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Interplay of off-diagonal random disorder and quasiperiodic potential in a one-dimensional Aubry-André model

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Abstract – We study the interplay of a random off-diagonal (hopping) disorder with the on-site quasiperiodic potential in a one-dimensional Aubry-André (AA) chain. There is evidence for the absence of delocalized states, at least for a finite lattice, in presence of a weak disorder, thereby removing the possibility of a sharp transition from extended to localized regime. This renders testimony for the presence of a weakly localized phase which we denote as the "critical" phase. We also evaluate whether the random disorder helps or hinders the quasiperiodic term on either side of the "duality" point in inducing a complete localization phenomenon via computing a few relevant quantities, such as the inverse participation ratio (IPR), which estimates the extent of localization, and an extensive multifractal analysis to assess the nature of the disordered states. We observe that a weak random disorder corresponding to the strength of the quasiperiodic term above the critical value ($\lambda_c = 2$) is more efficient in inducing the localization phenomenon as compared to a large disorder below the critical value. We also find that a large disorder is found to compete with the quasiperiodic term beyond its critical value in localizing the eigenstates, while it aids at strengths below the critical value, both of which are intuitively conceivable. Such a differential behavior of the random off-diagonal disorder and its interplay with the quasiperiodic potential albeit expected, have not been reported earlier in the literature. With regard to the multifractal analysis, we ascertain the nature of the critical phase and comment on the fractal dimension, the critical exponents and occurrence of rare events.

Introduction. – The localization phenomenon, being the property of states, is one of the fundamental topics in the transport of the quantum mechanical systems. After the prediction of the Anderson localization of electronic wave function for disordered quantum systems, the area has gained large attention [1,2]. Particularly after several observations have been reported through several experiments for a variety of systems, such as, light waves [3], electron gases [4], etc., the phenomenon of localization has been elucidated in many ways [5]. Moreover, a one-dimensional quasiperiodic potential [6–8] serves as a compelling candidate since the 1980s (represented by the well-known Aubry-André (AA) model [9,10]), which shows the localization transition similar to random potential in three-dimensional systems [1,10]. There have been different works available on this model for decades [11–13], along with generalization of the model, such as, exponential short-range hopping [11], the next-nearest-neighbor hopping [14,15], power-law hopping [16], coupled AA chain [17], modulated hopping [18], in higher dimensions [19].

Later, the developments of ultracold atoms in optical lattices [20–22] have opened a new doorway to implement these theories through experiment and motivate one to go beyond the known. The AA Hamiltonian [10] can be experimentally achievable with ultracold atoms loaded in optical lattices [23,24]. It has been noted that the quasiperiodic potential is mapped by using a bichromatic potential, which can be obtained from the superposition of two optical lattices of the form [23–25]

$$V(x) = s_1 E_{R_1} \sin^2(k_1 x) + s_2 E_{R_2} \sin^2(k_2 x + \phi), \qquad (1)$$

where $k_1 = 2\pi/\lambda_1$ and $k_2 = 2\pi/\lambda_2$ (λ_1 and λ_2 are wavelengths of the two optical lattice potentials). The role of the two lattices is described in the following. The potential with wavelength λ_1 is used to create the tight-binding primary lattice, which is weakly perturbed by another lattice potential, namely, the secondary lattice potential with a wavelength, λ_2. The amplitudes of the two lattice

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potentials are s_1 and s_2 which are expressed in units of the recoil energies E_{R_1} and E_{R_2} with ϕ being an arbitrary phase [25]. The hopping strength and the quasiperiodic potential of the AA Hamiltonian can independently be controlled in experiments by tuning s_1 and s_2, which give us the idea and opportunity to explore further by introducing random hopping into the system. While the random on-site potential has received significant attention in the literature, there has not been much emphasis on the off-diagonal (hopping) disorder and its interplay with the quasiperiodic potential.

In this work, we ask a very specific question: which of the two, namely, the on-site quasiperiodic potential or the off-diagonal random (hopping) disorder, is more dominant in inducing complete localization in a one-dimensional AA model? We provide multiple evidences to support our claim that the quasiperiodic term plays a more dominant role in this regard. A weak random (hopping) disorder above the critical value of the quasiperiodic potential (namely, the "duality" point [10]) causes a much larger degree of localization, than what a strong random disorder does below the critical point. To add further excitement to the story, we have observed that beyond the critical point, the weaker disorder seems to be more efficient in inducing localization as compared to a stronger disorder. Thus, while below the critical point, the random disorder aids in the localization phenomena, beyond the critical disorder, it starts competing with the quasiperiodic potential. This is one of the main results of our paper and we deliberate on it in details in the following section. Further, we perform an extensive multifractal analysis [26–33] to ascertain the critical nature of the wave function and compute the multifractal exponents and the fractal dimension. We also demonstrate interesting prospects of occurrence of "rare events".

