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Quantifying parallelism of vectors is the quantification of distributed *n*-party entanglement

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Abstract

The three way distributive entanglement is shown to be related to the parallelism of vectors. Using a measurement based approach we form a set of 2-dimensional vectors, representing the post measurement states of one of the parties. These vectors originate at the same point and have an angular distance between them. The area spanned by a pair of such vectors is a measure of the entanglement of formation. This leads to a geometrical manifestation of the 3-tangle in 2-dimensions, from an inequality in area which generalizes for n-qubits to reveal that the n-tangle also has a planar structure. Quantifying the genuine n-party entanglement in every $1|(n-1)$ bi-partition, we show that the genuine n-way entanglement does not manifest in n-tangle. A new quantity geometrically similar to 3-tangle is then introduced that represent the genuine n-way entanglement. Extending our formalism to 3-qutrits, we show that the non locality without entanglement arises from a condition under which the post measurement state vectors of a separable state show parallelism. A connection to non trivial sum uncertainty relation analogous to Maccone and Pati uncertainty relation (Maccone and Pati 2014 *Phys. Rev. Lett.* **113** 260401) is then shown using decomposition of post measurement state vectors along parallel and perpendicular direction of the pre-measurement state vectors.

Keywords: quantum entanglement, distributive entanglement, tangle, parallelism of vectors, geometry of quantum entanglement

(Some figures may appear in colour only in the online journal)

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1. Introduction

Non-separability of quantum states has been proven as the most useful resource in the field of quantum information and communication. It leads to quantum entanglement in multiparty systems [2], and lies at the origin of the well known EPR paradox [3] As a quantum resource it has found applications in quantum tasks like teleportation [4–6], quantum error correction [7], quantum dynamics [8] and quantum cryptography [9, 10]. Concurrence, first developed by Hill and Wooters [11], is monotonically related to the bi-partite entanglement of a quantum system and an exact measure of this resource. As shown in [12], concurrence of a physical system, consisting mixed states can be measured experimentally in the form of magnetic susceptibility. Though for two party systems this quantum resource has been studied rigorously [13, 14], entanglement in the higher dimensional systems are not well quantified. In [15], a measurement based separability criterion of quantum states was introduced using the parallelism of post measurement state vectors. The condition for non separability induced from this criterion was then used to co relate concurrence with the parallelism of state vectors of the post measurement systems. In a three particle system, the concurrence between two subsystems has been used to quantify the three way distributive entanglement between the parties [16]. This three way entanglement is termed as 3-tangle. Later 3-tangle was generalized for multipartite cases [17, 18]. The approach was then extended to the class of mixed states [19, 20]. Tangle is useful to other aspects of quantum information as well. It has been used to classify entanglement of all three qubit pure states [21, 22]. Tangle has found applications in classifying multi-qubit graph and hypergraph states [23] as well as in three-ferimionic states [24].

In this work, we provide a geometric interpretation of distributive entanglement present in n-qubit pure states along with a method to quantify the genuine n-way entanglement of a n-party system. A summary of the main contributions of this work is given below-

(i) We use a measurement based approach to form a set of vectors that originates from the same point and show that the distributive entanglement can be expressed in terms of the area of the parallelograms formed by those vectors. This approach is analogous to the Jones vector formalism in polarisation optics, leading to the famous Pancharatnam phase [25], but takes into account the area spanned by two such vectors as opposed to the area swept by the polarization vectors via local transport considered by Pancharatnam. As wedge product of two vectors gives rise to an oriented parallelogram in 2-dimension, we show that the 3-tangle arises from an inequality in the areas of such parallelograms and geometrically the distributive entanglement of 3-qubit pure states can be explained in a 2-dimensional plane. This naturally leads to the Coffman–Kundu–Wooters inequality and the invariance of 3-tangle under permutation of the qubit in focus. This approach provides a deep geometric understanding of multiparty entanglement. Interestingly, concurrence is related to the geometric Berry phase [26], the quantum analogue of Pancharatnam phase. The Berry phase is found to be exactly equal to the concurrence of system in case of an entangled state of two spin $\frac{1}{2}$ particles [27]. We further point that this approach has a connection with non locality induced from irreducibility [28], sum uncertainty relation as given by Pati and Maccone and entanglement of n-qudit pure states.

(ii) We emphasize that from the planner definition of 3-tangle presented here one can not comment on the positivity of the 3-tangle; instead it depends on the direction and magnitudes of the contributing area vectors. This accounts for the generality of this approach, as it has been reported that monogamy relation does not hold for higher dimensional objects [29].

(iii) We further describe the distributive entanglement of n-qubit pure states. Using this geometry, we point out the n-party generalization of tangle [16] is not similar to 3-tangle. It has been shown that n- tangle does not represent the residual n-party entanglement [30]. We show that neither it represents the genuine n- party entanglement present in a system.

(iv) We then introduce a quantity that is geometrically similar to the 3-tangle that quantifies the genuine n-party entanglement in n-qubit systems .

In section 2, we provide the geometrical interpretation of the definition of 3-qubit distributive entanglement from the parallelism of vectors in 2-dimensions. The geometry of n-tangle for odd and even qubits is presented in section 3. Section 4 describes the genuine n-way entanglement in an n-qubit system and the corresponding geometry. We extend the approach of vector parallelism in section 5 to 3-qutrit pure states and show that it can be used to detect non locality without entanglement. In section 6, a connection to the non trivial sum uncertainty relation has been shown. We conclude in section 7 with future directions.

2. Geometry of distributive entanglement from parallelism of vectors

In [15], the concurrence of a pure multipartite state has been discussed using the parallelism of vectors. As entanglement of formation is directly related to squared concurrence [11], one can see the significance of the vector parallelism in quantifying entanglement. The pure state of a 2-qubit system in computational basis can be represented as,

$$|\psi\rangle = a|0_A0_B\rangle + b|1_A0_B\rangle + c|0_A1_B\rangle + d|1_A1_B\rangle. \tag{1}$$

A separability criterion arises for 2-qubit pure states from here as,

$$|ad - bc| = 0. \tag{2}$$

A similar formalism was built in [31] to operationally well defined measures in bipartite quantum systems in arbitrary state space dimensions. From this separability criterion, Bhaskara and Panighrahi have defined the concurrence measure of entanglement (C_A) for 2-qubit systems as the area spanned by the vectors $\langle 0_A|\psi\rangle$ and $\langle 1_A|\psi\rangle$, i.e. $C_A = 2|\langle 0_A|\psi\rangle \wedge \langle 1_A|\psi\rangle|$. If the vectors are found to be parallel to each other, the area spanned by them is non-existent i.e. they are separable. This process for measurement of concurrence using the parallelism of vectors has been generalized for general n-qudit bi-partite systems [15].

If $|\psi\rangle$ be a pure state of a general 3-qubit system with qubits A, B, C, then one can write,

$$|\psi\rangle = a|000\rangle + b|001\rangle + c|010\rangle + d|011\rangle + p|100\rangle + q|101\rangle + r|110\rangle + s|111\rangle. \tag{3}$$

We then consider the bi-partition A| BC and measure system BC in the computational basis. The post measurement non-normalised vectors for system A will be, $\chi_0^A = a|0\rangle + p|1\rangle, \chi_1^A = b|0\rangle + q|1\rangle, \chi_2^A = c|0\rangle + r|1\rangle$, and $\chi_3^A = d|0\rangle + s|1\rangle$. The squared concurrence in this bi-partition of the system is given by [15],

$$\begin{aligned} C^2_{A|BC} = 4[&|\chi_0^A \wedge \chi_1^A|^2 + |\chi_0^A \wedge \chi_2^A|^2 + |\chi_0^A \wedge \chi_3^A|^2 \\ &+ |\chi_1^A \wedge \chi_2^A|^2 + |\chi_1^A \wedge \chi_3^A|^2 + |\chi_2^A \wedge \chi_3^A|^2] \end{aligned} \tag{4}$$

$C^2_{A|BC}$ is directly related to the entanglement of formation in this bi-partition. If all of the four $\chi_i^A, i \in [0, 3]$ vectors are parallel to each other, then the state is separable in this bi-partition.

