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Dynamics analysis and optimal control strategy for a SIRS epidemic model with two discrete time delays

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Abstract

In this paper, a SIRS epidemic model with nonlinear incidence rate, saturated treatment and two time delays is investigated. Firstly, by using the method of regeneration matrix, we have determined the basic regeneration number R_0 and demonstrated the existence of the positive equilibrium point. The permanence of the SIRS epidemic model is obtained by mathematical analysis. Moreover, by selecting time delay as the bifurcation parameter, we discuss the local asymptotic stability of the positive equilibrium point and the existence of Hopf bifurcation for six different situations. Afterwards, to minimize the spread of infectious diseases, we introduce an optimal control technique by the Pontryagin's maximum principle. Finally, we verify the correctness of theoretical analysis through numerical simulations.

Keywords: SIRS epidemic model, stability, hopf bifurcation, delays, optimal control strategy

(Some figures may appear in colour only in the online journal)

1. Introduction

To prevent and control the spread of disease, many mathematicians devote themselves to studying models of various infectious diseases. A lot of mathematical models of infectious diseases have been proposed. It can be found that many factors influence the dynamic behavior of the epidemic model, for example, incidence rate, recovery rate and so on. Control strategy also plays an important role in epidemic models. Thus, it is very necessary to investigate these factors.

There are many studies on incidence rate and treatment function in infectious disease models [1–6]. In [7], Wang *et al* investigated a SIR infectious disease model with constant treatment function and bilinear incidence rate, and presented the evidences of choosing a bilinear incidence rate to prove the existence of Hopf bifurcation. In [8], a SIR epidemic model with saturated treatment function was studied. According to the article [9], Liu introduced the SIRS infectious disease model with nonlinear incidence function, and proved the existence of Hopf bifurcation. Moreover, in order to give a better description of population growth under restricted conditions, Verhulst [10] proposed a logistic equation which was very appropriate to describe changes in the population of one animal species. In fact, the susceptible population in the study of infectious diseases is always considered to be subject to Logistic growth model according to the former literatures [11–17]. For example, Xu [13] gave a simple criterion for the existence of Hopf bifurcation for an SEIR epidemiological model with logistic growth. By introducing the saturated treatment and logistic growth rate into an SIR epidemic model, Teng [15] obtained the conditions of the backward bifurcation. Song [16] and Jin [17], by constructing reaction-diffusion models of infectious diseases with logistic growth, analyzed the pattern dynamics of infectious disease models. Inspired by the above articles, we will investigate a SIRS epidemic model with a saturated incidence rate and a

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saturated treatment function which is in the following

$$\begin{cases} \frac{dS(t)}{dt} = rS(t)\left(1 - \frac{S(t)}{k}\right) - \frac{\beta_1 S(t)I(t)}{1+\alpha_1 I(t)} + \sigma R(t), \\ \frac{dI(t)}{dt} = \frac{\beta_1 S(t)I(t)}{1+\alpha_1 I(t)} - (\mu+\varepsilon)I(t) - \frac{\beta_2 I(t)}{1+\alpha_2 I(t)}, \\ \frac{dR(t)}{dt} = \frac{\beta_2 I(t)}{1+\alpha_2 I(t)} - (\mu+\sigma)R(t), \end{cases} \tag{1.1}$$

where $S(t)$ denotes the density of the susceptible individual, $I(t)$ represents the density of the infected individual and $R(t)$ denotes the density of the recovered individual at time t. We assume that the susceptible individuals follow Logistic growth model. Logistic model describes the natural growth process of species, which is a comprehensive reflection of species' birth and death [15]. Thus, we're no longer looking at the natural mortality of the susceptible individuals alone in system (1.1). β_1, β_2, α_1, α_2, r, k, μ, ε, σ are positive constant. β_1 is the incidence rate between susceptible individual and infected individual, β_2 represents the recovery rate, α_1 and α_2 are the half saturation and the coefficient respectively, r represents the intrinsic growth rate, k represents the carrying capacity for $S(t)$, μ and ε denote natural mortality rate and disease mortality rate respectively, σ is the ratio that the recovered individual becomes susceptible individual. $\frac{\beta_1 S(t)I(t)}{1+\alpha_1 I(t)}$ is more realistic than the bilinear incidence rate which is affected by psychological factors. $\frac{\beta_2 I(t)}{1+\alpha_2 I(t)}$ is a saturated treatment function, which is limited by medical conditions and individual physical qualities.

With the development of infectious disease models, many scholars have paid attention to the time delay infectious disease model. Because it is closer to the reality of the problem. We find that time delay plays an important role in infectious disease model and affects the stability of equilibrium points. When we consider time delay in the system, the stability of equilibrium point may change, such as from stable to unstable [18, 19]. In recent years, more and more scholars increase time delay factor on the epidemic models with nonlinear incidence rate. For example, in [20], a time delay infectious disease model with saturated incidence was studied. In [21], Jin *et al* investigated a time delay epidemic model of vector transmission and its global stability. Xu *et al* [22] discussed the global stability of disease-free equilibrium point and the existence conditions of local epidemic stability. According to the article [23], Deng *et al* investigated a ratio-dependent predator-prey system which exists the stage structure and two time delays. Inspired by the above works, we will establish a new class of time-delay SIRS infectious diseases model based on system (1.1), which is in the following form

$$\begin{cases} \frac{dS(t)}{dt} = rS(t)\left(1 - \frac{S(t)}{k}\right) - \frac{\beta_1 S(t)I(t)}{1+\alpha_1 I(t)} + \sigma R(t), \\ \frac{dI(t)}{dt} = \frac{\beta_1 S(t-\tau_1)I(t-\tau_1)}{1+\alpha_1 I(t-\tau_1)} - (\mu+\varepsilon)I(t) - \frac{\beta_2 I(t)}{1+\alpha_2 I(t)}, \\ \frac{dR(t)}{dt} = \frac{\beta_2 I(t-\tau_2)}{1+\alpha_2 I(t-\tau_2)} - (\mu+\sigma)R(t), \end{cases} \tag{1.2}$$

with the initial conditions

$$S(\theta) > 0,\ I(\theta) > 0,\ R(\theta) > 0, \tag{1.3}$$

where $\theta \in [-\max(\tau_1, \tau_2), 0]$.

Each parameter has the same meaning as the parameters in system (1.1). When the susceptible individual is infected by the infected individual, the symptoms of the disease are not immediately apparent, and we still consider him (or her) to be susceptible. Therefore, there is a time delay defined by τ_1 in the process from the susceptible to the infectious. τ_2 is due to the limited medical level in each region, the recovery process of the disease, and the limit to the ability of each individual to resist disease. A time delay in the process from the infectious to the recovered is existent. In conclusion, it is reasonable to add two time delays in this model.

In order to better prevent and reduce the spread of infectious diseases, more and more scholars pay attention to optimal control [24–26]. In [24], an optimal control model for SIR infectious diseases was discussed. The existence of the control model was proved and an optimization system was obtained. In [27], because of the outbreak of infectious diseases Ebola, Finkenstädt established a realistic stochastic model to study its dynamic behavior and the influence of control behavior on the model. In [28], an optimal control method for a class of problems was proposed and its effectiveness was also illustrated. Inspired by the above articles, we will further establish a control model corresponding to (1.2) in our paper.

The structure of this article is as follows. In section 2, the existence of the positive equilibrium point is proved. In section 3, we prove the persistence of system (1.2). In section 4, by discussing the characteristic equation of the positive equilibrium point, we prove the local asymptotic stability of the positive equilibrium point, and give the sufficient conditions for the existence of Hopf bifurcation [29–31]. In section 5, an optimal control model of SIRS model with time delay is presented and we obtain the optimal control [32–34]. In section 6, we test the validity of some theorems and the effect of some parameters on disease spreading by numerical simulations. Finally, we have made a brief summary of this paper.

2. The existence of a positive equilibrium point

The basic regeneration number is obtained by using the spectral radius of the regeneration matrix, now we write system (1.2) as

$$\frac{dx}{dt} = F(x) - V(x),$$

where $x = (S(t), I(t), R(t))^T$, and

$$F(x) = \begin{pmatrix} 0 \\ \frac{\beta_1 S(t)I(t)}{1+\alpha_1 I(t)} \\ 0 \end{pmatrix}, \tag{2.1}$$

$$V(x) = \begin{pmatrix} -rS(t)\left(1 - \frac{S(t)}{k}\right) + \frac{\beta_1 S(t)I(t)}{1+\alpha_1 I(t)} - \sigma R(t) \\ (\mu+\varepsilon)I(t) + \frac{\beta_2 I(t)}{1+\alpha_2 I(t)} \\ -\frac{\beta_2 I(t)}{1+\alpha_2 I(t)} + (\mu+\sigma)R(t) \end{pmatrix}. \tag{2.2}$$

A simple calculation shows that

$$f = DF(x) = \frac{\beta_1 S}{(1+\alpha_1 I)^2}\Bigg|_{(k,0,0)} = \beta_1 k, \tag{2.3}$$

$$v = DV(x) = \mu + \varepsilon + \frac{\beta_2}{(1+\alpha_2 I)^2}\Bigg|_{(k,0,0)} = \mu + \varepsilon + \beta_2. \tag{2.4}$$

Therefore the basic regeneration number $R_0 = \rho(fv^{-1}) = \frac{\beta_1 k}{\mu+\varepsilon+\beta_2}$.

Lemma 1. *When $R_0 > 1$, system (1.2) has at least one positive equilibrium point, denoted by* $E^*(S^*, I^*, R^*) = \left(\left(\mu+\varepsilon+\frac{\beta_2}{1+\alpha_2 I^*}\right)\frac{1+\alpha_1 I^*}{\beta_1}, I^*, \frac{\beta_2 I^*}{(1+\alpha_2 I^*)(\mu+\sigma)}\right)$.

Proof. Let the three equations of system (1.2) equal zero, we know from the second and third equations of system (1.2) that $R = \frac{\beta_2 I}{(1+\alpha_2 I)(\mu+\sigma)}$, $S = \left(\mu+\varepsilon+\frac{\beta_2}{1+\alpha_2 I}\right)\frac{1+\alpha_1 I}{\beta_1}$, then put them into the first equation of system (1.2), we have

$$\begin{aligned}
&\frac{r}{\beta_1}(1+\alpha_1 I)\left(\mu+\varepsilon+\frac{\beta_2}{1+\alpha_2 I}\right)\\
&\quad\times\left[1-\left(\mu+\varepsilon+\frac{\beta_2}{1+\alpha_2 I}\right)\frac{1+\alpha_1 I}{\beta_1 k}\right]\\
&\quad-\left(\mu+\varepsilon+\frac{\beta_2}{1+\alpha_2 I}\right)I+\frac{\sigma\beta_2 I}{(1+\alpha_2 I)(\mu+\sigma)}\\
&=\frac{r}{\beta_1}[(\mu+\varepsilon)(1+\alpha_2 I)+\beta_2](1+\alpha_1 I)\\
&\quad\times\left[1+\alpha_2 I-((\mu+\varepsilon)(1+\alpha_2 I)+\beta_2)\frac{1+\alpha_1 I}{\beta_1 k}\right]\\
&\quad-[(\mu+\varepsilon)(1+\alpha_2 I)^2+\beta_2(1+\alpha_2 I)]I\\
&\quad+\frac{\sigma\beta_2}{\mu+\sigma}(1+\alpha_2 I)I\\
&=\frac{r}{\beta_1}\alpha_1\alpha_2^2(\mu+\varepsilon)I^3+\frac{r\alpha_2}{\beta_1}[\alpha_1(\mu+\varepsilon+\beta_2)\\
&\quad+(\mu+\varepsilon)(\alpha_1+\alpha_2)]I^2+\frac{r}{\beta_1}[(\mu+\varepsilon+\beta_2)(\alpha_1+\alpha_2)\\
&\quad+\alpha_2(\mu+\varepsilon)]I+\frac{r}{\beta_1}(\mu+\varepsilon+\beta_2)\\
&\quad-\frac{r}{\beta_1^2 k}\alpha_1^2\alpha_2^2(\mu+\varepsilon)^2 I^4\\
&\quad-\frac{r}{\beta_1^2 k}2\alpha_1\alpha_2(\mu+\varepsilon)[\alpha_1(\mu+\varepsilon+\beta_2)+\alpha_2(\mu+\varepsilon)]I^3\\
&-\frac{r}{\beta_1^2 k}[\alpha_1^2(\mu+\varepsilon+\beta_2)^2+4\alpha_1\alpha_2(\mu+\varepsilon+\beta_2)(\mu+\varepsilon)\\
&\quad+\alpha_2^2(\mu+\varepsilon)^2]I^2\\
&\quad-\frac{r}{\beta_1^2 k}2(\mu+\varepsilon+\beta_2)[\alpha_1(\mu+\varepsilon+\beta_2)\\
&+\alpha_2(\mu+\varepsilon)]I-\frac{r}{\beta_1^2 k}(\mu+\varepsilon+\beta_2)^2\\
&\quad-\alpha_2^2(\mu+\varepsilon)I^3-\alpha_2[2(\mu+\varepsilon)+\beta_2]I^2\\
&\quad-(\mu+\varepsilon+\beta_2)I+\frac{\sigma\alpha_2\beta_2}{\mu+\sigma}I^2+\frac{\sigma\beta_2}{\mu+\sigma}I\\
&=\frac{r}{\beta_1^2 k}\alpha_1^2\alpha_2^2(\mu+\varepsilon)^2 I^4\\
&\quad+\left[-\frac{r}{\beta_1}\alpha_1\alpha_2^2(\mu+\varepsilon)+\frac{r}{\beta_1^2 k}2\alpha_1\alpha_2(\mu+\varepsilon)\right.\\
&\quad\left.\times[\alpha_1(\mu+\varepsilon+\beta_2)+\alpha_2(\mu+\varepsilon)]+\alpha_2^2(\mu+\varepsilon)\right]I^3\\
&+\left[-\frac{r\alpha_2}{\beta_1}[\alpha_1(\mu+\varepsilon+\beta_2)+(\mu+\varepsilon)(\alpha_1+\alpha_2)]\right.\\
&\quad+\frac{r}{\beta_1^2 k}[\alpha_1^2(\mu+\varepsilon+\beta_2)^2+4\alpha_1\alpha_2(\mu+\varepsilon+\beta_2)\\
&\quad\times(\mu+\varepsilon)+\alpha_2^2(\mu+\varepsilon)^2]+\alpha_2[2(\mu+\varepsilon)+\beta_2]\\
&\quad\left.-\frac{\sigma\alpha_2\beta_2}{\mu+\sigma}\right]I^2\\
&+\left[-\frac{r}{\beta_1}[(\mu+\varepsilon+\beta_2)(\alpha_1+\alpha_2)+\alpha_2(\mu+\varepsilon)]\right.\\
&\quad+\frac{r}{\beta_1^2 k}2(\mu+\varepsilon+\beta_2)[\alpha_1(\mu+\varepsilon+\beta_2)+\alpha_2(\mu+\varepsilon)]\\
&\left.+\mu+\varepsilon+\beta_2-\frac{\sigma\beta_2}{\mu+\sigma}\right]I\\
&\quad+\frac{r}{\beta_1}(\mu+\varepsilon+\beta_2)\left(\frac{\mu+\varepsilon+\beta_2}{\beta_1 k}-1\right)=0.
\end{aligned}$$

