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New variational characterization of periodic waves in the fractional Korteweg–de Vries equation

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Abstract

Periodic waves in the fractional Korteweg–de Vries equation have been previously characterized as constrained minimizers of energy subject to fixed momentum and mass. Here we characterize these periodic waves as constrained minimizers of the quadratic form of energy subject to fixed cubic part of energy and the zero mean. This new variational characterization allows us to unfold the existence region of travelling periodic waves and to give a sharp criterion for spectral stability of periodic waves with respect to perturbations of the same period. The sharp stability criterion is given by the monotonicity of the map from the wave speed to the wave momentum similarly to the stability criterion for solitary waves.

Keywords: fractional Korteweg–de Vries equation, periodic traveling waves, existence, spectral stability, fold bifurcation
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(Some figures may appear in colour only in the online journal)

1. Introduction

We address the fractional Korteweg–de Vries (KdV) equation [8], which is written in the form:

$$u_t + 2uu_x - (D^\alpha u)_x = 0, \tag{1.1}$$

where $u(t,x)$ is a real function of $(t,x) \in \mathbb{R} \times \mathbb{R}$ and D^α represents the fractional derivative defined via Fourier transform as

$$\widehat{D^\alpha g}(\xi) = |\xi|^\alpha \widehat{g}(\xi), \quad \xi \in \mathbb{R}.$$

In what follows we consider the periodic traveling waves with the normalized period $T = 2\pi$, for which x is restricted on $\mathbb{T} := [-\pi, \pi]$ and ξ is restricted on $\mathbb{Z}$.

The fractional KdV equation (1.1) admits formally the following conserved quantities:

$$E(u) = \frac{1}{2}\int_{-\pi}^{\pi} (D^{\frac{\alpha}{2}} u)^2 - \frac{1}{3}\int_{-\pi}^{\pi} u^3 \mathrm{d}x, \tag{1.2}$$

$$F(u) = \frac{1}{2}\int_{-\pi}^{\pi} u^2 \mathrm{d}x, \tag{1.3}$$

and

$$M(u) = \int_{-\pi}^{\pi} u\,\mathrm{d}x, \tag{1.4}$$

which have meaning of energy, momentum, and mass respectively.

Local well-posedness of the Cauchy problem for the fractional KdV equation (1.1) was proven in [1] for the initial data in Sobolev space $H^s(\mathbb{R})$ or $H^s(\mathbb{T})$ for $s \geqslant \frac{3}{2}$. Local well-posedness in $H^s(\mathbb{R})$ for $s > \frac{3}{2} - \frac{3}{8}\alpha$ was proven in [31], where the authors also showed existence of weak global solutions in energy space $H^{\frac{\alpha}{2}}(\mathbb{R})$ for $\alpha > \frac{1}{2}$ and for $\alpha = \frac{1}{2}$ and small data. More recently, local well-posedness in $H^s(\mathbb{R})$ was proven in [33] for $\alpha > 0$ and $s > \frac{3}{2} - \frac{5}{4}\alpha$. Together with the conservation of energy, the latter result implies global well-posedness in the energy space $H^{\frac{\alpha}{2}}(\mathbb{R})$ for $\alpha > \frac{6}{7}$. Traveling solitary waves were characterized as minimizers of energy subject to the fixed momentum in [32] for $\alpha \in \left(\frac{1}{2}, 1\right)$ and in [2] for $\alpha \geqslant 1$.

Existence and stability of traveling periodic waves were analyzed by using perturbative [25], variational [10, 13, 24], and fixed-point [12] methods. From the variational point of view, the traveling periodic waves are characterized as constrained minimizers of energy $E(u)$ subject to fixed momentum $F(u)$ and mass $M(u)$ for every $\alpha \in \left(\frac{1}{3}, 2\right]$ [24]. Spectral stability of periodic waves with respect to perturbations of the same period follows from computations of eigenvalues of a 2-by-2 matrix involving derivatives of momentum and mass with respect to two parameters of the periodic waves, see [16, 22] for review.

The following two recent works are particularly important in the context of the present study. In [28], perturbative and fixed-point arguments for single-lobe periodic waves were reviewed and a threshold was found on bifurcations of the small-amplitude periodic waves at $\alpha = \alpha_0$, where

$$\alpha_0 := \frac{\log 3}{\log 2} - 1 \approx 0.585.$$

This threshold separates the supercritical pitchfork bifurcation of single-lobe periodic solutions from the constant solution for $\alpha > \alpha_0$ and the subcritical pitchfork bifurcation for

$\alpha < \alpha_0$. It is also confirmed in lemmas 2.2 and 2.3 of [28] that the small-amplitude periodic waves are constrained minimizers of energy for $\alpha > \alpha_0$ and $\alpha < \alpha_0$ subject to fixed momentum and mass, although the count of negative eigenvalues of the associated Hessian operator and the 2-by-2 matrix of constraints is different between the two cases.

In [21], the positive single-lobe periodic waves were constructed by minimizing the energy $E(u)$ subject to only one constraint of the fixed momentum $F(u)$. It was shown that for every $\alpha \in \left(\frac{1}{2}, 2\right]$ and for every positive value of the fixed momentum each such minimizer is degenerate only up to the translation symmetry and is spectrally stable. No derivatives of the momentum with respect to Lagrange multipliers is used in [21].

The main purpose of this work is to develop a new variational characterization of the periodic waves in the fractional KdV equation (1.1). These periodic waves are constrained minimizers of the quadratic part of the energy $E(u)$ subject to the fixed cubic part of the energy $E(u)$ and the zero mean value, see [29] for a similar approach in the context of the fifth-order KdV equation. The existence region of the periodic waves with the zero mean for α near α_0 is unfolded in the new variational characterization. Moreover, spectral stability of periodic waves with respect to perturbations of the same period is obtained from the sharp criterion of monotonicity of the map from the wave speed to the wave momentum similarly to the stability criterion for solitary waves, see [9, 26, 30, 35] for review.

Let us now explain the main formalism for existence and stability of traveling periodic waves. A traveling wave solution to the fractional KdV equation (1.1) is a solution of the form $u(t, x) = \psi(x - ct)$, where c is a real constant representing the wave speed and $\psi(x) : \mathbb{T} \to \mathbb{R}$ is a smooth 2π-periodic function satisfying the stationary equation:

$$D^\alpha \psi + c\psi - \psi^2 + b = 0, \tag{1.5}$$

where b is another real constant obtained from integrating equation (1.1) in x. If we require that $\psi(x) : \mathbb{T} \to \mathbb{R}$ be a periodic function with the zero mean value, then $b = b(c)$ is defined at an admissible solution ψ by

$$b(c) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi^2 \mathrm{d}x. \tag{1.6}$$

The solution ψ also depends on the speed parameter c but we often omit explicit reference to this dependence for notational simplicity. The momentum $F(u)$ and mass $M(u)$ computed at the solution ψ are given by

$$F(\psi) = \pi b(c), \quad M(\psi) = 0. \tag{1.7}$$

Note that the choice (1.6) is precisely the relation excluded from the statement of theorem 1 in [21]. The relation (1.6) closes the stationary equation (1.5) as the boundary-value problem

$$D^\alpha \psi + c\psi = \Pi_0 \psi^2, \quad \psi \in H^\alpha_{\mathrm{per}}(\mathbb{T}), \tag{1.8}$$

where $\Pi_0 f := f - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\mathrm{d}x$ is the projection operator reducing the mean value of 2π-periodic functions to zero.

Among all possible periodic waves satisfying the boundary-value problem (1.8), we are interested in the *single-lobe* periodic waves, according to the following definition.

Definition 1.1. We say that the periodic wave satisfying the boundary-value problem (1.8) has a single-lobe profile ψ if there exist only one maximum and minimum of ψ on $\mathbb{T}$. Without the loss of generality, the maximum of ψ is placed at $x = 0$.

The stationary equation (1.5) is the Euler–Lagrange equation for the augmented Lyapunov functional,

$$G(u) = E(u) + cF(u) + bM(u), \tag{1.9}$$

so that $G'(\psi) = 0$. Computing the Hessian operator from (1.9) yields the linearized operator around the wave ψ

$$\mathcal{L} := G''(\psi) = D^{\alpha} + c - 2\psi. \tag{1.10}$$

The linearized operator $\mathcal{L}$ determines the spectral and linear stability of the periodic wave with the profile ψ. By using $u(t,x) = \psi(x-ct) + v(t,x-ct)$ and substituting equation (1.5) for ψ, we obtain

$$v_t + 2vv_x + 2(\psi v)_x - cv_x - D^{\alpha}v_x = 0. \tag{1.11}$$

Replacing the nonlinear equation (1.11) by its linearization at the zero solution yields the linearized stability problem

$$v_t = \partial_x \mathcal{L} v, \tag{1.12}$$

where $\mathcal{L}$ is given by (1.10). Since ψ depends only on x, separation of variables in the form $v(t,x) = \mathrm{e}^{\lambda t}\eta(x)$ with some $\lambda \in \mathbb{C}$ and $\eta(x) : \mathbb{T} \to \mathbb{C}$ reduces the linear equation (1.12) to the spectral stability problem

$$\partial_x \mathcal{L}\eta = \lambda\eta. \tag{1.13}$$

The spectral stability of the periodic wave ψ is defined as follows.

Definition 1.2. The periodic wave $\psi \in H^{\alpha}_{\text{per}}(\mathbb{T})$ is said to be spectrally stable with respect to perturbations of the same period if $\sigma(\partial_x\mathcal{L}) \subset \mathrm{i}\mathbb{R}$ in $L^2_{\text{per}}(\mathbb{T})$. Otherwise, that is, if $\sigma(\partial_x\mathcal{L})$ in $L^2_{\text{per}}(\mathbb{T})$ contains a point λ with $\mathrm{Re}(\lambda) > 0$, the periodic wave ψ is said to be *spectrally unstable*.

In the periodic case, since ∂_x is not a one-to-one operator, the classical spectral stability theory as the one in [20] can not be applied. To overcome this difficulty, a constrained spectral problem was considered in [22]:

$$\partial_x \mathcal{L}\big|_{X_0}\eta = \lambda\eta, \tag{1.14}$$

where $\mathcal{L}\big|_{X_0} = \Pi_0\mathcal{L}\Pi_0$ is a restriction of $\mathcal{L}$ on the closed subspace X_0 of periodic functions with zero mean,

$$X_0 = \left\{ f \in L^2_{\text{per}}(\mathbb{T}) : \quad \int_{-\pi}^{\pi} f(x)\mathrm{d}x = 0 \right\}. \tag{1.15}$$

A specific Krein–Hamiltonian index formula for the constrained spectral problem (1.14) determines a sharp criterion for spectral stability of periodic waves [6, 16, 23, 35]. This theory has been applied to the generalized KdV equation of the form:

$$u_t + u^p u_x + u_{xxx} = 0, \tag{1.16}$$

where $p \in \mathbb{N}$. For nonlocal evolution equations, spectral stability of periodic traveling waves was studied in [5] in the context of the intermediate long-wave (ILW) equation,

$$u_t + uu_x + \upsilon^{-1}u_x - (\mathcal{T}_\upsilon u)_{xx} = 0, \qquad \upsilon > 0, \tag{1.17}$$

where $\mathcal{T}_\upsilon$ is the the linear operator is defined by

$$\mathcal{T}_\upsilon u(x) = \text{p.v.} \int_{-\pi}^{\pi} \Gamma_\upsilon(x-y)u(y)\mathrm{d}y,$$

with $\Gamma_\upsilon(\xi) = \frac{1}{2\pi i}\sum_{n\neq 0}\coth(n\upsilon)\,\mathrm{e}^{in\xi}$. In the limit $\upsilon \to 0$, the ILW equation reduces to the KdV equation (1.16) with $p = 1$, whereas in the limit $\upsilon \to \infty$, the ILW equation reduces to the Benjamin–Ono (BO) equation. Alternatively, these two limiting cases coincide with the fractional KdV equation (1.1) with $\alpha = 2$ and $\alpha = 1$ respectively. Stability of periodic waves for these limiting cases were previously considered in [7] by exploring the fact that the corresponding periodic waves are positive with positive Fourier transform. In [5], periodic waves of the ILW equation with $\upsilon \in (0, \infty)$ were considered under the zero mean constraint, whereas Galilean transformation was used to connect periodic waves with zero mean and periodic wave with positive Fourier transform.

Another important case of the fractional KdV equation (1.1) is the reduced Ostrovsky equation

$$(u_t + uu_x)_x = u \tag{1.18}$$

which corresponds to $\alpha = -2$. Periodic waves of the reduced Ostrovsky equation naturally have zero mean and smooth periodic waves exist in an admissible interval of the wave speeds for $\alpha = -2$ [17] and more generally for every $\alpha < -1$ [11]. Spectral stability of such periodic waves with zero mean was obtained for $\alpha = -2$ in [17] from a sharp criterion given by monotonicity of the map from the wave speed to the wave momentum. Interesting enough, the family of smooth periodic waves terminates for every $\alpha < -1$ at a peaked periodic wave [11, 18] and the peaked periodic wave was shown to be linearly and spectrally unstable [18, 19].

The following theorem presents the main results of this paper.

Theorem 1.3. *Fix $\alpha \in \left(\frac{1}{3}, 2\right]$. For every $c_0 \in (-1, \infty)$, there exists a solution to the boundary-value problem (1.8) with the even, single-lobe profile ψ_0, which is obtained from a constrained minimizer of the following variational problem:*

$$\inf_{u\in H^{\frac{\alpha}{2}}_{\text{per}}(\mathbb{T})} \left\{ \int_{-\pi}^{\pi} \left[(D^{\frac{\alpha}{2}}u)^2 + c_0 u^2\right] \mathrm{d}x : \quad \int_{-\pi}^{\pi} u^3 \mathrm{d}x = 1, \quad \int_{-\pi}^{\pi} u\mathrm{d}x = 0 \right\}. \tag{1.19}$$

Assuming that $\mathrm{Ker}(\mathcal{L}|_{X_0}) = \mathrm{span}(\partial_x\psi_0)$ *for the linearized operator $\mathcal{L}$ at ψ_0, there exists a C^1 mapping $c \mapsto \psi(\cdot, c) \in H^{\alpha}_{\text{per}}(\mathbb{T})$ in a local neighborhood of c_0 such that $\psi(\cdot, c_0) = \psi_0$ and the spectrum of $\mathcal{L}$ in $L^2_{\text{per}}(\mathbb{T})$ includes*

- *a simple negative eigenvalue and a simple zero eigenvalue if $c_0 + 2b'(c_0) > 0$,*
- *a simple negative eigenvalue and a double zero eigenvalue if $c_0 + 2b'(c_0) = 0$,*
- *two negative eigenvalues and a simple zero eigenvalue if $c_0 + 2b'(c_0) < 0$.*

The periodic wave ψ_0 is spectrally stable if $b'(c_0) \geqslant 0$ and is spectrally unstable with exactly one unstable (real, positive) eigenvalue of $\partial_x\mathcal{L}$ in $L^2_{\text{per}}(\mathbb{T})$ if $b'(c_0) < 0$.

