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Scalar quasinormal modes of nonlinear charged black holes in Rastall gravity

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Abstract – In this paper, the scalar quasinormal modes of nonlinear charged black hole metrics in Rastall gravity is investigated. The electromagnetic tensor presented in the background spacetime possesses a power-law form, and the system is assumed to be surrounded by a quintessence field. By utilizing the recently obtained analytic form of the metric, the quasinormal frequencies are obtained via the matrix method. The numerical values have been further compared against those evaluated by using the WKB approximation up to thirteenth order. Also, the finite difference method is utilized to study the temporal evolution of the scalar perturbations. In terms of calculations carried out for both massless and massive scalar fields, we discuss the properties of the resultant quasinormal frequencies, as well as their dependences on the model parameters describing the background black hole. To be specific, the effect of the electric charge, mass of the scalar field, and equation of state of the quintessence are examined. Besides, the quasinormal frequencies associated with an extremal black hole regarding the Nariai limit is explored, where the obtained results are found to be consistent with theoretical arguments. The black hole metric is found to be stable against scalar perturbations, in the presence of both linear and nonlinear electromagnetic fields.

Introduction. – The recent detections of gravitational wave events reported by LIGO and Virgo Collaborations [1–5] have opened up a new frontier in astronomy and astrophysics. These observations originated from the merger of binary systems established one final missing piece predicted by Einstein's theory of general relativity. Moreover, as an emerging branch of observational astronomy, the gravitational wave astronomy possesses the potential to provide unprecedented details about black holes and neutron stars. Subsequent progress in the near future might lead us to further intriguing evidence on important topics such as the discrimination between different models of modified gravity, or even experimental attestations regarding black hole perturbation theory. Concerning the gravitational wave signal emitted during the merger of binary systems, the waveform during the inspiral as well as merger phase can be satisfactorily described by numerical relativity [6–8]. During the merger phase, the two inspiraling bodies become so close that the relativistic effects in strong fields dominate the process. From a physical viewpoint, apart from numerical simulations, the nature of the merger waveform is mainly unknown to

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date. Thereafter, a sudden increase of in frequency, known as "chirp", takes place and subsequently the oscillations decay exponentially in time as the system settles down into a stable state. This last phase is referred to in the literature as ringdown. Regarding the measurements, however, the least amount of information has been extracted from the ringdown phase due to the signal strength as well as duration. On the theoretical side, however, the process is understood to be closely related to the quasinormal modes (QNMs) and its signal-to-noise ratio [9] of the resultant distorted black hole.

In practice, the specific approach to study the QNMs depends on the asymptotical behavior of the black hole metric. For instance, distinct spacetimes, such as asymptotically flat, de Sitter, and anti-de Sitter spacetimes, dictate the boundary conditions of the problem [10] differently. Also, various types of initial disturbances can be assumed which include those of the scalar [11–13], electromagnetic [14,15], and Dirac fields [12,16,17], in addition to the gravitational ones [15]. Moreover, aside from Einstein's general relativity, QNMs in modified gravity has also aroused much attention [18–20]. Among existing theories of modified gravity, the Rastall gravity [21] has undergone a significant surge in popularity, partly owing to its applications to cosmology [22–24] and other large-scale systems [25]. In cosmology, the results of the standard ΛCDM model can be reasonably reproduced by the Rastall gravity, for the background as well as linear level perturbations. Besides, additional effects have been explored regarding nonlinear corrections [24]. Among others, a power-law mass density distribution can be obtained for the inner region of early-type galaxies in Rastall gravity, which is not feasible in the framework of Einstein gravity [25].

Rastall's theory assumes that the covariant divergence of the energy momentum tensor does not vanish in curved spacetime. Alternatively, this conjecture can be viewed as a consequence of the non-minimal coupling between the curvature and matter. It is worth noting that the latter is a known feature in modified gravity [26,27]. Moreover, it can be readily shown that various theories of modified gravity, such as $f(R)$ and quadratic gravity, can be reformulated in terms of a generalized form of Rastall gravity [28,29].

