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φ-deformed boson algebra based on φ-deformed addition and non-classical properties of φ-deformed coherent states

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Abstract

In this paper the general deformed boson algebra called a φ-deformed boson algebra is discussed based on the φ-addition and φ-subtraction. The φ-derivative and the realization of φ-deformed boson algebra are also discussed. Besides, φ-deformed $su(2)$ algebra ($su_\varphi(2)$ algebra) and supersymmetric quantum mechanics are also discussed. Seven possible φ-deformed boson algebras are discussed and φ-coherent states and non-classical properties are investigated.

Keywords: deformed boson algebra, supersymmetric quantum mechanics, φ-coherent states

(Some figures may appear in colour only in the online journal)

1. Introduction

The ordinary boson algebra takes the following form

$$[a, a^\dagger] = 1, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger. \tag{1}$$

The first deformation of boson algebra was accomplished by Wigner [1], who considered the following deformed algebra

$$aa^\dagger - a^\dagger a = 1 + 2\nu(-1)^N, \quad \nu > 0. \tag{2}$$

After this, three types of q-boson algebras appeared in [2–5]. Arik and Coon [2] considered the following q-boson algebra

$$aa^\dagger - qa^\dagger a = 1 \tag{3}$$

and Biedenharn [3] and Macfarlane [4] considered the following q-boson algebra

$$aa^\dagger - qa^\dagger a = q^{-N} \tag{4}$$

and Odaka *et al* [5] considered the following q-boson algebra

$$aa^\dagger - qa^\dagger a = q^N. \tag{5}$$

These algebras were unified into the following general deformed boson algebra (GBA) [6–8]

$$[a, a^\dagger] = \varphi(N+1) - \varphi(N), \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \tag{6}$$

$$a^\dagger a = \varphi(N), \quad aa^\dagger = \varphi(N+1), \tag{7}$$

where $\varphi(x)$ is a positive analytic function called a structure function with $\varphi(0) = 0$, and N is the number operator. The structure function $\varphi(x)$ contains some deformation parameters and it reduces to x when the deformation parameters take some special values. Later another type of deformed boson algebras appeared in literatures: The q-deformed Wigner algebra [9] was presented in the form

$$aa^\dagger - qa^\dagger a = q^{-N}(1 + 2\nu(-1)^N). \tag{8}$$

Unification of three q-boson algebra [2–5] appeared in the following form [10]

$$aa^\dagger - qa^\dagger a = q^{\alpha N+\beta}. \tag{9}$$

More general form of the above algebra appeared in the following form [11]

$$aa^\dagger - q^\gamma a^\dagger a = q^{\alpha N+\beta}. \tag{10}$$

Recently, a new kind of the deformed boson algebra appeared by Rebesh *et al* [12] in the form

$$[a, a^\dagger] = \frac{N+1}{1+qN} - \frac{N}{1+q(N-1)}. \tag{11}$$

The deformation of ordinary boson algebra can be applied to the thermodynamics of deformed boson gas [13–21] or to investigation of non-classical properties of deformed coherent states [22, 23] when photon is regarded as a kind of deformed boson.

In this paper we discuss a general deformed boson algebra called φ-deformed boson algebra which is written in terms of the φ-addition and φ-subtraction for invertible structure function φ. This paper is organized as follows: in section 2 we discuss φ-deformed boson algebra. In section 3 we discuss φ-derivative, φ-integral and realization of φ-deformed boson algebra (infinite Fock space). In section 4 we discuss supersymmetric quantum mechanics (SQM). In section 5 we discuss possible structure functions. In section 6 we discuss φ-deformed boson algebra as a Hopf algebra. In section 7 we discuss φ-deformed $su(2)$ algebra ($su_\varphi(2)$ algebra). In section 8 we discuss φ-coherent states and non-classical properties.

2. φ-deformed boson algebra

The GBA (1) is written in terms of the commutator $[A, B] = AB - BA$. In this paper we will introduce the φ-addition and φ-subtraction to change the commutator into the φ-deformed commutator $AB \ominus BA$. We will show that all GBA can be written with a help of φ-deformed commutator in the form $aa^\dagger \ominus a^\dagger a = 1$. Here and from now on we demand that $\varphi(x)$ should obey

$$\varphi(0) = 0, \quad \varphi(1) = 1. \tag{12}$$

2.1. φ-addition and φ-subtraction

Now let us consider φ-deformation of an ordinary addition and subtraction.

Definition 2.1. For invertible structure function $\varphi(x)$, φ-addition is defined as

$$x \oplus y = \varphi(\varphi^{-1}(x) + \varphi^{-1}(y)). \tag{13}$$

Proposition 2.1. *The φ-addition obeys the following:*

$$x \oplus y = y \oplus x, \tag{14}$$

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z. \tag{15}$$

Proof. It is simple. □

Proposition 2.2. *The φ-additive identity is 0.*

Proof. Using the equation (12), we have $x \oplus 0 = \varphi(\varphi^{-1}(x) + 0) = \varphi(\varphi^{-1}(x)) = x$, which completes the proof. □

Definition 2.2. The φ-additive inverse of x denoted by $\ominus x$ is defined through

$$x \oplus (\ominus x) = 0. \tag{16}$$

Proposition 2.3. *The φ-additive inverse of x is given by*

$$\ominus x = \varphi(-\varphi^{-1}(x)). \tag{17}$$

Proof. It follows from $\varphi^{-1}(x) + \varphi^{-1}(\ominus x) = 0$. □

Definition 2.3. The φ-subtraction is defined as

$$x \ominus y = x \oplus (\ominus y) = \varphi(\varphi^{-1}(x) - \varphi^{-1}(y)). \tag{18}$$

2.2. φ-deformed boson algebra

In this subsection we will derive φ-deformed boson algebra in an algebraical way with a help of φ-addition and φ-subtraction.

Proposition 2.4. *All φ-deformed algebras are written as*

$$aa^\dagger \ominus a^\dagger a = 1, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \tag{19}$$

where

$$a^\dagger a = \varphi(N) \tag{20}$$

and $\varphi(0) = 0$, $\varphi(1) = 1$. Thus, the φ-deformed boson algebra can also be written as

$$aa^\dagger = \varphi(1 + \varphi^{-1}(a^\dagger a)). \tag{21}$$

Proof. From the relation (20) we have

$$N = \varphi^{-1}(a^\dagger a). \tag{22}$$

From the equation (19) we get

$$aa^\dagger = 1 \oplus a^\dagger a, \tag{23}$$

which gives

$$aa^\dagger = 1 \oplus \varphi(N) = \varphi(1 + N), \tag{24}$$

which completes the proof. □

Proposition 2.5. *When $\varphi(n) \geqslant 0$ for $n = 0, 1, 2, \cdots$, the Fock representation of the φ-deformed boson algebra is as follows:*

$$N|n\rangle = n|n\rangle, \quad n = 0, 1, 2, \cdots, \tag{25}$$

$$a|n\rangle = \sqrt{\varphi(n)}\, |n - 1\rangle, \tag{26}$$

$$a^\dagger|n\rangle = \sqrt{\varphi(n + 1)}\, |n + 1\rangle. \tag{27}$$

Thus, we have infinite Fock space.

