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Various kinds of high-order solitons to the Bogoyavlenskii–Kadomtsev–Petviashvili equation

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Abstract

In this work, the Bogoyavlenskii–Kadomtsev–Petviashvili equation which is used to describe the wave phenomenon in fluid mechanics is investigated. Based on the bilinear representation, perturbation method and Taylor expansion approach, we derive various kinds of high-order solitons including the N-kink soliton, n-order lump-type soliton and mixture solution of kink soliton and lump-type soliton. First, N-kink soliton solution is obtained by the bilinear representation and perturbation method. Second, by using the Taylor expansion approach for the $2n$-kink soliton solution, n-order lump-type soliton is obtained. Third, by mean of the Taylor expansion approach for $2n$-kink soliton solution in the N-kink soliton solution ($1 < 2n < N$), we construct the mixture solution consisting of $(N-2n)$-kink soliton and n-order lump-type soliton. Interestingly, the collision between kink soliton and lump-type soliton can give rise to a high-order lump-type soliton.

Keywords: Bogoyavlenskii–Kadomtsev–Petviashvili equation, perturbation method, Taylor expansion approach, soliton

1. Introduction

Soliton solutions play an important role in the research of nonlinear wave theory, these particular solutions are often used to help us understand the associated nonlinear localized excitation phenomenon in nonlinear physics science, such as solitary waves observed by Russell in water waves, shock waves in the gas and solids dynamics, Langmuir waves in plasmas, optical solitary waves in nonlinear optics and so on [1–4]. Since Korteweg–de Vries (KdV) equation and its soliton solutions were derived [2], seeking for soliton solutions and exploring their dynamical behavior for nonlinear wave equations have captured a lot of attention in the field of nonlinear wave. In the higher-dimensional nonlinear wave equations, soliton solutions of many equations have been studied in detail. For instance, (2+1)-dimensional Kadomtsev–Petviashvili equation [5], (2+1)-dimensional Sawada–Kotera equation [6], (3+1)-dimensional potential Yu–Toda–Sasa–Fukuyama equation [7], (3+1)-dimensional B-type Kadomtsev–Petviashvili equation [8], (4+1)-dimensional Fokas equation [9] and so on. So far, a great many effective methods are used to construct soliton solutions, such as the inverse scattering transformation method [1], Darboux transformation method [10], long wave limit technique [11], Hirota's bilinear method [12, 13] and so on. In this work, we will devote to studying two dimensional Bogoyavlenskii–Kadomtsev–Petviashvili (BKP) equation

$$u_{xxt} + u_{xxxxy} + 12u_{xx}u_{xy} + 8u_x u_{xxy} + 4u_{xxx}u_y = u_{yyy}, \quad (1.1)$$

where $u(x, y, t)$ is an analytic real scalar field and represents the amplitude of the relevant wave. This equation is an extension of the Bogoyavlenskii–Schiff equation and KP equation [14–22], it can be used as a model for evolutionary shallow water waves [14]. The BKP equation is a member of the higher dimensional KP hierarchy and has been discussed in the previous literatures [14–23]. In [15], this equation was derived by a reduction for the well-known three-dimensional Kadomtsev–Petviashvili equation [16] which describes the propagation of nonlinear waves in plasmas, fluid dynamics and electrical networks. If we neglect the dispersion effect term u_{yyy}, then equation (1.1) reduces to the Calogero–Bogoyavlenskii–Schiff equation [17] which describes the interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis. The Lax pair for the BKP equation was presented in [18], and the construction of Wronskian-type solution owes to the work [15]. Period traveling wave, soliton-like and rational function solutions were given in [19, 20]. The transformation groups, Kac–Moody–Virasoro algebras and conservation laws for the BKP equation have been presented by the generalized symmetry method and Ibragimov's theorem [21]. The authors have also derived the nonlocal symmetry, non-auto Bäcklund transformation and consistent Riccati expansion solvable of BKP equation, and investigated the interaction between kink wave and lump-type wave in [22]. The bilinear structures and multiple wave solutions have been deduced by means of the binary Bell polynomials method [23].

Through the logarithmic transformation [22, 23]

$$u = (\log F)_x = \frac{F_x}{F}, \tag{1.2}$$

the BKP equation (1.1) can be converted into the Hirota bilinear system

$$\begin{cases} H_1(D_x, D_y, D_t, D_s)F \cdot F = (D_x^4 - 3\alpha D_x D_s - 3D_y^2)F \cdot F = 0, \\ H_2(D_x, D_y, D_t, D_s)F \cdot F = (3D_x D_t + 2D_x^3 D_y + 3\alpha D_y D_s)F \cdot F = 0, \end{cases} \tag{1.3}$$

where $F = F\,(x,\,y,\,t;\,s)$ is an unknown function, s is an auxiliary parameter variable and α is a non-zero constant. Hence, if $F\,(x,\,y,\,t;\,s)$ solves the Hirota bilinear system (1.3), then (1.2) is the solution of the BKP equation (1.1).

Remark. Introducing the auxiliary parameter variable s equates to adding a spatial variable, then the (2+1)-dimensional BKP equation (1.1) can be transformed into the (3+1)-dimensional Hirota bilinear system (1.3) by the variable transformation (1.2). It is more concise to express the solution of the BKP equation (1.1) by the solution of the (3+1)-dimensional Hirota bilinear system. Besides, the higher dimensional systems have richer dynamical behaviors.

In the previous literature [14–23], general high-order lump-type soliton and higher-order mixture solution consisting of the kink soliton and lump-type soliton for the BKP equation (1.1) have not been investigated before. Therefore, our main goal is to construct general higher-order lump-type soliton and higher-order mixture solution consisting of the kink soliton and lump-type soliton solutions and to explore their interesting dynamical behaviors. Section 2 is mainly divided into three parts: (2.1) N-kink soliton solution is obtained by the bilinear representation and perturbation method; (2.2) by using the Taylor expansion approach for the $2n$-kink soliton solution, n-order lump-type soliton is obtained; (2.3) by mean of the Taylor expansion approach for $2n$-kink soliton solution in the N-kink soliton solution ($1 < 2n < N$), mixture solution consisting of $(N - 2n)$-kink soliton and n-order lump-type soliton is derived. We discuss the properties of interaction between kink soliton and lump-type soliton. Finally, conclusions are given in section 3.

2. Various kinds of high-order solitons of the BKP equation

In this section, we mainly consider the kink soliton, lump-type soliton solutions and their high order cases. The kink soliton is also known as the front wave [1, 2, 24] and also investigated as the shock wave [25–27]. This kind of soliton has been found in a variety of physical systems, such as the Burgers equation in fluid dynamics [24, 25], sine-Gordon equation in nonlinear optics [28], modified Kadomtsev–Petviashvili equation in plasma physics and electrodynamics [29], modified Zakharov–Kuznetsov equation in nonlinear electrical network [30] and so on. The lump-type soliton is another kind of rational soliton solution which decays algebraically to the background wave in space direction [31–43]. Compared with the kink soliton, they are completely different, because the kink soliton is exponentially nonlinear localized wave solution along the propagation direction. The lump-type soliton can be used to describe the rogue wave in Bose–Einstein condensate [31], internal rogue wave in density stratified flows [32], freak wave in the ocean [33], nonlinear localized wave in plasma [34], optical rogue wave phenomena in nonlinear optic media [35], etc. In these physical contexts, the lump-type soliton can play a role of elementary wave excitations. Moreover, the nontrivial internal interaction between lump-type solitons can be used to describe a model of a strong wave turbulence [34]. The high-order soliton can be used to describe the interaction between solitons and demonstrate the amplitude, velocity and wave shape relations between solitons before and after interaction. In general, if the amplitude, velocity and wave shape of solitons do not change after nonlinear collisions, then this kind of interaction is called a completely elastic interaction [29]. However, some interactions can lead to more interesting physical phenomena. For example, in the interaction process, several solitons can fuse to one after interaction with each other, or one soliton fission into several solitons. These two types of physical phenomena are called soliton fission and soliton fusion, respectively. These two types of interactions are also referred to as completely non-elastic interactions [44].