Remark 1.4. If $\mathcal{L}$ has a simple negative eigenvalue, we show that the assumption

$$\mathrm{Ker}(\mathcal{L}|_{X_0}) = \mathrm{span}(\partial_x\psi_0)$$

in theorem 1.3 is satisfied. Moreover, we show that if this assumption is not satisfied, then the periodic wave with the profile ψ_0 is spectrally unstable and $b(c)$ is not differentiable at c_0.

In section 2, we prove existence of solutions of the boundary-value problem (1.8) with an even, single-lobe profile ψ in the sense of definition 1.1 for every fixed $\alpha \in \left(\frac{1}{3}, 2\right]$ and $c \in (-1, \infty)$. This result is obtained from the existence of minimizers in the constrained variational problem (1.19) at every fixed $c_0 \in (-1, \infty)$ using classical tools of calculus of variations in the compact domain $\mathbb{T}$. Furthermore, we prove with the help of Lagrange multipliers that each constrained minimizer in $H^{\alpha/2}_{\mathrm{per}}(\mathbb{T})$ yields a proper solution ψ_0 to the boundary-value problem (1.8) for the same c_0. Moreover, the solution ψ_0 is smooth in $H^{\infty}_{\mathrm{per}}(\mathbb{T})$. The first assertion of theorem 1.3 is proven from theorem 2.1, corollary 2.2 and proposition 2.4.

In section 3, we characterize the number and multiplicity of negative and zero eigenvalues of the linearized operator $\mathcal{L}$ in $L^2_{\mathrm{per}}(\mathbb{T})$ The linearized operator $\mathcal{L}$ is considered for the periodic wave with the profile ψ_0 and the speed c_0. We find in lemma 3.8 a sharp condition $\mathrm{Ker}(\mathcal{L}|_{X_0}) = \mathrm{span}(\partial_x \psi_0)$ for continuation of the zero-mean solution ψ to the boundary-value problem (1.8) as a smooth family with respect to parameter c in a local neighborhood of c_0. For each value of $c_0 \in (-1, \infty)$, for which the family is a C^1 function of c, we show in lemma 3.14 that $\mathcal{L}$ has two negative eigenvalues if $c_0 + 2b'(c_0) < 0$ and one simple negative eigenvalue if $c_0 + 2b'(c_0) \geqslant 0$. In addition, $\mathcal{L}$ has a double zero eigenvalue if $c_0 + 2b'(c_0) = 0$ and a simple zero eigenvalue if $c_0 + 2b'(c_0) \neq 0$. The zero eigenvalue of $\mathcal{L}$ always exists due to the translational symmetry implying $\mathcal{L}\partial_x \psi_0 = 0$. The second assertion of theorem 1.3 is proven from lemma 3.8, corollary 3.11 and lemma 3.14.

The sharp characterization of negative and zero eigenvalues of the linearized operator $\mathcal{L}$ is one of *the most interesting applications* of the new variational formulation. It allows us to discuss the non-degeneracy result on simplicity of the zero eigenvalue obtained in proposition 3.1 of [24] based on an extension of Sturm's oscillation theory. The non-degeneracy result does not hold for $\alpha < \alpha_0$ because a continuation of the solution ψ to the stationary equation (1.5) with respect to parameters c and b passes a fold point in the sense of the following definition.

Definition 1.5. We say that the solution ψ to the stationary equation (1.5) is at the fold point if the linearized operator $\mathcal{L}$ at ψ has a double zero eigenvalue.

If $b = 0$ is fixed and c is labeled as ω with $c = \omega$, the fold point located at $\omega_0 \in (0, \infty)$ induces the fold bifurcation: no branches of single-lobe solutions exist for $\omega < \omega_0$ and two branches of single-lobe solutions exist for $\omega > \omega_0$. The linearized operator $\mathcal{L}$ has one negative eigenvalue for one branch of single-lobe solutions and two negative eigenvalues for the other branch. The fold bifurcation occurs if $\alpha < \alpha_0$, as follows from the Stokes expansions in [28]. We show that this fold bifurcation is unfolded in the boundary-value problem (1.8) so that only one branch of single-lobe solutions exists on the (c, b) parameter plane from both sides of the fold point. These results are discussed in remarks 2.8, 3.13 and 3.15 using the Galilean transformation in proposition 2.5 and the Stokes expansion in proposition 2.6.

In section 4, we present the spectral stability result which yields the last assertion of theorem 1.3. For each value of $c_0 \in (-1, \infty)$, for which the family is a C^1 function of c, we prove in lemma 4.1 that the periodic wave is spectrally stable in the sense of definition 1.2 if $b'(c_0) \geqslant 0$ and unstable if $b'(c_0) < 0$. Moreover, in the case of spectral instability, there exists exactly one unstable (real, positive) eigenvalue of $\partial_x \mathcal{L}$ in $L^2_{\mathrm{per}}(\mathbb{T})$. Thanks to the correspondence $F(\psi) = \pi b(c)$ in (1.7), the spectral stability result reproduces the criterion for stability of solitary waves [9, 26, 30, 35]. Note that this scalar criterion obtained from the new variational characterization of periodic waves replaces computations of a 2×2 matrix needed to establish if the periodic wave is a constrained minimizer of energy subject to fixed momentum and mass as in [24]. In particular, the sharp criterion based on the sign of $b'(c_0)$ works equally well in the cases when the linearized operator $\mathcal{L}$ has one or two negative eigenvalues, see remark 4.3.

We note that if $b'(c_0) > 0$ and the periodic wave with profile ψ_0 is spectrally stable, then it is also orbitally stable in $H^{\frac{\alpha}{2}}_{\mathrm{per}}(\mathbb{T})$ according to the standard technique from [3], assuming global well-posedness of the fractional KdV equation (1.1) in $H^s_{\mathrm{per}}(\mathbb{T})$ for $s > \frac{\alpha}{2}$. For such results on the orbital stability of the periodic wave, we do not need to use the non-degeneracy assumption on the 2-by-2 matrix of derivatives of momentum $F(\psi)$ and mass $M(\psi)$ with respect to parameters c and b stated in theorem 4.1 in [24].

We show the validity of remark 1.4 in lemma 4.4, corollary 4.5, lemmas 4.6 and 4.7. Because all constrained minimizers of energy subject to fixed momentum in [21] are characterized by only one simple negative eigenvalue of the linearized operator $\mathcal{L}$, the assumption $\mathrm{Ker}(\mathcal{L}|_{X_0}) = \mathrm{span}(\partial_x\psi_0)$ in theorem 1.3 is satisfied for all solutions in [21]. Based on the numerical evidence, we formulate the following conjecture.

Conjecture 1.6. Let $\psi_0 \in H^{\alpha}_{\mathrm{per}}(\mathbb{T})$ be the solution to the boundary-value problem (1.8) with $c = c_0$ obtained from theorem 1.3. For every $c_0 \in (-1,\infty)$ and every $\alpha \in \left(\frac{1}{3}, 2\right]$, $\mathrm{Ker}(\mathcal{L}|_{X_0}) = \mathrm{span}(\partial_x\psi_0)$.

For further comparison with the outcomes of the variational method in [21], we mention that our method allows us (i) to construct all single-lobe periodic solutions of the stationary equation (1.5) on the (c, b) parameter plane, (ii) to extend the results for every $\alpha \in \left(\frac{1}{3}, 2\right]$, (iii) to filter out the constant solution from the single-lobe periodic solutions, (iv) to find more spectrally stable branches of local minimizers, and (v) to unfold the fold point in definition 1.5.

As an illustrative example, we consider the simplest case $\alpha = 1$ (the BO equation). Figure 1 (left) shows the exact dependence $b(c) = c + 1$ computed for the mean-zero single-lobe periodic waves with the profile ψ satisfying the boundary-value problem (1.8).

In comparison, figure 1 (right) shows the outcome of the variational method in [21] on the parameter plane (ω, μ), where $b = 0$ and $c = \omega \in (0,\infty)$ is chosen in the stationary equation (1.5) and μ is the period-normalized momentum $F(\psi)$. Note that the periodic wave with the single-lobe profile ψ is positive and has nonzero mean if $b = 0$ and $\omega \in (1,\infty)$, see the exact solutions (5.1).

There exists a constrained minimizer of energy for every $\mu > 0$ as in theorem 1 in [21], however, it is given by the constant solution for $\mu \in (0, 1)$ and $\omega \in (0, 1)$ with the exact relation $\mu = \omega^2$ (solid black curve) and by the single-lobe periodic solution for $\mu \in (1,\infty)$ and $\omega \in (1,\infty)$ with the exact relation $\mu = \omega$ (solid blue curve). The constant solution is a saddle point of energy for $\mu \in (1,\infty)$ (dotted black curve). As a result, the family of constrained minimizers of energy is piecewise smooth and a transition between the two minimizers occur at $\mu = 1$. Only the single-lobe solutions are recovered on the parameter plane (c, b) shown on figure 1 (left). In the end of section 5, we show that the bifurcations of minimizers of energy become more complicated for $\alpha < 1$ with more branches of local minimizers and saddle points of energy, all are unfolded on the (c, b) parameter plane.

Spectral stability of solitary waves for the fractional KdV equation (1.1) was recently considered in [4] for $\alpha \in \left(\frac{1}{3}, 2\right]$. Solitary waves were found to be spectrally and orbitally stable if $\alpha > \frac{1}{2}$ and unstable if $\alpha < \frac{1}{2}$ with an open question on the borderline case $\alpha = \frac{1}{2}$. The result of [4] relies on the scaling invariance of the fractional KdV equation on infinite line $\mathbb{R}$. Since this scaling invariance is lost in the periodic domain, we have to rely on the numerical computations of the existence curve on the (c, b) plane in order to find the parameter regions where the periodic waves are spectrally stable or unstable.

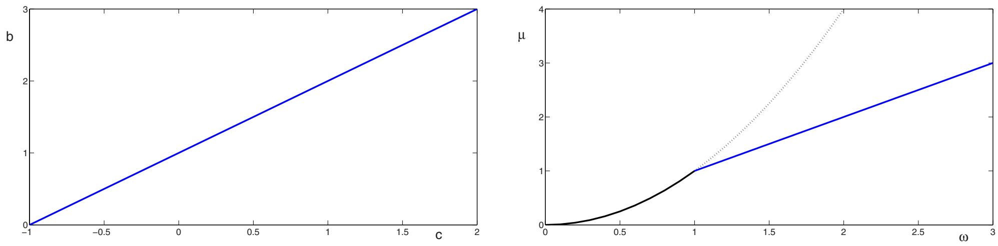

Figure 1. The dependence of b versus c (left) and μ versus ω (right) for $\alpha = 1$.

Numerical computations of the existence curve on the parameter plane (c, b) for different values of α are reported in section 5. For the integrable cases $\alpha = 1$ and $\alpha = 2$, the existence curve can be computed exactly. For $\alpha \in \left[\frac{1}{2}, 2\right]$, we show numerically that $b'(c) > 0$ for every $c \in (-1, \infty)$, hence the corresponding periodic waves are spectrally stable. For $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$, we show numerically that there exists $c_* \in (-1, \infty)$ such that $b'(c) > 0$ for $c \in (-1, c_*)$ and $b'(c) < 0$ for $c \in (c_*, \infty)$, hence the periodic waves are spectrally stable for $c \in (-1, c_*)$ and spectrally unstable for $c \in (c_*, \infty)$. These numerical results in the limit $c \to \infty$ agree with the analytical results of [4] for the solitary waves.

2. Existence via a new variational problem

Here we obtain solutions to the boundary-value problem (1.8) for $\alpha > \frac{1}{3}$. These solutions have an even, single-lobe profile ψ in the sense of definition 1.1 for $\alpha \leqslant 2$. Compared to the first assertion of theorem 1.3, we use the general notation ψ for the profile of the periodic wave satisfying the boundary-value problem (1.8) and c for the (fixed) wave speed.

For every fixed $c \in (-1, \infty)$, the existence of the periodic wave with profile ψ is established in three steps. First, we prove the existence of a minimizer of the following minimization problem

$$q_c = \inf_{u \in Y_0} \mathcal{B}_c(u), \quad \mathcal{B}_c(u) := \frac{1}{2} \int_{-\pi}^{\pi} \left[(D^{\frac{\alpha}{2}} u)^2 + cu^2 \right] \mathrm{d}x \tag{2.1}$$

in the constrained set

$$Y_0 := \left\{ u \in H^{\frac{\alpha}{2}}_{\mathrm{per}}(\mathbb{T}) : \quad \int_{-\pi}^{\pi} u^3 \mathrm{d}x = 1, \quad \int_{-\pi}^{\pi} u \mathrm{d}x = 0 \right\}. \tag{2.2}$$

Second, we use Lagrange multipliers to show that the Euler–Lagrange equation for (2.1) and (2.2) is equivalent to the stationary equation (1.5). Third, we use bootstrapping arguments to show that the solution ψ of the minimization problem (2.1) is actually smooth in $H^{\infty}_{\mathrm{per}}(\mathbb{R})$ so that it satisfies the boundary-value problem (1.8).

Theorem 2.1. *Fix $\alpha > \frac{1}{3}$. For every $c > -1$, there exists a ground state of the constrained minimization problem (2.1), that is, there exists $\phi \in Y_0$ satisfying*

$$\mathcal{B}_c(\phi) = \inf_{u \in Y_0} \mathcal{B}_c(u). \tag{2.3}$$

If $\alpha \leqslant 2$, the ground state has an even, single-lobe profile ϕ in the sense of definition 1.1.