Proof. It is simple. □

Proposition 2.6. *When $\varphi(n) \geqslant 0$ for $n = 0, 1, \cdots, p$ and $\varphi(p+1) = 0$ for $p = 1, 2, \cdots$, we have the $(p+1)$-dimensional Fock space. In this case, the Fock representation of the φ-deformed boson algebra is as follows:*

$$N|n\rangle = n|n\rangle, \quad n = 0, 1, \cdots, p \tag{28}$$

$$a|n\rangle = \sqrt{\varphi(n)}\,|n-1\rangle, \quad n = 0, 1, \cdots, p \tag{29}$$

$$a^\dagger|n\rangle = \sqrt{\varphi(n+1)}\,|n+1\rangle, \quad n = 0, 1, \cdots, p-1 \tag{30}$$

$$a^\dagger|p\rangle = 0. \tag{31}$$

Thus, we have finite Fock space. In this case we also have

$$a^{p+1} = (a^\dagger)^{p+1} = 0. \tag{32}$$

Proof. It is simple. □

3. φ-derivative, φ-integral and realization of φ-deformed boson algebra (infinite Fock space)

Now let us introduce φ-derivative to discuss realization of φ-deformed boson algebra. Here we will restrict our interest to infinite Fock space.

3.1. φ-derivative

Definition 3.1. The φ-derivative D_x^φ is defined as follows:

$$D_x^\varphi x^n = \varphi(n) x^{n-1}, \quad n = 0, 1, 2, \cdots \tag{33}$$

Imposing the equation (12) we have

$$D_x^\varphi(1) = 0. \tag{34}$$

Proposition 3.1. *The φ-derivative D_x^φ can be written as*

$$D_x^\varphi = \frac{1}{x}\varphi(x\partial) = \frac{1}{1 + x\partial}\varphi(1 + x\partial)\partial, \tag{35}$$

where

$$\partial = \frac{\mathrm{d}}{\mathrm{d}x}.$$

Proof. From $\varphi(0) = 0$, we can set

$$\varphi(x) = \sum_{k=1}^{\infty} c_k x^k. \tag{36}$$

Thus we have

$$\begin{aligned} D_x^\varphi x^n &= \frac{1}{x}\varphi(x\partial)x^n = \frac{1}{x}\sum_{k=1}^{\infty} c_k (x\partial)^k x^n \\ &= \frac{1}{x}\sum_{k=1}^{\infty} c_k n^k x^n = \varphi(n) x^{n-1}. \end{aligned} \tag{37}$$

Besides, we have

$$\begin{aligned} D_x^\varphi &= \frac{1}{x}\sum_{k=1}^{\infty} c_k (x\partial)^k = \sum_{k=1}^{\infty} c_k (\partial x)^{k-1}\partial \\ &= \frac{1}{\partial x}\sum_{k=1}^{\infty} c_k (\partial x)^k \partial = \frac{1}{1 + x\partial}\varphi(1 + x\partial)\partial. \end{aligned} \tag{38}$$

This completes the proof. □

Proposition 3.2. *The φ-derivative D_x^φ is linear*

Proof. We have $D_x^\varphi(aF(x) + bG(x)) = aD_x^\varphi F(x) + bD_x^\varphi G(x)$. □

3.2. φ-integral

Definition 3.2. The φ-integral is defined as

$$\begin{aligned} \int \mathrm{d}_\varphi x F(x) &= \frac{1}{\varphi(x\partial)}(xF(x)) \\ &= \int \mathrm{d}x \frac{1}{\varphi(1 + x\partial)}(1 + x\partial)F(x). \end{aligned} \tag{39}$$

Proposition 3.3. *The φ-integral and φ-derivative obey*

$$\int \mathrm{d}_\varphi x D_x^\varphi F(x) = D_x^\varphi \int \mathrm{d}_\varphi x F(x) = F(x). \tag{40}$$

Proof. From the definition of φ-integral and φ-derivative, we have

$$\int \mathrm{d}_\varphi x D_x^\varphi F(x) = \frac{1}{\varphi(x\partial)}(xD_x^\varphi F(x)) = F(x) \tag{41}$$

and

$$\begin{aligned} D_x^\varphi \int \mathrm{d}_\varphi x F(x) &= \frac{1}{1 + x\partial}\varphi(1 + x\partial)\partial \\ &\times \int \mathrm{d}x \frac{1}{\varphi(1 + x\partial)}(1 + x\partial)F(x) = F(x). \end{aligned} \tag{42}$$

This completes the proof. □

3.3. φ-exponential function

Definition 3.3. The φ-exponential function $e_\varphi(x)$ is defined as follows:

$$e_\varphi(x) = \sum_{n=0}^{\infty} \frac{1}{\varphi(n)!} x^n. \tag{43}$$

Proposition 3.4. *The φ-exponential function obeys*

$$D_x^\varphi e_\varphi(x) = e_\varphi(x). \tag{44}$$

Proof. It is simple. □

Proposition 3.5. *For the φ-exponential function the following holds:*

$$e_\varphi(x)e_\varphi(y) = e_\varphi((x+y)_\varphi), \tag{45}$$

where

$$e_\varphi((x+y)_\varphi) = \sum_{n=0}^{\infty} \frac{1}{\varphi(n)!}(x+y)_\varphi^n \tag{46}$$

and

$$(x+y)_\varphi^n = \sum_{k=0}^{n} \binom{n}{k}_\varphi x^k y^{n-k},$$
$$\binom{n}{k}_\varphi = \frac{\varphi(n)!}{\varphi(k)!\varphi(n-k)!}. \tag{47}$$

Proof. It is simple. □

From the definition of the φ-exponential function, we can define the following function:

$$e_\varphi(a\colon x) := [e_\varphi(x)]^a. \tag{48}$$

Then we have

$$D_x^\varphi e_\varphi(a\colon x) = a e_\varphi(a\colon x). \tag{49}$$

Definition 3.4. The φ-hyperbolic functions are defined as follows:

$$\cosh_\varphi(a\colon x) = \frac{1}{2}(e_\varphi(a\colon x) + e_\varphi(a\colon -x)), \tag{50}$$
$$\sinh_\varphi(a\colon x) = \frac{1}{2}(e_\varphi(a\colon x) - e_\varphi(a\colon -x)). \tag{51}$$

Proposition 3.6. *For the φ-hyperbolic functions, the following holds:*

$$D_x^\varphi \cosh_\varphi(a\colon x) = a \sinh_\varphi(a\colon x), \tag{52}$$
$$D_x^\varphi \sinh_\varphi(a\colon x) = a \cosh_\varphi(a\colon x). \tag{53}$$