We organize our paper in the following way. First, we shall describe the model Hamiltonian and the results. Hence we study the localization phenomena of the system via computing the inverse participation ratio (IPR) and normalized participation ratio (NPR). Next, we shall perform a multifractal analysis to decipher the nature of the critical phase.

The model and results. – The Hamiltonian denotes a one-dimensional tight-binding term along with a AA potential, and is written as

$$H = -\sum_i t_i(|i\rangle\langle i+1| + \text{h.c.}) + \lambda \sum_i \cos(2\pi\beta i)|i\rangle\langle i|, \quad (2)$$

where $|i\rangle$ is a Wannier state at lattice site i, t_i is the nearest-neighbor random hopping strength and λ is the disorder strength of the on-site quasiperiodic potential. Mathematically, an irrational value of β is necessary and sufficient for a phase transition in the quasiperiodic potential model. So here we choose $\beta = (\sqrt{5}-1)/2$ as an inverse of the "golden mean" and also a Diophantine number [10].

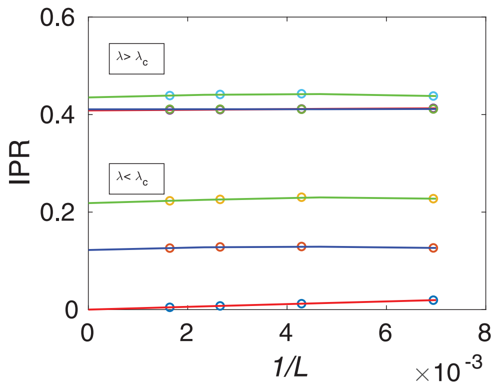

Fig. 1: IPR is shown as a function of $1/L$, where L are taken only from the Fibonacci sequence for (a) $\lambda = 1.5$ and (b) $\lambda = 2.5$ with corresponding colours being red, blue and green for $\sigma = 0$, 0.5, 1, respectively.

Now our main aim is to understand a variant of the AA model, where we introduce a random nearest-neighbor hopping in the lattice. We define a random distribution of hopping energies that are picked up from a uniform rectangular distribution of width σ and in the range from $-t'/2$ to $t'/2$, where $t' = t - 1$ (so that t is distributed uniformly about $t = 1$). A realistic scenario, such as this, can be created by changing the (primary) wavelength, λ_1 (see footnote [1]). It is worthwhile to mention that our disorder model can be derived from a 2D quantum Hall system on a lattice (Hofstadter model) with equal and opposite position-dependent phases in hopping specifically along the y-direction. Earlier Liu *et al.* [34] have considered tunable phase difference between the on-site and the hopping terms of the AA model which enables demonstration of an induced localization transition. While in our case, a transition from the delocalized to localized phase is absent, however a transition from a critical phase to a completely localized can be tuned via the disorder strength which is discussed below.

In particular, in our work we have considered 3 different values of σ ranging from small to moderately large disorder strengths, namely, $\sigma = 0.1,\ 0.5,$ and 1, such that a large disorder is compatible to other energy scales of the problem, that is t $(= 1)$ and λ (varied between 1.5 and 2.5). Thus the effect of random hopping disorder and the competition with the quasiperiodic term will be evaluated for σ to be roughly in the interval $[0.1\lambda : \lambda]$. We have considered 500 different disorder realizations whenever we needed configuration-averaged quantities. We have checked that this number suffices for our purpose even for the largest disorder, that is $\sigma = 1$. The $\sigma = 0$ case corresponds to the AA model, for which the results are known and will be used as a benchmark.

Inverse participation ratio (IPR). – To understand the localization characteristics, we consider the inverse

[1] Since the incommensurability parameter, β, is defined by λ_1/λ_2, a change in λ_1 would necessitate changing λ_2 as well.

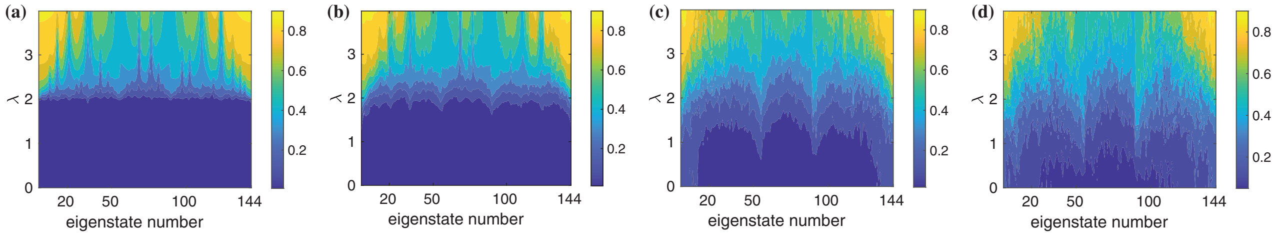

Fig. 2: Surface plots of IPR are shown as a function of λ and the eigenstate number. Here (a) $\sigma = 0$ represents the pure AA model and (b), (c), (d) correspond to disorder values $\sigma = 0.1,\ 0.5,\ 1$, respectively. The darker shade represents more extended states, while the light shade represents larger fraction of localized states.