Proof. It follows that $\mathcal{B}_c$ is a smooth functional bounded on $H^{\frac{\alpha}{2}}_{\text{per}}(\mathbb{T})$. Moreover, $\mathcal{B}_c$ is proportional to the quadratic form of the operator $c + D^\alpha$ with the spectrum in $L^2_{\text{per}}(\mathbb{T})$ given by $\{c + |m|^\alpha, \ \ m \in \mathbb{Z}\}$. Thanks to the zero-mass constraint in (2.2), for every $c > -1$, we have

$$\mathcal{B}_c(u) \geqslant \frac{1}{2}(c+1)\|u\|^2_{L^2_{\text{per}}(\mathbb{T})}, \quad u \in Y_0, \tag{2.4}$$

and by the standard Gårding's inequality, for every $c > -1$ there exists $C > 0$ such that

$$\mathcal{B}_c(u) \geqslant C\|u\|^2_{H^{\frac{\alpha}{2}}_{\text{per}}(\mathbb{T})}, \quad u \in Y_0.$$

Hence $\mathcal{B}_c$ is equivalent to the squared norm in $H^{\frac{\alpha}{2}}_{\text{per}}(\mathbb{T})$ for functions in Y_0, yielding $q_c \geqslant 0$ in (2.1). Let $\{u_n\}_{n\in\mathbb{N}}$ be a minimizing sequence for the constrained minimization problem (2.1), that is, a sequence in Y_0 satisfying

$$\mathcal{B}_c(u_n) \to q_c \quad \text{as} \quad n \to \infty.$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^{\frac{\alpha}{2}}_{\text{per}}(\mathbb{T})$, there exists $\phi \in H^{\frac{\alpha}{2}}_{\text{per}}(\mathbb{T})$ such that, up to a subsequence,

$$u_n \rightharpoonup \phi \quad \text{in } H^{\frac{\alpha}{2}}_{\text{per}}(\mathbb{T}), \quad \text{as} \quad n \to \infty.$$

For every $\alpha > \frac{1}{3}$, the energy space $H^{\frac{\alpha}{2}}_{\text{per}}(\mathbb{T})$ is compactly embedded in $L^3_{\text{per}}(\mathbb{T})$. Thus,

$$u_n \to \phi \quad \text{in } L^3_{\text{per}}(\mathbb{T}), \quad \text{as} \quad n \to \infty.$$

Using the estimate

$$\begin{aligned}\left|\int_{-\pi}^{\pi}(u_n^3 - \phi^3)\mathrm{d}x\right| &\leqslant \int_{-\pi}^{\pi}|u_n^3 - \phi^3|\mathrm{d}x \\ &\leqslant \left(\|\phi\|^2_{L^3_{\text{per}}} + \|\phi\|_{L^3_{\text{per}}}\|u_n\|_{L^3_{\text{per}}} + \|u_n\|^2_{L^3_{\text{per}}}\right)\|u_n - \phi\|_{L^3_{\text{per}}},\end{aligned}$$

it follows that $\int_{-\pi}^{\pi}\phi^3\mathrm{d}x = 1$. By a similar argument, since $H^{\frac{\alpha}{2}}_{\text{per}}(\mathbb{T})$ is also compactly embedded in $L^1_{\text{per}}(\mathbb{T})$, it follows that $\int_{-\pi}^{\pi}\phi\mathrm{d}x = 0$. Hence, $\phi \in Y_0$. Thanks to the weak lower semicontinuity of $\mathcal{B}_c$, we have

$$\mathcal{B}_c(\phi) \leqslant \liminf_{n\to\infty} \mathcal{B}(u_n) = q_c.$$

Therefore, $\mathcal{B}_c(\phi) = q_c$.

If $\alpha \in (0, 2]$, the symmetric decreasing rearrangements of u do not increase $\mathcal{B}_c(u)$ while leaving the constraints in Y_0 invariant thanks to the fractional Polya–Szegö inequality, see lemma A.1 in [14]. As a result, the minimizer $\phi \in Y_0$ of $\mathcal{B}_c(u)$ must decrease away symmetrically from the maximum point. By the translational invariance, the maximum point can be placed at $x = 0$, which yields an even, single-lobe profile for ϕ. □

Corollary 2.2. *For every $\alpha \in \left(\frac{1}{3}, 2\right]$, there exists a solution to the boundary-value problem (1.8) with an even, single-lobe profile ψ.*

Proof. By Lagrange's Multiplier Theorem, the constrained minimizer $\phi \in Y_0$ in theorem 2.1 satisfies the stationary equation

$$D^\alpha \phi + c\phi = C_1 \phi^2 + C_2, \tag{2.5}$$

for some constants C_1 and C_2. From the two constraints in Y_0, we have

$$C_1 = 2\mathcal{B}_c(\phi), \quad C_2 = -\frac{1}{2\pi}\left(\int_{-\pi}^{\pi} \phi^2 \mathrm{d}x\right) C_1. \tag{2.6}$$

The scaling transformation $\psi = C_1\phi$ maps the stationary equation (2.5) to the form (1.5) with $b = b(c)$ computed from ψ by (1.6). $\square$

The following lemma states that the infimum q_c in (2.1) is continuous in c for $c > -1$ and that $q_c \to 0$ as $c \to -1$.

Lemma 2.3. *Let $\phi \in Y_0$ be the ground state of the constrained minimization problem (2.1) in theorem 2.1 and $q_c = \mathcal{B}_c(\phi)$. Then q_c is continuous in c for $c > -1$ and $q_c \to 0$ as $c \to -1$.*

Proof. For a fixed $u \in Y_0$ and for every $c' > c > -1$, we have

$$0 \leqslant \mathcal{B}_{c'}(u) - \mathcal{B}_c(u) = \frac{1}{2}(c' - c)\|u\|^2_{L^2_{\mathrm{per}}} \leqslant \frac{c' - c}{c + 1}\mathcal{B}_c(u),$$

thanks to the bound (2.4). Let $\mathcal{B}_c(\phi) = q_c$ and $\mathcal{B}_{c'}(\phi') = q_{c'}$. Then, we have

$$q_{c'} - q_c = \mathcal{B}_{c'}(\phi') - \mathcal{B}_c(\phi') + \mathcal{B}_c(\phi') - \mathcal{B}_c(\phi) \geqslant \mathcal{B}_{c'}(\phi') - \mathcal{B}_c(\phi') \geqslant 0$$

and

$$q_{c'} - q_c = \mathcal{B}_{c'}(\phi') - \mathcal{B}_{c'}(\phi) + \mathcal{B}_{c'}(\phi) - \mathcal{B}_c(\phi) \leqslant \mathcal{B}_{c'}(\phi) - \mathcal{B}_c(\phi) \leqslant \frac{c' - c}{c + 1}\mathcal{B}_c(\phi).$$

From here, it is clear that $q_{c'} \to q_c$ as $c' \to c$, so that q_c is continuous in c for $c > -1$. It remains to show that $q_c \to 0$ as $c \to -1$. Consider the following family of two-mode functions in Y_0:

$$u_\mu(x) = \mu\cos(x) + \frac{2}{3\pi\mu^2}\cos(2x), \quad \mu > 0,$$

which satisfy the constraints in (2.2). Substituting u_μ into $\mathcal{B}_c(u)$ yields

$$\mathcal{B}_c(u_\mu) = \frac{\pi}{2}\left[\mu^2(1 + c) + \frac{4}{9\pi^2\mu^4}(2^\alpha + c)\right] \geqslant \frac{3\pi(2^\alpha + c)^{\frac{1}{3}}(1 + c)^{2/3}}{2(3\pi)^{2/3}},$$

where the lower bound is found from the minimization of $\mathcal{B}_c(u_\mu)$ in μ. Therefore, we obtain

$$0 \leqslant q_c \leqslant \frac{3\pi(2^\alpha + c)^{\frac{1}{3}}(1 + c)^{2/3}}{2(3\pi)^{2/3}},$$

which shows that $q_c \to 0$ as $c \to -1$. $\square$

The following proposition ensures that ψ is smooth in x and hence satisfies the boundary-value problem (1.8). Note that the result below is not original since similar results were reported in [15, 24, 28]. It is reproduced here for the sake of completeness.

Proposition 2.4. *Assume that $\psi \in H_{\mathrm{per}}^{\frac{\alpha}{2}}(\mathbb{T})$ is a solution of the stationary equation (1.5) with $c > -1$ and $b = b(c)$ in the sense of distributions. Then $\psi \in H_{\mathrm{per}}^{\infty}(\mathbb{T})$.*

Proof. In view of the embedding $H_{\mathrm{per}}^{s_2}(\mathbb{T}) \hookrightarrow H_{\mathrm{per}}^{s_1}(\mathbb{T})$, $s_2 \geqslant s_1 > 0$, it suffices to assume $\frac{1}{3} < \alpha < \frac{1}{2}$. First, we will prove that $\psi \in L_{\mathrm{per}}^{\infty}(\mathbb{T})$. Indeed, applying the Fourier transform in (1.5) yields

$$\widehat{\psi}(m) = \frac{\widehat{\psi^2}(m)}{|m|^\alpha + c}, \quad m \in \mathbb{Z}\backslash\{0\}.$$

Since $\psi \in H_{\mathrm{per}}^{\frac{\alpha}{2}}(\mathbb{T})$, it follows that $\psi \in L_{\mathrm{per}}^{p}(\mathbb{T})$ and $\psi^2 \in L_{\mathrm{per}}^{\frac{p}{2}}(\mathbb{T})$, for all $2 \leqslant p \leqslant \frac{2}{1-\alpha}$. Hence, by Hausdorff–Young inequality, we have $\widehat{\psi^2} \in \ell^q$ for all $\frac{1}{\alpha} \leqslant q \leqslant \infty$.

Since $c > -1$, we see that $(|m|^\alpha + c)^{-1} \in \ell^p$ for all $p > \frac{1}{\alpha}$. Let $\varepsilon > 0$ be a small number such that $1 \leqslant \frac{2}{1+\alpha+\varepsilon}$. Thus

$$\|\widehat{\psi}\|_{\ell^{\frac{2}{1+\alpha+\varepsilon}}}^{\frac{2}{1+\alpha+\varepsilon}} \leqslant \|(\widehat{\psi^2})^{\frac{2}{1+\alpha+\varepsilon}}\|_{\ell^q} \|\,(|m|^\alpha + c)^{-\frac{2}{1+\alpha+\varepsilon}}\,\|_{\ell^{q'}},$$

where $q, q' > 0$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Next, we consider the smallest q such that the first term on the right side is finite, that is, $q = \frac{1+\alpha+\varepsilon}{2\alpha}$, hence $q' = \frac{1+\alpha+\varepsilon}{1-\alpha+\varepsilon}$. The second term on the right side is finite if $\frac{1}{\alpha} < \frac{2q'}{1+\alpha+\varepsilon}$ which is true if $1 + \varepsilon < 3\alpha$. Note that for every $\alpha > \frac{1}{3}$, one can always find a suitable $\varepsilon > 0$. Under these constraints, we get $\widehat{\psi} \in \ell^{\frac{2}{1+\alpha+\varepsilon}}$ which implies that there exists $\xi \in L_{\mathrm{per}}^{\frac{2}{1-\alpha-\varepsilon}}(\mathbb{T})$ such that $\widehat{\xi} = \widehat{\psi}$ (see [36, page 190]). Hence, using [36, corollary 1.51] we obtain $\xi = \psi$ and so $\psi \in L_{\mathrm{per}}^{p}(\mathbb{T})$ for $2 \leqslant p \leqslant \frac{2}{1-\alpha-\varepsilon}$. An iterating procedure gives us $\widehat{\psi} \in \ell^1$ and thus $\psi \in L_{\mathrm{per}}^{\infty}(\mathbb{T})$.

Finally, one sees that

$$\|D^\alpha \psi\|_{L^2_{\mathrm{per}}} = \left\|(D^\alpha + c)^{-1} D^\alpha \psi^2\right\|_{L^2_{\mathrm{per}}} \leqslant \|\psi^2\|_{L^2_{\mathrm{per}}} \leqslant \|\psi\|_{L^\infty_{\mathrm{per}}} \|\psi\|_{L^2_{\mathrm{per}}},$$

which implies $\psi \in H_{\mathrm{per}}^{\alpha}(\mathbb{T})$. Furthermore, from the fact that $\widehat{\psi} \in \ell^1$, we have

$$\begin{aligned}\|D^{2\alpha}\psi\|_{L^2_{\mathrm{per}}} &= \left\|(D^\alpha + c)^{-1} D^{2\alpha}\psi^2\right\|_{L^2_{\mathrm{per}}} = \left\|\frac{|\cdot|^{2\alpha}\widehat{\psi^2}}{|\cdot|^\alpha + c}\right\|_{\ell^2} \leq \|(1 + |\cdot|^\alpha)(\widehat{\psi} * \widehat{\psi})\|_{\ell^2} \\ &\leq \|\hat{\psi}\|_{\ell^1}\|\hat{\psi}\|_{\ell^2} + \||\cdot|^\alpha \hat{\psi}\|_{\ell^2}\|\hat{\psi}\|_{\ell^1}.\end{aligned}$$

After iterations, we conclude that $\psi \in H_{\mathrm{per}}^{\infty}(\mathbb{T})$. $\square$

We show next that the periodic waves of the boundary-value problem (1.8) with an even, single-lobe profile ψ in the sense of definition 1.1 are given by the Stokes expansion for c near -1. Because we reuse the method of Lyapunov–Schmidt reductions from [25], the results on the Stokes expansion of the periodic wave ψ are restricted to the values of $\alpha > \frac{1}{2}$. Similar computations of the Stokes expansions are reported in theorem 2.1 of [28].

The small-amplitude (Stokes) expansion for single-lobe periodic waves of the boundary-value problem (1.8) is constructed in three steps. First, we present *Galilean transformation* between solutions of the stationary equation (1.5). Second, we obtain Stokes expansion of the normalized stationary equation. Third, we transform the Stokes expansion of the normalized stationary equation back to the solutions of the boundary-value problem (1.8).

Proposition 2.5. *Let $\psi \in H^{\alpha}_{\rm per}(\mathbb{T})$ be a solution to the stationary equation (1.5) with some (c, b). Then,*

$$\varphi := \psi - \frac{1}{2}\left(c - \sqrt{c^2 + 4b}\right) \tag{2.7}$$

is a solution of the stationary equation

$$D^{\alpha}\varphi + \omega\varphi - \varphi^2 = 0, \quad \varphi \in H^{\alpha}_{\rm per}(\mathbb{T}), \tag{2.8}$$

with $\omega := \sqrt{c^2 + 4b}$.

