



On the Inference of a Star’s Inclination Angle from its Rotation Velocity and Projected Rotation Velocity

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Received 2019 October 22; revised 2019 December 12; accepted 2019 December 23; published 2020 February 3

Abstract

It is possible to learn about the orientation of a star’s rotation axis by combining measurements of the star’s rotation velocity (v) and its projection onto our line of sight ($v \sin i$). This idea has found many applications, including the investigation of the obliquities of stars with transiting planets. Here, we present a method for the probabilistic inference of the inclination of the star’s rotation axis based on independent data sets that constrain v and $v \sin i$. We also correct several errors and misconceptions that appear in the literature.

Unified Astronomy Thesaurus concepts: [Exoplanet astronomy \(486\)](#); [Stellar rotation \(1629\)](#); [Astrostatistics \(1882\)](#)

1. Introduction

When studying a single unresolved star, the inclination of the star’s spin axis relative to the line of sight is often irrelevant. There are exceptions, though, in which the star’s orientation has a practical or physical significance. From a practical point of view, constraints on the stellar inclination can help to interpret interferometric observations of the stellar disk (see, e.g., Monnier et al. 2012). Knowledge of the inclination is also useful for interpreting photometric or spectroscopic variability due to starspots, by breaking some of the usual modeling degeneracies (Walkowicz et al. 2013). From a physical point of view, measuring the inclination angles of large samples of stars can be used to test for large-scale correlations in a spin orientation (Struve 1945; Abt 2001; Corsaro et al. 2017; Kuszewicz et al. 2019).

The inclination of a star is also of direct physical significance when it has a disk or an orbiting companion. In such cases, the stellar spin orientation is a fundamental geometric property that may relate to the formation and evolution of the system. Investigations have been undertaken to assess the angle between the stellar rotation axis and the orbital and spin axis of a stellar companion (Hale 1994; Albrecht et al. 2009), the plane of a surrounding disk (Watson et al. 2011; Greaves et al. 2014), and the orbital plane of a planetary companion (Schlaufman 2010). The application to planets has been especially productive because of the large sample of transiting planets that has recently become available. The orbit of a transiting planet always has an inclination close to 90° . Therefore, any constraint on the stellar inclination is also a constraint on the stellar obliquity.

A commonly used method for probing the stellar inclination angle is to combine measurements or estimates of three quantities: the stellar radius, R , the stellar rotation period, P , and the line-of-sight projection of the stellar rotation velocity, $v \sin i$. The radius might be based on the observed spectroscopic parameters, the Stefan–Boltzmann law, or eclipse observations. The rotation period comes from the observation of periodic photometric variability, or from empirical relations between rotation and activity indicators. The projected rotation velocity can be determined from the Doppler-rotational broadening of the star’s absorption lines. Neglecting the effects of

differential rotation, the inclination can be calculated as

$$i = \sin^{-1} \left(\frac{v \sin i}{v} \right) = \sin^{-1} \left(\frac{v \sin i}{2\pi R/P} \right). \quad (1)$$

This idea dates back at least as far as Abt et al. (1972), Doyle et al. (1984), and Campbell & Garrison (1985). The latter authors had in mind the Doppler technique for exoplanet detection, which does not permit the measurement of a planet’s mass m , but rather $m \sin i$. But if one is willing to assume that the planet’s orbital axis and the star’s spin axis are aligned, then any constraint on the stellar inclination can be used in conjunction with Doppler data to determine the planet’s mass, or place an upper limit stringent enough to confirm that the unseen companion is a planet. This was part of the evidence presented by Mayor & Queloz (1995) that 51 Peg b is a planet rather than a brown dwarf, although we now know that spin–orbit alignment is not guaranteed (see, e.g., Winn & Fabrycky 2015; Trianaud 2018).

The subject of this paper is the probabilistic inference of the stellar inclination angle for the case in which the measurement uncertainties cannot be neglected and may have nontrivial distributions. This is a common situation when dealing with Sun-like stars, for which $v \sin i$ is only a few kilometers per second, and the effects of rotation on the stellar line profiles are comparable to those of turbulence and instrumental broadening. Furthermore, for any type of star, the inference of the stellar radius can lead to asymmetric uncertainty intervals because of the nonlinear relationships between the observed spectroscopic parameters and the outputs of theoretical stellar-evolutionary models. The rotation period can be determined precisely from a periodic pattern of photometric variations. However, it is subject to large uncertainties if only a portion of a rotation cycle is observed, or if the rotation period must be estimated less directly via activity indicators.