In 2000, Coffman, Kundu and Wooters has shown if three particles A, B, and C are entangled, the sum of the squared concurrence between A and B ($C^2_{A|B}$) and the squared concurrence between A and C ($C^2_{A|C}$) obey the following Coffman–Kundu–Wooters inequality as [16],

$$C^2_{A|BC} \geqslant C^2_{A|B} + C^2_{A|B}. \tag{5}$$

A definition of the three way entanglement termed as 3-tangle was then followed [16],

$$\tau = C^2_{A|BC} - C^2_{A|B} - C^2_{A|B}. \tag{6}$$

Using the measurement based approach as followed in [15], one can define the concurrence between system A and B, while C is in the system as,

$$C^2_{A|B} = |C_{A|B_0} + C_{A|B_1}|^2,$$

where, $C_{A|B_i}$ is the concurrence between system A and B, when C is measured at state $|i\rangle$. Similarly, the concurrence between system A and C while B is in the system can be written as,

$$C^2_{A|C} = |C_{A|C_0} + C_{A|C_1}|^2.$$

From equation (3) and using the definitions of the post-measurement state vectors of system A, one finds,

$$C^2_{A|B} = 4|(\chi^A_0 \wedge \chi^A_2) + (\chi^A_1 \wedge \chi^A_3)|^2, \tag{7}$$

and,

$$C^2_{A|C} = 4|(\chi^A_0 \wedge \chi^A_1) + (\chi^A_2 \wedge \chi^A_3)|^2. \tag{8}$$

Implementation of equations (4), (7), and (8) in equation (6) leads us to a definition of 3-tangle using the post-measurement state vectors of system A as,

$$\begin{aligned}\tau = 4[|\chi^A_0 \wedge \chi^A_3|^2 + |\chi^A_1 \wedge \chi^A_2|^2 - 2(\chi^A_0 \wedge \chi^A_1) \cdot (\chi^A_2 \wedge \chi^A_3) \\ -2(\chi^A_0 \wedge \chi^A_2) \cdot (\chi^A_1 \wedge \chi^A_3)].\end{aligned} \tag{9}$$

We consider the well known Cauchy–Binet identity,

$$(a \wedge b) \cdot (c \wedge d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c),$$

where a, b, c, d are vectors. Implementation of this identity in equation (9) leads us to the definition of tangle as given by Coffman–Kundu–Wooters. Detailed calculations have been provided in appendix A.

Geometrically the wedge product between two vectors t and u, i.e. $t \wedge u$, is represented as a oriented parallelogram spanned by them [32].The magnitude of the wedge product can be straightforwardly seen as the area of the parallelogram and the orientation of the parallelogram can be understood as a rotation from the direction of u to the direction of t [32].

Considering bi-partition $A|BC$ of the general 3-qubit pure state in equation (3), one writes the post measurement states for system A after measuring the subsystem BC in computational basis as,

$$\chi^A_i = a_i|0\rangle + a_{2^2+i}|1\rangle. \tag{10}$$

Here i is the decimal equivalent to the qubit representation of the state where BC were measured and a_i is the coefficient of the $(i+1)$th term in the expression for $|\psi\rangle$. A coordinate system can be formed using $|0\rangle$ and $|1\rangle$ as the axes in 2-dimensions. The states χ^A_is can be

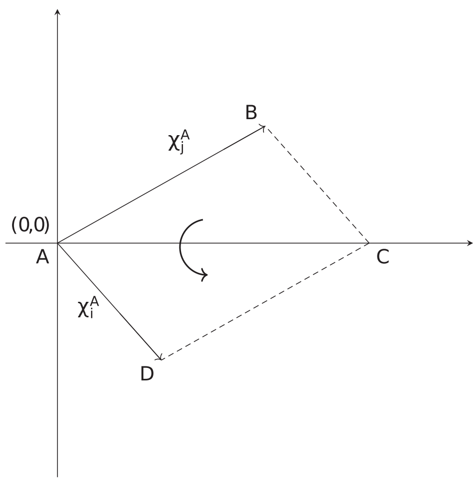

Figure 1. The parallelogram ABCD is spanned by vectors χ_i^A and χ_j^A. The arrow in the middle shows the orientation of the parallelogram.

represented in that co-ordinate systems as vectors (figure 1). The wedge product between two such vectors χ_i^A and χ_j^A can be identified as an oriented parallelogram spanned by them as depicted in figure 1.

From the definition of 3-tangle in equation (9), one can observe that the three way entanglement in system ABC can be represented by the area of several parallelograms as depicted in figure 2. The first two terms in the expression represent squared area of the oriented parallelograms formed by vectors χ_0^A , χ_3^A; and χ_1^A , χ_2^A. The third term, $2|(\chi_0^A \wedge \chi_1^A) \cdot (\chi_3^A \wedge \chi_2^A)|$ can be expressed geometrically as the projection of the oriented parallelogram formed by χ_0^A and χ_1^A on the oriented parallelogram formed by χ_3^A and χ_2^A. Similarly, the fourth term is the projection of the oriented parallelogram formed by χ_0^A and χ_2^A on the oriented parallelogram formed by χ_3^A and χ_1^A. It readily follows that 3-tangle, a property of a 8- dimensional system manifests naturally in 2-dimensions. Also, this definition does not comment on the general positivity of 3-tangle. It depends on the magnitude and directions of the contributing area vectors. However, for 3-qubit systems, the positivity of 3- tangle holds [33].

An interesting observation can be made from this planar representation of 3-tangle as well as distributive entanglement. The first two terms in the expression represents the total squared area of the parallelograms formed by the pairs of state vectors of system A- (χ_0^A, χ_3^A) and (χ_1^A, χ_2^A). The state vectors, χ_0^A and χ_3^A are the states of system A, when the combined system BC is measured in $|00\rangle$ and $|11\rangle$ respectively, i.e. two such states where both the qubits B and C are in mutually exclusive state. Similar observations can be made in case of the parallelogram formed by χ_1^A and χ_2^A. Both these parallelograms do not represent concurrence between any of the parties B or C with A after measuring out the other. On the contrary, the squared concurrence between A and any other party I in a 3-qubit system $C^2_{A|I}, I \in \{B, C\}$ consists of concurrences between A and I while the third party $J(j \in \{B, C\}; J \neq I)$ is measured in computational basis. These projected areas for $I = B$ and $I = C$ in fact adds the last two terms in the definition of the 3-tangle, as can be seen from equation (9). In light of this observation, one can infer that in a three qubit system, the three way entanglement consists of the squared area of the parallelograms that represents no individual bi-party entanglement and a negative contribution from the individual bipartite entanglement between any two parties after measuring out the third party. One can thus rewrite the three tangle as,

$$\tau_3 = 4[\sum_{i=0}^{1} |\chi_i \wedge \chi_{3-i}|^2 - 2\sum_{i=2}^{3}\prod_{j=0}^{1} C_{1|23/i,i=j}] = \tau_3' - \tau_3'', \tag{11}$$

here, $C_{1|23/i,i=j}$ is the concurrence between party 1 and any one of the other two parties, where the third party i is measured at a state that is binary equivalent of j.

Form invariance of 3-tangle under Permutation of qubits—According to [16], the residual entanglement or tangle is invariant under the permutation of qubits. This implies that the residual entanglement in the 3-qubit system is same if we choose bi-partition $B|AC$ of the system $|\psi\rangle$ instead of $A|BC$ and consider the entanglement of formation between qubit B, C and B, A, i.e. choose qubit B as the qubit of focus.

Equation (6) then takes the form,

$$\tau = C^2_{B|AC} - C^2_{B|A} - C^2_{B|C}. \tag{12}$$

Following the measurement approach, one derives the 3-tangle of the system $|\psi\rangle$ with B as qubit of focus as,

$$\tau_3 = 4[\sum_{i=0}^{1} |\chi_i \wedge \chi_{3-i}|^2 - 2\sum_{i=2}^{3}\prod_{j=0}^{1} C_{1|23/i,i=j}] = \tau_3' - \tau_3'',$$

one can easily see that the expression for 3-tangle keeping qubit B in focus respectively is similar to that keeping qubit A in focus. We further prove that they yield same value in terms of coefficients. Using the definitions of the state vectors representing system B after measuring system AC, one obtains the same value as Coffman–Kundu–Wooters. Detailed calculation are provided in appendix B. Similarly considering bi-partition $C|AB$ and entanglement of formation between qubits C, A and C, B, one can obtain the 3-tangle to have the same form as equation (13), and yield same value, which establishes that the definition of tangle achieved through this measurement based approach remains invariant under permutation of qubits.