Proof. The proof is given by direct substitution. □

Proposition 2.6. *For every $\alpha > \frac{1}{2}$, there exists $a_0 > 0$ such that for every $a \in (0, a_0)$ there exists a locally unique, even, single-lobe solution φ of the stationary equation (2.8) in the sense of definition 1.1. The pair $(\omega, \varphi) \in \mathbb{R} \times H^{\alpha}_{\rm per}(\mathbb{T})$ is smooth in a and is given by the following Stokes expansion:*

$$\varphi(x) = 1 + a\cos(x) + a^2\varphi_2(x) + a^3\varphi_3(x) + \mathcal{O}(a^4), \tag{2.9}$$

and

$$\omega = 1 + \omega_2 a^2 + \mathcal{O}(a^4), \tag{2.10}$$

where the corrections terms are defined in (2.11)–(2.13) below.

Proof. We give algorithmic computations of the higher-order coefficients to the periodic wave by using the classical Stokes expansion:

$$\varphi(x) = 1 + \sum_{k=1}^{\infty} a^k \varphi_k(x), \quad \omega = 1 + \sum_{k=1}^{\infty} \omega_{2k} a^{2k}.$$

The correction terms satisfy recursively,

$$\begin{cases} \mathcal{O}(a) : & (D^{\alpha} - 1)\varphi_1 = 0, \\ \mathcal{O}(a^2) : & (D^{\alpha} - 1)\varphi_2 + \omega_2 - \varphi_1^2 = 0, \\ \mathcal{O}(a^3) : & (D^{\alpha} - 1)\varphi_3 + \omega_2\varphi_1 - 2\varphi_1\varphi_2 = 0. \end{cases}$$

Since the periodic wave has a single-lobe profile φ with the global maximum at $x = 0$, we select uniquely $\varphi_1(x) = \cos(x)$ since ${\rm Ker}_{\rm even}(D^{\alpha} - 1) = {\rm span}\{\cos(\cdot)\}$ in the space of even functions in $L^2_{\rm per}(\mathbb{T})$. In order to select uniquely all other corrections to the Stokes expansion (2.9), we require the corrections terms $\{\varphi_k\}_{k\geqslant 2}$ to be orthogonal to φ_1 in $L^2_{\rm per}(\mathbb{T})$. Solving the inhomogeneous equation at $\mathcal{O}(a^2)$ yields the exact solution in $H^{\alpha}_{\text{per}}(\mathbb{T})$:

$$\varphi_2(x) = \omega_2 - \frac{1}{2} + \frac{1}{2(2^\alpha - 1)}\cos(2x), \tag{2.11}$$

where ω_2 is to be determined. The inhomogeneous equation at $\mathcal{O}(a^3)$ admits a solution $\varphi_3 \in H^{\alpha}_{\text{per}}(\mathbb{T})$ if and only if the right-hand side is orthogonal to φ_1, which selects uniquely the correction ω_2 by

$$\omega_2 = 1 - \frac{1}{2(2^\alpha - 1)}. \tag{2.12}$$

After the resonant term is removed, the inhomogeneous equation at $\mathcal{O}(a^3)$ yields the exact solution in $H^{\alpha}_{\text{per}}(\mathbb{T})$:

$$\varphi_3(x) = \frac{1}{2(2^\alpha - 1)(3^\alpha - 1)}\cos(3x). \tag{2.13}$$

Justification of the existence, uniqueness, and analyticity of the Stokes expansions (2.9) and (2.10) is performed with the method of Lyapunov–Schmidt reductions for $\alpha > \frac{1}{2}$, see lemma 2.1 and theorem A.1 in [25]. $\square$

Corollary 2.7. *For every $\alpha \in \left(\frac{1}{2}, 2\right]$, there exists $c_0 \in (-1, \infty)$ such that the solution of the boundary-value problem (1.8) for every $c \in (-1, c_0)$ with an even, single-lobe profile ψ in theorem 2.1 and corollary 2.2 is given by the following Stokes expansion:*

$$\psi = a\cos(x) + \frac{a^2}{2(2^\alpha - 1)}\cos(2x) + \frac{a^3}{2(2^\alpha - 1)(3^\alpha - 1)}\cos(3x) + \mathcal{O}(a^4) \tag{2.14}$$

with parameters

$$c = -1 + \frac{1}{2(2^\alpha - 1)}a^2 + \mathcal{O}(a^4) \tag{2.15}$$

and

$$b(c) = \frac{1}{2}a^2 + \mathcal{O}(a^4). \tag{2.16}$$

Proof. We apply the Galilean transformation (2.7) of proposition 2.5 to the Stokes expansion (2.9) and (2.10) in proposition 2.6. Therefore, we define

$$\psi = \Pi_0\varphi, \quad c = \omega - \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi\mathrm{d}x, \quad b(c) = \frac{1}{4}(\omega^2 - c^2) \tag{2.17}$$

and obtain the Stokes expansion (2.14)–(2.16) for solutions of the boundary-value problem (1.8).

It follows from (2.14) and (2.15) that $\|\psi\|_{L^2_{\text{per}}} \to 0$ as $c \to -1$. Since the Stokes expansion (2.9) for the even, single-lobe solution ψ is locally unique by proposition 2.6 and $\mathcal{B}_c(\phi) \to 0$

as $c \to -1$ by lemma 2.3 implies that $\|\psi\|_{L^2_{\rm per}} \to 0$ as $c \to -1$, the small-amplitude periodic wave (2.14) with an even, single-lobe profile ψ coincides as $c \to -1$ with the family of minimizers in theorem 2.1 and corollary 2.2 given by $\psi = 2\mathcal{B}_c(\phi)\phi$. $\square$

Remark 2.8. It follows from (2.12) that $\omega_2 > 0$ if and only if $\alpha > \alpha_0$, where

$$\alpha_0 := \frac{\log 3}{\log 2} - 1 \approx 0.585.$$

It follows from the expansions (2.14)–(2.16) that the threshold α_0 does not show up in the Stokes expansion of the solution ψ to the boundary-value problem (1.8).

Remark 2.9. Employing Krasnoselskii's Fixed Point Theorem, the existence and uniqueness of solutions φ to the stationary equation (2.8) with a positive, even, single-lobe profile ψ was proven for every $\alpha \in (\alpha_0, 2]$ and $\omega \in (1, \infty)$ in theorem 2.2 of [28]. The proof of theorem 2.2 in [28] relies on the assumption that the kernel of the Jacobian operator is one-dimensional. The latter assumption is proven in proposition 3.1 in [24] if the minimizers of energy $E(u)$ subject to fixed momentum $F(u)$ and mass $M(u)$ are smooth with respect to the Lagrange multipliers c and b. The latter condition is however false for $\alpha < \alpha_0$ (see remark 3.4).

3. Smooth continuation of periodic waves in c

Here we find a sharp condition for a smooth continuation of solutions ψ to the boundary-value problem (1.8) with respect to the parameter c in $(-1, \infty)$. Because we use the oscillation theory from [24], the results on the smooth continuation of periodic waves with respect to wave speed c are limited to the interval $\alpha \in (\frac{1}{3}, 2]$ and to the periodic waves with an even, single-lobe profile ψ.

Let $\psi \in H^\infty_{\rm per}(\mathbb{T})$ be a solution to the boundary-value problem (1.8) for some $c \in (-1, \infty)$ obtained with theorem 2.1, corollary 2.2 and proposition 2.4. The solution has an even, single-lobe profile ψ in the sense of definition 1.1. The linearized operator $\mathcal{L}$ at ψ is given by (1.10), which we rewrite again as the following self-adjoint operator:

$$\mathcal{L} = D^\alpha + c - 2\psi : \quad H^\alpha_{\rm per}(\mathbb{T}) \subset L^2_{\rm per}(\mathbb{T}) \to L^2_{\rm per}(\mathbb{T}). \tag{3.1}$$

For continuation of the solution $\psi \in H^\infty_{\rm per}(\mathbb{T})$ to the boundary-value problem (1.8) in c, we need to determine the multiplicity of the zero eigenvalue of $\mathcal{L}$ denoted as $z(\mathcal{L})$. For spectral stability of the periodic wave ψ, we also need to determine the number of negative eigenvalues of $\mathcal{L}$ with the account of their multiplicities denoted as $n(\mathcal{L})$.

It follows by direct computations from the boundary-value problem (1.8) that

$$\mathcal{L}\psi = -\psi^2 - b(c) \tag{3.2}$$

and

$$\mathcal{L}1 = -2\psi + c. \tag{3.3}$$

By the translational symmetry, we always have $\mathcal{L}\partial_x\psi = 0$. However, the main question is whether $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x\psi)$, that is, if $z(\mathcal{L}) = 1$. This question was answered in [24] for

$\alpha \in (\frac{1}{3}, 2]$, where the following result was obtained using Sturm's oscillation theory for fractional derivative operators.

Proposition 3.1. *Let $\alpha \in (\frac{1}{3}, 2]$ and $\psi \in H^\infty_{\mathrm{per}}(\mathbb{T})$ be an even, single-lobe periodic wave. An eigenfunction of $\mathcal{L}$ in (3.1) corresponding to the nth eigenvalue of $\mathcal{L}$ for $n = 1, 2, 3$ changes its sign at most $2(n-1)$ times over $\mathbb{T}$.*

Proof. The result is formulated as lemma 3.2 in [24] and is proved in appendix A [24]. □

Corollary 3.2. *Assume ψ be an even, single-lobe periodic wave obtained with theorem 2.1, corollary 2.2 and proposition 2.4 for $\alpha \in (\frac{1}{3}, 2]$ and $c \in (-1, \infty)$. Then, $n(\mathcal{L}) \in \{1, 2\}$ and $z(\mathcal{L}) \in \{1, 2\}$.*

Proof. It follows by (3.2) that

$$\langle \mathcal{L}\psi, \psi \rangle = -\int_{-\pi}^{\pi} \psi^3 \mathrm{d}x = -8\mathcal{B}_c(\phi)^3 < 0, \tag{3.4}$$

thanks to (2.2), (2.4) and (2.6). Therefore, $n(\mathcal{L}) \geqslant 1$. Thanks to the variational formulation (2.1) and (2.2) and theorem 2.1, $\psi \in H^\infty_{\mathrm{per}}(\mathbb{T})$ is a minimizer of $G(u)$ in (1.9) for every $c \in (-1, \infty)$ subject to two constraints in (2.2). Since $\mathcal{L}$ is the Hessian operator for $G(u)$ in (1.10), we have

$$\mathcal{L}\big|_{\{1,\psi^2\}^\perp} \geqslant 0. \tag{3.5}$$

By Courant's Mini-Max Principle, $n(\mathcal{L}) \leqslant 2$, so that $n(\mathcal{L}) \in \{1, 2\}$ is proven.

Since ψ is even, $L^2_{\mathrm{per}}(\mathbb{T})$ is decomposed into an orthogonal sum of an even and odd subspaces. By (L1) in lemma 3.3 in [24], 0 is the lowest eigenvalue of $\mathcal{L}$ in the subspace of odd functions in $L^2_{\mathrm{per}}(\mathbb{T})$ with the eigenfunction $\partial_x\psi$ with a single node. Hence, $z(\mathcal{L}) \geqslant 1$. In the subspace of even functions in $L^2_{\mathrm{per}}(\mathbb{T})$, the number of nodes is even. If $n(\mathcal{L}) = 1$, then 0 is the second eigenvalue of $\mathcal{L}$. By proposition 3.1, the corresponding even function may have at most two nodes, hence there may be at most one such eigenfunction of $\mathcal{L}$ for the zero eigenvalue in the subspace of even functions in $L^2_{\mathrm{per}}(\mathbb{T})$. If $n(\mathcal{L}) = 2$, then the second (negative) eigenvalue has an even eigenfunction with exactly two nodes, whereas 0 is the third eigenvalue of $\mathcal{L}$. By proposition 3.1, the corresponding even function for the zero eigenvalue may have at most four nodes, hence there may be at most one such eigenfunction of $\mathcal{L}$ in the subspace of even functions in $L^2_{\mathrm{per}}(\mathbb{T})$. In both cases, $z(\mathcal{L}) \leqslant 2$, so that $z(\mathcal{L}) \in \{1, 2\}$ is proven. □

Proposition 3.3. *Assume $\alpha \in (\frac{1}{3}, 2]$ and $\psi \in H^\infty_{\mathrm{per}}(\mathbb{T})$ be an even, single-lobe periodic wave. If $\{1, \psi, \psi^2\} \in \mathrm{Range}(\mathcal{L})$, then $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x\psi)$.*

Proof. The result is formulated as proposition 3.1 in [24] and is proven from the property $\{1, \psi, \psi^2\} \in \mathrm{Range}(\mathcal{L})$ claimed in (L3) of lemma 3.3 in [24]. □

Remark 3.4. The proof of (L3) in lemma 3.3 in [24] relies on the smoothness of minimizers of energy $E(u)$ subject to fixed momentum $F(u)$ and mass $M(u)$ with respect to Lagrange multipliers c and b. Unfortunately, this smoothness cannot be taken as granted and may be false. Indeed, $\mathrm{Ker}(\mathcal{L}) \neq \mathrm{span}(\partial_x\psi)$ for some periodic waves satisfying the stationary equation (1.5) for $\alpha < \alpha_0$ (see corollary 3.11, remarks 3.13 and 3.15).

The following lemma characterizes the kernel of $\mathcal{L}|_{X_0} = \Pi_0\mathcal{L}\Pi_0$, where Π_0 is defined in (1.8) and X_0 is defined in (1.15). The standard inner product in $L^2_{\mathrm{per}}(\mathbb{T})$ is denoted by $\langle\cdot,\cdot\rangle$.

Lemma 3.5. *Assume $\alpha \in (\frac{1}{3}, 2]$ and $\psi \in H^\infty_{\mathrm{per}}(\mathbb{T})$ be an even, single-lobe periodic wave. If there exists $f \in \mathrm{Ker}(\mathcal{L}|_{X_0})$ such that $\langle f, \partial_x\psi\rangle = 0$ and $f \neq 0$, then*

$$\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x\psi), \quad \langle f, \psi\rangle \neq 0, \quad \text{and} \quad \langle f, \psi^2\rangle = 0. \tag{3.6}$$

Proof. Since $f \in \mathrm{Ker}(\mathcal{L}|_{X_0})$, then $\langle 1,f\rangle = 0$ and f satisfies

$$0 = \mathcal{L}|_{X_0}f = \mathcal{L}f + \frac{1}{\pi}\int_{-\pi}^{\pi} f\psi\mathrm{d}x. \tag{3.7}$$

Either $\langle f, \psi\rangle = 0$ or $\langle f, \psi\rangle \neq 0$.