The statistical inference of inclination angles has been the goal of many studies, including those cited earlier in this paper. We have come to realize that many of the methods described in the literature suffer from errors. In fact, we have not found any cases that directly dealt with the probability distribution for i in which the analysis was 100% correct. Some investigators performed simple “quadrature sum” error propagation with

reference to Equation (1) assuming independent Gaussian uncertainties for $v \sin i$, R , and P . This is not correct when the uncertainties are large. It is also wrong because $v \sin i$ and $v \equiv 2\pi R/P$ are not statistically independent, i.e., knowledge of v provides some information about $v \sin i$, and vice versa. Many investigators who have performed Bayesian analyses have made the mistaken assumption that $v \sin i$ and v are statistically independent. In this paper, we present a mathematically correct procedure based on reasonable assumptions, and explain why some of the procedures described in the literature are incorrect.

2. Assumptions

We consider a situation in which we have two data sets, d_v and d_u , from which we have computed the likelihood functions for v and $u \equiv v \sin i$:

$$\mathcal{L}_v(v) \equiv p(d_v | v), \quad \mathcal{L}_u(u) \equiv p(d_u | u). \quad (2)$$

In practice, the constraint on v usually comes from multiple data sets, for example, a photometric time series from which the rotation period can be measured, and the parallax and spectral energy distribution from which the star's radius can be determined. These data can be fitted simultaneously to obtain the likelihood function for $v = 2\pi R/P$.³ For simplicity, we denote the combined data set by d_v , and assume that all the relevant information is incorporated into the likelihood function $\mathcal{L}_v(v)$.

Likewise, we denote by d_u the data set that constrains u . The type of data we have in mind is a high-resolution spectrum that shows the effects of Doppler-rotational broadening on the star's spectral lines. The amount of broadening depends only on the projected rotation rate and not the sense of rotation, i.e., it does not matter whether the north pole or the south pole is on the star's visible hemisphere. Consequently, the data do not allow i and $180^\circ - i$ to be distinguished. For this reason, we restrict our attention to values of i within the range from 0° to 90° . The values of $\sin i$ and $\cos i$ are also between 0 and 1.

Our goal is to combine the information on v and u to compute the marginal likelihood and posterior probability density function (PDF) for $\cos i$. We deal with $\cos i$ rather than i or $\sin i$ because the PDF for $\cos i$ reduces to a constant when the orientation of the spin axis is completely random.

We also make the following assumptions.

- (I) The data sets d_v and d_u are independent. By this, we mean that if the totality of data is $D = \{d_v, d_u\}$, then the likelihood function for the D is separable:

$$\begin{aligned} \mathcal{L}_{vu}(v, u) &= p(D | v, u) = p(d_v | v, u) p(d_u | v, u) \\ &= p(d_v | v) p(d_u | u) = \mathcal{L}_v(v) \mathcal{L}_u(u). \end{aligned} \quad (3)$$

- (II) The quantities v and i are a priori independent, i.e., the prior PDF $\mathcal{P}_{vi}(v, i)$ for these parameters is separable:

$$\mathcal{P}_{vi}(v, i) = \mathcal{P}_v(v) \mathcal{P}_i(i), \quad (4)$$

or equivalently,

$$p(v | i) = \mathcal{P}_v(v) \text{ and } p(i | v) = \mathcal{P}_i(i). \quad (5)$$

Assumption II is not trivial, and there are circumstances in which it would be false. For example, if there are populations of stars within which the stars tend to be spin-aligned, and the populations have systematically different rotation velocities (because of differences in age), then there would exist an overall correlation between v and i . A correlation might also arise from an observational bias: a sample of stars with measured rotation periods may be biased against low-inclination systems, because the amplitude of the photometric variability associated with rotation tends toward zero as the star approaches a pole-on orientation. As a result, in a signal-to-noise limited sample, the stars with the weakest variability—which tend to be the slower rotators—might preferentially have high inclinations. For this paper, though, we neglect any such effects.

