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Reduction and diversity of exact solutions for a class of generalized KP equations

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Abstract

We aim to investigate a class of generalized KP and BKP equations for which exist abundant exact solutions. The resulting two-wave solutions involve particular phase shifts and wave frequencies generating complexitons. Special reductions of the parameters present concrete examples of phase shifts and wave frequencies. It is also pointed out that the presented equations can be transformed into many well-known equations associated with bilinear forms. Moreover, a class of lump-type solutions is generated through symbolic computations with Maple. Three illustrative examples are given to show the diversity of exact solutions to the introduced equations.

Keywords: generalized KP and BKP equations, reduction, complexiton solutions, lump-type solutions, Hirota's bilinear form

(Some figures may appear in colour only in the online journal)

1. Introduction

It is of significant importance to investigate exact solutions of nonlinear evolution equations (NLEEs) in mathematical physics. Exact solutions might help us to better understand qualitative features of many phenomena and processes in various areas of natural science. NLEEs possess diverse solutions, such as solitary wave solutions, lump and complexiton solutions. Solitary wave solutions called solitons are analytic solutions exponentially decaying in all directions, while lump solutions are a kind of rational function solutions localized in all directions in space. Adding to the diversity of solitons, complexiton solutions, i.e. solutions involving two kinds of transcendental functions—exponential functions and trigonometric functions, have been presented in references [1–6].

The Kadomtsev–Petviashvilli (KP) hierarchy contains many different systems of equations [7, 8]. The basice equation at the bottom of the KP hierarchy is the KP equation, which can be written in the bilinear form as

$$(D_1^4 - 4D_1D_3 + 3D_2^2)f \cdot f = 0, \tag{1.1}$$

where the Hirota bilinear differential operators [9] are defined by

$$D_{x_1}^{n_1} D_{x_2}^{n_2} f \cdot g = (\partial_{x_1} - \partial_{x_1'})^{n_1} (\partial_{x_2} - \partial_{x_2'})^{n_2} \times f(x_1, x_2) g(x_1', x_2')|_{x_1'=x_1, x_2'=x_2}, \tag{1.2}$$

with n_1 and n_2 are arbitrary nonnegative integers. In nonlinear sciences, the KP equation is a completely integrable system that describes the motion of two-dimensional solitary waves.

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In addition to the KP hierarchy, there exists B-type Kadomtsev–Petviashvili (BKP) hierarchy. The BKP hierarchy, that is KP hierarchy of the B-type, was introduced by Date, Jimbo, Kashiwara, and Miwa [8]. The first equation of this hierarchy is expressed in the bilinear form as

$$[(D_3 - D_1^3)D_{-1} + 3D_1^2]f \cdot f = 0. \tag{1.3}$$

Various soliton equations were reproduced as reductions and generalizations of equations in these hierarchies, such as the the Hirota–Satsuma shallow water wave equation [9], the generalized Calogero–Bogoyavlenskii–Schiff equation [10], and the $(3+1)$-dimensional generalized KP and BKP equations [11–13]. For these nonlinear equations, many approaches have been conducted to seek exact solutions, which contain the Hirota's bilinear method [9, 14, 15], the Bäcklund transformation method [16, 17], the exp-function method [18, 19], the auxiliary equation method [20, 21], the Riemann–Hilbert method [22–25] and the Wronskian technique [26–30]. Among the existing methods, the multiple exp-function method [31, 32], as a generalization of Hirota's perturbation scheme, provides a direct algebraic approach for constructing multi-exponential wave solutions to nonlinear equations. The abundant multiple wave solutions to the $(3+1)$-dimensional generalized KP and BKP equations have been presented by Ma and Zhu with the help of Maple [32], applying the multiple exp-function algorithm.

In this paper, we will consider a class of NLEEs:

$$\begin{aligned} &u_{xxxy} + \chi(u_x u_y)_x + c_1 u_{xt} + c_2 u_{yt} + c_3 u_{xx} \\ &\quad + c_4 u_{yy} + c_5 u_{zz} = 0, \end{aligned} \tag{1.4}$$

where χ is a non-zero parameter, c_i, $1 \leqslant i \leqslant 5$, are all arbitrary real constants and $c_1^2 + c_2^2 \neq 0$. Under the transformation

$$u = \frac{6}{\chi}(\ln f)_x, \tag{1.5}$$

a direct computation tells that (1.4) can be expressed as a Hirota bilinear form

$$\begin{aligned} &(D_x^3 D_y + c_1 D_x D_t + c_2 D_y D_t + c_3 D_x^2 \\ &\quad + c_4 D_y^2 + c_5 D_z^2) f \cdot f = 0. \end{aligned} \tag{1.6}$$

If $\chi = 3$, $c_1 = c_4 = c_5 = 0$, $c_2 = -1$ and $c_3 = -3$, then (1.4) can be reduced to the $(2+1)$-dimensional BKP equation

$$u_{yt} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xx} = 0. \tag{1.7}$$

By the typical transformation $u = 2(\ln f)_x$, this can be written in Hirota form as (1.3) if we set $D_x = D_1$, $D_y = D_{-1}$ and $D_t = D_3$. The class of nonlinear equations defined by (1.4) is also a general generalization of the following $(3+1)$-dimensional generalized KP and BKP equations [12, 33–37]:

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0, \tag{1.8}$$

$$u_{yt} - u_{xxxy} - 3(u_x u_y)_x - 3\sigma u_{xx} + 3u_{zz} = 0,\ \sigma = \pm 1. \tag{1.9}$$

The equations (1.8) and (1.9) have been proven to possess Wronskian and Grammian determinant solutions [12, 37, 38]. More recent studies on exact solutions and integrable properties of (1.4) may be found in the literature [13, 39, 40].

The framework of this paper is as follows. In section 2, we will construct one-wave and two-wave solutions to (1.4) via the multiple exp-function method. Based on two-wave solutions with particular phase shifts, a kind of complexiton solutions will be derived. In section 3, we will present three-wave solutions and discuss their dimensional reductions. Moreover, it will be pointed out that any equation defined by (1.4) can be transformed into many well-known equations associated with bilinear forms, such as the Korteweg–de Vries (KdV) equation, the $(2+1)$-dimensional KP equation and the Boussinesq equation. In section 4, we would like to search for lump-type solutions to the associated bilinear equation, under the help of Maple. Our conclusion and remarks will be given at the end of the paper.