Assume first that $\langle f, \psi\rangle = 0$. It follows by (3.7) that $f \in \mathrm{Ker}(\mathcal{L})$ and by equality (3.2), we have $\langle f, \psi^2\rangle = 0$. By corollary 3.2, the kernel of $\mathcal{L}$ can be at most two-dimensional (2D), hence $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x\psi, f)$ and $\{1, \psi, \psi^2\} \in [\mathrm{Ker}(\mathcal{L})]^\perp$. By Fredholm theorem for self-adjoint operator (3.1), we have $\{1, \psi, \psi^2\} \in \mathrm{Range}(\mathcal{L})$ and by proposition 3.3, $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x\psi)$ in contradiction to the conclusion that $f \in \mathrm{Ker}(\mathcal{L})$. Therefore, assumption $\langle f, \psi\rangle = 0$ leads to contradiction.

Assume now that $\langle f, \psi\rangle \neq 0$. It follows by (3.7) that $1 \in \mathrm{Range}(\mathcal{L})$. Then, by (3.2) and (3.3), we have $\psi^2 \in \mathrm{Range}(\mathcal{L})$ and $\psi \in \mathrm{Range}(\mathcal{L})$ respectively. In other words, $\{1, \psi, \psi^2\} \in \mathrm{Range}(\mathcal{L})$ and by proposition 3.3, $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x\psi)$. In addition, by (3.2), we have

$$\langle f, \psi^2\rangle = -\langle f, \mathcal{L}\psi\rangle = -\langle \mathcal{L}f, \psi\rangle = \frac{1}{\pi}\langle f, \psi\rangle\langle 1, \psi\rangle = 0.$$

This yields (3.6). □

Corollary 3.6. *If f exists in lemma 3.5, then* $\mathrm{Ker}(\mathcal{L}|_{X_0}) = \mathrm{span}(\partial_x\psi, f)$.

Proof. Assume two orthogonal vectors $f_1, f_2 \in \mathrm{Ker}(\mathcal{L}|_{X_0})$ such that $\langle f_{1,2}, \partial_x\psi\rangle = 0$ and $f_{1,2} \neq 0$. Since $\langle f_{1,2}, \psi\rangle \neq 0$, there exists a linear combination of f_1 and f_2 in $\mathrm{Ker}(\mathcal{L})$ in contradiction with $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x\psi)$ in (3.6). □

Corollary 3.7. $\mathrm{Ker}(\mathcal{L}|_{X_0}) = \mathrm{Ker}(\mathcal{L}|_{\{1,\psi^2\}^\perp})$.

Proof. By using orthogonal projections, we write

$$\mathcal{L}|_{\{1,\psi^2\}^\perp}f = \mathcal{L}f + \frac{1}{\pi}\int_{-\pi}^{\pi} f\psi\mathrm{d}x - \alpha\Pi_0\psi^2, \quad \alpha = \frac{\langle \mathcal{L}f, \Pi_0\psi^2\rangle}{\langle \psi^2, \Pi_0\psi^2\rangle}, \tag{3.8}$$

where $\langle \psi^2, \Pi_0\psi^2\rangle = \|\psi\|^4_{L^4} - \frac{1}{2\pi}\|\psi\|^2_{L^2} > 0$ for every non-constant (single-lobe) ψ.

By lemma 3.5, if $f \in \mathrm{Ker}(\mathcal{L}|_{X_0})$, then $\langle f, \psi^2\rangle = 0$. Since $\langle 1, \Pi_0\psi^2\rangle = 0$, it follows from (3.7) and (3.8) that $f \in \mathrm{Ker}(\mathcal{L}|_{\{1,\psi^2\}^\perp})$.

In the opposite direction, assume that $f \in \mathrm{Ker}(\mathcal{L}|_{\{1,\psi^2\}^\perp})$, $\langle f, \partial_x \psi\rangle = 0$, and $f \neq 0$. Since $\langle f, 1\rangle = \langle f, \psi^2\rangle = 0$, we have by (3.2) that $0 = \langle f, \mathcal{L}\psi\rangle = \langle \mathcal{L}f, \psi\rangle = \alpha\langle \Pi_0\psi^2, \psi\rangle$. Since $\langle \Pi_0\psi^2, \psi\rangle = \langle \psi^2, \psi\rangle > 0$, thanks to (2.2), (2.4) and (2.6), we obtain $\alpha = 0$ which implies that $f \in \mathrm{Ker}(\mathcal{L}|_{X_0})$. $\square$

The following lemma provides a sharp condition for a smooth continuation of the periodic wave with profile ψ with respect to the wave speed c.

Lemma 3.8. *Assume $\alpha \in (\frac{1}{3}, 2]$ and ψ_0 be an even, single-lobe solution of the boundary-value problem (1.8) for a fixed $c_0 \in (-1, \infty)$ obtained with theorem 2.1, corollary 2.2 and proposition 2.4. Assume $\mathrm{Ker}(\mathcal{L}|_{X_0}) = \mathrm{span}(\partial_x\psi_0)$. Then, there exists a unique continuation of even solutions of the boundary-value problem (1.8) in an open interval $\mathcal{I}_c \subset (-1, \infty)$ containing c_0 such that the mapping*

$$\mathcal{I}_c \ni c \mapsto \psi(\cdot, c) \in H^\alpha_{\mathrm{per}}(\mathbb{T}) \cap X_0 \tag{3.9}$$

is C^1 and $\psi(\cdot, c_0) = \psi_0$.

Proof. Let $\psi_0 \in H^\alpha_{\mathrm{per}}(\mathbb{T}) \cap X_0$ be an even, single-lobe solution of the boundary-value problem (1.8) for $c_0 \in (-1, \infty)$. Let $\psi \in H^\alpha_{\mathrm{per}}(\mathbb{T}) \cap X_0$ be a solution of the boundary-value problem (1.8) for $c \in (-1, \infty)$ to be constructed from ψ_0 for c near c_0. Then, $\tilde{\psi} := \psi - \psi_0 \in H^\alpha_{\mathrm{per}}(\mathbb{T}) \cap X_0$ satisfies the following equation:

$$\mathcal{L}_0|_{X_0}\tilde{\psi} = -(c - c_0)(\psi_0 + \tilde{\psi}) + \Pi_0\tilde{\psi}^2, \tag{3.10}$$

where $\mathcal{L}_0$ is obtained from $\mathcal{L}$ in (3.1) at $c = c_0$ and $\psi = \psi_0$, whereas $\mathcal{L}_0|_{X_0}$ acts on $\tilde{\psi}$ by the same expressions as in (3.7).

Assume $\mathrm{Ker}(\mathcal{L}_0|_{X_0}) = \mathrm{span}(\partial_x\psi_0)$ and consider the subspace of even functions for which ψ_0 belongs. Then, $\mathcal{L}_0|_{X_0}$ is invertible on the subspace of even functions in $H^\alpha_{\mathrm{per}}(\mathbb{T}) \cap X_0$ so that we can rewrite (3.10) as the fixed-point equation:

$$\tilde{\psi} = -(c - c_0)\left(\mathcal{L}_0|_{X_0}\right)^{-1}(\psi_0 + \tilde{\psi}) + \left(\mathcal{L}_0|_{X_0}\right)^{-1}\Pi_0\tilde{\psi}^2. \tag{3.11}$$

By the implicit function theorem, there exist an open interval containing c_0, an open ball $B_r \in H^\alpha_{\mathrm{per}}(\mathbb{T}) \cap X_0$ of radius $r > 0$ centered at 0, and a unique C^1 mapping $\mathcal{I}_c \ni c \mapsto \tilde{\psi}(\cdot, c) \in B_r$ such that $\tilde{\psi}(\cdot, c)$ is an even solution to the fixed-point equation (3.11) for every $c \in \mathcal{I}_c$ and $\tilde{\psi}(\cdot, c_0) = 0$. In particular, we find that

$$\partial_c\psi(\cdot, c_0) := \lim_{c \to c_0} \frac{\psi - \psi_0}{c - c_0} = -\left(\mathcal{L}_0|_{X_0}\right)^{-1}\psi_0. \tag{3.12}$$

Hence, $\psi(\cdot, c)$ is an even solution of the boundary-value problem (1.8) for every $c \in \mathcal{I}_c$. $\square$

Remark 3.9. Although the solution ψ_0 is obtained from a global minimizer of the variational problem (2.1) and (2.2), the solution $\psi(\cdot, c)$ in lemma 3.8 is continued from the Euler–Lagrange equation (1.8). Therefore, even if the solution $\psi(\cdot, c)$ is C^1 with respect to c in $\mathcal{I}_c$ as in lemma 3.8, this solution may not coincide with the global minimizer of $\mathcal{B}_c$ in Y_0 for $c \neq c_0$, the existence of which is guaranteed by theorem 2.1 for every $c \in (-1, \infty)$. For example, the solution may only be a local minimizer of $\mathcal{B}_c$ in Y_0 for $c \neq c_0$ in $\mathcal{I}_c$. Similarly, we cannot guarantee that the solution $\psi(\cdot, c)$ has a single-lobe profile for $c \neq c_0$.

Remark 3.10. In what follows, we again use the general notation ψ for the solution to the boundary-value problem (1.8) and c for the (fixed) wave speed.

Corollary 3.11. *For every $c \in (-1, \infty)$ for which $\mathrm{Ker}(\mathcal{L}|_{X_0}) = \mathrm{span}(\partial_x \psi)$, we have*

$$\mathcal{L}\partial_c \psi = -\psi - b'(c), \tag{3.13}$$

where $b'(c) = \frac{1}{\pi}\int_{-\pi}^{\pi} \psi \partial_c \psi \mathrm{d}x$. If $c + 2b'(c) \neq 0$, then $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x \psi)$, whereas if $c + 2b'(c) = 0$, then $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x \psi, 1 - 2\partial_c \psi)$.

Proof. By lemma 3.8, equation (3.13) follows from (3.12) and the definition of $\mathcal{L}|_{X_0}$ in (3.7). The same equation can also be obtained by formal differentiation of the boundary-value problem (1.8) in c since ψ and b are C^1 with respect to c. It follows from (3.3) and (3.13) that

$$\mathcal{L}\,(1 - 2\partial_c \psi) = c + 2b'(c). \tag{3.14}$$

If $c + 2b'(c) = 0$, then $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x \psi, 1 - 2\partial_c \psi)$ by corollary 3.2. If $c + 2b'(c) \neq 0$, then $\{1, \psi, \psi^2\} \in \mathrm{Range}(\mathcal{L})$ by (3.2), (3.3) and (3.13), so that $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x \psi)$ by proposition 3.3. □

Remark 3.12. It follows from (3.2) and (3.13) that

$$-2\pi b(c) = \langle \mathcal{L}\partial_c \psi, \psi \rangle = \langle \partial_c \psi, \mathcal{L}\psi \rangle = -\frac{2\pi}{3}\gamma'(c),$$

so that $\gamma'(c) = 3b(c) > 0$, where $\gamma(c) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi^3 \mathrm{d}x$.

Remark 3.13. If $c_0 + 2b'(c_0) = 0$ for some $c_0 \in (-1, \infty)$, then φ and ω, which satisfy the stationary equation (2.8) after the Galilean transformation (2.7), are C^1 functions of c in $\mathcal{I}_c$ but not C^1 functions of ω at $\omega_0 := \sqrt{c_0^2 + 4b(c_0)}$. Indeed, differentiating the relation $\omega^2 = c^2 + 4b(c)$ in c yields

$$\omega \frac{\mathrm{d}\omega}{\mathrm{d}c} = c + 2b'(c),$$

so that $\frac{\mathrm{d}\omega}{\mathrm{d}c}|_{c=c_0} = 0$ and the C^1 mapping $\mathcal{I}_c \ni c \to \omega(c) \in \mathcal{I}_\omega$ is not invertible. Since the kernel of $\mathcal{L}$ at ψ_0 is 2D, the solution ψ_0 is at the fold point according to definition 1.5. The fold point yields the fold bifurcation of the solution φ with respect to parameter ω at ω_0.

The following lemma provides the explicit count of the number of negative eigenvalues $n(\mathcal{L})$ and the multiplicity of the zero eigenvalue $z(\mathcal{L})$ for the linearized operator $\mathcal{L}$ in (3.1).

Lemma 3.14. *Assume $\alpha \in (\frac{1}{3}, 2]$ and $\psi \in H^\infty_{\mathrm{per}}(\mathbb{T})$ be an even, single-lobe periodic wave for $c \in (-1, \infty)$ in lemma 3.8 with $\mathrm{Ker}(\mathcal{L}|_{X_0}) = \mathrm{span}(\partial_x \psi)$. Then, we have*

$$z(\mathcal{L}) = \begin{cases} 1, & c + 2b'(c) \neq 0, \\ 2, & c + 2b'(c) = 0, \end{cases} \tag{3.15}$$

and

$$n(\mathcal{L}) = \begin{cases} 1, & c + 2b'(c) \geqslant 0, \\ 2, & c + 2b'(c) < 0. \end{cases} \tag{3.16}$$

Proof. Thanks to (3.5), we have $n(\mathcal{L}|_{\{1,\psi^2\}^\perp}) = 0$. By corollary 3.7 and the assumption $\mathrm{Ker}(\mathcal{L}|_{X_0}) = \mathrm{span}(\partial_x\psi)$, we have $z(\mathcal{L}|_{\{1,\psi^2\}^\perp}) = 1$. By theorem 5.3.2 in [27] or theorem 4.1 in [34], we construct the following symmetric 2-by-2 matrix related to the two constraints in (3.5):

$$P(\lambda) := \begin{bmatrix} \langle(\mathcal{L} - \lambda I)^{-1}\psi^2, \psi^2\rangle & \langle(\mathcal{L} - \lambda I)^{-1}\psi^2, 1\rangle \\ \langle(\mathcal{L} - \lambda I)^{-1}1, \psi^2\rangle & \langle(\mathcal{L} - \lambda I)^{-1}1, 1\rangle \end{bmatrix}, \quad \lambda \notin \sigma(\mathcal{L}).$$