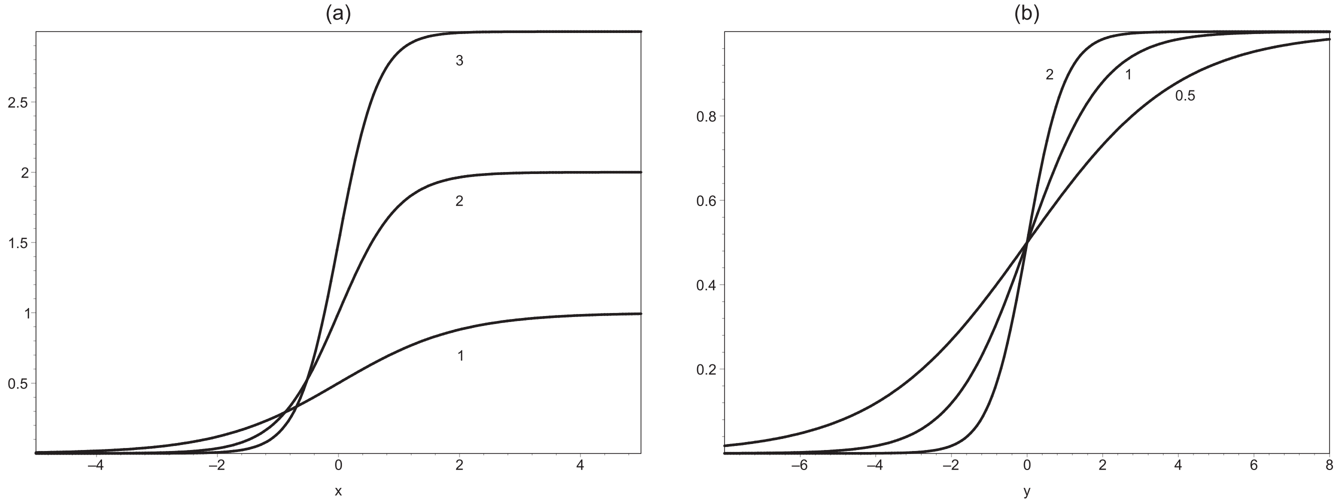

Figure 1. The profiles of the kink soliton: (a) the cross-sectional views at $y = 0$ with $\theta_1 = 0$ and $k_1 = 1, 2, 3$, respectively; (b) the cross-sectional views at $x = 0$ with $k_1 = 1$, $\theta_1 = 0$ and $l_1 = 0.5, 1, 2$, respectively. The parameter value of s is 0.

Therefore, it is a very meaningful research work to investigate various kinds of solitons and their high order cases.

2.1. High-order kink soliton solutions

This section we will employ the perturbation method [10] to derive the high-order kink soliton solution of (1.3). For this purpose, let us expand the function F into the following power series in ϵ

$$F = \sum_{n=0}^{\infty} \epsilon^n F_n, \tag{2.1}$$

where $F_0 = 1$ and $F_n = F_n(x, y, t; s)(n = 1, 2, 3,\cdots)$ are some unknown functions, they will be determined later. Inserting (2.1) into the Hirota bilinear system (1.3) we obtain

$$\begin{cases} \sum_{n=0}^{\infty}(\sum_{i=0}^{n} H_1(D_x, D_y, D_t, D_s)F_{n-i}\cdot F_i)\,\epsilon^n = 0,\\ \sum_{n=0}^{\infty}(\sum_{i=0}^{n} H_2(D_x, D_y, D_t, D_s)F_{n-i}\cdot F_i)\,\epsilon^n = 0.\end{cases} \tag{2.2}$$

Suppose the coefficients of each order of the parameter ϵ to zero, we get the set of equations

$$\begin{cases} \sum_{i=0}^{n} H_1(D_x, D_y, D_t, D_s)F_{n-i}\cdot F_i = 0,\\ \sum_{i=0}^{n} H_2(D_x, D_y, D_t, D_s)F_{n-i}\cdot F_i = 0,\end{cases} \quad n = 0, 1, 2, 3,\cdots. \tag{2.3}$$

Solving the equations (2.3) by applying the exponential function, a class of solution of the Hirota bilinear system (1.3) can be derived as follows

$$F_N(x, y, t; s) = \sum_{\sigma=0,1} \exp(\sum_{i=1}^{N}\sigma_i\xi_i + \sum_{i<j}^{N}\sigma_i\sigma_j\delta_{ij}),\ \xi_i = k_i x + l_i y + c_i t + d_i s + \theta_i, \tag{2.4}$$

where the parameters k_i, l_i, c_i, d_i satisfy the following conditions

$$c_i = \frac{l_i(l_i^2 - k_i^4)}{k_i^2},\ d_i = \frac{k_i^4 - 3l_i^2}{3\alpha k_i},$$
$$\mathrm{e}^{\delta_{ij}} = \frac{k_i^2 k_j^2 (k_i - k_j)^2 + \det(k, l)_{ij}^2}{k_i^2 k_j^2 (k_i + k_j)^2 + \det(k, l)_{ij}^2},\ 1 \leqslant i < j \leqslant N, \tag{2.5}$$

and k_i, l_i, θ_i are some arbitrary constants, $\det(k, l)_{ij} = \begin{vmatrix} k_i & l_i \\ k_j & l_j \end{vmatrix}(1 \leqslant i < j \leqslant N)$. In the expression (2.4), the perturbation parameter ϵ is merged into the corresponding phase constant θ_i, the first summation sign $\sum_{\sigma=0,1}$ means the summation over all possible combinations of $\sigma_1 = 0, 1$, $\sigma_2 = 0, 1, \cdots, \sigma_N = 0, 1$, and the third notation $\sum_{i<j}^{N}$ denotes the summation over all possible arrays (i, j) chosen the set $\{1, 2, \cdots, N\}$ with the condition $i < j$. Hence we can derive the N-kink soliton solution by inserting (2.4) with (2.5) into (1.2). The determinant $\det(k, l)_{ij}$ determines the types of interaction between two kink solitons. If $\det(k, l)_{ij} = 0$, the interaction between two kink solitons is parallel. Otherwise, if $\det(k, l)_{ij} \neq 0$, the interaction between two kink solitons is oblique.