2. Complexiton solutions

The multiple exp-function method is well known in the literature [31, 32]. Let us introduce one-wave solutions to (1.4)

$$u = \frac{6(a_1 + a_2\varepsilon_1 k_1 e^{\theta_1})}{\chi(1 + \varepsilon_1 e^{\theta_1})}, \tag{2.1}$$

where a_1, a_2 and ε_1 are constants and $\theta_1 = k_1 x + l_1 y + m_1 z - w_1 t$. Then, direct symbolic computations with Maple show

$$a_1 = (a_2 - 1)k_1,\ w_1 = \frac{k_1^3 l_1 + c_3 k_1^2 + c_4 l_1^2 + c_5 m_1^2}{c_1 k_1 + c_2 l_1}, \tag{2.2}$$

and so, the resulting one-wave solutions are given by

$$u = \frac{6(a_2 - 1)k_1 + 6a_2\varepsilon_1 k_1 e^{k_1 x + l_1 y + m_1 z - w_1 t}}{\chi(1 + \varepsilon_1 e^{k_1 x + l_1 y + m_1 z - w_1 t})}, \tag{2.3}$$

where a_2, ε_1, k_1, l_1 and m_1 are arbitrary and w_1 is defined by (2.2).

We next consider two-wave solutions of (1.4):

$$u = \frac{6}{\chi}(\ln f)_x, f = 1 + \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2} + a_{12}\varepsilon_1\varepsilon_2 e^{\theta_1 + \theta_2}, \tag{2.4}$$

where ε_i, $i = 1, 2$, are arbitrary and

$$\theta_i = k_i x + l_i y + m_i z - w_i t,\ i = 1, 2.$$

Applying the multiple exp-function algorithm by Maple, a direct computation tells us that

$$\begin{aligned} w_i &= \frac{k_i^3 l_i + c_3 k_i^2 + c_4 l_i^2 + c_5 m_i^2}{c_1 k_i + c_2 l_i},\ i = 1, 2 \text{ and} \\ a_{12} &= \frac{b_{12}}{c_{12}}, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} b_{12} = & 3c_1k_1k_2(k_1l_2 + k_2l_1)(c_1k_1k_2 + c_2k_1l_2 + c_2k_2l_1) \\ & - 3c_2^2k_1k_2l_1l_2(k_1 - k_2)(l_1 - l_2) \\ & - c_1^2k_1k_2(2k_1k_2^2l_2 + 2k_1^2k_2l_1 + k_1^3l_2 + k_2^3l_1) \\ & + (c_1^2c_4 + c_2^2c_3)(k_1l_2 - k_2l_1)^2 \\ & - c_1c_2[2k_1k_2l_1l_2(k_1^2 + k_2^2) + 3k_1^2k_2^2(l_1^2 + l_2^2) \\ & + k_1^4l_2^2 + k_2^4l_1^2] + c_5[c_1(k_1m_2 - k_2m_1) \\ & + c_2(l_1m_2 - l_2m_1)]^2, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} c_{12} = & -3c_1k_1k_2(k_1l_2 + k_2l_1)(c_1k_1k_2 + c_2k_1l_2 + c_2k_2l_1) \\ & - 3c_2^2k_1k_2l_1l_2(k_1 + k_2)(l_1 + l_2) \\ & - c_1^2k_1k_2(2k_1k_2^2l_2 + 2k_1^2k_2l_1 + k_1^3l_2 + k_2^3l_1) \\ & + (c_1^2c_4 + c_2^2c_3)(k_1l_2 - k_2l_1)^2 \\ & - c_1c_2[2k_1k_2l_1l_2(k_1^2 + k_2^2) + 3k_1^2k_2^2(l_1^2 + l_2^2) \\ & + k_1^4l_2^2 + k_2^4l_1^2] + c_5[c_1(k_1m_2 - k_2m_1) \\ & + c_2(l_1m_2 - l_2m_1)]^2. \end{aligned} \tag{2.7}$$

It is clear that the phase shift a_{12} depends on all coefficients k_i, l_i and m_i, $i = 1, 2$, of the spatial variables x, y and z, respectively. This shows that two-wave solutions are determined for k_i, l_i and m_i being free parameters. In the following, we would like to give a class of complexiton solutions of (1.4) by extending the parameters to the complex field.

Assuming

$$\begin{aligned} & k_1 = \bar{k}_2 = a + \mathrm{i}b,\ l_1 = \mathrm{i}\alpha k_1,\ l_2 = -\mathrm{i}\alpha k_2, \\ & m_1 = \bar{m}_2 = c + \mathrm{i}d, \\ & i = \sqrt{-1},\ \ \varepsilon_1 = \varepsilon_2,\ \ a,\ b,\ c,\ d,\ \alpha,\ \varepsilon_1,\ \varepsilon_2 \in \mathbb{R}, \end{aligned} \tag{2.8}$$

and noting expressions (2.4)–(2.7), we have

$$a_{12} = 1,\ \theta_1 = \bar{\theta}_2 = \Omega + \mathrm{i}\psi, \tag{2.9}$$

where $^-$ denotes the complex conjugate and

$$\begin{aligned} \Omega &= ax - \alpha by + cz - pt, \\ \psi &= bx + \alpha ay + dz - qt, \end{aligned} \tag{2.10}$$

with

$$\begin{aligned} p = & \frac{1}{(a^2 + b^2)(c_1^2 + \alpha^2c_2^2)}[\alpha c_1b(b^4 - 3a^4 - 2a^2b^2) \\ & + \alpha^2c_2a(a^4 - 3b^4 - 2a^2b^2) \\ & + \alpha c_2c_5(bd^2 + 2acd - bc^2) + c_1c_5(-ad^2 + 2bcd + ac^2) \\ & + (c_1c_3a + \alpha c_2c_3b - \alpha^3c_2c_4b - \alpha^2c_1c_4a)(a^2 + b^2)], \\ q = & \frac{1}{(a^2 + b^2)(c_1^2 + \alpha^2c_2^2)}[\alpha c_1a(a^4 - 3b^4 - 2a^2b^2) \\ & + \alpha^2c_2b(3a^4 - b^4 + 2a^2b^2) \\ & + \alpha c_2c_5(ad^2 - 2bcd - ac^2) + c_1c_5(bd^2 + 2acd - bc^2) \\ & + (c_1c_3b - \alpha c_2c_3a + \alpha^3c_2c_4a - \alpha^2c_1c_4b)(a^2 + b^2)]. \end{aligned}$$

Substituting (2.9) and (2.10) into (2.4), we can now compute

$$f = 1 + 2\varepsilon_1\mathrm{e}^{\Omega}\cos\psi + \varepsilon_1^2\mathrm{e}^{2\Omega}. \tag{2.11}$$

Thus, a class of explicit solutions of (1.4) is given as follows:

$$u = \frac{12(a\varepsilon_1\mathrm{e}^{\Omega}\cos\psi - b\varepsilon_1\mathrm{e}^{\Omega}\sin\psi + a\varepsilon_1^2\mathrm{e}^{2\Omega})}{\chi(1 + 2\varepsilon_1\mathrm{e}^{\Omega}\cos\psi + \varepsilon_1^2\mathrm{e}^{2\Omega})}, \tag{2.12}$$

where the real parameters a and b need to satisfy $a^2 + b^2 \neq 0$, and the others are suitably chosen. Based on the above results, we will present an illustrative example.