By corollary 3.11, we can use equation (3.13) in addition to equations (3.2) and (3.3). Assuming $c + 2b'(c) \neq 0$, we compute at $\lambda = 0$:

$$\begin{aligned} \langle\mathcal{L}^{-1}1, 1\rangle &= \frac{\langle 1 - 2\partial_c\psi, 1\rangle}{c + 2b'(c)} = \frac{2\pi}{c + 2b'(c)}, \\ \langle\mathcal{L}^{-1}1, \psi^2\rangle &= \frac{\langle 1 - 2\partial_c\psi, \psi^2\rangle}{c + 2b'(c)} = \frac{2\pi}{c + 2b'(c)}\left[b(c) - \frac{2}{3}\gamma'(c)\right], \\ \langle\mathcal{L}^{-1}\psi^2, 1\rangle &= -\langle\psi, 1\rangle - b(c)\frac{\langle 1 - 2\partial_c\psi, 1\rangle}{c + 2b'(c)} = -\frac{2\pi b(c)}{c + 2b'(c)}, \\ \langle\mathcal{L}^{-1}\psi^2, \psi^2\rangle &= -\langle\psi, \psi^2\rangle - b(c)\frac{\langle 1 - 2\partial_c\psi, \psi^2\rangle}{c + 2b'(c)} = -2\pi\gamma(c) - \frac{2\pi b(c)}{c + 2b'(c)}\left[b(c) - \frac{2}{3}\gamma'(c)\right], \end{aligned}$$

where $\gamma'(c) = 3b(c)$ holds by remark 3.12. Therefore, the determinant of $P(0)$ for $c + 2b'(c) \neq 0$ is computed as follows:

$$\det P(0) = -\frac{4\pi^2\gamma(c)}{c + 2b'(c)}. \tag{3.17}$$

Denote the number of negative and zero eigenvalues of $P(0)$ by n_0 and z_0 respectively. If $c + 2b'(c) = 0$, then $P(0)$ is singular, in which case denote the number of diverging eigenvalues of $P(\lambda)$ as $\lambda \to 0$ by z_∞. By theorem 4.1 in [34], we have the following identities:

$$\begin{cases} n(\mathcal{L}|_{\{1,\psi^2\}^\perp}) = n(\mathcal{L}) - n_0 - z_0, \\ z(\mathcal{L}|_{\{1,\psi^2\}^\perp}) = z(\mathcal{L}) + z_0 - z_\infty. \end{cases} \tag{3.18}$$

Since $\gamma(c) > 0$, it follows that $z_0 = 0$. Since $n(\mathcal{L}|_{\{1,\psi^2\}^\perp}) = 0$ we have $n(\mathcal{L}) = n_0$ by (3.18). It follows from the determinant (3.17) that $n_0 = 1$ if $c + 2b'(c) > 0$ and $n_0 = 2$ if $c + 2b'(c) < 0$. This yields (3.16) for $c + 2b'(c) \neq 0$.

Since $z(\mathcal{L}|_{\{1,\psi^2\}^\perp}) = 1$, we have $z(\mathcal{L}) = 1 + z_\infty$ by (3.18). If $c + 2b'(c) \neq 0$, then $z_\infty = 0$ so that $z(\mathcal{L}) = 1$. The determinant (3.17) implies that one eigenvalue of $P(\lambda)$ remains negative as $\lambda \to 0$, whereas the other eigenvalue of $P(\lambda)$ in the limit $\lambda \to 0$ jumps from positive side for $c + 2b'(c) > 0$ to the negative side for $c + 2b'(c) < 0$ through infinity at $c + 2b'(c) = 0$. Therefore, if $c + 2b'(c) = 0$, then $n_0 = 1$ and $z_\infty = 1$ so that $n(\mathcal{L}) = 1$ and $z(\mathcal{L}) = 2$. This yields (3.15) and (3.16) for $c + 2b'(c) = 0$. □

Remark 3.15. By proposition 2.5, we have invariance of the linearized operator $\mathcal{L}$ under the Galilean transformation (2.7):

$$\mathcal{L} = D^\alpha + c - 2\psi = D^\alpha + \omega - 2\varphi. \tag{3.19}$$

By using (2.15) and (2.16), we compute the small-amplitude expansion

$$c + 2b'(c) = 2^{\alpha+1} - 3 + \mathcal{O}(a^2).$$

Hence, for $\alpha > \alpha_0$ and small $a \in (0, a_0)$, we have $c + 2b'(c) > 0$ so that $n(\mathcal{L}) = 1$ in agreement with lemma 2.2 in [28], whereas for $\alpha < \alpha_0$ and small $a \in (0, a_0)$, we have $c + 2b'(c) < 0$ so that $n(\mathcal{L}) = 2$. In the continuation of the solution ψ in a for $\alpha < \alpha_0$ by corollary 2.7, there exists a fold point in the sense of definition 1.5 for which $c + 2b'(c) = 0$, see corollary 3.11 and remark 3.13.

4. Spectral stability

Here we consider the spectral stability problem (1.13). We assume that $\psi \in H^\infty_{\rm per}(\mathbb{T})$ is an even, single-lobe solution to the boundary-value problem (1.8) for some $c \in (-1, \infty)$ obtained with theorem 2.1, corollary 2.2 and proposition 2.4. Since ψ is smooth, the domain of $\partial_x \mathcal{L}$ in $L^2_{\rm per}(\mathbb{T})$ is $H^{1+\alpha}_{\rm per}(\mathbb{T})$.

If $\mathrm{Ker}(\mathcal{L}|_{X_0}) = \mathrm{span}(\partial_x \psi)$, then $\psi(\cdot, c)$ and $b(c)$ are C^1 functions in c by lemma 3.8. Therefore, we can use the three equations (3.2), (3.3) and (3.13) for the range of $\mathcal{L}$. We can also use the count of $n(\mathcal{L})$ and $z(\mathcal{L})$ in lemma 3.14. The following lemma provides a sharp criterion on the spectral stability of the periodic wave with profile ψ in the sense of definition 1.2.

Lemma 4.1. *Assume $\alpha \in (\frac{1}{3}, 2]$ and $\psi \in H^\infty_{\rm per}(\mathbb{T})$ be an even, single-lobe periodic wave for $c \in (-1, \infty)$ in lemma 3.8 with $\mathrm{Ker}(\mathcal{L}|_{X_0}) = \mathrm{span}(\partial_x \psi)$. The periodic wave ψ is spectrally stable if $b'(c) \geqslant 0$ and is spectrally unstable with exactly one unstable (real, positive) eigenvalue of $\partial_x \mathcal{L}$ in $L^2_{\rm per}(\mathbb{T})$ if $b'(c) < 0$.*

Proof. It is well-known [16, 22] that the periodic wave ψ is spectrally stable if it is a constrained minimizer of energy (1.2) under fixed momentum (1.3) and mass (1.4). Since $\mathcal{L}$ is the Hessian operator for $G(u)$ in (1.10), the spectral stability holds if

$$\mathcal{L}\big|_{\{1,\psi\}^\perp} \geqslant 0. \tag{4.1}$$

On the other hand, the periodic wave ψ is spectrally unstable with exactly one unstable (real, positive) eigenvalue of $\partial_x \mathcal{L}$ in $L^2_{\rm per}(\mathbb{T})$ if $n\left(\mathcal{L}\big|_{\{1,\psi\}^\perp}\right) = 1$.

By theorem 5.3.2 in [27] or theorem 4.1 in [34], we construct the following symmetric 2-by-2 matrix related to the two constraints in (4.1):

$$D(\lambda) := \begin{bmatrix} \langle (\mathcal{L} - \lambda I)^{-1} \psi, \psi \rangle & \langle (\mathcal{L} - \lambda I)^{-1} \psi, 1 \rangle \\ \langle (\mathcal{L} - \lambda I)^{-1} 1, \psi \rangle & \langle (\mathcal{L} - \lambda I)^{-1} 1, 1 \rangle \end{bmatrix}, \quad \lambda \notin \sigma(\mathcal{L}).$$

Assuming $c + 2b'(c) \neq 0$, we compute at $\lambda = 0$:

$$\begin{aligned}
\langle \mathcal{L}^{-1}1, 1\rangle &= \frac{2\pi}{c+2b'(c)},\\
\langle \mathcal{L}^{-1}1, \psi\rangle &= -\frac{2\pi b'(c)}{c+2b'(c)},\\
\langle \mathcal{L}^{-1}\psi, 1\rangle &= -\frac{2\pi b'(c)}{c+2b'(c)},\\
\langle \mathcal{L}^{-1}\psi, \psi\rangle &= -\pi b'(c) + \frac{2\pi [b'(c)]^2}{c+2b'(c)}.
\end{aligned}$$

Therefore, the determinant of $D(0)$ for $c+2b'(c)\neq 0$ is computed as follows:

$$\det D(0) = -\frac{2\pi^2 b'(c)}{c+2b'(c)}. \tag{4.2}$$

Denote the number of negative and zero eigenvalues of $D(0)$ by n_0 and z_0 respectively. If $c+2b'(c)=0$, then $D(0)$ is singular, in which case denote the number of diverging eigenvalues of $D(\lambda)$ as $\lambda\to 0$ by z_∞. By theorem 4.1 in [34], we have the following identities:

$$\begin{cases} n(\mathcal{L}|_{\{1,\psi\}^\perp}) = n(\mathcal{L}) - n_0 - z_0,\\ z(\mathcal{L}|_{\{1,\psi\}^\perp}) = z(\mathcal{L}) + z_0 - z_\infty. \end{cases} \tag{4.3}$$

By lemma 3.14, $n(\mathcal{L})=1$ if $c+2b'(c)\geqslant 0$ and $n(\mathcal{L})=2$ if $c+2b'(c)<0$, whereas $z(\mathcal{L})=1$ if $c+2b'(c)\neq 0$ and $z(\mathcal{L})=2$ if $c+2b'(c)=0$.

Assume first that $c+2b'(c)\neq 0$ so that $z_\infty=0$. If $b'(c)>0$, then $z_0=0$ whereas $n_0=1$ if $c+2b'(c)>0$ and $n_0=2$ if $c+2b'(c)<0$. In both cases, it follows from (4.3) that $n(\mathcal{L}|_{\{1,\psi\}^\perp})=0$ and $z(\mathcal{L}|_{\{1,\psi\}^\perp})=1$ which implies spectral stability of ψ.

If $b'(c)=0$, then $z_0=1$ whereas $n_0=0$ if $c+2b'(c)>0$ and $n_0=1$ if $c+2b'(c)<0$. In both cases, it follows from (4.3) that $n(\mathcal{L}|_{\{1,\psi\}^\perp})=0$ and $z(\mathcal{L}|_{\{1,\psi\}^\perp})=2$, which still implies spectral stability of ψ.

If $b'(c)<0$, then $z_0=0$ whereas $n_0=0$ if $c+2b'(c)>0$ and $n_0=1$ if $c+2b'(c)<0$. In both cases, it follows from (4.3) that $n(\mathcal{L}|_{\{1,\psi\}^\perp})=1$ and $z(\mathcal{L}|_{\{1,\psi\}^\perp})=1$, which implies spectral instability of ψ.

If $c+2b'(c)=0$, then $z_\infty=1$ and $z(\mathcal{L})=2$. Therefore, there is no change in the count compared to the previous cases. □

Corollary 4.2. *If $b'(c)\neq 0$, then $\mathrm{Ker}(\mathcal{L}|_{\{1,\psi\}^\perp})=\mathrm{span}(\partial_x\psi)$, whereas if $b'(c)=0$, then there exists $f\in\mathrm{Ker}(\mathcal{L}|_{\{1,\psi\}^\perp})$ such that $\langle f,\partial_x\psi\rangle=0$ and $f\neq 0$. In the latter case, $\langle f,\psi^2\rangle\neq 0$ and $\mathrm{Ker}(\mathcal{L})=\mathrm{span}(\partial_x\psi)$.*

Proof. It follows from (3.2) and (3.3) that for every $f\in\mathrm{dom}(\mathcal{L})$ satisfying $\langle f,1\rangle=\langle f,\psi\rangle=0$, we have

$$\mathcal{L}|_{\{1,\psi\}^\perp} f = \mathcal{L}f + \frac{\langle f,\psi^2\rangle}{\langle\psi,\psi\rangle}\psi. \tag{4.4}$$

If $f \in \mathrm{Ker}(\mathcal{L}|_{\{1,\psi\}^\perp})$ and $f \neq 0$, then either $\langle f, \psi^2 \rangle = 0$ or $\langle f, \psi^2 \rangle \neq 0$.

If $\langle f, \psi^2 \rangle = 0$, then $f \in \mathrm{Ker}(\mathcal{L})$ so that $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x \psi, f)$ by corollary 3.2. Then, $\{1, \psi, \psi^2\} \in [\mathrm{Ker}(\mathcal{L})]^\perp = \mathrm{Range}(\mathcal{L})$ and proposition 3.3 yields a contradiction with $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x \psi)$. Hence, $\langle f, \psi^2 \rangle \neq 0$.

If $\langle f, \psi^2 \rangle \neq 0$, then we have $\{1, \psi, \psi^2\} \in \mathrm{Range}(\mathcal{L})$ so that $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x \psi)$ by proposition 3.3. In addition, it follows from (3.13) that

$$0 = \langle f, \mathcal{L} \partial_c \psi \rangle = \langle \mathcal{L} f, \partial_c \psi \rangle = -\frac{\langle f, \psi^2 \rangle}{\langle \psi, \psi \rangle} \pi b'(c),$$

hence $b'(c) = 0$. This corresponds to the result $z(\mathcal{L}|_{\{1,\psi\}^\perp}) = 2$ if $b'(c) = 0$ in lemma 4.1. On the other hand, $z(\mathcal{L}|_{\{1,\psi\}^\perp}) = 1$ if $b'(c) \neq 0$ in lemma 4.1 so that $\mathrm{Ker}(\mathcal{L}|_{\{1,\psi\}^\perp}) = \mathrm{span}(\partial_x \psi)$ if $b'(c) \neq 0$. □

Remark 4.3. By using (2.15) and (2.16), we compute

$$b'(c) = 2^\alpha - 1 + \mathcal{O}(a^2),$$

which shows that the small-amplitude periodic waves are spectrally stable for small a and $\alpha > 0$ thanks to lemma 4.1. Since the fold point in the sense of definition 1.5 exists for $\alpha < \alpha_0$, see remark 3.15, the result of lemma 4.1 shows spectral stability of the periodic waves across the fold point as long as $b'(c) > 0$.

In the rest of this section, we address the possibility that the assumption $\mathrm{Ker}(\mathcal{L}|_{X_0}) = \mathrm{span}(\partial_x \psi_0)$ in lemma 3.8 is not satisfied at a particular point $c_0 \in (-1, \infty)$. The following lemma shows that this case corresponds to the linearized operator $\mathcal{L}$ with two negative eigenvalues.