The present work aims to investigate the QNMs of a charged black hole metric in Rastall gravity obtained recently [28]. The background black hole solution in question is surrounded by the quintessence field in the presence of linear or nonlinear electromagnetic fields. The present letter is organized as follows. In the following section, the black hole solution is presented. In particular, we discuss a second type of extremal black hole solution regarding the Nariai limit where the cosmological horizon coincides with the event horizon. In the third section, the QNMs for both massless and massive scalar perturbations are evaluated. The calculated quasinormal frequencies as functions of various parameters of the background metric are investigated. Also, the QNMs of the second type of extremal black hole are investigated. The results are shown to be consistent with the analytical arguments. The last section is devoted to further discussions and concluding remarks.

Charged black hole solutions for $(3+1)$ dimensional spacetimes in Rastall gravity. – In Rastall gravity, the divergence of energy-momentum tensor does not vanish in curved spacetime, and in particular, it is proportional to the gradient of the Ricci scalar:

$$T^{\nu}_{\mu;\nu} = \lambda R_{,\mu}. \tag{1}$$

The corresponding field equation reads

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa (T_{\mu\nu} - \lambda g_{\mu\nu} R). \tag{2}$$

The present study considers the black hole solution in $(3+1)$-dimensional spacetime in the presence of linear and nonlinear electromagnetic field surrounded by quintessence fluid. For the static and spherically symmetric case, the resultant metric is [28]

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2). \tag{3}$$

The linear or nonlinear electromagnetic field, described in terms of the vector potential, assumes the following *ansatz* of nonzero temporal component:

$$A_\mu = \delta^t_\mu A_t(r), \tag{4}$$

with

$$A_t(r) = A_0 + \frac{Q}{2s-3}(2sr - r)^{1-\frac{3-1}{2s-1}}. \tag{5}$$

The eletromagnetic field is nonlinear, except for $s = 1$, where the formulism falls back to the case of linear Maxwell field and one has $A_0 = 0$. Since Q is identified as the electric charge of the black hole for $s = 1$, it is subsequently understood as the generalized electric charge for $s \neq 1$. The energy-momentum tensor of electromagnetic field is

$$E^v_\mu = -(-\xi)^s(\mathcal{F})^{s-1}\left(2sF_{\sigma\mu}F^{\sigma v} - \frac{1}{2}\delta^v_\mu \mathcal{F}\right), \tag{6}$$

where $F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu}$.

For the quintessence field, one assumes the following barotropic equation of state $p = \omega_q \rho$. The corresponding energy-momentum tensor reads

$$\begin{aligned} T^{*t}_{t} &= T^{*r}_{r} = -\rho(r), \\ T^{*\theta_1}_{\theta_1} &= T^{*\theta_2}_{\theta_2} = \frac{1}{2}\rho(r)(3\omega + 1). \end{aligned} \tag{7}$$

Therefore, the energy-momentum tensor of the matter field is summed up to read

$$T_{\mu\nu} = E_{\mu\nu} + T^*_{\mu\nu}. \tag{8}$$

$$
\begin{aligned}
\rho &= C_\rho r^{-\frac{3(-1+4\kappa\lambda)(1+\omega_q)}{-1+3\kappa\lambda(1+\omega_q)}} + \frac{16Q^2(-1+s)(-1+2s)^{1+\frac{2}{-1+2s}}(r(-1+2s))^{\frac{4s}{1-2s}}(-1+2s)^{\frac{2}{1-2s}}\lambda}{-3(1+\omega_q)+s(2+6\omega_q)}, \\
f(r) &= 1 - \frac{2M}{r} + \frac{Q^2 r^{\frac{2}{1-2s}}(-1+2s)^{\frac{3-2s}{1-2s}}}{(3-2s)s} + \frac{C_\rho r^{\frac{1+3\omega_q-6\kappa\lambda(1+\omega_q)}{-1+3\kappa\lambda(1+\omega_q)}}\kappa(1-3\kappa\lambda(1+\omega_q))^2}{3(-1+4\kappa\lambda)(-\omega_q+\kappa\lambda(1+\omega_q))},
\end{aligned}
\tag{9}
$$