Even under assumption II, we note that v and u are in general prior dependent: $\mathcal{P}_{vu}(v, u) \neq \mathcal{P}_v(v) \mathcal{P}_u(u)$ and $p(u | v) \neq \mathcal{P}_u(u)$. Ignoring this dependence turns out to be the main source of errors in the literature. As a demonstration that u and v are often prior dependent, let us consider a mathematically simple case in which \mathcal{P}_v is uniform between 0 and v_{\max} , and all possible orientations for the spin axis are equally probable,

$$\mathcal{P}_{\cos i}(\cos i) = 1, \quad \mathcal{P}_{\sin i}(\sin i) = \frac{\sin i}{\sqrt{1 - \sin^2 i}}. \quad (6)$$

First,

$$\begin{aligned} p(u | v) &= p(v \sin i | v) = \frac{1}{v} p(\sin i | v) \\ &= \frac{1}{v} \mathcal{P}_{\sin i}(\sin i) = \frac{1}{v} \frac{u/v}{\sqrt{1 - (u/v)^2}}, \end{aligned} \quad (7)$$

where the third equality follows from assumption II. The function $p(u | v)$ rises monotonically as u approaches v . This is a restatement of the familiar fact that low inclinations are less probable than high inclinations for random spin-axis orientations. On the other hand, the prior PDF for $u = v \sin i$ is

$$\begin{aligned} \mathcal{P}_u(u) &= \int_{\frac{u}{v_{\max}}}^1 \frac{1}{s} \mathcal{P}_v\left(\frac{u}{s}\right) \mathcal{P}_{\sin i}(s) ds = \frac{1}{v_{\max}} \int_{\frac{u}{v_{\max}}}^1 \frac{ds}{\sqrt{1 - s^2}} \\ &= \frac{1}{v_{\max}} \left[\frac{\pi}{2} - \arcsin\left(\frac{u}{v_{\max}}\right) \right], \end{aligned} \quad (8)$$

where the first equality relies on assumption II. This is not the same as $\mathcal{P}_u(u)$ is nearly constant for $u \ll v_{\max}$, and decreases as u approaches v_{\max} .

3. Correct Procedure

From the likelihood function for the whole data set, $p(D | v, \cos i)$, we calculate the marginal likelihood for $\cos i$ as

$$\begin{aligned} p(D | \cos i) &= \int p(D | v, \cos i) p(v | \cos i) dv \\ &= \int \mathcal{L}_v(v) \mathcal{L}_u(u) p(v | \cos i) dv \\ &= \int \mathcal{L}_v(v) \mathcal{L}_u(v \sqrt{1 - \cos^2 i}) \mathcal{P}_v(v) dv. \end{aligned} \quad (9)$$

Here, the second equality relies on assumption I, and the last equality relies on assumption II that $p(v | \cos i)$ is equal to the prior PDF for v .

³ The identification of v with $2\pi R/P$ ignores the effects of latitudinal differential rotation, which would otherwise require us to distinguish between the equatorial rotation period and the rotation period of whichever features are producing the photometric variability.

We can obtain the (unnormalized) posterior PDF for $\cos i$ using Bayes' theorem

$$p(\cos i | D) \propto \mathcal{P}_{\cos i}(\cos i) \int \mathcal{L}_v(v) \mathcal{L}_u(v\sqrt{1 - \cos^2 i}) \mathcal{P}_v(v) dv. \quad (10)$$

Using $v = u/\sqrt{1 - \cos^2 i}$, we may also write this as an integral over u :

$$p(\cos i | D) \propto \frac{\mathcal{P}_{\cos i}(\cos i)}{\sqrt{1 - \cos^2 i}} \int \mathcal{L}_v\left(\frac{u}{\sqrt{1 - \cos^2 i}}\right) \times \mathcal{L}_u(u) \mathcal{P}_v\left(\frac{u}{\sqrt{1 - \cos^2 i}}\right) du. \quad (11)$$

Given uninformative measurements of v and $v \sin i$, these formulas should reduce to the prior PDF for $\cos i$. Indeed, if $\mathcal{L}_u(u)$ is a constant for the values of $u = v\sqrt{1 - \cos^2 i} \leq v$ for which both $\mathcal{L}_v(v)$ and $\mathcal{P}_v(v)$ have non-zero values, the integral in Equation (10) does not depend on $\cos i$ and therefore $p(\cos i | D) \propto \mathcal{P}_{\cos i}(\cos i)$.

4. Incorrect Procedures

Here we describe some incorrect procedures that have appeared in the literature, and explain why they are incorrect. In short, these errors stem from not taking into account the dependence of different variables.

