\tag{B.2}$$

with $E_\mu=\sum_{\alpha=1}^{n}\lambda_\alpha^2$. To evaluate (B.1) one needs $\Psi_\mu^*(x,..x)=n!\mathrm{e}^{-\mathrm{i}x\sum_\alpha\lambda_\alpha}$. For the Brownian IC the wavefunction of the initial replica state is:

$$\Phi_0(Y)=\langle y_1,..y_n|\Phi_0\rangle=\langle\prod_{\alpha=1}^{n}(\mathrm{e}^{w_Ly_\alpha}\mathrm{e}^{B_L(-y_\alpha)}\theta(-y_\alpha)+\mathrm{e}^{-w_Ry_\alpha}\mathrm{e}^{B_R(y_\alpha)}\theta(y_\alpha))\rangle_{B_L,B_R}\tag{B.3}$$

where $Y\equiv y_1,..y_n$. The half-Brownian is obtained setting $w_L\to+\infty$, and for the droplet IC, $\Phi_0(Y)=\prod_{\alpha=1}^{n}\delta(y_\alpha)$, obtained e.g. by further multiplying by w_R^n and sending $w_R\to+\infty$. One needs the overlap

$$\langle\Phi_0|\Psi_\mu\rangle=n!\int_{y_1<y_2<..<y_n}\Psi_\mu(Y)\Phi_0(Y)=n!\sum_{P\in S_n}A_P\int_{y_1<y_2<..<y_n}\mathrm{e}^{\mathrm{i}\sum_{\alpha=1}^{n}\lambda_{P_\alpha}y_\alpha}\Phi_0(Y).\tag{B.4}$$

A 'miracle' occurs in performing the sum over permutations, and one finds [17, 20, 21] that the overlap takes the very simple 'decoupled' form for the Brownian IC

$$\langle\Phi_0|\Psi_\mu\rangle=n!\frac{\prod_{j=1}^{n}(w_R+w_L-j)}{\prod_{j=1}^{n}(w_R-\frac{1}{2}-\mathrm{i}\lambda_j)\prod_{j=1}^{n}(w_L-\frac{1}{2}+\mathrm{i}\lambda_j)}\tag{B.5}$$

which leads to $\langle\Phi_0|\Psi_\mu\rangle=n!\frac{1}{\prod_{j=1}^{n}(w_R-\frac{1}{2}-\mathrm{i}\lambda_j)}$ for the half-Brownian, and simply $\langle\Phi_0|\Psi_\mu\rangle=n!$ for the droplet IC. In the infinite system size limit, each eigenstate is made of $1\leqslant n_s\leqslant n$ *strings* with rapidities $\lambda_{j,a}=k_j-\frac{\mathrm{i}}{2}(m_j+1-2a)$, $a=1,\dots,m_j$ (here k_j are real momenta and $m_j\geqslant1$ integers with $\sum_{j=1}^{n_s}m_j=n$). The overlap are thus, for half-Brownian IC (hb) and Brownian IC (b)

$$\langle\Phi_0|\Psi_\mu\rangle_{hb}=n!\prod_{j=1}^{n_s}s_{-k_j,m_j}^{w_R}\quad,\quad\langle\Phi_0|\Psi_\mu\rangle_b=n!\frac{\Gamma(w_L+w_R)}{\Gamma(w_L+w_R-n)}$$
$$\prod_{j=1}^{n_s}s_{-k_j,m_j}^{w_R}s_{k_j,m_j}^{w_L}\quad,\quad s_{k,m}^{w}=\frac{\Gamma\left(w+\mathrm{i}k-\frac{m}{2}\right)}{\Gamma\left(w+\mathrm{i}k+\frac{m}{2}\right)}\tag{B.6}$$

with $S_{k,1}^{w}=\frac{1}{w+\mathrm{i}k-\frac{1}{2}}$ and so on. To remove the extra factor in the full Brownian case (and allow for a FD expression) one defines (in that case only) the modified partition sum $\tilde{Z}(x,t)$ as

$$\langle\tilde{Z}^n(x,t)\rangle=\frac{\langle Z^n(x,t)\rangle}{\prod_{j=1}^{n}(w_R+w_L-j)}\quad,\quad\tilde{Z}(x,t)=\mathrm{e}^{\tilde{h}(x,t)}=\mathrm{e}^{h(x,t)+\chi}\quad,\quad\langle\mathrm{e}^{n\chi}\rangle=\frac{\Gamma(w_L+w_R-n)}{\Gamma(w_L+w_R)}\tag{B.7}$$

i.e. one defines [20, 21] a randomly shifted height field $\tilde{h}(x,t)=h(x,t)+\chi$ (recalling that $Z(x,t)=\mathrm{e}^{h(x,t)}$), where χ a log-gamma variable of parameter $\gamma=w_L+w_R$, independent of h. Finally after inserting the known expression for the norms $||\mu||^2$ of the eigenstates, and changing $k_j\to-k_j$ one obtains the formula (137) of the text, which we reproduce here