Based on the result given by (2.4), N-kink soliton solutions can be presented by choosing different values of N. For example, when $N = 1$, the expression (2.4) becomes

$$F_1(\xi_1) = 1 + \mathrm{e}^{\xi_1}, \tag{2.6}$$

with

$$\xi_1 = k_1\left(x - \frac{l_1(k_1^4 - l_1^2)}{k_1(k_1^2 + l_1^2)}t\right) + l_1\left(y - \frac{l_1^2(k_1^4 - l_1^2)}{k_1^2(k_1^2 + l_1^2)}t\right) + \frac{k_1^4 - 3l_1^2}{3\alpha k_1}s + \theta_1. \tag{2.7}$$

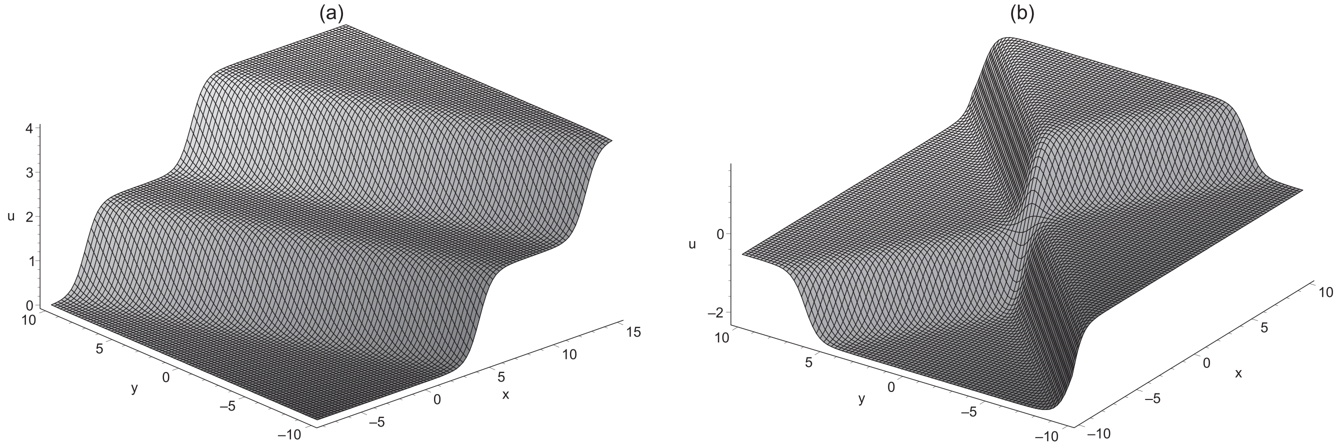

Figure 2. The spatial structures of interaction solution between two kink solitons at $t = 0$: (a) parallel kink solitons; (b) crossed kink solitons. The parameters in the formula (2.13) are (a) $(k_1, k_2, l_1, l_2, \alpha, \theta_1, \theta_2) = (2, 2.002, 1, 1.001, 1, 0, 0)$ and (b) $(k_1, k_2, l_1, l_2, \alpha, \theta_1, \theta_2) = (-1.5, 1, 1.5, 1.5, 1, 0, 0)$, respectively. The parameter value of s is 0.

Inserting the solution (2.6) with (2.7) into the transformation (1.2), we get the single kink soliton solution

$$u(\xi_1) = \frac{1}{2}k_1\left(1 + \tanh\frac{1}{2}\xi_1\right). \tag{2.8}$$

This solution $u(\xi_1)$ is a traveling wave solution in the BKP equation (1.1). The amplitude function u has the kink shape of a hyperbolic tangent, see figure 1. Therefore, it is called kink soliton solution and its kink front is determined by $\xi_1 = 0$. By the asymptotic state analysis, it is easy to deduce that the single kink soliton solution tends to k_1 as $\xi_1 \to +\infty$, and approaches to zero as $\xi_1 \to -\infty$. Two different asymptotic states are presented, hence the solution (2.8) is a solitary wave and its amplitude is $|k_1|$. From (2.7) we can also see that the kink soliton propagates in the (x, y)-plane with the velocity

$$\mathbf{v} = (v_x, v_y) = \left(\frac{l_1(k_1^4 - l_1^2)}{k_1(k_1^2 + l_1^2)}, \frac{l_1^2(k_1^4 - l_1^2)}{k_1^2(k_1^2 + l_1^2)}\right). \tag{2.9}$$

Figure 1 exhibits the sectional diagram of the single kink soliton along the x-axis and the y-axis at $t = 0$. Since $u_x = \frac{k_1^2}{4}\operatorname{sech}^2\frac{1}{2}\xi_1$, the steepness of the single kink soliton along the x-axis depends on the amplitude $|k_1|$. The kink front will become steeper as the amplitude gets bigger, see figure 1(a). However, when the amplitude is fixed, the steepness of the kink soliton can be controlled by the wave number l_1. Since $u_y = \frac{k_1 l_1}{4}\operatorname{sech}^2\frac{1}{2}\xi_1$, when $k_1 l_1 > 0$, with the increasing of the parameter $|l_1|$, the steepness of the kink front will increase, see figure 1(b).

When we take $N = 2$ in the formula (2.4), then we gain the following function

$$F_2(\xi_1, \xi_2) = 1 + e^{\xi_1} + e^{\xi_2}(1 + e^{\xi_1 + \delta_{12}}), \tag{2.10}$$

with

$$e^{\delta_{12}} = \frac{k_1^2 k_2^2 (k_1 - k_2)^2 + \det(k, l)_{12}^2}{k_1^2 k_2^2 (k_1 + k_2)^2 + \det(k, l)_{12}^2},\ c_i = \frac{l_i(l_i^2 - k_i^4)}{k_i^2},$$
$$d_i = \frac{k_i^4 - 3l_i^2}{3\alpha k_i},\ i = 1, 2, \tag{2.11}$$

where the determinant $\det(k, l)_{12}$ is given by

$$\det(k, l)_{12} = \begin{vmatrix} k_1 & l_1 \\ k_2 & l_2 \end{vmatrix}. \tag{2.12}$$

Inserting the function (2.10) with (2.11) and (2.12) into (1.2), two-kink soliton solution of equation (1.1) can be presented as follows

$$u(\xi_1, \xi_2) = \frac{k_1 e^{\xi_1}(1 + e^{\xi_2 + \delta_{12}}) + k_2 e^{\xi_2}(1 + e^{\xi_1 + \delta_{12}})}{1 + e^{\xi_1} + e^{\xi_2}(1 + e^{\xi_1 + \delta_{12}})}. \tag{2.13}$$

This solution $u(\xi_1, \xi_2)$ is a kind of interaction solution which describes the elastic collision between two kink solitons. The determinant $\det(k, l)_{12}$ determines the type of interaction. If $\det(k, l)_{12} = 0$, the solution (2.13) presents the interaction of two parallel kink solitons, see figure 2(a). If $\det(k, l)_{12} \neq 0$, the solution (2.13) describes the oblique interaction of two kink solitons, the propagation of the two solitons is crossed, see figure 2(b). Now we consider the asymptotic property of the 2-kink soliton solution (2.13) as $t \to \pm\infty$. Let $\xi_2 - \xi_1 = \tau$ and $c_2 > c_1$, then we get that $\tau \to \pm\infty$ corresponds to $t \to \pm\infty$. Hence we can obtain that when the distance between two kink solitons increases, two-kink soliton can be expressed as a sum of two noninteracting kink solitons

$$u(\xi_1, \xi_2) \simeq \begin{cases} u(\xi_1 + \delta_{12}) + u(\xi_2), & \tau \to +\infty, \\ u(\xi_1) + u(\xi_2 + \delta_{12}), & \tau \to -\infty, \end{cases} \tag{2.14}$$

where $u(\xi_1) = \frac{1}{2}k_1\left(1 + \tanh\frac{1}{2}\xi_1\right)$ and $u(\xi_2) = \frac{1}{2}k_2(1 + \tanh\frac{1}{2}\xi_2)$ are two single kink soliton solutions, δ_{12} is a phase shift parameter. This result indicates that the interaction between two kink solitons does not change the dynamic behaviors of two kink solitons except for a phase shift. Therefore, two-kink soliton (2.13) demonstrates an elastic interaction between two kink solitons. Besides, since the phase shift δ_{12} is greater than zero, the interaction between two kink solitons can not form the resonant kink soliton. Similarly, we can also investigate the dynamical behaviors of N-kink soliton.