Lemma 4.4. *Assume that for some $c_0 \in (-1, \infty)$ there exists $f \in \mathrm{Ker}(\mathcal{L}|_{X_0})$ such that $\langle f, \partial_x \psi_0 \rangle = 0$ and $f \neq 0$. Then, $n(\mathcal{L}) = 2$ and $z(\mathcal{L}) = 1$.*

Proof. The assertion $z(\mathcal{L}) = 1$ is proven in lemma 3.5. It follows from (3.7) that $\mathcal{L} f = -\frac{1}{\pi} \langle f, \psi_0 \rangle$ with $\langle f, 1 \rangle = 0$, $\langle f, \psi_0 \rangle \neq 0$, and $\langle f, \psi_0^2 \rangle = 0$. By normalizing

$$f_0 := \frac{\pi f}{\langle f, \psi_0 \rangle}$$

so that $\langle f_0, \psi_0 \rangle = \pi$, we use (3.2) and (3.3) to write

$$\mathcal{L} f_0 = -1, \quad \mathcal{L}\left(\psi_0 - b(c_0) f_0\right) = -\psi_0^2, \quad \mathcal{L}\left(1 + c_0 f_0\right) = -2\psi_0. \tag{4.5}$$

Thanks to the facts $\langle f_0, 1 \rangle = \langle f_0, \psi_0^2 \rangle = 0$, direct computations yield

$$\langle \mathcal{L}^{-1} 1, 1 \rangle = 0, \quad \langle \mathcal{L}^{-1} 1, \psi_0^2 \rangle = \langle \mathcal{L}^{-1} \psi_0^2, 1 \rangle = 0, \quad \langle \mathcal{L}^{-1} \psi_0^2, \psi_0^2 \rangle = -2\pi \gamma(c_0).$$

Since $\gamma(c_0) > 0$, we have $n_0 = 1$ and $z_0 = 1$ in the proof of lemma 3.14, so that the identities (3.18) yield

$$\begin{cases} n(\mathcal{L}) = n(\mathcal{L}\big|_{\{1,\psi_0^2\}^\perp}) + n_0 + z_0 = 2, \\ z(\mathcal{L}) = z(\mathcal{L}\big|_{\{1,\psi_0^2\}^\perp}) - z_0 = 1, \end{cases} \tag{4.6}$$

where we have used $n(\mathcal{L}\big|_{\{1,\psi_0^2\}^\perp}) = 0$ by theorem 2.1 and $z(\mathcal{L}\big|_{\{1,\psi_0^2\}^\perp}) = 2$ by corollary 3.7. □

By lemma 4.4, we obtain immediately the following corollary.

Corollary 4.5. *If* $n(\mathcal{L}) = 1$*, then* $\mathrm{Ker}(\mathcal{L}|_{X_0}) = \mathrm{span}(\partial_x\psi_0)$.

The following lemma shows that the exceptional case in lemma 4.4 corresponds to the spectrally unstable periodic wave with the profile ψ_0.

Lemma 4.6. *Under the same assumption as in lemma 4.4, the periodic wave* ψ_0 *is spectrally unstable with exactly one unstable (real, positive) eigenvalue of* $\partial_x\mathcal{L}$ *in* $L^2_{\mathrm{per}}(\mathcal{T})$.

Proof. Let f_0 be the same as in lemma 4.4 and define

$$\tilde{f}_0 := f_0 - \frac{\psi_0}{2b(c_0)}.$$

Then, $\langle \tilde{f}_0, 1\rangle = \langle \tilde{f}_0, \psi_0\rangle = 0$, and

$$\langle \mathcal{L}\tilde{f}_0, \tilde{f}_0\rangle = \frac{\langle \mathcal{L}\psi_0, \psi_0\rangle}{4b(c_0)^2} < 0,$$

thanks to (3.4). Therefore, $\mathcal{L}|_{\{1,\psi_0\}^\perp}$ is not positive definite and the periodic wave ψ_0 is spectrally unstable. Alternatively, one can compute directly

$$\langle \mathcal{L}^{-1}1, 1\rangle = 0, \quad \langle \mathcal{L}^{-1}1, \psi_0\rangle = \langle \mathcal{L}^{-1}\psi_0^2, 1\rangle = -\pi, \quad \langle \mathcal{L}^{-1}\psi_0, \psi_0\rangle = \frac{\pi c_0}{2},$$

so that we have $n_0 = 1$ and $z_0 = 0$ in the proof of lemma 4.1. and the identities (4.3) yield

$$\begin{cases} n(\mathcal{L}\big|_{\{1,\psi_0\}^\perp}) = n(\mathcal{L}) - n_0 - z_0 = 1, \\ z(\mathcal{L}\big|_{\{1,\psi_0\}^\perp}) = z(\mathcal{L}) + z_0 = 1. \end{cases} \tag{4.7}$$

Hence, the periodic wave ψ_0 is spectrally unstable with exactly one unstable (real, positive) eigenvalue of $\partial_x\mathcal{L}$ in $L^2_{\mathrm{per}}(\mathcal{T})$. □

Finally, we show that the condition $\mathrm{Ker}(\mathcal{L}|_{X_0}) = \mathrm{span}(\partial_x\psi_0)$ for the C^1 continuation of the single-lobe periodic wave with profile ψ_0 in lemma 3.8 is sharp in the sense that if $\mathrm{Ker}(\mathcal{L}|_{X_0}) \neq \mathrm{span}(\partial_x\psi_0)$, then the mapping (3.9) is not differentiable at c_0, in particular, $b'(c_0)$ does not exist.

Lemma 4.7. *Assume* $\mathrm{Ker}(\mathcal{L}|_{X_0}) \neq \mathrm{span}(\partial_x\psi_0)$*. Then,* $\psi(\cdot, c)$ *and* $b(c)$ *are not* C^1 *functions in* c *at* c_0.

Proof. Assume $\mathrm{Ker}(\mathcal{L}|_{X_0}) = \mathrm{span}(\partial_x\psi_0, f_0)$. Then, $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x\psi_0)$ and $\langle f_0, \psi_0\rangle \neq 0$ by lemma 3.5. Hence, equation (3.10) cannot be solved by inverting the operator $\mathcal{L}|_{X_0}$.

By using the Galilean transformation (2.7) of proposition 2.5, let $\varphi_0 \in H^\alpha_{\rm per}(\mathbb{T})$ be an even solution of the normalized equation (2.8) for parameter ω_0, where $\varphi_0 := \psi_0 - \frac{1}{2}(c_0 - \omega_0)$ and $\omega_0 := \sqrt{c_0^2 + 4b(c_0)}$. Let $\varphi \in H^\alpha_{\rm per}(\mathbb{T})$ be a solution of the normalized equation (2.8) for ω near ω_0. Then, $\tilde{\varphi} := \varphi - \varphi_0 \in H^\alpha_{\rm per}(\mathbb{T})$ satisfies the following equation:

$$\mathcal{L}\tilde{\varphi} = -(\omega - \omega_0)(\varphi_0 + \tilde{\varphi}) + \tilde{\varphi}^2, \tag{4.8}$$

where $\mathcal{L}$ is given by (3.19) at φ_0 and ω_0. (For simplicity of notations, we do not relabel this linearized operator as $\mathcal{L}_0$, compared to the proof of lemma 3.8.)

Since $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x \psi_0)$, applying the same argument as in lemma 3.8 yields the existence of the unique C^1 mapping $\mathcal{I}_\omega \ni \omega \mapsto \tilde{\varphi}(\cdot, \omega) \in \tilde{B}_r \subset H^\alpha_{\rm per}(\mathbb{T})$ such that $\mathcal{I}_\omega$ is an open interval containing ω_0 and $\tilde{\varphi}(\cdot, \omega)$ is an even solution to equation (4.8) for every $\omega \in \mathcal{I}_\omega$ and $\tilde{\varphi}(\cdot, \omega_0) = 0$. In particular, we have

$$\partial_\omega \varphi(\cdot, \omega_0) = -\mathcal{L}^{-1}\varphi_0. \tag{4.9}$$

Hence, $\varphi(\cdot, \omega)$ is an even solution of the boundary-value problem (2.8) for every $\omega \in \mathcal{I}_\omega$.

It follows from the transformation formulas

$$\psi(\cdot, \omega) = \Pi_0 \varphi(\cdot, \omega), \quad c(\omega) = \omega - \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi \mathrm{d}x, \quad b(\omega) = \frac{1}{4}(\omega^2 - c^2) \tag{4.10}$$

that $\psi(\cdot, \omega)$, $c(\omega)$, and $b(\omega)$ are C^1 functions of ω for every $\omega \in \mathcal{I}_\omega$. It follows from (2.7), (3.3) and (4.9) that

$$\mathcal{L}\left(\partial_\omega \varphi(\cdot, \omega_0) - \frac{1}{2}\right) = -\frac{\omega_0}{2}, \quad \Rightarrow \quad \mathcal{L}\left(\partial_\omega \psi(\cdot, \omega_0) - \frac{1}{2}c'(\omega_0)\right) = -\frac{\omega_0}{2}.$$

Let $f_0 \in \mathrm{Ker}(\mathcal{L}|_{X_0})$ be normalized from (4.5) so that $\mathcal{L}f_0 = -1$. Therefore, in the subspace of even functions, we have

$$\partial_\omega \psi(\cdot, \omega_0) - \frac{1}{2}c'(\omega_0) = \frac{\omega_0}{2} f_0,$$

which implies $c'(\omega_0) = 0$ because $\partial_\omega \psi(\cdot, \omega_0)$ and f_0 are periodic functions with zero mean. Hence, the C^1 mapping $\mathcal{I}_\omega \ni \omega \to c(\omega) \in \mathcal{I}_c$ is not invertible. Consequently, $\psi(\cdot, c)$ and $b(c)$ are not C^1 functions of c at c_0. In particular, the relation $\omega = (c + 2b'(c))c'(\omega)$ for $\omega \in \mathcal{I}_\omega$ implies that $b'(c_0)$ does not exist. $\square$

5. Numerical approximations of periodic waves

Here we compute the existence curve for the single-lobe periodic solutions of the boundary-value problem (1.8) on the parameter plane (c, b) for $\alpha \in \left(\frac{1}{3}, 2\right]$.

For the integrable BO equation ($\alpha = 1$), the single-lobe periodic solution to the boundary-value problem (2.8) is known in the exact form:

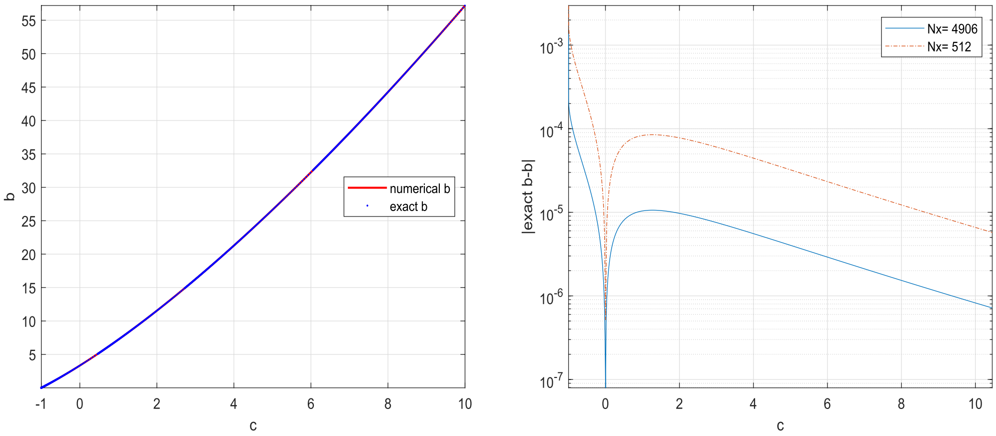

Figure 2. Left: the dependence of b versus c for $\alpha = 2$. Right: the difference between the numerical and exact values of b versus c.

$$\omega = \coth\gamma, \quad \varphi(x) = \frac{\sinh\gamma}{\cosh\gamma - \cos x}, \tag{5.1}$$

where $\gamma \in (0, \infty)$ is a free parameter of the solution. Since $\int_0^\pi \varphi(x)\mathrm{d}x = \pi$, we compute explicitly $c = \omega - 2$ and $b = \frac{1}{4}(\omega^2 - c^2) = \omega - 1$. Eliminating $\omega \in (1, \infty)$ yields $b(c) = c + 1$ shown on figure 1 (left).

For the integrable KdV equation ($\alpha = 2$), the single-lobe periodic solution to the boundary-value problem (2.8) is known in the exact form:

$$\omega = \frac{4K(k)^2}{\pi^2}\sqrt{1 - k^2 + k^4} \tag{5.2}$$

and

$$\varphi(x) = \frac{2K(k)^2}{\pi^2}\left[\sqrt{1 - k^2 + k^4} + 1 - 2k^2 + 3k^2\mathrm{cn}^2\left(\frac{K(k)}{\pi}x; k\right)\right], \tag{5.3}$$

where the elliptic modulus $k \in (0, 1)$ is a free parameter of the solution. Since

$$\int_0^\pi \varphi(x)\mathrm{d}x = \frac{2K(k)^2}{\pi}\left[\sqrt{1 - k^2 + k^4} + 1 - 2k^2\right] + \frac{6K(k)}{\pi}\left[E(k) + (k^2 - 1)K(k)\right],$$

where $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kinds, respectively, we compute explicitly

$$c = \frac{4K(k)^2}{\pi^2}\left[2 - k^2 - \frac{3E(k)}{K(k)}\right] \tag{5.4}$$

and

$$b = \frac{4K(k)^4}{\pi^4}\left[-3(1 - k^2) + (2 - k^2)\frac{6E(k)}{K(k)} - \frac{9E(k)^2}{K(k)^2}\right]. \tag{5.5}$$

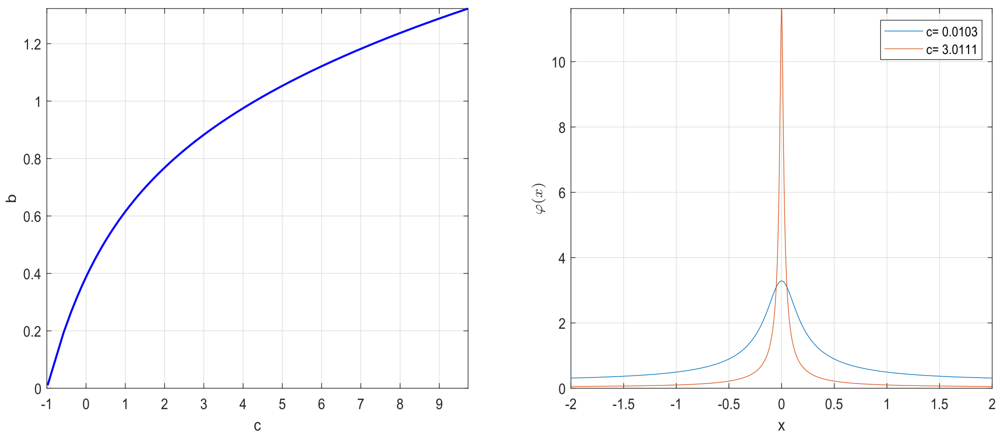

Figure 3. Left: the dependence of b versus c for $\alpha = 0.6$ obtained with the Petviashvili's method. Right: profiles of φ for two values of c.