4.1. Incorrect Monte Carlo Sampling

It is *incorrect* to calculate $p(\cos i | D)$ by sampling v and $v \sin i$ independently from their respective PDFs and constructing the resulting distribution of $\cos i = \sqrt{1 - (v \sin i / v)^2}$. This is because v and $v \sin i$ are not statistically independent, as noted in Section 2. For example, it is always the case that $v \sin i \leq v$. Also, if we assume the isotropic priors represented in Equation (6), then $v \sin i$ is more likely to be closer to v than to 0. Independently sampling from the PDFs of v and $v \sin i$ does not take into account these correlations.

This type of mistake has been made frequently (see, e.g., Henry & Winn 2008). Two particular studies deserve mention. Morton & Winn (2014) noted that this procedure is incorrect but did not supply the reason, and then proceeded to use a different incorrect method (see Section 4.3). Likewise, Hirano et al. (2014) used a correct formula for $p(i | D)$ in one part of their study to display the PDFs of i (their Equation (10)), but then went on to perform an incorrect Monte Carlo sampling (their Section 4.3).

4.2. Incorrect Use of the Product Rule

It is *incorrect* to compute the PDF of $\sin i = v \sin i / v$ applying the well-known product rule for $z = xy$,

$$p_z(z) = \int p_x(x) p_y\left(\frac{z}{x}\right) \frac{1}{|x|} dx, \quad (12)$$

where p_x and p_y are the PDFs of x and y . This is because the preceding formula is derived under the assumption that x and y are statistically independent variables, which is not the case for v and u (Section 2). When they are dependent, the correct

formula is

$$p_z(z) = \int p_x(x) p_y\left(\frac{z}{x} \middle| x\right) \frac{1}{|x|} dx. \quad (13)$$

For $x = v^{-1}$ and $y = v \sin i$, Equation (13) yields

$$p(\sin i | D) = \int p(v^{-1} | D) p(\sin i / v^{-1} | v^{-1}, D) \frac{1}{v^{-1}} dv^{-1} \\ = \int p(\sin i | v, D) p(v | D) dv, \quad (14)$$

which is simply the marginal PDF for $\sin i$. Application of Bayes' theorem gives

$$p(\sin i | D) \propto \int p(\sin i | v, d_v, d_u) \mathcal{L}_v(v) \mathcal{P}_v(v) dv \\ \propto \int p(d_u | v, d_v, \sin i) \mathcal{P}_{\sin i}(\sin i) \mathcal{L}_v(v) \mathcal{P}_v(v) dv \\ \propto \mathcal{P}_{\sin i}(\sin i) \int \mathcal{L}_v(v) \mathcal{L}_u(u) \mathcal{P}_v(v) dv. \quad (15)$$

Thus, we obtain

$$p(\cos i | D) = p(\sin i | D) \left| \frac{d \sin i}{d \cos i} \right| \\ \propto \mathcal{P}_{\cos i}(\cos i) \int \mathcal{L}_v(v) \mathcal{L}_u(u) \mathcal{P}_v(v) dv, \quad (16)$$

which recovers Equation (10).

The work by Muñoz & Perets (2018) included this type of error in combination with a second error. The first error was to confuse the likelihood function $p(D | \sin i)$ with the PDF $p(\sin i | D)$. The former is the probability to obtain the data D for a fixed value of $\sin i$, while the latter is the probability density for $\sin i$ per unit $\sin i$. The second error was to apply the product rule incorrectly, as described in this section, to compute $p(\sin i | D)$. The effects of these two errors partly canceled each other out, as we will show in Section 5. Had they made only the single error of applying the incorrect product rule to calculate $p(\sin i | D)$, and then correctly transformed the PDF to obtain $p(\cos i | D)$, they would have arrived at the incorrect Monte Carlo result described in the previous section. This reflects the fact that using the product rule given by Equation (12) is equivalent to ignoring the statistical dependence of v and $v \sin i$.

4.3. Incorrect Marginalization

It is correct to write down the joint likelihood function for the entire data set as

$$p(D | u, \cos i) = \mathcal{L}_v\left(\frac{u}{\sqrt{1 - \cos^2 i}}\right) \mathcal{L}_u(u). \quad (17)$$

However, it is *incorrect* to derive the marginal likelihood as

$$(\text{incorrect}) \quad p(D | \cos i) = \int p(D | u, \cos i) \mathcal{P}_u(u) du \\ = \int \mathcal{L}_v\left(\frac{u}{\sqrt{1 - \cos^2 i}}\right) \mathcal{L}_u(u) \mathcal{P}_u(u) du \quad (18)$$

because $u = v \sin i$ and $\cos i$ are not statistically independent as described in Section 2. This incorrect formula was used by

