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Breathers and breather-rogue waves on a periodic background for the derivative nonlinear Schrödinger equation

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Abstract

In this paper, we give the solutions on a periodic background in terms of the determinant form for the derivative nonlinear Schrödinger equation. Because its rogue wave on a periodic background has been studied, we investigate only the breather and breather-rogue wave on a periodic background for the derivative nonlinear Schrödinger equation. We obtain Kuznetsov–Ma breather, Akhmediev breather and spatio-temporal breather on a periodic background for this equation. In addition, we mainly focus on three types of the breather-rogue wave on a periodic background: (1) the interaction between a Peregrine soliton and a breather; (2) the interaction between a Peregrine soliton and two breathers; (3) the interaction between a second-order rogue wave and a breather. For the first type, we analyse the effects of the free parameters on its dynamical behaviour. The second type is described as 'rogue wave quanta' on a periodic background. The third type has two spatial-temporal distribution structures: the fundamental structure and the triangular structure.

Keywords: derivative nonlinear Schrödinger equation, dynamical behaviour, breather on a periodic background, breather-rogue wave on a periodic background

(Some figures may appear in colour only in the online journal)

1. Introduction

In the nonlinear science field, nonlinear evolution equations play an important role and their solutions have been a hot research spot, including soliton, rogue wave (RW), lump wave and others [1–29]. As one of the most important nonlinear waves, RW has been widely applied in various fields, such as oceanography [7, 8], plasma physics [9–11], financial system [12], geophysical fluids [13, 14], Bose–Einstein condensate [15, 16] and nonlinear optics [17, 18]. There are various structures of RWs, including the fundamental, four-peaked, triangular, ring and semi-rational structures [7–25]. In addition, the above RWs are obviously on a constant background. Recently, RW on a periodic background, called rogue periodic wave, has been reported for some soliton equations, for example, the nonlinear Schrödinger equation [30–34], the modified KdV equation [35], the derivative nonlinear Schrödinger equation [36] and Gerdjikov–Ivanov equation [37]. Many methods of constructing rogue periodic wave have been presented, e.g. Jacobi elliptic functions [30, 32, 35], integrable equations with variable coefficients [31] and vector form [34], $\mathcal{PT}$-symmetric [33], the odd-th fold Darboux transformation (DT) [36, 37].

The derivative nonlinear Schrödinger equation

$$q_t + iq_{xx} + (|q|^2 q)_x = 0, \tag{1.1}$$

plays a significant role in plasma physics and nonlinear optics [10, 11, 22, 23, 36, 38, 39]. All kinds of RW solutions on a constant wave background have been obtained for the derivative nonlinear Schrödinger equation [10, 11, 22, 23]. Moreover, Liu, Zhang and He showed the fundamental structure and the triangular structure for RWs on a periodic background by means of the odd-th order DT [36].

Subsequently, Ding, Gao and Li used this method to obtain the richer structures of RWs on a periodic background for the Gerdjikov–Ivanov equation [37]. More recently, Xu, He and Mihalache discussed a new mechanism of RW generation through multiphase solutions degeneration for the derivative nonlinear Schrödinger equation [39]. To our best knowledge, the breathers and breather-RWs on a periodic background have not been obtained for the derivative nonlinear Schrödinger equation (1.1). In this paper, we will give the solution on a periodic background in terms of the determinant form for equation (1.1). As an application, we obtain Kuznetsov–Ma breather (KM), Akhmediev breather (AB) and spatio-temporal breather (STB) on a periodic background. More importantly, we shall exhibit three types of the breather-rogue periodic waves: (1) the interaction between a Peregrine soliton (PS) and a breather; (2) the interaction between a PS and two breathers; (3) the interaction between a second-order RW and a breather. We analyse the dynamics of the first type by taking the different values for the free parameters. The second type can be referred as 'rogue wave quanta' on a periodic background [40]. For the third type, we show two spatial-temporal distribution structures: the fundamental structure and the triangular structure.

This paper is organized as follows. In section 2, we will present the formula of the local wave on a periodic background by modifying the odd-th order DT for equation (1.1). In section 3, we will consider the breathers and breather-RWs on a periodic background for equation (1.1) and display their dynamical behaviour.

2. The formula of the solution on a periodic background

In this section, we will give the formula of the solution on a periodic background for equation (1.1), which includes RWs, breathers, breather-RWs. To this end, the following spectral problems are gave

$$\Psi_x = U\Psi,\quad U = \begin{pmatrix} i\lambda^2 & \lambda q \\ -\lambda q^* & -i\lambda^2 \end{pmatrix},$$
$$\Psi_t = V\Psi,$$
$$V = \begin{pmatrix} 2i\lambda^4 - i\lambda^2|q|^2 & 2\lambda^3 q - \lambda(|q|^2 q + iq_x) \\ -2\lambda^3 q^* + \lambda(|q|^2 q^* - iq_x^*) & -2i\lambda^4 + i\lambda^2|q|^2 \end{pmatrix}, \tag{2.1}$$

which can derive the derivative nonlinear Schrödinger equation (1.1). Here, the classical odd-th fold DT of equation (1.1) is recalled in the form of a theorem without considering the first-fold DT.

Theorem 2.1. *([11], Xu et al) Let $(2n+1)$ distinct eigenfunctions $\Psi_k = (\phi_k, \varphi_k)^T$ ($n \in \mathbb{N}_+$) associated with the eigenvalue λ_k and the seed solution q_0 for the spectral problems (2.1) have the following properties:*

$$\phi_k^* = \varphi_{k+1},\quad \varphi_k^* = \phi_{k+1},\quad \lambda_k^* = -\lambda_{k+1};$$
$$\phi_{2n+1}^* = \varphi_{2n+1},\quad \lambda_{2n+1} = -\lambda_{2n+1}^*,\quad k = 1, 3, \ldots, 2n-1.$$

Then, equation (1.1) has the $(2n+1)$-order solutions:

$$q^{[2n+1]} = \frac{\tilde{\Theta}_{11}^2}{\tilde{\Theta}_{21}^2} q_0 + 2i\frac{\tilde{\Theta}_{11}\tilde{\Theta}_{12}}{\tilde{\Theta}_{21}^2},$$