Example 2.1. Let us set

$$\chi = 3,\ c_1 = c_3 = c_4 = 0,\ c_2 = 2,\ c_5 = -3,$$

and then we arrive at a (3 + 1)-dimensional Jimbo–Miwa type equation

$$u_{xxxy} + 3(u_xu_y)_x + 2u_{yt} - 3u_{zz} = 0, \tag{2.13}$$

which has been introduced by Ma [41, 42]. Through the dependent variable transformation $u = 2(\ln f)_x$, the Hirota bilinear form of (2.13) is written as

$$(D_x^3D_y + 2D_yD_t - 3D_z^2)f\cdot f = 0. \tag{2.14}$$

The corresponding two-wave solutions to (2.13) read

$$u = \frac{2[k_1\varepsilon_1\mathrm{e}^{\theta_1} + k_2\varepsilon_2\mathrm{e}^{\theta_2} + a_{12}(k_1 + k_2)\varepsilon_1\varepsilon_2\mathrm{e}^{\theta_1+\theta_2}]}{1 + \varepsilon_1\mathrm{e}^{\theta_1} + \varepsilon_2\mathrm{e}^{\theta_2} + a_{12}\varepsilon_1\varepsilon_2\mathrm{e}^{\theta_1+\theta_2}}, \tag{2.15}$$

where ε_1 and ε_2 are arbitrary and

$$\begin{aligned} \theta_i &= k_ix + l_iy + m_iz - \frac{k_i^3l_i - 3m_i^2}{2l_i}t,\ i = 1, 2, \\ a_{12} &= \frac{k_1k_2l_1l_2(k_1 - k_2)(l_1 - l_2) + (l_1m_2 - l_2m_1)^2}{k_1k_2l_1l_2(k_1 + k_2)(l_1 + l_2) + (l_1m_2 - l_2m_1)^2}. \end{aligned} \tag{2.16}$$

Moreover, we can also obtain a class of complexiton solutions to (2.13) as follows:

$$u = \frac{4(a\varepsilon_1\mathrm{e}^{\Omega}\cos\psi - b\varepsilon_1\mathrm{e}^{\Omega}\sin\psi + a\varepsilon_1^2\mathrm{e}^{2\Omega})}{1 + 2\varepsilon_1\mathrm{e}^{\Omega}\cos\psi + \varepsilon_1^2\mathrm{e}^{2\Omega}}, \tag{2.17}$$

where

$$\begin{aligned} \Omega = & ax - \alpha by + cz \\ & - \frac{\alpha^2a(a^4 - 3b^4 - 2a^2b^2) - 3\alpha(bd^2 + 2acd - bc^2)}{2\alpha^2(a^2 + b^2)}t, \\ \psi = & bx + \alpha ay + dz \\ & - \frac{\alpha^2b(3a^4 - b^4 + 2a^2b^2) - 3\alpha(ad^2 - 2bcd - ac^2)}{2\alpha^2(a^2 + b^2)}t, \end{aligned}$$

with $\alpha(a^2 + b^2) \neq 0$. Figure 1 shows the propagation of the complexiton solution by the expression (2.17) with special parameters in the xz-plane. It is generated by the interaction between the soliton and the periodic wave.

3. Three-wave solutions and their dimensional reductions

We now consider three-wave functions:

$$u = \frac{6}{\chi}(\ln f)_x, \tag{3.1}$$

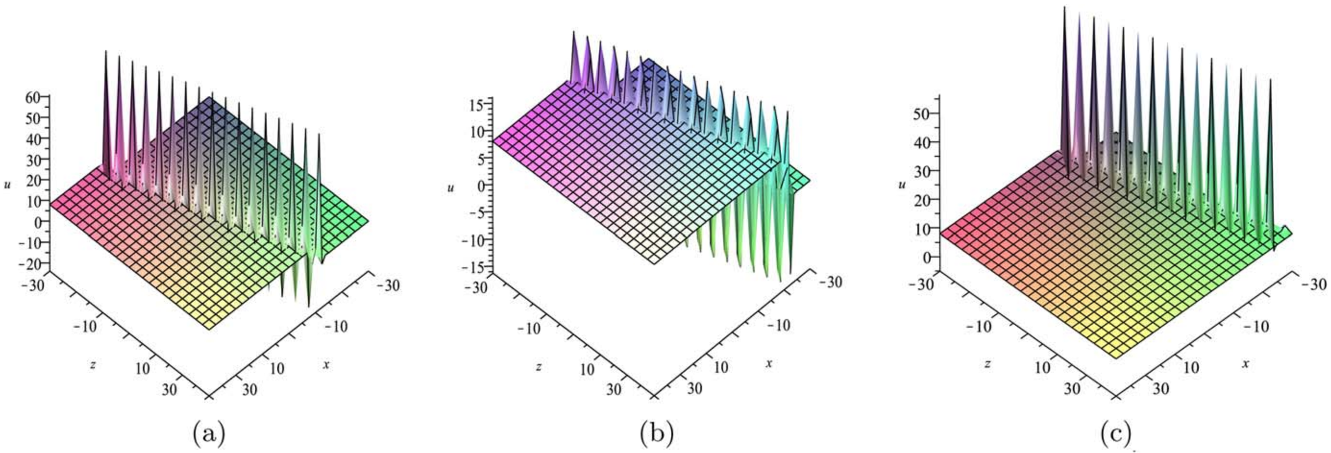

Figure 1. The complexiton solution (2.17) with $\alpha = 1$, $a = 2$, $b = 12$, $c = \frac{2}{3}$, $d = 4$, $\varepsilon_1 = 10$, $y = 0$ and (a) $t = 0$, (b) $t = 0.05$, (c) $t = 0.1$.