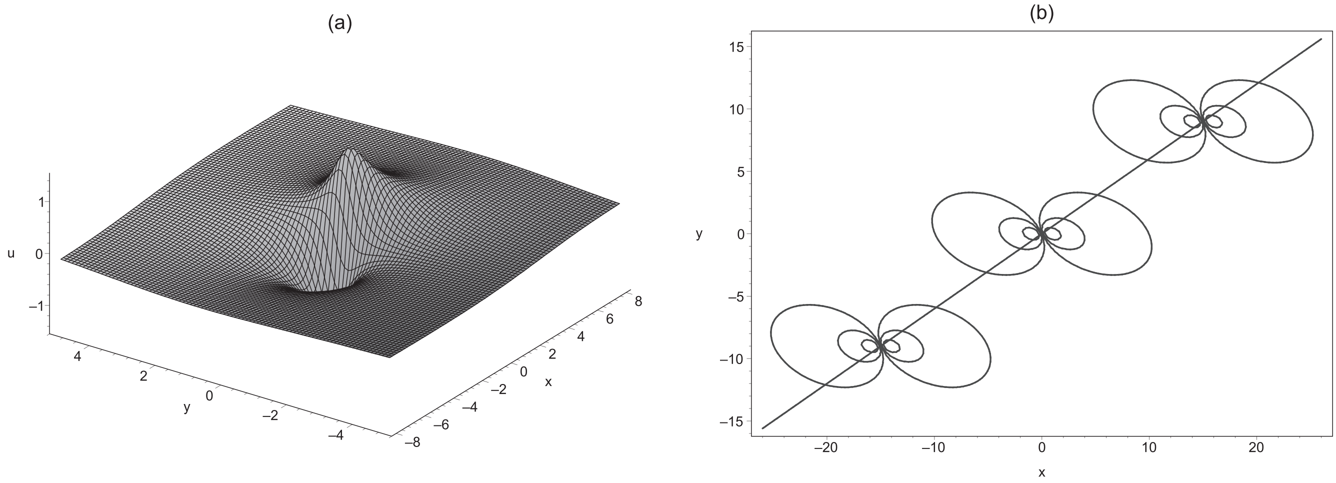

Figure 3. The spatial structures of lump-type soliton solution: (a) 3D profile at $t = 0$; (b) contour plot with $t = -6, 0, 6$, respectively. The parameters are selected with $(k_1, l_1, \theta_1, \alpha) = (1 + \mathbf{i}, -1 + 2\mathbf{i}, 0, 1)$. The parameter value of s is 0.

2.2. High-order lump-type soliton solutions

The section 2.1 derives and investigates the N-kink soliton solution of equation (1.1). Indeed, based on the N-kink soliton solution and the Taylor expansion method, we can derive another kind of higher-order lump-type soliton solution.

First, we derive the single lump-type soliton solution from the 2-kink soliton solution. Let $\theta_j \to p_j\theta_j + \pi\mathbf{i}$ and $(k_j, l_j) \to p_j(k_j, l_j)(j = 1, 2)$ in the formula (2.10), where p_1, p_2 are arbitrary parameters and $\mathbf{i}^2 = -1$. Further we consider the Taylor expansion of the function $F_2(\xi_1, \xi_2)$ at the point $(p_1, p_2) = (0, 0)$, then the polynomial solution of the Hirota bilinear system (1.3) can be obtained

$$F_{1L}(\Theta_1, \Theta_2) = \Theta_1\Theta_2 + R_{12}, \tag{2.15}$$

with

$$R_{12} = -\frac{4k_1^3k_2^3}{\det(k, l)_{12}^2}, \ \Theta_i = k_ix + l_iy + \frac{l_i^3}{k_i^2}t - \frac{l_i^2}{\alpha k_i}s + \theta_i, \ i = 1, 2, \tag{2.16}$$

where k_1, k_2, l_1, l_2 are arbitrary non-zero constants and $\det(k, l)_{12} \neq 0$. In general, the function $F_{1L}(\Theta_1, \Theta_2)$ has zero, the corresponding solution $u = (\ln F_{1L}(\Theta_1, \Theta_2))_x$ is a singular solution. In order to obtain the nonsingular solution, the parameters k_1, l_1, k_2, l_2 need to meet the following conditions

$$l_2 = l_1^*, \ k_2 = k_1^*, \ \theta_2 = \theta_1^*, \tag{2.17}$$

where * represents the complex conjugate. In terms of the constraint conditions (2.17), substituting the solution (2.15) into (1.2), we can obtain the nonsingular rational solution

$$u(\Theta_1, \Theta_1^*) = \frac{k_1\Theta_1^* + k_1^*\Theta_1}{\Theta_1\Theta_1^* - \frac{4(k_1k_1^*)^3}{(k_1l_1^* - k_1^*l_1)^2}}. \tag{2.18}$$

Figure 3 shows the spatial structure of the nonsingular rational solution (2.18) at $t = 0$. To understand the spatial structure more clearly, let us demonstrate their properties through the mathematical theory and method. This solution with parameters $(k_1, l_1, \theta_1, \alpha, s) = (1 + \mathbf{i}, -1 + 2\mathbf{i}, 0, 1, 0)$ is expressed as

$$u(x, y, t) = \frac{4x + 2y - 13t}{(x - y - t)^2 + \left(x + 2y - \frac{11}{2}t\right)^2 + \frac{8}{9}}.$$

Obviously, the propagation path of this solution is given by $(x(t), y(t)) = \left(\frac{5}{2}t, \frac{3}{2}t\right)$, that is, in the (x,y)-plane, this solution propagates along the line $y = \frac{3}{5}x$, see figure 3(b). By using the extremum principle, this solution has only two extremum points $\left(\frac{5}{2}t_0 \pm \frac{2}{3}, \frac{3}{2}t_0\right)$ for a fixed time $t = t_0$, where $\left(\frac{5}{2}t_0 + \frac{2}{3}, \frac{3}{2}t_0\right)$ is a local maximum point, and $\left(\frac{5}{2}t_0 - \frac{2}{3}, \frac{3}{2}t_0\right)$ is a local minimum point. By the asymptotic state analysis, this solution tend to zero as $(|x|, |y|) \to (\infty, \infty)$ for a fixed time $t = t_0$. Hence, this solution has only one maximum value and one minimum value. These properties imply that this solution is localized in all directions in the space. As can be seen from the figure 3, this solution has two parts. The first part is the background plane wave $u = 0$. The second part is the localized lump-type wave which has one upper peak and one down valley, see figure 3(a). Obviously, the amplitude of the upward and downward lump-type waves are equal. The peak and valley are divided by the plane wave $u = 0$. This solution is called the single lump-type soliton solution [31–43].