Regard the left half of the above equation as $f(I)$. When $R_0 > 1$, we have $\frac{r}{\beta_1}(\mu+\varepsilon+\beta_2)\left(\frac{\mu+\varepsilon+\beta_2}{\beta_1 k}-1\right) < 0$. That is to say $f(0) < 0$. It's easy to know $\lim_{I\to+\infty} f(I) = +\infty$. Therefore $f(I) = 0$ has at least one positive root I^*. In other words, system (1.2) has at least one positive equilibrium point

$$E^*(S^*, I^*, R^*) = \left(\left(\mu+\varepsilon+\frac{\beta_2}{1+\alpha_2 I^*}\right)\frac{1+\alpha_1 I^*}{\beta_1}, I^*, \frac{\beta_2 I^*}{(1+\alpha_2 I^*)(\mu+\sigma)}\right).$$

3. Permanence

Now we will prove that system (1.2) has the permanence.

Lemma 2. *For any positive solution $(S(t), I(t), R(t))$ of system(1.2) with initial conditions (1.3), there exist positive constants M_1, M_2, M_3 which satisfy:*

$$\lim_{t\to+\infty} sup S(t) \leqslant M_1,\ \lim_{t\to+\infty} sup I(t) \leqslant M_2,\ \lim_{t\to+\infty} sup R(t) \leqslant M_3.$$

Proof. We assume that $(S(t), I(t), R(t))$ is an arbitrary positive solution of system (1.2) with the initial conditions (1.3). We prove lemma 2 by contradiction. Assume that when $t\to+\infty$, we have $I\to+\infty$.

Define

$$V(t) = S(t-\tau_1) + I(t) + R(t). \tag{3.1}$$

Thus, if we take the derivative of V with respect to t along the positive solution, then

$$\begin{aligned}
\frac{dV(t)}{dt} &= \frac{dS(t-\tau_1)}{dt} + \frac{dI(t)}{dt} + \frac{dR(t)}{dt} \\
&= rS(t-\tau_1)\left(1 - \frac{S(t-\tau_1)}{k}\right) + \sigma \times R(t-\tau_1) \\
&\quad - (\mu+\varepsilon)I - \frac{\beta_2 I}{1+\alpha_2 I} \\
&\quad + \frac{\beta_2 I(t-\tau_2)}{1+\alpha_2 I(t-\tau_2)} - (\mu+\sigma)R \\
&\leqslant rS(t-\tau_1)\left(1 - \frac{S(t-\tau_1)}{k}\right) \\
&\quad + \sigma R(t-\tau_1) - bS(t-\tau_1) + bS(t-\tau_1) \\
&\quad - bI - \frac{\beta_2 I}{1+\alpha_2 I} + \frac{\beta_2 I(t-\tau_2)}{1+\alpha_2 I(t-\tau_2)} - bR \\
&= -bV(t) + rS(t-\tau_1)\left(1 + \frac{b}{r} - \frac{S(t-\tau_1)}{k}\right) \\
&\quad + \sigma R(t-\tau_1) \\
&\quad - \frac{\beta_2 I}{1+\alpha_2 I} + \frac{\beta_2 I(t-\tau_2)}{1+\alpha_2 I(t-\tau_2)},
\end{aligned}$$

where $b = \min\{\mu+\varepsilon, \mu+\sigma\}$.

Because

$$\lim_{t\to+\infty} \frac{\beta_2 I}{1+\alpha_2 I} = \lim_{I\to+\infty} \frac{\beta_2 I}{1+\alpha_2 I} = \frac{\beta_2}{\alpha_2}$$

and

$$\lim_{t\to+\infty} \frac{\beta_2 I(t-\tau_2)}{1+\alpha_2 I(t-\tau_2)} = \lim_{I\to+\infty} \frac{\beta_2 I(t-\tau_2)}{1+\alpha_2 I(t-\tau_2)} = \frac{\beta_2}{\alpha_2},$$

therefore $\forall \xi > 0$, $\exists T_0$, when $t > T_0$, we have

$$\frac{\beta_2}{\alpha_2} - \xi < \frac{\beta_2 I}{1+\alpha_2 I} < \frac{\beta_2}{\alpha_2} + \xi$$

and

$$\frac{\beta_2}{\alpha_2} - \xi < \frac{\beta_2 I(t-\tau_2)}{1+\alpha_2 I(t-\tau_2)} < \frac{\beta_2}{\alpha_2} + \xi.$$

And when $t > T_0$, a direct calculation shows that

$$\begin{aligned}
\frac{dV(t)}{dt} &\leqslant -bV(t) + rS(t-\tau_1)\left(\frac{r+b}{r} - \frac{S(t-\tau_1)}{k}\right) \\
&\quad + \sigma R(t-\tau_1) - \left(\frac{\beta_2}{\alpha_2} - \xi\right) \\
&\quad + \left(\frac{\beta_2}{\alpha_2} + \xi\right) \\
&\leqslant -(b-\sigma)V(t) + rS(t-\tau_1)\left(\frac{r+b}{r} - \frac{S(t-\tau_1)}{k}\right) \\
&\quad + 2\xi \\
&\leqslant -(b-\sigma)V(t) + \frac{(b+r)^2 k}{4r} + 2\xi,
\end{aligned}$$

where $b = \min\{\mu+\varepsilon, \mu+\sigma\} > \sigma$. Considering

$$\frac{dV(t)}{dt} = -(b-\sigma)V(t) + \frac{(b+r)^2 k}{4r} + 2\xi,$$

we calculated that

$$V(t) = ce^{-(b-\sigma)t} + \frac{\frac{(b+r)^2 k}{4r} + 2\xi}{b-\sigma}.$$

According to the comparison principle, we have

$$\lim_{t\to+\infty} supV(t) \leqslant \frac{\frac{(b+r)^2 k}{4r} + 2\xi}{b-\sigma}. \tag{3.2}$$

On the other hand, $V(t) = S(t-\tau_1) + I(t) + R(t)$, then take the limit on both sides of this equation, and we can get that

$$\begin{aligned}
\lim_{t\to+\infty} supV(t) &= \lim_{t\to+\infty} supS(t-\tau_1) \\
&\quad + \lim_{t\to+\infty} supI(t) + \lim_{t\to+\infty} supR(t) = +\infty.
\end{aligned}$$

Clearly, it is a contradiction. Therefore, when $t > T_0$, $I(t)$ is bounded. In other words, $\lim_{t\to+\infty} supI(t) \leqslant M_2$, where M_2 is a positive constant.

Thus, when $t > T_0$, we further obtain that

$$\begin{aligned}
\frac{dV(t)}{dt} &\leqslant -(b-\sigma)V(t) + rS(t-\tau_1)\left(\frac{r+b}{r} - \frac{S(t-\tau_1)}{k}\right) \\
&\quad - \frac{\beta_2 I}{1+\alpha_2 I} + \frac{\beta_2 I(t-\tau_2)}{1+\alpha_2 I(t-\tau_2)} \\
&\leqslant -(b-\sigma)V(t) + \frac{(b+r)^2 k}{4r} + \beta_2 M_2.
\end{aligned}$$

According to the comparison principle, we have

$$\lim_{t\to+\infty} supV(t) \leqslant \frac{\frac{(b+r)^2 \times k}{4\times r} + \beta_2 M_2}{b-\sigma} = M,$$

where M is a positive constant.

Therefore, there exist nonnegative constants M_1 and M_3 $(M_1 \leqslant M, M_2 \leqslant M, M_3 \leqslant M)$ such that for any positive solution of system (1.2) with the initial conditions (1.3), the following conditions are always true, namely,

$$\lim_{t\to+\infty} supS(t) \leqslant M_1, \ \lim_{t\to+\infty} supR(t) \leqslant M_3.$$

In conclusion, lemma 2 has been proved.

Lemma 3. *For any positive solution $(S(t), I(t), R(t))$ of system(1.2) with the initial conditions (1.3), there exist positive constants m_1, m_2, m_3 which satisfy :*

$$\lim_{t\to+\infty} infS(t) \geqslant m_1, \ \lim_{t\to+\infty} infI(t) \geqslant m_2, \ \lim_{t\to+\infty} infR(t) \geqslant m_3.$$

Proof. We assume that $(S(t), I(t), R(t))$ is an arbitrary positive solution of system (1.2) with the initial conditions (1.3). Due to $\lim_{t\to+\infty} supI(t) \leqslant M_2$, $\lim_{t\to+\infty} supR(t) \leqslant M_3$, therefore $\forall \xi > 0$, $\exists T_0$, and when $t > T_0$, we have

$$M_2 - \xi < I < M_2 + \xi, M_3 - \xi < R < M_3 + \xi,$$

and then

$$\begin{aligned}\frac{dS}{dt} &\geqslant rS\left(1-\frac{S}{k}\right)-\frac{\beta_1(M_2+\xi)S}{1+\alpha_1(M_2-\xi)}+\sigma(M_3-\xi)\\ &\geqslant S\left(r-\frac{rS}{k}\right)-\frac{\beta_1(M_2+\xi)S}{1+\alpha_1(M_2-\xi)}\\ &=S\left(-\frac{rS}{k}+r-\frac{\beta_1(M_2+\xi)}{1+\alpha_1(M_2-\xi)}\right).\end{aligned}$$

We calculated that

$$\lim_{t\to+\infty} infS(t)\geqslant\frac{k}{r}\left(r-\frac{\beta_1(M_2+\xi)}{1+\alpha_1(M_2-\xi)}\right). \tag{3.3}$$

Denote

$$m_1=\max\left\{\frac{k}{r}\left(r-\frac{\beta_1(M_2+\xi)}{1+\alpha_1(M_2-\xi)}\right),0\right\},$$

where m_1 is a nonnegative constant.