3. Extension of the geometric definition of tangle in *n*-qubit systems

The representation of tangle in 2-dimensions is further extended for n-qubit states as follows. In [17], a generalized definition of tangle for even n qubits was given as,

$$\tau_{1,2,\ldots,n} = 2|a_{\alpha_1\alpha_2\ldots\alpha_n} a_{\beta_1\beta_2\ldots\beta_n} a_{\gamma_1\gamma_2\ldots\gamma_n} a_{\delta_1\delta_2\ldots\delta_n} \times \epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\ldots\epsilon_{\alpha_{n-1}\beta_{n-1}}\epsilon_{\gamma_1\delta_1}\epsilon_{\gamma_2\delta_2}\ldots\epsilon_{\gamma_{n-1}\delta_{n-1}}\epsilon_{\alpha_n\gamma_n}\epsilon_{\beta_n\delta_n}|, \tag{13}$$

where a terms are the coefficients of the state of the system expressed in the computational basis, $\epsilon_{00} = -\epsilon_{11} = 0$, and $\epsilon_{01} = -\epsilon_{10} = 1$. The tangle for even n-qubit states can be rewritten as [30],

$$\tau_{1,2,\ldots,n} = 4|\sum_{i=0}^{2^{n-1}-1} (-1)^{N(i)} a_{2i} a_{(2^n-1)-2i}|^2, \tag{14}$$

where $N(i)$ is the number of 1s in the binary representation of i. This expression can be reduced as,

$$\tau_{1,2,\ldots,n} = 4[\sum_{i=0}^{2^{n-2}-1} |\chi_i \wedge \chi_j|]^2 \tag{15}$$

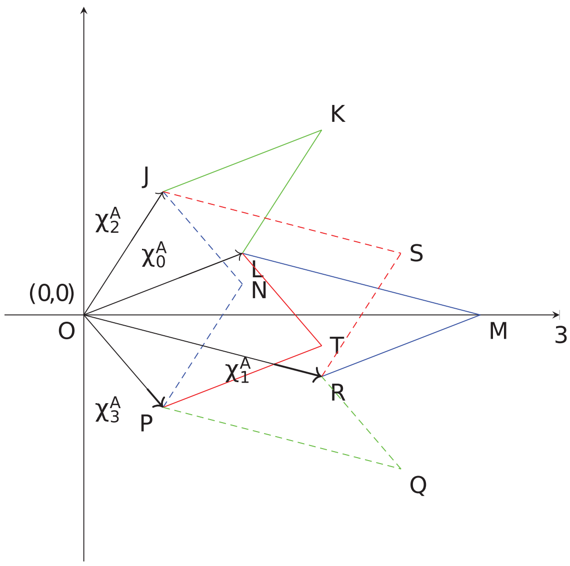

Figure 2. This figure represents the geometrical visualization of the 3-tangle. The post measurement state vectors of qubit A are shown as vectors $\chi_0^A = \vec{OL}$, $\chi_1^A = \vec{OR}$, $\chi_0^A = \vec{OJ}$, $\chi_0^A = \vec{OP}$. The parallelogram OLTP (solid red) is spanned by vectors χ_0^A and χ_3^A represents the first term in the expression for tangle; The second term in the expression can be visualized as parallelogram OJSR (dashed red). The first interaction term in the expression is between concurrence $C_{A|B_0}$ and $C_{A|B_0}$. The geometric representation for $C_{A|B_0}$ can be visualized as parallelogram OJKL (solid green) and $C_{A|B_1}$ as OPQR (dashed green). Similarly, concurrences $C_{A|C_0}$ and $C_{A|C_1}$ in the second interaction term can be visualized as parallelograms OLMR (solid blue) and OPNJ (dashed blue).

where $j = 2^{(n-1)} - 1 - i$, $k = 2^{(n-1)} - 1 - l$, $k' = 2^{(n-1)} - 1 - l'$ and χ_i^A s are the post measurement state vectors of party A while the other bi-partition is measured in a state that is the binary equivalent of i.

From this description of n-tangle it is clear that $\tau_{1,2,..,n}$ can be expressed in a 2-dimensional plane as a sum of squared and projected areas of parallelograms formed by the state vectors of party 1..

Following the definition of n-tangle for odd n-qubits as [18], one can express it as,

$$\tau_{1,2,\ldots,n} = 4[\sum_{i=0}^{2^{n-2}-1} |\chi_i \wedge \chi_l|^2 + \sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} \sum_{j=0,i<j}^{2^{n-2}-1} (-1)^{N(j)} [(\chi_i \wedge \chi_j) \cdot (\chi_k \wedge \chi_l) + (\chi_i \wedge \chi_k) \cdot (\chi_j \wedge \chi_l)]], \quad (16)$$

where $l = 2^{(n-1)} - 1 - i$ and $k = 2^{(n-1)} - 1 - j$. Detailed calculations have been provided in appendix C.

Equation (16) reflects that for n-qubit systems n-tangle can be represented in terms of the squared or projected areas of the parallelograms formed by the post-measurement state vectors of the qubit in focus. Moreover, n-tangle can be expressed in a plane i.e. 2-dimensions. However, comparing the forms of equations (9), (15), and (16), one can easily see that the geometrically, 3-tangle is not similar to the n-tangle. Also, as found by, Li and Li n-tangle is not the residual entanglement for any even $n \geqslant 4$ qubits [30]. Furthermore, neither n-tangle, nor residual entanglement give any idea on genuine n-way entanglement in a system with $n > 3$ qubits. Residual entanglement and n-tangle for qubits deals with the relation between the total entanglement in a bi-partition of a n-party system and the individual entanglement between each parties, i,e,

$$\tau_{1,2,...n} = C^2_{1|2...n} - (C^2_{1|2} + C^2_{1|3} + ... + C^2_{1|n}).$$

But n-tangle does not comment on the difference of entanglement in $1|(n-1)$ bi-partition and sum of entanglement in the $1|(n-2)$ bi-partitions, which is also defined by 3-tangle in 3-party systems. Using the measurement based approach presented in this work, we now define a quantity that measures the n-way entanglement in an n-party system, i.e. the relation between entanglement of formation in a bi-partition $1|23..(n-1)n$ and the sum of the total entanglement in that bi-partition when one of n parties are not considered. We further show that this quantity is geometrically similar to 3-tangle, i,e, the correlation like $C^2_{1|2..n} - (C^2_{1|23...(n-1)} + C^2_{1|23...(n-2)n} + ... + C^2_{1|34..n})$ does not manifest in n-tangle. We name this correlation the genuine n- way entanglement.

4. Genuine *n*-way entanglement in *n*-qubit system

We consider a 4-qubit system first. For a system with 4-parties 1, 2, 3 and 4, we define the relation between the total entanglement in the bi-partition $1|234$ and the entanglement between the parties $1|23$, $1|34$ and $1|24$ and call the difference 'the genuine 4-way entanglement (τ_4)' of that bi-partition as,

$$\tau_4 = [2C^2_{1|234} - (C^2_{1|23} + C^2_{1|24} + C^2_{1|34})]. \tag{17}$$

Using the definition of concurrence in this bi-partition [15], it can be shown that,

$$\tau_4 = 4[\sum_{i=0}^{2^2-1} 2|\chi_i \wedge \chi_{2^3-1-i}|^2 - 2\sum_{i=2}^{4}\prod_{j=0}^{1} C_{1|234/i,i=j}] = 2\tau_4' - \tau_4''. \tag{18}$$

Detailed calculations have been provided in appendix D. Here χ_i is the post measurement state vector of party 1, while the rest of the system is measured a state that is binary equivalent of i. This genuine 4-party entanglement is composed of two terms. Geometrically, the first term $2\tau_4'$ represents twice the total area of the parallelograms formed by the state vectors of party 1, when the rest of the system is measured at 4 particular pairs of states($|000\rangle, |111\rangle$; $|001\rangle, |110\rangle$; $|010\rangle, |101\rangle$; and $|011\rangle, |100\rangle$). The superposition of the states of each of these pairs represents maximal entanglement in a 3-party system. Each qubit is at mutually exclusive states in these pairs. The second term, τ_4'' represents twice the total projected area of total entanglement between party 1 and any two parties (i, j) in the system when the fourth party (k) is measured at $|0\rangle$ on the area of total entanglement in bi-partition $1|ij$, while k is measured at $|1\rangle$. The genuine 4-party entanglement is the difference of these two areas. One can not comment if τ_4 is positive; it depends entirely on the length and direction of the concerned area vectors. One can check, for 4-qubit GHZ state, $\tau_4 = 2$, and for 4-qubit W state, $\tau_4 = 0$; implying GHZ is genuinely 4-party entangled and W is not. In [34], it has been shown that 4-qubit systems can be entangled in nine different ways under SLOCC, of which the generic family gives rise to true 4-party entanglement like GHZ state. The 3-tangle is used to distinguish between the SLOCC classes of entanglement in 3-party systems. From the geometrical symmetry of 3-tangle and genuine 4-party entanglement we hope that this quantity can be useful in differentiating the SLOCC classes in 4-qubit systems.