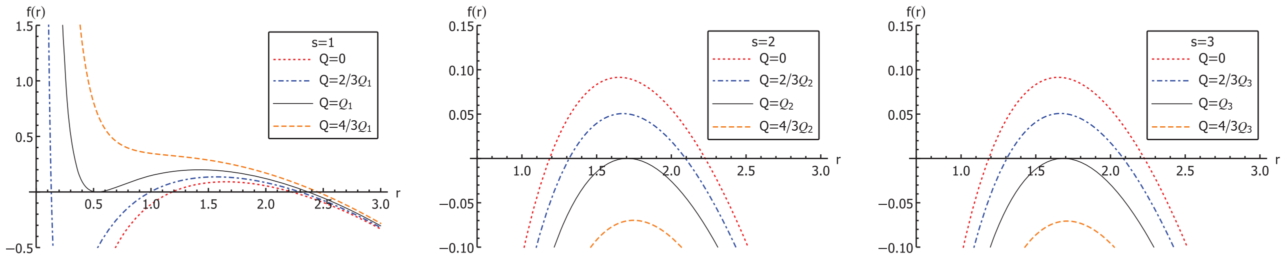

Fig. 1: The functions $f_1(r)$ (left), $f_2(r)$ (middle), and $f_3(r)$ (right) *vs.* r evaluated for $s = 1$, 2, and 3, respectively.

By substituting the above pieces into the field equation (2), the resultant metric, eq. (3), is found to be [28]

see eq. (9) *above*

where C_ρ, M are constants of integration. Physically, M is associated with the mass of black hole. For $\omega_q = -1$, the metric, dictated by eq. (9), describes the asymptotically de Sitter spacetimes when $C_\rho < 0$ and the anti-de Sitter spacetime for $C_\rho > 0$. In the remaining of the paper, we consider $C_\rho < 0$ and choose $\omega_q = -1$, $C_\rho\kappa = -1$, $\lambda\kappa = 1$, and $M = 1/2$ for the numerical calculations. In this case, it is straightforward to show that eq. (9) simplifies to

$$
f(r) = 1 - \frac{1}{r} - \frac{r^2}{9} + \frac{Q^2 r^{\frac{2}{1-2s}}(-1+2s)^{\frac{3-2s}{1-2s}}}{(3-2s)s}. \tag{10}
$$

An extremal black hole is a black hole with the maximal possible amount of charge with a given mass and angular momentum. For the present metric, one can demonstrate that there are two types of extremal black hole solutions that are physically distinct. As discussed below, they correspond to the ones with linear and nonlinear electromagnetic fields, respectively. For $s = 1$, which corresponds to the case of the linear electromagnetic field, eq. (10) implies there is an inner horizon which sits inside the black hole event horizon. If one increases the charge of the black hole for a given mass, the inner and the event horizons gradually approach each other. At the moment when the inner horizon coincides with the event horizon, the charge reaches a critical value, which will be denoted as Q_1. To further increase the value of the charge is physically prohibited by the cosmic censorship hypothesis. This is the usual scenario for an extremal black hole, as indicated in the left plot of fig. 1. On the other hand, for $s > 1$, which corresponds to the case of the nonlinear electromagnetic field, the situation is quite different. One can readily show that there is an event horizon and a cosmological horizon, according to eq. (10). Again, as the charge increases, the two horizons approach each other. When the event horizon comes very close to the cosmological one, a second type of extremal black hole is formed, which is known as the Nariai limit [30]. For the observer which is sandwiched between the event horizon and cosmological horizons, as the two horizons approach each other, it is not difficult to show that the proper distance between them is well defined and finite. In fact, the metric on the causal patch between the two horizons can be approximately described as a patch of the Nariai solution. However, the metric becomes unphysical as the charge further increases and exceeds the critical value. We denote the corresponding critical charge as Q_s. Two examples of the second type of extremal black hole solution are shown in the middle and right plots of fig. 1.

The quasinormal modes for massless and massive scalar perturbations. – The radial part of the Klein-Gordon equation for scalar field can be derived by using the method of separation of variables. By substituting $\Phi = \frac{\phi(r)}{r}Y(\theta,\varphi)e^{-i\omega t}$, one finds

$$
\frac{\mathrm{d}^2\phi}{\mathrm{d}r_*^2} + [\omega^2 - V(r)]\phi = 0, \tag{11}
$$

where $r_* = \int \frac{\mathrm{d}r}{f(r)}$ is the so-called tortoise coordinate and the effective potential $V(r)$ is given by

$$
V(r) = f(r)\left(\frac{rf'(r) + \ell(\ell+1) + r^2m^2}{r^2}\right). \tag{12}
$$