In order to derive the second-order lump-type soliton solution, let us consider the 4-kink soliton solution with $\theta_j \to p_j\theta_j + \pi\mathbf{i}$ and $(k_j, l_j) \to p_j(k_j, l_j)(j = 1, 2, 3, 4)$. From the Taylor series generated by the function $F_4(x, y, t; s)$ at the point $(p_1, p_2, p_3, p_4) = (0, 0, 0, 0)$, then we can derive the

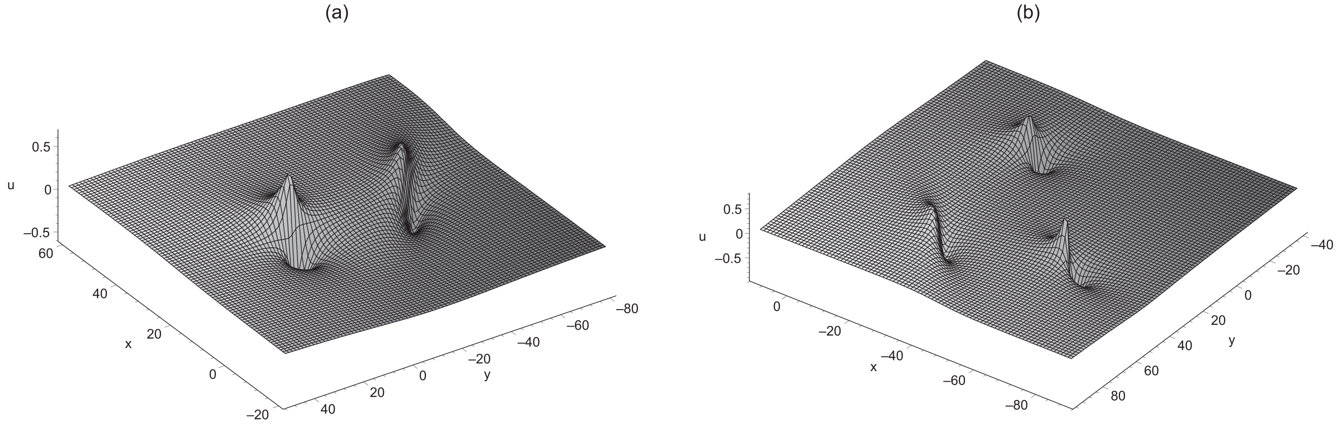

Figure 4. The spatial structures of lump-type soliton solution: (a) second-order lump-type soliton; (b) third-order lump-type soliton, respectively. The parameters are selected with: (a) $(k_1, l_1, k_2, l_2, \theta_1, \theta_2, \alpha, t) = (3 + \mathbf{i}, 0.5 - 2\mathbf{i}, 2 + 0.1\mathbf{i}, 1 + \mathbf{i}, -20 + 25\mathbf{i}, 10 + 40\mathbf{i}, 1, 0)$; (b) $(k_1, l_1, k_2, l_2, k_5, l_5, \theta_1, \theta_2, \theta_5, \alpha, t) = (3 + \mathbf{i}, 0.5 - 2\mathbf{i}, 2 + 0.1\mathbf{i}, 1 + \mathbf{i}, 1.5 + 0.1\mathbf{i}, 0.5 + 1.5\mathbf{i}, 75 + 12\mathbf{i}, -3 - 57\mathbf{i}, 71 - 61\mathbf{i}, 1, 0)$, respectively. The parameter value of s is 0.

following polynomial solution

$$F_{2L}(\Theta_1, \Theta_2, \Theta_3, \Theta_4) = \prod_{i=1}^{4} \Theta_i + \frac{1}{2}\sum_{i,j}^{4} R_{ij} \prod_{l\neq i,j}^{4} \Theta_l + \frac{1}{2!2^2}\sum_{i,j,k,l}^{4} R_{ij}R_{kl}, \tag{2.19}$$

with

$$R_{ij} = -\frac{4k_i^3k_j^3}{D_{ij}^2}, \Theta_i = k_ix + l_iy + \frac{l_i^3}{k_i^2}t - \frac{l_i^2}{\alpha k_i}s + \theta_i, (i, j = 1, 2, 3, 4), \tag{2.20}$$

where k_j, $l_j (j = 1, 2, 3, 4)$ are arbitrary non-zero constants and $\det(k, l)_{ij} \neq 0$. In the formula (2.19), the summation sign $\sum_{i,j,k,l}^{4}$ indicates the summation over all possible arrays (i, j, k, l) taken from the set $\{1, 2, 3, 4\}$, and i, j, k, l are all different.

Generally speaking, the function $F_{2L}(\Theta_1,\cdots,\Theta_4)$ has zero, the corresponding solution $u = (\ln F_{1L}(\Theta_1,\cdots,\Theta_4))_x$ is a singular solution. In order to obtain the nonsingular solution, the parameters k_j, l_j $(j = 1, 2, 3, 4)$ need to meet the following conditions

$$k_1 = k_3^*, k_2 = k_4^*, l_1 = l_3^*, l_2 = l_4^*, \theta_1 = \theta_3^*, \theta_2 = \theta_4^*. \tag{2.21}$$

In terms of the constraint conditions (2.21), substituting the solution (2.19) with (2.20) into (1.2), we can obtain the second-order nonsingular rational solution. Similar to the mathematical analysis method of first-order lump-type soliton, the solution given by (2.19) is localized in all directions in the space and has two single lump-type soliton solutions. Hence this solution is called the second-order lump-type soliton solution. Figure 4(a) shows the spatial structure of second-order lump-type soliton solution which has two single lump-type soliton solutions. Now we investigate the asymptotic form of the second-order lump-type soliton solution as $t \to \pm\infty$. At first, we consider $|\Theta_2|^2 =$ constant, then when $t \to \pm\infty$, we can obtain $|\Theta_1|^2 \to \infty$. Hence, $F_{2L}(\Theta_1, \Theta_2, \Theta_3, \Theta_4)$ has the asymptotic properties

$$F_{2L}(\Theta_1, \Theta_2, \Theta_3, \Theta_4) \simeq |\Theta_1|^2(|\Theta_2|^2 + R_{24}),\ \ t \to \pm\infty.$$

From the transformation (1.2) we know that the solution given by $|\Theta_1|^2(|\Theta_2|^2 + R_{24})$ is equal to the lump-type solution given by $|\Theta_2|^2 + R_{24}$ in the limit of $t \to \pm\infty$. Thus, the second-order lump-type soliton solution asymptotically reduces to the form of a single lump-type soliton solution

$$u \simeq (\ln(|\Theta_2|^2 + R_{24}))_x,\ \ t \to \pm\infty, |\Theta_2|^2 = \text{constant}.$$

Similarly, we can also derive that another asymptotic state of the second-order lump-type soliton solution is still the form of a single lump-type soliton solution

$$u \simeq (\ln(|\Theta_1|^2 + R_{13}))_x,\ \ t \to \pm\infty, |\Theta_1|^2 = \text{constant}.$$

The above asymptotic analysis suggests that there is no phase shift when two lump-type solitons interact with each other. Thus, this second-order lump-type soliton solution describes a completely elastic interaction between two lump-type solitons.

In order to derive the n-order lump-type wave solution, we consider the $2n$-kink soliton solution with $\theta_j \to p_j\theta_j + \pi\mathbf{i}$ and $(k_j, l_j) \to p_j(k_j, l_j)(j = 1, 2,\cdots,2n)$. From the Taylor series generated by the function $F_{2n}(x, y, t; s)$ at the point $(p_1, p_2,\cdots,p_{2n}) = (0, 0,\cdots,0)$, the higher order polynomial solution can be obtained

$$F_{nL}(\Theta_1,\cdots,\Theta_{2n}) = \prod_{i=1}^{2n} \Theta_i + \frac{1}{2}\sum_{i,j}^{2n} R_{ij} \prod_{l\neq i,j}^{2n} \Theta_l + \cdots + \frac{1}{S!2^S}\sum_{i,j,\cdots,r,s}^{2n} \overbrace{R_{ij}R_{kl}\cdots R_{rs}}^{S} \prod_{p\neq i,j,\cdots,r,s}^{2n} \Theta_p + \cdots, \tag{2.22}$$

with

$$\Theta_i = k_ix + l_iy + \frac{l_i^3}{k_i^2}t - \frac{l_i^2}{\alpha k_i}s + \theta_i, R_{ij} = -\frac{4k_i^3k_j^3}{\det(k, l)_{ij}^2}, 1 \leqslant i, j \leqslant 2n, \tag{2.23}$$

where k_i, $l_i(1, 2,\cdots,2n)$ are arbitrary non-zero constants and $\det(k, l)_{ij} \neq 0$, the summation sign $\sum_{i,j,\cdots,r,s}^{2n}$ denotes the summation over all possible arrays $(i, j,\cdots,r, s)$ chosen from the set $\{1, 2, \cdots, 2n\}$, and $i, j, \cdots, r, s$ are completely different. Since the function $F_{nL}(\Theta_1,\cdots,\Theta_{2n})$ has zero, the solution $u = (\ln F_{1L}(\Theta_1,\cdots,\Theta_{2n}))_x$ has singularity. In order to derive the nonsingular solution, we have to add the constraint conditions

$$k_{n+i} = k_i^*,\ l_{n+i} = l_i^*,\ \theta_{n+i} = \theta_i^*,\ i = 1, 2,\cdots,n. \tag{2.24}$$