where

$$\tilde{\Theta}_{11} = \begin{vmatrix} \lambda_1^{2n}\varphi_1 & \lambda_1^{2n-1}\phi_1 & \lambda_1^{2n-2}\varphi_1 & \cdots & \lambda_1\phi_1 & \varphi_1 \\ \lambda_2^{2n}\varphi_2 & \lambda_2^{2n-1}\phi_2 & \lambda_2^{2n-2}\varphi_2 & \cdots & \lambda_2\phi_2 & \varphi_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_{2n+1}^{2n}\varphi_{2n+1} & \lambda_{2n+1}^{2n-1}\phi_{2n+1} & \lambda_{2n+1}^{2n-2}\varphi_{2n+1} & \cdots & \lambda_{2n+1}\phi_{2n+1} & \varphi_{2n+1} \end{vmatrix},$$
$$\tilde{\Theta}_{21} = \begin{vmatrix} \lambda_1^{2n}\phi_1 & \lambda_1^{2n-1}\varphi_1 & \lambda_1^{2n-2}\phi_1 & \cdots & \lambda_1\varphi_1 & \phi_1 \\ \lambda_2^{2n}\phi_2 & \lambda_2^{2n-1}\varphi_2 & \lambda_2^{2n-2}\phi_2 & \cdots & \lambda_2\varphi_2 & \phi_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_{2n+1}^{2n}\phi_{2n+1} & \lambda_{2n+1}^{2n-1}\varphi_{2n+1} & \lambda_{2n+1}^{2n-2}\phi_{2n+1} & \cdots & \lambda_{2n+1}\varphi_{2n+1} & \phi_{2n+1} \end{vmatrix},$$
$$\tilde{\varpi} = (\lambda_1^{2n+1}\phi_1, \lambda_2^{2n+1}\phi_2, \ldots, \lambda_{2n+1}^{2n+1}\phi_{2n+1})^T,$$

and $\tilde{\Theta}_{12}$ is obtained from $\tilde{\Theta}_{11}$ by replacing its first column with $\tilde{\varpi}$.

According to theorem 2.1, we can only select $(n+1)$ different eigenfunctions $\Psi_k(x, t; \lambda_k) = (\phi_k, \varphi_k)^T$ ($n \in \mathbb{N}_+$) related to the seed solution q_0 for the spectral problems (2.1), which meet

$$(1)\ \lambda_k \neq -\lambda_k^*,\ \text{as } k = 1, 2, \ldots, n;$$
$$(2)\ \phi_{n+1}^* = \varphi_{n+1},\quad \lambda_{n+1} = -\lambda_{n+1}^*.$$

Subsequently, we can get a $(2n+1)$-order solution for equation (1.1):

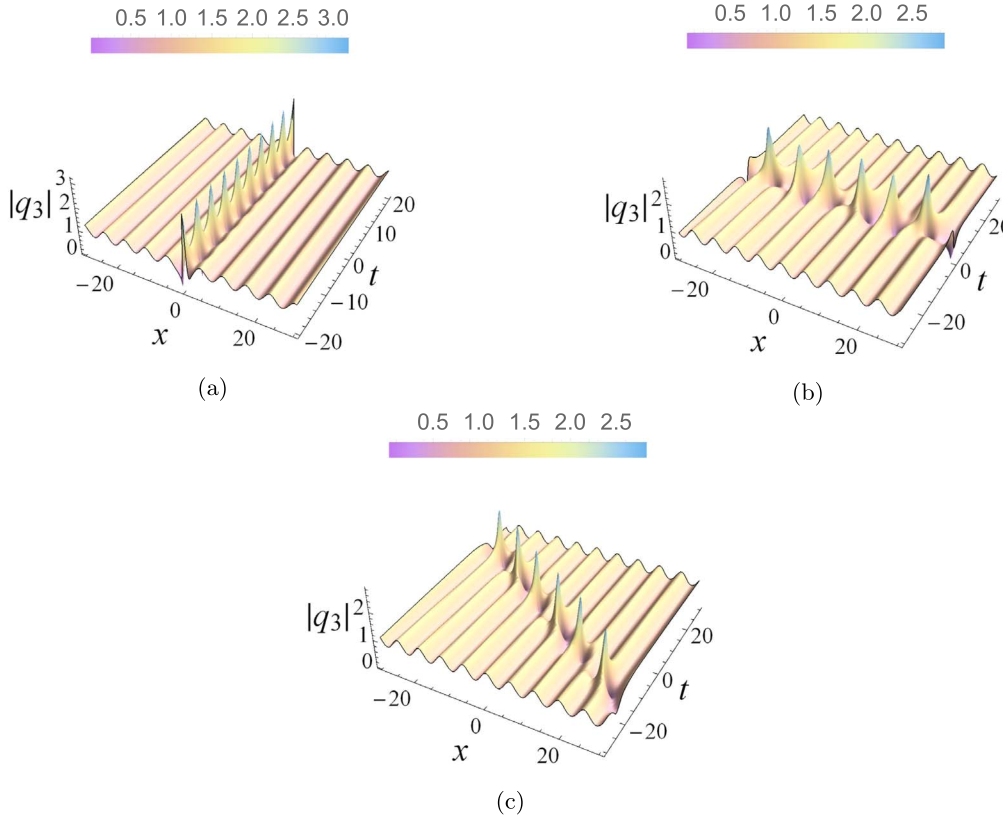

Figure 1. The third-order solution of equation (1.1) with $s_{01} = s_{02} = 0$ and $\beta_2 = 1/10$: (a) a KMB on a periodic background as $\mu_1 = \nu_1 = 2^{-3/4}$; (b) an AB on a periodic background as $\mu_1 = \nu_1 = 2^{-5/4}$; (c) a STB on a periodic background as $\mu_1 = 27/50$ and $\nu_1 = 9/20$.

$$q^{[2n+1]} = \frac{\hat{\Theta}_{11}^2}{\hat{\Theta}_{21}^2} q_0 + 2i \frac{\hat{\Theta}_{11}\hat{\Theta}_{12}}{\hat{\Theta}_{21}^2}, \tag{2.2}$$

where

$$\hat{\Theta}_{11} = \begin{vmatrix} \lambda_1^{2n}\varphi_1 & \lambda_1^{2n-1}\phi_1 & \lambda_1^{2n-2}\varphi_1 & \cdots & \lambda_1\phi_1 & \varphi_1 \\ \lambda_1^{*2n}\phi_1^* & -\lambda_1^{*(2n-1)}\varphi_1^* & \lambda_1^{*(2n-2)}\phi_1^* & \cdots & -\lambda_1^*\varphi_1^* & \phi_1^* \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_n^{2n}\varphi_n & \lambda_n^{2n-1}\phi_n & \lambda_n^{2n-2}\varphi_n & \cdots & \lambda_n\phi_n & \varphi_n \\ \lambda_n^{*2n}\phi_n^* & -\lambda_n^{*(2n-1)}\varphi_n^* & \lambda_n^{*(2n-2)}\phi_n^* & \cdots & -\lambda_n^*\varphi_n^* & \phi_n^* \\ \lambda_{n+1}^{2n}\varphi_{n+1} & \lambda_{n+1}^{2n-1}\phi_{n+1} & \lambda_{n+1}^{2n-2}\varphi_{n+1} & \cdots & \lambda_{n+1}\phi_{n+1} & \varphi_{n+1} \end{vmatrix},$$