We now generalize the genuine 3-and 4-party entanglement for pure n-qubit systems. Using mathematical induction, one can find a general expression for genuine n way entanglement in bi-partition $1|23..n$ from the expressions for τ_3 and τ_4 in equations (11) and (18) respectively,

$$\tau_n = 4[\sum_{i=0}^{2^{(n-2)}-1}(n-2)|\chi_i \wedge \chi_{2^{(n-1)}-1-i}|^2 - 2\sum_{i=2}^{n}\prod_{j=0}^{1}C_{1|23....(n-1)n/i,i=j}] \\ = (n-2)\tau_n' - \tau_n''. \tag{19}$$

χ_i is the post measurement state vector of party 1, while the rest of the system is measured a state that is binary equivalent of i. Detailed calculations have been provided in appendix E. We also show that the genuine n-way entanglement is form invariant under permutation of qubits, i.e.

$$(\tau_n)_k = 4[\sum_{i=0}^{2^{(n-2)}-1}(n-2)|\chi_i \wedge \chi_{2^{(n-1)}-1-i}|^2 \\ - 2\sum_{i=1,i\neq k}^{n}\prod_{j=0}^{1}C_{k|12...(k-1)(k+1)...(n-1)n/i,i=j}] = (n-2)(\tau_n')_k - (\tau_n'')_k,$$

here $(\tau_n)_k$ is the genuine n-way entanglement in bi-partition $k|123..(k-1)(k+1)...n$ and χ_i is the post measurement state vector of party k, while the rest of the system is measured a state that is binary equivalent of i. Detailed calculations have been provided in appendix F.

The genuine n-way entanglement τ_n, like 3-tangle and the genuine 4-way entanglement, consists of two terms, $(n-2)\tau_n'$ and τ_n''. The first term always represents $(n-2)$ times the total squared area of the parallelograms formed by the pairs state vectors of the system 1, where the rest of the qubits are at a state that is mutually exclusive. The term τ_n'' is given by twice the total projected area of total entanglement between party 1 and all other parties in the system excluding i, when the ith party is measured at $|0\rangle$ on the area of total entanglement in same bi-partition while i is measured at $|1\rangle$. This area can add to area represented by τ_n' or can be subtracted from it depending on the length and direction of the area vectors. Thus, the genuine n-tangle can be along the direction of the total entanglement in the bi-partition or along the direction of the τ_n''. Based on this properties, one can assume that geometrically, the 3-tangle is a special case of the genuine n-party entanglement. τ_n is 0 for general n-qubit W state, and $(n-2)$ for general n-qubit GHZ states.

5. Extension to 3-qutrit pure states to detect non-locality

We consider a 3-qutrit pure state as,

$$|\psi\rangle_{123} = \sum_{i=0}^{3^3-1} a_i|i\rangle,$$

here i represent the decimal equivalent of the state of the system. Post measurement vectors of system 1 after measuring the system 23 in basis $|ij\rangle, i \in [0,2], j \in [0,2]$, can be written as,

$$\chi_i = a_i|0\rangle + a_{3^2+i}|1\rangle + a_{2\times 3^2+i}|2\rangle.$$

χ_i can thus be seen as a vector in 3-dimensions. The entanglement in bi-partition $1|23$ can be written as [15],

$$C_{1|23}^2 = 4\sum_{i=0}^{7}\sum_{j=1,j>i}^{8}|\chi_i \wedge \chi_j|^2.$$

Each term in the expression of $C_{1|23}^2$ here can be seen as squared area in 3 dimensions and the entanglement in this bi-partition being the combined squared length of the said vector. Thus the extension of parallelism of vectors in arbitrary n-qudit states is possible. This approach can find applications in not only quantifying n-party entanglement but also detecting non-locality in a set of quantum states. In [28], it has been shown that a strong quantum non locality exists in $1-23$ bi-partition of a orthogonal product basis (OPB) using the notion of local irreducible sets. The OPB defined on 3-qutrit system stated to be locally irreducible, thus strongly non-local in [28] is given as,

$$\begin{array}{ccc}
|1\rangle|2\rangle|1\pm2\rangle & |2\rangle|1\pm2\rangle|1\rangle & |1\pm2\rangle|1\rangle|2\rangle \\
|1\rangle|3\rangle|1\pm3\rangle & |3\rangle|1\pm3\rangle|1\rangle & |1\pm3\rangle|1\rangle|3\rangle \\
|2\rangle|3\rangle|1\pm2\rangle & |3\rangle|1\pm2\rangle|2\rangle & |1\pm2\rangle|2\rangle|3\rangle \\
|3\rangle|2\rangle|1\pm3\rangle & |2\rangle|1\pm3\rangle|3\rangle & |1\pm3\rangle|3\rangle|2\rangle \\
\end{array}$$
$$|1\rangle|1\rangle|1\rangle \qquad |2\rangle|2\rangle|2\rangle \qquad |3\rangle|3\rangle|3\rangle.$$

We note that after performing multiple orthogonality preserving measurement on one of the parties (party 1), the state vectors of the rest of the system are non parallel for each measurement outcome of 1, the system can be considered non local. We also consider the locally reducible GHZ basis, and note that in this case, the post measurement state vectors are found to be parallel. Thus, there is a co-relation between non-local quantum correlations and non-parallelism of post measurement vectors.

The OPB defined on 3-qutrit system stated to be strongly non-local in [28] can be expressed as,

$$\begin{array}{ccc}
|1\rangle|2\rangle|1\pm2\rangle & |2\rangle|1\pm2\rangle|1\rangle & |1\pm2\rangle|1\rangle|2\rangle \\
|1\rangle|3\rangle|1\pm3\rangle & |3\rangle|1\pm3\rangle|1\rangle & |1\pm3\rangle|1\rangle|3\rangle \\
|2\rangle|3\rangle|1\pm2\rangle & |3\rangle|1\pm2\rangle|2\rangle & |1\pm2\rangle|2\rangle|3\rangle \\
|3\rangle|2\rangle|1\pm3\rangle & |2\rangle|1\pm3\rangle|3\rangle & |1\pm3\rangle|3\rangle|2\rangle \\
\end{array}$$
$$|1\rangle|1\rangle|1\rangle \qquad |2\rangle|2\rangle|2\rangle \qquad |3\rangle|3\rangle|3\rangle.$$

One consider a orthogonality preserving local measurement on party 1, and finds the post measurement states of composite system of parties 23 to be either at $|21\pm22\rangle$ or $|31\pm13\rangle$ or at $|11\rangle$ if system 1 is at $|1\rangle$. Similarly, if party 1 is at $|2\rangle$, the rest of the system will be at a state from the set

$$\{|11\pm21\rangle, |13\pm33\rangle, |31\pm32\rangle, |22\rangle\}$$

and from the set

$$\{|11\pm31\rangle, |21\pm23\rangle, |12\pm22\rangle, |33\rangle\}$$

if the party 1-is at $|3\rangle$. For the other four possible measurement outcomes of party 1, $|1\pm2\rangle$, and $|1\pm3\rangle$, system 23 is at a state from the set

$$\{|12\rangle, |23\rangle\}$$

and

$$\{|13\rangle, |32\rangle\}$$

respectively. One notices that none of the post measurement state vectors while party 1 is measured in different states are parallel to each other.