In terms of the constraint conditions (2.24), substituting the solution (2.22) with (2.23) into (1.2), we can obtain the n-order lump-type soliton solution. By using the above asymptotic analysis method, we can obtain that the n-order lump-type soliton solution describes the multiple elastic interaction of n lump-type solitons. Figure 4(b) shows the spatial structure of 3-order lump-type soliton solution which has three single lump-type soliton solutions. The third-order lump-type soliton solution is localized in all directions in the space.

2.3. High-order mixture solution of the kink soliton and lump-type soliton

In the sections 2.1 and 2.2, we derive the high-order kink soliton and lump-type soliton solutions, respectively. Now we investigate the mixture solution consisting of the kink soliton and lump-type soliton. Base on the solution $F_N(x, y, t; s)$ given by (2.4), and taking $\theta_j \to p_j\theta_j + \pi\mathbf{i}$ and $(k_j, l_j) \to p_j(k_j, l_j)(j = 1, 2,\cdots,2n)$ in $F_N(x, y, t; s)$, where $1 < 2n < N$. Further, from the Taylor series generated by $F_N(x, y, t; s)$ at the point $(p_1, p_2,\cdots,p_{2n}) = (0, 0,\cdots,0)$ we can derive the high-order mixture solution consisting of $(N - 2n)$-kink soliton and n-order lump-type soliton. For example, in order to deduce the mixture solution consisting of a single kink soliton and a first-order lump-type soliton, we consider the 3-kink soliton solution. If we take $N = 3$ in the formula (2.4), then we gain the following function

$$F_3(\xi_1, \xi_2, \xi_3) = F_2(\xi_1, \xi_2) + \mathrm{e}^{\xi_3}F_2(\xi_1 + \delta_{13}, \xi_2 + \delta_{23}), \tag{2.25}$$

where ξ_3, δ_{13}, δ_{23} are given by (2.4) and (2.5), and the function $F_2(\xi_1, \xi_2)$ is defined by (2.10). This solution describes the interaction of three kink soliton solutions on the (x,y)-plane as the development of time t. However, if we take $\theta_j \to p_j\theta_j + \pi\mathbf{i}$ and $(k_j, l_j) \to p_j(k_j, l_j)(j = 1, 2)$ in the expression (2.25), and consider further the Taylor expansion of the function $F_3(\xi_1, \xi_2, \xi_3)$ at the point $(p_1, p_2) = (0, 0)$, then we can obtain the following mixture solution

$$\begin{aligned} F_{11}(\Theta_1, \Theta_2, \xi_3) &= F_{1L}(\Theta_1, \Theta_2) \\ &+ e^{\xi_3}F_{1L}(\Theta_1 - \theta_{13}, \Theta_2 - \theta_{23}), \end{aligned} \tag{2.26}$$

where $F_{1L}(\Theta_1, \Theta_2)$ is given by (2.15) and

$$\theta_{13} = \frac{4k_1^3k_3^3}{k_1^2k_3^4 + \det(k, l)_{13}^2},\ \theta_{23} = \frac{4k_2^3k_3^3}{k_2^2k_3^4 + \det(k, l)_{23}^2}, \tag{2.27}$$

k_j, l_j $(j = 1, 2, 3)$ are arbitrary non-zero constants. When the parameters k_1, l_1, k_2, l_2 satisfy the conditions $k_2 = k_1^*$, $l_2 = l_1^*$, $\theta_2 = \theta_1^*$, and inserting the solution (2.26) into (2.2), we can obtain the nonsingular mixture solution

$$u(\Theta_1, \Theta_1^*, \xi_3) = \frac{(k_3(\Theta_1 - \theta_{13})(\Theta_1^* - \theta_{13}^*) + k_1(\Theta_1^* - \theta_{13}^*) + k_1^*(\Theta_1 - \theta_{13}))e^{\xi_3} + k_1\Theta_1^* + k_1^*\Theta_1}{\Theta_1\Theta_1^* - \frac{4(k_1k_1^*)^3}{(k_1l_1^* - k_1^*l_1)^2} + e^{\xi_3}((\Theta_1 - \theta_{13})(\Theta_1^* - \theta_{13}^*) - \frac{4(k_1k_1^*)^3}{(k_1l_1^* - k_1^*l_1)^2})}. \tag{2.28}$$

This solution is made up of two polynomial functions and an exponential function, it describes the interaction between a single kink soliton and a first-order lump-type soliton on the (x, y)-plane as the development of time t. By the asymptotic state analysis, away from the interaction region, we can obtain the asymptotic behaviors with $c_3 > 0$ as follows

$$u(x, y, t) \simeq \begin{cases} u^-(x, y, t) + u_s(x, y, t), & t \to -\infty, \\ u_s(x, y, t) + u^+(x, y, t), & t \to +\infty, \end{cases}$$

where

$$\begin{aligned} u^-(x, y, t) &= (\ln F_{1L}(\Theta_1, \Theta_2))_x,\ u^+(x, y, t) \\ &=(\ln F_{1L}(\Theta_1 - \theta_{13}, \Theta_2 - \theta_{23}))_x, \end{aligned}$$

are two single lump-type solitons and $u_s(x, y, t) = \frac{k_3}{2}(1 + \tanh\frac{\xi_3}{2})$ is a kink soliton. Indeed, $u_l(x, y, t)$ and $u_r(x, y, t)$ are the same lump-type soliton except for a phase shift. This implies that the interaction between a lump-type soliton and a kink soliton is a completely elastic interaction. In order to illustrate more clearly the interaction process of kink soliton and first-order lump-type soliton, here we display the figure of the potential function $U = u_x$, see figure 5. From figure 5 we can see that when the lump-type soliton gets close to the line soliton, the lump-type soliton exchanges the energy with the line soliton. Some energy of lump-type soliton is absorbed by the line soliton, which triggers a new lump-type soliton from the line soliton. The amplitude of the original lump-type soliton becomes small. Hence, in the region of interaction, two lump-type soliton are presented, see figure 5(b). However, as time goes on, the amplitude of new lump-type soliton becomes bigger and separated completely from the line soliton, the original lump-type soliton is completely absorbed by the line soliton finally. We call this phenomenon an emit-absorb interaction. Throughout the process, we find that the interaction between a lump-type soliton and a line soliton is completely elastic, i.e. before and after collisions, the