Proof. It is simple. □

3.4. Realization of φ-deformed boson algebra

In order to have a functional realization of this representation, we consider the space $\mathcal{P}$ of all polynomials in one supplementary variable x, and introduce its basis of monomials;

$$|n\rangle \leftrightarrow e_n(x) = \frac{x^n}{\sqrt{\varphi(n)!}}, \tag{54}$$

where

$$\varphi(n)! = \varphi(n)\varphi(n-1)\cdots\varphi(2)\varphi(1) \tag{55}$$

and

$$\varphi(0)! = 0! = 1. \tag{56}$$

Proposition 3.7. *Acted on analytic function $\psi(x) \in \mathcal{P}$, the operators of the φ-deformed boson algebra with infinite Fock space can be realized as follows;*

$$N = x\partial, \quad a = D_x^\varphi, \quad a = x. \tag{57}$$

Then we have

$$[D_x^\varphi, x] = \varphi(x\partial + 1) - \varphi(x\partial). \tag{58}$$

Proof. Using the equation (54) we have

$$Ne_n(x) = x\partial\frac{x^n}{\sqrt{\varphi(n)!}} = n\frac{x^n}{\sqrt{\varphi(n)!}} = ne_n(x), \tag{59}$$

$$ae_n(x) = x\frac{x^n}{\sqrt{\varphi(n)!}} = \frac{x^{n+1}}{\sqrt{\varphi(n)!}} = \sqrt{\varphi(n+1)}\, e_{n+1}(x), \tag{60}$$

$$a^\dagger e_n(x) = D_x^\varphi\frac{x^n}{\sqrt{\varphi(n)!}} = \varphi(n)\frac{x^{n-1}}{\sqrt{\varphi(n)!}} = \sqrt{\varphi(n)}\, e_{n-1}(x). \tag{61}$$

The derivation of the equation (58) is as follows:

$$Dx = \frac{1}{x}\varphi(x\partial)x = \frac{1}{x}\sum_{k=1}^{\infty} c_k(x\partial)^k x = \sum_{k=1}^{\infty} c_k(\partial x)^k$$
$$= \sum_{k=1}^{\infty} c_k(1+x\partial)^k = \varphi(1+x\partial), \tag{62}$$

which completes the proof. □

4. Supersymmetric quantum mechanics (SQM)

The simplest SQM for φ-boson and ordinary fermion is expressed in terms of two supercharges

$$Q_+ = \sqrt{\hbar w}\, af^\dagger, \quad Q_- = \sqrt{\hbar w}\, a^\dagger f, \tag{63}$$

where $a, a^\dagger$ are step operators of φ-deformed boson algebra and $f, f^\dagger$ are fermion's step operators obeying $\{f, f^\dagger\} = 1, f^2 = (f^\dagger)^2 = 0$.

Proposition 4.1. *For φ-boson and fermion, the SQM algebra is given by*

$$Q_\pm^2 = 0, \quad \{Q_+, Q_-\} = H, \quad [H, Q_\pm] = 0, \tag{64}$$

where

$$H = \hbar w \varphi(N_\varphi + N_F) \tag{65}$$

and N_φ and N_F denote number operators of φ-boson and fermion, respectively.

Proof. We have

$$\{Q_+, Q_-\} = \hbar w(\varphi(N_\varphi + 1)N_F + \varphi(N_\varphi)(1 - N_F))$$
$$= \hbar w[\varphi(N_\varphi) + (\varphi(N_\varphi + 1) - \varphi(N_\varphi))N_F]$$
$$= \hbar w\varphi(N_\varphi + N_F) \tag{66}$$

which completes the proof. □

5. Possible structure functions for infinite Fock space

In this section we discuss some interesting choices of $\varphi(x)$ for infinite Fock space.

5.1. Case I: boson algebra

Now let us consider the boson algebra

$$aa^\dagger - a^\dagger a = 1,\ \ [N, a^\dagger] = a^\dagger,\ \ [N, a] = -a. \tag{67}$$

This algebra has infinite Fock space and we have $N = a^\dagger a$. The first relation of the equation (67) is the same as

$$aa^\dagger \ominus a^\dagger a = 1, \tag{68}$$

where $\ominus$ is the ordinary subtraction.

5.2. Case II [2]

Now let us consider the following q-boson algebra [2]

$$aa^\dagger - qa^\dagger a = 1,\ \ [N, a^\dagger] = a^\dagger,\ \ [N, a] = -a. \tag{69}$$

This algebra has infinite Fock space for $q > 0$. The Fock space is not well defined for complex q because $a^\dagger$ cannot be regarded as Hermitian adjoint of a. The algebra (69) gives the following relation

$$a^\dagger a = \frac{q^N - 1}{q - 1}, \tag{70}$$

which implies that the structure function and its inverse are given by

$$\varphi(x) = \frac{q^x - 1}{q - 1},\quad \varphi^{-1}(x) = \frac{1}{\ln q}\ln[1 + (q - 1)x]. \tag{71}$$

Then, φ-addition and φ-subtraction is defined as

$$x \oplus y = x + y + (q - 1)xy, \tag{72}$$
$$x \ominus y = \frac{x - y}{1 + (q - 1)y}. \tag{73}$$

The first relation of the equation (69) is the same as

$$aa^\dagger \ominus a^\dagger a = 1 \tag{74}$$

because we have

$$aa^\dagger = 1 \oplus a^\dagger a \tag{75}$$
$$=1 \oplus \varphi(N) \tag{76}$$
$$=1 + \varphi(N) + (q - 1)\varphi(N) \tag{77}$$
$$=1 + a^\dagger a + (q - 1)a^\dagger a \tag{78}$$
$$=1 + qa^\dagger a. \tag{79}$$

5.2.1. φ-deformed derivative and φ-deformed integral.

The φ-deformed derivative is given

$$D_x^\varphi = \frac{1}{(q - 1)x}(T_q - 1), \tag{80}$$

where

$$T_q = q^{x\partial},\qquad T_qF(x) = F(qx). \tag{81}$$

The deformed Leibnitz rule for φ-deformed derivative is

$$D_x^\varphi(F(x)G(x)) = (D_x^\varphi F(x))G(qx) + F(x)D_x^\varphi G(x). \tag{82}$$

The φ-deformed integral is given by

$$\int \mathrm{d}_\varphi xF(x) = (1 - q)(1 - T_q)^{-1}(xF(x))$$
$$= (1 - q)\sum_{n=0}^{\infty} q^n xF(q^n x). \tag{83}$$

The φ-deformed exponential function obeys the following relation

$$e_\varphi(qx) = (1 + (q - 1)x)e_\varphi(x). \tag{84}$$

5.3. Case III [3, 4]

Now let us consider the following q-boson algebra [3, 4]

$$aa^\dagger - qa^\dagger a = q^{-N},\ \ [N, a^\dagger] = a^\dagger,\ \ [N, a] = -a. \tag{85}$$

This algebra has infinite Fock space for $q > 0$. When $q = \mathrm{e}^{\pi \mathrm{i}/(p+1)}$, $p = 1, 2, 3, \cdots$ is a complex number, we have finite dimensional Fock space because $\varphi(p + 1) = 0$. The algebra (85) gives the following relation

$$a^\dagger a = \frac{q^N - q^{-N}}{q - q^{-1}}, \tag{86}$$

which implies that the structure function and its inverse are given by

$$\varphi(x) = \frac{q^x - q^{-x}}{q - q^{-1}},$$
$$\varphi^{-1}(x) = \frac{1}{\ln q}\ln\left[\frac{1}{2}((q - q^{-1})x + \sqrt{(q - q^{-1})^2x^2 + 4})\right]. \tag{87}$$