Hence, we have

$$\lim_{t\to+\infty} infS(t)\geqslant m_1. \tag{3.4}$$

By $\frac{dR}{dt}=\frac{\beta_2 I(t-\tau_2)}{1+\alpha_2 I(t-\tau_2)}-(\mu+\sigma)R$, when $t>T_0$, we can get

$$\frac{dR}{dt}\geqslant-(\mu+\sigma)R+\frac{\beta_2(M_2-\xi)}{1+\alpha_2(M_2+\xi)}. \tag{3.5}$$

According to the comparison principle, we can obtain that

$$\lim_{t\to+\infty} infR(t)\geqslant\frac{\beta_2(M_2-\xi)}{[1+\alpha_2(M_2+\xi)](\mu+\sigma)}. \tag{3.6}$$

Denote

$$m_3=\frac{\beta_2(M_2-\xi)}{[1+\alpha_2(M_2+\xi)](\mu+\sigma)},$$

where m_3 is a positive constant. Therefore, we have

$$\lim_{t\to+\infty} infR(t)\geqslant m_3. \tag{3.7}$$

By $\frac{dI}{dt}=\frac{\beta_1 S(t-\tau_1)I(t-\tau_1)}{1+\alpha_1 I(t-\tau_1)}-(\mu+\varepsilon)I-\frac{\beta_2 I}{1+\alpha_2 I}$, when $t>T_0$, we can get

$$\begin{aligned}\frac{dI}{dt}\geqslant&-\left(\mu+\varepsilon+\frac{\beta_2}{1+\alpha_2(M_2-\xi)}\right)I\\ &+\frac{\beta_1(M_1-\xi)(M_2-\xi)}{1+\alpha_2(M_2+\xi)}.\end{aligned} \tag{3.8}$$

Like the one above, we obtain

$$\begin{aligned}\lim_{t\to+\infty} infI(t)\geqslant&\frac{\beta_1(M_1-\xi)(M_2-\xi)}{1+\alpha_2(M_2+\xi)}\\ &\times\left(\mu+\varepsilon+\frac{\beta_2}{1+\alpha_2(M_2-\xi)}\right)^{-1}.\end{aligned} \tag{3.9}$$

Denote

$$m_2=\frac{\beta_1(M_1-\xi)(M_2-\xi)}{1+\alpha_2(M_2+\xi)}\left(\mu+\varepsilon+\frac{\beta_2}{1+\alpha_2(M_2-\xi)}\right)^{-1},$$

where m_2 is a positive constant. Hence, we have

$$\lim_{t\to+\infty} infI(t)\geqslant m_2. \tag{3.10}$$

In conclusion, lemma 3 is proved. Therefore, we conclude that system (1.2) has permanence.

4. Local stability and Hopf bifurcations

In epidemic models, Hopf bifurcation occurs when the complex conjugate set of eigenvalues of a linear system becomes a pure imaginary root at a fixed point. A point may change from stable to unstable when there exists a Hopf bifurcation. In this section, we study the local asymptotic stability Hopf bifurcation of system (1.2) at the positive equilibrium point E^*.

We linearized system (1.2) at the positive equilibrium point E^*, the result is as follows

$$\begin{cases}\frac{dS(t)}{dt}=a_{11}S(t)+a_{12}I(t)+a_{13}R(t),\\ \frac{dI(t)}{dt}=b_{21}S(t-\tau_1)+a_{22}I(t)+b_{22}I(t-\tau_1),\\ \frac{dR(t)}{dt}=b_{31}I(t-\tau_2)+a_{33}R(t),\end{cases} \tag{4.1}$$

where

$$\begin{aligned}&a_{11}=r-\frac{2rS^*}{k}-\frac{\beta_1 I^*}{1+\alpha_1 I^*},\ a_{12}=-\frac{\beta_1 S^*}{(1+\alpha_1 I^*)^2},\\ &a_{13}=\sigma,\ a_{22}=-(\mu+\varepsilon)-\frac{\beta_2}{(1+\alpha_2 I^*)^2},\\ &a_{33}=-\mu-\sigma,\ b_{21}=\frac{\beta_1 I^*}{1+\alpha_1 I^*},\\ &b_{22}=\frac{\beta_1 S^*}{(1+\alpha_1 I^*)^2},\ b_{31}=\frac{\beta_2}{(1+\alpha_2 I^*)^2}.\end{aligned}$$

The characteristic matrix of the linearized system (4.1) is

$$A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ b_{21}e^{-\lambda\tau_1} & a_{22}+b_{22}e^{-\lambda\tau_1} & 0\\ 0 & b_{31}e^{-\lambda\tau_2} & a_{33}\end{pmatrix}. \tag{4.2}$$

Thus, the characteristic equation of the linearized system (4.1) is

$$\begin{aligned}&\lambda^3+m_1\lambda^2+m_2\lambda+m_3+(n_1+n_2\lambda+n_3\lambda^2)e^{-\lambda\tau_1}\\ &\quad+qe^{-\lambda(\tau_1+\tau_2)}=0,\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}&m_1=-a_{11}-a_{22}-a_{33},\ m_2=a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33},\\ &m_3=-a_{11}a_{22}a_{33},\ n_1=a_{12}a_{33}b_{21}-a_{11}b_{22}a_{33},\\ &n_2=b_{22}a_{33}+a_{11}b_{22}-a_{12}b_{21},\ n_3=-b_{22},\ q=-a_{13}b_{21}b_{31}.\end{aligned}$$

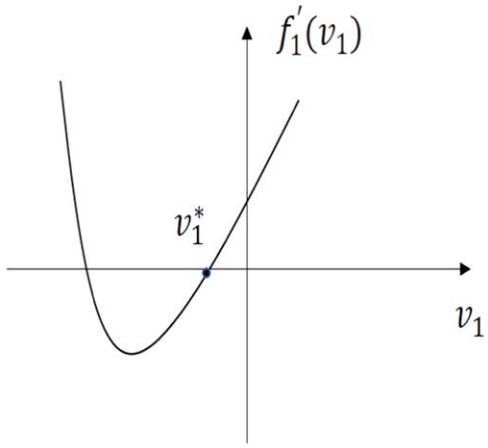


Figure 1. $\Delta > 0$, $v_1^* < 0$.

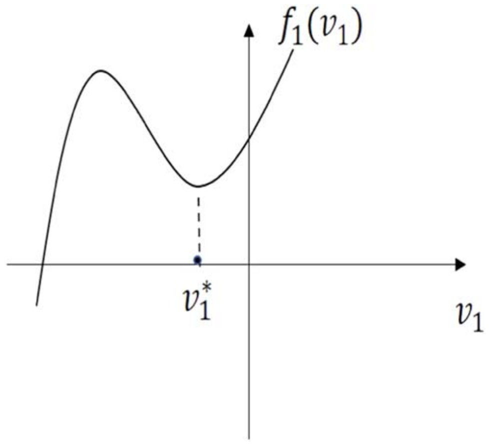


Figure 2. $m_{23} > 0$.

Case 1: When both delays are zero, i.e. $\tau_1 = 0$, $\tau_2 = 0$, the characteristic equation (4.3) becomes

$$\lambda^3 + m_{12}\lambda^2 + m_{11}\lambda + m_{10} = 0, \tag{4.4}$$

where

$$m_{12} = m_1 + n_3,\ m_{11} = m_2 + n_2,\ m_{10} = m_3 + n_1 + q.$$

According to the Huiwitz criterion, if the following condition holds: (H_{11}): $m_{12} > 0$, $m_{12}m_{11} - m_{10} > 0$, $m_{10} > 0$, then equation (4.4) has negative real roots. Thus, system (1.2) is locally asymptotically stable at the positive equilibrium point E^*.

Case 2: When $\tau_1 > 0$, $\tau_2 = 0$, equation (4.3) becomes

$$\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3 + (q + n_1 + n_2\lambda + n_3\lambda^2)e^{-\lambda\tau_1} = 0. \tag{4.5}$$

We assume $\lambda = i\omega_1(\omega_1 > 0)$ is a solution of equation (4.5), then put it in equation (4.5), separating real and imaginary parts we obtain

$$\begin{cases} -m_1\omega_1^2 + m_3 + (n_1 + q - n_3\omega_1^2)\cos(\omega_1\tau_1) + n_2\omega_1\sin(\omega_1\tau_1) = 0,\\ -\omega_1^3 + m_2\omega_1 + n_2\omega_1\cos(\omega_1\tau_1) - (n_1 + q - n_3\omega_1^2)\sin(\omega_1\tau_1) = 0. \end{cases} \tag{4.6}$$

Squaring and adding the two equations of (4.6), we can get that

$$\omega_1^6 + m_{21}\omega_1^4 + m_{22}\omega_1^2 + m_{23} = 0, \tag{4.7}$$

where

$$m_{21} = m_1^2 - 2m_2 - n_3^2,\ m_{22} = -2m_1m_3 + m_2^2 + 2(n_1 + q)n_3 - n_2^2,\ m_{23} = m_3^2 - (n_1 + q)^2.$$

Let $v_1 = \omega_1^2$, then we have

$$v_1^3 + m_{21}v_1^2 + m_{22}v_1 + m_{23} = 0. \tag{4.8}$$

Denote

$$f_1(v_1) = v_1^3 + m_{21}v_1^2 + m_{22}v_1 + m_{23}, \tag{4.9}$$

we calculate that

$$f_1'(v_1) = 3v_1^2 + 2m_{21}v_1 + m_{22}. \tag{4.10}$$

Then we have the following results.

Lemma 4. *(1) If* $(H_{21}) : m_{23} \geqslant 0,\ \Delta = 4(m_{21}^2 - 3m_{22}) \leqslant 0$, *or* $(H_{22}) : m_{23} > 0,\ \Delta > 0,\ v_1^* = \frac{-m_{21} + \sqrt{m_{21}^2 - 3m_{22}}}{3} < 0$, *or* $(H_{23}) : m_{23} \geqslant 0,\ \Delta > 0,\ v_1^* > 0, f_1(v_1^*) > 0$ *hold, then* $f_1(v_1)$ *has no positive real roots.*

(2) If $(H_{24}) : m_{23} \geqslant 0,\ \Delta > 0,\ v_1^* > 0, f_1(v_1^*) < 0$, *or* $(H_{25}) : m_{23} < 0$ *hold, then* $f_1(v_1)$ *has at least one positive real root.*

Proof. By the properties of quadratic functions, when (H_{21}) holds, it's obvious that $f_1(v_1)$ is increasing on the positive half axis.

As shown in figures 1 and 2, when (H_{22}) is satisfied, $f_1'(v_1)$ has no positive root, and $f_1(v_1) > 0$ on the positive half axis is always true.

As shown in figures 3 and 4, when (H_{23}) is satisfied, $f_1'(v_1)$ has at least a positive root v_1^*, where v_1^* is the larger root, and $f_1(v_1^*) > 0$. Therefore $f_1(v_1^*) > 0$ on the positive half axis is always true.

As shown in figures 5 and 6, when (H_{24}) is satisfied, $f_1'(v_1)$ has at least a positive root v_1^*, where v_1^* is the larger root, and $f_1(v_1^*) < 0$. Therefore $f_1(v_1^*) > 0$ has at least one positive root on the positive half axis.

If (H_{25}) is satisfied, in other words $m_{23} < 0$, then we have $f_1(v_1) < 0$. Moreover, clearly $f_1(v_1) \to +\infty$ when $v_1 \to +\infty$. Because of the continuity of the function $f_1(v_1)$, it must have a positive root on the positive half axis.

Without loss of generality, we assume that equation (4.9) has three positive real roots v_{11}, v_{12}, v_{13}, then equation (4.7) also has three real roots $\omega_{11} = \sqrt{v_{11}}$, $\omega_{12} = \sqrt{v_{12}}$, $\omega_{13} = \sqrt{v_{13}}$.

The critical value of the time delay τ_{1k}^j satisfies

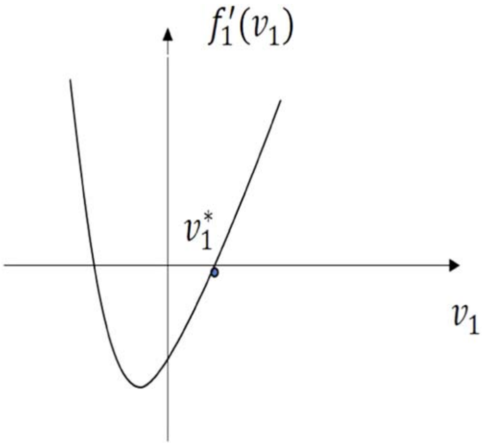


Figure 3. $\Delta > 0$, $v_1^* > 0$.

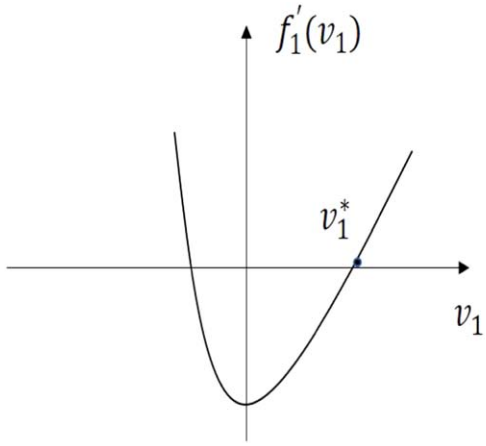


Figure 5. $\Delta > 0$, $v_1^* > 0$.

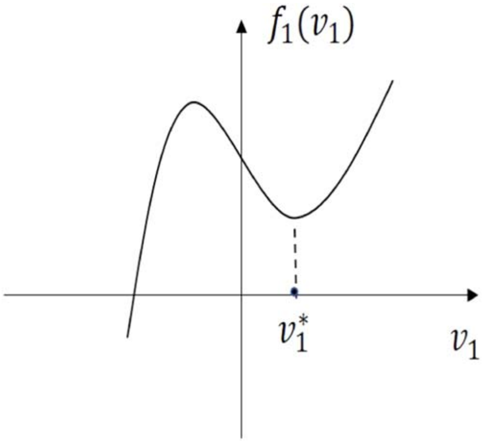


Figure 4. $m_{23} \geqslant 0, f_1(v_1^*) > 0$.