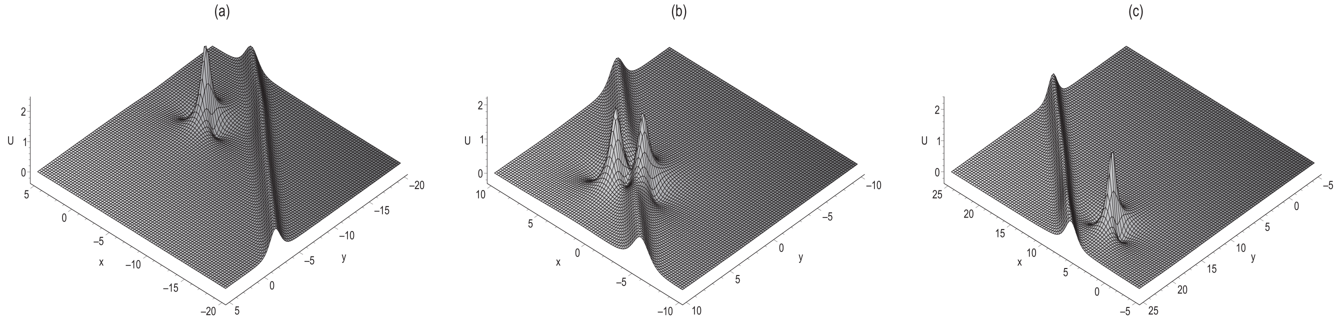

Figure 5. Interaction between a lump-type soliton and a line soliton. The parameters are selected with $(k_1, l_1, k_3, l_3, \theta_1, \theta_3, \alpha) = (1 + 1.2\mathbf{i}, 1.5 - \mathbf{i}, 1.8, 2, 0, 0, 1)$. (a) $t = -10$; (b) $t = 2.5$; (c) $t = 15$. The parameter value of s is 0.

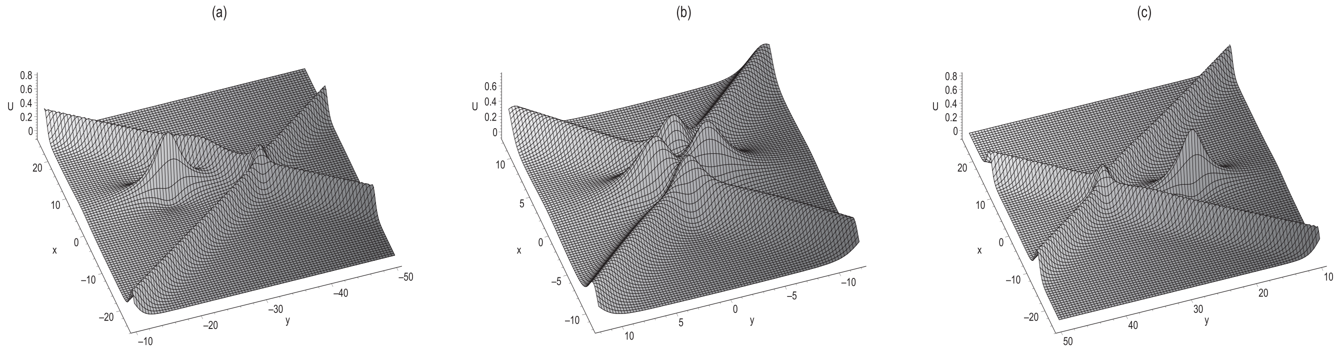

Figure 6. Interaction between a lump-type soliton and two line soliton. The parameters are selected with $(k_1, l_1, k_3, l_3, k_4, l_4, \theta_1, \theta_3, \theta_4, \alpha) = (3 + \mathbf{i}, 0.5 - 2\mathbf{i}, 1.2, 1.1, 1.2, -1.1, 0, 0, 0, 1)$. (a) t = −60; (b) t = 0; (c) t = 60. The parameter value of s is 0.

dynamical properties of lump-type soliton and line soliton do not change. In addition, we also observe an interesting phenomenon that the collision between a kink soliton and a first-order lump-type soliton can create a second-order lump-type soliton at the moment of collision, they do not produce the superposition of amplitudes, see figure 5(b).

Further, if we take $\theta_j \to p_j\theta_j + \pi\mathbf{i}$ and $(k_j, l_j) \to p_j(k_j, l_j)(j = 1, 2)$ for the 4-kink soliton solution, and consider the Taylor series generated by the function $F_4(x, y, t; s)$ at the point $(p_1, p_2) = (0, 0)$, we can obtain the following mixture solution

$$\begin{aligned} F_{12}(\Theta_1, \Theta_2, \xi_3, \xi_4) &= F_{1L}(\Theta_1, \Theta_2) \\ &+ \mathrm{e}^{\xi_3} F_{1L}(\Theta_1 - \theta_{13}, \Theta_2 - \theta_{23}) \\ &+ \mathrm{e}^{\xi_4} F_{1L}(\Theta_1 - \theta_{14}, \Theta_2 - \theta_{24}) \\ &+ \mathrm{e}^{\xi_3+\xi_4+\delta_{34}} F_{1L}(\Theta_1 - \theta_{13} - \theta_{14}, \Theta_2 - \theta_{23} - \theta_{24}), \end{aligned} \tag{2.29}$$

where ξ_3, ξ_4 and $\mathrm{e}^{\delta_{34}}$ are given by (2.4) and (2.5), $F_{1L}(\Theta_1, \Theta_2)$ is defined by (2.15), and

$$\theta_{ij} = \frac{4k_i^3 k_j^3}{k_i^2 k_j^4 + \det(k, l)_{ij}^2}, \; i = 1, 2, j = 3, 4, \tag{2.30}$$

k_j, l_j $(j = 1, 2, 3, 4)$ are arbitrary non-zero constants. When the parameters k_1, l_1, k_2, l_2 meet the conditions $k_2 = k_1^*$, $l_2 = l_1^*$, $\theta_2 = \theta_1^*$, and inserting the solution (2.29) into (2.2), we can obtain the nonsingular mixture solution consisting of two kink soliton and one lump-type soliton. This solution describes the interaction between two kink solitons and first-order lump-type soliton, see figure 6. As can be seen from figure 6, two different types of interactions are contained in the process of interaction. One is the interaction of two line solitons; the other is the interaction between lump-type soliton and line soliton. In the process of interaction, the lump-type soliton exchanges the energy with two line soliton. In the region of interaction, the structure with four peaks are presented, see figure 6(b). Indeed, the collision between two kink solitons and a first-order lump-type soliton creates three lump-type solitons at the moment of collision. Besides, in the region of interaction, the collision between two kink solitons leads to a superposition of amplitudes. Therefore, we can observe the structure with four peaks in figure 6(b). Throughout the process, we find that the interaction between a lump-type soliton and two line soliton is completely elastic.