J. Stat. Mech. (2020) 043201

$$\langle Z(x,t)^n\rangle = n!\sum_{n_s=1}^{n}\frac{1}{n_s!}\sum_{(m_1,\dots,m_{n_s})}\prod_{j=1}^{n_s}\int\frac{\mathrm{d}k_j}{2\pi m_j}\mathrm{e}^{\mathrm{i}x\sum_{j=1}^{n_s}m_jk_j}\mathrm{e}^{t\sum_{j=1}^{n_s}[\frac{1}{12}(m_j^3-m_j)-m_jk_j^2]}$$
$$\times\prod_{j=1}^{n_s}S_{k_j,m_j}\prod_{1\leqslant i<j\leqslant n_s}\frac{4(k_i-k_j)^2+(m_i-m_j)^2}{4(k_i-k_j)^2+(m_i+m_j)^2} \tag{B.8}$$

with $S_{k,m}=1$ for the droplet IC, $S_{k,m}=s^w_{k,m}$ for the half-Brownian IC, and $S_{k,m}=s^{w_R}_{k,m}s^{w_L}_{-k,m}$ for the Brownian, where it is implicit here and below that in that case the lhs of (B.8) must be replaced by $\langle\tilde{Z}(x,t)^n\rangle$, the moments of the modified partition sum.

If the KP equation property holds, the cumulants must satisfy the equations (131). We want to understand the mechanism for this property on the form (B.8). Let us start with the first two cumulants obtained from (B.8). They read

$$Z_1(x,t)=\langle Z(x,t)\rangle=\int\frac{\mathrm{d}k}{2\pi}\mathrm{e}^{-\mathrm{i}xk}\mathrm{e}^{-tk^2}S_{k,1} \tag{B.9}$$

$$Z_2(x,t)=\langle Z(x,t)^2\rangle-\langle Z(x,t)\rangle^2=\mathrm{e}^{\frac{t}{2}}\int\frac{\mathrm{d}k}{2\pi}\mathrm{e}^{-2\mathrm{i}xk}\mathrm{e}^{-2tk^2}S_{k,2}$$
$$+\int\frac{\mathrm{d}k_1}{2\pi}\int\frac{\mathrm{d}k_2}{2\pi}\mathrm{e}^{-\mathrm{i}x(k_1+k_2)}\mathrm{e}^{-t(k_1^2+k_2^2)}[\frac{(k_1-k_2)^2}{(k_1-k_2)^2+1}-1]S_{k_1,1}S_{k_2,1}. \tag{B.10}$$

In the expression for Z_2 the first integral is the contribution of the single string state which contains two bosons, $n_s=1$, $m_1=2$, and the second integral the contribution of the two string state, $n_s=2$, $m_1=1$, $m_2=1$ (these strings are just 'particles' since their length is unity).

Let us now check that the first equation in (131) is obeyed

$$\partial_t Z_2(x,t)-\frac{1}{2}Z_2(x,t)-\frac{1}{2}\partial_x^2Z_2(x,t)=\frac{1}{2}Z_1(x,t)^2. \tag{B.11}$$

We note that the differential operator $D_2=\partial_t-\frac{1}{2}\partial_x^2-\frac{1}{2}$ gives zero on the first term in (B.10). It is the 1-string contribution $n_s=1$ and, as mentioned in the text, it is a general property that this term obeys the *linear* part of the equation (B.11) (and more generally the linear part of (131)). Acting on the second term in (B.10) the operator D_2 multiplies the integrand by

$$D_2(k_1,k_2)=-(k_1^2+k_2^2)+\frac{1}{2}(k_1+k_2)^2-\frac{1}{2}=-\frac{1}{2}((k_1-k_2)^2+1). \tag{B.12}$$

Hence in the integrand we have the simplification

$$D(k_1,k_2)[\frac{(k_1-k_2)^2}{(k_1-k_2)^2+1}-1]=D(k_1,k_2)\frac{-1}{(k_1-k_2)^2+1}=\frac{1}{2} \tag{B.13}$$

which leads to factorization of the double integral in two factors Z_1 given by (B.9), which thus implies that (B.11) holds.