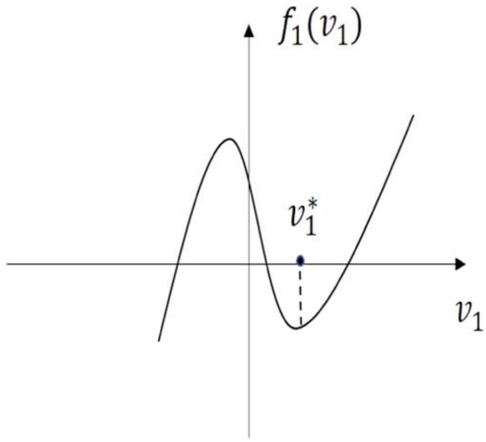


Figure 6. $m_{23} \geqslant 0, f_1(v_1^*) < 0$.

$$\tau_{1k}^j = \frac{1}{\omega_{1k}}\arccos\frac{(m_1\omega_{1k}^2 - m_3)(n_1 + q - n_3\omega_{1k}^2) + n_2\omega_{1k}^2(\omega_{1k}^2 - m_2)}{(n_1 + q - n_3\omega_{1k}^2)^2 + n_2^2\omega_{1k}^2} + \frac{2j\pi}{\omega_{1k}}, k = 1, 2, 3, j = 0, 1, 2, \cdots. \tag{4.11}$$

Hence $\pm i\omega_{1k}$ are two purely imaginary roots of equation (4.5) with $\tau_1 = \tau_{1k}^j$. Let $\tau_{10} = \min_{k\in 1,2,3}\{\tau_{1k}^0\}$, and $\omega_{10} = \omega_{1k_0}$

On account of the Hopf bifurcation theorem, we want to find the transversality condition of equation (4.5). Notice that λ is a function of τ_1. Now, taking the derivative of equation (4.5) respect to τ_1, we obtain that

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1} = -\frac{3\lambda^2 + 2m_1\lambda + m_2}{\lambda(\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3)} + \frac{2n_3\lambda + n_2}{\lambda(n_3\lambda^2 + n_2\lambda + n_1 + q)} - \frac{\tau_1}{\lambda}. \tag{4.12}$$

Extract the real part of (4.12), we can obtain

$$Re\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\Bigg|_{\lambda=i\omega_{10}} = \frac{3\omega_{10}^4 + (2m_1^2 - 4m_2 - 2n_3^2)\omega_{10}^2 + m_2^2 + 2n_3(n_1 + q) - 2m_1m_3 - n_2^2}{(\omega_{10}^3 - m_2\omega_{10})^2 + (-m_1\omega_{10}^2 + m_3)^2}. \tag{4.13}$$

Clearly, $Re\left(\frac{d\lambda}{d\tau_1}\right)^{-1}$ and $Re\left(\frac{d\tau_1}{d\lambda}\right)^{-1}$ have the same notation. Let

$$g_1(\omega_{10}) = 3\omega_{10}^4 + (2m_1^2 - 4m_2 - 2n_3^2)\omega_{10}^2 + m_2^2 + 2n_3(n_1 + q) - 2m_1m_3 - n_2^2,$$

and $v_{10} = \omega_{10}^2$, then

$$g_1(v_{10}) = 3v_{10}^2 + a_{21}v_{10} + a_{22}, \tag{4.14}$$

where $a_{21} = 2m_1^2 - 4m_2 - 2n_3^2$, $a_{22} = m_2^2 + 2n_3(n_1 + q) - 2m_1 m_3 - n_2^2$, and

$$Re\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\Bigg|_{\lambda = i\omega_{10}} = \frac{g_1(v_{10})}{(\omega_{10}^3 - m_2\omega_{10})^2 + (-m_1\omega_{10}^2 + m_3)^2}. \tag{4.15}$$

Lemma 5. *If* $(H_{26}): g_1(v_{10}) > 0$ *holds, we can say that* $Re\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\Big|_{\lambda = i\omega_{10}} > 0$.

Theorem 1. *For system (1.2), when* $\tau_1 > 0$, $\tau_2 = 0$, *assume that* (H_{11}) *holds, we have the following results.*

(1) If (H_{21}) or (H_{22}) or (H_{23}) holds, $\forall\, \tau_1 > 0$ the positive equilibrium $E^*(S^*, I^*, R^*)$ is locally asymptotically stable.

(2) If (H_{24}), (H_{26}) or (H_{25}), (H_{26}) or holds,there is a positive constant τ_{10} that makes E^* locally asymptotically stable when $\tau_1 \in [0, \tau_{10})$ and unstable when $\tau_1 \in (\tau_{10}, +\infty)$, furthermore, system (1.2) undergoes a Hopf bifurcation at E^* when $\tau_1 = \tau_{10}$.

Case 3: When $\tau_1 = 0$, $\tau_2 > 0$, equation (4.3) becomes

$$\lambda^3 + (m_1 + n_3)\lambda^2 + (m_2 + n_2)\lambda + m_3 + n_1 + qe^{-\tau_2} = 0. \tag{4.16}$$

We assume $\lambda = i\omega_2(\omega_2 > 0)$ is a solution of equation (4.16), then put it in equation (4.16). Separating real and imaginary parts we obtain

$$\begin{cases} -(m_1 + n_3)\omega_2^2 + m_3 + n_1 \\ \quad + q\cos(\omega_2\tau_2) = 0, \\ \quad -\omega_2^3 + (m_2 + n_2)\omega_2 - q\sin(\omega_2\tau_2) = 0. \end{cases} \tag{4.17}$$

Squaring and adding the two equations of (4.17), we can get that

$$\omega_2^6 + ((m_1 + n_3)^2 - 2(m_2 + n_2))\omega_2^4 + (-2(m_1 + n_3) + (m_3 + n_1)(m_1 + n_3)^2)\omega_2^2 + (m_3 + n_1)^2 - q^2 = 0. \tag{4.18}$$

$$Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\Bigg|_{\lambda = i\omega_{20}} = \frac{3\omega_{20}^4 + (2(m_1 + n_3)^2 - 4(m_2 + n_2))\omega_{20}^2 + (m_2 + n_2)^2 - 2(m_1 + n_3)(m_3 + n_1)}{(\omega_{20}^3 - (m_2 + n_2)\omega_{20})^2 + (-(m_1 + n_3)\omega_{20}^2 + (m_1 + n_3))^2}. \tag{4.23}$$

Let $v_2 = \omega_2^2$, then we have

$$v_2^3 + m_{31}v_2^2 + m_{32}v_2 + m_{33} = 0. \tag{4.19}$$

Denote

$$f_2(v_2) = v_2^3 + m_{31}v_2^2 + m_{32}v_2 + m_{33}, \tag{4.20}$$

where

$$m_{31} = (m_1 + n_3)^2 - 2(m_2 + n_2),$$
$$m_{32} = -2(m_1 + n_3)(m_3 + n_1) + (m_2 + n_2)^2,$$
$$m_{33} = (m_3 + n_1)^2 - q^2.$$

Lemma 6. *(1) If* $(H_{31}): m_{33} \geqslant 0$, $\Delta = 4(m_{31}^2 - 3m_{32}) \leqslant 0$, *or* $(H_{32}): m_{33} > 0$, $\Delta > 0$, $v_2^* = \frac{-m_{31} + \sqrt{m_{31}^2 - 3m_{32}}}{3} < 0$, *or* $(H_{33}): m_{33} \geqslant 0$, $\Delta > 0$, $v_2^* > 0$, $f_2(v_2^*) > 0$ *holds, then* $f_2(v_2)$ *has no positive real roots.*

(2) If $(H_{34}): m_{33} \geqslant 0$, $\Delta > 0$, $v_2^* > 0$, $f_2(v_2^*) < 0$, *or* $(H_{35}): m_{33} < 0$ *holds, then* $f_2(v_2)$ *has at least one positive real root.*

Proof. The proof of the lemma is similar to lemma 4.

Without loss of generality, we assume that equation (4.20) has three positive real roots v_{21}, v_{22}, v_{23}, equation (4.18) also has three real roots $\omega_{21} = \sqrt{v_{21}}$, $\omega_{22} = \sqrt{v_{22}}$, $\omega_{23} = \sqrt{v_{23}}$.

The critical value of the time delay τ_{2k}^j satisfies

$$\tau_{2k}^j = \frac{1}{\omega_{2k}}\arccos\frac{(m_1 + n_3)\omega_{2k}^2 - (m_3 + n_1)}{q} + \frac{2j\pi}{\omega_{2k}},\ k = 1, 2, 3, j = 0, 1, 2, \cdots. \tag{4.21}$$

Hence $\pm i\omega_{2k}$ are two purely imaginary roots of equation (4.16) with $\tau_2 = \tau_{2k}^j$. Let $\tau_{20} = \min_{k \in 1,2,3}\{\tau_{2k}^0\}$, and $\omega_{20} = \omega_{2k_0}$.

On account of the Hopf bifurcation theorem, we want to find the transversality condition of equation (4.16). Now, taking the derivative of equation (4.16) with respect to τ_2, we have

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1} = -\frac{3\lambda^2 + 2(m_1 + n_3)\lambda + (m_2 + n_2)}{\lambda(\lambda^3 + (m_1 + n_3)\lambda^2 + (m_2 + n_2)\lambda + (m_3 + n_1))} - \frac{\tau_2}{\lambda}. \tag{4.22}$$

Extract the real part of (4.22), we can obtain

Obviously, $Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}$ and $Re\left(\frac{d\tau_2}{d\lambda}\right)^{-1}$ have the same notation. Let

$$g_2(\omega_{20}) = 3\omega_{20}^4 + (2(m_1 + n_3)^2 - 4(m_2 + n_2))\omega_{20}^2 + (m_2 + n_2)^2 - 2(m_1 + n_3)(m_3 + n_1),$$

and $v_{20} = \omega_{20}^2$, then

$$g_2(v_{20}) = 3v_{20}^2 + a_{31}v_{10} + a_{32}, \tag{4.24}$$

where $a_{31} = 2(m_1 + n_3)^2 - 4(m_2 + n_2)$, $a_{32} = (m_2 + n_2)^2$.
$- 2(m_1 + n_3)(m_3 + n_1)$

We can calculate that

$$Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\Bigg|_{\lambda=i\omega_{20}} = \frac{g_2(v_{20})}{(\omega_{20}^3 - (m_2 + n_2)\omega_{20})^2 + (-(m_1 + n_3)\omega_{20}^2 + (m_1 + n_3))^2}. \tag{4.25}$$

Lemma 7. *If* $(H_{36}) : g_2(v_{20}) > 0$ *holds, then* $Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\Big|_{\lambda=i\omega_{20}} > 0.$

Theorem 2. *For system (1.2), when* $\tau_1 = 0$, $\tau_2 > 0$, *assume that* (H_{11}) *holds, we have the following results*

(1) If (H_{31}) *or* (H_{32}) *or* (H_{33}) *holds, the positive equilibrium point* $E^*(S^*, I^*, R^*)$ *is locally asymptotically stable for* $\forall \tau_2 > 0$.

(2) If (H_{34}), (H_{36}) *or* (H_{35}), (H_{36}) *holds, there is a positive constant* τ_{20} *that makes the positive equilibrium point* E^* *is locally asymptotically stable when* $\tau_2 \in [0, \tau_{20})$ *and unstable when* $\tau_2 \in (\tau_{20}, +\infty)$. *Furthermore, system (1.2) undergoes a Hopf bifurcation at* E^* *when* $\tau_2 = \tau_{20}$.

