Emergent metric and geodesic analysis in cosmological solutions of (torsion-free) polynomial affine gravity

Oscar Castillo-Felisola[1,2,4], **José Perdiguero**[1],
Oscar Orellana[3] **and Alfonso R Zerwekh**[1,2]

[1] Departamento de Física, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile
[2] Centro Científico Tecnológico de Valparaíso, Casilla 110-V, Valparaí so, Chile
[3] Departamento de Matemáticas, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile

E-mail: o.castillo.felisola@gmail.com

CrossMark

Abstract

Starting from an affinely connected space, we consider a model of gravity whose fundamental field is the connection. We build up the action using as sole premise the invariance under diffeomorphisms, and study the consequences of a cosmological ansatz for the affine connection in the torsion-free sector. Although the model is built without requiring a metric, we show that the nondegenerated Ricci curvature of the affine connection can be interpreted as an *emergent* metric on the manifold. We show that there exists a parametrization in which the (r, φ)-restriction of the geodesics coincides with that of the Friedman–Robertson–Walker model. Additionally, for connections with nondegenerated Ricci we are able to distinguish between space-, time- and null-like self-parallel curves, providing a way to differentiate *trajectories* of massive and massless particles.

Keywords: affine connection, affine gravity, self-parallel curves, geodesics, cosmological models

(Some figures may appear in colour only in the online journal)

1. Introduction

General relativity was proposed by Einstein as an attempt to compatibilise the gravitational interactions with the postulates of special relativity [1–5]. The ground-breaking idea behind

[4] Author to whom any correspondence should be addressed.

the proposal was the interpretation of the gravitational interaction as the effect of properties of the spacetime, represented by a nontrivial geometry. The spacetime is modelled by a Riemannian manifold, whose geometric properties are determined by the metric tensor, and therefore it is the natural field describing the dynamics of the spacetime. The properties of the *matter* distribution are encoded in the energy–momentum tensor. Einstein's field equations are the extrema of the Einstein–Hilbert action when varied with respect to the metric.

General relativity has been tested extensively with magnificent agreement with the experimental data, as one can appreciate in the excellent review [6]. The most recent triumph of the theory was the direct measurement of gravitational waves by the LIGO-Virgo collaborations [7, 8].

Although general relativity is, by far, the most successful theory of gravitational interactions, there is an increasing interest in alternative models of gravity, particularly driven for the lack of a complete framework of quantum gravity [9–15], and the necessity of hypothesising a dark sector that accounts for approximately 96% of the energy content of the Universe [16–20].

The existence of these problems is a signal of *new physics*, and their solutions require either including new fields or changing the gravitational theory. The latter suggests that Einstein's theory is an effective theory of gravity and, therefore, one may consider alternative models. Among the generalisations one encounters for example: the Einstein–Cartan theory, which extends general relativity by allowing a non-symmetric connection, but considers the same action [21–24]; models with extra dimensions, firstly proposed by Kaluza and Klein [25, 26]; Lovelock models, which are build under the same premises than General Relativity, but in any dimension [27]; the metric-affine models, in which the conditions of metricity and vanishing torsion are generally dropped [28]; the Lovelock–Cartan model, which are the extension of Lovelock models with a non-symmetric connection [29]; and many others.

Inspired by the fact that fundamental interactions (other than gravity) are described by gauge theories whose dynamical field is a connection, seems reasonable to search for a model of gravity described solely by a connection. The first affine model of gravity was proposed by Eddington, who considered an action defined by the square root of the Ricci tensor [30] (See also [31]). Moderns attempts to describe gravity as a theory for the affine connection have been proposed in [32–40].

Recently, a novel model has been proposed, dubbed polynomial affine gravity, which is built out with polynomial terms of the irreducible components of the connection [41–44], assuming invariance under the group of diffeomorphims and no explicit use of a metric, i.e. even when the spacetime is metric this field plays no role in the mediation of gravitational interactions[5]. The action of polynomial affine gravity has very interesting features: (i) It is power-counting renormalisable[6]; (ii) No other term can be added, i.e. all vacuum solutions of general relativity are solution of polynomial affine model of Gravity, and particularly it is not possible to add counter-terms; (iii) All the couplings are dimensionless, suggesting that the model is a conformal theory at tree level; (iv) The torsion-free sector is a consistent truncation compatible with general relativity, i.e. all vacuum solutions of general relativity are solution of polynomial affine model of gravity, and particularly it passes the classical test of gravity; and (v) The structure of the model yields no three-point graviton vertices, which might allow to overcome the no-go theorems found in [45, 46][7].

---

[5] In general relativity the metric plays a double role: it is the instrument that serves to measure distances, and also it is the field that mediates gravity. The idea behind metric-affine models is that those roles are played by different fields, but both fields are dynamic. In our construction, we manage to build a model of gravity without the need of an instrument that measures distances.

[6] We highlight that this is a necessary but not sufficient condition for the model to be renormalisable.

[7] Regarding the aforementioned no-go theorem, a similar phenomenon happens in the case of a massive spin-1 field coupled to a non-Abelian gauge field. A necessary condition for making the theory consistent with perturbative unitarity is the absence of a three-point vetex for the massive field [47].

There are several conceptual subtleties one has to rethink about when working with an affine model. The most recurrent question is: How do we measure distances if the model lacks a metric? In this paper we aim to broaden current knowledge on those subtleties. With this in mind, in section 2 we briefly review the polynomial affine model of gravity, and reformulate the model in terms of geometrical objects with simpler interpretation. In addition, we argue that the limit of zero torsion is well defined, i.e. it is a consistent *truncation* of the model, and show that the field equations on this sector are a known generalisation of the Einstein field equations. Then, in section 3 we proceed—by restricting ourselves to the torsion-free sector—to find cosmological solutions of the field equations [43], that extend the results reported in [44]. In the absence of a metric there is no concept of geodesic, however the concept of self-parallel curve is still valid. Assuming that the trajectories of *free falling* test particles are self-parallel curves, we analyse them in section 4 and show that, there exist a parametrisation in which the (r, φ)-restriction of the equations is nothing but the expected from the Friedman–Robertson–Walker model. At this point, the obstruction is that without a metric it is not possible to differentiate between trajectories of massive and massless particles. In section 5 we show that under certain conditions the Ricci tensor is a well-behaved (emergent) metric[8], allowing to define *space-like*, *time-like* or *light-like* vectors, and providing the arena for an affine definition of Einstein manifolds. Then, in section 6 we mention how some basic cosmological quantities are defined in terms of the parametric functions of the connection. In section 7 we conclude with a discussion of the results. For the sake of completeness we include some appendixes. In appendix A we review the dubbed *dimensional analysis* that allowed us to build the action. In appendix B we show the explicit contribution of each term of the action to the field equations.

2. The model of polynomial affine gravity

The polynomial affine gravity is an alternative theory of gravitation, whose sole fundamental field is the affine connection, $\hat{\Gamma}$. Notice that without the use of a metric, one calculates the curvature and the Ricci tensors, but not the curvature scalar. Therefore, it is not possible to write an equivalent action to the Einstein–Hilbert action.

Our goal is to build up the most general action which is invariant under the group of diffeomorphims. Firstly, one could try to use the connection as a whole, however this yields no interesting models, since the obtained terms are topological invariants[9],

$$S[\hat{\Gamma}] = \int \left(a_1 \hat{\mathcal{R}}_{\mu\nu}{}^\alpha{}_\beta \hat{\mathcal{R}}_{\lambda\rho}{}^\beta{}_\alpha + a_2 \hat{\mathcal{R}}_{\mu\nu}{}^\alpha{}_\alpha \hat{\mathcal{R}}_{\lambda\rho}{}^\beta{}_\beta \right) \mathrm{d}V^{\mu\nu\lambda\rho}, \tag{1}$$

which are the four-dimensional Pontryagin density and the product of (generalised) two-dimensional Pontryagin densities. In the above equation we have introduced the *natural* volume form, defined as the wedge product of the coordinates, i.e. $\mathrm{d}V^{\alpha\beta\gamma\delta} = J(x)\, \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \wedge \mathrm{d}x^\delta$ for an arbitrary nonvanishing—within a chart—function $J(x)$.

The irreducible components of the connection, used to build the action, are defined as[10]

---

[8] We prefer to call this metric 'emergent', because it is a derived instead of a fundamental geometrical object.

[9] All other possible terms are related to these, up to boundary terms.

[10] In a model with a metric, the connection can always be decomposed into the Levi-Cività component and the distorsion. The upper index of the later can be lowered with the metric, and since the distorsion is a tensor, it decomposes according with the Young projection, i.e. it has a completely symmetric and antisymmetric parts plus a component with mixed symmetries. A detailed analysis of the irreducible components of the connection in the metric case can be found in [48, 49]. The central difference with our approach is that without the use of a metric, only the lower indices can be decomposed through Young projection, as shown in equation (2).

$$
\begin{aligned}
\hat{\Gamma}_\mu{}^\lambda{}_\nu &= \hat{\Gamma}_{(\mu}{}^\lambda{}_{\nu)} + \hat{\Gamma}_{[\mu}{}^\lambda{}_{\nu]} \\
&= \Gamma_\mu{}^\lambda{}_\nu + \epsilon_{\mu\nu\sigma\kappa}\mathcal{T}^{\lambda,\sigma\kappa} + \mathcal{A}_{[\mu}\delta^\lambda_{\nu]} \\
&= \Gamma_\mu{}^\lambda{}_\nu + \mathcal{B}_\mu{}^\lambda{}_\nu + \delta^\lambda_{[\mu}\mathcal{A}_{\nu]},
\end{aligned}
\tag{2}
$$

where we have first separated the symmetric and antisymmetric part in the lower indices (the latter is nothing but the torsion of the connection), in the second line we have renamed the symmetric part of the affine connection $\hat{\Gamma}_{(\mu}{}^\lambda{}_{\nu)} \equiv \Gamma_\mu{}^\lambda{}_\nu$ and provided a reparametrisation of the torsion in terms of its trace ($\mathcal{A}_\mu$) and a *dual* of a Curtright-like tensor[11], while in the third line the $\mathcal{B}$ field is just the traceless part of the torsion. All these elements transform as tensors under diffeomorphism, except for the symmetric part of the affine connection, $\Gamma_\mu{}^\lambda{}_\nu$, which consequently must be included in the action solely through the covariant derivative[12].

In order to build the most general action using the irreducible fields, $\Gamma_\mu{}^\lambda{}_\nu$, $\mathcal{B}_\mu{}^\lambda{}_\nu$ and $\mathcal{A}_\mu$, the strategy to write down the action is to define the most general scalar density, where the dynamics is given by the covariant derivative with respect to the symmetric part of the connection.

Using the second parametrisation in equation (2), the most general action is

$$
\begin{aligned}
S = \int \mathrm{d}V^{\alpha\beta\gamma\delta} &\Big[B_1 \mathcal{R}_{\mu\nu}{}^\mu{}_\rho \mathcal{B}_\alpha{}^\nu{}_\beta \mathcal{B}_\gamma{}^\rho{}_\delta + B_2 \mathcal{R}_{\alpha\beta}{}^\mu{}_\rho \mathcal{B}_\gamma{}^\nu{}_\delta \mathcal{B}_\mu{}^\rho{}_\nu + B_3 \mathcal{R}_{\mu\nu}{}^\mu{}_\alpha \mathcal{B}_\beta{}^\nu{}_\gamma \mathcal{A}_\delta \\
&+ B_4 \mathcal{R}_{\alpha\beta}{}^\sigma{}_\rho \mathcal{B}_\gamma{}^\rho{}_\delta \mathcal{A}_\sigma + B_5 \mathcal{R}_{\alpha\beta}{}^\rho{}_\rho \mathcal{B}_\gamma{}^\sigma{}_\delta \mathcal{A}_\sigma + C_1 \mathcal{R}_{\mu\alpha}{}^\mu{}_\nu \nabla_\beta \mathcal{B}_\gamma{}^\nu{}_\delta + C_2 \mathcal{R}_{\alpha\beta}{}^\rho{}_\rho \nabla_\sigma \mathcal{B}_\gamma{}^\sigma{}_\delta \\
&+ D_1 \mathcal{B}_\nu{}^\mu{}_\lambda \mathcal{B}_\mu{}^\nu{}_\alpha \nabla_\beta \mathcal{B}_\gamma{}^\lambda{}_\delta + D_2 \mathcal{B}_\alpha{}^\mu{}_\beta \mathcal{B}_\mu{}^\lambda{}_\nu \nabla_\lambda \mathcal{B}_\gamma{}^\nu{}_\delta + D_3 \mathcal{B}_\alpha{}^\mu{}_\nu \mathcal{B}_\beta{}^\lambda{}_\gamma \nabla_\lambda \mathcal{B}_\mu{}^\nu{}_\delta \\
&+ D_4 \mathcal{B}_\alpha{}^\lambda{}_\beta \mathcal{B}_\gamma{}^\sigma{}_\delta \nabla_\lambda \mathcal{A}_\sigma + D_5 \mathcal{B}_\alpha{}^\lambda{}_\beta \mathcal{A}_\sigma \nabla_\lambda \mathcal{B}_\gamma{}^\sigma{}_\delta + D_6 \mathcal{B}_\alpha{}^\lambda{}_\beta \mathcal{A}_\gamma \nabla_\lambda \mathcal{A}_\delta + D_7 \mathcal{B}_\alpha{}^\lambda{}_\beta \mathcal{A}_\lambda \nabla_\gamma \mathcal{A}_\delta \\
&+ E_1 \nabla_\rho \mathcal{B}_\alpha{}^\rho{}_\beta \nabla_\sigma \mathcal{B}_\gamma{}^\sigma{}_\delta + E_2 \nabla_\rho \mathcal{B}_\alpha{}^\rho{}_\beta \nabla_\gamma \mathcal{A}_\delta + F_1 \mathcal{B}_\alpha{}^\mu{}_\beta \mathcal{B}_\gamma{}^\sigma{}_\delta \mathcal{B}_\mu{}^\lambda{}_\rho \mathcal{B}_\sigma{}^\rho{}_\lambda \\
&+ F_2 \mathcal{B}_\alpha{}^\mu{}_\beta \mathcal{B}_\gamma{}^\nu{}_\lambda \mathcal{B}_\delta{}^\lambda{}_\rho \mathcal{B}_\mu{}^\rho{}_\nu + F_3 \mathcal{B}_\nu{}^\mu{}_\lambda \mathcal{B}_\mu{}^\nu{}_\alpha \mathcal{B}_\beta{}^\lambda{}_\gamma \mathcal{A}_\delta + F_4 \mathcal{B}_\alpha{}^\mu{}_\beta \mathcal{B}_\gamma{}^\nu{}_\delta \mathcal{A}_\mu \mathcal{A}_\nu \Big]
\end{aligned}
\tag{3}
$$

where the covariant derivation and the curvature are defined with respect to the symmetric connection, i.e. $\nabla = \nabla^\Gamma$ and $\mathcal{R} = \mathcal{R}^\Gamma$. The action is defined up to boundary and topological terms. Although the dropped terms are relevant when studying global aspects of the model, they do not contribute to the equations of motion. In order to write down the action we used a variation of the *dimensional analysis* method introduced in [42]. Details of the dimensional analysis are shown in appendix A.

Interestingly, all of the coupling constant are dimensionless, which from the view point of quantum field theory is desirable if one is interested in trying to quantise the model. In addition, as shown by the dimensional analysis in [42], there is a finite number of possible terms contributing to the action (once those ignored in equation (3) are included), which we interpret as a *rigidity* of the model, given than in the hypothetical scenario of quantisation of polynomial affine gravity all the counter-terms should have the form of terms already present in the original action.