Let us check whether this mechanism, which uses only properties of the factor arising from the norm, $\frac{(k_1-k_2)^2}{(k_1-k_2)^2}$, and not of the factor $S_{k,m}$, holds to higher order. Let us write the third cumulant from (B.8). We see that the substractions, arising from the

J. Stat. Mech. (2020) 043201

definition of a cumulant, result in 'counterterms' inside each contribution, which we have written in a symmetric form

$$
\begin{aligned}
&Z_3(x,t) = \langle Z(x,t)^3\rangle - 3\langle Z(x,t)^2\rangle\langle Z(x,t)\rangle + 2\langle Z(x,t)\rangle^3 = 2\mathrm{e}^{2t}\int \frac{\mathrm{d}k}{2\pi}\mathrm{e}^{-3\mathrm{i}xk}\mathrm{e}^{-3tk^2}S_{k,3}\\
&+3\mathrm{e}^{t/2}\int \frac{\mathrm{d}k_1}{2\pi}\frac{\mathrm{d}k_2}{2\pi}\mathrm{e}^{-\mathrm{i}x(2k_1+k_2)-(2k_1^2+k_2^2)t}S_{k_1,2}S_{k_2,1}[\frac{4(k_1-k_2)^2+1}{4(k_1-k_2)^2+9}-1]\\
&+\int \frac{\mathrm{d}k_1}{2\pi}\frac{\mathrm{d}k_2}{2\pi}\frac{\mathrm{d}k_3}{2\pi}\mathrm{e}^{-\mathrm{i}x(k_1+k_2+k_3)-(k_1^2+k_2^2+k_3^2)t}S_{k_1,1}S_{k_2,1}S_{k_3,1}\\
&\times\left[\frac{(k_1-k_2)^2}{(k_1-k_2)^2+1}\frac{(k_1-k_3)^2}{(k_1-k_3)^2+1}\frac{(k_3-k_3)^2}{(k_2-k_3)^2+1}-\frac{(k_1-k_2)^2}{(k_1-k_2)^2+1}-\frac{(k_1-k_3)^2}{(k_1-k_3)^2+1}-\frac{(k_2-k_3)^2}{(k_2-k_3)^2+1}+2\right].
\end{aligned}
\tag{B.14}
$$

Again we have written first the contribution $n_s = 1$, then $n_s = 2$ and finally the $n_s = 3$ term. Let us check that the second equation from (131) is obeyed.

$$
\partial_t Z_3(x,t) - 2Z_3(x,t) - \frac{1}{3}\partial_x^2 Z_3(x,t) = 4Z_1(x,t)Z_2(x,t). \tag{B.15}
$$

The differential operator $D_3 = \partial_t - \frac{1}{3}\partial_x^2 - 2$, gives again zero when applied on the first term $n_s = 1$. On the second term $n_s = 2$, it multiplies the integrand by

$$
D_3 \to -(2k_1^2+k_2^2) + \frac{1}{3}(2k_1+k_2)^2 - \frac{3}{2} = -\frac{1}{6}(4(k_1-k_2)^2+9) \tag{B.16}
$$

thereby producing exactly the first term in the rhs of (B.15) which reads explicitly

$$
\begin{aligned}
4Z_1(x,t)Z_2(x,t) &= 4\mathrm{e}^{t/2}\int \frac{\mathrm{d}k_1}{2\pi}\frac{\mathrm{d}k_2}{2\pi}\mathrm{e}^{-\mathrm{i}x(2k_1+k_2)-(2k_1^2+k_2^2)t}S_{k_1,2}S_{k_2,1}\\
&+4\int \frac{\mathrm{d}k_1}{2\pi}\frac{\mathrm{d}k_2}{2\pi}\frac{\mathrm{d}k_3}{2\pi}\mathrm{e}^{-\mathrm{i}x(k_1+k_2+k_3)-(k_1^2+k_2^2+k_3^2)t}S_{k_1,1}S_{k_2,1}S_{k_3,1}[\frac{(k_1-k_2)^2}{(k_1-k_2)^2+1}-1].
\end{aligned}
\tag{B.17}
$$

Finally on the third term its action on the integrand gives exactly the second term in (B.17) using that

$$
\begin{aligned}
&(-k_1^2-k_2^2-k_3^2-\frac{1}{3}(k_1+k_2+k_3)^2-2)\\
&\times\left[\frac{(k_1-k_2)^2}{(k_1-k_2)^2+1}\frac{(k_1-k_3)^2}{(k_1-k_3)^2+1}\frac{(k_3-k_3)^2}{(k_2-k_3)^2+1}-\frac{(k_1-k_2)^2}{(k_1-k_2)^2+1}-\frac{(k_1-k_3)^2}{(k_1-k_3)^2+1}-\frac{(k_2-k_3)^2}{(k_2-k_3)^2+1}+2\right]\\
&=\frac{4}{3}[\frac{(k_1-k_2)^2}{(k_1-k_2)^2+1}-1+2\ \mathrm{perm}].
\end{aligned}
\tag{B.18}
$$

Hence (B.15) is obeyed.