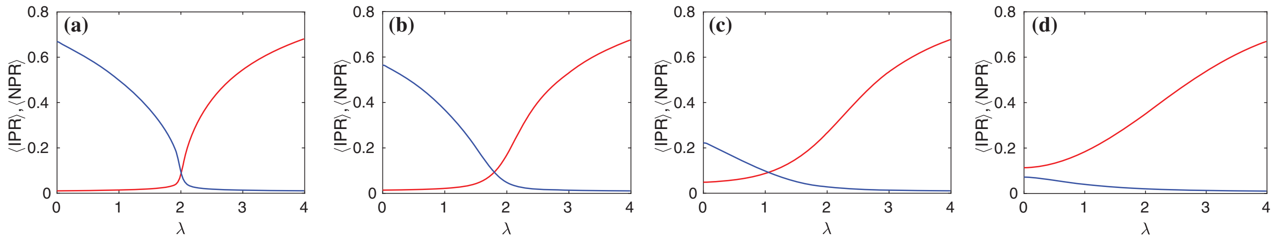

Fig. 3: $\langle$IPR$\rangle$ and $\langle$NPR$\rangle$ averaged over the eigenstate number as a function of λ are shown. Here we plot $\sigma = 0$ for the pure AA model, in (a) together with three different values of $\sigma = 0.1,\ 0.5,\ 1$, in (b), (c), (d) where the plot in red corresponds to $\langle$IPR$\rangle$, and blue corresponds to the $\langle$NPR$\rangle$.

participation ratio (IPR) [13,35,36] which is defined by

$$\mathrm{IPR}_{(n)} = \frac{\sum_i |\psi_i^n|^4}{\left(\sum_i |\psi_i^n|^2\right)^2}, \tag{3}$$

where i represents the number of lattice sites and ψ_i^n is the eigenstate at site i corresponding to the eigenvalue index, n. The system becomes completely localized when IPR reaches its maximum value (IPR $= 1$), for which the ψ_i^n (almost) becomes a delta function, thereby implying vanishing overlap even among the neighboring sites. In the opposite limit, that is when IPR $= 0$, the eigenstates become extended and we have, $|\psi_i^n| \sim 1/\sqrt{L}$, L being the system dimension.

We now comment on the dependence of our results on the system dimension, L. To do that we have considered a Fibonacci sequence of L, namely, $L = 144,\ 233,\ 377,\ 610$, etc., in the presence of disorder (see fig. 1) both for $\lambda > \lambda_c$ and $\lambda < \lambda_c$ (where λ_c is the "duality" point). It may be observed that the IPR as a function of $1/L$ has weak site dependence for $\sigma = 0$ (red curve), while for $\sigma = 0.5$ and 1, the plots are flat even when extrapolated to $L \to \infty$, indicating negligible (or none) size dependencies. Thus we have performed all our calculations corresponding to $L = 144$ for both below ($\lambda = 1.5$) and above ($\lambda = 2.5$) the duality point.

In fig. 2 we present surface plots of IPR for all the eigenvalue indices, n as functions of the strength of the quasiperiodic potential, λ and eigenstate number for a few strengths of the random hopping disorder, σ. Inclusion of a little disorder $\sigma = 0.1$ shows an entirely different picture. A sharp transition from delocalized to localized phases ceases to exist, and the states become energy-dependent at the critical point ($\lambda_c = 2$) [30,37] (see fig. 2(b)). This indicates the presence of an intermediate phase of coexisting localized and delocalized states around the critical transition point. To make this point more lucid, we explore another quantity, namely, the normalized participation ratio (NPR), which is defined by [13]

$$\mathrm{NPR}_{(n)} = \left[L \sum_i |\psi_i^n|^4 \right]^{-1}. \tag{4}$$

NPR shows complete localization for $\mathrm{NPR}_n = 0$ value and complete delocalization for values $\sim$1.