with f being defined by

$$\begin{aligned} f = 1 &+ \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2} + \varepsilon_3 e^{\theta_3} + \varepsilon_1\varepsilon_2 a_{12} e^{\theta_1+\theta_2} \\ &+ \varepsilon_1\varepsilon_3 a_{13} e^{\theta_1+\theta_3} + \varepsilon_2\varepsilon_3 a_{23} e^{\theta_2+\theta_3} \\ &+ \varepsilon_1\varepsilon_2\varepsilon_3 a_{123} e^{\theta_1+\theta_2+\theta_3}, \quad a_{123} = a_{12}a_{13}a_{23}, \end{aligned} \tag{3.2}$$

where ε_i, $1 \leqslant i \leqslant 3$, are arbitrary and

$$\theta_i = k_i x + l_i y + m_i z - w_i t, \ 1 \leqslant i \leqslant 3. \tag{3.3}$$

We would like to search for three-wave soliton type solutions with the selection of

$$w_i = \frac{k_i^3 l_i + c_3 k_i^2 + c_4 l_i^2 + c_5 m_i^2}{c_1 k_i + c_2 l_i}, \ 1 \leqslant i \leqslant 3 \tag{3.4}$$

and

$$a_{ij} = \frac{b_{ij}}{c_{ij}}, \ 1 \leqslant i < j \leqslant 3, \tag{3.5}$$

where

$$\begin{aligned} b_{ij} = &\, 3c_1 k_i k_j (k_i l_j + k_j l_i)(c_1 k_i k_j + c_2 k_i l_j + c_2 k_j l_i) \\ &- 3c_2^2 k_i k_j l_i l_j (k_i - k_j)(l_i - l_j) \\ &- c_1^2 k_i k_j (2k_i k_j^2 l_j + 2k_i^2 k_j l_i + k_i^3 l_j + k_j^3 l_i) \\ &+ (c_1^2 c_4 + c_2^2 c_3)(k_i l_j - k_j l_i)^2 \\ &- c_1 c_2 [2k_i k_j l_i l_j (k_i^2 + k_j^2) + 3k_i^2 k_j^2 (l_i^2 + l_j^2) \\ &+ k_i^4 l_j^2 + k_j^4 l_i^2] + c_5 [c_1 (k_i m_j - k_j m_i) \\ &+ c_2 (l_i m_j - l_j m_i)]^2, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} c_{ij} = &-3c_1 k_i k_j (k_i l_j + k_j l_i)(c_1 k_i k_j + c_2 k_i l_j + c_2 k_j l_i) \\ &- 3c_2^2 k_i k_j l_i l_j (k_i + k_j)(l_i + l_j) \\ &- c_1^2 k_i k_j (2k_i k_j^2 l_j + 2k_i^2 k_j l_i + k_i^3 l_j + k_j^3 l_i) \\ &+ (c_1^2 c_4 + c_2^2 c_3)(k_i l_j - k_j l_i)^2 \\ &- c_1 c_2 [2k_i k_j l_i l_j (k_i^2 + k_j^2) + 3k_i^2 k_j^2 (l_i^2 + l_j^2) \\ &+ k_i^4 l_j^2 + k_j^4 l_i^2] + c_5 [c_1 (k_i m_j - k_j m_i) \\ &+ c_2 (l_i m_j - l_j m_i)]^2. \end{aligned} \tag{3.7}$$

Due to the class of nonlinear equations defined by (1.4) is not completely integrable [34], we need to determine conditions to guarantee the existence of the three-wave soliton solutions.

We begin with the choice

$$l_1 = \alpha k_1, \ l_2 = \alpha k_2, \ l_3 = -\alpha k_3, \tag{3.8}$$

where α is a constant, which leads to the wave frequencies

$$\begin{aligned} w_i &= \frac{k_i^4 \alpha + c_3 k_i^2 + c_4 \alpha^2 k_i^2 + c_5 m_i^2}{c_1 k_i + c_2 \alpha k_i}, \ i = 1, 2, \\ w_3 &= \frac{-k_3^4 \alpha + c_3 k_3^2 + c_4 \alpha^2 k_3^2 + c_5 m_3^2}{c_1 k_3 - c_2 \alpha k_3}, \end{aligned} \tag{3.9}$$

and the phase shifts

$$\begin{aligned} a_{12} &= \frac{-3\alpha k_1^2 k_2^2 (k_1 - k_2)^2 + c_5 (k_1 m_2 - k_2 m_1)^2}{-3\alpha k_1^2 k_2^2 (k_1 + k_2)^2 + c_5 (k_1 m_2 - k_2 m_1)^2}, \\ a_{13} &= 1, \ a_{23} = 1. \end{aligned} \tag{3.10}$$

Therefore we have a class of three-wave soliton solutions (3.1) associated with (3.9) and (3.10) to (1.4).

Secondly, we would here like to focus on some dimensional reductions of the three-wave solutions.