Although we have not established it $n \geqslant 4$, it is already clear on the cases $n = 2, 3$, that the mechanism of 'simplification' which transforms the nth cumulant onto a sum of lower cumulants, upon application of the differential linear operator, works only from some combinatoric property of the norm factor, and does not involve $S_{k,m}$. Only the decoupled form of the overlap is crucial, hence it works *in exactly the same way* for droplet, half-Brownian, and Brownian (in the latter case using the modified partition sum $\tilde{Z}$). Of course this decoupled form is also the reason for a simple FD formula to exist (when summing up the moments in the generating function G) but it is useful to see how it works on the cumulants.

J. Stat. Mech. (2020) 043201

Remark B.1. One can ask how this mechanism works on the nested contour integral representation. Consider any solution of the form

$$\langle Z(x,t)^n\rangle = \prod_j [\int_{C_j} \frac{\mathrm{d}t_j}{2\mathrm{i}\pi} \mathrm{e}^{t z_j^2 + x z_j} g(z_j)] \prod_{1\leqslant i<j\leqslant n} \frac{z_i - z_j}{z_i - z_j - \bar{c}}. \tag{B.19}$$

This formula holds for the droplet IC, with $\bar{c}=1$ in our units, where the C_j are parallel to the imaginary axis with real parts such that $\mathrm{Re}(z_i - z_j) > \bar{c}$ (see formula (6.6) in [65]). In that case $g(z)=1$ but we consider here the more general case. Note that the case $\bar{c}<0$ is also of interest as it provides a solution for the repulsive delta Bose gas [64][8].

Let us set $\bar{c}=1$ (but the property extends for any $\bar{c}$). For $n=2$ the property is very simple. The operator $D_2 = \partial_t - \frac{1}{2}\partial_x^2 - \frac{1}{2}$ leads to the multiplication of the integrand by

$$z_1^2 + z_2^2 - \frac{1}{2}(z_1+z_2)^2 - \frac{1}{2} = \frac{1}{2}(z_1 - z_2 - 1)(z_1 - z_2 + 1). \tag{B.20}$$

The following simplification thus occurs in the integrand

$$\frac{1}{2}(z_1 - z_2 - 1)(z_1 - z_2 + 1)(\frac{z_1 - z_2}{z_1 - z_2 - 1} - 1) = \frac{1}{2}(z_1 - z_2 + 1) \to \frac{1}{2}. \tag{B.21}$$

The last step arises from the symmetry of the integrand, which can now be used, since the poles have disappeared and the contours C_j can be brought together. Hence (B.11) holds. It works quite similarly to the previous paragraph, although the factors are slightly different.

For $n=3$, from the definition of the third cumulant in (B.14), we implement the subtractions in a symmetric way, which leads to the following factor in the integrand of Z_3

$$Z_3 \equiv \prod_{1\leqslant i<j\leqslant n} \frac{z_i - z_j}{z_i - z_j - 1} - \frac{z_1 - z_2}{z_1 - z_2 - 1} - \frac{z_1 - z_3}{z_1 - z_3 - 1} - \frac{z_2 - z_3}{z_2 - z_3 - 1} + 2 = \frac{2}{(z_1 - z_2 - 1)(z_2 - z_3 - 1)} \tag{B.22}$$

which, we note, simplifies. The differential linear operator D_3 acts by multiplying the integrand by $D_3(z_1,z_2,z_3) = z_1^2 + z_2^2 + z_3^2 - \frac{1}{3}(z_1+z_2+z_3)^2 - 2$ and an important property is that at the double pole of (B.22), $D_3(z_1,z_2,z_3)|_{z_1=z_2+1, z_2=z_3+1} = 0$. One also checks the following symmetrization property

$$D_3(z_1,z_2,z_3)\mathrm{sym}_{z_1,z_2,z_3}[\frac{2}{(z_1 - z_2 - 1)(z_2 - z_3 - 1)}] = 4\mathrm{sym}_{z_1,z_2,z_3}[\frac{z_1 - z_2}{z_1 - z_2 - 1} - 1] \tag{B.23}$$

which is a necessary condition for (B.15) to hold. It would be sufficient for $\bar{c}<0$ but here for $\bar{c}>0$ one need to examine the poles to make sure it holds also for the nested contours. Since we did that in the previous paragraph, we know that it must work, and we will not pursue it here. It seems clear that there is a general mechanism for the equations on the cumulant to hold, provided, again, that the overlap is factorized. It would be interesting to establish it for any value of n.

[8] See footnote 7.