$$\hat{\Theta}_{21} = \begin{vmatrix} \lambda_1^{2n}\phi_1 & \lambda_1^{2n-1}\varphi_1 & \lambda_1^{2n-2}\phi_1 & \cdots & \lambda_1\varphi_1 & \phi_1 \\ \lambda_1^{*2n}\varphi_1^* & -\lambda_1^{*(2n-1)}\phi_1^* & \lambda_1^{*(2n-2)}\varphi_1^* & \cdots & -\lambda_1^*\phi_1^* & \varphi_1^* \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_n^{2n}\phi_n & \lambda_n^{2n-1}\varphi_n & \lambda_n^{2n-2}\phi_n & \cdots & \lambda_n\varphi_n & \phi_n \\ \lambda_n^{*2n}\varphi_n^* & -\lambda_n^{*(2n-1)}\phi_n^* & \lambda_n^{*(2n-2)}\varphi_n^* & \cdots & -\lambda_n^*\phi_n^* & \varphi_n^* \\ \lambda_{n+1}^{2n}\phi_{n+1} & \lambda_{n+1}^{2n-1}\varphi_{n+1} & \lambda_{n+1}^{2n-2}\phi_{n+1} & \cdots & \lambda_{n+1}\varphi_{n+1} & \phi_{n+1} \end{vmatrix},$$

$$\hat{\varpi} = (\lambda_1^{2n+1}\phi_1, -\lambda_1^{*(2n+1)}\varphi_1^*, \ldots, \lambda_n^{2n+1}\phi_n, -\lambda_n^{*(2n+1)}\varphi_n^*, \lambda_{n+1}^{2n+1}\phi_{n+1})^T,$$

and $\hat{\Theta}_{12}$ is obtained from $\hat{\Theta}_{11}$ by replacing its first column with $\hat{\varpi}$.

Obviously, equation (2.2) is a $(2n + 1)$-fold DT of equation (1.1). In order to obtain the generalized DT, we

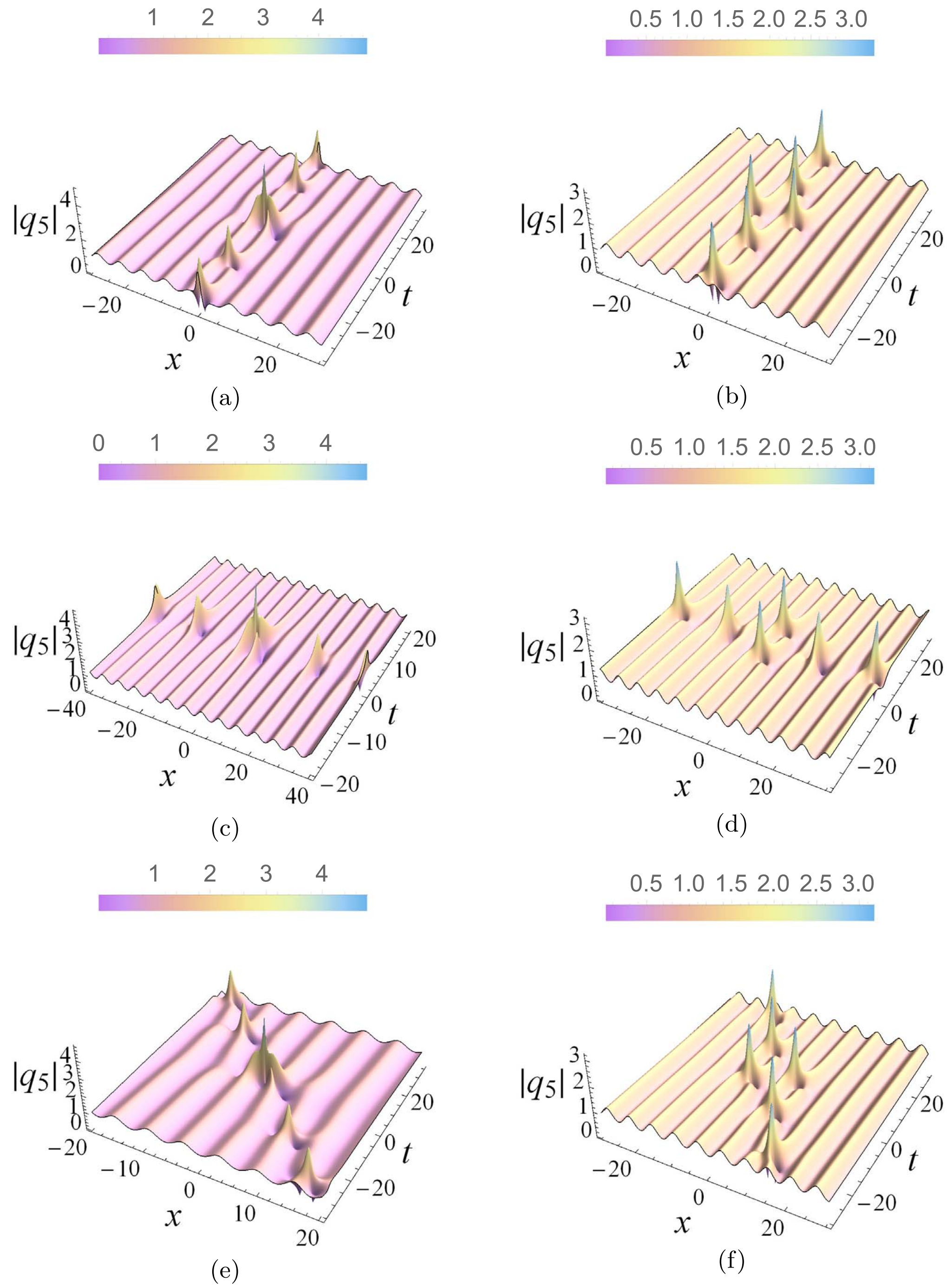

Figure 2. The fifth-order solution of equation (1.1) with $s_{01} = s_{02} = s_{03} = 0$ and $\beta_3 = 1/10$: (a) the superposition of a PS and a KMB with a second-order central RW on a periodic background as $\mu_2 = -\nu_2 = 13/25$; (b) the superposition of a PS and a KMB without a second-order central RW on a periodic background as $\mu_2 = \nu_2 = 13/25$; (c) the superposition of a PS and an AB with a second-order central RW on a periodic background as $\mu_2 = -\nu_2 = 12/25$; (d) the superposition of a PS and an AB without a second-order central RW on a periodic background as $\mu_2 = \nu_2 = 12/25$; (e) the superposition of a PS and a STB with a second-order central RW on a periodic background as $\mu_2 = 53/100$ and $\nu_2 = -1/2$; (f) the superposition of a PS and a STB without a second-order central RW on a periodic background as $\mu_2 = 53/100$ and $\nu_2 = 1/2$.