2.1. Field equations

In what follows, we will obtain the field equations of the model using the standard variational principle. It is well-known that to ensure the well-posedness of the variational problem no

---

[11] This was the parametrisation of the decomposition utilised in [41–43].
[12] Notice that under similar requirements but in odd dimensions, there are Chern–Simons terms, in which the connection enters explicitly in the action. See for example the three-dimensional construction in [41].

second-class constraints should be present, or higher derivative terms might appear in the field equations. In general relativity, one needs to add the Gibbons–Hawking–York term to solve this problem. Although the absence of second-class constraints in polynomial affine gravity has not been proven yet, the structure of the action in equation (3) suggests that the variational problem is well-posed[13].

Under the assumption that no Gibbons–Hawking–York term is needed, and since the action contains up to first derivatives of the fields, the field equations are obtained through the Euler–Lagrange equations,

$$
\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \left(\partial_\mu \Gamma_\nu{}^\lambda{}_\rho \right)} \right) - \frac{\partial \mathscr{L}}{\partial \Gamma_\nu{}^\lambda{}_\rho} = 0,
$$
$$
\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \left(\partial_\mu \mathcal{B}_\nu{}^\lambda{}_\rho \right)} \right) - \frac{\partial \mathscr{L}}{\partial \mathcal{B}_\nu{}^\lambda{}_\rho} = 0,
$$
$$
\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \left(\partial_\mu \mathcal{A}_\nu \right)} \right) - \frac{\partial \mathscr{L}}{\partial \mathcal{A}_\nu} = 0.
\tag{4}
$$

A simple way of dealing with the field equations was introduced by Kijowski in [32], and we will reviewed in what follows[14].

In the following, we present an alternative form of writing the Euler–Lagrange equations presented above, in a way that the calculations are easier to follow. Nonetheless, the explicit calculations are given in appendix B.

2.1.1. Field equations for the symmetric connection. The canonically conjugated momenta of the connection are defined by

$$
\Pi_\Gamma{}^{\mu\nu}{}_\lambda{}^\rho = \frac{\partial \mathscr{L}}{\partial \left(\partial_\mu \Gamma_\nu{}^\lambda{}_\rho \right)} \equiv \frac{\partial \mathscr{L}}{\partial \Gamma_{\mu\nu}{}^\lambda{}_\rho},
\tag{5}
$$

and since the derivative of the connection appears only in the curvature tensor, it follows that

$$
\Pi_\Gamma{}^{\mu\nu}{}_\lambda{}^\rho = \frac{\partial \mathscr{L}}{\partial \mathcal{R}_{\alpha\beta}{}^\gamma{}_\delta} \frac{\partial \mathcal{R}_{\alpha\beta}{}^\gamma{}_\delta}{\partial \Gamma_{\mu\nu}{}^\lambda{}_\rho} \equiv z_\Gamma{}^{\alpha\beta}{}_\gamma{}^\delta \frac{\partial \mathcal{R}_{\alpha\beta}{}^\gamma{}_\delta}{\partial \Gamma_{\mu\nu}{}^\lambda{}_\rho}.
\tag{6}
$$

The last term in equation (6) can be calculated explicitly from the definition of the curvature tensor,

$$
\frac{\partial \mathcal{R}_{\alpha\beta}{}^\gamma{}_\delta}{\partial \Gamma_{\mu\nu}{}^\lambda{}_\rho} = 4 \, \delta^\gamma_\lambda \delta^\mu_{[\alpha} \delta^{(\nu}_{\beta]} \delta^{\rho)}_\delta,
\tag{7}
$$

from which it follows that

$$
\Pi_\Gamma{}^{\mu\nu}{}_\lambda{}^\rho = 2 z_\Gamma{}^{[\mu\nu]}{}_\lambda{}^\rho + 2 z_\Gamma{}^{[\mu\rho]}{}_\lambda{}^\nu.
\tag{8}
$$

The last equation implies that the canonical momenta satisfy the Jacobi–Bianchi identity,

$$
\Pi_\Gamma{}^{[\mu\nu}{}_\lambda{}^{\rho]} = 0.
\tag{9}
$$

---

[13] Analysis of affine analogues to the Gibbons–Hawking–York term can be found in [50–55].
[14] Our notation is inspired by that of the cited work, but it might differ in both symbols and signs. We advise to be careful if you would like to compare the results.

On the other hand, the second term in the Euler–Lagrange equations for Γ yields

$$\frac{\partial \mathscr{L}}{\partial \Gamma_{\nu}{}^{\lambda}{}_{\rho}} = \frac{\partial \mathscr{L}}{\partial \mathcal{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}} \frac{\partial \mathcal{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}}{\partial \Gamma_{\nu}{}^{\lambda}{}_{\rho}}. \tag{10}$$

Once again the last term can be calculated from the definition of curvature

$$\frac{\partial \mathcal{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}}{\partial \Gamma_{\nu}{}^{\lambda}{}_{\rho}} = 4 \left[\delta_{\lambda}^{\gamma} \delta_{[\alpha}^{(\nu} \Gamma_{\beta]}{}^{\rho)}{}_{\delta} + \delta_{\delta}^{\rho} \delta_{[\beta}^{(\nu} \Gamma_{\alpha]}{}^{\gamma)}{}_{\lambda} \right], \tag{11}$$

and then,

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial \Gamma_{\nu}{}^{\lambda}{}_{\rho}} &= 2 \left[z_{\Gamma}{}^{[\nu\beta]}{}_{\lambda}{}^{\delta} \Gamma_{\beta}{}^{\rho}{}_{\delta} + z_{\Gamma}{}^{[\rho\beta]}{}_{\lambda}{}^{\delta} \Gamma_{\beta}{}^{\nu}{}_{\delta} \right. \\
&\qquad \left. + z_{\Gamma}{}^{[\beta\nu]}{}_{\gamma}{}^{\rho} \Gamma_{\beta}{}^{\gamma}{}_{\lambda} + z_{\Gamma}{}^{[\beta\rho]}{}_{\gamma}{}^{\nu} \Gamma_{\beta}{}^{\gamma}{}_{\lambda} \right] \\
&= \frac{1}{2} \Pi_{\Gamma}{}^{\nu\beta}{}_{\lambda}{}^{\delta} \Gamma_{\beta}{}^{\rho}{}_{\delta} + \frac{1}{2} \Pi_{\Gamma}{}^{\rho\beta}{}_{\lambda}{}^{\delta} \Gamma_{\beta}{}^{\nu}{}_{\delta} + \Pi_{\Gamma}{}^{\beta\nu}{}_{\gamma}{}^{\rho} \Gamma_{\beta}{}^{\gamma}{}_{\lambda} \\
&= -\Pi_{\Gamma}{}^{\mu\nu}{}_{\lambda}{}^{\delta} \Gamma_{\mu}{}^{\rho}{}_{\delta} - \Pi_{\Gamma}{}^{\mu\rho}{}_{\lambda}{}^{\delta} \Gamma_{\mu}{}^{\nu}{}_{\delta} + \Pi_{\Gamma}{}^{\mu\nu}{}_{\gamma}{}^{\rho} \Gamma_{\mu}{}^{\gamma}{}_{\lambda}.
\end{aligned} \tag{12}$$

Therefore, the field equations for the symmetric part of the connection are

$$\nabla_{\mu} \Pi_{\Gamma}{}^{\beta\nu}{}_{\lambda}{}^{\rho} = \frac{\partial^{*} \mathscr{L}}{\partial \Gamma_{\nu}{}^{\lambda}{}_{\rho}}. \tag{13}$$

In this last equation, we have used the fact that the canonical momenta are densities, thus there are two term (which seems to be missing above) that cancel themselves[15]. The asterisk on the right-hand side of equation (13) denotes the partial derivative with respect to the connection that is not contained in the curvature tensor.

Notice that there are seven term in which the symmetric part of the connection enters through the curvature tensor, while it enters through the covariant derivative of the tensors $\mathcal{B}$ and $\mathcal{A}$ in eleven terms. Noteworthily, there are only two terms in which the symmetric part of the connection enters in both ways, these are the terms in the action with coupling constants C_1 and C_2. Furthermore, the terms in the action with couplings C_1 and C_2 are the only which are linear in either $\mathcal{B}$ and $\mathcal{A}$ fields. Hence, these are the terms which could possibly contribute to the field equations in the sector of vanishing torsion, i.e. $\mathcal{A} \to 0$ and $\mathcal{B} \to 0$.

2.1.2. Field equations for the $\mathcal{B}$ field. Using the relations

$$\frac{\partial \nabla_{\alpha} \mathcal{B}_{\beta}{}^{\gamma}{}_{\delta}}{\partial (\partial_{\mu} \mathcal{B}_{\nu}{}^{\lambda}{}_{\rho})} = 2 \delta_{\alpha}^{\mu} \delta_{\lambda}^{\gamma} \delta_{\beta}^{[\nu} \delta_{\delta}^{\rho]} \tag{14}$$

and

$$\frac{\partial \nabla_{\alpha} \mathcal{B}_{\beta}{}^{\gamma}{}_{\delta}}{\partial \mathcal{B}_{\nu}{}^{\lambda}{}_{\rho}} = -2 \Gamma_{\alpha}{}^{[\nu}{}_{\beta} \delta_{\delta}^{\rho]} \delta_{\lambda}^{\gamma} - 2 \Gamma_{\alpha}{}^{[\rho}{}_{\delta} \delta_{\beta}^{\nu]} \delta_{\lambda}^{\gamma} + 2 \Gamma_{\alpha}{}^{\gamma}{}_{\lambda} \delta_{\beta}^{[\nu} \delta_{\delta}^{\rho]} \tag{15}$$

---

[15] The covariant derivative of this tensor density is given by the expression,

$$\nabla_{\sigma} \Pi_{\Gamma}{}^{\mu\nu}{}_{\lambda}{}^{\rho} = \partial_{\sigma} \Pi_{\Gamma}{}^{\mu\nu}{}_{\lambda}{}^{\rho} + \Gamma_{\sigma}{}^{\mu}{}_{\tau} \Pi_{\Gamma}{}^{\tau\nu}{}_{\lambda}{}^{\rho} + \Gamma_{\sigma}{}^{\nu}{}_{\tau} \Pi_{\Gamma}{}^{\mu\tau}{}_{\lambda}{}^{\rho} - \Gamma_{\sigma}{}^{\tau}{}_{\lambda} \Pi_{\Gamma}{}^{\mu\nu}{}_{\tau}{}^{\rho} + \Gamma_{\sigma}{}^{\rho}{}_{\tau} \Pi_{\Gamma}{}^{\mu\nu}{}_{\lambda}{}^{\tau} - \Gamma_{\sigma}{}^{\tau}{}_{\tau} \Pi_{\Gamma}{}^{\mu\nu}{}_{\lambda}{}^{\rho},$$

then, when one contract the σ and μ indices, the second and sixth terms in the right-hand side cancel each other.

it is straightforward to show that the equations of motion for the $\mathcal{B}$ field are

$$\nabla_\mu \Pi_{\mathcal{B}}{}^{\mu\nu}{}_\lambda{}^\rho = \frac{\partial \mathscr{L}}{\partial \mathcal{B}_\nu{}^\lambda{}_\rho}. \tag{16}$$

2.1.3. Field equations for the $\mathcal{A}$ field. The field equations for the $\mathcal{A}$ field are simpler to calculate, but a rigorous approach as in the previous sections, yields the field equations,

$$\nabla_\mu \Pi_{\mathcal{A}}{}^{\mu\nu} = \frac{\partial \mathscr{L}}{\partial \mathcal{A}_\nu}. \tag{17}$$

2.2. The torsion-less limit

An important result, obtained in [42], is that within the torsion-less sector of the connection, the field equations admit all vacuum solutions of Einstein's gravity, as solutions of the polynomial affine gravity.

The sector of vanishing torsion is equivalent to the limit $\mathcal{A} \to 0$ and $\mathcal{B} \to 0$. Clearly, such limit cannot be taken at the level of the action, but at the equation of motions. In the appendix B, the explicit field equations are shown from equations (B.1)–(B.45), and it can be checked that the mentioned limit is well-defined. In the torsion-free sector, the only nontrivial field equations are equations (B.21) and (B.22), i.e.

$$\nabla_\mu \left(\mathcal{R}_{\sigma\alpha}{}^\sigma{}_\lambda \, \mathrm{d}V^{\mu\nu\rho\alpha} + C\mathcal{R}_{\alpha\beta}{}^\sigma{}_\sigma \delta^\mu_\lambda \, \mathrm{d}V^{\nu\rho\alpha\beta} \right) = 0, \tag{18}$$

where C is the ratio of the original parameters of the model, $C = \frac{C_2}{C_1}$. The second term in equation (18) is proportional to the trace of the curvature two-form, which vanishes in general relativity. Also, the volume form is not necessarily compatible with the connection. However, if one restricts oneself to connections that preserve a volume form, such connections are dubbed *equiaffine* connections [56], the trace of the curvature two-form is ensured and the field equations simplify further to

$$\nabla_{[\mu} \mathcal{R}_{\nu]\lambda} = 0, \tag{19}$$

which can be written as

$$\nabla_\rho \mathcal{R}_{\mu\nu}{}^\rho{}_\lambda = 0, \tag{20}$$

after using the second Bianchi identity. Equations (19) and (20) are part of a set of well-known generalisations of Einstein's field equations (see for example chapter 16 of [57]).

Particularly, equation (20) can be obtained as the field equation for the connection of a gravitational Yang–Mills theory,

$$S_{\mathrm{SKY}} = \int \mathrm{d}^4 x \, \sqrt{g} \mathcal{R}_{\mu\nu}{}^\lambda{}_\rho \mathcal{R}^{\mu\nu\rho}{}_\lambda. \tag{21}$$

The above model is known as Stephenson–Kilmister–Yang (or SKY for short) [58–60], but its structure requires the inclusion of the metric in order to build the action. Therefore, besides equation (20) there is a field equation for the metric, and it spoils desirable features of the Stephenson–Kilmister–Yang model [61, 62][16].

---

[16] Notice for example that in terms of the connection, equation (19) is a set of second order differential equations, while if we would interpret them as equations for the metric, become a set of third order differential equations.

The solutions to equation (19) are classified in three categories: (i) Ricci-flat connections, $R_{\mu\nu} = 0$; (ii) connections with parallel Ricci, $\nabla_\lambda R_{\mu\nu} = 0$; and (iii) connections with harmonic curvature, $\nabla_\rho \mathcal{R}_{\mu\nu}{}^\rho{}_\lambda = 0$. Interestingly, among the possible solutions of equation (19) one encounters the vacuum solutions to the Einstein field equations, dubbed Einstein manifolds. A key difference in polynomial affine gravity is that unlike general relativity the cosmological constant appears as an integration constant. The same feature occurs in other generalisations of general relativity, e.g. unimodular gravity [63].

2.3. General relativity as a limiting case

The general theory of relativity is a dynamical model of the spacetime, which is assumed to be a pseudo-Riemannian (Lorentzian) manifold. A pseudo-Riemannian manifold is an affinely connected space, (M, Γ), endowed with a metric structure, i.e. a nondegenerated (invertible), symmetric, $\binom{0}{2}$-tensor g, which is parallel with respect to the affine connection $\nabla^\Gamma g = 0$.

Additional restrictions in general relativity are to consider the unique torsion-free connection compatible with the metric structure, dubbed the Levi-Civita connection, and the volume element $(\mathrm{d}V_g)$ as the unique tensor density such that, for any orthonormal basis $\{ X_i \} \in T_x M$, $\mathrm{d}V_g(X_1, \cdots, X_n) = 1$.

From the aforementioned considerations it follows that the volume element is the sole constant section on the bundle $\Lambda^n T^* M \otimes \mathfrak{o}_M$, where $\mathfrak{o}_M$ is the orientation line bundle.