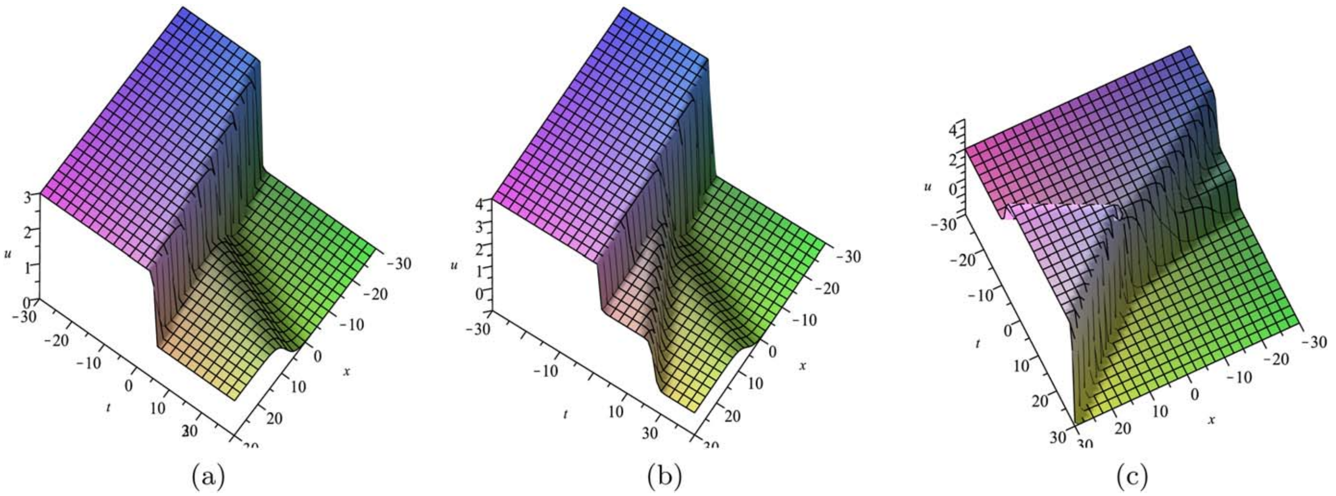

Figure 2. The plots of the resonant two-, three- and four-wave solutions (3.24). The involved parameters are, respectively, set as: $\beta = 2$, $y = -2$, $z = -1$ and (a) $N = 2$, $\varepsilon_1 = \varepsilon_2 = 1$, $k_1 = 0.5$, $k_2 = 1.5$; (b) $N = 3$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$, $k_1 = 1$, $k_2 = 2$, $k_3 = -0.5$; (c) $N = 4$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$, $k_1 = -0.5$, $k_2 = 1$, $k_3 = -1.2$, $k_4 = 2$.

Case 1 The choice

$$l_i = \alpha k_i,\ m_i = \beta k_i,\ 1 \leqslant i \leqslant 3, \tag{3.11}$$

where α and β are two constants, presents a class of three-wave soliton type solutions defined by (3.1) with

$$\theta_i = k_i\left(x + \alpha y + \beta z - \frac{c_3 + c_4\alpha^2 + c_5\beta^2}{c_1 + c_2\alpha}t\right) - \frac{\alpha k_i^3}{c_1 + c_2\alpha}t,\ 1 \leqslant i \leqslant 3 \tag{3.12}$$

and

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_j + k_j)^2},\ 1 \leqslant i < j \leqslant 3. \tag{3.13}$$

Let

$$x' = x + \alpha y + \beta z - \frac{c_3 + c_4\alpha^2 + c_5\beta^2}{c_1 + c_2\alpha}t,$$
$$t' = \frac{\alpha}{c_1 + c_2\alpha}t. \tag{3.14}$$

It is direct to see that this kind of three-wave solutions defined by (3.1) with (3.11)–(3.13) reduces to the one presented for the KdV equation in previous literature [3, 26]. In fact, the invertible linear transform (3.14) of x, y, z and t can transform (1.6) into

$$(D_{x'}^4 + D_{x'}D_{t'})f \cdot f = 0, \tag{3.15}$$

which presents the bilinear KdV equation.

Case 2 Let $c_5 \neq 0$. The choice

$$l_i = \alpha k_i,\ m_i = \beta k_i^2,\ 1 \leqslant i \leqslant 3, \tag{3.16}$$

where α and β are two real constants, generates a class of three-wave soliton type solutions defined by (3.1) with

$$\theta_i = k_i x + \alpha k_i y + \beta k_i^2 z - \frac{(c_3 + c_4\alpha^2)k_i + (\alpha + c_5\beta^2)k_i^3}{c_1 + c_2\alpha}t,\ 1 \leqslant i \leqslant 3 \tag{3.17}$$

and

$$a_{ij} = \frac{(-3\alpha + c_5\beta^2)(k_i - k_j)^2}{-3\alpha(k_i + k_j)^2 + c_5\beta^2(k_i - k_j)^2},\quad 1 \leqslant i < j \leqslant 3. \tag{3.18}$$

Further taking

$$\alpha = \frac{c_5\beta^2}{3}, \tag{3.19}$$

then we have

$$\theta_i = k_i\left(x + \frac{c_5\beta^2}{3}y - \frac{9c_3 + c_4c_5^2\beta^4}{9c_1 + 3c_2c_5\beta^2}t\right) + \beta k_i^2 z - \frac{4c_5\beta^2}{3c_1 + c_2c_5\beta^2}k_i^3 t,\ 1 \leqslant i \leqslant 3,$$
$$a_{ij} = 0,\ \ 1 \leqslant i < j \leqslant 3, \tag{3.20}$$

which gives a kind of resonant solutions to (1.4):

$$f = 1 + \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2} + \varepsilon_3 e^{\theta_3}. \tag{3.21}$$

We also find that the bilinear equation (1.6) reduces to the following bilinear KP equation:

$$(D_{x'}^4 + 4D_{x'}D_{t'} + 3D_{y'}^2)f \cdot f = 0, \tag{3.22}$$

under the invertible linear transform of independent variables

$$x' = x + \frac{c_5\beta^2}{3}y - \frac{9c_3 + c_4c_5^2\beta^4}{3(3c_1 + c_2c_5\beta^2)}t,$$
$$y' = \beta z,\ t' = \frac{4c_5\beta^2}{3c_1 + c_2c_5\beta^2}t. \tag{3.23}$$

As a special example, the $(3+1)$-dimensional bilinear equation (2.14) has the following resonant multiple wave solution:

$$f = 1 + \sum_{i=1}^{N} \varepsilon_i (k_i x - \beta^2 k_i y + \beta k_i^2 z - 2k_i^3 t), \tag{3.24}$$

where β and the ε_is and k_is are arbitrary constants. The resonant two-, three- and four-wave solutions (3.24) with special parameters are plotted in figure 2. In two resonant cases, the length of resonant soliton will enlarge and this will develop a triad as displayed in figure 2(a). It is easy to see the interaction will become even more complicated with the increase of the number of solitons. Additionally, the invertible linear transform of x, y, z and t

$$x' = x - \beta^2 y,\ y' = \beta z,\ t' = 2t, \tag{3.25}$$

can transform (2.14) into the bilinear equation (3.22).

Case 3 Let $c_5(c_3^2 + c_4^2) \neq 0$. We still consider the above three-wave soliton type solutions defined by (3.1) with (3.17) and the phase shifts (3.18). Under the selection of

$$\alpha = -c_5\beta^2, \tag{3.26}$$

where β is suitably chosen, expressions (3.17) and (3.18) become

$$\theta_i = k_i\left(x - c_5\beta^2 y - \frac{c_3 + c_4 c_5^2\beta^4}{c_1 - c_2 c_5\beta^2}t\right) + \beta k_i^2 z,\ 1 \leqslant i \leqslant 3 \tag{3.27}$$

and

$$a_{ij} = \frac{k_i^2 - 2k_i k_j + k_j^2}{k_i^2 + k_i k_j + k_j^2},\ 1 \leqslant i < j \leqslant 3, \tag{3.28}$$

respectively. Similar1y, if we suppose

$$x' = x - c_5\beta^2 y - \frac{c_3 + c_4 c_5^2\beta^4}{c_1 - c_2 c_5\beta^2}t,\ \ t' = \beta z, \tag{3.29}$$

then the invertible linear transform (3.29) can transform (1.6) into

$$(D_{x'}^4 - D_{t'}^2) f\cdot f = 0, \tag{3.30}$$

which yields the bilinear Boussinesq equation.