introduce the following functions

$$\begin{aligned}&(\lambda_k + \delta_k)^s \phi_k(x,\, t;\, \lambda_k + \delta_k)\\ &\quad = \phi_{s,k,0} + \phi_{s,k,1}\delta_k + \phi_{s,k,2}\delta_k^2 + \cdots + \phi_{s,k,j}\delta_k^j + \cdots,\\ &(\lambda_k + \delta_k)^s \varphi_k(x,\, t;\, \lambda_k + \delta_k)\\ &\quad = \varphi_{s,k,0} + \varphi_{s,k,1}\delta_k + \varphi_{s,k,2}\delta_k^2 + \cdots + \varphi_{s,k,j}\delta_k^j + \cdots,\end{aligned} \tag{2.3}$$

where

$$\phi_{s,k,j} = \frac{1}{j!}\frac{\partial^j}{\partial \delta_k^j}[(\lambda_k + \delta_k)^s \phi_k(x,\, t;\, \lambda_k + \delta_k)],$$

$$\varphi_{s,k,j} = \frac{1}{j!}\frac{\partial^j}{\partial \delta_k^j}[(\lambda_k + \delta_k)^s \varphi_k(x,\, t;\, \lambda_k + \delta_k)],$$

δ_k is a real small parameter and $s, j \in \mathbb{N}$. According to [11, 23, 29, 36], we can obtain the following $(2N+1)$-order solution for equation (1.1) by the limit technique and Taylor expansion:

$$q_{2N+1} = \frac{\Delta_{1N}^2}{\Delta_{3N}^2} q_0 + 2i\frac{\Delta_{1N}\Delta_{2N}}{\Delta_{3N}^2}, \tag{2.4}$$

with

$$\Delta_{1N} = \begin{vmatrix} \varphi_{2N,1,0} & \phi_{2N-1,1,0} & \varphi_{2N-2,1,0} & \cdots & \phi_{1,1,0} & \varphi_{0,1,0} \\ \phi^*_{2N,1,0} & -\varphi^*_{2N-1,1,0} & \phi^*_{2N-2,1,0} & \cdots & -\varphi^*_{1,1,0} & \phi^*_{0,1,0} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \varphi_{2N,1,m_1} & \phi_{2N-1,1,m_1} & \varphi_{2N-2,1,m_1} & \cdots & \phi_{1,1,m_1} & \varphi_{0,1,m_1} \\ \phi^*_{2N,1,m_1} & -\varphi^*_{2N-1,1,m_1} & \phi^*_{2N-2,1,m_1} & \cdots & -\varphi^*_{1,1,m_1} & \phi^*_{0,1,m_1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \varphi_{2N,l,0} & \phi_{2N-1,l,0} & \varphi_{2N-2,l,0} & \cdots & \phi_{1,l,0} & \varphi_{0,l,0} \\ \phi^*_{2N,l,0} & -\varphi^*_{2N-1,l,0} & \phi^*_{2N-2,l,0} & \cdots & -\varphi^*_{1,l,0} & \phi^*_{0,l,0} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \varphi_{2N,l,m_l} & \phi_{2N-1,l,m_l} & \varphi_{2N-2,l,m_l} & \cdots & \phi_{1,l,m_l} & \varphi_{0,l,m_l} \\ \phi^*_{2N,l,m_l} & -\varphi^*_{2N-1,l,m_l} & \phi^*_{2N-2,l,m_l} & \cdots & -\varphi^*_{1,l,m_l} & \phi^*_{0,l,m_l} \\ \lambda_{l+1}^{2N}\varphi_{l+1} & \lambda_{l+1}^{2N-1}\phi_{l+1} & \lambda_{l+1}^{2N-2}\varphi_{l+1} & \cdots & \lambda_{l+1}\phi_{l+1} & \varphi_{l+1} \\ \lambda_{l+1}^{*2N}\phi^*_{l+1} & -\lambda_{l+1}^{*(2N-1)}\varphi^*_{l+1} & \lambda_{l+1}^{*(2N-2)}\phi^*_{l+1} & \cdots & -\lambda^*_{l+1}\varphi^*_{l+1} & \phi^*_{l+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_n^{2N}\varphi_n & \lambda_n^{2N-1}\phi_n & \lambda_n^{2N-2}\varphi_n & \cdots & \lambda_n\phi_n & \varphi_n \\ \lambda_n^{*2N}\phi^*_n & -\lambda_n^{*(2N-1)}\varphi^*_n & \lambda_n^{*(2N-2)}\phi^*_n & \cdots & -\lambda^*_n\varphi^*_n & \phi^*_n \\ \lambda_{n+1}^{2N}\varphi_{n+1} & \lambda_{n+1}^{2N-1}\phi_{n+1} & \lambda_{n+1}^{2N-2}\varphi_{n+1} & \cdots & \lambda_{n+1}\phi_{n+1} & \varphi_{n+1} \end{vmatrix},$$

$$\Delta = (\phi_{2N+1,1,0}, -\varphi^*_{2N+1,1,0}, \ldots, \phi_{2N+1,1,m_1}, -\varphi^*_{2N+1,1,m_1}, \ldots, \phi_{2N+1,l,0}, -\varphi^*_{2N+1,l,0}, \ldots,$$
$$\phi_{2N+1,l,m_l}, -\varphi^*_{2N+1,l,m_l}, \lambda_{l+1}^{2N+1}\phi_{l+1}, -\lambda_{l+1}^{*(2N+1)}\varphi^*_{l+1}, \ldots, \lambda_n^{2N+1}\phi_n, -\lambda_n^{*(2N+1)}\varphi^*_n,$$
$$\lambda_{n+1}^{2N+1}\phi_{n+1})^T,$$

$$l = 0, 1, \ldots, n, \quad j = 1, 2, \ldots, l, \quad m_j \in \mathbb{N}, \quad N = n + \sum_{j=1}^{l} m_j,$$