Then, φ-addition and φ-subtraction is defined as

$$x \oplus y = \frac{1}{2}(x\sqrt{4 + (q - q^{-1})^2y^2} + y\sqrt{4 + (q - q^{-1})^2x^2}), \tag{88}$$

$$x \ominus y = \frac{1}{2}(x\sqrt{4 + (q - q^{-1})^2y^2} - y\sqrt{4 + (q - q^{-1})^2x^2}). \tag{89}$$

The first relation of the equation (85) is the same as

$$aa^\dagger \ominus a^\dagger a = 1 \tag{90}$$

because we have

$$aa^\dagger = 1 \oplus a^\dagger a \tag{91}$$
$$=1 \oplus \varphi(N) \tag{92}$$
$$= \frac{1}{2}(\sqrt{4 + (q - q^{-1})^2\varphi(N)^2} + \varphi(N)(q + q^{-1})) \tag{93}$$
$$= \frac{1}{2}(\sqrt{4 + (q^N - q^{-N})^2} + a^\dagger a(q + q^{-1})) \tag{94}$$
$$= \frac{1}{2}(q^N + q^{-N} + 2qa^\dagger a - (q - q^{-1})a^\dagger a) \tag{95}$$
$$= qa^\dagger a + \frac{1}{2}(q^N + q^{-N} - (q^N - q^{-N})) \tag{96}$$
$$= qa^\dagger a + q^{-N}. \tag{97}$$

5.3.1. φ-deformed derivative and φ-deformed integral.

The φ-deformed derivative is given

$$D_x^\varphi = \frac{1}{(q - q^{-1})x}(T_q - T_q^{-1}). \tag{98}$$

The deformed Leibnitz rule for φ-deformed derivative is

$$D_x^\varphi(F(x)G(x)) = (D_x^\varphi F(x))G(qx) + F(q^{-1}x)D_x^\varphi G(x). \tag{99}$$

The φ-deformed integral is given by

$$\int \mathrm{d}_\varphi xF(x) = (q - q^{-1})(T_q - T_q^{-1})^{-1}(xF(x))$$
$$= (q - q^{-1})\sum_{n=0}^{\infty} q^{-2n-1}xF(q^{-2n-1}x). \tag{100}$$

The φ-deformed exponential function obeys the following relation

$$e_\varphi(qx) - e_\varphi(q^{-1}x) = (q - q^{-1})xe_\varphi(x). \tag{101}$$

5.4. Case IV [5]

Now let us consider the following q-boson algebra [5]

$$aa^\dagger - qa^\dagger a = q^N,\ \ [N, a^\dagger] = a^\dagger,\ \ [N, a] = -a. \tag{102}$$

This algebra has infinite Fock space for $q > 0$. The Fock space is not well defined for complex q because $a^\dagger$ cannot be regarded as Hermitian adjoint of a. The algebra (102) gives the following relation

$$a^\dagger a = Nq^{N-1}, \tag{103}$$

which implies that the structure function and its inverse are given by

$$\varphi(x) = xq^{x-1},\quad \varphi^{-1}(x) = \frac{1}{\ln q}W(qx\ln q), \tag{104}$$

where $W(x)$ is Lambert function, which is defined as

$$x = ye^y \ \Rightarrow\ y = W(x). \tag{105}$$

Then, φ-addition and φ-subtraction is defined as

$$x \oplus y = xe^{W(qy\ln q)} + ye^{W(qx\ln q)}, \tag{106}$$
$$x \ominus y = e^{-2W(qy\ln q)}(xe^{W(qy\ln q)} - ye^{W(qx\ln q)}), \tag{107}$$

where we used

$$W(x) = \ln\frac{x}{W(x)}. \tag{108}$$

The first relation of the equation (102) is the same as

$$aa^\dagger \ominus a^\dagger a = 1 \tag{109}$$

because we have

$$aa^\dagger = 1 \oplus a^\dagger a \tag{110}$$
$$=1 \oplus \varphi(N) \tag{111}$$
$$= e^{W(q\ln q\varphi(N))} + \varphi(N)e^{W(q\ln q)} \tag{112}$$
$$= e^{W(Nq^N\ln q)} + a^\dagger ae^{W(q\ln q)} \tag{113}$$
$$= e^{N\ln q} + qa^\dagger a \tag{114}$$
$$= q^N + qa^\dagger a, \tag{115}$$

where we used

$$W(q\ln q) = \ln q. \tag{116}$$

5.4.1. φ-deformed derivative and φ-deformed integral.

The φ-deformed derivative is given

$$D_x^\varphi = T_q\partial. \tag{117}$$

The deformed Leibnitz rule for φ-deformed derivative is

$$D_x^\varphi(F(x)G(x)) = (D_x^\varphi F(x))G(qx) + F(qx)D_x^\varphi G(x). \tag{118}$$

The φ-deformed integral is given by

$$\int \mathrm{d}_\varphi xF(x) = \int \mathrm{d}xF(q^{-1}x). \tag{119}$$

The φ-deformed exponential function obeys

$$T_q\partial e_\varphi(x) = e_\varphi(x) \tag{120}$$

or

$$\partial e_\varphi(x) = e_\varphi(q^{-1}x). \tag{121}$$

Here we have

$$e_\varphi(x) = \sum_{n=0}^{\infty}\frac{q^{-\frac{1}{2}n(n-1)}}{n!}x^n. \tag{122}$$

5.5. Case V [12]

Now let us consider the following q-boson algebra [12]

$$[a, a^\dagger] = \frac{N+1}{1+qN} - \frac{N}{1+q(N-1)}. \tag{123}$$

This algebra has infinite Fock space for $q > 0$. The algebra (123) gives the following relation

$$a^\dagger a = \frac{N}{1 + q(N-1)}, \tag{124}$$

which implies that the structure function and its inverse are given by

$$\varphi(x) = \frac{x}{1 + q(x-1)}, \quad \varphi^{-1}(x) = \frac{(1-q)x}{1 - qx}. \tag{125}$$

Then, φ-addition and φ-subtraction is defined as

$$x \oplus y = \frac{x + y - 2qxy}{1 - q^2 xy}, \tag{126}$$

$$x \ominus y = \frac{x - y}{1 - 2qy + q^2 xy}. \tag{127}$$

The algebra (123) can be written as

$$aa^\dagger \ominus a^\dagger a = 1, \tag{128}$$

which gives

$$aa^\dagger + (2q-1)a^\dagger a = 1 + q^2 a(a^\dagger)^2 a. \tag{129}$$

The algebra (123) can also be written as

$$(1 + qN)aa^\dagger - (1 + q(N-1))a^\dagger a = 1. \tag{130}$$

5.5.1. φ-deformed derivative and φ-deformed integral. The φ-deformed derivative is given

$$D_x^\varphi = \frac{1}{x}\left(\frac{x\partial}{1 + q(x\partial - 1)}\right) = \frac{1}{1 + qx\partial}\partial. \tag{131}$$

The φ-deformed integral is given by

$$\int \mathrm{d}_\varphi x F(x) = \int \mathrm{d}x(1 + qx\partial)F(x). \tag{132}$$

The φ-deformed exponential function obeys

$$e_\varphi(x) = (1 - qx)^{-\frac{1}{q}}. \tag{133}$$

5.6. Case VI

Now let us consider the following structure function

$$\varphi(x) = \frac{1}{1+q}x(1 + qx), \tag{134}$$

$$\varphi^{-1}(x) = (1+q)\left(\frac{-1 + \sqrt{1 + 4qx}}{2q}\right). \tag{135}$$