Case 4: When $\tau_1 = \tau_2 = \tau > 0$, equation (4.3) becomes

$$\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3 + (n_1 + n_2\lambda + n_3\lambda^2)e^{-\lambda\tau} + qe^{-2\lambda\tau} = 0. \tag{4.26}$$

Multiply both sides of equation (4.26) by $e^{\lambda\tau}$, we can obtain

$$n_3\lambda^2 + n_2\lambda + n_1 + (\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3)e^{\lambda\tau} + qe^{-\lambda\tau} = 0. \tag{4.27}$$

Suppose that equation (4.27) has a pure imaginary root $\lambda = i\omega(\omega > 0)$, then put it in equation (4.27) the real part and the imaginary part of equation (4.27) are separated as follows

$$\begin{cases} E_{41}\sin(\omega\tau) + E_{42}\cos(\omega\tau) = E_{45}, \\ E_{43}\sin(\omega\tau) + E_{44}\cos(\omega\tau) = E_{46}. \end{cases} \tag{4.28}$$

where

$$E_{41} = \omega^3 - m_2\omega,\ E_{42} = -m_1\omega^2 + m_3 + q,$$
$$E_{43} = -m_1\omega^2 + m_3 - q,$$
$$E_{44} = -\omega^3 + m_2\omega,\ E_{45} = n_3\omega^2 - n_1,\ E_{46} = -n_2\omega.$$

From equation (4.28) we can obtain that

$$\begin{cases} \sin(\omega\tau) = \frac{A_{41}\omega^5 + A_{42}\omega^3 + A_{43}\omega}{\omega^6 + B_{41}\omega^4 + B_{42}\omega^2 + B_{43}}, \\ \cos(\omega\tau) = \frac{A_{44}\omega^4 + A_{45}\omega^2 + A_{46}}{\omega^6 + B_{41}\omega^4 + B_{42}\omega^2 + B_{43}}. \end{cases} \tag{4.29}$$

where

$$A_{41} = n_3,\ A_{42} = m_1m_2 - n_1 - m_2n_3,$$
$$A_{43} = m_2n_1 - n_2(m_3 + q),$$
$$B_{41} = m_1^2 - 2m_2,\ B_{42} = m_2^2 - 2m_1m_3,$$
$$B_{43} = m_3^2 - q^2,\ A_{44} = n_2 - m_1n_3,$$
$$A_{45} = n_3(m_3 - q) + n_1m_1 - n_2m_2,\ A_{46} = n_1(q - m_3).$$

Squaring and adding the two equations of equation (4.29), we can get that

$$\omega^{12} + e_{45}\omega^{10} + e_{44}\omega^8 + e_{43}\omega^6 + e_{42}\omega^4 + e_{41}\omega^2 + e_{40} = 0, \tag{4.30}$$

where

$$e_{45} = 2B_{41} - A_{41}^2,$$
$$e_{44} = B_{41}^2 + 2B_{42} - 2A_{41}A_{42} - A_{44}^2,$$
$$e_{43} = 2B_{43} + 2B_{41}B_{42} - A_{42}^2 - 2A_{41}A_{43} - 2A_{44}A_{45},$$
$$e_{42} = B_{42}^2 + 2B_{41}B_{43} - 2A_{42}A_{43} - A_{45}^2 - 2A_{44}A_{46},$$
$$e_{41} = 2B_{42}B_{43} - A_{43}^2 - 2A_{45}A_{46},\ e_{40} = B_{43}^2 - A_{46}^2.$$

Let $v = \omega^2$, we can obtain that

$$v^6 + e_{45}v^5 + e_{44}v^4 + e_{43}v^3 + e_{42}v^2 + e_{41}v + e_{40} = 0. \tag{4.31}$$

Denote

$$f(v) = v^6 + e_{45}v^5 + e_{44}v^4 + e_{43}v^3 + e_{42}v^2 + e_{41}v + e_{40}. \tag{4.32}$$

Without loss of generality, we assume that $f(v) = 0$ has six positive real roots v_k, $k = 1, 2, \cdots, 6$, then equation (4.30) also has six real roots $\omega_k = \sqrt{v_k}$, $k = 1, 2,\cdots,6$. The critical value of the time delay τ_k^j satisfies

$$\tau_k^{(j)} = \frac{1}{\omega_k}\arccos\frac{A_{44}\omega_k^4 + A_{45}\omega_k^2 + A_{46}}{\omega_k^6 + B_{41}\omega_k^4 + B_{42}\omega_k^2 + B_{43}} + \frac{2j\pi}{\omega_k},\ k = 1, 2,\cdots,6,\ j = 0, 1, 2,\cdots. \tag{4.33}$$

Hence $\pm i\omega_k$ is a pair of pure imaginary roots of (4.26) with $\tau = \tau_k^j$, let $\tau_0 = \min_{k\in 1,2,\cdots,6}\{\tau_k^0\}$, and $\omega_0 = \omega_{k_0}$.

Notice that λ is a function of τ. Now taking the derivative of (4.26) with respect to τ, we obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2n_3\lambda + n_2 + (3\lambda^2 + 2m_1\lambda + m_2)e^{\lambda\tau}}{-\lambda(\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3)e^{\lambda\tau} + q\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda}. \tag{4.34}$$

Extract the real part of equation (4.34), we can obtain

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\lambda=i\omega_0} = \frac{Q_RP_R + Q_IP_I}{P_R^2 + P_I^2}, \tag{4.35}$$

where

$$\begin{aligned}
Q_R &= n_2 + (-3\omega_0^2 + m_2)\cos(\omega_0\tau_0) - 2m_1\omega_0\sin(\omega_0\tau_0),\\
Q_I &= 2n_3\omega_0 + (-3\omega_0^2 + m_2)\sin(\omega_0\tau_0) + 2m_1\omega_0\cos(\omega_0\tau_0),\\
P_R &= (-\omega_0^4 + m_2\omega_0^2)\cos(\omega_0\tau_0)\\
&\quad + (q\omega_0 - m_1\omega_0^3 + m_3\omega_0)\sin(\omega_0\tau_0),\\
P_I &= (q\omega_0 + m_1\omega_0^3 - m_3\omega_0)\cos(\omega_0\tau_0)\\
&\quad + (-\omega_0^4 + m_2\omega_0^2)\sin(\omega_0\tau_0).
\end{aligned}$$

Thus, when (H_{41}): $Q_RP_R + Q_IP_I > 0$ holds, we have $Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\Big|_{\lambda=i\omega_0} > 0$.

Theorem 3. *For $\tau_1 = \tau_2 = \tau > 0$ in system (1.2), if (H_{11}) holds, and assume further that (H_{41}) satisfies. Then there is a positive constant τ_0 such that the positive equilibrium point E^* of system (1.2) is locally asymptotically stable when $\tau \in [0, \tau_0)$ and unstable when $\tau \in (\tau_0, +\infty)$. That is to say, system (1.2) undergoes a Hopf bifurcation at $\tau = \tau_0$.*

Case 5: When $\tau_1 \in [0, \tau_{10})$, $\tau_2 > 0$, and $\tau_1 \neq \tau_2$, equation (4.3) becomes

$$\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3 + (n_1 + n_2\lambda + n_3\lambda^2)e^{-\lambda\tau_1} + qe^{-\lambda(\tau_1+\tau_2)} = 0. \tag{4.36}$$

Suppose that equation (4.36) has a pure imaginary root $\lambda = i\omega_3 (\omega_3 > 0)$, then taking the real and imaginary parts of equation (4.36), we can obtain that

$$\begin{cases} E_{51}\sin(\omega_3\tau_2) - E_{52}\cos(\omega_3\tau_2) = E_{53},\\ E_{51}\cos(\omega_3\tau_2) + E_{52}\sin(\omega_3\tau_2) = E_{54},\end{cases} \tag{4.37}$$

where

$$\begin{aligned}
E_{51} &= q\sin(\omega_3\tau_1),\\
E_{52} &= q\cos(\omega_3\tau_1),\\
E_{53} &= -m_1\omega_3^2 + m_3 + (-n_3\omega_3^2 + n_1)\cos(\omega_3\tau_1)\\
&\quad + n_2\omega_3\sin(\omega_3\tau_1),\\
E_{54} &= -(-n_3\omega_3^2 + n_1)\sin(\omega_3\tau_1) + n_2\omega_3\cos(\omega_3\tau_1)\\
&\quad - \omega_3^3 + m_2\omega_3.
\end{aligned}$$

Squaring and adding the two equations of equation (4.37), we can get that

$$\begin{aligned}
&\omega_3^6 + e_{58}\omega_3^4 + e_{57}\omega_3^2 + e_{56} + (e_{55}\omega_3^4 + e_{54}\omega_3^2\\
&\quad + e_{53})\cos(\omega_3\tau_1)\\
&\quad + (e_{52}\omega_3^5 + e_{51}\omega_3^3 + e_{50}\omega_3)\sin(\omega_3\tau_1) = 0,
\end{aligned} \tag{4.38}$$

where

$$\begin{aligned}
e_{50} &= 2(m_3n_2 - n_1m_2),\\
e_{51} &= 2(-m_1n_2 + n_1 + m_2n_3),\ e_{52} = -2n_3,\\
e_{53} &= 2m_3n_1,\ e_{54} = 2(-m_1n_1 - m_3n_3 + m_2n_2),\\
e_{55} &= 2(m_1n_3 - n_2),\\
e_{56} &= m_3^2 + n_1^2 - q^2,\ e_{57} = -2m_1m_3 + m_2^2 - 2n_1n_3\\
&\quad + n_2^2,\ e_{58} = m_1^2 - 2m_2 + n_3^2.
\end{aligned}$$

Assume that equation (4.38) has finite positive roots $\omega_{31}, \omega_{32},\cdots,\omega_{3k}$, for each ω_{3i}, $i = 1, 2,\cdots,k$, then there is a corresponding delay threshold $\tau_{2i}^{(j)}$, $j = 1, 2,\cdots$, where

$$\tau_{2i}^{(j)} = \frac{1}{\omega_{3i}}\arccos\frac{E_{51}E_{54} - E_{53}E_{52}}{E_{51}^2 + E_{52}^2} + \frac{2j\pi}{\omega_{3i}},\quad j = 0, 1, 2,\cdots, i = 1, 2,\cdots,k. \tag{4.39}$$

Let $\tau_{20}' = \min\{\tau_{2i}^{(0)}, i = 1, 2,\cdots,k\}$, at the same time ω_{30} and τ_{20}' are correspond.

On account of the Hopf bifurcation theorem, we want to find the transversality condition of equation Now taking the derivative of (4.36) with respect to τ_2, we obtain

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1} = \frac{(3\lambda^2 + 2m_1\lambda + m_2) + ((2n_3\lambda + n_2) - (n_3\lambda^2 + n_2\lambda + n_1)\tau_1)e^{-\lambda\tau_1}}{\lambda qe^{-\lambda(\tau_1+\tau_2)}} - \frac{\tau_1 + \tau_2}{\lambda}. \tag{4.40}$$

Extract the real part of (4.34), we can obtain

$$Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\Bigg|_{\lambda=i\omega_{30}} = \frac{\widetilde{Q_R}\widetilde{P_R} + \widetilde{Q_I}\widetilde{P_I}}{\widetilde{P_R^2} + \widetilde{P_I^2}}, \tag{4.41}$$

where

$$\begin{aligned}\widetilde{Q_R} &= -3\omega_{30}^2 + m_2 + (n_2 - (-n_3\omega_{30}^2 + n_1)\tau_1)\cos(\omega_{30}\tau_1)\\ &\quad + (2n_3\omega_{30} - n_2\omega_{30}\tau_1)\sin(\omega_{30}\tau_1),\\ \widetilde{Q_I} &= 2m_1\omega_{30} + (2n_3 - n_2\tau_1)\omega_{30}\cos(\omega_{30}\tau_1)\\ &\quad + (-n_3\omega_{30}^2 - n_2 + n_1)\sin(\omega_{30}\tau_1),\\ \widetilde{P_R} &= q\omega_{30}(\cos(\omega_{30}\tau_1)\sin(\omega_{30}\tau_2) + \sin(\omega_{30}\tau_1)\cos(\omega_{30}\tau_2)),\\ \widetilde{P_I} &= q\omega_{30}(\cos(\omega_{30}\tau_1)\cos(\omega_{30}\tau_2) - \sin(\omega_{30}\tau_1)\sin(\omega_{30}\tau_2)).\end{aligned}$$

Clearly, when $(H_{51}): \widetilde{Q_R}\widetilde{P_R} + \widetilde{Q_I}\widetilde{P_I} > 0$ holds, we have $Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\Big|_{\lambda=i\omega_{30}} > 0$.

Theorem 4. *For $\tau_1 \in [0, \tau_{10})$, $\tau_2 > 0$ and $\tau_1 \neq \tau_2$ in system (1.2), if (H_{11}) holds, and assume further that (H_{51}) satisfies. Then there exists a positive constant τ_{20}' such that the positive equilibrium point E^* of system (1.2) is locally asymptotically stable when $\tau_2 \in [0, \tau_{20}')$ and unstable when $\tau_2 \in (\tau_{20}', +\infty)$. Furthermore, system (1.2) occurs a Hopf bifurcation at E^* when $\tau_2 = \tau_{20}'$.*

Case 6: When $\tau_1 > 0$, $\tau_2 \in [0, \tau_{20})$, and $\tau_1 \neq \tau_2$, equation (4.3) becomes

$$\begin{aligned}&\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3 + (n_1 + n_2\lambda + n_3\lambda^2)e^{-\lambda\tau_1}\\ &\quad + qe^{-\lambda(\tau_1+\tau_2)} = 0.\end{aligned} \tag{4.42}$$

Suppose that equation (4.42) has a pure imaginary root $\lambda = i\omega_4(\omega_4 > 0)$, then put it in equation (4.42), and separate real and imaginary parts, we obtain

$$\begin{cases} E_{61}\cos(\omega_4\tau_1) + E_{62}\sin(\omega_4\tau_1) = E_{63},\\ -E_{61}\sin(\omega_4\tau_1) + E_{62}\cos(\omega_4\tau_1) = E_{64},\end{cases} \tag{4.43}$$

where

$$\begin{aligned}E_{61} &= -n_3\omega_4^2 + n_1 + q\cos(\omega_4\tau_2),\\ E_{62} &= n_2\omega_4 - q\sin(\omega_4\tau_2),\\ E_{63} &= m_1\omega_4^2 - m_3,\ E_{64} = \omega_4^3 - m_2\omega_4.\end{aligned}$$

Squaring and adding the two equations of (4.43), we can get that

$$\begin{aligned}&\omega_4^6 + e_{65}\omega_4^4 + e_{64}\omega_4^2 + e_{63} + (e_{62}\omega_4^2 + e_{61})\cos(\omega_4\tau_2)\\ &\quad + e_{60}\omega_4\sin(\omega_4\tau_2) = 0,\end{aligned} \tag{4.44}$$

where

$$\begin{aligned}e_{65} &= m_1^2 - 2m_2 - n_3^2,\ e_{64} = -2m_1m_3 + m_2^2\\ &\quad + 2n_1n_3 - n_2^2,\ e_{63} = m_3^2 - n_1^2 - q^2,\\ e_{62} &= 2n_3q,\ e_{61} = -2n_1q,\ e_{60} = 2n_2q.\end{aligned}$$

Assume that equation (4.44) has finite positive roots $\omega_{41}, \omega_{42},\cdots,\omega_{4l}$, for each ω_{4i}, $i = 1, 2,\cdots,l$, there is a corresponding delay threshold $\tau_{1i}^{(j)}$, $j = 0, 1, 2,\cdots$, and

$$\begin{aligned}\tau_{1i}^{(j)} &= \frac{1}{\omega_{4i}}\arccos\frac{E_{61}E_{63} + E_{62}E_{64}}{E_{61}^2 + E_{62}^2} + \frac{2j\pi}{\omega_{4i}},\\ j &= 0, 1, 2,\cdots, i = 1, 2,\cdots,l.\end{aligned} \tag{4.45}$$

Let $\tau_{10}' = \min\{\tau_{1i}^{(0)}, i = 1, 2,\cdots,l\}$, at the same time ω_{40} and τ_{10}' are correspond.