Finally, we take an average of the IPR_n and NPR_n over all the eigenstates of the system and define them as $\langle$IPR$\rangle$ and $\langle$NPR$\rangle$, respectively. In fig. 3 we plot $\langle$IPR$\rangle$ and $\langle$NPR$\rangle$ as a function of λ for different disorder. For the case of $\sigma = 0$ (see fig. 3(a)), there is a sharp transition from a delocalized to a localized phase at $\lambda = \lambda_c$, a well-known result [10] and is indicated by a very narrow region overlap. While for $\sigma \neq 0$, the region over which the transition occurs broadens with disorder. These are indicated in figs. 3(b)–(d). These regions denote a critical phase where the states are neither localized, nor extended [13]. We may also conclude that a complete delocalized phase does not exist for any non-zero σ value corresponding to a finite lattice.

Next, we shall study the physical properties of the system in its critical phase. In fig. 4 we show the plots of IPR as a function of the eigenstate number for two different λ values, namely, $\lambda < \lambda_c$ and $\lambda > \lambda_c$, corresponding to different values of random hopping disorder. Corresponding to $\lambda < \lambda_c$ for $\sigma = 0$, all the states display extended behavior, while for the second condition, namely,

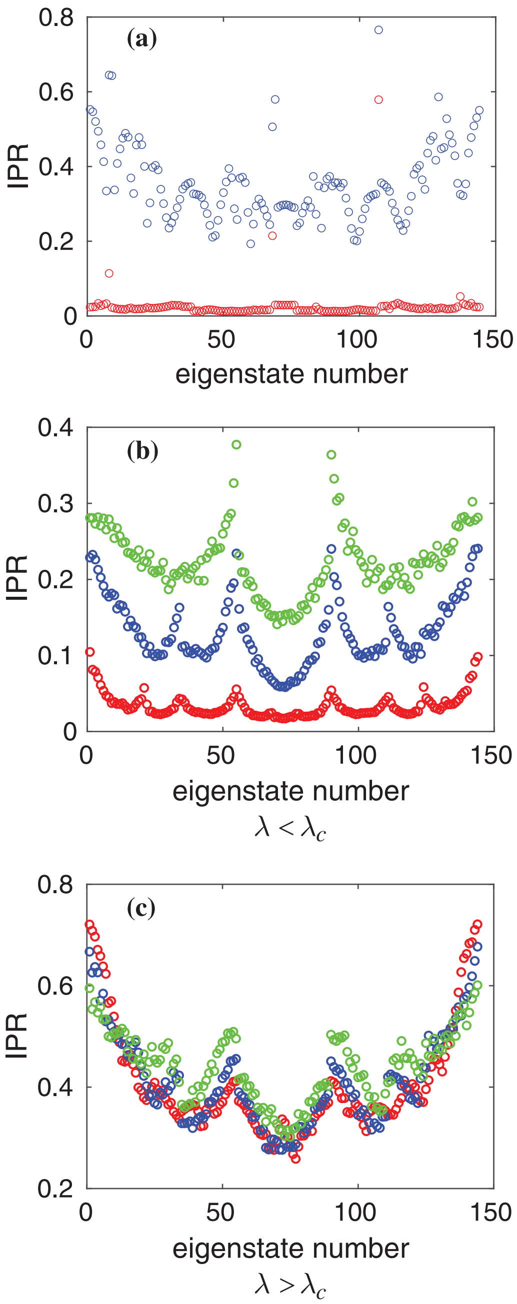

Fig. 4: IPR is shown as a function of the eigenstate number. Here, we plot in (a) the AA model where $\lambda = 1.5,\ 2.5$, the corresponding colours are red and blue, respectively. (b) and (c) correspond to $\sigma = 0.1,\ 0.5,\ 1$, shown in red, blue and green for the same values of λ, respectively.

$\lambda > \lambda_c$ all the states are localized (see fig. 4(a)) [10]. Thus by introducing random hopping disorder corresponding to $\lambda < \lambda_c$, the localization phenomena become more prominent (see fig. 4(b)). A weak disorder implies a relatively larger number of extended states, which, however decreases with the strength of disorder. Further, a higher number of states accumulate at the edges (see fig. 4(b)). Also it shows that the degree of localization is larger towards the center (IPR values are relatively smaller at the edges). In the other case, $\lambda > \lambda_c$, we see a very different picture, as demonstrated in fig. 4(c). At lower values of randomness, the localization phenomenon is more prominent, which subsides by a small amount as the randomness increases. In addition, unlike the previous case ($\lambda < \lambda_c$), the localization starts setting in at the edges, as opposed to the center (note that the IPR has large values at the edges and low values at the center). Thus it seems that weaker values of randomness are more effective in inducing localization phenomena corresponding to a strength of the quasiperiodic potential greater than its critical value demarcating the localized to extended phase transition.