Then φ-addition and φ-subtraction read

$$x \oplus y = \frac{1}{4q}(\sqrt{1+4qx} + \sqrt{1+4qy}) \times (\sqrt{1+4qx} + \sqrt{1+4qy} - 2), \tag{136}$$

$$x \ominus y = \frac{1}{4q}(\sqrt{1+4qx} - \sqrt{1+4qy}) \times (\sqrt{1+4qx} - \sqrt{1+4qy} + 2). \tag{137}$$

From the φ-deformed boson algebra

$$aa^\dagger \ominus a^\dagger a = 1, \tag{138}$$

we have the φ-deformed boson algebra

$$[a, a^\dagger] = 1 + \frac{2q}{1+q}N. \tag{139}$$

When $q > 0$, all states in the Fock space have positive-definite norm for all n, hence we have infinite dimensional Fock space. When $q = -\frac{1}{p+1} < 0$, states have positive-definite norm for $n \leqslant p$ and $\varphi(p+1) = 0$, hence we have finite dimensional Fock space.

5.6.1. φ-deformed derivative and φ-deformed integral. The φ-deformed derivative is given

$$D_x^\varphi = \frac{1}{1+q}\partial(1 + qx\partial) = \left(1 + \frac{q}{1+q}x\partial\right)\partial. \tag{140}$$

The φ-deformed integral is given by

$$\int \mathrm{d}_\varphi x F(x) = \int \mathrm{d}x \frac{1}{1 + \frac{q}{1+q}x\partial}F(x). \tag{141}$$

The φ-deformed exponential function obeys

$$\partial e_\varphi(x) + \frac{q}{1+q}x\partial^2 e_\varphi(x) = e_\varphi(x). \tag{142}$$

5.7. Case VII

Now let us consider the following structure function

$$\varphi(x) = \frac{\tanh(qx)}{\tanh q}, \tag{143}$$

$$\varphi^{-1}(x) = \frac{1}{q}\tanh^{-1}(x\tanh q), \quad q \in \mathbb{R}. \tag{144}$$

The φ-addition and φ-subtraction read

$$x \oplus y = \frac{1}{q}\tanh(\tanh^{-1}qx + \tanh^{-1}qy), \tag{145}$$

$$x \ominus y = \frac{1}{q}\tanh(\tanh^{-1}qx - \tanh^{-1}qy). \tag{146}$$

From the φ-deformed boson algebra

$$aa^\dagger \ominus a^\dagger a = 1, \tag{147}$$

we have the φ-deformed boson algebra

$$[a, a^\dagger] = \frac{\cosh q}{\cosh q(N+1)\cosh qN}. \tag{148}$$

This gives the infinite dimensional Fock space.

5.7.1. φ-deformed derivative and φ-deformed integral. The φ-deformed derivative is given

$$D_x^\varphi = \frac{1}{x\tanh q}\tanh(qx\partial) = \frac{1}{x\tanh q}\left(\frac{T_e^q - T_e^{-q}}{T_e^q + T_e^{-q}}\right). \tag{149}$$

The φ-deformed integral is given by

$$\int d_\varphi x F(x) = \tanh q \int dx \frac{1}{\tanh(q + qx\partial)}(1 + x\partial)F(x)$$
$$= (\tanh q)xF(x) + 2\tanh q \sum_{n=0}^{\infty}(-1)^n e^{-2qn} xF(e^{-2qn}x). \tag{150}$$

The φ-deformed exponential function obeys

$$\tanh(qx\partial)e_\varphi(x) = (\tanh q)xe_\varphi(x). \tag{151}$$

6. φ-deformed boson algebra as a Hopf algebra

In this section we investigate the Hopf algebraic structure for seven types of φ-deformed boson algebras given in the previous section. For a given associative algebra A with unit, we call A a Hopf algebra if we can define three operations in A: the co-multiplication Δ, antipod S and co-unit ϵ

$$\begin{aligned} &\Delta\colon A \to A \times A, \Delta(ab) = \Delta(a)\Delta(b)\\ &S\colon A \to A, \qquad S(ab) = S(b)S(a)\\ &\epsilon\colon A \to \mathbb{C}, \qquad \epsilon(ab) = \epsilon(a)\epsilon(b), \end{aligned} \tag{152}$$

where $a, b \in A$ and $\mathbb{C}$ is the field of complex numbers, and $\times$ denotes tensor product. The operations must be consistent, i.e.

$$\begin{aligned} (id \times \Delta)\Delta(a) &= (\Delta \times id)\Delta(a)\\ m(id \times S)\Delta(a) &= m(S \times id)\Delta(a)\\ (\epsilon \times id)\Delta(a) &= (id \times \epsilon)\Delta(a). \end{aligned} \tag{153}$$

For φ-deformed boson algebra, the co-multiplication Δ should obey

$$\Delta(aa^\dagger \ominus a^\dagger a) = I \times I. \tag{154}$$

It is well known [10] that φ-deformed boson algebras for cases I, II, III, IV are Hopf algebra. For case VI the φ-deformed boson algebra is Hopf algebra with co-multiplication

$$\begin{aligned} \Delta(N) &= N \times I + I \times N + \left(\frac{1+q}{2q}\right)I \times I\\ \Delta(a) &= a \times I + I \times a\\ \Delta(a^\dagger) &= a^\dagger \times I + I \times a^\dagger \end{aligned} \tag{155}$$

and antipod

$$S(N) = -N - \frac{1+q}{2q}, \quad S(a) = -a, \quad S(a^\dagger) = -a^\dagger \tag{156}$$

and co-unit

$$\epsilon(N) = -\frac{1+q}{2q}, \quad \epsilon(a) = \epsilon(a^\dagger) = 0. \tag{157}$$

For the cases V and VII, we cannot find co-multiplication Δ, antipod S and co-unit ϵ obeying the equations (152) and (153). Thus, neither case V nor VII is Hopf algebra.

7. φ-deformed *su*(2) algebra (*su*$_\varphi$(2) algebra)

In this section we discuss the multi mode φ-deformed boson algebra and φ-deformed $su(2)$ algebra. Some studies on the deformed $su(2)$ algebra are given in [24–30].

Definition 7.1. The multi mode φ-deformed boson algebra is defined as follows:

$$\begin{aligned} &[a_i, a_j^\dagger] = \delta_{ij}(\varphi(N_i + 1) - \varphi(N_i)),\\ &[N_i, a_j] = -\delta_{ij}a_j, [N_i, a_j^\dagger] = \delta_{ij}a_j^\dagger, \end{aligned} \tag{158}$$

$$a_i^\dagger a_i = \varphi(N_i), \quad a_i a_i^\dagger = \varphi(N_i + 1). \tag{159}$$

In order to obtain $su_\varphi(2)$ algebra, we use a two mode realization called Jordan–Schwinger realization.