As the effective potential vanishes on the event and cosmological horizons $f(r_h) = f(r_c) = 0$, the boundary conditions of the wave function ϕ are dictated by the fact that

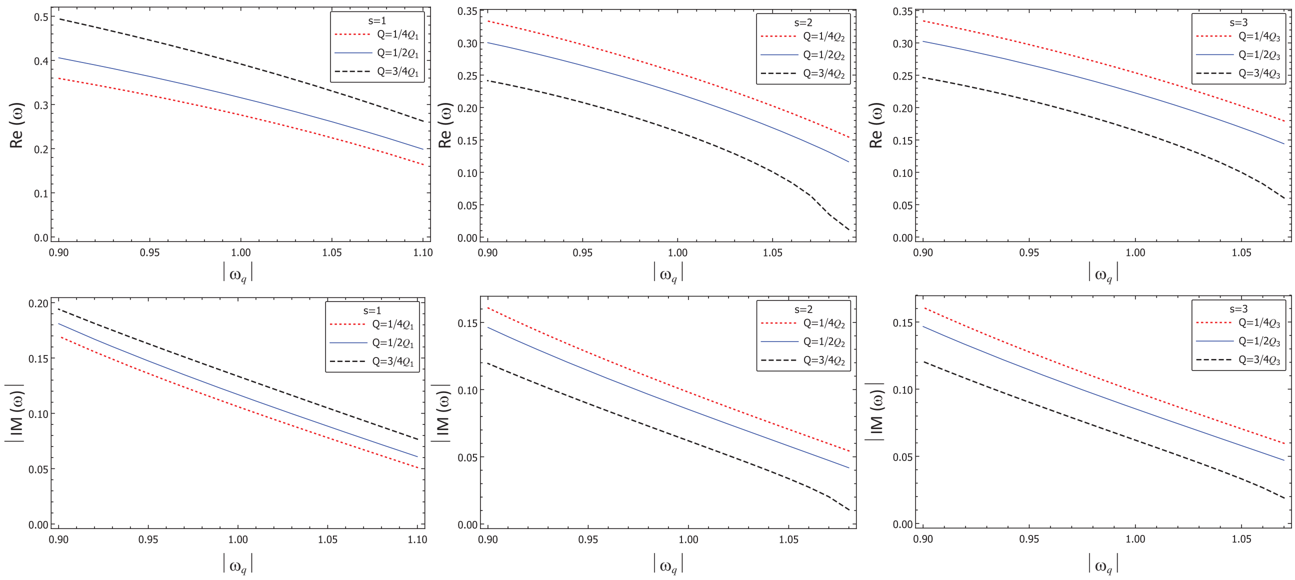

Fig. 2: The quasinormal frequencies evaluated for linear as well as nonlinear electromagnetic fields with $s = 1, 2$ and 3, different charges Q, and quintessence fluild equation of state ω_q. The calculations are carried out by using the matrix method for both massless and massive scalar perturbations.

the waves are asymptotically ingoing on the event horizon, while outgoing on the cosmological horizon.

In what follows, we study the quasinormal modes by using the matrix method [31,32] and the more venerable WKB [33–35] approximation. By expanding the function f near the event and cosmological horizons by Taylor series, the boundary conditions of the radial equation, when substituting the definition of the tortoise coordiante, assume the following form [32]:

$$\begin{aligned} \phi(r_h) &\sim e^{-i\omega r_*} \sim (r-r_h)^{-\frac{i\omega}{f'(r_h)}}, \\ \phi(r_c) &\sim e^{i\omega r_*} \sim (r-r_c)^{\frac{i\omega}{f'(r_c)}}. \end{aligned} \tag{13}$$

We then introduce the coordinate transformation $x = \frac{r-r_h}{r_c-r_h}$, and rewrite the wave function of the scalar field as $\phi = (1-x)^{\frac{i\omega}{f'(r_c)}} x^{-\frac{i\omega}{f'(r_h)}} R(x)$. To simplify the boundary conditions, we further define $\sigma(x) = (1-x)xR(x)$. The resulting master equation regarding $\sigma(x)$ is a second-order ordinary differential equation which possesses the desired boundary conditions

$$\sigma(0) = \sigma(1) = 0. \tag{14}$$

According to the matrix method, the derivatives are rewritten in terms of the function values at the grids, so that the second-order differential equations and their boundary conditions can be transformed into a matrix equation. Subsequently, the quasinormal frequencies can be obtained by solving a nonlinear algebraic equation.