This is somewhat expected because of the following reason. In the delocalized regime ($\lambda < 2$), all the states are extended. Random scattering (due to hopping disorder) introduces backscattering which leads to the localization. The localization will become stronger with larger disorder. While in the localized regime (λ just larger than λ_c), an intrinsic incommensurability (that exists due to the interplay of the AA potential and period of the lattice) produces backscattering which gives the localized states in this regime in the original AA model. Inducing a random hopping on top of that, the phenomenon of backscattering and the scattering due to random hopping amplitudes impede a constructive interference, which finally results in a lesser degree of localization. The above scenario becomes more discernible at larger values of disorder. However, for large values of λ (well beyond $\lambda = 2$), complete localization prevails.

To put things in perspective, we also plot a histogram of the IPR values for the same scenario as in fig. 5 to understand the behavior on either side of the duality point ($\lambda_c = 2$) more precisely. For the case of $\lambda < \lambda_c$, the distribution of IPR has a monotonic variation with the disorder strength, that is, it grows larger with increasing disorder. While for $\lambda > \lambda_c$ the distribution, albeit having larger values, shows non-monotonicity with regard to the disorder strength. It can be seen that a small σ, namely, $\sigma = 0.1$ (shown by red color) shows larger values of IPR (implying the presence of more localized states) than that for $\sigma = 1$ (shown by green colour). Thus we may conclude that the random off-diagonal disorder competes with the quasiperiodic potential above duality, while it is aided by for $\lambda < \lambda_c$.

Finally we show a phase diagram in fig. 6 where the IPR is plotted in the λ-σ plane. At small values of σ and λ, delocalized phases are observed (shown via darker shades), while, at larger disorder, the presence and extent of the critical phase are shown via a reddish shade. Finally the regime shown by yellow shade corresponds to a completely localized phase which occurs for large values of λ irrespective of the value of disorder. It may be observed that the boundary of the critical phase (not a sharp one though) bends downward, that is, towards lower values of λ as σ is made larger which supports our preceding discussion. Thus a transition from a weakly localized phase to a critical phase and that from a critical to a

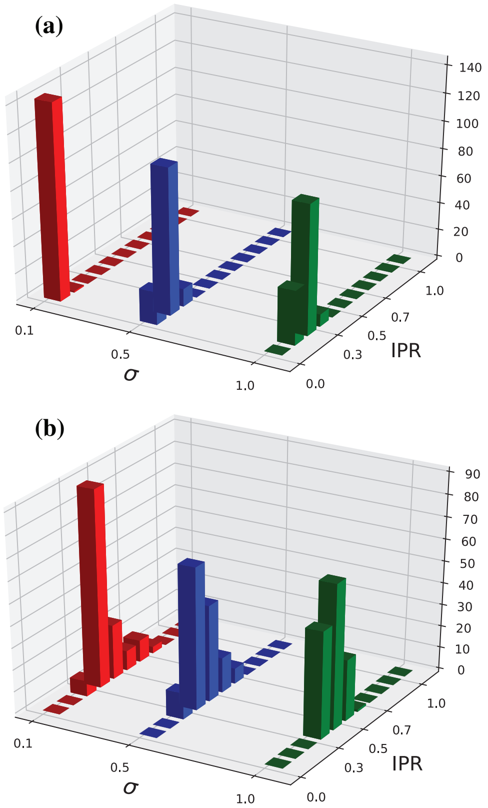

Fig. 5: Here we plot histogram of IPR, where in (a) $\lambda = 1.5$ and $\sigma = 0.1,\ 0.5,\ 1$, and in (b) $\lambda = 2.5$ and $\sigma = 0.1, 0.5, 1$, with the corresponding colours being red, blue and green, respectively.

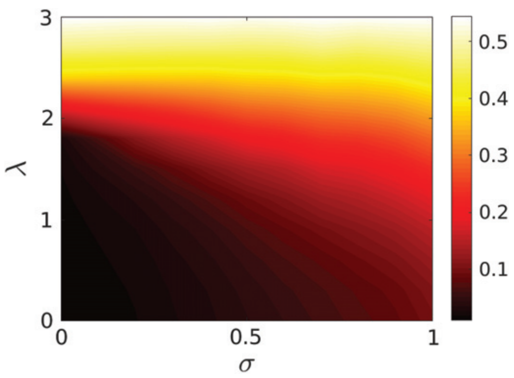

Fig. 6: Here we plot IPR as a function of λ and σ. The black shade corresponds to delocalized phase (IPR values $\leq$0.05), the reddish shade corresponds to critical phase (IPR in the range 0.1–0.3) and the yellow shade corresponds to localized phase (IPR values $\geq$ 0.3).

localized phase can be achieved via tuning the strength of disorder.