We now consider the 3-qubit GHZ basis as,

$$|000\rangle \pm |111\rangle, |011\rangle \pm |100\rangle, |001\rangle \pm |110\rangle, |010\rangle \pm |101\rangle.$$

After performing a local measurement on party 1, one finds the post measurement state vectors for system 23 to be at a state from the state

$$\{|00\rangle, |11\rangle\}$$

for both the cases when party 1 is measured at $|0\rangle$ and $|1\rangle$. In this case, the post measurement state vectors of system 23 can be parallel to each other after the first party is measured in two different states. This means, from non-parallelism of post measurement state vectors the strong non locality in the OPB [28] can be detected, i.e. using parallelism of vectors one can identify the irreducibility induced non locality in a multiparty system. If after multiple measurements on one of the parties (party 1), the state vectors of the rest of the system are non parallel for each measurement outcome of 1, the system can be considered non local.

6. Connection to non trivial sum uncertainty relation

The post measurement state vector of a system remains parallel to the state vector prior to the measurement if and only if the state is a eigenstate of the measuring observable. The Heisenberg uncertainty relation [35] is given by,

$$\Delta A^2 \Delta B^2 \geqslant |\frac{1}{2}\langle[A, B]\rangle|.$$

This relation becomes trivial when the state of the system is an eigenstate of any of the operators, i.e. the state vector of the system is parallel to the state vector after measuring any of the operators involved. Using Vaidman's formula [36] and considering a sum of uncertainty between the observables A and B when state of the system is at an eigenstate of only one observable one obtains,

$$\Delta A^2 + \Delta B^2 \geqslant \max(\Delta A^2, \Delta B^2).$$

This uncertainty relation is analogous to the uncertainty relation given by Maccone and Pati [1]. From Vaidman's formula [36], for any hermitian operator A acting on a quantum state $|\psi\rangle$, the resultant state can always be decomposed as,

$$H|\psi\rangle = a|\psi\rangle + b|\psi_\perp\rangle,$$

where $|\psi_\perp\rangle$ is the orthogonal state to $|\psi\rangle$. This means, if one performs a measurement on a state of a quantum system, the post measurement state vectors will have two components, one along the direction of the pre-measurement state vector and another along the direction perpendicular to it. We now consider two measurement operators A and B acting on the system which is at an eigenstate $|\phi\rangle$ of operator A, i.e.

$$A|\phi\rangle = a|\phi\rangle.$$

The post measurement state in this case is entirely parallel to the direction of the pre measurement state. From [36],

$$B|\phi\rangle = b_1|\phi\rangle + b_2|\phi_\perp\rangle.$$

One finds,

$$\Delta A^2 = \langle A^2 \rangle - \langle A \rangle^2 = 0.$$

As the post measurement state vector is parallel to the pre measurement state vector in this case, the expected average of A^2 yields a^2. The squared expected average of A also yields a^2, so uncertainty in measuring A in this system is 0.

However, for operator B one finds,

$$\Delta B^2 = \langle B^2 \rangle - \langle B \rangle^2 = b^2 = |\langle \phi_\perp | B | \phi \rangle|^2.$$

The product uncertainty relation as given by Heisenberg, [35],

$$\Delta A^2 \Delta B^2 \geqslant |\frac{1}{2}\langle [A, B] \rangle|$$

becomes 0 when one considers the post measurement state of any one operator to be parallel to the pre measurement state. Considering a sum of uncertainty between the observables A and B and the state of the system to be an eigenstate of A as above one obtains,

$$\Delta A^2 + \Delta B^2 = b^2 = |\langle \phi_\perp | B | \phi \rangle|^2.$$

This makes the sum uncertainty relation non trivial as the lower bound becomes

$$\Delta A^2 + \Delta B^2 \geqslant \max(\Delta A^2, \Delta B^2),$$

for any state of the system other than a simultaneous eigenstate of both the operators. This uncertainty relation is analogous to that given by Maccone and Pati [1].

7. Conclusions

We showed that vector parallelism can lead to a geometrical representation of the three-way entanglement in 2-dimensions. This geometric interpretation of distributive entanglement explains Coffman–Kundu–Wootters inequality as an inequality in area and the natural manifestation of 3-tangle from it. We generalized the same geometry for n-qubit pure entangled states to observe that the n-tangle also has a planar structure, but the geometry of n-tangle is not similar to that of 3-tangle. While 3-tangle measures both distributive and the genuine 3-party entanglement of a 3-qubit system, n-tangle represents none of them. We then proposed a new quantity, geometrically similar to 3-tangle, that quantifies the genuine n-party entanglement of a n-qubit system. Akin to the physical convenience of the concurrence and tangle we also hope to report more about the distributed n-party entanglement in future. For 4-qubit graph states, the genuine 4-party entanglement is able to establish a connection between the entanglement of formation at of a party and the number of edges at the vertex of the graph representing it [33]. The geometrical formalism based on parallelism of post measurement state vectors is itself a main result of this work. In [37], the quantum geometric phase has been connected to quantum many body systems using Majorana's stellar representation and it has been further connected to 3-tangle and the entanglement of the symmetric qubit pure states [38]. Since the formalism presented here is similar to Jones matrix formalism of polarization optics leading to the Pancharatnam phase, it may be possible to extend the relation between the geometric phases and the genuine n-way entanglement. The connection of this formalism with non locality can be explored to quantify quantum steering [2, 39], and quantum noncontextuality [40]. Product uncertainty relation has been connected with inseparability criterion using PPT states[41–44]. The formalism presented here can be further explored to connect the stronger and non-trivial Pati Maccone uncertainty relation with inseparability criterion.

Appendix A. Expression of 3-tangle in terms of area

$|\psi\rangle$ be a pure state of a general 3-qubit system with qubits A, B, C, then one can write,

$$|\psi\rangle = a|000\rangle + b|001\rangle + c|010\rangle + d|011\rangle \\ +p|100\rangle + q|101\rangle + r|110\rangle + s|111\rangle.$$

In terms of the coefficients of the pure state $|\psi\rangle$ the 3-tangle can be written as [16],

$$\tau = 4|d_1 - 2d_2 + 4d_3|,$$

where,

$d_1 = a^2s^2 + b^2r^2 + c^2q^2 + p^2d^2,$
$d_2 = asdp + ascq + asbr + cqdp + cqbr + dpbr$
and, $d_3 = bscp + ardq.$

The post measurement state vectors of qubit A after measuring system BC in computational basis are,

$\chi_0^A = a|0\rangle + p|1\rangle$
$\chi_1^A = b|0\rangle + q|1\rangle$
$\chi_2^A = c|0\rangle + r|1\rangle$
$\chi_3^A = d|0\rangle + s|1\rangle$

We consider the wedge product definition of tangle,

$$4|\chi_0^A \wedge \chi_3^A|^2 + |\chi_1^A \wedge \chi_2^A|^2 - 2(\chi_0^A \wedge \chi_1^A) \cdot (\chi_2^A \wedge \chi_3^A) \\ -2(\chi_0^A \wedge \chi_2^A) \cdot (\chi_1^A \wedge \chi_3^A)| = 4[\alpha - 2\beta - 2\gamma],$$

where,

$\alpha = |\chi_0^A \wedge \chi_3^A|^2 + |\chi_1^A \wedge \chi_2^A|^2$
$\beta = |(\chi_0^A \wedge \chi_1^A) \cdot (\chi_2^A \wedge \chi_3^A)|$
$\gamma = |(\chi_0^A \wedge \chi_2^A) \cdot (\chi_1^A \wedge \chi_3^A)|.$

Expanding α in terms of the coefficients,

$$\alpha = |(as - dp)^2| + |(cq - br)^2| \\ = |a^2s^2 + d^2p^2 - 2asdp + c^2q^2 + b^2r^2 - 2cqbr|.$$

From the expression of β,

$$\beta = |(\chi_0^A \wedge \chi_1^A) \cdot (\chi_2^A \wedge \chi_3^A)|.$$

Using Binet–Cauchy identity

$$(a \wedge b) \cdot (c \wedge d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$$

one can have,

$$\beta = (\chi_0^A \cdot \chi_2^A)(\chi_1^A \cdot \chi_3^A) - (\chi_0^A \cdot \chi_3^A)(\chi_1^A \cdot \chi_2^A).$$