The WKB approximation [36] is a well-known semi-analytic method for solving linear differential equations. Besides the application for the time-independent Schrödinger equation, the method can also be employed for the master equation of black hole perturbation theory. Recently, the WKB method has been extended to 13th order [35]. We proceed to evaluate the quasinormal frequencies by using both the WKB and matrix method. Also, we investigate how the numerical results behave by studying their dependence on the order of the expansion for both approaches. It is found that both approaches give reasonably good precision. To be specific, the matrix method is found to give convergent results when the grid number is bigger than seven. The numerical results are compared against those by employing the WKB approximation up to the thirteenth order, and satisfactory consistency is observed.

We first evaluate the quasinormal frequencies for massless scalar perturbations. This is done by taking $m = 0$ in the above equations. In fig. 2, we present the resulting quasinormal frequencies for black holes with linear and nonlinear electromagnetic fields for different charges and equations of state of the quintessence fluid. Accordingly, the real and imaginary parts of the frequencies are shown as a function of ω_q, with different values of s and Q. Since $\omega_q = -1$ is mostly consistent with the observed accelarating Universe [37], the calculations are carried out in the vicinity of this value. It is found that the quasinormal frequency decreases with increasing ω_q. The parameter ω_q describes the stiffness of the equation of state. Therefore, the above results imply that for a given amount of initial perturbation, a stiffer quintessence fluid is harder to be dragged along as the oscillation period becomes larger while the disturbance decays away more slowly. Also, the observed dependence of quasinormal frequency on ω_q is mostly found to be linear. In a few cases, the linearity becomes less strict as the charge approaches that of the extremal value Q_s. On the other hand, the results indicate that in the case of the linear electromagnetic field for

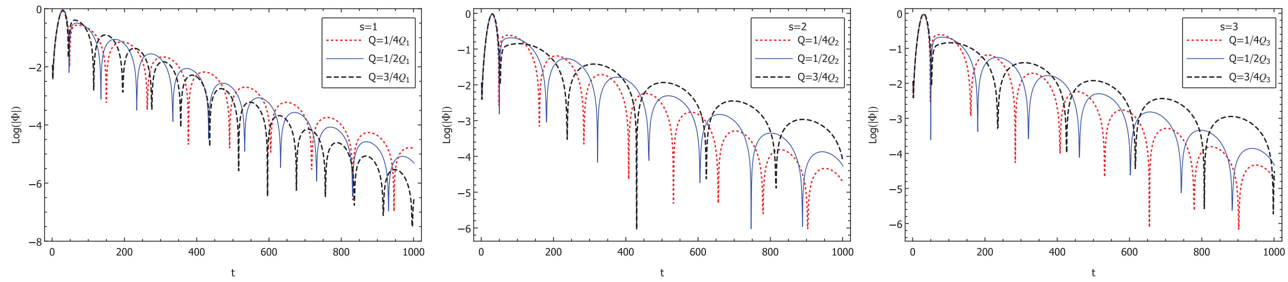

Fig. 3: The calculated temporal evolution of massless scalar perturbation for linear and nonlinear eletromagnetic fields and charges.

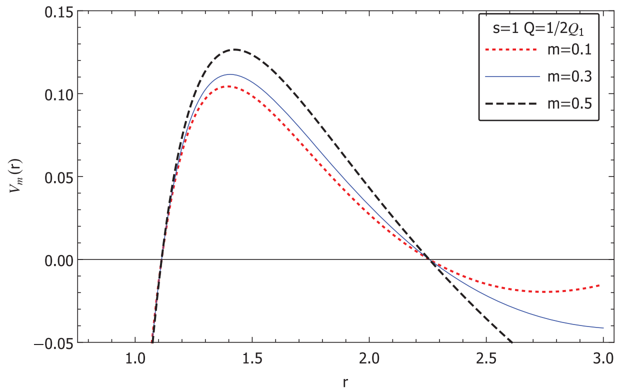

Fig. 4: The effective potential evaluated for different masses of the scalar field.

given ω_q, as the charge approaches that of the extremal black hole, the real part of the quasinormal frequency increases monotonically, while the the imaginary part increases and then decreases. On the contrary, in the cases of nonlinear electromagnetic fields, both the temporal oscillation frequency and decay rate decrease as the black hole charge increases toward the corresponding extremal value. Moreover, regarding a given percent variation of Q, the corresponding change of the quasinormal frequencies is more significant in the cases of the nonlinear electromagnetic field.