Multifractal analysis. – It is important to study the critical phase of the model with disorder, especially since the physical characteristics of these states are intermediate to have completely localized and completely extended features. The spatial density distribution of such states has a multifractal nature which can be denoted by the generalized IPR corresponding to the q-th moment of the wave function having a form [13,33]

$$P_q = \int |\psi(x)|^{2q} \mathrm{d}x. \tag{5}$$

At criticality, the average of P_q shows an anomalous scaling with the system size, L which is given by [13]

$$\langle P_q \rangle \sim L^{-\tau_q}, \tag{6}$$

where τ_q emphasizes the critical behavior of the wave function. It is also conventional to introduce the fractal dimension, D_q and relate it to τ_q via $\tau_q = D_q(q-1)$. For a one-dimensional system, $D_q = 1$ for the extended states and $D_q = 0$ for the localized states, while for the critical states, D_q is a non-trivial function of q, which is an indication of the multifractal behavior of the wave function. Thus, for $q = 2$ (IPR), the average moment $\langle P_q \rangle$ scales as $\frac{1}{L}$ (since $\tau_q = 1$) for the extended states, remains independent of the system size for localized states, while it acquires a value in the interval $[0:1]$ for the critical states.

In the following we plot τ_q as a function of the moment index, q corresponding to both $\lambda < \lambda_c$ and $\lambda > \lambda_c$ (figs. 7(a) and (b)) which are also shown for different disorder strengths. At all values of σ, τ_q increases till a certain value of q ($q = 8$) thereby signaling a critical nature, and beyond which it saturates, thereby showing localization effects. However the values of τ_q are much larger for $\lambda < \lambda_c$ compared to those for $\lambda > \lambda_c$. This implies a large criticality of the states for $\lambda < \lambda_c$ and is thus more discernible than those for $\lambda > \lambda_c$. Further the plots corresponding to different random disorder strengths are further apart for $\lambda < \lambda_c$, while they show bunching together for $\lambda > \lambda_c$, which is most noticeable for $q = 2$ (this feature is also clear from the IPR studied earlier in fig. 4(c), which shows reverse trends as explained earlier).

Further, we plot the fractal dimension (see figs. 7(c) and (d)), D_q, which as a function of q shows intermediate values between zero and one for all values of disorder. They further show a decay for larger q along with suppressed values for $\lambda > \lambda_c$ compared to that of $\lambda < \lambda_c$, the consequences of which are similar to what has been stated above. This is expected for a distribution demonstrating multifractal behavior [13].

A deeper analysis of the fractal singularities can further be done by noting that the average distribution function can also be written as [31,33,38]

$$\langle P_q \rangle = L^d \langle |\psi^{2q}| \rangle \sim \int \mathrm{d}\alpha L^{-q\alpha + f(\alpha)}, \tag{7}$$

where $\alpha = -\ln(|\psi^2|)/\ln(L)$ and the other symbols have the usual meaning. The integral can be evaluated using a saddle point method (valid in the thermodynamic limit) which would yield eq. (6) [33]. τ_q is related to the singularity spectrum, $f(\alpha)$ via the Legendre transformation,