Definition 7.2. The Jordan–Schwinger realization is given by

$$\begin{aligned} &J_+ = a_1^\dagger a_2, \quad J_- = a_2^\dagger a_1, \quad J_0 = \frac{1}{2}(N_1 - N_2),\\ &C = \frac{1}{2}(N_1 + N_2), \end{aligned} \tag{160}$$

where

$$[C, J_i] = 0, \quad for \;\; i = \pm, 0. \tag{161}$$

Then, $su_\varphi(2)$ algebra reads

$$[J_0, J_\pm] = \pm J_\pm, \tag{162}$$

$$[J_+, J_-] = \Phi(J_0, C), \tag{163}$$

where

$$\begin{aligned} \Phi(J_0, C) = &\varphi(C - J_0 + 1)\varphi(C + J_0)\\ &-\varphi(C - J_0)\varphi(C + J_0 + 1). \end{aligned} \tag{164}$$

For $su_\varphi(2)$ algebra we have the following representation:

$$\begin{aligned} J_0|n_1, n_2\rangle &= \frac{1}{2}(n_1 - n_2)|n_1, n_2\rangle,\\ C|n_1, n_2\rangle &= \frac{1}{2}(n_1 + n_2)|n_1, n_2\rangle,\\ J_-|n_1, n_2\rangle &= \sqrt{\varphi(n_1)\varphi(n_2 + 1)}\,|n_1 - 1, n_2 + 1\rangle\\ J_+|n_1, n_2\rangle &= \sqrt{\varphi(n_1 + 1)\varphi(n_2)}\,|n_1 + 1, n_2 - 1\rangle. \end{aligned} \tag{165}$$

By introducing

$$n_1 = j + m, \quad n_2 = j - m \tag{166}$$

and

$$|j, m\rangle \coloneqq |n_1, n_2\rangle, \tag{167}$$

we have the following representation :

$$\begin{aligned} &J_0|j, m\rangle = m|j, m\rangle, \quad C|j, m\rangle = j|j, m\rangle\\ &J_-|j, m\rangle = \sqrt{\varphi(j + m)\varphi(j - m + 1)}\,|j, m - 1\rangle\\ &J_+|j, m\rangle = \sqrt{\varphi(j + m + 1)\varphi(j - m)}\,|j, m + 1\rangle. \end{aligned} \tag{168}$$

Because $J_+|j, j\rangle = J_-|j, -j\rangle = 0$, the representation is bounded below and above and the possible value of m is given by

$$m = -j, -j+1, -j+2, \cdots, j-2, j-1, j, \qquad (169)$$

where $j = 0, 1/2, 1, 3/2, 2, \cdots$. So we have the finite dimensional Fock space. Applying J_+ to the lowest state $|j, -j\rangle$ $(2j+1)$ times, we have

$$(J_+)^{2j+1}|j, -j\rangle = 0, \qquad (170)$$

and applying J_- to the highest state $|j, j\rangle$ $(2j+1)$ times, we have

$$(J_-^{\dagger})^{2j+1}|j, j\rangle = 0. \qquad (171)$$

For seven cases we have the following $su_\varphi(2)$ algebras.

Case I: $[J_+, J_-] = 2J_0$.

Case II: $[J_+, J_-] = q^{C-J_0}\varphi(2J_0)$.

Case III: $[J_+, J_-] = \varphi(2J_0)$.

Case IV: $[J_+, J_-] = q^{2C-1}(2J_0)$.

Case V: $[J_+, J_-] = \frac{(1-q)(1+2qC)(2J_0)}{(1-\mu(1+J_0-C))(1-q(J_0-C))(1-q(1-J_0-C))(1+q(J_0+C))}$.

Case VI : $[J_+, J_-] = \frac{2}{(1+q)^2}J_0[1 + q(1+2C) + 2q^2(C(C+1) - J_0^2)]$.

Case VII: $[J_+, J_-] = \frac{1}{(\tanh q)^2}(\tanh q(1+C-J_0)\tanh q(C+J_0) - \tanh q(C-J_0)\tanh q(1+C+J_0))$.

It is well known [31–34] that φ-deformed boson algebras for cases I, II, III, IV are Hopf algebra. But, the cases V, VI and VII do not give Hopf algebra structure.

8. φ-coherent states and non-classical properties

In quantum mechanics, the minimum possible product of uncertainties is characteristic of the coherent states, one of whose definitions includes the annihilation operators for the oscillator algebra. Now we will investigate the φ-deformed coherent state for φ-boson algebra in a similar way.

8.1. Infinite Fock space

In this case the φ-deformed coherent state is defined as a eigenvector of the annihilation operator as follows:

$$a|z\rangle = z|z\rangle, \qquad (172)$$

where z is a complex number. The normalized φ-deformed coherent states are then

$$|z\rangle = \frac{1}{\sqrt{e_\varphi(x)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\varphi(n)!}}|n\rangle, \quad x = |z|^2. \qquad (173)$$

Now let us show that the coherent state $|z\rangle$ forms a complete set of states. To establish this, we invoke the completeness relation;

$$\int |z\rangle\,\mu(x)\,\langle z|\mathrm{d}_\varphi x = \frac{1}{\pi}I, \qquad (174)$$

where $\mu(x)$ is a weight function. Inserting the equation (141) into the equation (142), we obtain

$$\sum_{n=0}^{\infty}\frac{1}{\varphi(n)!}|n\rangle\langle n|\int_0^{\infty}\frac{\mu(x)}{e_\varphi(x)}x^n\mathrm{d}_\varphi x = \frac{1}{\pi}I, \qquad (175)$$

which is satisfied if

$$\mu(x) = \frac{1}{\pi}e_\varphi(x)e_\varphi(-x) \qquad (176)$$

and

$$\int_0^{\infty}\mathrm{d}_\varphi x e_\varphi(-x)x^n = \varphi(n)!. \qquad (177)$$

The relation (145) is satisfied when $\varphi(-x) = -\varphi(x)$ which corresponds to cases I, III, VII. Now we will prove this simply. From the relation

$$\int_0^{\infty}\mathrm{d}_\varphi x e_\varphi(-ax) = a^{-1}, \quad a > 0, \qquad (178)$$

we φ-differentiate it n times with respect to a to obtain

$$\int_0^{\infty}\mathrm{d}_\varphi x e_\varphi(-ax)(-x)^n = a^{-n-1}\prod_{k=1}^{n}\varphi(-k). \qquad (179)$$

Inserting $a = 1$ we get the equation (145).

8.1.1. Mandel parameter. Now let us discuss non-classical properties of φ-deformed coherent states. We deal with super-/sub-Poissonian structure for the φ-deformed coherent states. Commonly, photon-counting statistics of the φ-deformed coherent states can be investigated by evaluating the Mandel parameter Q. The φ-deformed coherent states for which $Q = 0$, $Q < 0$ and $Q > 0$, respectively correspond to Poissonian, sub-Poissonian (non-classical) and super-Poissonian state. The Mandel parameter is defined as

$$Q = \frac{\langle N(N-1)\rangle}{\langle N\rangle} - \langle N\rangle, \qquad (180)$$

where

$$\langle A\rangle = \langle z|A|z\rangle. \qquad (181)$$

Now let us the φ-deformed coherent states with small deformation and small x for seven cases. For cases II, III and IV, let us set $q = 1 + \epsilon$ where ϵ is small. For cases V, VI and VII let us consider that q is small.