Appendix C. Large time large deviation for more general $g(x)$, and linear statistics of the Airy process

As noted in remark 2, for any function $g(x)$, the function $\phi(t,r) = \partial_r^2 q_{t,\beta}(\sigma = -\mathrm{e}^{-r})$, where $q_{t,\beta}(\sigma = -\mathrm{e}^{-r})$ is the FD (32), must obey the KP equation with a more general initial condition $\phi(t,r) \simeq_{t\to 0} \frac{1}{\sqrt{t}} p_0'(r)$ with

$$p_0'(r) = \mathcal{L}_1(-\mathrm{e}^{-r}) = \frac{\beta}{\pi}(\partial_r)^2 \int_0^{+\infty} \mathrm{d}x \sqrt{x} g(-\mathrm{e}^{-x-r}). \tag{C.1}$$

One can now choose a more general function $g(x)$, as in equation (214) of [34], with

$$\beta g(-\mathrm{e}^{-r}) = \Gamma(1+\gamma)\mathrm{Li}_\gamma(-\mathrm{e}^{-r}) \simeq_{r\to-\infty} -(-r)_+^\gamma \tag{C.2}$$

except that we do not include the factor $t^{1-\gamma}$ of (214), i.e. we must choose $g(x)$ to be time-independent. The value $\gamma = 1$ corresponds to the KPZ case, $g_{\mathrm{KPZ}}(x) = -\log(1-x)$. Then one has, from equation (218) of [34],

$$p_0'(r) = \mathcal{L}_1(-\mathrm{e}^{-r}) \simeq_{r\to-\infty} -\frac{\Omega}{2}(-r)^{\gamma-\frac{1}{2}} \quad , \quad \Omega = \frac{\Gamma(1+\gamma)}{\sqrt{\pi}\Gamma(\frac{1}{2}+\gamma)} \tag{C.3}$$

which is a monomial at large negative r.

We now ask about the large time large deviation regime. We will use the analysis of cumulants of [34], extended here in section 2.5. Let us perform the counting of powers of time. We write, from (56) and (60)

$$\log \hat{G} = \sum_{n\geqslant 1} \frac{\kappa_n(t,r)}{n!} = \sum_{n\geqslant 1} \frac{1}{n!}\left[t^{\frac{n}{2}-1} 2^{n-1}(-1)^{n-1}(\partial_r)^{n-3}(p_0'(r))^n + t^{\frac{n}{2}}(\partial_r)^{n-2}(p_0')^{n-2} + \ldots\right] \tag{C.4}$$

where the second (subleading) term is written only schematically. Because what we do is slightly different than in [34] we must scale $r \sim t^\alpha$ and determine α later. We thus set $r = zt^\alpha$ with fixed $z < 0$. Then the powers of time in the first term are $t^{\frac{n}{2}-1-\alpha(n-3)+n\alpha(\gamma-\frac{1}{2})}$. To make it independent of n we must choose $\alpha = \frac{1}{3-2\gamma}$ (we will restrict here to $\gamma < 3/2$). The power of time of each term is then $\log \hat{G} \sim t^{3\alpha-1} = t^{\frac{2\gamma}{3-2\gamma}}$. The subleading term (second term) then scales as $\log \hat{G} \sim t^{\frac{n}{2}-\alpha(n-2)+(n-2)\alpha(\gamma-\frac{1}{2})} = t$, hence it is indeed subdominant at large time for $\gamma > 3/4$, which is $\alpha > 2/3$. Now we note that although the power counting in t is different, all coefficients being the same, the summation of the leading term should lead to the same function as in [34], i.e

$$\log \hat{G}(t,r) \simeq -t^{3\alpha-1}\Phi_-(z) \quad , \quad \Phi_-(z) = -\frac{1}{2}\sum_{n\geqslant 1} \frac{(-\Omega)^n}{\Gamma(n+1)} \frac{\Gamma(n(\gamma-\frac{1}{2})+1)}{\Gamma(4-n(\frac{3}{2}-\gamma))}(-z)^{3-n(\frac{3}{2}-\gamma)}. \tag{C.5}$$

On the other hand, we can directly search for a solution of the reduced KP equation (13) which scales for large negative r as

$$\log \hat{G}(t,r) = -t^{3\alpha-1}\Phi_-(\frac{r}{t^\alpha}) \quad , \quad \psi(t,r) = \partial_r^2 \log \hat{G}(t,r) = t^{\alpha-1} H_0(\frac{r}{t^\alpha}) \tag{C.6}$$

J. Stat. Mech. (2020) 043201

with $H_0 = -\Phi''_-$. It leads to a generalisation of (38), to which it reduces for $\alpha = 1$

$$(\alpha - \frac{1}{2})H_0(z) - \alpha z H_0'(z) + H_0(z)H_0'(z) = 0. \tag{C.7}$$

This equation is solved by the change of variable $H_0(z) = zh(z)$ and $z = -e^u$ which leads to $\frac{dh}{du} = \frac{h(\frac{1}{2}-h)}{h-a}$ leading to

$$-Kz = (1 - 2\frac{H_0(z)}{z})^{2\alpha-1}(\frac{z}{H_0(z)})^{2\alpha} \tag{C.8}$$

where K is an integration constant. For $\alpha = 1$ one recovers $H_0(z) = \frac{1}{K}(1 - \sqrt{1 - Kz})$ with $K = \pi^2$ for the KPZ equation.