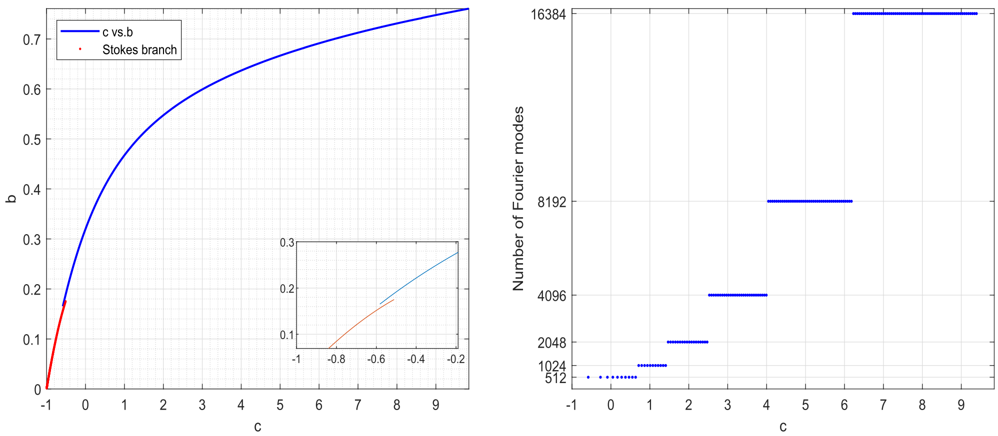

Figure 4. Left: the dependence of b versus c for $\alpha = 0.55$ obtained with the Petviashvili's method. Right: the number of Fourier modes versus c.

Figure 2 (left) shows the existence curve (5.4) and (5.5) on the parameter plane (c, b). It follows that the function $b(c)$ is monotonically increasing in c. In the limit $k \to 1$, for which $K(k) \to \infty$ and $E(k) \to 1$, we compute from (5.4) and (5.5) the asymptotic behavior

$$b(c) \sim \frac{3}{\pi} c^{3/2} \quad \text{as} \quad c \to \infty,$$

which coincides with the behavior of KdV solitons.

The existence curve on the (c, b) plane is also computed numerically by using the Petviashvili's method from [28] for the stationary equation (2.8) with $\omega \in (1, \infty)$ and applying the transformation formula (2.17). Figure 2 (left) also shows the numerically obtained existence curve (invisible from the theoretical curve). The right panel of figure 2 shows the error between the numerical and exact curves for two computations different by the number N of Fourier modes in the approximation of periodic solutions (for $N = 512$ by red curve and $N = 4906$ by blue curve). The more Fourier modes are included, the smaller is the error.

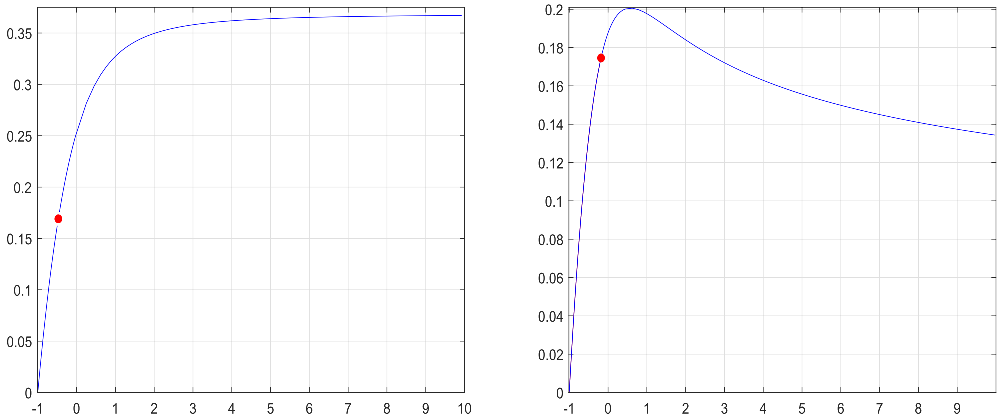

Figure 5. The dependence of b versus c for $\alpha = 0.5$ (left) and $\alpha = 0.45$ (right) obtained with the Newton's method.

For other values of α in $\left(\frac{1}{3}, 1\right)$, we only compute the existence curve numerically. Figure 3 shows the existence curve (left) and two profiles of the numerically computed φ in the stationary equation (2.8) (right) in the case $\alpha = 0.6 > \alpha_0$. The function $b(c)$ is still monotonically increasing in c and the values of $c \in (-1, \infty)$ are obtained monotonically from the values of $\omega \in (1, \infty)$ in the stationary equation (2.8). We also note that the greater is the wave speed c, the larger is the amplitude of the periodic wave and the smaller is its characteristic width.

Figure 4 (left) shows the existence curve in the case $\alpha = 0.55 < \alpha_0$ computed numerically (blue curve) and by using Stokes expansions (2.15) and (2.16) (red curve). The insert displays the mismatch between the red and blue curves with a small gap. The reason for mismatch is the lack of numerical data for $c \in (-1, -0.6)$ due to the fold point discussed in remarks 2.8, 3.13 and 3.15. The function $\omega(c)$ is not monotonically increasing near the fold point and there exist two single-humped solutions for $\omega < 1$. Only the solution with $n(\mathcal{L}) = 1$ can be approximated with the Petviashvili's method as in [28], whereas the other solution with $n(\mathcal{L}) = 2$ is unstable in the iterations of the Petviashvili's method which then converge to a constant solution instead of the single-lobe solution. This is why we augmented the existence curve on figure 4 (left) with the Stokes expansion given by (2.15) and (2.16).

The right panel of figure 4 shows the number of Fourier modes used in our numerical computations as the wave speed c increases. We have to increase the number of Fourier modes in order to control the accuracy of the numerical approximations and to ensure that the strongly compressed solution with the wave profile φ is properly resolved. It follows from the Heisenberg's uncertainty principle that the narrower is the characteristic width of the wave profile, the weaker is the decay of the Fourier transform at infinity. We compute the maximum of the Fourier transform at the last ten Fourier modes and increase the number of Fourier modes every time the maximum becomes bigger than a certain tolerance level of the size 10^{-8}. The computational time slows down for larger values of the wave speed, nevertheless, it is clear that the function $b(c)$ is still monotonically increasing in c.

In order to overcome the computational problem seen on figure 4 (left), we have developed the Newton's method for the solutions φ to the stationary equation (2.8) near the fold point that exists for $\alpha < \alpha_0$. With the initial guess from the Stokes expansion in (2.9) and (2.10), we were able to find the branch of solutions with $n(\mathcal{L}) = 2$ and connect it with the branch of solutions with $n(\mathcal{L}) = 1$. As a result, the mismatch seen on the insert of figure 4 for $\alpha = 0.55$ has been eliminated by using the Newton's method (not shown).

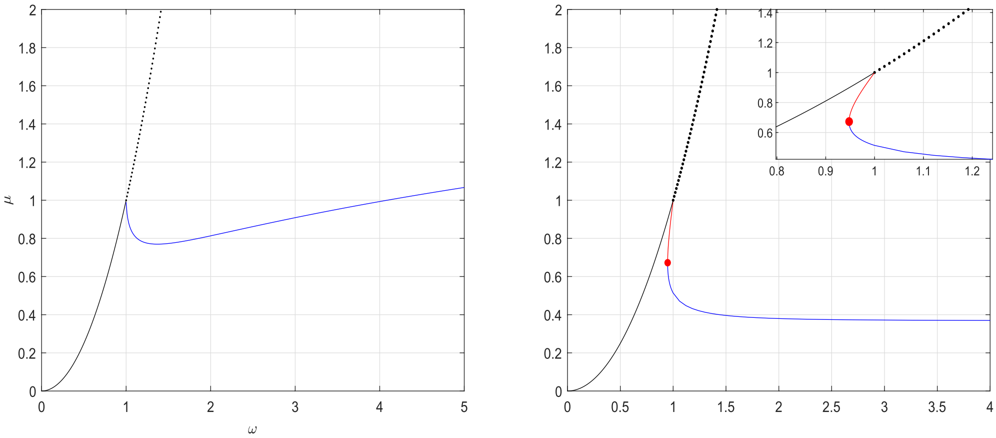

Figure 6. The dependence of μ versus ω for $\alpha = 0.6$ (left) and $\alpha = 0.5$ (right) obtained with the Newton's method.

Figure 5 shows the existence curve on the parameter plane (c, b) in the cases $\alpha = 0.5$ (left) and $\alpha = 0.45$ (right) obtained with the Newton's method. It is obvious that the function $b(c)$ is monotonically increasing in c for $\alpha = 0.5$ and approaches to the horizontal asymptote as $c \to \infty$, whereas the function $b(c)$ is not monotone in c for $\alpha = 0.45$ and is decreasing for large values of c. This coincides with the conclusion of [4] on the solitary waves which correspond to the limit of $c \to \infty$.

By the stability result of theorem 1.3, we conjecture based on our numerical results that the single-lobe periodic waves are spectrally stable for $\alpha \in \left[\frac{1}{2}, 2\right]$ since $b'(c) > 0$ for every $c \in (-1, \infty)$. On the other hand, for $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$, there exists $c_* \in (-1, \infty)$ such that $b'(c) > 0$ for $c \in (-1, c_*)$ and $b'(c) < 0$ for $c \in (c_*, \infty)$, hence the periodic waves are spectrally stable for $c \in (-1, c_*)$ and spectrally unstable for $c \in (c_*, \infty)$.

Finally, we reproduce the same results but on the parameter plane (ω, μ), where ω is the Lagrange multiplier in the boundary-value problem (2.8) and $\mu := \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi^2 \mathrm{d}x$ is the period-normalized momentum computed at the periodic wave φ. The parameter plane corresponds to the minimization of the energy $E(u)$ subject to the fixed momentum $F(u)$ with $a = 0$ used in [21].

The boundary-value problem (2.8) always has the constant solution given by $\varphi(x) = \omega$ for which $\mu = \omega^2$. As is shown in [28], the constant solution is a constrained minimizer of energy for $\mu \in (0, 1)$ and is a saddle point of energy for $\mu \in (1, \infty)$. It is shown by solid black curve for $\mu \in (0, 1)$ and by dashed black curve for $\mu \in (1, \infty)$.

For $\alpha = 1$, the exact solution (5.1) for the single-lobe periodic wave φ can be used to compute explicitly $\mu = \omega$ for $\omega \in (1, \infty)$ shown on figure 1 (right) by solid blue curve. The slope of μ along the branch for single-lobe periodic waves at $\omega = 1$ can be found directly from the Stokes expansion (2.9) and (2.15) as

$$\lim_{\omega \searrow 1} \mu'(\omega) = 2 - \frac{1}{2\omega_2} = \frac{3 \cdot 2^\alpha - 5}{2 \cdot 2^\alpha - 3}.$$

The slope becomes horizontal at $\alpha = \alpha_* = \frac{\log 5 - \log 3}{\log 2} \approx 0.737$, negative for $\alpha \in (\alpha_0, \alpha_*)$, vertical at $\alpha = \alpha_0 = \frac{\log 3}{\log 2} - 1 \approx 0.585$, and positive for $\alpha < \alpha_0$. Figure 6 shows the bifurcation diagram on the parameter plane (ω, μ) for $\alpha = 0.6$ (left) and $\alpha = 0.5$ (right).

For $\alpha = 0.6$, see figure 6 (left), two single-lobe periodic waves (blue curve) coexist for the same value of μ below 1. The right branch is a local minimizer of energy $E(u)$ subject to fixed momentum $F(u)$, whereas the left branch is a saddle point of energy subject to fixed momentum and is a local minimizer of energy $E(u)$ subject to two constraints of momentum $F(u)$ and mass $M(u)$. This folded picture is unfolded on figure 3 (left), which contains all the single-lobe periodic waves and none of the constant solutions.

For $\alpha = 0.5$, see figure 6 (right), the folded diagram on the (ω, μ) plane becomes more complicated because two single-lobe periodic waves coexist for ω below 1 (red and blue curves) and two periodic waves coexist for μ below 1. The red (blue) curve on figure 6 (right) corresponds to the part of the curve on figure 5 (left) below (above) the red point. Both branches are resolved well by using the Newton's method. The branch shown by the red curve corresponds to $n(\mathcal{L}) = 2$, nevertheless, it is a local minimizer of energy $E(u)$ subject to two constraints of momentum $F(u)$ and mass $M(u)$. At the fold point $\omega_0 \in (0, 1)$, the linearized operator $\mathcal{L}$ is degenerate with $z(\mathcal{L}) = 2$. The branch is continued below the fold point and then to the right with $n(\mathcal{L}) = 1$. The decreasing and increasing parts of the branch have the same variational characterization as those on figure 6 (left). The folded picture is again unfolded on figure 5 (left) on the parameter plane (c, b), where the scalar condition $b'(c) > 0$ for spectral stability of the single-lobe periodic waves implies that every point on the folded bifurcation diagram on the (ω, μ) parameter plane correspond to spectrally stable periodic waves. The fold point on figure 6 (right), where the linearized operator $\mathcal{L}$ is degenerate and the momentum and mass are not smooth with respect to Lagrange multipliers, appears to be an internal point on the branch on figure 5 (left) which remains smooth with respect to the only parameter of the wave speed c.

Thus, we conclude that the new variational characterization of the zero-mean single-lobe periodic waves in the fractional KdV equation (1.1) allows us to unfold all the solution branches on the parameter plane (c, b) and to identify the stable periodic waves using the scalar criterion $b'(c) > 0$.

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