Notice that the above considerations yield to the field equations in equation (19), and a solution to the parallel Ricci category is given by

$$\mathcal{R}_{\mu\nu} = \Lambda g_{\mu\nu} \text{ for } \Lambda \in \mathbb{R}, \tag{22}$$

provided by the existence of the metric structure. In section 5, a sort of generalisation of the above equation (which defines Einstein manifolds) is obtained when one asks for nondegenerated solutions of the parallel Ricci equation.

Equation (22) are the field equations of general relativity in absence of matter, and the appearance of these equations from polynomial affine gravity serves to sustain that (vacuum) general relativity is a limiting case. The *Einstenian* limit of polynomial affine gravity was first noticed in [42], where in addition it was demonstrated that coupling a (free and massles) scalar field with the torsion-less affine model yields to an equivalent set of equations to the Einstein–Klein–Gordon system[17]. In the polynomial affine model of gravity no other source for gravity has been considered, but currently we are developing a methodology for dealing with the subtleties that appear when trying to couple matter and gravity in this context.

3. Cosmological solutions on the torsion-free sector

In order to solve the equation (19) one proceeds—just as in general relativity—by giving an ansatz compatible with the symmetries of the problem. Using the Lie derivative, we have found the most general torsion-free connection compatible with the cosmological principle [43]. Since we shall restrict ourselves to the torsion-free sector, the nonvanishing coefficients of the connection are

$$\begin{aligned} \Gamma_t{}^t{}_t &= f(t), & \Gamma_i{}^t{}_j &= g(t)\, S_{ij}, \\ \Gamma_t{}^i{}_j &= h(t)\, \delta_j^i, & \Gamma_i{}^j{}_k &= \gamma_i{}^j{}_k, \end{aligned} \tag{23}$$

---

[17] In [64] the authors showed (under certain conditions) the equivalence between purely affine, purely metric and metric-affine formulations of gravity, considering matter fields coupled to these formulations of gravity.

where f, g and h are functions of time, while S_{ij} and $\gamma_i{}^j{}_k$ are the three-dimensional rank two symmetric tensor and connection compatible with isotropy and homogeneity, defined by

$$S_{ij} = \begin{pmatrix} \frac{1}{1-\kappa r^2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}, \tag{24}$$

and

$$\gamma_r{}^r{}_r = \frac{\kappa r}{1-\kappa r^2}, \qquad\qquad \gamma_\theta{}^r{}_\theta = \kappa r^3 - r,$$

$$\gamma_\varphi{}^r{}_\varphi = \left(\kappa r^3 - r\right)\sin^2\theta, \qquad \gamma_r{}^\theta{}_\theta = \frac{1}{r},$$

$$\gamma_\varphi{}^\theta{}_\varphi = -\cos\theta\sin\theta, \qquad\qquad \gamma_r{}^\varphi{}_\varphi = \frac{1}{r},$$

$$\gamma_\theta{}^\varphi{}_\varphi = \frac{\cos\theta}{\sin\theta}. \tag{25}$$

With the connection above, one can calculate the curvature,

$$\mathcal{R}_{ti}{}^t{}_j = -\mathcal{R}_{it}{}^t{}_j = \left(\dot{g} + (f-h)g\right)S_{ij}$$

$$\mathcal{R}_{mt}{}^n{}_t = -\mathcal{R}_{tm}{}^n{}_t = -\delta^n_m \left(\dot{h} + h^2 - fh\right)$$

$$\mathcal{R}_{mi}{}^n{}_j = -\mathcal{R}_{im}{}^n{}_j = (gh - \kappa)\,\delta^n_m S_{ij}, \tag{26}$$

and the Ricci tensor,

$$\mathcal{R}_{tt} = -3\left(\dot{h} + h^2 - fh\right)$$

$$\mathcal{R}_{ij} = \left(\dot{g} + (f+h)g + 2\kappa\right)S_{ij}. \tag{27}$$

The covariant derivative of the Ricci yields

$$\nabla_t \mathcal{R}_{tt} = \partial_t \mathcal{R}_{tt} - 2\Gamma_t{}^t{}_t \mathcal{R}_{tt} = -3\left[\ddot{h} + 2h\dot{h} - 3f\dot{h} - \dot{f}h - 2fh(f-h)\right],$$

$$\nabla_t \mathcal{R}_{ij} = \partial \mathcal{R}_{ij} - 2\Gamma_t{}^k{}_i \mathcal{R}_{kj} = \left[\ddot{g} + \dot{f}g + \dot{h}g + (f-h)\dot{g} - 2(f+h)gh - 4\kappa h\right]S_{ij},$$

$$\nabla_i \mathcal{R}_{tj} = -\Gamma_i{}^k{}_t \mathcal{R}_{kj} - \Gamma_i{}^t{}_j \mathcal{R}_{tt} = \left[3g\dot{h} - \dot{g}h - 4fg + 2gh^2 - 2\kappa h\right]S_{ij}. \tag{28}$$

Finally, the harmonic curvature expression has a single independent component,

$$\nabla_{[t}\mathcal{R}_{i]j} = \partial_t \mathcal{R}_{ij} - \Gamma_t{}^k{}_i \mathcal{R}_{kj} + \Gamma_i{}^t{}_j \mathcal{R}_{tt}$$
$$= \left(\ddot{g} + \dot{f}g + f\dot{g} - 2g\dot{h} + 2fgh - 4gh^2 - 2\kappa h\right)S_{ij}. \tag{29}$$

In the remaining of this section we will solve the equation (19). Firstly, notice that the Levi-Cività connection from Friedman–Robertson–Walker models is obtained from equation (23) by setting $f = 0$, $g = a\dot{a}$ and $h = \frac{\dot{a}}{a}$, implying that all (vacuum) cosmological models in general relativity are in the space of solutions of polynomial affine gravity. Moreover, it was shown in [44], that within this space of solutions, suitable deviations from the vacuum Friedman–Robertson–Walker connection mimic the behaviour of matter content, even though equation (19) aim to describe geometric properties of the manifold exclusively. Secondly, our classification of solutions—into Ricci-flat, parallel Ricci and harmonic curvature—is

hierarchic, in the sense that once a condition is satisfied, the remaining are satisfied as well. Therefore, in order to find a proper solution of the parallel Ricci equations, we have to ensure that the connection is not Ricci-flat; and in order to find a proper solution of the harmonic curvature equations, neither the Ricci-flat or parallel Ricci condition should be satisfied.

3.1. Cosmological solutions with vanishing Ricci

A first kind of solutions can be found by solving the system of equations determined by vanishing Ricci. From equation (27) the differential equations to solve are

$$\dot{h} - (f - h)h = 0, \tag{30}$$

$$\dot{g} + (f + h)g + 2\kappa = 0. \tag{31}$$

Since f is not a dynamical function, the system can be solved in terms of f (see [44]),

$$h(t) = \frac{\exp\left(F(t)\right)}{C_h + \int \mathrm{d}t\, \exp(F)}, \tag{32}$$

$$g(t) = \exp(-\Sigma(t))\left(C_g - 2\kappa\left(\int \mathrm{d}t\, \exp(\Sigma(t))\right)\right), \tag{33}$$

where $F = \int \mathrm{d}t\, f$ and $\Sigma(t) = \int \mathrm{d}t\,(f(t) + h(t))$ are integrals of the defining functions, while C_h and C_g are integration constants.

3.1.1. Friedman–Robertson–Walker-like models. In particular, Friedman–Robertson–Walker-like models are obtained by setting $f = 0$, and besides the trivial solution—$h = g = \kappa = 0$—, yielding a parametric family of nontrivial solutions,

$$g(t) = \frac{1}{t + C_h}\left(C_g - \kappa(t + C_h)^2\right), \qquad h(t) = \frac{1}{t + C_h}. \tag{34}$$

Unlike in general relativity—whose sole Ricci-flat cosmological solution is a flat manifold—, the above solution is not flat in general, since its curvature tensor has nonvanishing components

$$\mathcal{R}_{ti}{}^t{}_j = -\mathcal{R}_{it}{}^t{}_j = -2C_g h^2 S_{ij},$$
$$\mathcal{R}_{mi}{}^n{}_j = -\mathcal{R}_{im}{}^n{}_j = C_g h^2 S_{ij}\delta_m^n. \tag{35}$$

3.1.2. Case $h = f$. In these particular subspace, the solutions to equations (30) and (31) are given by

$$f = C_h \qquad h = C_h \qquad g = C_g \exp(-2C_h t) - \frac{\kappa}{C_h}. \tag{36}$$

3.1.3. Case $h = -f$. Equations (30) and (31) are solved by

$$f = -\frac{1}{2t + C_h} \qquad h = \frac{1}{2t + C_h} \qquad g = C_g - 2\kappa t. \tag{37}$$

3.1.4. Case $h = 0$ and a given f. In this case, equation (30) becomes an identity, and equation (31) is solved by

$$g(t) = \exp(-F(t))\left(C_g - 2\kappa\left(\int \mathrm{d}t\, \exp(F(t))\right)\right). \tag{38}$$

3.1.5. Case g = 0 and a given f.　In this case, equation (31) requires $\kappa = 0$, and h can still be solved for a given function f as

$$h(t) = \frac{\exp\left(F(t)\right)}{C_h + \int \mathrm{d}t \; \exp(F)}.\tag{39}$$

3.2. Cosmological solutions with parallel Ricci

In this section we solve the equation (28), under the condition that the Ricci tensor is nonzero, i.e. $\nabla_\lambda \mathcal{R}_{\mu\nu} = 0$ but $\mathcal{R}_{\mu\nu} \neq 0$. The strategy to solve these equations is to propose an ansatz for the Ricci tensor, and solve for f, g and h accordingly. Nonetheless, a broaden ansatz is useless, thus we focus in the two simpler cases.

3.2.1. Parallel time-independent Ricci.　A time-independent Ricci has the form,

$$\mathcal{R}_{tt} = R_1, \qquad\qquad \mathcal{R}_{ij} = R_2 S_{ij},\tag{40}$$

with constant R_1 and R_2. For this proposal of the Ricci tensor the equation (28) yield the constraints

$$\nabla_t \mathcal{R}_{tt} = 0 \qquad\qquad \Rightarrow f = 0 \vee R_1 = 0,\tag{41}$$

$$\nabla_t \mathcal{R}_{ij} = 0 \qquad\qquad \Rightarrow h = 0 \vee R_2 = 0,\tag{42}$$

$$\nabla_i \mathcal{R}_{tj} = 0 \qquad\qquad \Rightarrow h = 0 \vee (g = 0 \wedge \kappa = 0).\tag{43}$$

Notice that $h = 0$ implies $R_1 = 0$, and $g = 0 \wedge \kappa = 0$ implies $R_2 = 0$. Therefore, there is no solution of the equation (28) for a time-independent nondegenerated Ricci.

Given the above conditions, there are solely two solutions with degenerated nonvanishing Ricci. Firstly, for vanishing h, one has

$$h = 0,$$
$$f = \text{undetermined function},$$
$$g = \exp(-F)\left(C_g + (R_2 - 2\kappa)\int \mathrm{d}t \; \exp(F)\right).\tag{44}$$

Similarly, for $g = 0 \wedge \kappa = 0 \wedge f = 0$, the solution is given by

$$h = \sqrt{\frac{R_1}{3}} \tanh\left(\sqrt{\frac{R_1}{3}}(t - t_*)\right),$$
$$f = 0,$$
$$g = 0.\tag{45}$$

3.2.2. Parallel Ricci with a 'scale factor'.　We now consider an ansatz for the Ricci tensor with the form of a Friedman–Robertson–Walker metric, given that this has the required symmetries, i.e.

$$\mathcal{R}_{tt} = -R_1, \qquad \mathcal{R}_{ij} = A(t)S_{ij}.\tag{46}$$

With the above ansatz, the parallel Ricci equations require,

$$
\begin{aligned}
\nabla_t \mathcal{R}_{tt} &= 2fR_1 & \Rightarrow f = 0 \vee R_1 = 0, \\
\nabla_t \mathcal{R}_{ij} &= S_{ij}(\dot{A} - 2hA) & \Rightarrow A = C_A \exp(2H), \\
\nabla_i \mathcal{R}_{tj} &= -S_{ij}(hA - R_1 g) & \Rightarrow g = \frac{hA}{R_1}; \ (R_1 \neq 0).
\end{aligned}
\tag{47}
$$

Unlike the previous case, equation (47), accept a nondegenerated solution given by

$$
\begin{aligned}
f &= 0, \\
h &= \sqrt{\frac{R_1}{3}} \tanh\left(\sqrt{\frac{R_1}{3}}(t - t_*)\right), \\
A &= C_A \cosh^2\left(\sqrt{\frac{R_1}{3}}(t - t_*)\right), \\
g &= \frac{1}{\sqrt{12R_1}} \sinh\left(\sqrt{\frac{4R_1}{3}}(t - t_*)\right).
\end{aligned}
\tag{48}
$$

Notice that degenerated solutions, with $R_1 = 0$, require either vanishing C_A or a constant A for $h = 0$, which are part of previously considered cases.

3.3. Cosmological solutions with harmonic curvature

3.3.1. Harmonic curvature with time-independent Ricci. For a time-independent Ricci tensor, see equation (40), the harmonic curvature condition, equation (29), becomes

$$
\nabla_{[t} \mathcal{R}_{i]j} = -(gR_1 + hR_2)S_{ij},
\tag{49}
$$

i.e. the harmonic curvature requires

$$
g = -\frac{R_2}{R_1}h.
\tag{50}
$$

Therefore, the consistency equations for the Ricci are rewritten, after using equation (50), as

$$
\dot{h} + h^2 = \beta_1, \qquad \beta_1 = \frac{R_1}{3R_2}(3\kappa - R_2),
\tag{51}
$$

$$
f = \frac{\beta_2}{h}, \qquad \beta_2 = \frac{R_1}{3R_2}(3\kappa - 2R_2).
\tag{52}
$$

Notice that the general solution is parameterised by the function h, and for vanishing β_2 the function f is zero, however this solution is degenerated for vanishing κ.

We assume that the product of the constants R_1 and R_2 is positive, i.e. $R_1 R_2 > 0$. Then, the solution is

$$
h = \begin{cases} \omega \tanh\left(\omega(t - t_*)\right) & \omega^2 = \beta_1 > 0, \\ -\omega \tan\left(\omega(t - t_*)\right) & -\omega^2 = \beta_1 < 0, \\ \frac{1}{t - C_h} & \beta_1 = 0, \end{cases}
$$

$$
g = -\frac{R_2}{R_1} h,
$$

$$
f = \frac{\beta_2}{h}. \tag{53}
$$

3.3.2. Harmonic curvature from Ricci with a 'scale factor'. Finally, the harmonic curvature condition for a Ricci with a *scale factor*, obtained from (47), is

$$
\nabla_{[t} \mathcal{R}_{i]j} = (\dot{A} - gR_1 - hA)S_{ij}, \tag{54}
$$

whose solution for g is

$$
g = \frac{\dot{A} - hA}{R_1}. \tag{55}
$$

Substituting equation (55) into the consistency equations for the Ricci, we obtain

$$
\dot{h} + h^2 - fh = \frac{R_1}{3}, \tag{56}
$$

$$
\ddot{A} + f\dot{A} - \left(2fh + \frac{4R_1}{3}\right)A = -2\kappa R_1. \tag{57}
$$

Equation (56) can be solved for a constant function $f = C_f$, therefore, we shall restrict ourselves to that case. In this particular case, equation (56) can be rewritten as

$$
\dot{h} + \left(h - \frac{C_f}{2}\right)^2 = \beta_3 \qquad \beta_3 = \frac{1}{12}(3C_f^2 + 4R_1), \tag{58}
$$

and its solutions are

$$
h = \begin{cases} \omega \tanh(\omega(t - t_*)) + \frac{C_f}{2} & ; \quad \omega^2 = \beta_3 > 0, \\ -\omega \tan(\omega(t - t_*)) + \frac{C_f}{2} & ; -\omega^2 = \beta_3 < 0, \\ \frac{1}{t - t_*} + \frac{C_f}{2} & ; \quad \beta_3 = 0, \\ \pm\omega + \frac{C_f}{2} & ; \quad \omega^2 = \beta_3 > 0. \end{cases} \tag{59}
$$

Worth to highlight, the last case is a constant solution for h.