At large negative z, from (C.8) one has that $H_0(z) \simeq -K^{-\frac{1}{2\alpha}}(-z)^{1-\frac{1}{2\alpha}}$. If we set $\alpha = \frac{1}{3-2\gamma}$ as suggested by the above cumulant analysis, we obtain $H_0(z) \simeq -K^{\gamma-\frac{3}{2}}(-z)^{\gamma-\frac{1}{2}}$. This is indeed the behavior predicted in (C.5) from the leading term $n = 1$ (for $\gamma < 3/2$ which we assume here), that is for $z \to -\infty$

$$\Phi_-(z) \simeq \frac{\Gamma(1+\gamma)}{\sqrt{4\pi}\Gamma(\frac{5}{2}+\gamma)}(-z)^{\gamma+\frac{3}{2}} \quad , \quad H_0(z) = -\Phi''_-(z) \simeq -\frac{\Omega}{2}(-z)^{\gamma-\frac{1}{2}} \tag{C.9}$$

hence we identify $K = (\frac{\Omega}{2})^{\frac{-2}{3-2\gamma}}$ which reproduces $K = \pi^2$ for the KPZ case $\gamma = 1$ (with $\Omega = 2/\pi$). From (C.5) the (large $|z|$) series expansion predicted by the cumulants reads is

$$\frac{H_0(z)}{z} = -\frac{\Phi''_-(z)}{z} = -\frac{1}{2}\sum_{n\geqslant 1}\frac{(-1)^n}{\Gamma(n+1)}\frac{\Gamma(n(\gamma-\frac{1}{2})+1)}{\Gamma(2-n(\frac{3}{2}-\gamma))}(\Omega(-z)^{-(\frac{3}{2}-\gamma)})^n \tag{C.10}$$

which we can compare with the small $y, h = \frac{H_0}{z}$ expansion of the equation $y = h(1-2h)^{\frac{1}{2\alpha}-1}$ with $y = 1/(-Kz)^{1/(2\alpha)}$. Setting $\alpha = \frac{1}{3-2\gamma}$ one has $y = \frac{\Omega}{2}(-z)^{-(\frac{3}{2}-\gamma)}$ and the equation becomes $y = h(1-2h)^{\frac{1}{2}-\gamma}$. It is then easy to check with Mathematica that the two series coincide.

Hence the cumulant method and the KP equation once again agree, now for a larger class of functions g, i.e. a larger class of linear statistics of the Airy$_2$ point process. One recovers then from the KP equation the results for the large deviation function $\Phi_-(z)$ of [34] and [36] for monomials x_+^γ, although in a slightly different setting.

Let us close by showing how the above results can be equivalently be derived using the semi-classical expansion discussed in section 2.5, which, to leading order, maps to the Burgers equation. We can use the solution of the Burgers equation (66), in the form $\psi_0(t,r) = \frac{1}{\sqrt{t}}p_0'(r - 2t\psi_0(t,r))$. In the large time limit we insert the scaling form $\psi_0(t,r) = t^{\alpha-1}H_0(r/t^\alpha)$ and use the asymptotics (C.3) of $p_0'(r)$ at large negative $r = zt^\alpha$. One sees that the powers of t cancel out only if $\alpha = 1/(3-2\gamma)$, in agreement with the above results, and we obtain

$$H_0 = -\frac{\Omega}{2}(-z + 2H_0)^{\gamma-\frac{1}{2}}. \tag{C.11}$$

It is easy to see that this equation is equivalent to the equation (C.8) with $K = (\frac{\Omega}{2})^{\frac{-2}{3-2\gamma}}$ in full agreement with the above results. This already indicates that the Burgers

J. Stat. Mech. (2020) 043201

equation solution is equivalent to the self-consistent equation (15) found in [36], but we leave the full analysis of these connections to a future work.

Appendix D. Cumulants for the half-Brownian IC in the small time large deviation regime

We give explicit expressions for the cumulants at short time for the half-Brownian IC. One can solve directly the linear equations for the cumulants in the small time large deviation regime, from the main text

$$[\partial_t - \frac{n^3-n}{12} - \frac{1}{n}\partial_x^2] Z_n(x,t) = 0. \tag{D.1}$$

One can look for solutions of the form

$$Z_n(x,t) \simeq \frac{1}{\sqrt{\pi t}} t^{n/2} F_n(y = x/\sqrt{t}). \tag{D.2}$$