On account of the Hopf bifurcation theorem, now we want to find the transversality condition of equation (4.42). Notice that λ is a function of τ_1, we take the derivative of equation (4.42) with respect to τ_1, then

$$\begin{aligned}&\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\\ &= \frac{3\lambda^2 + 2m_1\lambda + m_2 + (2n_3\lambda + n_2)e^{-\lambda\tau_1} - q\tau_2e^{-\lambda(\tau_1+\tau_2)}}{\lambda(n_3\lambda^2 + n_2\lambda + n_1)e^{-\lambda\tau_1} + q\lambda e^{-\lambda(\tau_1+\tau_2)}}\\ &- \frac{\tau_1}{\lambda}.\end{aligned} \tag{4.46}$$

Extract the real part of equation (4.46), we can obtain

$$Re\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\Bigg|_{\lambda=i\omega_{40}} = \frac{\overline{Q_R}\,\overline{P_R} + \overline{Q_I}\,\overline{P_I}}{\overline{P_R}^2 + \overline{P_I}^2}, \tag{4.47}$$

where

$$\begin{aligned}\overline{Q_R} &= -3\omega_{40}^2 + m_2 + n_2\cos(\omega_{40}\tau_1) + 2n_3\omega_{40}\sin(\omega_{40}\tau_1)\\ &\quad - q\tau_2\cos(\omega_{40}(\tau_1 + \tau_2)),\\ \overline{Q_I} &= 2m_1\omega_{40} - n_2\sin(\omega_{40}\tau_1) + 2n_3\omega_{40}\cos(\omega_{40}\tau_1)\\ &\quad + q\tau_2\sin(\omega_{40}(\tau_1 + \tau_2)),\\ \overline{P_R} &= -n_2\omega_{40}^2\cos(\omega_{40}\tau_1) + (-n_3\omega_{40}^3 + n_1\omega_{40})\sin(\omega_{40}\tau_1)\\ &\quad + q\omega_{40}\sin(\omega_{40}(\tau_1 + \tau_2)),\\ \overline{P_I} &= (-n_3\omega_{40}^3 + n_1\omega_{40})\cos(\omega_{40}\tau_1) + n_2\omega_{40}^2\sin(\omega_{40}\tau_1)\\ &\quad + q\omega_{40}\cos(\omega_{40}(\tau_1 + \tau_2)).\end{aligned}$$

Hence, when $(H_{61}): \overline{Q_R}\,\overline{P_R} + \overline{Q_I}\,\overline{P_I} > 0$ holds, we have $Re\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\Big|_{\lambda=i\omega_{40}} > 0$.

Theorem 5. *For $\tau_1 > 0$, $\tau_2 \in [0, \tau_{20})$ and $\tau_1 \neq \tau_2$ in system (1.2), if (H_{11}) holds, and assume further that (H_{61}) satisfies. Then there exists a positive constant τ_{10}' such that the positive equilibrium point E^* of system (1.2) is locally asymptotically stable when $\tau_1 \in [0, \tau_{10}')$ and unstable when $\tau_1 \in (\tau_{10}', +\infty)$. Meanwhile, the Hopf bifurcation occurs when $\tau_1 = \tau_{10}'$. That is, system (1.2) has a branch of periodic solutions bifurcating from the positive equilibrium point E^* near $\tau_1 = \tau_{10}'$.*

5. Optimal control techniques in a delayed model

Based on model (1.2), we establish an optimal control model, the main purpose of this model is to propose effective control methods to control the spread of infectious diseases. We assume that $\tau_1 = \tau_2 = \tau > 0$, the optimal control model is as follows

$$\begin{cases} \frac{dS(t)}{dt} = rS(t)\left(1 - \frac{S(t)}{k}\right) - \frac{\beta_1 S(t)I(t)}{1+\alpha_1 I(t)} + \sigma R(t) + \theta u(t)I(t), \\ \frac{dI(t)}{dt} = \frac{\beta_1 S(t-\tau)I(t-\tau)}{1+\alpha_1 I(t-\tau)} - (\mu + \varepsilon)I(t) - \frac{\beta_2 I(t)}{1+\alpha_2 I(t)} - u(t)I(t), \\ \frac{dR(t)}{dt} = \frac{\beta_2 I(t-\tau)}{1+\alpha_2 I(t-\tau)} - (\mu + \sigma)R(t) + (1-\theta)u(t)I(t), \end{cases} \tag{5.1}$$

with initial condition (1.3), where θ is a positive constant, and $\theta \in [0, 1]$.

Denote

$$U = \{u(t) \in L^2(0, T)\colon 0 \leqslant t \leqslant T, 0 \leqslant u(t) \leqslant 1\}, \tag{5.2}$$

where $u(t)$ is the control variable, and it's Lebesgue measurable.

The biological significance of $u(t)$ is to reduce the number of infected individuals and increase the number of susceptible individuals by decreasing contact between infected individuals and susceptible individuals. Hence, the optimal control treatment $u(t)$ transfers part of infected individuals to susceptible individuals and recovered individuals. Our goal is to achieve optimal control of the disease by maximizing the number of susceptible individuals and recovered individuals and minimizing the number of infected individuals. At the same time, we also expect to minimize the cost of control. Based on the above ideas, we establish the following objective function

$$J(u) = \int_0^T [I(t) + \frac{c}{2}u(t)^2]dt, \tag{5.3}$$

where c is a positive weighting factor.

Next, we will look for the minimum value of the Lagrangian function $L(I, u) = I(t) + \frac{c}{2}u^2(t)$. Let

$$\begin{aligned} x(t) &= (S(t), I(t), R(t))^T, \ \ x_\tau(t) = (S_\tau(t), I_\tau(t), R_\tau(t))^T \\ &= (S(t-\tau), I(t-\tau), R(t-\tau))^T, \\ \lambda(t) &= (\lambda_1(t), \lambda_2(t), \lambda_3(t))^T, \end{aligned}$$

and define the Hamiltonian function of the optimal control problem as follows

$$\begin{aligned} &H(x, x_\tau, u, \lambda) = L(I, u) + \lambda_1(t)\frac{dS}{dt} + \lambda_2(t)\frac{dI}{dt} \\ &+\lambda_3(t)\frac{dR}{dt} = I(t) + \frac{c}{2}u(t)^2 + \lambda_1(t)\left(rS(t)\left(1 - \frac{S(t)}{k}\right)\right. \\ &\left.- \frac{\beta_1 S(t)I(t)}{1+\alpha_1 I(t)} + \sigma R(t) + \theta u(t)I(t)\right) \\ &+ \lambda_2(t)\left(\frac{\beta_1 S(t-\tau_1)I(t-\tau_1)}{1+\alpha_1 I(t-\tau_1)} - (\mu+\varepsilon)I(t)\right. \\ &\left.- \frac{\beta_2 I(t)}{1+\alpha_2 I(t)} - u(t)I(t)\right) + \lambda_3(t)\left(\frac{\beta_2 I(t-\tau_2)}{1+\alpha_2 I(t-\tau_2)}\right. \\ &\left.- (\mu+\sigma)R(t) + (1-\theta)u(t)I(t)\right). \end{aligned} \tag{5.4}$$

Theorem 6. *For system (5.1), given objective function (5.3) with the initial condition (1.3), there is an optimal control $u^* \in U$ which makes $J(u^*) = \min_{u\in U} J(u)$.*

Proof. We take advantage of the result of [20] to demonstrate the existence of an optimal control. We have proved the non-negativity of covariant variables and the control $0 \leqslant u(t) \leqslant 1$. In this minimization problem, the objective function in $u(t)$ is convex. By the definition, $u(t) \in U$ is convex and closed. Because the optimal system is bounded, the optimal system is the compact support. Furthermore, the integrand $I\left(t\right) + \frac{c}{2}u\left(t\right)^2$ is convex on the dominating set U. Finally, it is easy to know that there exist the constants $\rho > 1, \eta_1 > 0$ and $\eta_2 > 0$, which satisfies $J(u) \geqslant \eta_2 + \eta_1(|u|^2)^{\frac{\rho}{2}}$. In conclusion, this theorem has been proved.

According to the Pontryagin Maximum Principle, there exists a continuous function $\lambda(t) \in [0, T]$, which satisfies the following three equations.

(i) The state equation

$$\frac{dx(t)}{dt} = H_\lambda(x, x_\tau, u, \lambda)(t), \tag{5.5}$$

(ii) The optimality condition

$$0 = H_u(x, x_\tau, u, \lambda)(t), \tag{5.6}$$

(iii) The adjoint equation

$$-\frac{d\lambda}{dt} = H_x(x, x_\tau, u, \lambda)(t) + \lambda(t+\tau)H_{x_\tau}(x, x_\tau, u, \lambda)(t), \tag{5.7}$$

where H_λ denotes the derivative with respect to λ, H_u denotes the derivative with respect to u, H_x denotes the derivative with respect to x, H_{x_τ} denotes the derivative with respect to x_τ.

Theorem 7. *Given an optimal control u^* and the corresponding optimal solution $S^*(t)$, $I^*(t)$, $R^*(t)$ of system (5.1), then there are the adjoint variable $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$, which satisfies*

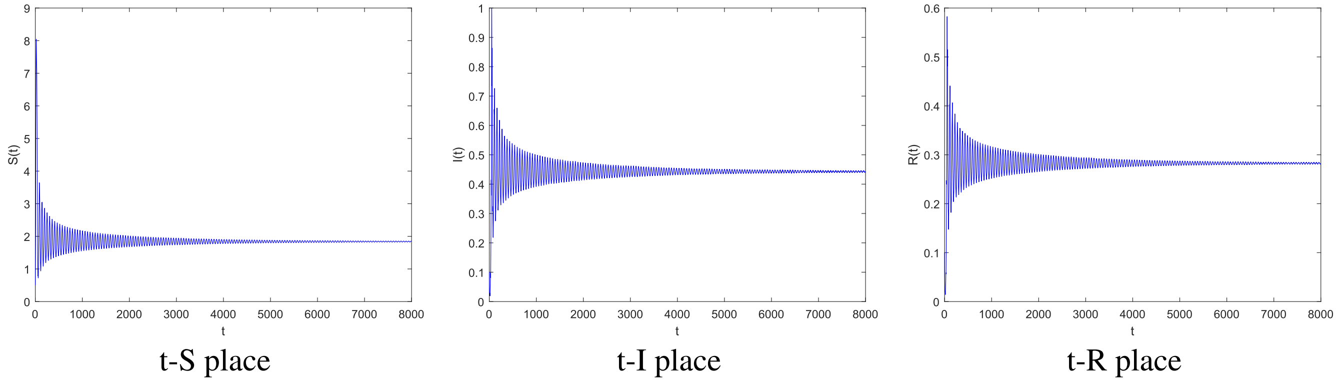


Figure 7. When $\tau_1 = 13 < \tau_{10}$, E^* is locally asymptotically stable.