Next, we use the finite difference method to explore the temporal evolution of scalar perturbations. The results are shown in fig. 3. One observes that the dependence of the quasinormal frequencies on the charge Q is consistent with those obtained by using semi-analytic method obtained above.

To investigate the quasinormal modes of massive scalar perturbations, we first show in fig. 4 the effective potential evaluted for different masses of the scalar field. The calculations are carried out by using the parameters $s = 1$, $\omega_q = -1$, and $Q = \frac{1}{2}Q_1$. Owing to eq. (12), the potential vanishes at the horizons which are entirely determined by the function f. As the mass of the scalar field increases, the maximum of the effective potential slightly increases, together with the corresponding location r where the maximum is attained. However, in general, the effective potential is not much affected by the mass. The corresponding quasinormal frequencies are presented in fig. 5. For given s, ω_q, and Q, both the real and imaginary parts of the quasinormal frequency increase with increasing mass.

Last but not least, in fig. 6, we study the calculated QNMs as the charge approaches that of the extremal black hole. From the figure, one observes that the quasinormal frequencies behave differently in linear and nonlinear electromagnetic fields. In particular, in the case of the nonlinear electromagnetic field, both the real and imaginary of the quasinormal frequency are found to approach zero at the limit of the extremal black hole. The above numerical results can be understood analytically as follows. As the charge increases to approach the extremal value, the coordinates of the event and cosmological horizons coincide. Now, the general solution of the master equation must satisfy the boundary conditions that it is an asymptotically ingoing wave at event horizon r_h and an outgoing wave at cosmological horizon r_c. As $r_c \to r_h$, one may study the limit of the quasinormal frequency. However, it is not difficult to observe in order that the frequency to satisfy both conditions, it must trivially vanish. This is in agreement with what have been obtained numerically in fig. 6.

Concluding remarks. – In this work, we investigate the QNMs scalar perturbation of a charged black hole metric proposed recently in Rastall gravity. Indeed, the presence of the quintessence field is a vital factor, as it is one of the significant candidates for dark energy. Relevant topics have already aroused much attention in the literature. The static spherically symmetric black hole solution surrounding by the quintessence in Einstein's gravity was first investigated by Kiselev [38]. Subsequently, the related QNMs for scalar, electromagnetic, spinor, and gravitational perturbations were explored by many authors [39–43]. These works were further extended to the case of Rastall gravity [44,45]. Regarding the charged black hole, the QNMs of the Reissner-Nordstrom black hole in the presence of quintessence were first investigated in Einstein gravity [46,47]. In this context, the present work extends the study of QNMs to a more general case with linear and nonlinear electromagnetic fields in Rastall gravity. To be specific, the black hole solution in question is surrounded by a quintessence fluid in the presence of linear or nonlinear Maxwell field. The quasinormal

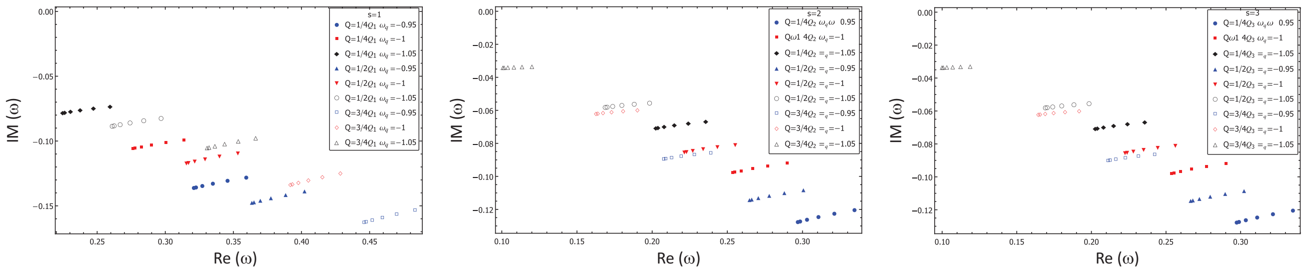

Fig. 5: The calculated quasinormal frequencies for quantum numbers $n = 0$ and $\ell = 1$ of massive scalar perturbations. The calculations are carried out for different masses of the scalar field, ranging from 0 to 0.5 with an interval of 0.1. The latter correspond to the scatter points of the same group, represented by the same symbols, which are located from left to right in the plot.