4. Lump-type solutions

In recent years, it has become an interesting research topic to investigate lump solutions or lump-type solutions, rationally localized solutions in almost all directions in space [43–52], to NLEEs, via the Hirota bilinear formulation. As we know, positive quadratic functions can generate lump or lump-type solutions of NLEEs under the dependent variable transformations $u = \gamma(\ln f)_x$ and $u = \gamma(\ln f)_{xx}$, where γ is a constant and x is one spatial variable. Thus lump solutions or lump-type solutions are a kind of special nonsingular rational solutions, which describe diverse nonlinear phenomena in nature. In this section, we will search for positive quadratic function solutions to the associated bilinear equation (1.6) through direct Maple computations.

To search for quadratic function solutions to the $(3+1)$-dimensional bilinear equation (1.6), we assume

$$f = g_1^2 + g_2^2 + a_{11}, \tag{4.1}$$

with

$$\begin{aligned} g_1 &= a_1 x + a_2 y + a_3 z + a_4 t + a_5,\\ g_2 &= a_6 x + a_7 y + a_8 z + a_9 t + a_{10}, \end{aligned}$$

where a_i, $1 \leqslant i \leqslant 11$, are real parameters to be determined. With the help of Maple symbolic computations, we can obtain a set of constraining equations for the parameters and the coefficients as follows:

$$\begin{aligned} \Big\{\{a_1 &= a_1,\ a_2 = a_2,\ a_3 = a_3,\ a_4 = \frac{-\lambda_1}{(c_1a_1 + c_2a_2)^2 + (c_1a_6 + c_2a_7)^2},\ a_5 = a_5,\\ a_6 &= a_6,\ a_7 = a_7,\ a_8 = a_8,\ a_9 = \frac{-\lambda_2}{(c_1a_1 + c_2a_2)^2 + (c_1a_6 + c_2a_7)^2},\ a_{10} = a_{10},\\ a_{11} &= \frac{-3(a_1a_2 + a_6a_7)(a_1^2 + a_6^2)[(c_1a_1 + c_2a_2)^2 + (c_1a_6 + c_2a_7)^2]}{\lambda_3}\Big\}, \end{aligned} \tag{4.2}$$

with

$$\begin{aligned} \lambda_1 = {} & c_1[(a_1^3 + a_1a_6^2)c_3 + (2a_2a_6a_7 - a_1a_7^2 + a_1a_2^2)c_4\\ & + (a_1a_3^2 + 2a_3a_6a_8 - a_1a_8^2)c_5]\\ & + c_2[(2a_1a_6a_7 - a_2a_6^2 + a_1^2a_2)c_3 + (a_2^3 + a_2a_7^2)c_4\\ & + (2a_3a_7a_8 + a_2a_3^2 - a_2a_8^2)c_5],\\ \lambda_2 = {} & c_1[(a_1^2a_6 + a_6^3)c_3 + (a_6a_7^2 + 2a_1a_2a_7 - a_2^2a_6)c_4\\ & + (a_6a_8^2 - a_3^2a_6 + 2a_1a_3a_8)c_5]\\ & + c_2[(a_6^2a_7 - a_1^2a_7 + 2a_1a_2a_6)c_3 + (a_7^3 + a_2^2a_7)c_4\\ & + (a_7a_8^2 - a_3^2a_7 + 2a_2a_3a_8)c_5],\\ \lambda_3 = {} & c_5[a_8(c_1a_1 + c_2a_2) - a_3(c_1a_6 + c_2a_7)]^2\\ & + (c_1^2c_4 + c_2^2c_3)(a_1a_7 - a_2a_6)^2, \end{aligned} \tag{4.3}$$

where all involved parameters and coefficients are arbitrary provided that the expressions make sense. Note that

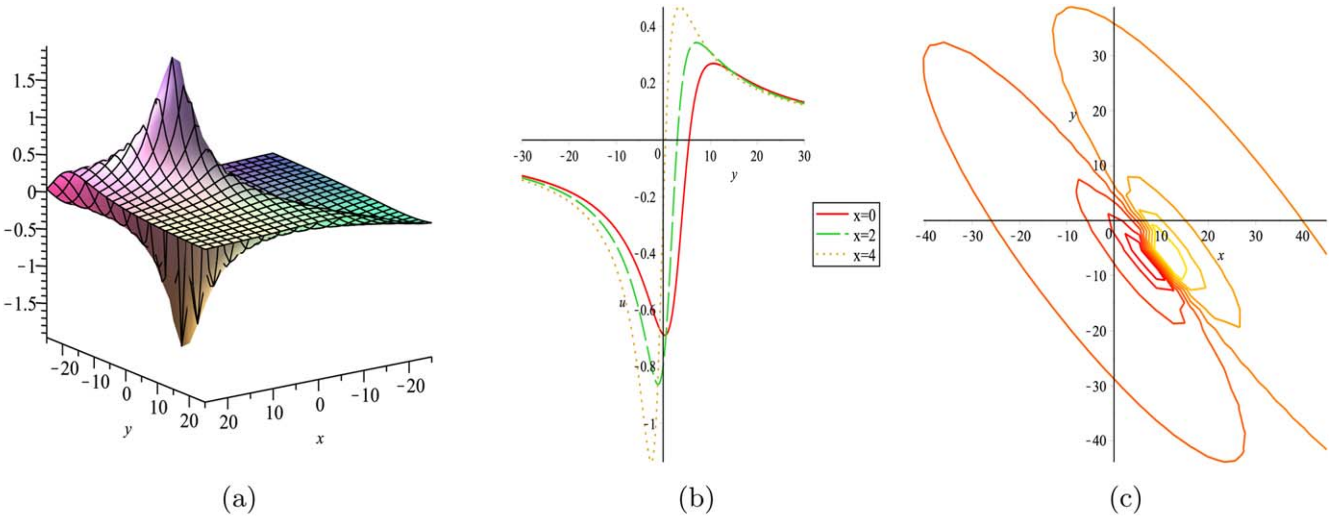

Figure 3. The plots of the expression (4.9) with special parameters: $z = 0$, $t = 2$. (a) Three-dimensional plot. (b) The y-curves. (c) The contour plot.