Expanding in terms of coefficients of the post-measurement state vectors of qubit A,

$$\beta = dpbr + ascq - ardq - bscp.$$

Similarly implementation of Binet–Cauchy identity in the definition of γ will yield,

$$\gamma = asbr + aqdp - ardq - bscp.$$

The wedge product definition of tangle therefore gives three tangle in terms of coefficients as,

$$\alpha - 2\beta - 2\gamma = a^2s^2 + c^2q^2 + b^2r^2 + d^2p^2$$
$$- 2(asdp + ascq + asbr + aqdp + cqbr + dpbr) + 4(bscp + ardq).$$

Appendix B. Invariance of 3-tangle under permutation of the qubit in focus

The post measurement state vectors of qubit B after measuring system AC in the computational basis are,

$$\chi_0^B = \langle 00|\psi\rangle = a|0\rangle + c|1\rangle$$
$$\chi_1^B = \langle 01|\psi\rangle = b\,|0\rangle + d\,|1\rangle$$
$$\chi_2^B = \langle 10|\psi\rangle = p\,|0\rangle + r\,|1\rangle$$
$$\chi_3^B = \langle 11|\psi\rangle = q\,|0\rangle + s\,|1\rangle\,.$$

The wedge product representation of 3-tangle keeping qubit B in focus is,

$$\tau = 4[|\chi_0^B \wedge \chi_3^B|^2 + |\chi_1^B \wedge \chi_2^B|^2 + 2|(\chi_0^B \wedge \chi_1^B)\cdot(\chi_3^B \wedge \chi_2^B)|$$
$$+2|(\chi_0^B \wedge \chi_2^B)\cdot(\chi_3^B \wedge \chi_1^B)|].$$

It is easily seen that,

$$|\chi_0^B \wedge \chi_3^B|^2 + |\chi_1^B \wedge \chi_2^B|^2 = |(as - cq)^2 + (br - dp)^2|$$
$$= |a^2s^2 + c^2q^2 + b^2r^2 + d^2p^2 - 2(ascq + dpbr)|.$$

Implementation of Binet- Cauchy identity in the interaction terms yield,

$$|(\chi_0^B \wedge \chi_1^B)\cdot(\chi_3^B \wedge \chi_2^B)| = |ardq + bscp - cqbr - asdp|,$$

and,

$$|(\chi_0^B \wedge \chi_2^B)\cdot(\chi_3^B \wedge \chi_1^B)| = |ardq + bscp - cqdp - asbr|.$$

The 3-tangle with qubit B in focus hence takes the form,

$$\tau = 4|a^2s^2 + c^2q^2 + b^2r^2 + d^2p^2$$
$$- 2(asdp + ascq + asbr + aqdp + cqbr + dpbr) + 4(bscp + ardq)|.$$

Appendix C. Generalisation of 3-tangle to the *n*-qubit system

For an n-qubit state, it can be easily seen that the post measurement state vectors of the first qubit after measuring the rest of the bi-partition in computational basis can be represented as,

$$\chi_i = a_i|0\rangle + a_l|1\rangle,$$

where $l = 2^{(n-1)} + i$.

n-tangle for $n =$ even qubits is given by [30],

$$\tau_{1,2,..,n} = 4|\sum_{i=0}^{2^{n-1}-1}(-1)^{N(i)}a_{2i}a_{(2^n-1)-2i}|^2,$$

$N(i)$ is the number of 1s in the binary representation of the decimal number i.

This definition can be rewritten as,

$$\tau_{1,2,..,n} = 4|\sum_{i=0}^{2^{n-1}-1}(-1)^{N(i)}a_i a_{2^{(n-1)}-1-i}|^2.$$

Where the R.H.S can found to be,

$$4|\sum_{i=0}^{2^{n-2}-1}(-1)^{N(i)}a_i a_{2^{(n-1)}+j} - a_j a_{2^{(n-1)}+i}|^2,$$

considering $j = 2^{(n-1)} - 1 - i$.

One can now see,

$$a_i a_{2^{(n-1)}+j} - a_j a_{2^{(n-1)}+i} = \chi_i \wedge \chi_j$$

.

Thus the n-tangle for even n qubits follows,

$$\tau_{1,2,..,n} = 4[\sum_{i=0}^{2^{n-2}-1}(-1)^{N(i)}|\chi_i \wedge \chi_j|]^2.$$

For odd n-qubits the n-tangle can be expressed as [18],

$$\tau_{1,2,...,n} = 4|T^2 - PQ|,$$

where,

$$T = \sum_{i=0}^{2^{n-1}-1}(-1)^{N(i)}a_i a_{2^n-1-i},$$

$$P = 2\sum_{i=0}^{2^{n-2}-1}(-1)^{N(i)}a_{2i}a_{2^{n-1}-1-2i},$$

and

$$Q = 2\sum_{i=0}^{2^{n-2}-1}(-1)^{N(i)}a_{2^{n-1}+2i}a_{2^n-1-2i}.$$

One can rewrite T as,

$$T = \sum_{i=0}^{2^{n-2}-1}(-1)^{N(i)}[a_i a_{2^{n-1}+l} + a_{2^{n-1}+i}a_l],$$

where $l = 2^{n-1} - 1 - i$.

Thus,

$$\begin{aligned}
T^2 = &\sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} |a_i a_{2^{n-1}+l} + a_{2^{n-1}+i} a_l|^2 \\
&+ 2 \sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} \sum_{j=0, i<j}^{2^{n-2}-1} (-1)^{N(j)} [a_i a_{2^{n-1}+l} + a_{2^{n-1}+i} a_l][a_j a_{2^{n-1}+k} + a_{2^{n-1}+j} a_k] \\
&= M_1 + W_1.
\end{aligned}$$

Here $k = 2^{n-1} - 1 - j$; M_1 denotes the first summation and W_1 denotes the second summation.

Rewriting P and Q one can see,

$$\begin{aligned}
PQ &= 4 \sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} [a_i a_l] \sum_{j=0}^{2^{n-2}-1} (-1)^{N(j)} [a_{2^{(n-1)}+i} a_{2^{(n-1)}+l}] \\
&= 4 \sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} [a_i a_l a_{2^{(n-1)}+i} a_{2^{(n-1)}+l}] \\
&+ 4 \sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} \sum_{j=0, i<j}^{2^{n-2}-1} (-1)^{N(j)} [a_i a_l a_{2^{(n-1)}+j} a_{2^{(n-1)}+k} + a_j a_k a_{2^{(n-1)}+i} a_{2^{(n-1)}+l}] \\
&= M_2 + W_2,
\end{aligned}$$

Here M_2 denotes the first summation and W_2 denotes the second summation.

The expression for n-tangle for odd n qubits takes the form,

$$\tau_{1,2,\ldots,n} = 4[M + W],$$

where $M = M_1 - M_2$ and $W = W_1 - W_2$.

One can see from the expression of M,

$$\begin{aligned}
M &= \sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} [|a_i a_{2^{n-1}+l} + a_{2^{n-1}+i} a_l|^2 - 4|a_i a_l a_{2^{(n-1)}+i} a_{2^{(n-1)}+l}|] \\
&= \sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} |a_i a_{2^{n-1}+l} - a_{2^{n-1}+i} a_l|^2 \\
&= \sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} |\chi_i \wedge \chi_j|^2.
\end{aligned}$$

Similarly the expression of W takes the form,

$$\begin{aligned}
W = &\sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} \sum_{j=0, i<j}^{2^{n-2}-1} (-1)^{N(j)} [2(a_i a_{2^{n-1}+l} a_j a_{2^{n-1}+k} + a_{2^{n-1}+i} a_l a_j a_{2^{n-1}+k} \\
&+ a_{2^{n-1}+i} a_l a_{2^{n-1}+j} a_k + a_i a_{2^{n-1}+l} a_{2^{n-1}+j} a_k) - 4(a_i a_l a_{2^{(n-1)}+j} a_{2^{(n-1)}+k} \\
&+ a_j a_k a_{2^{(n-1)}+i} a_{2^{(n-1)}+l})] \\
&= \sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} \sum_{j=0, i<j}^{2^{n-2}-1} (-1)^{N(j)} [(\chi_i \wedge \chi_j) \cdot (\chi_k \wedge \chi_l) + (\chi_i \wedge \chi_k) \cdot (\chi_j \wedge \chi_l)].
\end{aligned}$$

The n-tangle for n = odd qubits is thus,

$$\tau_{1,2,\ldots,n} = 4[\sum_{i=0}^{2^{n-2}-1} |\chi_i \wedge \chi_l|^2 + \sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} \sum_{j=0,i<j}^{2^{n-2}-1} (-1)^{N(j)} [(\chi_i \wedge \chi_j)\cdot(\chi_k \wedge \chi_l) + (\chi_i \wedge \chi_k)\cdot(\chi_j \wedge \chi_l)]].$$

Appendix D. Derivation of genuine 4-party entanglement in 4-qubit system

To define the genuine 4-way entanglement for a pure 4-qubit state in bi-partition $1-234$, we consider the sum of concurrences between $1-23$, $1-24$, and $1-34$ to be M, where,

$$M = C^2_{1|23} + C^2_{1|24} + C^2_{1|34},$$

where $C_{i|j}$ is the concurrence between the bi-partitions i and j.