$$\tau_q = \alpha f'(\alpha) - f(\alpha), \tag{8}$$

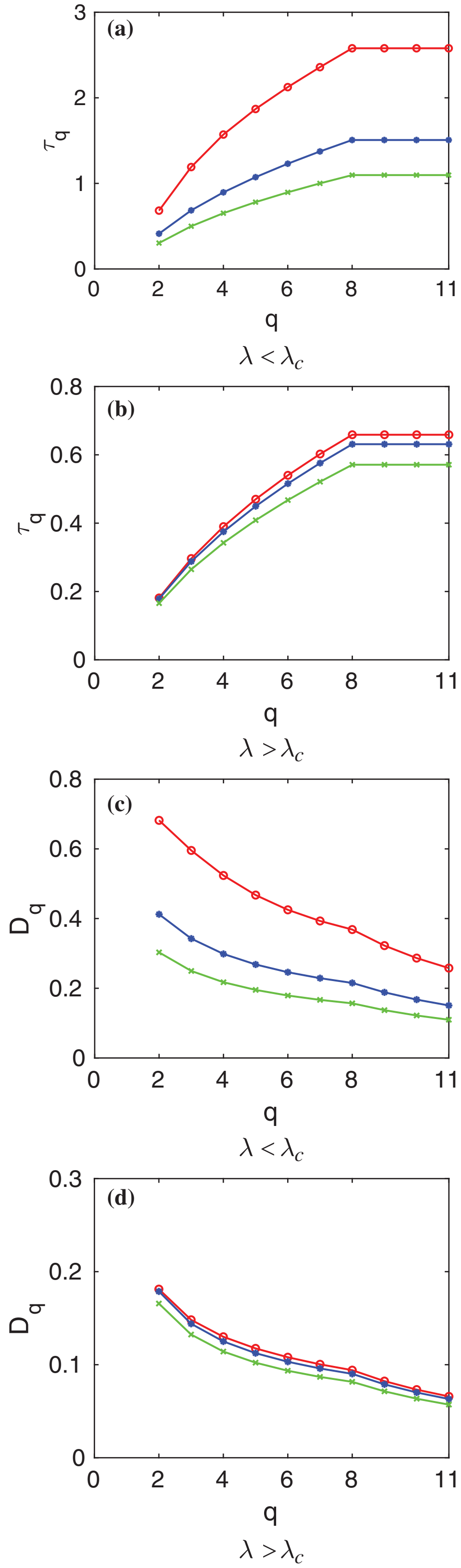

Fig. 7: Here in (a), (b) τ_q is shown as a function of q for $\lambda = 1.5$ and 2.5, respectively, and in (c), (d) D_q is shown as a function of q again for $\lambda = 1.5$ and 2.5 . All the plots are shown for three disorder values, namely, $\sigma = 0.1,\ 0.5,\ 1$, with corresponding colours being red, blue and green, respectively.

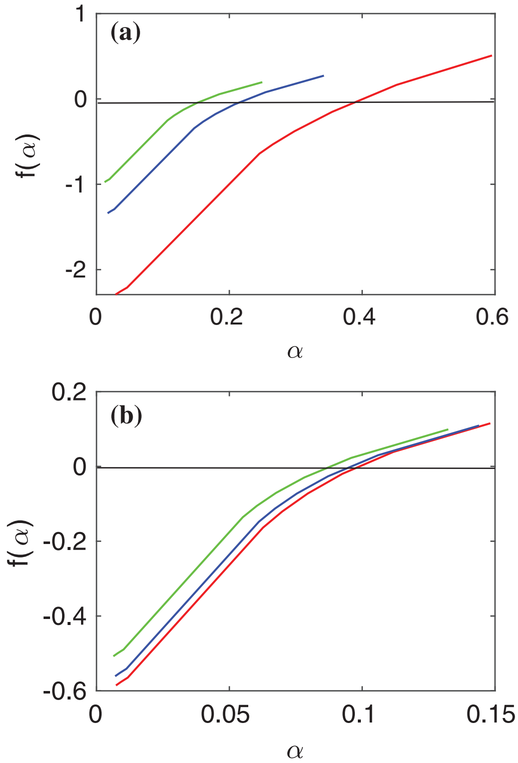

Fig. 8: Here we plot $f(\alpha)$ as a function of α where (a) corresponds to $\lambda = 1.5$ and (b) corresponds to $\lambda = 2.5$. Both the plots include disorder values $\sigma = 0.1,\ 0.5,\ 1$, with corresponding colours being red, blue and green, respectively.

where $f'(\alpha) = \frac{\mathrm{d}f(\alpha)}{\mathrm{d}\alpha}$ and $\alpha = \frac{\mathrm{d}\tau_q}{\mathrm{d}q}$. Thus $f(\alpha)$ is the fractal dimension of the number of sites whose squared wave function satisfies $|\psi^2| \sim L^{-\alpha}$. Corresponding to states being extended, $f(\alpha)$ is a sharply peaked function about a particular value of α. However, the scenario changes in the presence of localized states, where $f(\alpha)$ acquires a broad distribution, with regions, where it becomes negative. The above is related to an interesting phenomenon described in the following. To ascertain the presence of "rare events" (values of the wave function amplitudes occurring rarely in a system) [33] in the spectrum, $f(\alpha)$ is plotted (see fig. 8) as a function of α for both $\lambda < \lambda_c$ and $\lambda > \lambda_c$. The values of α for which $f(\alpha)$ is negative indicate the presence of rare events which are much more prominent for $\lambda < \lambda_c$ than that for $\lambda > \lambda_c$. Thus it seems that large values of the quasiperiodic potential suppresses the rare events. More work is required to evaluate the role of disorder in the occurrence of rare events.

Conclusion. – In this work, we have studied the interplay of nearest-neighbor random hopping disorder with the quasiperiodic potential in a one-dimensional Aubry-André (AA) chain. We have observed that the disorder seems to be more effective in inducing localization effects below the "duality" point ($\lambda_c = 2$) than above it. Also, in the critical phase, the extended and localized states are mixed in a way that a complete delocalized phase (or a complete localized phase) is absent. Thus in a way, random disorder and the quasiperiodic potential help each other below the critical value, while for values above this, they compete with each other. We have also done a detailed multifractal analysis of the eigenstates in the critical phase, where we have studied the fractal dimension and the critical exponent for λ values above and below λ_c. Suppression of rare events for $\lambda > \lambda_c$ seems to be an important consequence of our random disorder model.