$a_1a_7 - a_2a_6 \neq 0$ leads to $(c_1a_1 + c_2a_2)^2 + (c_1a_6 + c_2a_7)^2 \neq 0$, which guarantees a_4 and a_9 in (4.2) are well defined. Under the nonzero condition

$$a_1a_7 - a_2a_6 \neq 0, \tag{4.4}$$

the solutions in (4.1) are positive quadratic functions if and only if the parameter $a_{11} > 0$, and so, we require a positivity condition for a_{11} as

$$\frac{(a_1a_2 + a_6a_7)}{\lambda_3} < 0, \tag{4.5}$$

where λ_3 is defined by (4.3). It is easy to see that the condition $c_5(c_1^2c_4 + c_2^2c_3) \geqslant 0$ guarantees λ_3 or $-\lambda_3$ is non-negative. The resulting quadratic function solutions yield a class of lump-type solutions to the (3 + 1)-dimensional nonlinear equations defined by (1.4) through the transformation (1.5):

$$u = \frac{12(a_1g_1 + a_6g_2)}{\chi(g_1^2 + g_2^2 + a_{11})}, \tag{4.6}$$

where the functions g_1, g_2 and a_{11} are defined by (4.1) and (4.2). Obviously, the constraints (4.4) and (4.5) guarantee both analyticity and localization of the solutions in (4.1). If the sum of squares $g_1^2 + g_2^2 \to \infty$, then the above solutions in (4.1) may tend to zero, but cannot uniformly approach zero in all space directions.

Below we present two application examples in (3 + 1) dimensions, to shed light on lump-type solutions of (1.4).

Example 4.1. We consider the (3 + 1)-dimensional Jimbo–Miwa type equation (2.13), which has $c_5(c_1^2c_4 + c_2^2c_3) = 0$. Based on the presented solutions, we have

$$\left\{\begin{aligned} &a_1 = a_1,\ a_2 = a_2,\ a_3 = a_3,\\ &a_4 = \frac{3(a_2a_3^2 + 2a_3a_7a_8 - a_2a_8^2)}{2(a_2^2 + a_7^2)},\ a_5 = a_5,\\ &a_6 = a_6,\ a_7 = a_7,\ a_8 = a_8,\\ &a_9 = \frac{3(a_7a_8^2 + 2a_2a_3a_8 - a_7a_3^2)}{2(a_2^2 + a_7^2)},\ a_{10} = a_{10},\\ &a_{11} = \frac{(a_1a_2 + a_6a_7)(a_1^2 + a_6^2)(a_2^2 + a_7^2)}{(a_3a_7 - a_2a_8)^2} \end{aligned}\right\}, \tag{4.7}$$

which needs to satisfy the condition $a_3a_7 - a_2a_8 \neq 0$. When $a_{11} > 0$, namely

$$a_1a_2 + a_6a_7 > 0,$$

the corresponding quadratic function f defined by (4.1) is positive. In turn, through the transformation $u = 2\ln(f)_x$, a class of lump-type solutions to (2.13) can be expressed as:

$$u = \frac{4(a_1g_1 + a_6g_2)}{g_1^2 + g_2^2 + a_{11}}, \tag{4.8}$$

where g_1, g_2 and the involved parameters are defined by (4.1) and (4.2), respectively. Further taking

$$a_1 = 1,\ a_2 = 2,\ a_3 = -3,\ a_5 = 1,\ a_6 = 3,\ a_7 = 2,\\ a_8 = 1,\ a_{10} = -5,$$

then we have the following lump-type solution to (2.13):

$$u = \frac{2(20x + 16y - 30t - 28)}{\left(x + 2y - 3z + \frac{3}{4}t + 1\right)^2 + \left(3x + 2y + z - \frac{21}{4}t - 5\right)^2 + 10}. \tag{4.9}$$

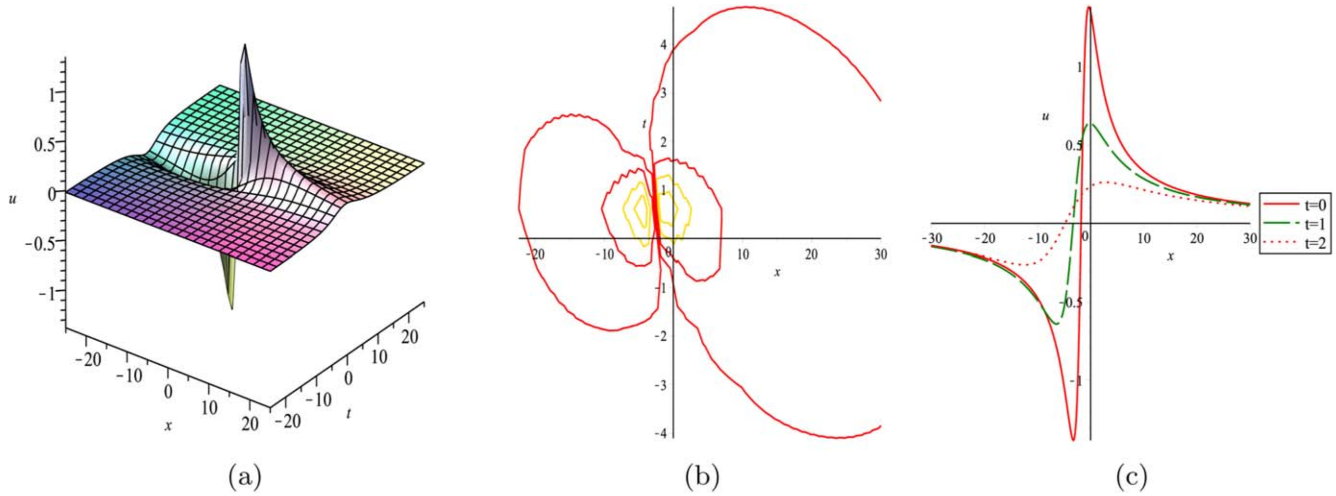

Figure 4. The plots of the expression (4.15) with special parameters: $y = 0$, $z = 2$. (a) Three-dimensional plot. (b) The contour plot. (c) The x-curves.

The plots of (4.9) are depicted in figure 3.