We consider the concurrence between party 1 and 23 in this 4-party system, $C_{1|23}$ as,

$$C^2_{1|23} = (C_{1|23_{4=0}} + C_{1|23_{4=1}})^2 = C^2_{1|23_{4=0}} + C^2_{1|23_{4=1}} + 2(C_{1|23_{4=0}}.C_{1|23_{4=1}}).$$

$C^2_{1|23_{4=0}}$ can be written as,

$$\begin{aligned} C^2_{1|23_{4=0}} &= C^2_{1|2_{4=0}} + C^2_{1|3_{4=0}} + \tau_3|_{1|23_{4=0}} \\ &= C^2_{1|2_{34=00}} + C^2_{1|2_{34=10}} + C^2_{1|3_{24=00}} + C^2_{1|2_{24=10}} + \tau_{x3}|_{1|23_{4=0}}, \end{aligned}$$

Here, $\tau_3|_{1|23_{4=0}}$ is the 3-tangle between the bi-partition $1-23$ when 4 is measures at state $|0\rangle$ and $\tau_{x3}|_{1|23_{4=0}}$ is the positive part of the 3-tangle in that bi-partition when 4 is measured at $|0\rangle$.

Similarly,

$$M = 2\sum_{i=2}^{4}\sum_{j=0}^{3} C^2_{1|i_j} + \sum_{i=2}^{4}\sum_{j=0}^{1} \tau_{x3}|_{1|234/i,i=j} + 2\sum_{i=2}^{4}\prod_{j=0}^{1} C_{1|234/i,i=j},$$

here, $C^2_{1|i_j}$ is the concurrence between party 1 and i, where the other two parties are measured at a state that is the binary equivalent of j. $C_{1|234/i,i=j}$ and $\tau_{x3}|_{1|234/i,i=j}$ refer respectively to the concurrence and the positive part of 3-tangle between party 1 and two parties from the other three $(2,3,4)$ excluding i, when i is measured at a state that is binary equivalent of j. M represents the total 3-party entanglement in the bi-partition $1-234$. We now define the genuine 4-party entanglement in this bi-partition as,

$$\begin{aligned} \tau_4 &= [2C^2_{1|234} - (C^2_{1|23} + C^2_{1|24} + C^2_{1|34})] \\ &= 4[\sum_{i=0}^{2^2-1} 2|\chi_i \wedge \chi_{2^3-1-i}|^2 - 2\sum_{i=2}^{4}\prod_{j=0}^{1} C_{1|234/i,i=j}] = 2\tau_4' - \tau_4''. \end{aligned}$$

Appendix E. Derivation of genuine *n*-party entanglement in *n*-qubit system

The genuine 3-way entanglement for 3-party systems is expressed as,

$$\tau_3 = 4[\sum_{i=0}^{2^1-1} |\chi_i \wedge \chi_{2^2-1-i}|^2 - 2\sum_{i=2}^{3}\prod_{j=0}^{1} C_{1|23/i,i=j}] = \tau_3' - \tau_3''.$$

The 4-party genuine entanglement is found as,

$$\tau_4 = 4[\sum_{i=0}^{2^2-1} 2|\chi_i \wedge \chi_{2^3-1-i}|^2 - 2\sum_{i=2}^{4}\prod_{j=0}^{1} C_{1|234/i,i=j}]$$
$$= 2\tau_4' - \tau_4''.$$

One can assume a general expression for genuine $(n-1)$ way entanglement from the expressions for τ_3 and τ_4 using as,

$$\tau_{n-1} = 4[\sum_{i=0}^{2^{((n-1)-2)}-1} (n-1-2)|\chi_i \wedge \chi_{2^{(n-1-1)}-1-i}|^2 - 2\sum_{i=2}^{n}\prod_{j=0}^{1} C_{1|23....(n-2)(n-1)/i,i=j}]$$
$$= (n-1-2)\tau_{(n-1)}' - \tau_{(n-1)}'',$$

here, $C_{1|23....(n-2)(n-1)/i,i=j}]$ is the concurrence between party 1-and the rest of the $(n-2)$ parties excluding i, when i is measured at a state that is binary equivalent to j.

One can define the genuine n-party entanglement using the expression for genuine $(n-1)$ party entanglement. We consider the total concurrences in the bi-partitions $1-23...(n-1)$, $1-23...(n-2)n$,......, $1-34...(n-1)n$ as,

$$M_n = C^2_{1|23...(n-1)} + C^2_{1|23...(n-2)n} + ... + C^2_{1|34...(n-1)n}.$$

One can write $C^2_{1|23...(n-1)}$ as,

$$C^2_{1|23...(n-1)} = C^2_{1|23...(n-1)}|_{n=0} + C^2_{1|23...(n-1)}|_{n=1} + 2C_{1|23...(n-1)}|_{n=0}.C_{1|23...(n-1)}|_{n=1}.$$

We consider the first two terms of the R.H.S of the previous equation,

$$C^2_{1|23...(n-1)}|_{n=0} + C^2_{1|23...(n-1)}|_{n=1}$$
$$= \sum_{j=2}^{n-1}\sum_{k=0}^{1} C^2_{1|23...(n-1)/j}|_{jn=k0} + \tau_{(n-1)}'|_{n=0} + \sum_{j=2}^{n-1}\sum_{k=0}^{1} C^2_{1|23...(n-1)/j}|_{jn=k1} + \tau_{(n-1)}'|_{n=1},$$

here, $C^2_{1|23...(n-1)/j}|_{jn=ki}$ is the squared concurrence between party 1 and any $(n-2)$ parties among the rest $(n-1)$ qubits excluding j, where the qubit j is measured at a state binary equivalent of k, and the nth qubit is measured at $|i\rangle, i \in \{0,1\}$. The term $\tau_{n-1_{n=i}}'$ represents the positive part of genuine $(n-1)$ entanglement, when the n-th qubit is measured at $|i\rangle, i \in \{0,1\}$.

The previous equation can also be expressed as,

$$C^2_{1|23...(n-1)}|_{n=0} + C^2_{1|23...(n-1)}|_{n=1}$$
$$= \sum_{j=2}^{n-1}\sum_{i=0}^{3} C^2_{1|23...(n-1)/j}|_{jn=i} + \sum_{i=0}^{1} \tau_{(n-1)}'|_{n=i},$$

where, the combined system of qubits j and n is measured at a state that is binary equivalent of i. Expanding further, one can have,

$$\begin{aligned}
&C^2_{1|23\ldots(n-1)}|_{n=0} + C^2_{1|23\ldots(n-1)}|_{n=1} \\
&\quad= \sum_{k=2}^{n-1}\sum_{j=2}^{n-1}\sum_{i=0}^{7} C^2_{1|23\ldots(n-1)/j,k}|_{kjn=i} \\
&\qquad+ \sum_{j=2}^{n-1}\sum_{i=0}^{3} \tau'_{(n-2)}|_{jn=i} + \sum_{i=0}^{1} \tau'_{(n-1)}|_{n=i},
\end{aligned}$$

Expanding similarly, one can finally arrive at,

$$\begin{aligned}
&C^2_{1|23\ldots(n-1)}|_{n=0} + C^2_{1|23\ldots(n-1)}|_{n=1} \\
&\quad= \sum_{i=2}^{n-1}\sum_{k=0}^{2^{n-2}-1} C^2_{1|i}|_{jn=k} + \sum_{k'=3}^{(n-1)}\sum_{i_k=2}^{(n-1)}\sum_{j=0}^{2^{(n-k')}-1} \tau'_{k'}|_{(\prod_{k=1}^{(n-k'-1)} i_k).n=j} \\
&\qquad \forall k_1 \leqslant k_2, i_{k_1} \leqslant i_{k_2}.
\end{aligned}$$

where, $C^2_{1|i}|_{jn}$ is the squared concurrence between party 1 and i, when the rest of the system jn is measured at the binary equivalent of k and $\tau'_{k'}$ is the positive part of genuine k' entanglement.