For the simplest solutions of h—a constant function—the second consistency condition, equation (57), can be integrated. Define the constant

$$
\alpha = 2C_f C_h + \frac{4R_1}{3},
$$

where $h = C_h = \pm\omega + C_f/2$ is the constant determined from equation (59) above. Therefore, the exact solutions for the *scale factor* is

$$
A = \mathbb{A}_1 \exp\left((2C_h - C_f)t\right) + \mathbb{A}_2 \exp\left(-2C_h t\right) + \frac{2\kappa R_1}{\alpha}. \tag{60}
$$

Additionally, second simplest solution to the equation (57) is obtained for $\beta_3 = 0$, in whose case the R_1 constant is determined by the value of C_f, and the function $h = \frac{1}{t} + \frac{C_f}{2}$. The solution for A is

$$A = t(C_f t + 2)\left[\mathbb{A}_1 + \mathbb{A}_2 C_f \Gamma(0, C_f t)\right] - \mathbb{A}_2 e^{-C_f t}(C_f t + 1) + \frac{3}{4}\kappa C_f^2 t^2, \quad (61)$$

with $\Gamma(0, C_f t)$ an incomplete gamma function defined by the expression

$$\Gamma(0, C_f t) = \int_1^\infty dx \; \frac{e^{-x C_f t}}{x}.$$

The *scale factor* can be obtained for the other choices of h when one sets the $C_f = 0$. In those cases, the solutions to equation (57) is

$$A = \mathbb{A}_1 \exp\left(\sqrt{\frac{4R_1}{3}} t\right) + \mathbb{A}_2 \exp\left(-\sqrt{\frac{4R_1}{3}} t\right) + \frac{3\kappa}{2}, \quad (62)$$

where the behaviour of the exponential functions is managed by the sign of the constant R_1.

4. The affine self-parallel curves

In general relativity, given that the gravitational connection is the one of Levi-Cività, the concepts of geodesic and self-parallel curve are equivalent, however, for generic connections these concepts differ. Geodesics play an important role in general relativity, since they represent the trajectories followed by free falling particles.

Although it is not possible to define geodesics in (purely) affine models of gravity, we shall postulate that trajectories of free falling particles are described by self-parallel curves

$$\ddot{x}^\mu + \Gamma_\nu{}^\mu{}_\lambda \dot{x}^\nu \dot{x}^\lambda = 0, \quad (63)$$

where now Γ is a generic connection, and the *dot* represents—unlike in the previous sections—derivation with respect to the affine parameter of the curve, τ.

With the coefficients of the cosmological affine connection, equation (23), the self-parallel curves are given by

$$\ddot{t} + f\dot{t}^2 + g S_{ij}\dot{x}^i \dot{x}^j = 0,$$
$$\ddot{x}^i + 2h\dot{t}\dot{x}^i + \gamma_j{}^i{}_k \dot{x}^j \dot{x}^k = 0.$$

The term that mixes the time with the spacial coordinates can be eliminated by a redefinition of the affine parameter as

$$\frac{d}{dl} = C_T e^{2H} \frac{d}{d\tau} = C_T e^{2\int dt\, h} \frac{d}{d\tau},$$

with C_T an arbitrary constant. In terms of the new parameter (were now the dot derivative is with respect to the parameter l) the self-parallel equations become

$$\ddot{t} + (f - 2h)\dot{t}^2 + g S_{ij}\dot{x}^i \dot{x}^j = 0,$$
$$\ddot{x}^i + \gamma_j{}^i{}_k \dot{x}^j \dot{x}^k = 0. \quad (64)$$

The three-dimensional restriction of equation (64) are the same as the spacial geodesic equations for a Friedman–Robertson–Walker model in General Relativity [65].

The above shows that there exists a parametrisation of the self-parallel curves from polynomial affine gravity, in which the restriction to the spatial coordinates coincides with the geodesic of the Friedman–Robertson–Walker models on general relativity.

For the sake of completeness, we remind to the readers that due to the isotropy, one can set $\theta = \frac{\pi}{2}$, and thus the spatial part of equation (64) are

$$\ddot{r} + \frac{kr}{1 - kr^2} \dot{r}^2 + \left(kr^3 - r \right) \dot{\varphi}^2 = 0$$

$$\ddot{\varphi} + 2\frac{\dot{r}\dot{\varphi}}{r} = 0. \tag{65}$$

Hence, in terms of the new affine parameter, the geometry of the $r(\varphi)$-curves is determined by

$$r(\varphi) = \begin{cases} \sin(l) & \text{for } \kappa = 1 \\ l & \text{for } \kappa = 0 \\ \sinh(l) & \text{for } \kappa = -1 \end{cases}, \tag{66}$$

for $\dot{\varphi} = 0$, i.e. for radial self-parallels, and

$$r(\varphi) = \frac{1}{\sqrt{\kappa + B^2 \cos^2(\varphi + \beta)}}, \tag{67}$$

for $\dot{\varphi} \neq 0$.

Interestingly, in this context, although the equation that describes the geometry of the orbits is the same, in polynomial affine gravity there is no way to differentiate among the orbits of massive and massless particles, since the absence of a metric precludes the classification of vectors into time-like, space-like or null-like.

5. The Ricci tensor as a metric

From the last section, we understood that—generically—the geometry of the self-parallel curves (or geodesics) is solely determined by the notion of parallelism, however, their relation with physical notions (as trajectories, equivalence principle and principles of relativity) requires the existence of a metric. Nonetheless, our proposal stands on the idea that no metric is required to formulate the model.

A coruscating fact is that, unlike in general relativity, a metric field is not necessarily a fundamental geometric object. In what follows, we highlight that under certain conditions the Ricci tensor could play the role of a *emergent* (or derived) metric. Therefore, we shall consider the affine connection as the fundamental field of polynomial affine gravity, while the (non-degenerated) symmetric part of the Ricci tensor serves as metric. Let us first remind the formal definition of a metric.

Definition 5.1 (Metric). Let M be an m-dimensional differential manifold. A pseudo-Riemannian metric on M is a $\binom{0}{2}$-tensor field g on M satisfying:

(i) g is a symmetric tensor field, and
(ii) g is nondegenerated, i.e. $\forall X \in TM$ the quantity $g(X, Y) = 0$ if and only if $TM \ni Y = 0$.

A pair (M, g) is called a pseudo-Riemannian manifold.

Notice that the second condition in definition 5.1 ensures that as a map, $g : TM \to T^*M$, the metric field has trivial kernel and thus it is invertible.
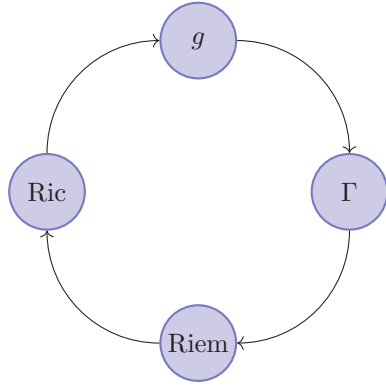
Figure 1. Closed chain of derived quantities for (Riemannian) Einstein manifolds.

The metric is the fundamental field in general relativity, and all other geometrical properties of the manifold—connection, curvature, etc—are derived from it,

$$g \rightarrow \Gamma \rightarrow \text{Riem} \rightarrow \text{Ric} \rightarrow \mathcal{R}. \tag{68}$$

A peculiar, and particularly important type of manifolds are the Einstein spaces, since the chain of derived quantities closes, see figure 1.

The closure condition for Einstein manifolds is expressed as

$$\mathcal{R}_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{69}$$

It is worth mentioning that in the vanishing Ricci case, despite the equation (69) is satisfied, one cannot say that the chain really closes, since

$$g \rightarrow \Gamma \rightarrow \text{Riem} \rightarrow \text{Ric} = 0.$$

In our model, we start with an affinely connected manifold (M, Γ), were the ansatz of the affine connection is determined by the symmetries of the problem[18]. Without the use of a metric, the chain of derided products stops at the Ricci tensor,

$$\Gamma \rightarrow \text{Riem} \rightarrow \text{Ric}. \tag{70}$$

Nevertheless, the above chain could close if the Ricci tensor—as in the case of Einstein manifolds—satisfies the conditions of a metric, i.e. conditions in definition 5.1. Assuming such, the chain of derived quantities closes, see figure 2, and therefore it is possible to define an affine analogue of Einstein manifolds.

Noticeable, along the developing of this work we restricted ourselves to *locally equiaffine* connections, which ensure the symmetry of the Ricci tensor (see proposition 3.1 in [56]). This is precisely the requirement to fulfil the first condition in definition 5.1. The second condition the Ricci tensor should satisfy in order to be a good metric field is not to posses a zero eigenvalue.

Given the classification of solutions of the field equation (19), one concludes that: (i) as in general relativity, the relation between the Ricci tensor and the metric interpretation is lost for Ricci-flat connections; (ii) in the case of parallel Ricci—if nondegenerated—it is a *good* metric field, which additionally satisfies the 'metricity' condition; (iii) in the case of proper

---

[18] A noticeable difference with the approach in General Relativity is that (in principle) a choice of the connection does not determine the signature of the manifold.
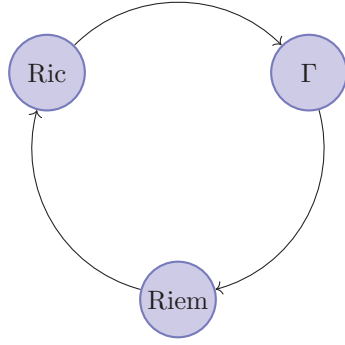
Figure 2. Closed chain of derived quantities for affine Einstein manifolds.

solutions of the harmonic curvature, the Ricci tensor accept interpretation of a metric, but it is not compatible with the connection, i.e. does not satisfy the metricity condition.

Since we started from an affinely connected space, (M, Γ), and a derived geometric object, the Ricci tensor, introduces a metric structure, we call it an *emergent* metric.

It can be shown with ease, given a locally equiaffine connection whose Ricci tensor is parallel and nondegenerated, that

$$\Gamma_\mu{}^\lambda{}_\nu = \frac{1}{2}\mathcal{R}^{\lambda\sigma}\left(\partial_\mu \mathcal{R}_{\sigma\nu} + \partial_\nu \mathcal{R}_{\mu\sigma} - \partial_\sigma \mathcal{R}_{\mu\nu}\right). \tag{71}$$

Consequently, in the analysed case the geometry is naturally Riemannian, and we can relate the Ricci tensor to a *canonical* covariantly constant $\binom{0}{2}$-tensor, g, by

$$\mathcal{R}_{\mu\nu} = \Lambda g_{\mu\nu}, \text{ with } \Lambda \in \mathbb{R}^*, \tag{72}$$

i.e. the space is an Einstein manifold. An additional comment is that given the relation in equation (72) it follows that

$$\Gamma(\mathcal{R}) = \Gamma(g).$$

The fact that the Ricci tensor—or its symmetric part—could play the role of a metric was (somehow) anticipated by Schrödinger [31].

6. Cosmological quantities

In the previous section we argue that even when in principle the space does not posses a metric structure, under certain conditions a metric structure emerges through the symmetric component of the Ricci tensor.

A consequence of the emergence of a metric is that one can now distinguish between null-like and time-like self-parallel curves, which could be interpreted as the trajectories of free-falling particles—as mentioned in section 4. In particular, it is possible to define a null-like self-parallel curve by the equations

$$\ddot{x}^\mu + \Gamma_\lambda{}^\mu{}_\rho \dot{x}^\lambda \dot{x}^\rho = 0,$$
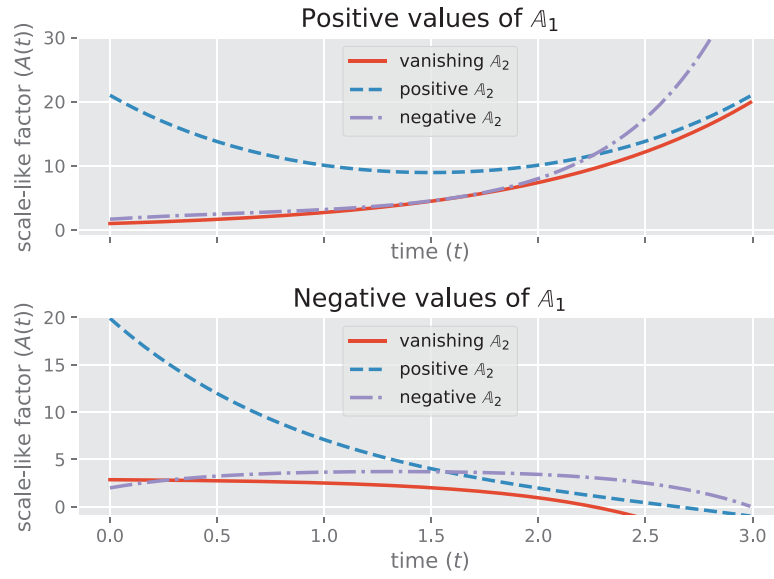$$\mathcal{R}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0. \tag{73}$$

Figure 3. Behaviour of the function $A(t)$ for certain values of the parameters $(\mathbb{A}_1, \mathbb{A}_2, \kappa, C_h, C_f)$.
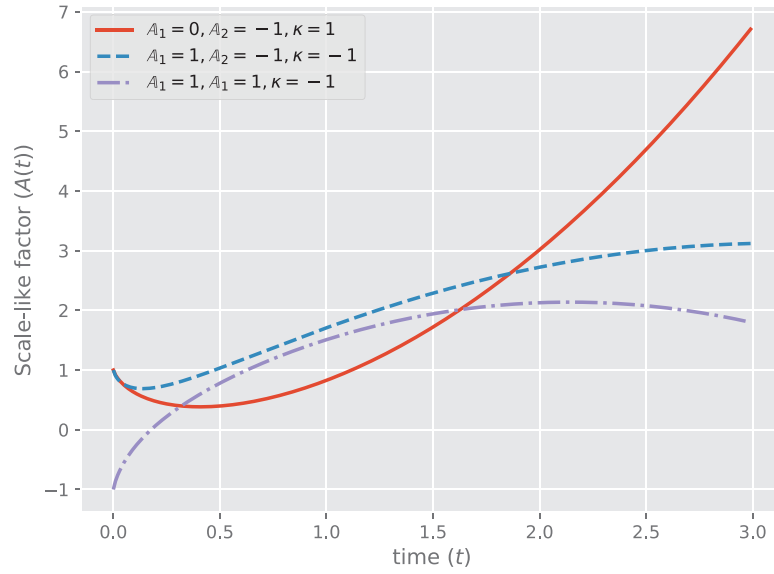
Figure 4. Behaviour of the function $A(t)$ in equation (61), for certain values of the parameters $(\mathbb{A}_1, \mathbb{A}_2, \kappa, C_f)$.