Similarly, if we take $\theta_j \to p_j\theta_j + \pi\mathbf{i}$ and $(k_j, l_j) \to p_j(k_j, l_j)(j = 1, 2, 3, 4)$ for the 5-kink soliton solution, and consider the Taylor series generated by the function $F_5(x, y, t; s)$ at the point $(p_1, p_2, p_3, p_4) = (0, 0, 0, 0)$, we can derive the following mixture solution

$$\begin{aligned} F_{21}(\Theta_1, \Theta_2, \Theta_3, \Theta_4, \xi_5) &= F_{2L}(\Theta_1, \Theta_2, \Theta_3, \Theta_4) \\ &+ e^{\xi_5} F_{2L}(\Theta_1 - \theta_{15}, \Theta_2 - \theta_{25}, \Theta_3 - \theta_{35}, \Theta_4 - \theta_{45}) \end{aligned} \tag{2.31}$$

where ξ_5 is determined by (2.4) and (2.5), $F_{2L}(\Theta_1,\cdots,\Theta_4)$ is defined by (2.19), and

$$\theta_{i5} = \frac{4k_i^3 k_5^3}{k_i^2 k_5^4 + \det(k, l)_{i5}^2}, \; i = 1, 2, 3, 4, \tag{2.32}$$

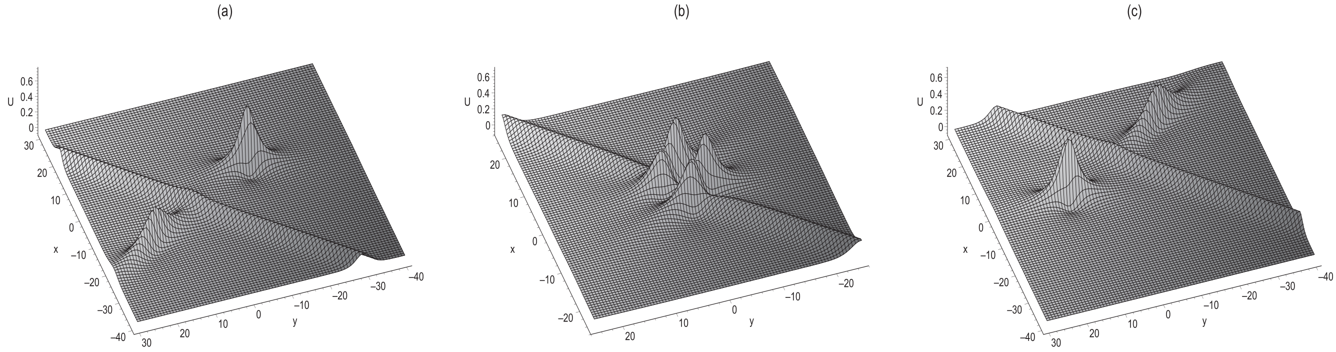

Figure 7. Interaction between two lump-type soliton and one line soliton. The parameters are selected with $(k_1, l_1, k_3, l_3, k_5, l_5, \theta_1, \theta_3, \theta_5, \alpha) = (3 + \mathbf{i}, 0.5 - 2\mathbf{i}, 2 + 0.1\mathbf{i}, 1 + \mathbf{i}, 0.8, -0.8, 0, 0, 0, 1)$. (a) $t = -30$; (b) $t = 0$; (c) $t = 30$. The parameter value of s is 0.

k_j, l_j $(j = 1, 2, 3, 4, 5)$ are arbitrary non-zero constants. When the parameters k_j, l_j $(j = 1, 2, 3, 4)$ meet the conditions $k_2 = k_1^*$, $l_2 = l_1^*$, $k_3 = k_4^*$, $l_3 = l_4^*$, $\theta_2 = \theta_1^*$, $\theta_3 = \theta_4^*$, and inserting the solution (2.31) into (2.2), we can derive the non-singular mixture solution consisting of a kink soliton and two lump-type solitons. This solution describes the interaction between a kink solitons and two lump-type soliton solutions, see figure 7. As can be seen from figure 7, two different types of interactions are contained in the process of interaction. One is the interaction of two lump-type solitons; the other is the interaction between lump-type soliton and line soliton. In the process of interaction, the lump-type soliton exchanges the energy with line soliton. In the region of interaction, four lump-type solitons are presented, see figure 7(b). Throughout the process, we find that the interaction between two lump-type soliton and one line soliton is completely elastic. Similarly, we can also investigate the interaction between high-order lump-type soliton and high-order line soliton.

In order to derive the higher-order mixture solution consisting of $(N - 2n)$-kink soliton and n-order lump-type soliton, we consider the N-kink soliton solution with $\theta_j \to p_j\theta_j + \pi\mathbf{i}$ and $(k_j, l_j) \to p_j(k_j, l_j)(j = 1, 2,\cdots,2n)$, where $1 < 2n < N$. From the Taylor series generated by the function $F_N(x, y, t; s)$ at the point $(p_1, p_2, \cdots, p_{2n}) = (0, 0, \cdots,0)$, the higher order mixture solution can be obtained

$$\begin{aligned} &F_{n(N-2n)}(\Theta_1,\cdots,\Theta_{2n}, \xi_{2n+1},\cdots,\xi_N) \\ &= \sum_{\sigma=0,1} F_{nL}(\Theta_1 - \sum_{j=2n+1}^{N} \sigma_j\theta_{1j},\cdots,\Theta_{2n} \\ &- \sum_{j=2n+1}^{N} \sigma_j\theta_{2nj})\exp\left(\sum_{j=2n+1}^{N} \sigma_j\xi_j + \sum_{j<l}^{N}\sigma_j\sigma_l\delta_{jl}\right), \end{aligned} \quad (2.33)$$

where $\xi_{2n+1},\cdots,\xi_N$ are determined by (2.4) and (2.5), $F_{nL}(\Theta_1,\cdots,\Theta_{2n})$ is defined by (2.22), and

$$\theta_{ij} = \frac{4k_i^3k_j^3}{k_i^2k_j^4 + \det(k, l)_{ij}^2}, \; i = 1, 2,\cdots,2n, j = 2n + 1,\cdots,N, \quad (2.34)$$

k_j, l_j $(j = 1, 2, \cdots, N)$ are arbitrary non-zero constants. The summation notations in (2.33) are similar to those in (2.4), they have the same meaning. When the parameters k_j, l_j $(j = 1, 2, \cdots, 2n)$ meet the conditions $k_{n+i} = k_i^*$, $l_{n+i} = l_i^*$, $\theta_{n+i} = \theta_i^*$, $i = 1, 2,\cdots,n$, and inserting the solution (2.33) into (2.2), we can derive the nonsingular higher-order mixture solution consisting of $(N - 2n)$-kink soliton and n-order lump-type soliton. This solution describes the interaction between $(N - 2n)$-kink soliton and n-order lump-type soliton.

3. Conclusion

In this work, we have derived and investigated various kinds of high-order solitons of the BKP equation by employing the bilinear representation, perturbation method and Taylor expansion approach, including the N-kink soliton, n-order lump-type soliton and mixture solution consisting of kink soliton and lump-type soliton. The dynamical behaviors of these high-order soliton solutions are investigated and displayed analytically and graphically, see figures 1–7. Interaction phenomenon of various kinds of high-order solitons are shown graphically in figures 2–7. Collisions between two kink solitons can be divided into two classes by the determinant $\det(k, l)_{ij}$: if $\det(k, l)_{ij} = 0$, the interaction between two kink solitons is parallel; if $\det(k, l)_{ij} \neq 0$, the interaction between two kink solitons is oblique, see figure 2. Collisions between kink soliton and lump-type soliton are shown in figures 5–7. From figures 2–7, we can observe that these interactions are elastic. Furthermore, we find that the collision between kink soliton and lump-type soliton can give rise to a high-order lump-type soliton. In the previous literature [37–43], the researchers investigate also the collision between kink soliton and lump-type soliton. The obtained results show that when kink soliton and lump-type soliton collide, lump-type soliton is completely absorbed by kink soliton, the interaction between them can not excite a lump-type soliton. Compared with our results, three kinds of high-order solitons are derived, some novel characteristic of interaction between kink soliton and lump-type soliton are presented. In particular, when kink soliton and lump-type soliton collide, two soliton do not produce the superposition of amplitudes. At the moment of collision, we can observe a higher-order lump-type soliton, see figure 5(b). It is hoped that these results derived in this work will provide some valuable information for understanding the dynamic behaviors of nonlinear waves.

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