Inserting we find that it is solved by hypergeometric functions. One finds, for $w = 0$

$$F_n(y) = 2^{n-2} n^{\frac{n-3}{2}} \left(\Gamma\left(\frac{n}{2}\right) {}_1F_1\left(\frac{1-n}{2}; \frac{1}{2}; -\frac{ny^2}{4}\right) + \sqrt{n} y \Gamma\left(\frac{n+1}{2}\right) {}_1F_1\left(1 - \frac{n}{2}; \frac{3}{2}; -\frac{ny^2}{4}\right)\right) \tag{D.3}$$

leading to the explicit forms for e.g. the lowest cumulants

$$Z_1(x,t) \simeq \frac{1}{2}\left(\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) + 1\right) \tag{D.4}$$

$$Z_2(x,t) \simeq \frac{1}{2} x \left(\operatorname{erf}\left(\frac{x}{\sqrt{2}\sqrt{t}}\right) + 1\right) + \frac{\sqrt{t} e^{-\frac{x^2}{2t}}}{\sqrt{2\pi}} \tag{D.5}$$

$$Z_3(x,t) \simeq \frac{1}{2}\left(2t + 3x^2\right)\left(\operatorname{erf}\left(\frac{\sqrt{3}x}{2\sqrt{t}}\right) + 1\right) + \sqrt{\frac{3}{\pi}} \sqrt{t} x e^{-\frac{3x^2}{4t}} \tag{D.6}$$

and one can check that they have the correct $t = 0$ limits (143). For $w > 0$ one finds

$$\begin{aligned} Z_n(x,t) &\simeq \frac{1}{\sqrt{\pi t}} t^{n/2} F_n(y = x/\sqrt{t}) \\ F_n(y) &= 2^{n-2} n^{\frac{n-3}{2}} e^{-\frac{ny^2}{4}} \left(\Gamma\left(\frac{n}{2}\right) {}_1F_1\left(\frac{n}{2}; \frac{1}{2}; \frac{1}{4} n (y - 2\tilde{w})^2\right)\right. \\ &\quad \left. + \sqrt{n} \Gamma\left(\frac{n+1}{2}\right) (y - 2\tilde{w})\, {}_1F_1\left(\frac{n+1}{2}; \frac{3}{2}; \frac{1}{4} n (y - 2\tilde{w})^2\right)\right). \end{aligned} \tag{D.7}$$

Appendix E. Fredholm determinant and KP equation

In a seminal paper [67, 68] Pöppe and Sattinger found a family of Fredholm determinants (FD) which satisfy the KP hierarchy. Following their (redundant) notation, consider $X=(x_1,x_2,x_3,..)$, $Z=(z_1,x_2,x_3,..)$ and a kernel $F(X,Z)$ which satisfies the linear equation (equation (2.4) in [67])

$$\partial_{x_n}F-\partial_{x_1}^n F+(-)^n\partial_{z_1}^n F=0 \quad , \quad n=2,3... \tag{E.1}$$

Note that x_n for $n>1$ occurs in both X and Z, hence ∂_{x_n} acts on both arguments of the kernel. One then defines the FD noted $D(X)$ on $\mathbb{L}^2[x_1,+\infty[$, with $x_2,x_3,\ldots$ being parameters

$$D(X)=\mathrm{Det}(I+P_{[x_1,+\infty[}F)=\sum_{n=0}^{+\infty}\frac{1}{n!}\prod_{a=1}^{n}\int_{x_1}^{+\infty}\mathrm{d}y_1^a \det_{1\leqslant b,c\leqslant n}F(Y_b,Y_c) \quad , \quad Y_a=(y_1^a,x_2,x_3,\ldots) \tag{E.2}$$

and $P_{[x_1,+\infty[}$ the projector on the interval $[x_1,+\infty[$. In other words $D(X)$ is the standard FD on $\mathbb{L}^2[x_1,+\infty[$ for the operator $\tilde{F}$ with kernel $\tilde{F}(y,y')=F((y,x_2,\ldots),(y',x_2,\ldots))$. Then (Theorem 3.1 in [67]) the function

$$u(X)=2\partial_{x_1}^2\log D(X) \tag{E.3}$$

satisfies the KP hierarchy, $D(X)$ being the tau function. The lowest member is the KP equation, obtained for $X=(x_1,x_2,x_3)$, which reads (in the conventions of [67])

$$\partial_{x_3}u-\frac{1}{4}(\partial_{x_1}^3 u+6u\partial_{x_1}u)=\frac{3}{4}\partial_{x_1}^{-1}\partial_{x_2}^2 u. \tag{E.4}$$

To connect with the present paper we set

$$x_1=r \quad , \quad x_2=\frac{x}{2} \quad , \quad x_3=-\frac{t}{3} \quad , \quad u(x_1,x_2,x_3)=2\phi(x,t,r) \quad , \quad D(x_1,x_2,x_3)=G(x,t,r). \tag{E.5}$$