Δ_{2N} is obtained from Δ_{1N} by replacing its first column with Δ, and

$$\Delta_{3N} = \begin{vmatrix} \phi_{2N,1,0} & \varphi_{2N-1,1,0} & \phi_{2N-2,1,0} & \cdots & \varphi_{1,1,0} & \phi_{0,1,0} \\ \varphi^*_{2N,1,0} & -\phi^*_{2N-1,1,0} & \varphi^*_{2N-2,1,0} & \cdots & -\phi^*_{1,1,0} & \varphi^*_{0,1,0} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{2N,1,m_1} & \varphi_{2N-1,1,m_1} & \phi_{2N-2,1,m_1} & \cdots & \varphi_{1,1,m_1} & \phi_{0,1,m_1} \\ \varphi^*_{2N,1,m_1} & -\phi^*_{2N-1,1,m_1} & \varphi^*_{2N-2,1,m_1} & \cdots & -\phi^*_{1,1,m_1} & \varphi^*_{0,1,m_1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{2N,l,0} & \varphi_{2N-1,l,0} & \phi_{2N-2,l,0} & \cdots & \varphi_{1,l,0} & \phi_{0,l,0} \\ \varphi^*_{2N,l,0} & -\phi^*_{2N-1,l,0} & \varphi^*_{2N-2,l,0} & \cdots & -\phi^*_{1,l,0} & \varphi^*_{0,l,0} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_{2N,l,m_l} & \varphi_{2N-1,l,m_l} & \phi_{2N-2,l,m_l} & \cdots & \varphi_{1,l,m_l} & \phi_{0,l,m_l} \\ \varphi^*_{2N,l,m_l} & -\phi^*_{2N-1,l,m_l} & \varphi^*_{2N-2,l,m_l} & \cdots & -\phi^*_{1,l,m_l} & \varphi^*_{0,l,m_l} \\ \lambda_{l+1}^{2N}\phi_{l+1} & \lambda_{l+1}^{2N-1}\varphi_{l+1} & \lambda_{l+1}^{2N-2}\phi_{l+1} & \cdots & \lambda_{l+1}\varphi_{l+1} & \phi_{l+1} \\ \lambda_{l+1}^{*2N}\varphi^*_{l+1} & -\lambda_{l+1}^{*(2N-1)}\phi^*_{l+1} & \lambda_{l+1}^{*(2N-2)}\varphi^*_{l+1} & \cdots & -\lambda^*_{l+1}\phi^*_{l+1} & \varphi^*_{l+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_n^{2N}\phi_n & \lambda_n^{2N-1}\varphi_n & \lambda_n^{2N-2}\phi_n & \cdots & \lambda_n\varphi_n & \phi_n \\ \lambda_n^{*2N}\varphi^*_n & -\lambda_n^{*(2N-1)}\phi^*_n & \lambda_n^{*(2N-2)}\varphi^*_n & \cdots & -\lambda^*_n\phi^*_n & \varphi^*_n \\ \lambda_{n+1}^{2N}\phi_{n+1} & \lambda_{n+1}^{2N-1}\varphi_{n+1} & \lambda_{n+1}^{2N-2}\phi_{n+1} & \cdots & \lambda_{n+1}\varphi_{n+1} & \phi_{n+1} \end{vmatrix}.$$

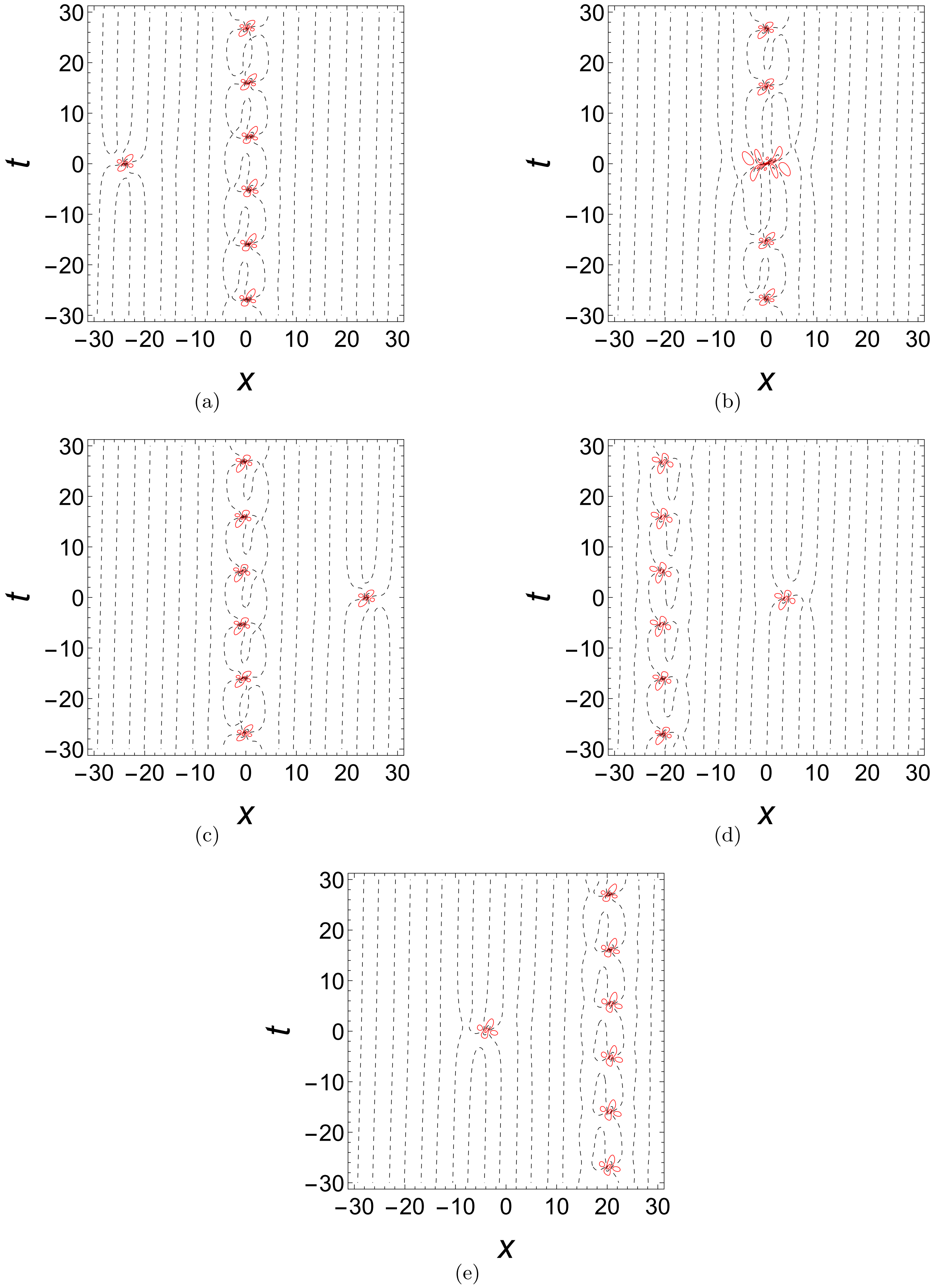

Figure 3. The contour plots for the fifth-order solution of equation (1.1) with $\mu_2 = -\nu_2 = 53/100$, $s_{03} = 0$ and $\beta_3 = 1/10$: (a) $s_{01} = 10$ and $s_{02} = 0$; (b) $s_{01} = s_{02} = 0$; (c) $s_{01} = -10$ and $s_{02} = 0$; (d) $s_{01} = 0$ and $s_{02} = 10$; (e) $s_{01} = 0$ and $s_{02} = -10$.