Example 4.2. Taking

$$\chi = 3,\ c_1 = c_4 = 0,\ c_2 = -1,\ c_3 = 3\sigma,\ c_5 = -3,\ \sigma = \pm 1, \tag{4.10}$$

then (1.4) reduces to the $(3 + 1)$-dimensional BKP equation (1.9). Under the typical transformation $u = 2(\ln f)_x$, this equation itself has a Hirota bilinear form

$$(D_t D_y - D_x^3 D_y - 3\sigma D_x^2 + 3D_z^2) f \cdot f = 0,\ \ \sigma = \pm 1. \tag{4.11}$$

By the above results in (4.2) and (4.3), a set of constraining relations for the parameters is

$$\begin{aligned}\Bigg\{ & a_1 = a_1,\ a_2 = a_2,\ a_3 = a_3,\ a_4 = \frac{3\sigma(a_1^2 a_2 - a_2 a_6^2 + 2a_1 a_6 a_7) - 3(a_2 a_3^2 - a_2 a_8^2 + 2a_3 a_7 a_8)}{a_2^2 + a_7^2},\\ & a_5 = a_5,\ a_6 = a_6,\ a_7 = a_7,\ a_8 = a_8,\\ & a_9 = \frac{3\sigma(a_6^2 a_7 - a_1^2 a_7 + 2a_1 a_2 a_6) - 3(a_7 a_8^2 - a_3^2 a_7 + 2a_2 a_3 a_8)}{a_2^2 + a_7^2},\\ & a_{10} = a_{10},\ a_{11} = \frac{(a_1 a_2 + a_6 a_7)(a_1^2 + a_6^2)(a_2^2 + a_7^2)}{(a_3 a_7 - a_2 a_8)^2 - \sigma(a_1 a_7 - a_2 a_6)^2}\Bigg\}.\end{aligned} \tag{4.12}$$

Note that $(a_3 a_7 - a_2 a_8)^2 - \sigma(a_1 a_7 - a_2 a_6)^2 \neq 0$ leads to $a_2^2 + a_7^2 \neq 0$, which guarantees a_4 and a_9 in (4.12) are well defined. Therefore, when $\sigma = -1$, the condition for guaranteeing lump-type solutions is

$$(a_3 a_7 - a_2 a_8)^2 + (a_1 a_7 - a_2 a_6)^2 \neq 0,\ a_1 a_2 + a_6 a_7 > 0, \tag{4.13}$$

which guarantees that a_{11} defined in (4.12) is positive. In addition, when $\sigma = 1$, the condition is

$$[(a_3 a_7 - a_2 a_8)^2 - (a_1 a_7 - a_2 a_6)^2](a_1 a_2 + a_6 a_7) > 0, \tag{4.14}$$

which guarantees that the quadratic function f defined by (4.1) is positive. For the case of $\sigma = 1$, we point out that the parameter a_6 is arbitrary in (4.12), and so the resulting positive quadratic function solutions above cover the second quadratic function solutions recently presented in [33].

Associated with

$$\sigma = 1,\ a_1 = 1,\ a_2 = 2,\ a_3 = 1,\ a_5 = 4,\ a_6 = 3,\\ a_7 = -1,\ a_8 = -2,\ a_{10} = 8,$$

the transformation $u = 2\ln(f)_x$ with (4.1) provides a lump-type solution to the corresponding $(3 + 1)$-dimensional BKP equation (1.9):

$$u = \frac{2(20x - 2y - 10z + 30t + 56)}{(x + 2y + z - 12t + 4)^2 + (3x - y - 2z + 9t + 8)^2 + \frac{5}{4}}. \tag{4.15}$$

In figure 4, three-dimensional and contour plots of this lump-type solution with special parameters are made through Maple plot tools. We can easily observe that the lump-type wave has a peak and a valley, and is called the bright-dark lump-type wave [53].

5. Concluding remarks

To conclude, we discussed a class of generalized KP and BKP equations (1.4), and computed multiple wave solutions with generic phase shifts and wave frequencies. The newly presented generic phase shifts and wave frequencies in this paper lead to the richness and diversity of exact solutions. Using the Hirota bilinear technique, some interesting reductions of three-wave solutions were explored. We showed that any nonlinear equation defined by (1.4) can be transformed into the KdV equation (3.15) and its dimensional reductions. Therefore, under the dimensional reduction, abundant exact solutions can be worked out to (1.4). We also remark that the phase shifts (3.28) are the same with the ones in Wazwaz's work [6, 54]. By extending the parameters to the complex field, nonsingular complexiton solutions can be generated from multiple wave solutions. Moreover, it is worth noting that there exist resonant phenomena in the presented solitons. The choices (3.16) and (3.19) imply the corresponding phase shifts $a_{ij}=0$. The phase shifts defined by (3.5) with (3.6) and (3.7) show some resonant phenomena, and so we can obtain the resonant multiple wave solutions u of (1.4) associated with the form $f=1+\sum_{i=1}^{n}\varepsilon_i e^{\theta_i}$ if taking $c_5\neq 0$. To our best knowledge, this has not been revealed in previous studies.

In addition, we presented a class of lump-type solutions to (1.4) with the help of Maple. The constraints of parameters guarantee the analyticity and localization to the resulting lump-type solutions. Two application examples were presented in $(3+1)$ dimensions, to shed light on lump-type solutions of (1.4). To a certain extent, the results presented in this letter are generalizations of previous studies, since many nonlinear equations of mathematical physics can be used as special cases of (1.4). For example, the set (4.2) of solutions can also generate a class of lump solutions of the $(2+1)$-dimensional BKP equation (1.7), which provides a supplement to existing literatures on related equations [53, 55, 56].

Generally speaking, Hirota bilinear forms play a key role in constructing multiple wave and lump-type solutions. To our knowledge, there are various discussions in terms of the Hirota bilinear forms to (1.4). For example, we hope to find positive quadratic function solutions to the following generalized bilinear equations

$$\begin{aligned}&(D_{3,x}^3D_{3,y}+c_1D_{3,x}D_{3,t}+c_2D_{3,y}D_{3,t}\\&\quad+c_3D_{3,x}^2+c_4D_{3,y}^2+c_5D_{3,z}^2)f\cdot f=0,\\&(D_{5,x}^3D_{5,y}+c_1D_{5,x}D_{5,t}+c_2D_{5,y}D_{5,t}\\&\quad+c_3D_{5,x}^2+c_4D_{5,y}^2+c_5D_{5,z}^2)f\cdot f=0,\end{aligned}$$

where $D_{3,x}, D_{3,y}, D_{3,z}, D_{3,t}$ and $D_{5,x}, D_{5,y}, D_{5,z}, D_{5,t}$ are two kinds of generalized bilinear derivatives [57, 58]. More research questions will be studied in our future works.

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