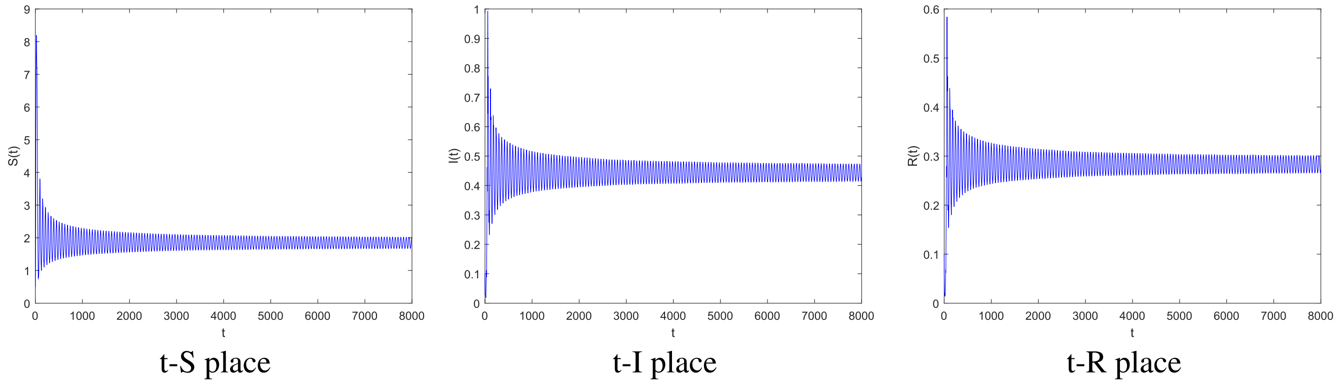


Figure 8. When $\tau_1 = 14 > \tau_{10}$, E^* is unstable.

$$\begin{cases} -\frac{d\lambda_1(t)}{dt} = \lambda_1(t)\left(r - \frac{2r}{k}S^* - \frac{\beta_1 I^*}{1+\alpha_1 I^*}\right) + \lambda_1(t+\tau)\lambda_2(t)\frac{\beta_1 I^*}{1+\alpha_1 I^*}, \\ -\frac{d\lambda_2(t)}{dt} = -\lambda_1(t)\left(\frac{\beta_1 S^*}{(1+\alpha_1 I^*)^2} - \theta u\right) - \lambda_2(t)\left(\mu + \varepsilon + \frac{\beta_2}{(1+\alpha_2 I^*)^2} + u\right) + \lambda_3(t)(1-\theta)u \\ \qquad + \lambda_2(t+\tau)\lambda_2(t)\frac{\beta_1 S^*}{(1+\alpha_1 I^*)^2} + \lambda_2(t+\tau)\lambda_3(t)\frac{\beta_2}{(1+\alpha_2 I^*)^2}, \\ -\frac{d\lambda_3(t)}{dt} = \lambda_1(t)\sigma - \lambda_3(t)(\mu+\sigma), \end{cases} \tag{5.8}$$

with boundary conditions

$$\lambda_i(T) = 0,\ i = 1, 2, 3. \tag{5.9}$$

Then, the optimal control u^* satisfies

$$u^* = \max\left\{\min\left\{\frac{I^*}{c}(-\lambda_1(t)\theta + \lambda_2(t) - \lambda_3(t)(1-\theta)),\ 1\right\},\ 0\right\}. \tag{5.10}$$

Proof. We use the Hamiltonian (5.4) to calculate the boundary conditions and the adjoint equation. According to the adjoint equation (5.7), let $x(t) = x^*(t)$ and $x_\tau(t) = x_\tau^*(t)$, we can obtain

$$-\frac{d\lambda_1(t)}{dt} = H_{S^*}(t) + \lambda_1(t+\tau)H_{S_\tau^*}(t), \tag{5.11}$$

$$-\frac{d\lambda_2(t)}{dt} = H_{I^*}(t) + \lambda_2(t+\tau)H_{I_\tau^*}(t), \tag{5.12}$$

$$-\frac{d\lambda_3(t)}{dt} = H_{R^*}(t) + \lambda_3(t+\tau)H_{R_\tau^*}(t). \tag{5.13}$$

If we put the specific expression of Hamiltonian (5.4) into (5.11)–(5.13), we can get the adjoint equations (5.8). By the optimality condition (5.6), we obtain

$$H_u(x, x_\tau, u, \lambda)(t) = cu^* + \lambda_1(t)\theta I^* - \lambda_2(t)I^* + \lambda_3(t)(1-\theta)I^* = 0.$$

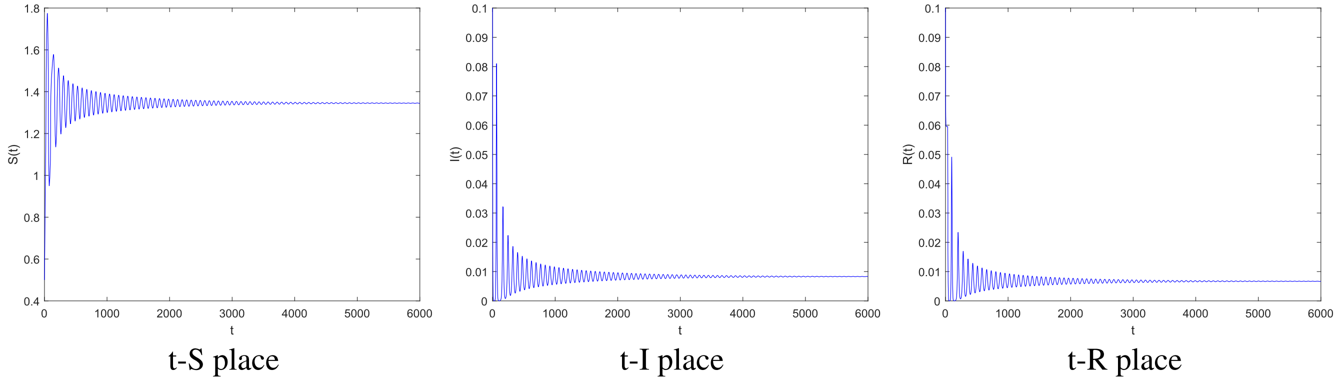


Figure 9. When $\tau_2 = 34.5 < \tau_{20}$, E^* is locally asymptotically stable.

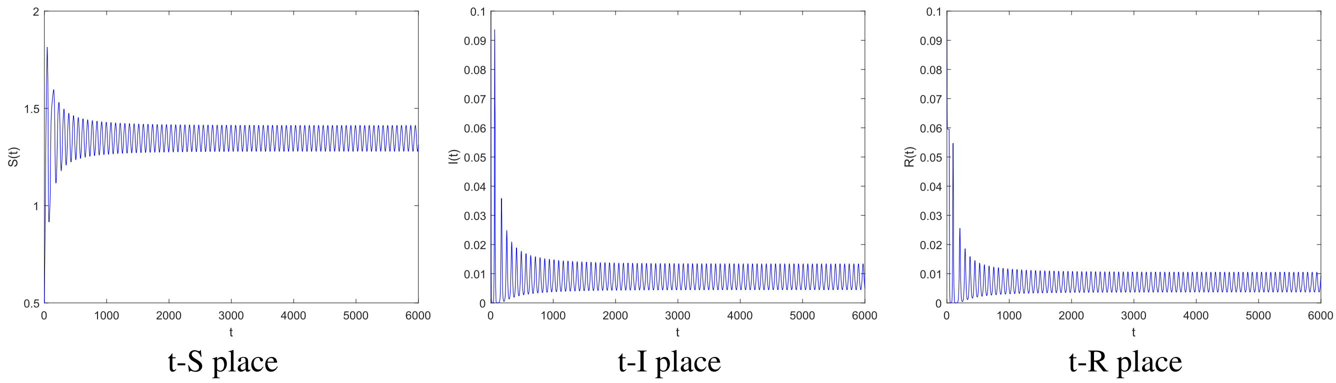


Figure 10. When $\tau_2 = 36 > \tau_{20}$, E^* is unstable.

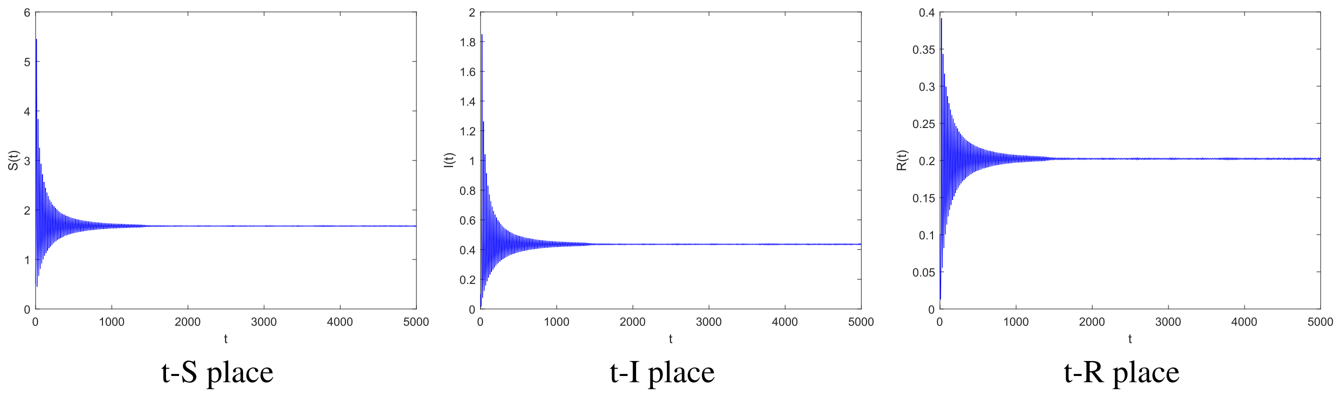


Figure 11. When $\tau = 1.3 < \tau_0$, E^* is locally asymptotically stable.

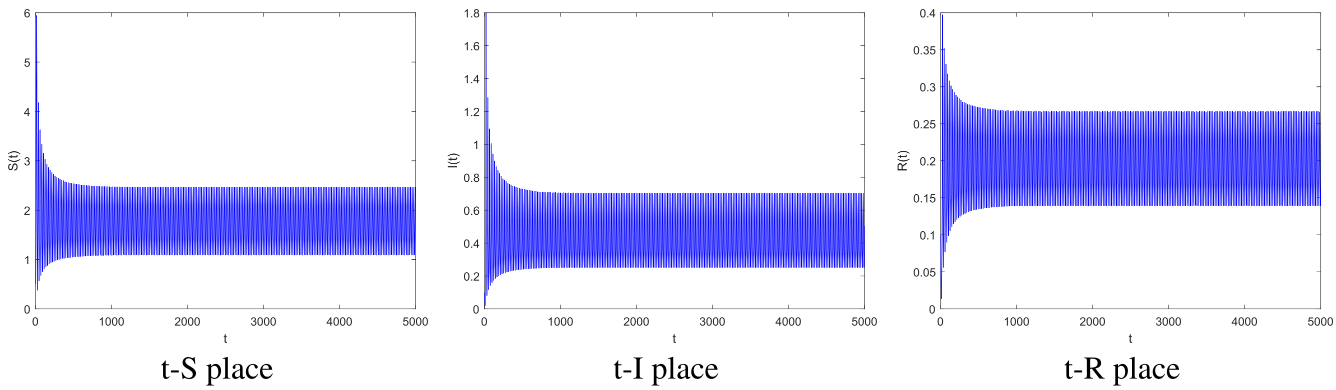


Figure 12. When $\tau = 2 > \tau$, E^* is unstable.

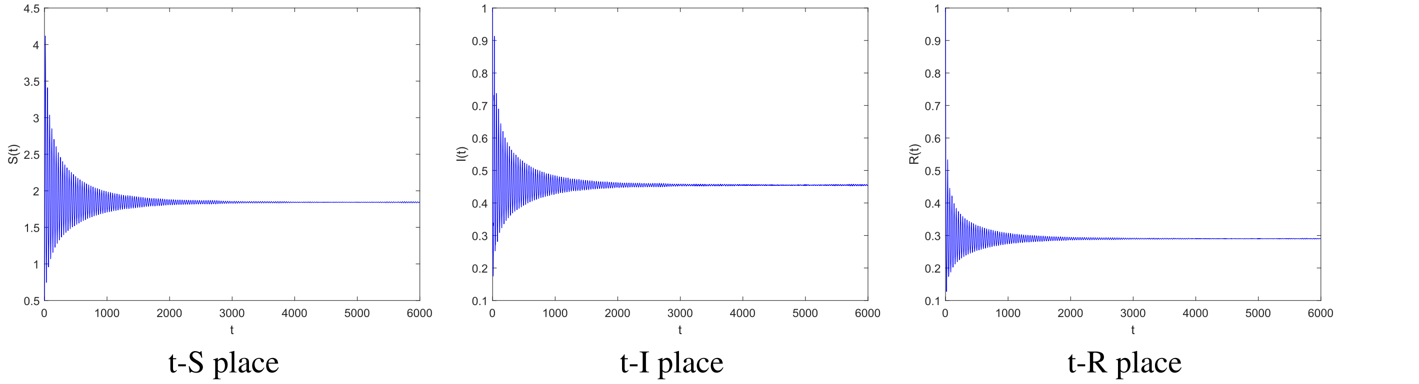


Figure 13. When $\tau_2 = 5.5 < \tau_{20}$, E^* is locally asymptotically stable.