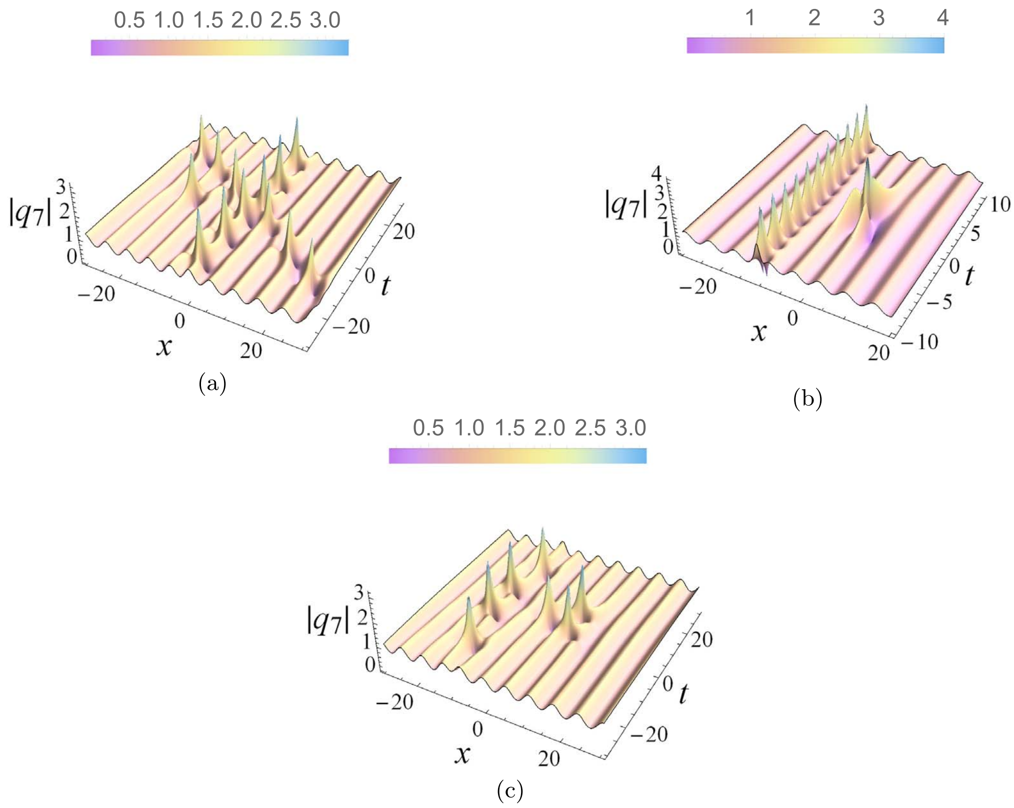

Figure 4. The seventh-order solution of equation (1.1): (a) the superposition of a PS, a STB and a KMB as $\lambda_1 = \lambda_0 + \epsilon_1^2$, $\lambda_2 = 53(1+i)/100$, $\lambda_3 = (11-9i)/20$, $\lambda_4 = i/10$, $s_{01} = 5$ and $s_{02} = s_{03} = s_{04} = 0$; (b) the superposition of a fundamental RW and a KMB on a periodic background as $\lambda_1 = \lambda_0 + \epsilon_1^2$, $\lambda_2 = 7(1-i)/10$, $\lambda_3 = i/10$, $m_1 = 1$, $s_{01} = s_{03} = 0$ and $s_{02} = 5$; (c) the superposition of a triangular RW and a KMB on a periodic background as $\lambda_1 = \lambda_0 + \epsilon_1^2$, $\lambda_2 = 13(1+i)/25$, $\lambda_3 = i/10$, $m_1 = 1$, $s_{01} = s_{03} = 0$ and $s_{02} = 5$.

Table 1. Structures of the breather-rogue periodic wave as $s_{01} = s_{02} = 0$, $l = 1$, $m_1 = 0$ and $n = 2$.

Parameters	Structures on a periodic background
$\lvert\mu_2\rvert = \lvert\nu_2\rvert > 1/2$, $\nu_2 < 0$,	PS+KMB with a second-order central RW in figure 2(a)
$\lvert\mu_2\rvert = \lvert\nu_2\rvert > 1/2$, $\nu_2 > 0$,	PS+KMB without a second-order central RW in figure 2(b)
$0 \neq \lvert\mu_2\rvert = \lvert\nu_2\rvert < 1/2$, $\nu_2 < 0$,	PS+AB with a second-order central RW in figure 2(c)
$0 \neq \lvert\mu_2\rvert = \lvert\nu_2\rvert < 1/2$, $\nu_2 > 0$,	PS+AB without a second-order central RW in figure 2(d)
$0 \neq \lvert\mu_2\rvert \neq \lvert\nu_2\rvert$, $\nu_2 < 0$,	PS+STB with a second-order central RW in figure 2(e)
$0 \neq \lvert\mu_2\rvert \neq \lvert\nu_2\rvert$, $\nu_2 > 0$,	PS+STB without a second-order central RW in figure 2(f)

On account of $n \in \mathbb{N}_+$, we can easily obtain $N \in \mathbb{N}_+$. Thus, equation (2.4) can not obtain the first-order solution of equation (1.1), which appeared as a periodic solution with a constant amplitude in [11, 36]. When the seed solution q_0 is considered a plane wave solution, we can get various solutions of equation (1.1) by utilizing the above formula (2.4): (1) breathers on a periodic background as $l = 0$; (2) RWs on a periodic background as $l = n$; (3) breather-RWs on a periodic as $0 < l < n$. Comparing with [29, 36], we can obtain all kinds of the breather-RWs or the breathers on a periodic background for the derivative nonlinear Schrödinger equation (1.1) by the formula (2.4). Similar to [37], we can also give the fundamental, triangular, ring or ring-triangular structure for higher-order RWs on a periodic background. Hence, we pay main attention to the breathers and breather-RWs on a periodic background ($0 \leqslant l < n$) for equation (1.1) in this paper, which have not been fully studied in the previous literature.

3. Dynamics of the breathers and breather-RWs on a periodic background

In this section, we will obtain the breathers and breather-RWs on a periodic background of equation (1.1) by utilizing the formula (2.4). Similar to [36], we still take the seed solution $q_0 = ce^{i\rho}$ with $\rho = ax + (a^2 - ac^2)t$ for equation (1.1) and choose the following eigenfunctions of the spectral problems

(2.1)

$$\Psi_k = \begin{pmatrix} \phi_k \\ \varphi_k \end{pmatrix} = \begin{pmatrix} e^{\frac{1}{2}i(\theta_k+\rho)} - \frac{2i\lambda_k^2 - ia - ih_k}{2c\lambda_k} e^{-\frac{1}{2}i(\theta_k-\rho)} \\ e^{-\frac{1}{2}i(\theta_k+\rho)} - \frac{2i\lambda_k^2 - ia - ih_k}{2c\lambda_k} e^{\frac{1}{2}i(\theta_k-\rho)} \end{pmatrix}, \tag{3.1}$$

where

$$\theta_k = h_k\left[x + (2\lambda_k^2 + a - c^2)t + 2\sum_{j=0}^{k-1} s_{jk}\epsilon_k^j\right],$$
$$h_k = \sqrt{4c^2\lambda_k^2 + 4\lambda_k^4 - 4a\lambda_k^2 + a^2}, \quad a,\ c,\ s_{jk} \in \mathbb{R},$$