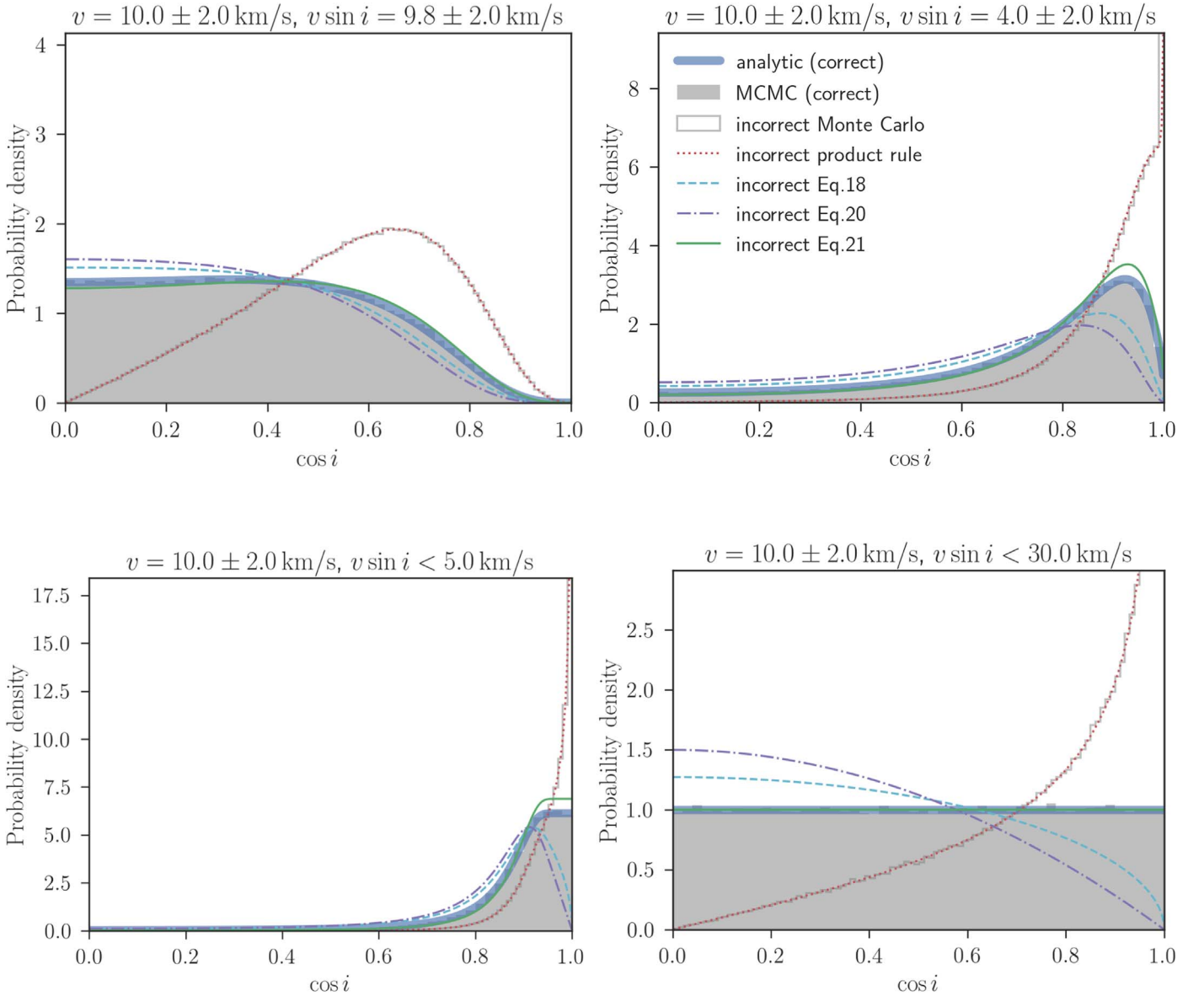


Figure 1. Comparison of different methods to derive the posterior probability density function for $\cos i$.

Morton & Winn (2014). The correct marginal likelihood is

$$\begin{aligned}
 p(D | \cos i) &= \int p(D | u, \cos i) p(u | \cos i) du \\
 &= \int p(D | u, \cos i) \frac{1}{\sin i} \mathcal{P}_v(v) du = \int \mathcal{L}_v(v) \mathcal{L}_u(u) \mathcal{P}_v(v) dv,
 \end{aligned}
 \tag{19}$$

which reproduces Equation (9).