Considering all the terms in the expression of M_n,

$$M_n = (n-2)\sum_{i=2}^{n}\sum_{j=0}^{2^{n-2}-1} C^2_{1|i}|_j + \sum_{k'=3}^{(n-1)}\sum_{i_k=2}^{(n-1)}\sum_{j=0}^{2^{(n-k')}-1} \tau'_{k'}|_{(\prod_{k=1}^{(n-k')} i_k)=j} \quad \forall k_1 \leqslant k_2, i_{k_1} \leqslant i_{k_2}.$$

The genuine n-party entanglement can then be defined as,

$$\begin{aligned}
\tau_n &= [(n-2)C^2_{1|23\ldots.(n-1)n} - M_n] \\
&= 4[\sum_{i=0}^{2^{(n-2)}-1} (n-2)|\chi_i \wedge \chi_{2^{(n-1)}-1-i}|^2 - 2\sum_{i=2}^{n}\prod_{j=0}^{1} C_{1|23\ldots.(n-1)n/i,i=j}] \\
&= (n-2)\tau'_n - \tau''_n.
\end{aligned}$$

Appendix F. Form invariance of genuine *n*-way entanglement under permutation of qubits

One can assume a general expression for genuine $(n-1)$ way entanglement in the bi-partition $2|134\ldots n$,

$$\begin{aligned}
\tau_{n-1} &= 4[\sum_{i=0}^{2^{((n-1)-2)}-1} (n-1-2)|\chi_i \wedge \chi_{2^{(n-1-1)}-1-i}|^2 - 2\sum_{i=1,3}^{n}\prod_{j=0}^{1} C_{2|13\ldots.(n-2)(n-1)/i,i=j}] \\
&= (n-1-2)\tau'_{(n-1)} - \tau''_{(n-1)},
\end{aligned}$$

here, $C_{2|13....(n-2)(n-1)/i,i=j}$] is the concurrence between party 2 and the rest of the $(n-2)$ parties excluding i, when i is measured at a state that is binary equivalent to j.

One can define the genuine n-party entanglement using the expression for genuine $(n-1)$ party entanglement. We consider the total concurrences in the bi-partitions $2-13...(n-1)$, $2-13...(n-2)n$,......, $2-34...(n-1)n$ as,

$$M_n = C^2_{2|13...(n-1)} + C^2_{2|13...(n-2)n} + ... + C^2_{2|34...(n-1)n}.$$

One can write $C^2_{2|13...(n-1)}$ as,

$$C^2_{2|13...(n-1)} = C^2_{2|13...(n-1)}|_{n=0} + C^2_{2|13...(n-1)}|_{n=1} + 2C_{2|13...(n-1)}|_{n=0}.C_{2|13...(n-1)}|_{n=1}.$$

We consider the first two terms of the R.H.S of the previous equation,

$$C^2_{2|13...(n-1)}|_{n=0} + C^2_{2|13...(n-1)}|_{n=1} = \sum_{j=1,3}^{n-1}\sum_{k=0}^{1} C^2_{2|13...(n-1)/j}|_{jn=k0} + \tau'_{(n-1)}|_{n=0}$$
$$+ \sum_{j=1,3}^{n-1}\sum_{k=0}^{1} C^2_{2|13...(n-1)/j}|_{jn=k1} + \tau'_{(n-1)}|_{n=1},$$

here, $C^2_{2|13...(n-1)/j}|_{jn=ki}$ is the squared concurrence between party 2 and any $(n-2)$ parties among the rest $(n-1)$ qubits excluding j, where the qubit j is measured at a state binary equivalent of k, and the nth qubit is measured at $|i\rangle, i \in \{0,1\}$. The term $\tau'_{n-1_{n=i}}$ represents the positive part of genuine $(n-1)$ entanglement, when the n-th qubit is measured at $|i\rangle, i \in \{0,1\}$.

The previous equation can also be expressed as,

$$C^2_{2|13...(n-1)}|_{n=0} + C^2_{2|13...(n-1)}|_{n=1} = \sum_{j=1,3}^{n-1}\sum_{i=0}^{3} C^2_{2|13...(n-1)/j}|_{jn=i} + \sum_{i=0}^{1}\tau'_{(n-1)}|_{n=i},$$

where, the combined system of qubits j and n is measured at a state that is binary equivalent of i. Expanding further, one can have,

$$C^2_{2|13...(n-1)}|_{n=0} + C^2_{2|13...(n-1)}|_{n=1}$$
$$= \sum_{k=1,3}^{n-1}\sum_{j=1,3}^{n-1}\sum_{i=0}^{7} C^2_{2|13...(n-1)/j,k}|_{kjn=i} + \sum_{j=1,3}^{n-1}\sum_{i=0}^{3}\tau'_{(n-2)}|_{jn=i} + \sum_{i=0}^{1}\tau'_{(n-1)}|_{n=i}.$$

Expanding similarly, one can finally arrive at,

$$C^2_{2|13...(n-1)}|_{n=0} + C^2_{2|13...(n-1)}|_{n=1}$$
$$= \sum_{i=1,3}^{n-1}\sum_{k=0}^{2^{n-2}-1} C^2_{2|i}|_{jn=k} + \sum_{k'=3}^{(n-1)}\sum_{i_k=1,3}^{(n-1)}\sum_{j=0}^{2^{(n-k')}-1} \tau'_{k'}|_{(\prod_{k=1}^{(n-k'-1)} i_k).n=j}$$
$$\forall k_1 \leqslant k_2, i_{k_1} \leqslant i_{k_2}$$

where, $C^2_{2|i}|_{jn}$ is the squared concurrence between party 2 and i, when the rest of the system jn is measured at the binary equivalent of k and $\tau'_{k'}$ is the positive part of genuine k' entanglement.

Considering all the terms in the expression of M_n,

$$M_n = (n-2)\sum_{i=1,3}^{n}\sum_{j=0}^{2^{n-2}-1} C^2_{1|i}|_j + \sum_{k'=3}^{(n-1)}\sum_{i_k=1,3}^{(n-1)}\sum_{j=0}^{2^{(n-k')}-1} \tau'_{k'}|_{(\prod_{k=1}^{(n-k')} i_k)=j}$$
$$\forall k_1 \leqslant k_2, i_{k_1} \leqslant i_{k_2}.$$

The genuine n-party entanglement can then be defined as,

$$(\tau_n)_2 = [(n-2)C^2_{2|13....(n-1)n} - M_n] = 4[\sum_{i=0}^{2^{(n-2)}-1}(n-2)|\chi_i \wedge \chi_{2^{(n-1)}-1-i}|^2$$
$$-2\sum_{i=1,3}^{n}\prod_{j=0}^{1} C_{2|13....(n-1)n/i,i=j}] = (n-2)(\tau'_n)_2 - (\tau''_n)_2.$$

χ_i is the post measurement state vector of party 2, while the rest of the system is measured a state that is binary equivalent of i. Proceeding in a similar way, one can show, for any arbitrary party k, the genuine n-party entanglement retains its form, i.e.

$$(\tau_n)_k = [(n-2)C^2_{k|12...(k-1)(k+1)....(n-1)n} - (M_n)_k]$$
$$= 4[\sum_{i=0}^{2^{(n-2)}-1}(n-2)|\chi_i \wedge \chi_{2^{(n-1)}-1-i}|^2 - 2\sum_{i=1,i\neq k}^{n}\prod_{j=0}^{1} C_{k|12...(k-1)(k+1)...(n-1)n/i,i=j}]$$
$$= (n-2)(\tau'_n)_k - (\tau''_n)_k.$$

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