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REFERENCES

[1] Anderson P. W., *Phys. Rev.*, **109** (1958) 1492.
[2] Lee P. A. and Ramakrishnan T. V., *Rev. Mod. Phys.*, **57** (1985) 287.
[3] Wiersma D. S., Bartolini P., Lagendijk A. and Righini R., *Nature*, **390** (1997) 671.
[4] Akkermans E. and Montambaux G., *Mesoscopic Physics of Electrons and Photons* (Cambridge Univ. Press, Cambridge, UK) 2006.
[5] Kramer B. and MacKinnon A., *Rep. Prog. Phys.*, **56** (1993) 1469.
[6] Modugno G., *Rep. Prog. Phys.*, **73** (2010) 102401.
[7] Das Sarma S., He S. and Xie X. C., *Phys. Rev. Lett.*, **61** (1988) 2144.
[8] Das Sarma S., He S. and Xie X. C., *Phys. Rev. B*, **41** (1990) 5544.
[9] Harper P. G., *Proc. R. Soc. London, Ser. A*, **68** (1955) 874.
[10] Aubry S. and André G., *Ann. Israel Phys. Soc.*, **3** (1980) 133.
[11] Biddle J. and Das Sarma S., *Phys. Rev. Lett.*, **104** (2010) 070601.
[12] Ray S., Pandey M., Ghosh A. and Sinha S., *New J. Phys.*, **18** (2016) 013013.
[13] Li X., Li X. and Das Sarma S., *Phys. Rev. B*, **96** (2017) 085119.
[14] Biddle J., Wang B., Priour D. J. and Das Sarma S., *Phys. Rev. A*, **80** (2009) 021603.
[15] Biddle J., Priour D. J. jr., Wang B. and Das Sarma S., *Phys. Rev. B*, **83** (2011) 075105.
[16] Gopalakrishnan S., *Phys. Rev. B*, **96** (2017) 054202.
[17] Rossignolo M. and Dell'Anna L., *Phys. Rev. B*, **99** (2019) 054211.
[18] Yahyavi M., Hetnyi B. B. and Tanatar B., *Phys. Rev. B*, **100** (2019) 064202.
[19] Devakul T. and Huse D. A., *Phys. Rev. B*, **96** (2017) 214201.
[20] Damski B., Zakrzewski J., Santos L., Zoller P. and Lewenstein M., *Phys. Rev. Lett.*, **91** (2003) 080403.
[21] Lewenstein M., Sanpera A., Ahufinger V., Damski B., Sen(De) A. and Sen U., *Adv. Phys.*, **56** (2007) 243.
[22] Boers D. J., Goedeke B., Hinrichs D. and Holthaus M., *Phys. Rev. A*, **75** (2007) 063404.
[23] Lye J. E., Fallani L., Fort C., Guarrera V., Modugno M., Wiersma D. S. and Inguscio M., *Phys. Rev. A*, **75** (2007) 061603.
[24] Roati G., D'Errico C., Fallani L., Fattori M., Fort C., Zaccanti M., Modugno G., Modugno M. and Inguscio M., *Nature*, **45** (2008) 3895.
[25] Modugno M., *New J. Phys.*, **11** (2009) 033023.
[26] Soukoulis C. M. and Economou E. N., *Phys. Rev. Lett.*, **52** (1985) 565.
[27] Halsey T. C., Jensen M. H., Kadanoff L. P., Procaccia I. and Shraiman B. I., *Phys. Rev. A*, **33** (1986) 1141.
[28] Wegner F., *Z. Phys. B*, **36** (1980) 209.
[29] Pietronero L. and Siebesma A. P., *Phys. Rev. Lett.*, **57** (1986) 1098.
[30] Siebesma A. P. and Pietronero L., *Europhys. Lett.*, **4** (1987) 597.
[31] Hiramoto H. and Kohmoto M., *Int. J. Mod. Phys. B*, **06** (1992) 281.
[32] Mirlin A. D. and Evers F., *Phys. Rev. B*, **62** (2000) 12.
[33] Evers F. and Mirlin A. D., *Rev. Mod. Phys.*, **80** (2008) 1355.
[34] Liu F., Ghosh S. and Chong Y. D., *Phys. Rev. B*, **91** (2015) 014108.
[35] Thouless D. J., *Phys. Rep.*, **13** (1974) 93.
[36] Schreiber M., *J. Phys. C*, **18** (1985) 2493.
[37] Kohmoto M., *Phys. Rev. Lett.*, **51** (1983) 1198.
[38] Kohmoto M. and Tobe D., *Phys. Rev. B*, **77** (2008) 134204.