8.1.2. Case I. In this case we have $Q = 0$. Hence φ-deformed coherent states is Poissonian.

8.1.3. Case II. In this case we have

$$\varphi(n) \approx n\left(1 + \frac{1}{2}(n-1)\epsilon\right). \qquad (182)$$

For small ϵ and small x, we have

$$Q \approx -\frac{1}{2}\epsilon x. \qquad (183)$$

For small x and small ϵ, the φ-deformed coherent states is sub-Poissonian when $\epsilon > 0$ while it is super-Poissonian when $\epsilon < 0$.

8.1.4. Case III. In this case we have

$$\varphi(n) \approx n\left(1 + \frac{1}{6}(n^2 - 1)\epsilon^2\right). \tag{184}$$

For small ϵ and small x, we have

$$Q \approx -\frac{1}{2}\epsilon^2 x. \tag{185}$$

For small x and small ϵ, the φ-deformed coherent states is sub-Poissonian for all non-zero ϵ.

8.1.5. Case IV. In this case we have

$$\varphi(n) \approx n(1 + (n - 1)\epsilon). \tag{186}$$

For small ϵ and small x, we have

$$Q \approx -\epsilon x. \tag{187}$$

For small x and small ϵ, the φ-deformed coherent states is sub-Poissonian when $\epsilon > 0$ while it is super-Poissonian when $\epsilon < 0$.

8.1.6. Case V. In this case we have

$$\varphi(n) \approx n(1 - (n - 1)q). \tag{188}$$

For small q and small x, we have

$$Q \approx qx. \tag{189}$$

For small x and small q, the φ-deformed coherent states is sub-Poissonian when $q < 0$ while it is super-Poissonian when $q > 0$.

8.1.7. Case VI. In this case we have

$$\varphi(n) \approx n(1 + (n - 1)q). \tag{190}$$

For small q and small x, we have

$$Q \approx -qx. \tag{191}$$

For small x and small q, the φ-deformed coherent states is sub-Poissonian when $q > 0$ while it is super-Poissonian when $q < 0$.

8.1.8. Case VII. In this case we have

$$\varphi(n) \approx n\left(1 - \frac{1}{3}(n^2 - 1)q^2\right). \tag{192}$$

For small q and small x, we have

$$Q \approx q^2 x. \tag{193}$$

For small x and small q, the φ-deformed coherent states is super-Poissonian for all non-zero q.

8.1.9. Bunching or anti-bunching effect. To investigate bunching or anti-bunching effects, second-order correlation function, defined as

$$g^{(2)}(0) = \frac{\langle (a^\dagger)^2 a^2 \rangle}{\langle a^\dagger a \rangle^2}, \tag{194}$$

where $g^{(2)}(0) > 1$ and $g^{(2,M)}(0) < 1$ respectively indicates bunching and anti-bunching effects. The case $g^{(2)}(0) = 1$ corresponds particularly to the canonical coherent states. For seven cases, we have $g^{(2)}(0) = 1$, hence there is neither bunching effect nor anti-bunching effect.

8.2. Finite Fock space

The finite Fock space can be obtained for case III with $q = \mathrm{e}^{\frac{\pi \mathrm{i}}{p+1}}$ and case VI with $q = -\frac{1}{p+1}$ where $p = 1, 2, 3, \cdots$. In this case we have $(p + 1)$-dimensional Fock space.

In $(p + 1)$-dimensional Fock space, the normalized φ-deformed coherent state is given by

$$|z, p\rangle = c_p \sum_{n=0}^{p} \frac{z^n}{\sqrt{\varphi(n)!}} |n\rangle, \tag{195}$$

where the normalization is

$$c_p = [e_{\varphi,p}(x)]^{-1/2}, \quad x = |z|^2 \geqslant 0 \tag{196}$$

and

$$e_{\varphi,p}(x) = \sum_{n=0}^{p} \frac{x^n}{\varphi(n)!}. \tag{197}$$

The φ-deformed coherent state then obeys

$$a|z, p\rangle = z\frac{c_p}{c_{p-1}}|z, p - 1\rangle. \tag{198}$$

Now let us show that the coherent state $|z, p\rangle$ forms a complete set of states. To establish this, we invoke the completeness relation;

$$\frac{1}{2}\int\int |z, p\rangle \mu_p(x) \langle z, p| \mathrm{d}x\mathrm{d}\theta = I, \tag{199}$$

where $z = |z|\mathrm{e}^{\mathrm{i}\theta}$ and $\mu_p(x)$ is a weight function. Inserting the equation (195) into the equation (199), we obtain

$$\int_0^\infty [e_p(x)]^{-1} \mu_p(x) x^n \mathrm{d}x = \frac{1}{\pi}\varphi(n)!. \tag{200}$$

Now let us set

$$\mu_p(x) = \frac{1}{\pi} e_p(x) e^{-x} \left(\sum_{k=0}^{p} a_k x^k\right) \tag{201}$$

hence we get

$$\sum_{k=0}^{p} (n + k)! a_k = \varphi(n)!. \tag{202}$$

The coefficients a_k's can be determined from inversion of matrix. For the first few p's we have the following.

Case of $p = 1$: In this case the equation (202) gives

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{203}$$

which gives $a_0 = 1$, $a_1 = 0$. Thus the weighting function is

$$\mu_1(x) = \frac{1}{\pi} e_{\varphi,1}(x) e^{-x} \tag{204}$$

Case of $p = 2$: In this case the equation (202) gives

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 6 \\ 2 & 6 & 24 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{205}$$

which gives the weighting function

$$\mu_2(x) = \frac{1}{\pi} e_{\varphi,2}(x) e^{-x} \left(\frac{\varphi(2)}{2} + (2 - \varphi(2))x + \left(-\frac{1}{2} + \frac{\varphi(2)}{4} \right) x^2 \right). \tag{206}$$

Case of $p = 3$: In this case the equation (202) gives

$$\begin{pmatrix} 1 & 1 & 2 & 6 \\ 1 & 2 & 6 & 24 \\ 2 & 6 & 24 & 120 \\ 6 & 24 & 120 & 720 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \varphi(2) \\ \varphi(2)\varphi(3) \end{pmatrix}, \tag{207}$$

which gives the weighting function

$$\mu_3(x) = \frac{1}{\pi} e_{\varphi,3}(x) e^{-x} \left[\left(-2 + 2\varphi(2) - \frac{1}{6}\varphi(2)\varphi(3) \right) + \left(8 - \frac{11}{2}\varphi(2) + \frac{1}{1}\varphi(2)\varphi(3) \right) x + \left(-\frac{7}{2} + \frac{5}{2}\varphi(2) - \frac{1}{4}\varphi(2)\varphi(3) \right) x^2 + \left(\frac{1}{3} - \frac{1}{4}\varphi(2) + \frac{1}{36}\varphi(2)\varphi(3) \right) x^3 \right]. \tag{208}$$

8.2.1. Mandel parameter. The Mandel parameter depends on p and is given by

$$Q_p = \frac{(x\partial_x)^2 e_{\varphi,p}(x)}{x\partial_x e_{\varphi,p}(x)} - x\partial_x \ln e_{\varphi,p}(x). \tag{209}$$

For a small x, Q_p behaves like Q_1 while for a large x, Q_p approaches zero.