Let us now define the following kernel on $\mathbb{L}^2[0,+\infty[$ $(v,v'>0)$

$$K_{xtr}(v,v')=K_{xt}(r+v,r+v')=F((v+r,\frac{x}{2},-\frac{t}{3}),(v'+r,\frac{x}{2},-\frac{t}{3})) \tag{E.6}$$

where x,t are parameters. By construction it satisfies

$$\partial_r K_{xtr}(v,v')=(\partial_v+\partial_{v'})K_{xtr}(v,v') \tag{E.7}$$

and the conditions obtained from (E.1) read

$$\partial_t K=-\frac{1}{3}(\partial_v^3+\partial_{v'}^3)K \quad , \quad \partial_x K=\frac{1}{2}(\partial_v^2-\partial_{v'}^2)K. \tag{E.8}$$

Hence, if these conditions are satisfied, one has that

$$\phi(x,t,r)=\partial_r^2\mathrm{Det}(1+\alpha P_{[0,+\infty[}K_{xtr}) \tag{E.9}$$

satisfies the KP equation (5) for any α such that the FD is well-defined. Equations (E.7) and (E.8) are the conditions given in [1].

The generating function for the droplet IC can be written as

$$G(x,t,r)=\mathrm{Det}(I-M_{xtr})|_{\mathbb{L}^2(\mathbb{R}^+)} \tag{E.10}$$

$$M_{xtr}(v,v') = \int \mathrm{d}u\, \Sigma(t^{1/3}u - r)\, \mathrm{Ai}(u + v + \frac{x^2}{4t^{4/3}})\mathrm{Ai}(u + v' + \frac{x^2}{4t^{4/3}}) \quad , \quad \Sigma(z) = \frac{1}{1+\mathrm{e}^{-z}} \tag{E.11}$$

Performing the shift $u \to u + t^{-1/3}r$, using the integral representation of both Airy functions, rescaling $z, w \to t^{1/3}z, t^{1/3}w$, $u \to t^{-1/3}u$, and, in a second stage, using the translation $z \to z + \frac{x}{2t}$, $w \to w - \frac{x}{2t}$ we see that M is equivalent under a similarity transformation (which does not change the FD) to the kernel

$$K_{xtr}(v,v') = \int_{-\infty}^{+\infty} \mathrm{d}u\Sigma(u) \int_{C^2} \frac{\mathrm{d}t\mathrm{d}w}{(2\mathrm{i}\pi)^2} \mathrm{e}^{t(\frac{z^3}{3}+\frac{w^3}{3})-z(v+r+u)-w(v'+r+u)+x(\frac{z^2}{2}-\frac{w^2}{2})} \tag{E.12}$$

with $M_{xtr}(v,v') = t^{1/3}K_{xtr}(vt^{1/3}, v't^{1/3})\mathrm{e}^{\frac{x}{2t}(v'-v)}$. The kernel K manifestly satisfies the above conditions equations (E.7) and (E.8). This establishes that for the droplet IC, $\phi(x,t,r) = \partial_r^2 G(x,t,r)$ satisfies the KP equation.

Furthermore, the same conditions equations (E.7) and (E.8) are also satisfied for any choice of $\Sigma(u)$ in (E.12) (for which we assume the FD to be well defined). For the choice $\Sigma(u) = 1 - \mathrm{e}^{\beta g(-e^u)}$ one recovers exactly the FD considered in the remark 2 (and in [34]). Indeed one then has $\mathrm{Det}(I - M_{xtr})|_{\mathbb{L}^2(\mathbb{R}^+)} = \mathrm{Det}[I - (1 - \mathrm{e}^{\beta \hat{g}_{t,\sigma}})K_{\mathrm{Ai}}]$ where $\hat{g}_{t,\sigma}(u) = g(\sigma \mathrm{e}^{t^{1/3}u})$ and $\sigma = -\mathrm{e}^{-r}$, as in (32) (this is easily seen e.g. expanding in traces and using the cyclic property). Hence the whole class of FD (32), useful to evaluate linear statistics of the Airy point process, satisfy the KP property, as claimed in the text.

Finally note that the N soliton solution of the KP hierarchy is obtained from a linear superposition of a particular solution of (E.1)

$$F = \sum_{j=1}^{N} a_j \mathrm{e}^{x_1 p_j - z_1 q_j + \sum_{n\geqslant 2} x_n (p_j^n - q_j^n)}. \tag{E.13}$$

One finds that

$$D(x) = \det_{N\times N}\left(\delta_{ij} - \frac{a_i}{p_i - q_j}\mathrm{e}^{x_1(p_i - q_j) + \sum_{n\geqslant 2} x_n (p_i^n - q_j^n)}\right). \tag{E.14}$$

More general solutions are obtained from the continuous superposition

$$F = \int \mathrm{d}\mu(p,q)\mathrm{e}^{x_1 p - z_1 q + \sum_{n\geqslant 2} x_n (p^n - q^n)} \tag{E.15}$$

for some weight measure $\mathrm{d}\mu(p,q)$, leading to kernels generalizing (E.12) and which obey the KP hierarchy property.

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