The special case of a light ray coming in the radial direction yields

$$
\mathrm{d}t = \pm \sqrt{\frac{\mathcal{R}_{rr}}{\mathcal{R}_{tt}}} \frac{\mathrm{d}r}{\sqrt{1 - \kappa r^2}},
$$

(74)

which by renaming

$$a(t) = \sqrt{\frac{\mathcal{R}_{rr}}{\mathcal{R}_{tt}}},$$

(75)

is the same equation that serves to define the cosmological redshift, i.e.

$$1 + z = \frac{a(t_0)}{a(t_1)} = \sqrt{\frac{\mathcal{R}_{rr}(t_0)\mathcal{R}_{tt}(t_1)}{\mathcal{R}_{tt}(t_0)\mathcal{R}_{rr}(t_1)}}.$$

(76)

Not surprisingly, after the emergence of the metric, the physical properties of the space are determined by a single function, i.e. the scale factor. If we focus on the nondegenerated Ricci tensors found in section 3, the standard scale factor is defined as

$$a(t) = \sqrt{A(t)},$$

where $A(t)$ is the function defining the Ricci tensor in equation (46).

7. Conclusions and remarks

In this paper we have extended the set of known solutions to the parallel Ricci and harmonic curvature equations, when these are defined in terms of a connection instead of a metric[19]. We also analysed the equation of self-parallel curves and noticed that, without the aid of a metric, it is not possible to distinguishing between the (yet hypothetical) trajectories followed by massive or massless free-falling particles. Then, we show that the Ricci derived from the connection would, under certain conditions, be a good metric. In these cases, it is possible to surpass the mentioned limitation, and distinguish trajectories followed by massive and massless free-falling particles. Moreover, the emergent metric allows us to make contact with the standard cosmological quantities such as the redshift, scale factor, Hubble and des-acceleration parameters, etc.

We first would like to highlight the fact that a very important step toward the simplification of the polynomial affine model of gravity was due to the change of the field decomposition of the connection, equation (2), which alleviate both the geometric interpretation of the irreducible components of the connection and the process of finding the complete field equations (see appendix B). An advantage of having the complete field equations at hand, is that readers can convince themselves that the torsion-free sector is a consistent truncation of the model.

Despite the field equations of polynomial affine gravity are well-defined in the torsion-free sector, the limit $\mathcal{A} \to 0$ and $\mathcal{B} \to 0$ is meaningless at the action level because all the terms are at least linear in either $\mathcal{A}$ or $\mathcal{B}$. Hence, the Feynman rules for the model lack vertices with only *gravitons*[20]. Furthermore, the effective action from which one can derive the field equation (19) contains vertices with three and four gravitons.

In the formulation of the polynomial affine model of gravity there is not something such as a cosmological constant. Nevertheless, the integration constant included in the process of solving equation (19), plays the role of cosmological constant. This changes the paradigm on the *cosmological problem*, similarly as in unimodular gravity models. The most relevant feature of Unimodular Gravity is that vacuum fluctuations of the energy–momentum tensor

---

[19] For solutions of the mentioned equations in terms of the metric, we refer the readers to chapter 16 of [57]. In addition, a detailed exposition of the SKY model of gravity can be found in Chap. 7 of [66].

[20] Here we call gravitons to the spin-2 field within the symmetric part of the connection, $\Gamma_\mu{}^\lambda{}_\nu$. This feature suggests that the polynomial affine model of gravity could bypass the no-go theorems found in [45, 46].

do not gravitate [67], removing the discrepancy between the observed and estimated values of the vacuum energy [68–70][21].

Although cosmological solutions, in the torsion free sector of the polynomial affine model of Gravity, were found in [44], in this work we were able of developing further arguments that allows us to obtain explicit solutions in cases that were previously unexplored.

The field equations of general relativity—without cosmological constant—in vacuum are equivalent to Ricci-flat manifolds. In general these solutions are curved spacetimes (i.e. Schwarzschild space), however, once one asks for cosmological solutions the field equations require the manifold to be flat. In the analysis of the solutions of polynomial affine gravity field equations, we notice that they accept Ricci-flat cosmological solutions which are not flat. Let us work out the interpretation of this situation.

Remember first that a connection, ∇, on a vector bundle $\pi : E \rightarrow M$ assigns to each vector field, X, a map ∇_X from the space of sections $C^\infty(E)$ to itself. Therefore, for a given direction, the connection represents an endomorphism on the space of sections, i.e. $C^\infty(\mathrm{End}(E))$. Similarly, the curvature of a linear connection, $\mathcal{R}^\nabla$, on a vector bundle $\pi : E \rightarrow M$ is a two-form on M with values in $C^\infty(\mathrm{End}(E))$[22]. In the gravitational case, the vector bundle is the tangent bundle, TM, and in particular when one considers pseudo-Riemannian geometries the group structure of $\mathrm{End}(TM)$ is a subgroup of the orthogonal group, $O(p,q)$[23].

Now, from equations (26) and (27), it follows that the Ricci-flat condition allows the components $\mathcal{R}_{ti}{}^t{}_j = -\mathcal{R}_{it}{}^t{}_j$ and $\mathcal{R}_{mi}{}^n{}_j = -\mathcal{R}_{im}{}^n{}_j$ of the curvature not to vanish. Thus, the group structure underling the endomorphisms of the tangent bundle is the homogeneous Carroll group [74][24]. The homogeneous Carroll groups can be obtained from the Lorentz group through the Inönü–Wigner contraction in the limit $c \rightarrow 0$.

Notice that the nonvanishing components of curvature tensor for the Ricci-flat manifolds obtained in section 3.1 are

$$
\begin{aligned}
\mathcal{R}_{ti}{}^t{}_j &= -\mathcal{R}_{it}{}^t{}_j = -2S_{ij}\big(gh + \kappa\big), \\
\mathcal{R}_{mi}{}^n{}_j &= -\mathcal{R}_{im}{}^n{}_j = S_{ij}\,\delta^n_m\,\big(gh + \kappa\big).
\end{aligned}
\tag{77}
$$

Thus, for example the solution in equation (34) is flat—irrespective of the value of κ—if and only if C_g vanishes.

The family of solutions in section 3.1.2 has vanishing curvature for $C_g = 0$, while the family from section 3.1.3 requires $C_g = -\kappa C_h$. Solutions in section 3.1.4 and 3.1.5 are flat solely for $\kappa = 0$.

The second class of solutions of the field equations are those connections with parallel Ricci tensor, $\nabla_\lambda \mathcal{R}_{\mu\nu} = 0$. In order to solve the field equations we required the Ricci tensor to be compatible with the cosmological principle, i.e. to preserve the isotropic and homogeneity symmetries. We analyse the time-independent and Friedman–Robertson–Walker-like cases.

Noticeable, there are no connections with parallel, nondegenerated, time-independent Ricci tensor. However, we found solutions with degenerated Ricci (see equations (44) and (45)).

There are connections with parallel, nondegenerated Friedman–Robertson–Walker-like Ricci tensor. In this cases the Ricci represents an emergent metric, and thus the underling structure of the manifold is Riemannian. From equation (48) one notices that depending on the sign of the

---

[21] Very recently, it has been pointed that from an affine point of view, the nature of the cosmological constant is related with the volume preserving property (instead of being related to the sectional curvature) [71].
[22] See [57, 72, 73].
[23] We are using the standard notation that $n = p + q$ is the dimension of the pseudo-Riemannian manifold, p the number of space-like coordinates and q the number of time-like dimensions.
[24] We thank Dr Zurab Silagadze for help us with the identification of this group.

constant R_1 the manifold is a sphere- or hyperbolic-like space[25]. It can be checked that the solution in equation (48) satisfies—as the standard Friedman–Robertson–Walker solutions—:

$$f = 0, \qquad a = \sqrt{A},$$
$$g = a\dot{a}, \qquad h = \frac{\dot{a}}{a}.$$

Such result, as expected from the discussion in section 5 is not very interesting from the view point of the polynomial affine model of Gravity.

The third class of solutions to the field equations, i.e. connections with harmonic curvature, does not accept degenerated solutions for the time-independent Ricci case, since either f or g diverge when $R_1 \vee R_2$ vanish. The solutions shown in equation (53) assume that $R_1 R_2 > 0$, which ensures, since the Ricci tensor endows the manifold with a metric structure, a Lorentzian signature of the metric. Worth mentioning that it is possible to set $\beta_2 = 0$ which implies that f vanishes (similar to the behaviour of standard Friedman–Robertson–Walker models).

Even more interesting are the solutions of the harmonic curvature equations, whose Ricci contains a scale factor. We found solution with nondegenerated Ricci tensor, which again endows the manifold with a metric structure. Although this metric structure has a similar form than the expected from the Friedman–Robertson–Walker models, the cosmological scale factor, $a = \sqrt{A}$, admits richer behaviour in the cosmological evolution. Of course, one has to remind that that these are (so far) vacuum solutions in polynomial affine gravity.

Notice for example that for constant $f = C_f$ and $h = C_h$, equation (57) becomes the equation of motion of a damped harmonic oscillator on which a constant force is exerted. This case is the simplest case of a cosmological model in polynomial affine gravity with an emergent metric, with no equivalent in general relativity. These kind of solutions generalise the metric solutions to the field equations $\nabla_{[\mu} \mathcal{R}_{\nu]\lambda} = 0$ reported in [44]. The behaviour of the A function for certain set of values of the parameters $(\mathbb{A}_1, \mathbb{A}_2, \kappa, C_h, C_f)$ is shown in figure 3.

We were able to solve the field equations for the harmonic curvature in the case $\beta_3 = 0$, which introduces interesting models of cosmologies through the appearance of the incomplete gamma function within the scale factor (see equation (61)). In figure 4 the behaviour of the scale-like factor $A(t)$ is shown for a set of values of the parameters $(\mathbb{A}_1, \mathbb{A}_2, \kappa, C_f)$. Notice the complex behaviour for the particular case with dashed lines.

It is noticeable that connections with harmonic curvature could provide an emergent metric (i.e. Ricci tensor) corresponding to the standard metric on a Eucliean or Minkowskian space, even though their curvature do not vanish.

Further studies, which take the nature of the dark sector into account, will need to be undertaken. Interesting proposals—in models other than the polynomial affine model of Gravity—could be found in [71, 75–81].

Acknowledgments

---

[25] If one demand the signature to be Lorentzian, the solutions are de Sitter and Anti de Sitter. However, as mentioned previously the signature of the space is not fixed from the affine structure.

Appendix A. Building the simplified polynomial affine action

Our goal is to build the most general action using the irreducible fields, $\Gamma_\mu{}^\lambda{}_\nu$, $\mathcal{B}_\mu{}^\lambda{}_\nu$ and $\mathcal{A}_\mu$, where the dynamics is given by the covariant derivative with respect to the symmetric part of the connection, i.e. $\nabla = \nabla^\Gamma$. Now, the strategy to write down the action is to define the most general scalar density. For that sake, inspired by the procedure described in [42], we introduced a two operators, $\mathcal{N}$ and $\mathcal{W}$, that count the number of free indices and the weight density—respectively—of a given term.

The action of the operators $\mathcal{N}$ and $\mathcal{W}$ on the irreducible components of the connection,

$$
\begin{aligned}
\mathcal{N}(\mathcal{A}) &= -1, & \mathcal{N}(\mathcal{B}) &= -1, \\
\mathcal{N}(\nabla) &= -1, & \mathcal{N}(\mathrm{d}V) &= 4, \\
\mathcal{W}(\mathcal{A}) &= 0, & \mathcal{W}(\mathcal{B}) &= 0, \\
\mathcal{W}(\nabla) &= 0, & \mathcal{W}(\mathrm{d}V) &= 1.
\end{aligned}
\tag{A.1}
$$

As example of how the dimensional analysis works, we consider a general term of the form $\mathcal{O} = \mathcal{A}^m \mathcal{B}^n \nabla^p \, \mathrm{d}V^q$, the action of the $\mathcal{N}$ and $\mathcal{W}$ operators on the term yield the equations

$$
\begin{aligned}
\mathcal{N}(\mathcal{O}) &= 4q - m - n - p, \\
\mathcal{W}(\mathcal{O}) &= q.
\end{aligned}
\tag{A.2}
$$

We are interested in the Lagrangian, i.e. a scalar density. Equation (A.2) require $q = 1$ and $m + n + p = 4$. The terms contributing to this construction are shown in table A1.

From table A1 one can straightforwardly read terms that vanish, e.g. the term with four $\mathcal{A}$ does not contribute to the action since its contraction with the volume element is identically zero. Whenever two covariant derivatives are contracted with the volume form they give a curvature tensor, and since the curvature is defined for the symmetric component of the connection, such curvature satisfy the torsion-free Bianchi identities, which relate some of the several possible contractions of indices. An additional argument that helps to drop contraction of indices is that $\mathcal{B}$ is traceless. Finally, the terms contributing to the action come from

$$
\begin{aligned}
\mathcal{A}\mathcal{A}\mathcal{B}\mathcal{B} &\to F_4 & \mathcal{A}\mathcal{A}\mathcal{B}\nabla &\to D_6, D_7 \\
\mathcal{A}\mathcal{B}\mathcal{B}\mathcal{B} &\to F_3 & \mathcal{A}\mathcal{B}\mathcal{B}\nabla &\to D_4, D_5 \\
\mathcal{A}\mathcal{B}\nabla\nabla &\to B_3, B_4, B_5, E_2 & \mathcal{B}\mathcal{B}\mathcal{B}\mathcal{B} &\to F_1, F_2 \\
\mathcal{B}\mathcal{B}\mathcal{B}\nabla &\to D_1, D_2, D_3 & \mathcal{B}\mathcal{B}\nabla\nabla &\to B_1, B_2, E_1 \\
\mathcal{B}\nabla\nabla\nabla &\to C_1, C_2. & &
\end{aligned}
$$

Table A1. Possible terms contributing to the action of polynomial affine gravity.

m	n	p	Terms
4	0	0	$\mathcal{A}\mathcal{A}\mathcal{A}\mathcal{A}$
3	1	0	$\mathcal{A}\mathcal{A}\mathcal{A}\mathcal{B}$
3	0	1	$\mathcal{A}\mathcal{A}\mathcal{A}\nabla$
2	2	0	$\mathcal{A}\mathcal{A}\mathcal{B}\mathcal{B}$
2	1	1	$\mathcal{A}\mathcal{A}\mathcal{B}\nabla$
2	0	2	$\mathcal{A}\mathcal{A}\nabla\nabla$
1	3	0	$\mathcal{A}\mathcal{B}\mathcal{B}\mathcal{B}$
1	2	1	$\mathcal{A}\mathcal{B}\mathcal{B}\nabla$
1	1	2	$\mathcal{A}\mathcal{B}\nabla\nabla$
1	0	3	$\mathcal{A}\nabla\nabla\nabla$
0	4	0	$\mathcal{B}\mathcal{B}\mathcal{B}\mathcal{B}$
0	3	1	$\mathcal{B}\mathcal{B}\mathcal{B}\nabla$
0	2	2	$\mathcal{B}\mathcal{B}\nabla\nabla$
0	1	3	$\mathcal{B}\nabla\nabla\nabla$
0	0	4	$\nabla\nabla\nabla\nabla$

Appendix B. Explicit calculation of the field equations

In order to obtain the field equations we proceed as proposed in [32][26], where an auxiliary field is introduced to ease the process. Additionally, we calculate the contribution of each term in the action to the field equations separately, since this allows us to obtain manageable expressions, and to check explicitly the possible *consistent truncations* of the model.

B.1. Field equations for $\Gamma_\mu{}^\lambda{}_\nu$

We shown in section 2.1.1 that the field equations for the symmetric part of the connection are

$$\nabla_\mu \Pi_\Gamma{}^{\mu\nu}{}_\lambda{}^\rho = \frac{\partial^* \mathscr{L}}{\partial \Gamma_\nu{}^\lambda{}_\rho}.$$

The asterisk on the right-hand side denotes the partial derivative with respect to the connection that is not contained in the curvature tensor.