and ϵ_k are real small parameters, $j \in \mathbb{N}$, $k = 1, 2,\cdots,n+1$. Note that $h_k = 0$ as $\lambda_k = (\sqrt{2a - c^2} - ic)/2 \triangleq \lambda_0$. For equation (2.4), we can select $\lambda_{r_1} = \lambda_0 + \epsilon_{r_1}^2$, $\lambda_{r_2} = \mu_{r_2} + i\nu_{r_2}$, $\lambda_{n+1} = i\beta_{n+1}$, $s_{0,n+1} = 0$ and expand $\lambda_{r_1}^s \Psi_{r_1}$ at $\epsilon_{r_1} = 0$ via the Taylor expansion where $\mu_{r_2},\ \nu_{r_2},\ \beta_{n+1} \in \mathbb{R}\backslash\{0\}$, $s \in \mathbb{N}$, $r_1 = 1, 2, \cdots, l$ and $r_2 = l+1,\cdots,n$. All kinds of the breather (-RW) on a periodic background for equation (1.1) can be obtained by taking the limit $\epsilon_k \to 0$ $(k = 1, 2,\ldots,n+1)$. For simplicity, we only consider the case of $l = 0,1$ as $a = c = 1$.

For $N = 1$ in equation (2.4), when $l = n = 1$, we can get the same results with [36], namely, a PS on a periodic background. Hence, we discuss here only the case of $l = 0$ and $n = 1$. Under this case, a third-order solution of equation (1.1) is obtained directly by means of the formula (2.4) and the eigenfunctions (3.1) as $\lambda_1 = \mu_1 + i\nu_1$ and $\lambda_2 = i\beta_2$. Specifically, when $\pm\mu_1 = \nu_1 = 1/2$ and $s_{01} = 0$, a periodic solution is achieved

$$q_{periodic} = -\frac{2e^{ix(\omega-1)}[4\beta_2^3(2\beta_2^2 + \omega + 1)e^{2i\beta_2^2 t\omega} - 2\beta_2^2(2\beta_2^2 + \omega)e^{i\omega(4\beta_2^2 t - x)} + (\omega+1)e^{ix\omega}]}{(H_R + iH_I)^2}, \tag{3.2}$$

with $\omega = \sqrt{1 + 4\beta_2^4}$, $H_R = (2\beta_2^2 + \omega + 1)\cos(2\beta_2^2 t\omega) - 2\beta_2\cos(x\omega)$, $H_I = (2\beta_2^2 + \omega + 1)\sin(2\beta_2^2 t\omega) - 2\beta_2\sin(x\omega)$. Owing to $\beta_2 \neq 0$, we have

$$\begin{aligned} H_R^2 + H_I^2 &= (2\beta_2^2 + \omega + 1)^2 + 4\beta_2^2 \\ &\quad - 4\beta_2(2\beta_2^2 + \omega + 1)\cos(2\beta_2^2 t\omega - x\omega) \\ &\geqslant (2\beta_2^2 + \omega + 1)^2 + 4\beta_2^2 - 4|\beta_2|(2\beta_2^2 + \omega + 1) \\ &= (2\beta_2^2 + \omega + 1 - 2|\beta_2|)^2 \\ &= [\beta_2^2 + \omega + (1 - |\beta_2|)^2]^2 \\ &> 0, \end{aligned}$$

which implies equation (3.2) is nonsingular. Interestingly, the periodic solution (3.2) is similar to the case of $n = 1$, $s_1 = 0$ and $\lambda_1 = i\beta_1$ in [36]. However, the breather on a periodic has not been well studied for the derivative nonlinear Schrödinger equation in the previous research. More importantly, we can also obtain a breather on a periodic background: a KMB as $|\mu_1| = |\nu_1| > 1/2$ in figure 1(a); an AB as $|\mu_1| = |\nu_1| < 1/2$ in figure 1(b); a STB as $|\mu_1| \neq |\nu_1|$ in figure 1(c). As shown in figure 1, we take $s_{01} = 0$ because the parameter has indeed no real influence upon the structure of breather, except in the position of the breather. Considering the complexity, we only show the expression as $s_{01} = s_{02} = 0$ and $\mu_1 = \nu_1 = 2^{-3/4}$, i.e. the KMB on a periodic background displayed in figure 1(a):

$$q_{kmb\ periodic} = \frac{\Delta_{1,kmb}^2}{\Delta_{3,kmb}^2}e^{ix} + 2i\frac{\Delta_{1,kmb}\Delta_{2,kmb}}{\Delta_{3,kmb}^2}, \tag{3.3}$$

where

$$\begin{aligned} \Delta_{1,kmb} &= (-1)^{3/4}e^{i[\pi - 4t(\beta_2^2\omega + \sqrt{2}) - 2x\omega - 2(1-2i)x]/4} \\ &\times \{[(-1)^{1/8}2(\sqrt{2} - 1)^{1/2}e^{i\sqrt{2}t} \\ &+ (2 - \sqrt{2})e^{2i\sqrt{2}t + x} \\ &+ \sqrt{2}e^x + 2^{3/4}(e^{i\pi/4} - i)e^{i\sqrt{2}t + 2x}]\beta_2^2 \\ &\times [2\beta_2 e^{2it\omega\beta_2^2} + (1 + 2\beta_2^2 + \omega)e^{ix\omega}] \\ &+ [e^{x + 2i\sqrt{2}t} + 2^{1/4} \\ &\times (1 + e^{3i\pi/4})e^{i\sqrt{2}t + 2x - i\pi/4} + 2^{1/4}(e^{i\pi/4} - i)e^{i\sqrt{2}t} \\ &+ (\sqrt{2} - 1)e^x][2\beta_2 e^{2i\beta_2^2 t\omega} + (2\beta_2^2 + \omega \\ &+ 1)e^{ix\omega}] - [2^{3/4}(e^{i\pi/4} - i)e^{2x + i\sqrt{2}t} + 2e^{x + 2i\sqrt{2}t} \\ &- 2^{3/4}(e^{3i\pi/4} - i)e^{i\sqrt{2}t} + 2(\sqrt{2} - 1)e^x] \\ &\times \beta_2[(2\beta_2^2 + \omega + 1)e^{2i\beta_2^2 t\omega} + 2\beta_2 e^{ix\omega}]\}/(2\sqrt{2}\beta_2), \end{aligned}$$