8.2.2. Bunching or anti-bunching effect. In this case second-order correlation function is

$$g_p^{(2)}(0) = \frac{e_{\varphi,p}(x) e_{\varphi,p-2}(x)}{e_{\varphi,p-1}^2(x)}. \tag{210}$$

8.2.3. Case III with $q = e^{\frac{\pi i}{p+1}}$. In this case structure function is

$$\varphi(n) = \frac{\sin \frac{\pi n}{p+1}}{\sin \frac{\pi}{p+1}} \tag{211}$$

and φ-exponential function is

$$e_{\varphi,p}(x) = \sum_{n=0}^{p} \frac{\left(x \sin \frac{\pi}{p+1} \right)^n}{\prod_{k=1}^{n} \sin \frac{\pi k}{p+1}}. \tag{212}$$

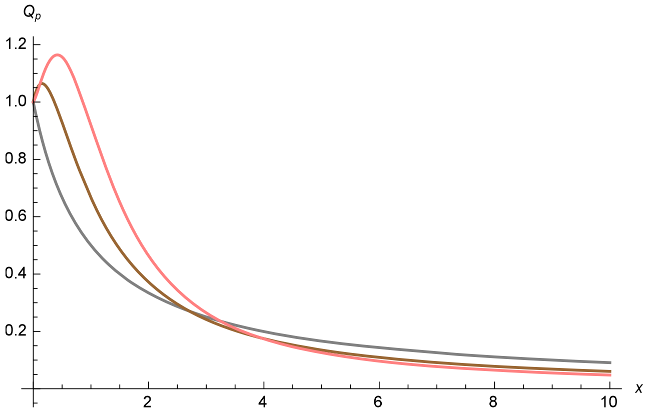

Figure 1. Plot of plot of Q_p versus x for $p = 1$ (Gray), $p = 2$ (Brown) and $p = 3$ (Pink).

For the first few p's, φ-exponential functions are

$$e_1(x) = 1 + x, \tag{213}$$

$$e_2(x) = 1 + x + x^2, \tag{214}$$

$$e_3(x) = 1 + x + \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{2}}. \tag{215}$$

Mandel parameters for $p = 1, 2, 3$ are

$$Q_1 = \frac{1}{1+x} > 0, \tag{216}$$

$$Q_2 = \frac{1 + 4x + x^2}{(1+2x)(1+x+x^2)} > 0, \tag{217}$$

$$Q_3 = \frac{2(2 + 4\sqrt{2}x + 10\sqrt{2}x^2 + 4\sqrt{2}x^3 + x^4)}{(1+x)(2+\sqrt{2}x^2)(2 + 2\sqrt{2}x + 3\sqrt{2}x^2)} > 0, \tag{218}$$

which shows super-Poissonian distribution. Figure 1 shows plot of Q_p versus x for $p = 1$ (Gray), $p = 2$ (Brown) and $p = 3$ (Pink).

The second-order correlation function for $p = 1, 2, 3$ are

$$g_1^{(2)}(0) = 0 < 1, \tag{219}$$

$$g_2^{(2)}(0) = \frac{1 + x + x^2}{(1+x)^2} < 1, \tag{220}$$

$$g_3^{(2)}(0) = \frac{(1+x)\left(1 + x + \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{2}}\right)}{(1+x+x^2)^2} < 1, \tag{221}$$

which shows anti-bunching effect.

8.2.4. Case VI with $q = -\frac{1}{p+1}$. In this case structure function is

$$\varphi(n) = \frac{n}{p}(p + 1 - n) \tag{222}$$

and φ-exponential function is

$$e_{\varphi,p}(x) = {}_0F_1^{(p)}(;-p;-px), \tag{223}$$

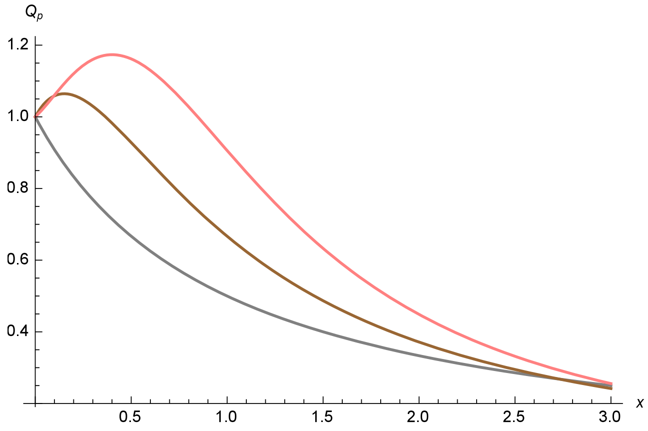

Figure 2. Plot of plot of Q_p versus x for $p = 1$ (Gray), $p = 2$ (Brown) and $p = 3$ (Pink).

where

$$ {}_0F_1^{(p)}(;a;\, t) = \sum_{n=0}^{p} \frac{1}{n!(a)_n} t^n. \tag{224} $$

For the first few p's, φ-exponential functions are

$$ e_1(x) = 1 + x, \tag{225} $$

$$ e_2(x) = 1 + x + x^2, \tag{226} $$

$$ e_3(x) = 1 + x + \frac{3x^2}{4} + \frac{3x^3}{4}. \tag{227} $$

Mandel parameters for $p = 1, 2, 3$ are

$$ Q_1 = \frac{1}{1 + x} > 0, \tag{228} $$

$$ Q_2 = \frac{1 + 4x + x^2}{(1 + 2x)(1 + x + x^2)} > 0, \tag{229} $$

$$ Q_3 = \frac{16 + 48x + 120x^2 + 48x^3 + 9x^4)}{(4 + 6x + 9x^2)(4 + 4x + 3x^2 + 3x^3)} > 0, \tag{230} $$

which shows super-Poissonian distribution. Figure 2 shows plot of Q_p versus x for $p = 1$ (Gray), $p = 2$ (Brown) and $p = 3$ (Pink).

The second-order correlation function for $p = 1, 2, 3$ are

$$ g_1^{(2)}(0) = 0 < 1 \tag{231} $$

$$ g_2^{(2)}(0) = \frac{1 + x + x^2}{(1 + x)^2} < 1 \tag{232} $$

$$ g_3^{(2)}(0) = \frac{(1 + x)\left(1 + x + \frac{3x^2}{4} + \frac{3x^3}{4}\right)}{(1 + x + x^2)^2} < 1, \tag{233} $$

which shows anti-bunching effect.

9. Conclusion

In this paper we discussed general deformed boson algebra called φ-deformed boson algebra. We introduced φ-addition and φ-subtraction for invertible structure function φ to derive the φ-deformed boson algebra. We considered two types of φ-deformed boson algebra; one has infinite Fock space and another finite Fock space. For φ-deformed boson algebra with infinite Fock space, we introduced φ-derivative and obtained realization of φ-deformed boson algebra. Besides we introduced φ-exponential function and φ-hyperbolic functions and investigated some of their properties. We discussed multi mode φ-deformed boson algebra and φ-deformed $su(2)$ algebra. We discussed the SQM for φ-boson and fermion and constructed the Hamiltonian. As examples, we considered seven possible structure functions which are invertible. Here cases III and VI have infinite and finite Fock spaces while others have infinite Fock space only. As a physical example we dealt with non-classical properties for the φ-deformed coherent states.

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