B.1.1. Calculation of the $\Pi_\Gamma{}^{\mu\nu}{}_\lambda{}^\rho$. We start calculating the $z_\Gamma{}^{\mu\nu}{}_\lambda{}^\rho$ and then use the equation (8), to obtain

$$B_1: \quad \left[2\delta_\lambda^{[\mu} \mathcal{B}_\alpha{}^{\nu]}{}_\beta \mathcal{B}_\gamma{}^\rho{}_\delta + 2\delta_\lambda^{[\mu} \mathcal{B}_\alpha{}^{\rho]}{}_\beta \mathcal{B}_\gamma{}^\nu{}_\delta \right] \mathrm{d}V^{\alpha\beta\gamma\delta}$$

$$B_2: \quad 2\mathcal{B}_\gamma{}^\sigma{}_\delta \mathcal{B}_\sigma{}^{(\rho}{}_\lambda \mathrm{d}V^{\nu)\mu\gamma\delta}$$

$$B_3: \quad 2\delta_\lambda^{[\mu} \mathcal{B}_\beta{}^{\nu]}{}_\gamma \mathcal{A}_\delta \mathrm{d}V^{\rho\beta\gamma\delta} + 2\delta_\lambda^{[\mu} \mathcal{B}_\beta{}^{\rho]}{}_\gamma \mathcal{A}_\delta \mathrm{d}V^{\nu\beta\gamma\delta}$$

$$B_4: \quad -2\mathcal{B}_\gamma{}^{(\rho}{}_\delta \mathcal{A}_\lambda \mathrm{d}V^{\nu)\mu\gamma\delta}$$

$$B_5: \quad -2\mathcal{B}_\gamma{}^\sigma{}_\delta \mathcal{A}_\sigma \delta_\lambda^{(\rho} \mathrm{d}V^{\nu)\mu\gamma\delta}$$

$$C_1: \quad 2\nabla_\beta \mathcal{B}_\gamma{}^\rho{}_\delta \delta_\lambda^{[\mu} \mathrm{d}V^{\nu]\beta\gamma\delta} + 2\nabla_\beta \mathcal{B}_\gamma{}^\nu{}_\delta \delta_\lambda^{[\mu} \mathrm{d}V^{\rho]\beta\gamma\delta}$$

$$C_2: \quad -2\nabla_\sigma \mathcal{B}_\gamma{}^\sigma{}_\delta \delta_\lambda^{(\rho} \mathrm{d}V^{\nu)\mu\gamma\delta}.$$

[26] Notice that the original method was proposed by Tulczyjew in [86–88] and developed further by Kijowski and Tulczyjew in [89].

B.1.2. Calculation of $\frac{\partial^* \mathscr{L}}{\partial \Gamma_\nu{}^\lambda{}_\rho}$. In equation (3), the connection appears explicitly (non in the curvature tensor) in the covariant derivative. However, the covariant derivative contains different terms depending on the field it is acting on. The two different terms in which the connection appears are

$$\nabla_\alpha \mathcal{B}_\beta{}^\gamma{}_\delta = \partial_\alpha \mathcal{B}_\beta{}^\gamma{}_\delta + \Gamma_\alpha{}^\lambda{}_\beta \mathcal{B}_\lambda{}^\gamma{}_\delta + \Gamma_\alpha{}^\lambda{}_\delta \mathcal{B}_\beta{}^\gamma{}_\lambda - \Gamma_\alpha{}^\gamma{}_\lambda \mathcal{B}_\beta{}^\lambda{}_\delta$$

and

$$\nabla_\alpha \mathcal{A}_\beta = \partial_\alpha \mathcal{A}_\beta - \Gamma_\alpha{}^\lambda{}_\beta \mathcal{A}_\lambda.$$

Now their partial derivatives with respect to the connection are

$$\frac{\partial \nabla_\alpha \mathcal{B}_\beta{}^\gamma{}_\delta}{\partial \Gamma_\nu{}^\lambda{}_\rho} = 2\delta_\alpha^{(\nu} \delta_\beta^{\rho)} \mathcal{B}_\lambda{}^\gamma{}_\delta + 2\delta_\alpha^{(\nu} \delta_\delta^{\rho)} \mathcal{B}_\beta{}^\gamma{}_\lambda - 2\delta_\lambda^c \delta_\alpha^{(\nu} \mathcal{B}_\beta{}^{\rho)}{}_\delta$$

and

$$\frac{\nabla_\alpha \mathcal{A}_\beta}{\partial \Gamma_\nu{}^\lambda{}_\rho} = -2\delta_\alpha^{(\nu} \delta_\beta^{\rho)} \mathcal{A}_\lambda.$$

The terms coming from these derivatives are

$$C_1: \qquad 2\mathcal{R}_{\mu\lambda}{}^\mu{}_\alpha \mathcal{B}_\gamma{}^{(\rho}{}_\delta \, \mathrm{d}V^{\nu)\alpha\gamma\delta}$$

$$C_2: \qquad 2\mathcal{R}_{\alpha\beta}{}^\sigma{}_\sigma \left[2\mathcal{B}_\lambda{}^{(\nu}{}_\delta \, \mathrm{d}V^{\rho)\alpha\beta\delta} - \delta_\lambda^{(\nu} \mathcal{B}_\gamma{}^{\rho)}{}_\delta \, \mathrm{d}V^{\alpha\beta\gamma\delta} \right]$$

$$D_1: \qquad 2\mathcal{B}_\tau{}^\sigma{}_\lambda \mathcal{B}_\sigma{}^\tau{}_\alpha \mathcal{B}_\gamma{}^{(\rho}{}_\delta \, \mathrm{d}V^{\nu)\alpha\gamma\delta}$$

$$D_2: \qquad 2\mathcal{B}_\alpha{}^\sigma{}_\beta \mathcal{B}_\sigma{}^{(\nu}{}_\tau \left[2\mathcal{B}_\lambda{}^\tau{}_\delta \, \mathrm{d}V^{|\rho)\alpha\beta\delta} - \delta_\lambda^\tau \mathcal{B}_\gamma{}^{|\rho)}{}_\delta \, \mathrm{d}V^{\alpha\beta\gamma\delta} \right]$$

$$D_3: \qquad 2\mathcal{B}_\alpha{}^\sigma{}_\tau \mathcal{B}_\beta{}^{(\nu}{}_\gamma \left[\delta_\sigma^{\rho)} \mathcal{B}_\lambda{}^\tau{}_\delta + \delta_\delta^{\rho)} \mathcal{B}_\sigma{}^\tau{}_\lambda - \delta_\lambda^\tau \mathcal{B}_\sigma{}^{|\rho)}{}_\delta \right] \mathrm{d}V^{\alpha\beta\gamma\delta}$$

$$D_4: \qquad -2\mathcal{B}_\alpha{}^\nu{}_\beta \mathcal{B}_\gamma{}^\rho{}_\delta \mathcal{A}_\lambda \, \mathrm{d}V^{\alpha\beta\gamma\delta}$$

$$D_5: \qquad 2\mathcal{B}_\alpha{}^{(\nu}{}_\beta \mathcal{A}_\sigma \left[2\mathcal{B}_\lambda{}^\sigma{}_\delta \, \mathrm{d}V^{|\rho)\alpha\beta\delta} - \delta_\lambda^\sigma \mathcal{B}_\gamma{}^{|\rho)}{}_\delta \, \mathrm{d}V^{\alpha\beta\gamma\delta} \right]$$

$$D_6: \qquad 2\mathcal{B}_\alpha{}^{(\nu}{}_\beta \mathcal{A}_\gamma \mathcal{A}_\lambda \, \mathrm{d}V^{\rho)\alpha\beta\gamma}$$

$$D_7: \qquad 0$$

$$E_1: \qquad 4\nabla_\sigma \mathcal{B}_\alpha{}^\sigma{}_\beta \left[2\mathcal{B}_\lambda{}^{(\rho}{}_\delta \, \mathrm{d}V^{\nu)\alpha\beta\delta} - \delta_\lambda^{(\nu} \mathcal{B}_\gamma{}^{\rho)}{}_\delta \, \mathrm{d}V^{\alpha\beta\gamma\delta} \right]$$

$$E_2: \qquad 2\mathcal{F}_{\alpha\beta} \left[2\mathcal{B}_\lambda{}^{(\rho}{}_\delta \, \mathrm{d}V^{\nu)\alpha\beta\delta} - \delta_\lambda^{(\nu} \mathcal{B}_\gamma{}^{\rho)}{}_\delta \, \mathrm{d}V^{\alpha\beta\gamma\delta} \right].$$

Notice that in the last set of contributions, the one coming from the term accompanied by the coupling D_7 yields zero. This is because the antisymmetrization of $\nabla \mathcal{A}$ is nothing but the field strength of the potential $\mathcal{A}$ (or the curvature of an Abelian one-form). The strength does not depend on the symmetric connection.

B.1.3. Complete contribution.

$$B_1: \quad \nabla_\mu \left(\left[2\delta_\lambda^{[\mu} \mathcal{B}_\alpha{}^{\nu]}{}_\beta \mathcal{B}_\gamma{}^\rho{}_\delta + 2\delta_\lambda^{[\mu} \mathcal{B}_\alpha{}^{\rho]}{}_\beta \mathcal{B}_\gamma{}^\nu{}_\delta \right] \mathrm{d}V^{\alpha\beta\gamma\delta} \right) = 0 \tag{B.1}$$

$$B_2: \quad \nabla_\mu \left(2\mathcal{B}_\gamma{}^\sigma{}_\delta \mathcal{B}_\sigma{}^{(\rho}{}_\lambda \, \mathrm{d}V^{\nu)\mu\gamma\delta} \right) = 0 \tag{B.2}$$

24

$$B_3: \quad \nabla_\mu \left(2\delta_\lambda^{[\mu} \mathcal{B}_\beta{}^{\nu]}{}_\gamma \mathcal{A}_\delta \, dV^{\rho\beta\gamma\delta} + 2\delta_\lambda^{[\mu} \mathcal{B}_\beta{}^{\rho]}{}_\gamma \mathcal{A}_\delta \, dV^{\nu\beta\gamma\delta} \right) = 0 \tag{B.3}$$

$$B_4: \quad \nabla_\mu \left(-2\mathcal{B}_\gamma{}^{(\rho}{}_\delta \mathcal{A}_\lambda \, dV^{\nu)\mu\gamma\delta} \right) = 0 \tag{B.4}$$

$$B_5: \quad \nabla_\mu \left(-2\mathcal{B}_\gamma{}^\sigma{}_\delta \mathcal{A}_\sigma \delta_\lambda^{(\rho} \, dV^{\nu)\mu\gamma\delta} \right) = 0 \tag{B.5}$$

$$C_1: \quad \nabla_\mu \left(2\nabla_\beta \mathcal{B}_\gamma{}^\rho{}_\delta \delta_\lambda^{[\mu} \, dV^{\nu]\beta\gamma\delta} + 2\nabla_\beta \mathcal{B}_\gamma{}^\nu{}_\delta \delta_\lambda^{[\mu} \, dV^{\rho]\beta\gamma\delta} \right) = 2\mathcal{R}_{\mu\lambda}{}^\mu{}_\alpha \mathcal{B}_\gamma{}^{(\rho}{}_\delta \, dV^{\nu)\alpha\gamma\delta} \tag{B.6}$$

$$C_2: \quad \nabla_\mu \left(-2\nabla_\sigma \mathcal{B}_\gamma{}^\sigma{}_\delta \delta_\lambda^{(\rho} \, dV^{\nu)\mu\gamma\delta} \right) = 2\mathcal{R}_{\alpha\beta}{}^\sigma{}_\sigma \left[2\mathcal{B}_\lambda{}^{(\nu}{}_\delta \, dV^{\rho)\alpha\beta\delta} - \delta_\lambda^{(\nu} \mathcal{B}_\gamma{}^{\rho)}{}_\delta \, dV^{\alpha\beta\gamma\delta} \right] \tag{B.7}$$

$$D_1: \quad 2\mathcal{B}_\tau{}^\sigma{}_\lambda \mathcal{B}_\sigma{}^\tau{}_\alpha \mathcal{B}_\gamma{}^{(\rho}{}_\delta \, dV^{\nu)\alpha\gamma\delta} = 0 \tag{B.8}$$

$$D_2: \quad 2\mathcal{B}_\alpha{}^\sigma{}_\beta \mathcal{B}_\sigma{}^{(\nu}{}_\tau \left[2\mathcal{B}_\lambda{}^\tau{}_\delta \, dV^{|\rho)\alpha\beta\delta} - \delta_\lambda^\tau \mathcal{B}_\gamma{}^{|\rho)}{}_\delta \, dV^{\alpha\beta\gamma\delta} \right] = 0 \tag{B.9}$$

$$D_3: \quad 2\mathcal{B}_\alpha{}^\sigma{}_\tau \mathcal{B}_\beta{}^{(\nu}{}_\gamma \left[\delta_\sigma^{\rho)} \mathcal{B}_\lambda{}^\tau{}_\delta + \delta_\delta^{\rho)} \mathcal{B}_\sigma{}^\tau{}_\lambda - \delta_\lambda^\tau \mathcal{B}_\sigma{}^{|\rho)}{}_\delta \right] dV^{\alpha\beta\gamma\delta} = 0 \tag{B.10}$$

$$D_4: \quad -2\mathcal{B}_\alpha{}^\nu{}_\beta \mathcal{B}_\gamma{}^\rho{}_\delta \mathcal{A}_\lambda \, dV^{\alpha\beta\gamma\delta} = 0 \tag{B.11}$$

$$D_5: \quad 2\mathcal{B}_\alpha{}^{(\nu}{}_\beta \mathcal{A}_\sigma \left[2\mathcal{B}_\lambda{}^\sigma{}_\delta \, dV^{|\rho)\alpha\beta\delta} - \delta_\lambda^\sigma \mathcal{B}_\gamma{}^{|\rho)}{}_\delta \, dV^{\alpha\beta\gamma\delta} \right] = 0 \tag{B.12}$$

$$D_6: \quad 2\mathcal{B}_\alpha{}^{(\nu}{}_\beta \mathcal{A}_\gamma \mathcal{A}_\lambda \, dV^{\rho)\alpha\beta\gamma} = 0 \tag{B.13}$$

$$E_1: \quad 4\nabla_\sigma \mathcal{B}_\alpha{}^\sigma{}_\beta \left[2\mathcal{B}_\lambda{}^{(\rho}{}_\delta \, dV^{\nu)\alpha\beta\delta} - \delta_\lambda^{(\nu} \mathcal{B}_\gamma{}^{\rho)}{}_\delta \, dV^{\alpha\beta\gamma\delta} \right] = 0 \tag{B.14}$$

$$E_2: \quad 2\mathcal{F}_{\alpha\beta} \left[2\mathcal{B}_\lambda{}^{(\rho}{}_\delta \, dV^{\nu)\alpha\beta\delta} - \delta_\lambda^{(\nu} \mathcal{B}_\gamma{}^{\rho)}{}_\delta \, dV^{\alpha\beta\gamma\delta} \right] = 0. \tag{B.15}$$

B.2. Field equations for $\boldsymbol{B_\mu{}^\lambda{}_\nu}$ In section 2.1.2 we show that the field equations for the $\mathcal{B}$ field are

$$\nabla_\mu \Pi_\mathcal{B}{}^{\mu\nu}{}_\lambda{}^\rho = \frac{\partial \mathscr{L}}{\partial \mathcal{B}_\nu{}^\lambda{}_\rho}.$$

We now calculate explicitly the contribution of each term in the action to the above field equation.