$$\begin{aligned} \Delta_{2,kmb} &= ie^{-i[2t(\beta_2^2\omega + \sqrt{2}) + x\omega - (1+2i)x]/2}\{[-2(\sqrt{2} - 1)e^{x + 2i\sqrt{2}t} \\ &- i2^{3/4}(1 - e^{i\pi/4})e^{2x + i\sqrt{2}t} + i2^{3/4} \\ &\times (1 - e^{-i\pi/4})e^{i\sqrt{2}t} - 2e^x][2\beta_2 e^{2i\beta_2^2 t\omega} \\ &+ (2\beta_2^2 + \omega + 1)e^{ix\omega}] + [2^{3/4}(1 - e^{i\pi/4})ie^{2x + i\sqrt{2}t} \\ &+ \sqrt{2}e^{x + 2i\sqrt{2}t} + 2^{3/4}(e^{i\pi/4} - i)e^{i\sqrt{2}t} \\ &- (\sqrt{2} - 2)e^x]\beta_2[(2\beta_2^2 + \omega + 1)e^{2i\beta_2^2 t\omega} + 2\beta_2 e^{ix\omega}] \\ &- \sqrt{2}e^{i\pi/4}[2^{3/4}(1 - e^{i\pi/4})e^{2x + i\sqrt{2}t} \\ &+ (\sqrt{2} - 1)\sqrt{2}e^{x + 2i\sqrt{2}t - i\pi/4} + 2^{3/4}(e^{i\pi/4} - i)e^{i\sqrt{2}t} \\ &+ e^{x - i\pi/4}\sqrt{2}]\beta_2^3[(2\beta_2^2 + \omega + 1)e^{2i\beta_2^2 t\omega} \\ &+ 2\beta_2 e^{ix\omega}]\}/(4\beta_2), \end{aligned}$$

$$\begin{aligned} \Delta_{3,kmb} &= e^{-i[2t(\beta_2^2\omega + \sqrt{2}) + x\omega - (1+2i)x)]/2}\{\sqrt{2}e^{i\pi/4} \\ &\times [(2\sqrt{2} - 2)e^{x + 2i\sqrt{2}t - i\pi/4} \\ &+ 2^{3/4}(e^{i\pi/4} - i)e^{2x + i\sqrt{2}t} \end{aligned}$$

$$
\begin{aligned}
&+ 2^{3/4}(1 - e^{i\pi/4})e^{i\sqrt{2}t} - 2e^{x+3i\pi/4}]\beta_2 \\
&\times [2\beta_2 e^{2i\beta_2^2 t\omega} + (2\beta_2^2 + \omega + 1)e^{ix\omega}] \\
&+ [(\sqrt{2} - 2)e^{2i\sqrt{2}t} \\
&\times e^x + i2^{3/4}(1 - e^{-i\pi/4})e^{2x+i\sqrt{2}t} \\
&- i2^{3/4}(1 - e^{i\pi/4})e^{i\sqrt{2}t} - \sqrt{2}e^x] \\
&\times [(2\beta_2^2 + \omega + 1)e^{2i\beta_2^2 t\omega} \\
&+ 2\beta_2 e^{ix\omega}] + 2e^{i\pi/4}[2^{1/4}(1 - e^{-i\pi/4})ie^{2x+i\sqrt{2}t} \\
&+ 2^{1/4}(e^{i\pi/4} - 1)e^{i\sqrt{2}t} - e^{x-i\pi/4}(e^{2i\sqrt{2}t} \\
&+ \sqrt{2} - 1)]\beta_2^2 \\
&\times [(2\beta_2^2 + \omega + 1)e^{2i\beta_2^2 t\omega} + 2\beta_2 e^{ix\omega}]\}/(4\beta_2).
\end{aligned}
$$

Obviously, the expression (3.3) is more complex than the periodic solution (3.2). Through complex and direct calculation, $\Delta_{3,kmb}$ has the real part F_R and imaginary part F_I, i.e. $\Delta_{3,kmb} = F_R + iF_I$. With the aid of Mathematica software, the minimum of $F_R^2 + F_I^2$ is about 0.438 when (x, t, β) is approximately $(-0.084, -2.175, -0.250)$. Thus, $\Delta_{3,kmb} \neq 0$. That is to say, equation (3.3) is nonsingular.

For $N \geqslant 2$ in equation (2.4), we shall not discuss the case of $l = 0$, i.e. the breather(s) on a periodic background. Next, we will focus on the breather-RW on a periodic background as $l = 1$ and $N \geqslant 2$ for the formula (2.4). Besides, we set $\beta_{n+1} = 1/10$ because the effect of β_{n+1} on RWs on a periodic background was investigated [36].

For $N = 2$ in equation (2.4), let $\lambda_1 = \lambda_0 + \epsilon_1^2$, $\lambda_2 = \mu_2 + i\nu_2$, $\lambda_3 = i/10$, $l = 1$, $m_1 = 0$ and $n = 2$ as described above ($\mu_2, \nu_2 \in \mathbb{R}\setminus\{0\}$). Under this case, the fifth-order solution of equation (1.1) is a PS interacting with a breather on a periodic background, as shown in figure 2. Note that there are two structures for the fifth-order solution when the parameters are taken different values: the interaction with a second-order central RW in figures 2(a), (c), (e); the interaction without a second-order central RW in figures 2(b), (d), (f). Taking no account of the parameters s_{01} and s_{02}, table 1 shows the effects of the two free parameters μ_2 and ν_2 when $(|\mu_2|, |\nu_2|)$ is in a appropriately small neighborhood of $(1/2, 1/2)$. The spatial-temporal structure of the fifth-order solution shall become a soliton wave as $(|\mu_2|, |\nu_2|) \to (\infty, \infty)$ while it can degenerate into a PS on a periodic background as $\mu_2^2 + \nu_2^2 \to 0$. Additionally, figure 3 clearly indicates that the PS shall move gradually with the change of the parameter s_{01} as well as how the parameter s_{02} affects the KMB for the breather-RW on a periodic background. Furthermore, the parameters s_{01} and s_{02} have the similar effect on other structures in table 1. These dynamics are consistent with the breather-RWs on a constant background [29].

For $N = 3$ and $l = 1$ in equation (2.4), there are roughly two structures for the seventh-order solution of equation (1.1) when we ignore the structures of the breathers and whether a third-order central RW appears or not. One is the interaction between a PS and two breathers on a periodic background as $l = 1$, $m_1 = 0$ and $n = 3$ in figure 4(a). The hybrid can be called as 'rogue wave quanta' on a periodic background [40]. The other is the nonlinear superposition of a second-order RW with a breather on a periodic background as $l = 1$, $m_1 = 1$ and $n = 2$. Under this case, we display the fundamental structure in figure 4(b) and the triangular structure in figure 4(c).

4. Conclusions

In this paper, we give the expression (2.4) of the odd-th solution for the derivative nonlinear Schrödinger equation (1.1). On the basis of equation (2.4), we not only get the same results in [36], but also can obtain the distinct results, i.e. the breather and the breather-RW on a periodic background. For the derivative nonlinear Schrödinger equation (1.1), [11, 22, 23, 29] showed the soliton, RW and breather on a constant background, but this paper displays the breather and the breather-RW on a periodic background. As applications, we show the structures and study the dynamics for these solution of equation (1.1). Naturally, we can obtain 'rogue wave quanta' and the fundamental (or triangular or ring or ring-triangular) pattern for higher-order breather-RW on a periodic background like figure 4.

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