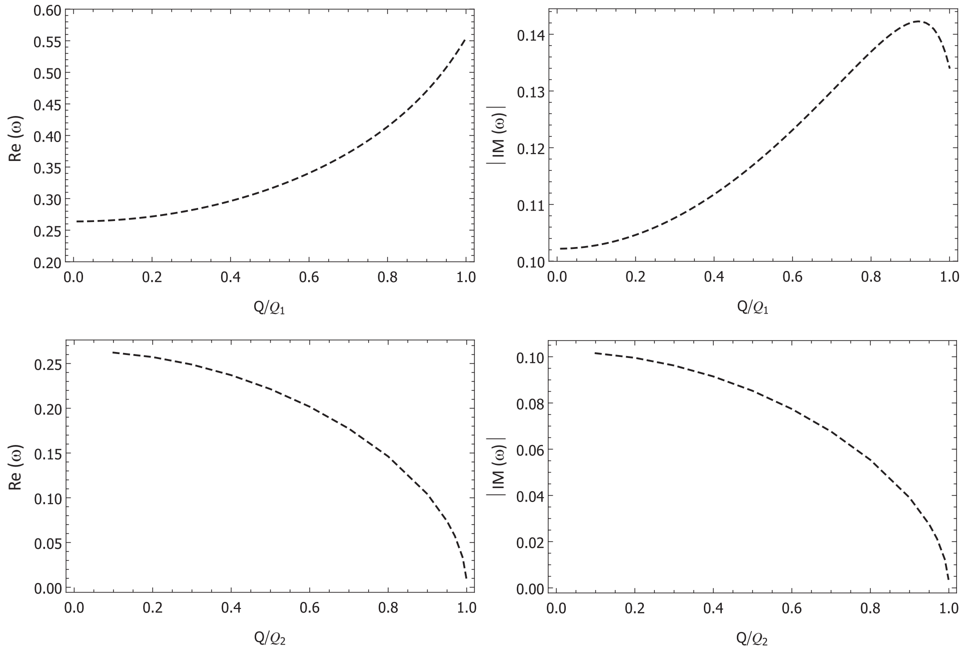

Fig. 6: Massless scalar quasinormal frequencies as a function of the ratio of the charge to that of the corresponding extremal black hole. The upper row presents the results in the case of the linear electromagnetic field with $s = 1$. The bottom row gives those for nonlinear electromagnetic field with $s = 2$.

frequencies are then evaluated semi-analytically by employing the matrix method and the WKB approximation. The results obtained by the two methods are consistent with each other, as the results are found to be stable regarding different orders or interpolation points. Also, the finite difference method is utilized to study the temporal evolution of small scalar perturbations. In the case of linear Maxwell field, as the charge increases, the oscillation period becomes smaller while the corresponding magnitude decays faster. Conversely, for the case of nonlinear Maxwell field, the obtained QNMs follow an exactly opposite trend. Besides, it is observed that as the mass of the scalar field increases, the oscillation frequency becomes more substantial, and the magnitude decays more slowly. In addition, we studied the QNMs of the extremal black hole associtated with the Nariai limit, where the event horizon collides with the cosmological one. Our numerical calculations show that the quasinormal frequency approaches zero at this limit, which is consistent with the analytic arguments. For the present black hole metric, the impact of the quintessence fluid is meaningful regarding its role as an essential candidate for the dark energy. From our calculation results, one concludes that both the real and imaginary parts of the quasinormal frequency increase with increasing ω_q. Our calculations with different metric parameters indicate that the black hole spacetime is likely stable against scalar perturbations. Although the gravitational wave detection, at its present stage, is not yet sufficiently robust to furnish satisfactory signal-to-noise ratio measurements for the ringdown phase. The study of QNMs in Rastall gravity, as well as other modified gravity, may hopefully contribute to the ongoing endeavor in gravitational-wave physics and astronomy.

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