B.2.1. Calculation of $\mathbf{\Pi}_{\mathcal{B}}{}^{\mu\nu}{}_{\lambda}{}^{\rho}$*.* The explicit calculation of $\Pi_{\mathcal{B}}{}^{\mu\nu}{}_{\lambda}{}^{\rho}$ yield

$$
\begin{aligned}
C_1 &: \qquad -2\mathcal{R}_{\sigma\alpha}{}^{\sigma}{}_{\lambda}\,\mathrm{d}V^{\mu\nu\rho\alpha}\\
C_2 &: \qquad 2\mathcal{R}_{\alpha\beta}{}^{\sigma}{}_{\sigma}\delta^{\mu}_{\lambda}\,\mathrm{d}V^{\nu\rho\alpha\beta}\\
D_1 &: \qquad -2\mathcal{B}_{\sigma}{}^{\theta}{}_{\lambda}\mathcal{B}_{\theta}{}^{\sigma}{}_{\alpha}\,\mathrm{d}V^{\mu\nu\rho\alpha}\\
D_2 &: \qquad 2\mathcal{B}_{\alpha}{}^{\sigma}{}_{\beta}\mathcal{B}_{\sigma}{}^{\mu}{}_{\lambda}\,\mathrm{d}V^{\nu\rho\alpha\beta}\\
D_3 &: \qquad -2\mathcal{B}_{\alpha}{}^{\nu}{}_{\lambda}\mathcal{B}_{\beta}{}^{\mu}{}_{\gamma}\,\mathrm{d}V^{\rho\alpha\beta\gamma}\\
D_5 &: \qquad 2\mathcal{B}_{\alpha}{}^{\mu}{}_{\beta}\mathcal{A}_{\lambda}\,\mathrm{d}V^{\nu\rho\alpha\beta}\\
E_1 &: \qquad 4\delta^{\mu}_{\lambda}\nabla_{\sigma}\mathcal{B}_{\alpha}{}^{\sigma}{}_{\beta}\,\mathrm{d}V^{\nu\rho\alpha\beta}\\
E_2 &: \qquad 2\delta^{\mu}_{\lambda}\mathcal{F}_{\alpha\beta}\,\mathrm{d}V^{\nu\rho\alpha\beta}\,.
\end{aligned}
$$

B.2.2. Calculation of $\frac{\partial\mathscr{L}}{\partial\mathcal{B}_{\nu}{}^{\lambda}{}_{\rho}}$*.* The contributions to the right-hand side of the field equations for the $\mathcal{B}$-field are,

$$
\begin{aligned}
B_1 &: \quad 4\mathcal{R}_{\mu(\sigma}{}^{\mu}{}_{\lambda)}\mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta}\,\mathrm{d}V^{\nu\rho\gamma\delta}\\
B_2 &: \quad 2\mathcal{R}_{\alpha\beta}{}^{\mu}{}_{\sigma}\mathcal{B}_{\mu}{}^{\sigma}{}_{\lambda}\,\mathrm{d}V^{\nu\rho\alpha\beta}+2\mathcal{R}_{\alpha\beta}{}^{[\nu}{}_{\lambda}\mathcal{B}_{\gamma}{}^{\rho]}{}_{\delta}\,\mathrm{d}V^{\alpha\beta\gamma\delta}\\
B_3 &: \quad 2\mathcal{R}_{\mu\lambda}{}^{\mu}{}_{\alpha}\mathcal{A}_{\beta}\,\mathrm{d}V^{\nu\rho\alpha\beta}\\
B_4 &: \quad 2\mathcal{R}_{\alpha\beta}{}^{\sigma}{}_{\lambda}\mathcal{A}_{\sigma}\,\mathrm{d}V^{\nu\rho\alpha\beta}\\
B_5 &: \quad 2\mathcal{R}_{\alpha\beta}{}^{\tau}{}_{\tau}\mathcal{A}_{\lambda}\,\mathrm{d}V^{\nu\rho\alpha\beta}\\
D_1 &: \quad 2\mathcal{B}_{\lambda}{}^{[\nu}{}_{\alpha}\nabla_{\beta}\mathcal{B}_{\gamma}{}^{\rho]}{}_{\delta}\,\mathrm{d}V^{\alpha\beta\gamma\delta}+2\mathcal{B}_{\lambda}{}^{[\nu|}{}_{\sigma}\nabla_{\beta}\mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta}\,\mathrm{d}V^{|\rho]\beta\gamma\delta}\\
D_2 &: \quad 2\mathcal{B}_{\lambda}{}^{\mu}{}_{\sigma}\nabla_{\mu}\mathcal{B}_{\alpha}{}^{\sigma}{}_{\beta}\,\mathrm{d}V^{\nu\rho\alpha\beta}+2\mathcal{B}_{\alpha}{}^{[\nu}{}_{\beta}\nabla_{\lambda}\mathcal{B}_{\gamma}{}^{\rho]}{}_{\delta}\,\mathrm{d}V^{\alpha\beta\gamma\delta}\\
D_3 &: \quad 2\mathcal{B}_{\beta}{}^{\mu}{}_{\gamma}\nabla_{\mu}\mathcal{B}_{\lambda}{}^{[\rho}{}_{\delta}\,\mathrm{d}V^{\nu]\beta\gamma\delta}+2\mathcal{B}_{\gamma}{}^{\mu}{}_{\sigma}\nabla_{\lambda}\mathcal{B}_{\mu}{}^{\sigma}{}_{\delta}\,\mathrm{d}V^{\nu\rho\gamma\delta}\\
D_4 &: \quad 4\mathcal{B}_{\alpha}{}^{\sigma}{}_{\beta}\nabla_{(\lambda}\mathcal{A}_{\sigma)}\,\mathrm{d}V^{\nu\rho\alpha\beta}\\
D_5 &: \quad 2\nabla_{\lambda}\mathcal{B}_{\alpha}{}^{\sigma}{}_{\beta}\mathcal{A}_{\sigma}\,\mathrm{d}V^{\nu\rho\alpha\beta}\\
D_6 &: \quad 2\mathcal{A}_{\gamma}\nabla_{\lambda}\mathcal{A}_{\delta}\,\mathrm{d}V^{\nu\rho\gamma\delta}\\
D_7 &: \quad 2\mathcal{A}_{\lambda}F_{\gamma\delta}\,\mathrm{d}V^{\nu\rho\gamma\delta}\\
F_1 &: \quad 4\mathcal{B}_{\alpha}{}^{\mu}{}_{\beta}\mathcal{B}_{\mu}{}^{\sigma}{}_{\tau}\mathcal{B}_{\lambda}{}^{\tau}{}_{\sigma}\,\mathrm{d}V^{\nu\rho\alpha\beta}+4\mathcal{B}_{\alpha}{}^{\mu}{}_{\beta}\mathcal{B}_{\gamma}{}^{[\nu}{}_{\delta}\mathcal{B}_{\mu}{}^{\rho]}{}_{\lambda}\,\mathrm{d}V^{\alpha\beta\gamma\delta}\\
F_2 &: \quad 2\mathcal{B}_{\alpha}{}^{\mu}{}_{\sigma}\mathcal{B}_{\beta}{}^{\sigma}{}_{\tau}\mathcal{B}_{\lambda}{}^{\tau}{}_{\mu}\,\mathrm{d}V^{\nu\rho\alpha\beta}+2\mathcal{B}_{\alpha}{}^{\mu}{}_{\beta}\mathcal{B}_{\mu}{}^{\sigma}{}_{\lambda}\mathcal{B}_{\gamma}{}^{[\rho}{}_{\sigma}\,\mathrm{d}V^{\nu]\alpha\beta\gamma}\\
&\quad -2\mathcal{B}_{\alpha}{}^{\mu}{}_{\beta}\mathcal{B}_{\gamma}{}^{\sigma}{}_{\lambda}\mathcal{B}_{\mu}{}^{[\rho}{}_{\sigma}\,\mathrm{d}V^{\nu]\alpha\beta\gamma}+2\mathcal{B}_{\alpha}{}^{[\nu}{}_{\beta}\mathcal{B}_{\gamma}{}^{\rho]}{}_{\sigma}\mathcal{B}_{\delta}{}^{\sigma}{}_{\lambda}\,\mathrm{d}V^{\alpha\beta\gamma\delta}\\
F_3 &: \quad 2\mathcal{B}_{\lambda}{}^{[\nu}{}_{\alpha}\mathcal{B}_{\beta}{}^{\rho]}{}_{\gamma}\mathcal{A}_{\delta}\,\mathrm{d}V^{\alpha\beta\gamma\delta}+2\mathcal{B}_{\alpha}{}^{\sigma}{}_{\beta}\mathcal{A}_{\gamma}\mathcal{B}_{\lambda}{}^{[\nu}{}_{\sigma}\,\mathrm{d}V^{\rho]\alpha\beta\gamma}\\
&\quad +2\mathcal{B}_{\sigma}{}^{\mu}{}_{\lambda}\mathcal{B}_{\mu}{}^{\sigma}{}_{\alpha}\mathcal{A}_{\beta}\,\mathrm{d}V^{\nu\rho\alpha\beta}\\
F_4 &: \quad 4\mathcal{B}_{\alpha}{}^{\mu}{}_{\beta}\mathcal{A}_{\mu}\mathcal{A}_{\lambda}\,\mathrm{d}V^{\nu\rho\alpha\beta}\,.
\end{aligned}
$$

B.2.3. Complete contribution.

$$
B_1: \quad 4\mathcal{R}_{\mu(\sigma}{}^{\mu}{}_{\lambda)}\mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta}\,\mathrm{d}V^{\nu\rho\gamma\delta}=0 \tag{B.16}
$$

$$
B_2: \quad 2\mathcal{R}_{\alpha\beta}{}^{\mu}{}_{\sigma}\mathcal{B}_{\mu}{}^{\sigma}{}_{\lambda}\,\mathrm{d}V^{\nu\rho\alpha\beta}+2\mathcal{R}_{\alpha\beta}{}^{[\nu}{}_{\lambda}\mathcal{B}_{\gamma}{}^{\rho]}{}_{\delta}\,\mathrm{d}V^{\alpha\beta\gamma\delta}=0 \tag{B.17}
$$

$B_3:$ $2\mathcal{R}_{\mu\lambda}{}^{\mu}{}_{\alpha}\mathcal{A}_{\beta}\,\mathrm{d}V^{\nu\rho\alpha\beta} = 0$ (B.18)

$B_4:$ $2\mathcal{R}_{\alpha\beta}{}^{\sigma}{}_{\lambda}\mathcal{A}_{\sigma}\,\mathrm{d}V^{\nu\rho\alpha\beta} = 0$ (B.19)

$B_5:$ $2\mathcal{R}_{\alpha\beta}{}^{\tau}{}_{\tau}\mathcal{A}_{\lambda}\,\mathrm{d}V^{\nu\rho\alpha\beta} = 0$ (B.20)

$C_1:$ $\nabla_{\mu}\left(-2\mathcal{R}_{\sigma\alpha}{}^{\sigma}{}_{\lambda}\,\mathrm{d}V^{\mu\nu\rho\alpha}\right) = 0$ (B.21)

$C_2:$ $\nabla_{\mu}\left(2\mathcal{R}_{\alpha\beta}{}^{\sigma}{}_{\sigma}\delta^{\mu}_{\lambda}\,\mathrm{d}V^{\nu\rho\alpha\beta}\right) = 0$ (B.22)

$D_1:$ $\nabla_{\mu}\left(-2\mathcal{B}_{\sigma}{}^{\theta}{}_{\lambda}\mathcal{B}_{\theta}{}^{\sigma}{}_{\alpha}\,\mathrm{d}V^{\mu\nu\rho\alpha}\right) = 2\mathcal{B}_{\lambda}{}^{[\nu}{}_{\alpha}\nabla_{\beta}\mathcal{B}_{\gamma}{}^{\rho]}{}_{\delta}\,\mathrm{d}V^{\alpha\beta\gamma\delta} + 2\mathcal{B}_{\lambda}{}^{[\nu|}{}_{\sigma}\nabla_{\beta}\mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta}\,\mathrm{d}V^{|\rho]\beta\gamma\delta}$ (B.23)

$D_2:$ $\nabla_{\mu}\left(2\mathcal{B}_{\alpha}{}^{\sigma}{}_{\beta}\mathcal{B}_{\sigma}{}^{\mu}{}_{\lambda}\,\mathrm{d}V^{\nu\rho\alpha\beta}\right) = 2\mathcal{B}_{\lambda}{}^{\mu}{}_{\sigma}\nabla_{\mu}\mathcal{B}_{\alpha}{}^{\sigma}{}_{\beta}\,\mathrm{d}V^{\nu\rho\alpha\beta} + 2\mathcal{B}_{\alpha}{}^{[\nu}{}_{\beta}\nabla_{\lambda}\mathcal{B}_{\gamma}{}^{\rho]}{}_{\delta}\,\mathrm{d}V^{\alpha\beta\gamma\delta}$ (B.24)

$D_3:$ $\nabla_{\mu}\left(-2\mathcal{B}_{\alpha}{}^{\nu}{}_{\lambda}\mathcal{B}_{\beta}{}^{\mu}{}_{\gamma}\,\mathrm{d}V^{\rho\alpha\beta\gamma}\right) = 2\mathcal{B}_{\beta}{}^{\mu}{}_{\gamma}\nabla_{\mu}\mathcal{B}_{\lambda}{}^{[\rho}{}_{\delta}\,\mathrm{d}V^{\nu]\beta\gamma\delta} + 2\mathcal{B}_{\gamma}{}^{\mu}{}_{\sigma}\nabla_{\lambda}\mathcal{B}_{\mu}{}^{\sigma}{}_{\delta}\,\mathrm{d}V^{\nu\rho\gamma\delta}$ (B.25)

$D_4:$ $4\mathcal{B}_{\alpha}{}^{\sigma}{}_{\beta}\nabla_{(\lambda}\mathcal{A}_{\sigma)}\,\mathrm{d}V^{\nu\rho\alpha\beta} = 0$ (B.26)

$D_5:$ $\nabla_{\mu}\left(2\mathcal{B}_{\alpha}{}^{\mu}{}_{\beta}\mathcal{A}_{\lambda}\,\mathrm{d}V^{\nu\rho\alpha\beta}\right) = 2\nabla_{\lambda}\mathcal{B}_{\alpha}{}^{\sigma}{}_{\beta}\mathcal{A}_{\sigma}\,\mathrm{d}V^{\nu\rho\alpha\beta}$ (B.27)

$D_6:$ $2\mathcal{A}_{\gamma}\nabla_{\lambda}\mathcal{A}_{\delta}\,\mathrm{d}V^{\nu\rho\gamma\delta} = 0$ (B.28)

$D_7:$ $2\mathcal{A}_{\lambda}F_{\gamma\delta}\,\mathrm{d}V^{\nu\rho\gamma\delta} = 0$ (B.29)

$E_1:$ $\nabla_{\mu}\left(4\delta^{\mu}_{\lambda}\nabla_{\sigma}\mathcal{B}_{\alpha}{}^{\sigma}{}_{\beta}\,\mathrm{d}V^{\nu\rho\alpha\beta}\right) = 0$ (B.30)

$E_2:$ $\nabla_{\mu}\left(2\delta^{\mu}_{\lambda}\mathcal{F}_{\alpha\beta}\,\mathrm{d}V^{\nu\rho\alpha\beta}\right) = 0$ (B.31)

$F_1:$ $4\mathcal{B}_{\alpha}{}^{\mu}{}_{\beta}\mathcal{B}_{\mu}{}^{\sigma}{}_{\tau}\mathcal{B}_{\lambda}{}^{\tau}{}_{\sigma}\,\mathrm{d}V^{\nu\rho\alpha\beta} + 4\mathcal{B}_{\alpha}{}^{\mu}{}_{\beta}\mathcal{B}_{\gamma}{}^{[\nu}{}_{\delta}\mathcal{B}_{\mu}{}^{\rho]}{}_{\lambda}\,\mathrm{d}V^{\alpha\beta\gamma\delta} = 0$ (B.32)