5. Examples

We have calculated $p(\cos i | D)$ for four illustrative cases, both by evaluating Equation (10) directly, and by performing a Markov Chain Monte Carlo (MCMC) sampling of the posterior assuming the likelihood given by Equation (3).⁴ Throughout

⁴ We note that the MCMC method is not needed to solve this two-dimensional problem: it is more efficient to integrate Equation (9) directly. The MCMC method might be worth the computational effort if one deals directly with the respective likelihood functions for $v \sin i$, R , and P .

this section, we adopt uniform, normalized, and separable prior PDFs for v and $\cos i$. The four panels of Figure 1 show the results. The top two panels show cases in which \mathcal{L}_v and \mathcal{L}_u are Gaussian functions specified by the central value and dispersion. The bottom two panels show cases in which \mathcal{L}_u is non-zero and flat below a certain threshold value, and zero otherwise. This is meant to be a mathematically simplified model of a situation in which only an upper limit on $v \sin i$ is available. In reality, the likelihood function \mathcal{L}_u would need to be determined from the data.

In all of these cases, the distribution of the MCMC samples (filled gray histograms) agrees with the output of Equation (10; thick blue lines), as expected. The correct formula recovers the uninformative PDF for $\cos i$ when the $v \sin i$ measurement has no constraining power, as shown in the bottom right panel. The panels also show the results of some of the incorrect procedures that have been used in the literature. The light gray unfilled histogram is based on Monte Carlo sampling under the faulty assumption that v and

$v \sin i$ are independent (Section 4.1).⁵ The dotted red line is based on the incorrect application of the product rule for PDFs (Section 4.2), which is equivalent to the incorrect Monte Carlo sampling. The dashed blue line was calculated via incorrect marginalization (Equation (18)) after multiplying by $\mathcal{P}_{\cos i}$ to convert it into a PDF. The dotted-dashed purple line shows the application of the incorrect formula

$$(\text{incorrect}) \quad p(\cos i|D) \propto \int u \mathcal{L}_u(u) \mathcal{L}_v\left(\frac{u}{\sqrt{1 - \cos^2 i}}\right) du, \quad (20)$$

based on Equation (4) of Winn et al. (2017) after multiplying by $\mathcal{P}_{\cos i}$, and also fixing a typographical error that led to the subscripts 1 and 2 being swapped. The solid green line shows the application of the incorrect formula

$$(\text{incorrect}) \quad p(\cos i|D) \propto \int v \mathcal{L}_u(v\sqrt{1 - \cos^2 i}) \mathcal{L}_v(v) dv, \quad (21)$$

from Muñoz & Perets (2018) as discussed in Section 4.2.

In general, both the incorrect Monte Carlo sampling and the incorrect application of the product rule lead to a severe overestimation of the probability density for $\cos i \approx 1$ and an underestimation of the probability density for $\cos i \approx 0$. Incorrect marginalization tends to overestimate the probability density at smaller values of $\cos i$ and significantly underestimate the density around larger values of $\cos i$. The formula by Muñoz & Perets (2018) is fairly accurate due to the partial cancellation of errors mentioned previously, but it does tend to overestimate the probability density for $\cos i \approx 1$.

6. Summary

We have described a procedure to combine measurements of v and $v \sin i$ to derive the marginal likelihood and posterior PDF for $\cos i$, and have explained why various formulas presented in the literature are incorrect. The errors originate from the incorrect treatment for PDFs of the variables that are not statistically independent from each other. We have also compared the resulting PDFs from incorrect formulas to the

correct PDF for several simplified models of v and $v \sin i$ measurements.

Based on Figure 1, we suspect that correcting the errors made in previous studies using the methods in Section 4 would not change the qualitative conclusions of those studies, although it would make quantitative differences. Regardless of the size of the error, it is always preferred to use the correct formula if the marginal likelihood or PDF for i is to be computed analytically.

We appreciate the referee's constructive comments which helped to improve the clarity of this paper. K.M. gratefully acknowledges the support by the W. M. Keck Foundation Fund. Work by J.N.W. was partly supported by a NASA Keck PI data award (contract NNN12AA01C).

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⁵ For this simulation, we discarded the samples that imply $\sin i \geq 1$. In some prior works, such samples were adjusted to give $\sin i = 1$, producing a sharp peak at $\cos i = 0$.