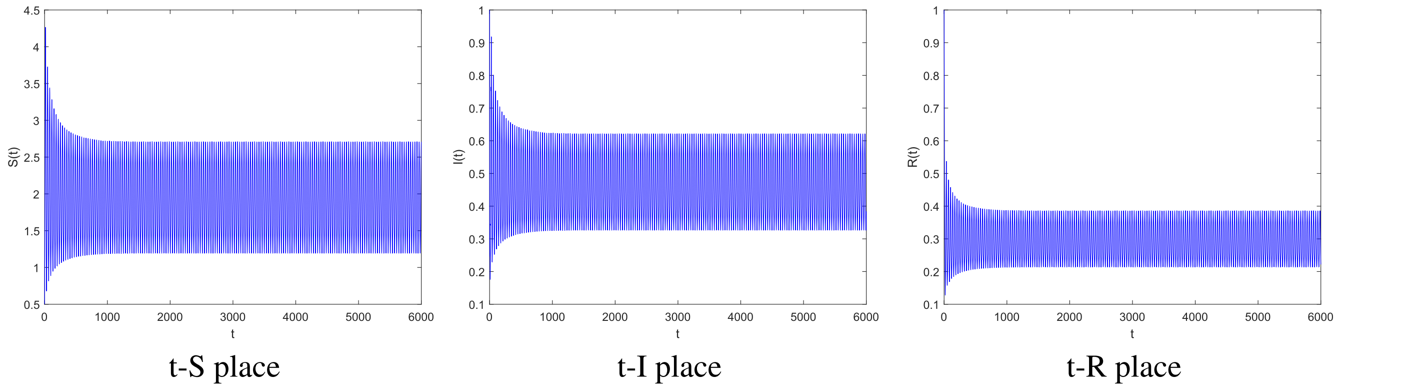


Figure 14. When $\tau_2 = 6 > \tau_{20}$, E^* is unstable.

That is to say,

$$u^* = \frac{I^*}{c}(-\lambda_1(t)\theta + \lambda_2(t) - \lambda_3(t)(1 - \theta)).$$

Thus,

$$u^* = \begin{cases} 0, & \text{if} \frac{I^*}{c}(-\lambda_1(t)\theta + \lambda_2(t) - \lambda_3(t)(1 - \theta)) \leqslant 0, \\ \frac{I^*}{c}(-\lambda_1(t)\theta + \lambda_2(t) - \lambda_3(t)(1 - \theta)), & \text{if } 0 < \frac{I^*}{c}(-\lambda_1(t)\theta + \lambda_2(t) - \lambda_3(t)(1 - \theta)) < 1, \\ 1, & \text{if } \frac{I^*}{c}(-\lambda_1(t)\theta + \lambda_2(t) - \lambda_3(t)(1 - \theta)) \geqslant 1. \end{cases}$$

That is

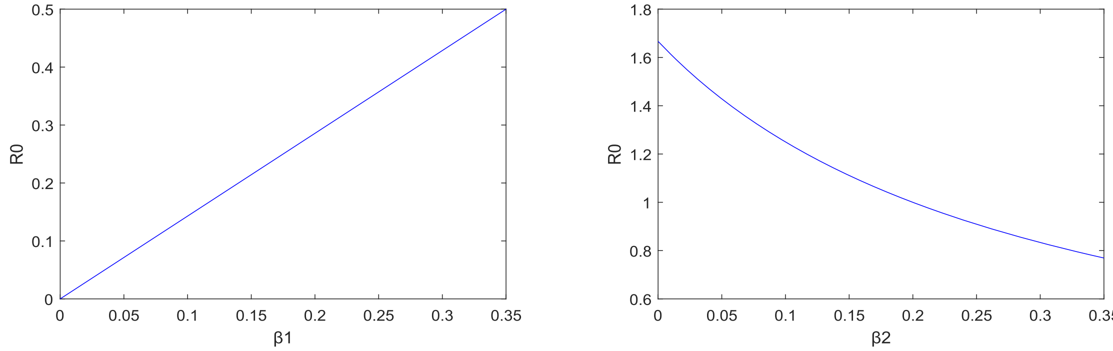
$$u^* = \max\left\{\min\left\{\frac{I^*}{c}(-\lambda_1(t)\theta + \lambda_2(t) - \lambda_3(t)(1 - \theta)), 1\right\}, 0\right\}.$$

6. Numerical simulation

Next, we verify the above conclusions by some simulation by some numerical simulation and analysis of the impact of parameters β_1 and β_2 on the basic regeneration number R_0.

Example 1. Consider the situation of $\tau_1 > 0$, $\tau_2 = 0$. When we choose parameters $r = 0.3$, $k = 9$, $\beta_1 = 0.7$, $\alpha_1 = 0.2$, $\sigma = 0.3$, $\mu = 0.3$, $\varepsilon = 0.5$, $\beta_2 = 0.4$, $\alpha_2 = 0.1$, it is not difficult to get $R_0 = 5.25 > 1$, the equilibrium point $E^*(1.839\,7, 0.442\,9, 0.282\,7)$ and $\tau_{10} = 14.706\,6$. At the same time, the conditions H_{25}, H_{26} are satisfied. According to theorem 1, we know when $\tau_1 < \tau_{10}$, the rumor spreading equilibrium point E^* is locally asymptotically stable and when $\tau_1 > \tau_{10}$, E^* is unstable. As shown in figures 7 and 8, E^* is locally asymptotically stable for $\tau_1 = 13 < \tau_{10}$ and unstable for $\tau_1 = 15 > \tau_{10}$. That is to say, the simulation results are consistent with the above theory.

Example 2. In system (1.2), when $\tau_1 = 0$, $\tau_2 > 0$, we choose $r = 0.01$, $k = 2$, $\beta_1 = 0.7$, $\alpha_1 = 1$, $\sigma = 0.5$, $\mu = 0.1$, $\varepsilon = 0.35$, $\beta_2 = 0.5$, $\alpha_2 = 4$, then we obtain that



(a) The relationship between R_0 and β_1, where $k=1, \mu=0.1, \varepsilon=0.2, \beta_2=0.4$.

(b) The relationship between R_0 and β_2, where $k=1, \mu=0.1, \varepsilon=0.2, \beta_1=0.5$.

Figure 15. The relationship between R_0 and parameters.

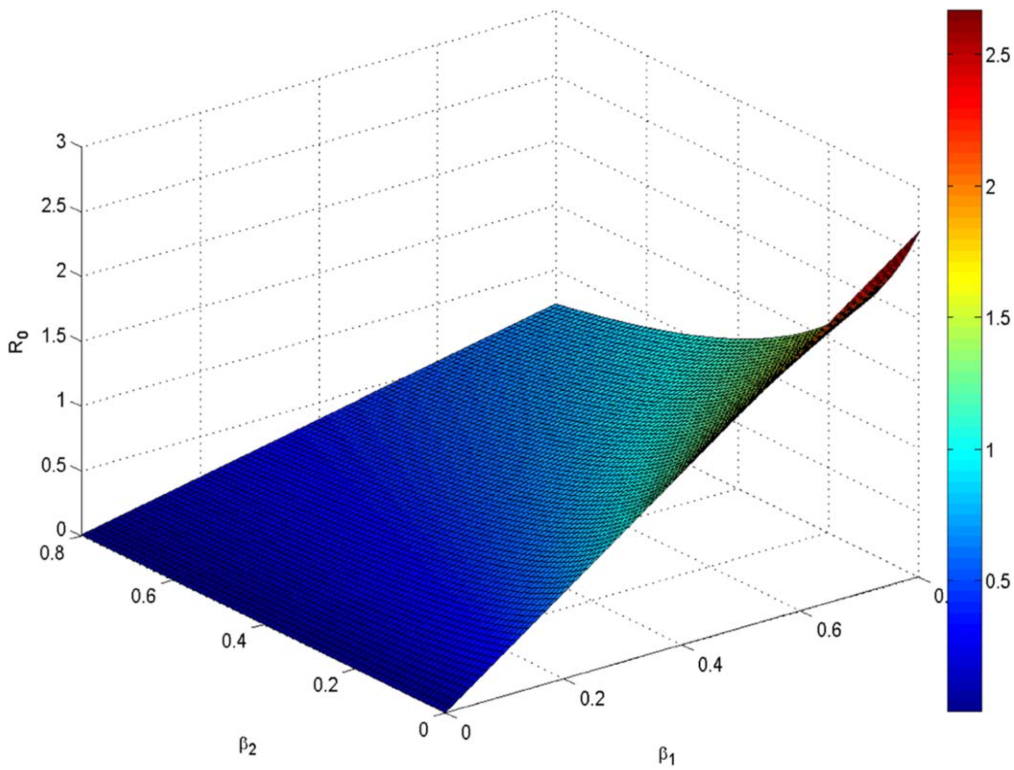


Figure 16. The relationship among R_0, β_1 and β_2, where $k = 1$, $\mu = 0.1$, $\varepsilon = 0.2$.

$R_0 = 1.473\,7 > 1$. Moreover, the conditions H_{34} and H_{36} are satisfied at the positive equilibrium point $E^* = (1.345\,3,\ 0.008\,3,\ 0.006\,7)$. Meanwhile, a simple calculation shows that $\tau_{20} = 35.260\,3$. As shown in figures 9 and 10, the positive equilibrium point E^* is locally asymptotically stable when $\tau_2 = 34.5 < \tau_{20}$, and it is unstable when $\tau_2 = 36 > \tau_{20}$, which is consistent with theorem 2.

Example 3. For $\tau_1 = \tau_2 = \tau$ and $r = 0.3$, $k = 9$, $\beta_1 = 0.7$, $\alpha_1 = 0.2$, $\sigma = 0.3$, $\mu = 0.3$, $\varepsilon = 0.5$, $\beta_2 = 0.4$, $\alpha_2 = 1$ which satisfy H_{41}, we obtain $R_0 = 5.25 > 1$, $E^*(1.675\,1,\ 0.435\,4,\ 0.202\,2)$ and $\tau_0 = 1.789\,2$. By theorem 3, when the value of $\tau < \tau_0$, the equilibrium point E^* is asymptotically stable and when $\tau > \tau_0$, the equilibrium point E^* is unstable. We choose $\tau = 1.3 < \tau_0$ and $\tau = 2 > \tau$ to verify the theorem, the corresponding figure is shown as figures 11 and 12.

Example 4. Consider case 5, that is $\tau_1 \in [0, \tau_{10})$, $\tau_2 > 0$. The parameter is selected as $\tau_1 = \tau_2 = \tau$ and $r = 0.3$, $k = 10$, $\beta_1 = 0.7$, $\alpha_1 = 0.2$, $\sigma = 0.3$, $\mu = 0.3$, $\varepsilon = 0.5$, $\beta_2 = 0.4$, $\alpha_2 = 0.1$, we obtain $R_0 = 5.833\,3 > 1$, $E^* = (1.843\,2,\ 0.455\,0,\ 0.290\,1)$ and $\tau'_{10} = 7.684\,1$. Choose the parameter $\tau_1 = 7 < \tau'_{10}$, by calculation we can get $\tau_{20} = 6.260\,6$. Combined theorem 4, E^* is locally asymptotically stable for $\tau_2 < \tau_{20}$ and when τ_2 exceeds τ_{20}, Hopf bifurcation occurs. When $\tau_2 = 5.5 < \tau_{20}$ and $\tau_2 = 6 > \tau_{20}$ is selected, the images are as follows (figures 13 and 14).

Example 5. In this example, we discuss how R_0 varies with the parameters β_1 and β_2. As shown in figure 15(a), we can find that R_0 and β_1 have a direct proportion relationship, and as shown in figure 15(b), we find that as β_2 increases, R_0 decreases. Moreover, we further study the relationship among R_0, β_1 and β_2. As shown in figure 16, we choose parameters $k = 1$, $\mu = 0.1$, $\varepsilon = 0.2$, $\beta_1 \in [0, 0.8]$ and $\beta_2 \in [0, 0.8]$. It shows that as β_1 increases and β_2 decreases, R_0 is gradually increasing. High incidence rate between susceptible individual and infected individual and low recovery rate are more conducive to the spread of epidemic diseases.

7. Conclusions

In this paper, a SIRS epidemic model with nonlinear incidence rate, saturated treatment and two time delays is investigated, in which the nonlinear incidence rate can reflect the psychology of people facing infectious diseases, and the nonlinear recovery rate can reflect the local medical level and personal physical quality. We use the method of regeneration matrix to find out the basic regeneration number R_0, and show that there is at least one positive equilibrium point in system (1.2), we also have proved system (1.2) has permanence. Meanwhile, two time delays are selected as bifurcation parameters and the corresponding characteristic equations are given. The sufficient conditions for the local stability of positive equilibrium and the existence of Hopf bifurcation are obtained. When there is no delays, we have obtained the condition for local stability of the positive equilibrium point $E^*(S^*, I^*, R^*)$. When there exist time delays, by taking $\tau_1 > 0$ $(\tau_2 = 0)$ and $\tau_2 > 0$ $(\tau_1 = 0)$ as bifurcation parameter, we have obtained the Hopf bifurcation conditions and a critical value of delay τ_{10} and τ_{20}, by taking $\tau_1 = \tau_2 = \tau > 0$ we have obtained the corresponding Hopf bifurcation conditions and a critical value of delay τ_0. Similarly, we have obtain the Hopf bifurcation conditions and a critical value of delay of $\tau_2 > 0$, $\tau_1 \in [0, \tau_{10})$, $\tau_1 \neq \tau_2$ and $\tau_1 > 0$, $\tau_2 \in [0, \tau_{20})$, $\tau_1 \neq \tau_2$. As we can see, for time delays, there is a threshold under which the positive equilibrium is stable, but if the delay is greater than the threshold, sustained oscillations will occur. Furthermore, to minimize the spread of infectious diseases, we have studied the optimal control techniques by the Pontryagin's maximum principle. Finally, we conduct a series of numerical simulations to prove some of the conclusions in this paper, and also investigate the effect of some parameters on R_0.

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