$F_2:$ $2\mathcal{B}_{\alpha}{}^{\mu}{}_{\sigma}\mathcal{B}_{\beta}{}^{\sigma}{}_{\tau}\mathcal{B}_{\lambda}{}^{\tau}{}_{\mu}\,\mathrm{d}V^{\nu\rho\alpha\beta} + 2\mathcal{B}_{\alpha}{}^{\mu}{}_{\beta}\mathcal{B}_{\mu}{}^{\sigma}{}_{\lambda}\mathcal{B}_{\gamma}{}^{[\rho}{}_{\sigma}\,\mathrm{d}V^{\nu]\alpha\beta\gamma}$

$\qquad - 2\mathcal{B}_{\alpha}{}^{\mu}{}_{\beta}\mathcal{B}_{\gamma}{}^{\sigma}{}_{\lambda}\mathcal{B}_{\mu}{}^{[\rho}{}_{\sigma}\,\mathrm{d}V^{\nu]\alpha\beta\gamma} + 2\mathcal{B}_{\alpha}{}^{[\nu}{}_{\beta}\mathcal{B}_{\gamma}{}^{\rho]}{}_{\sigma}\mathcal{B}_{\delta}{}^{\sigma}{}_{\lambda}\,\mathrm{d}V^{\alpha\beta\gamma\delta} = 0$ (B.33)

$F_3:$ $2\mathcal{B}_{\lambda}{}^{[\nu}{}_{\alpha}\mathcal{B}_{\beta}{}^{\rho]}{}_{\gamma}\mathcal{A}_{\delta}\,\mathrm{d}V^{\alpha\beta\gamma\delta} + 2\mathcal{B}_{\alpha}{}^{\sigma}{}_{\beta}\mathcal{A}_{\gamma}\mathcal{B}_{\lambda}{}^{[\nu}{}_{\sigma}\,\mathrm{d}V^{\rho]\alpha\beta\gamma} + 2\mathcal{B}_{\sigma}{}^{\mu}{}_{\lambda}\mathcal{B}_{\mu}{}^{\sigma}{}_{\alpha}\mathcal{A}_{\beta}\,\mathrm{d}V^{\nu\rho\alpha\beta} = 0$ (B.34)

$F_4:$ $4\mathcal{B}_{\alpha}{}^{\mu}{}_{\beta}\mathcal{A}_{\mu}\mathcal{A}_{\lambda}\,\mathrm{d}V^{\nu\rho\alpha\beta} = 0.$ (B.35)

B.3. Field equations for $\mathcal{A}_\mu$　　In this section, we show the *complete* Euler–Lagrage equations for the $\mathcal{A}$ field.

$$B_3: \quad \mathcal{R}_{\sigma\tau}{}^\sigma{}_\alpha \mathcal{B}_\beta{}^\tau{}_\gamma \, \mathrm{d}V^{\alpha\beta\gamma\nu} = 0 \tag{B.36}$$

$$B_4: \quad \mathcal{R}_{\alpha\beta}{}^\nu{}_\sigma \mathcal{B}_\gamma{}^\sigma{}_\delta \, \mathrm{d}V^{\alpha\beta\gamma\delta} = 0 \tag{B.37}$$

$$B_5: \quad \mathcal{R}_{\alpha\beta}{}^\sigma{}_\sigma \mathcal{B}_\gamma{}^\nu{}_\delta \, \mathrm{d}V^{\alpha\beta\gamma\delta} = 0 \tag{B.38}$$

$$D_4 \quad \nabla_\mu \left[\mathcal{B}_\alpha{}^\mu{}_\beta \mathcal{B}_\gamma{}^\nu{}_\delta \, \mathrm{d}V^{\alpha\beta\gamma\delta} \right] = 0 \tag{B.39}$$

$$D_5: \quad \mathcal{B}_\alpha{}^\sigma{}_\beta \nabla_\sigma \mathcal{B}_\gamma{}^\nu{}_\delta \, \mathrm{d}V^{\alpha\beta\gamma\delta} = 0 \tag{B.40}$$

$$D_6: \quad \nabla_\mu \left[\mathcal{B}_\alpha{}^\mu{}_\beta \mathcal{A}_\gamma \, \mathrm{d}V^{\alpha\beta\gamma\nu} \right] + \mathcal{B}_\alpha{}^\mu{}_\beta \nabla_\mu \mathcal{A}_\gamma \, \mathrm{d}V^{\alpha\beta\gamma\nu} = 0 \tag{B.41}$$

$$D_7: \quad \nabla_\mu \left[\mathcal{B}_\alpha{}^\sigma{}_\beta \mathcal{A}_\sigma \, \mathrm{d}V^{\alpha\beta\mu\nu} \right] + \mathcal{B}_\alpha{}^\nu{}_\beta \mathcal{F}_{\gamma\delta} \, \mathrm{d}V^{\alpha\beta\gamma\delta} = 0 \tag{B.42}$$

$$E_2: \quad \nabla_\mu \left[\nabla_\sigma \mathcal{B}_\alpha{}^\sigma{}_\beta \, \mathrm{d}V^{\alpha\beta\mu\nu} \right] = 0 \tag{B.43}$$

$$F_3: \quad \mathcal{B}_\sigma{}^\tau{}_\lambda \mathcal{B}_\tau{}^\sigma{}_\alpha \mathcal{B}_\beta{}^\lambda{}_\gamma \, \mathrm{d}V^{\alpha\beta\gamma\nu} = 0 \tag{B.44}$$

$$F_4: \quad 2\mathcal{B}_\alpha{}^\sigma{}_\beta \mathcal{B}_\gamma{}^\nu{}_\delta \mathcal{A}_\sigma \, \mathrm{d}V^{\alpha\beta\gamma\delta} = 0. \tag{B.45}$$

ORCID iDs

Oscar Castillo-Felisola ⓘ https://orcid.org/0000-0002-1097-2976

References

[1] Einstein A 1915 *Preussische Akad. Wiss. Sitzungsberichte* **315** 778–86
[2] Einstein A 1915 *Sitzungsber. Preuss. Akad. Wiss.* **1** 778
[3] Einstein A 1915 *Sitzungsber. Preuss. Akad. Wiss.* **47** 831
[4] Einstein A 1915 *Sitzungsber. Preuss. Akad. Wiss.* **1** 844
[5] Einstein A 1916 *Ann. Phys., NY* **49** 284–339
[6] Will C M 2014 *Living Rev. Relativ.* **17** 4
[7] Abbott B P *et al* (Virgo, LIGO Scientific) 2016 *Phys. Rev. Lett.* **116** 241103
[8] Abbott B P *et al* 2017 *Astrophys. J.* **848** L13
[9] DeWitt B S 1967 *Phys. Rev.* **160** 1113
[10] DeWitt B S 1967 *Phys. Rev.* **162** 1195
[11] Deser S and van Nieuwenhuizen P 1974 *Phys. Rev.* D **10** 401
[12] Deser S and van Nieuwenhuizen P 1974 *Phys. Rev.* D **10** 411
[13] 't Hooft G and Veltman M J G 1974 *Ann. Poincare Phys. Theor.* A **20** 69
[14] Ashtekar A 1986 *Phys. Rev. Lett.* **57** 2244–7
[15] Ashtekar A 1987 *Phys. Rev.* D **36** 1587
[16] Zwicky F 1937 *Astrophys. J.* **86** 217
[17] Rubin V C, Kent J and Ford W 1970 *Astrophys. J.* **159** 379
[18] Sofue Y and Rubin V 2001 *Ann. Rev. Astron. Astrophys.* **39** 137

[19] Riess A G *et al* (Supernova Search Team) 1998 *Astron. J.* **116** 1009
[20] Perlmutter S *et al* (Supernova Cosmology Project) 1999 *Astrophys. J.* **517** 565
[21] Cartan E 1922 *C. R. Acad. Sci., Paris* **174** 593
[22] Cartan E 1923 *Ann. Ecole Norm. Super.* **40** 325
[23] Cartan E 1924 *Ann. Ecole Norm. Super.* **41** 1
[24] Cartan E 1925 *Ann. Ecole Norm. Super.* **42** 17
[25] Kaluza T 1921 *Sitzungsber. Preuss. Akad. Wiss. Berl.* **1921** 966
[26] Klein O 1926 *Z. Phys.* **37** 895
[27] Lovelock D 1971 *J. Math. Phys.* **12** 498–501
[28] Hehl F W, McCrea J D, Mielke E W and Ne'eman Y 1995 *Phys. Rep.* **258** 1–171
[29] Mardones A and Zanelli J 1991 *Class. Quantum Grav.* **8** 1545
[30] Eddington A S 1923 *The Mathematical Theory of Relativity* (Cambridge: Cambridge University Press)
[31] Schrödinger E 1950 *Space-time Structure* (Cambridge: Cambridge University Press)
[32] Kijowski J 1978 *Gen. Relativ. Gravit.* **9** 857
[33] Krasnov K 2006 (arXiv:hep-th/0611182)
[34] Krasnov K 2007 *Mod. Phys. Lett.* A **22** 3013–26
[35] Krasnov K 2008 *Class. Quantum Grav.* **25** 025001
[36] Krasnov K and Shtanov Y 2008 *Class. Quantum Grav.* **25** 025002
[37] Krasnov K 2011 *Phys. Rev. Lett.* **106** 251103
[38] Popławski N J 2007 (arXiv:0705.0351)
[39] Popławski N J 2007 *Mod. Phys. Lett.* A **22** 2701
[40] Popławski N J 2014 *Gen. Relativ. Gravit.* **46** 1625
[41] Castillo-Felisola O and Skirzewski A 2015 *Rev. Mex. Fis.* **61** 421
[42] Castillo-Felisola O and Skirzewski A 2018 *Class. Quantum Grav.* **35** 055012
[43] Castillo-Felisola O 2018 Beyond Einstein: a polynomial affine model of gravity *Gravity* (London: IntechOpen) pp 183–201
[44] Castillo-Felisola O, Perdiguero J and Orellana O 2019 Cosmological solutions to polynomial affine gravity in the torsion-free sector *Redifining Standard Model Cosmology* (London: IntechOpen)
[45] McGady D A and Rodina L 2014 *Phys. Rev.* D **90** 084048
[46] Camanho X O, Edelstein J D, Maldacena J and Zhiboedov A 2016 *J. High Energy Phys.* JHEP02(2016)020
[47] Zerwekh A R 2013 *Int. J. Mod. Phys.* A **28** 1350054
[48] Iosifidis D 2019 Metric-affne gravity and cosmology: aspects of torsion and non-metricity in gravity theories *PhD Thesis* Institute of Theoretical Physics—Physics Department of Aristotle University of Thessaloniki
[49] Iosifidis D 2019 *Class. Quantum Grav.* **36** 085001
[50] Parattu K, Chakraborty S, Majhi B and Padmanabhan T 2016 *Gen. Relativ. Gravit.* **48** 94
[51] Krishnan C, Maheshwari S and Subramanian P 2017 *J. Phys.: Conf. Ser.* **883** 012011
[52] Krishnan C and Raju A 2017 *Mod. Phys. Lett.* A **32** 1750077
[53] Lehner L, Myers R, Poisson E and Sorkin R 2016 *Phys. Rev.* D **94** 084046
[54] Hopfmüller F and Freidel L 2017 *Phys. Rev.* D **95** 104006
[55] Jubb I, Samuel J, Sorkin R and Surya S 2017 *Class. Quantum Grav.* **34** 065006
[56] Nomizu K and Sasaki T 1994 *Affine Differential Geometry* (Cambridge: Cambridge University Press)
[57] Besse A L 2007 *Einstein Manifolds* (New York: Springer)
[58] Stephenson G 1958 *Nuovo Cimento* **9** 263–9
[59] Kilmister C W and Newman D J 1961 The use of algebraic structures in physics *Mathematical Proceedings of the Cambridge Philosophical Society* vol 57 (Cambridge: Cambridge University Press) p 851
[60] Yang C N 1974 *Phys. Rev. Lett.* **33** 445
[61] Pavelle R 1975 *Phys. Rev. Lett.* **34** 1114
[62] Thompson A H 1975 *Phys. Rev. Lett.* **35** 320
[63] Einstein A 1919 *Sitzungsber. Preuss. Akad. Wiss. Berl.* 349–56
[64] Ferraris M and Kijowski J 1982 *Gen. Relativ. Gravit.* **14** 165
[65] Garfinkle D, Mead L R and Ringermacher H I 2018 The shape of the orbit in FLRW spacetimes *J. Phys. Commun.* **2** 111001

[66] Mielke E W 2017 *Geometrodynamics of Gauge Fields* (*Mathematical Physics Studies*) (New York: Springer) p 373
[67] Weinberg S 1989 *Rev. Mod. Phys.* **61** 1–23
[68] Ng Y J and van Dam H 1991 *J. Math. Phys.* **32** 1337–40
[69] Smolin L 2009 *Phys. Rev.* D **80** 124017
[70] Ellis G F R, van Elst H, Murugan J and Uzan J P 2011 *Class. Quantum Grav.* **28** 225007
[71] Boskoff W G and Capozziello S 2019 *Int. J. Geom. Methods M* **16** 1950161
[72] Ivey T A and Landsberg J M 2003 *Cartan for Beginners* (*Graduate Studies in Mathematics* vol 61) (Providence, RI: American Mathematical Society)
[73] Baez J 1994 *Gauge Fields, Knots, and Gravity* (Singapore: World Scientific)
[74] Lévy-Leblond J M 1965 *Ann. l'I.H.P. Phys. Théor.* **3** 1–12
[75] Josset T, Perez A and Sudarsky D 2017 *Phys. Rev. Lett.* **118** 021102
[76] Brensinger S, Heitritter K, Rodgers V G J, Stiffler K and Whiting C A 2019 *Class. Quantum Grav.* **37** 055003
[77] Cervantes-Cota J L and Liebscher D E 2016 *Gen. Relativ. Gravit.* **48** 108
[78] Barrow J D, Tsagas C G and Fanaras G 2019 *Eur. Phys. J.* C **79** 764
[79] Kilic D 2019 (arXiv:1907.08850)
[80] Roberts C W 1995 *Differ. Geom. Appl.* **5** 237–55
[81] Chothe H R, Dutta A and Sur S 2019 *Int.l J. Mod. Phys.* D **28** 1950174
[82] Stein W A *et al* 2019 *Sage Mathematics Software (Version 8.6)* (The Sage Development Team) (www.sagemath.org)
[83] Gourgoulhon E *et al* 2019 *Sage Manifolds (Version 8.6)* (SageManifolds Development Team) (http://sagemanifolds.obspm.fr/index.html)
[84] Gourgoulhon E, Bejger M and Mancini M 2015 *J. Phys.: Conf. Ser.* **600** 012002
[85] Gourgoulhon É and Mancini M 2018 *Les Cours CIRM* **6** 1–54
[86] Tulczyjew W M 1974 *Hamiltonian Systems, Lagrangian Systems and the Legendre Tranformation* (*Symp. Mathematica* vol 15) (New York: Academic) pp 247–58
[87] Tulczyjew W M 1975 A symplectic formulation of particle dynamics *Conf. on Differential Geometrical Methods in Mathematical Physics Bonn (Germany*, *1–4 July 1975) pp* 457–63
[88] Tulczyjew W M 1975 A symplectic formulation of field dynamics *Conf. on Differential Geometrical Methods in Mathematical Physics Bonn (Germany*, *1–4 July 1975) pp* 464–8
[89] Kijowski J and Tulczyjew W M 1979 *A Symplectic Framework for Field Theories* (New York: Springer)