IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **53** (2020) 095002 (17pp) https://doi.org/10.1088/1751-8121/ab69a5

Asymptotic degeneracy and subdiffusivity

Sida Yuan[1] **and Reiichiro Kawai**

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia

E-mail: syua6813@uni.sydney.edu.au and reiichiro.kawai@sydney.edu.au

Received 30 October 2019, revised 22 December 2019
Accepted for publication 9 January 2020
Published 4 February 2020

Abstract

The time-changing of processes by inverse subordinators is widely studied in the literature as models of subdiffusion driven by sporadic trapping. We reveal that increments of such subdiffusion processes may exhibit asymptotic degeneracy depending on the underlying stochastic temporal progression. Various surprising aspects of time-changed processes are discussed in light of asymptotic degeneracy, including diffusive dynamics and optimal inference. We also motivate asymptotic degeneracy as an important criterion for determining appropriateness of such processes for modelling real-world phenomena.

Keywords: anomalous diffusion, degenerate increments, eventual rest, Fisher information, inverse subordinators

(Some figures may appear in colour only in the online journal)

1. Introduction

Subdiffusion describes processes with mean squared displacement proportional to t^α for $\alpha \in (0, 1)$. In the recent decades, there has been a growing interest in the literature towards subordinated processes as models of subdiffusion in various fields including statistical physics, biophysics, geophysics and finance. These processes are characterised by an outer process subjected to stochastic temporal progression via an inner inverse subordinator. Since subordinators are pure jump processes, inverse subordinators have random periods of immobility. By subordinating the evolution of time through the inverse subordinator, these periods of immobility are present in the resulting subdiffusion process. Thus, these processes are often used to model subdiffusive dynamics as a result of a trapping mechanism which causes particles to be sporadically trapped with independent and identically distributed waiting times. Some applications include the modelling of transport of solar magnetic flux elements [25], diffusion in the plasma membrane of living cells [26], various European interbank rates [10] and stock prices [11].

[1] Author to whom any correspondence should be addressed.

The main aim of this paper is to show that time-changed processes may exhibit asymptotically degenerate increments depending on the inner process. Specifically, we say that a process $\{Y_t : t \geqslant 0\}$ has *asymptotically degenerate increments* if for every $\Delta > 0$ it holds that $\mathbb{P}(Y_{t+\Delta} - Y_t = 0) \to 1$ as $t \to +\infty$. For brevity, we also refer to this property as *asymptotic degeneracy*, which as we will see later, is not equivalent to eventual rest but rather a counterintuitive generalisation. We discuss a variety of surprising and practical aspects of this asymptotic degeneracy. To this end, let us first summarise the subordination of processes as models of subdiffusion. Recall that a subordinator $\{U_s : s \geqslant 0\}$ is an increasing pure jump Lévy process. For every $t \geqslant 0$, denote $S_t := \inf\{s > 0 : U_s > t\}$ as the first passage time of the subordinator. Then, the process $\{S_t : t \geqslant 0\}$ is called the corresponding inverse subordinator. While our ultimate aim is to draw conclusions on general time-changed processes, our initial focus will be on stable subdiffusion processes to introduce asymptotic degeneracy of increments. Let $\{U_s : s \geqslant 0\}$ be a stable subordinator with stability parameter $\alpha \in (0, 1)$ such that the Laplace transform of its unit-time marginal is given by $\mathbb{E}[\mathrm{e}^{-yU_1}] = \mathrm{e}^{-y^\alpha}$ for every $y > 0$. For instance, in the case of time-changed Gaussian Ornstein–Uhlenbeck processes, we have the Langevin equations

$$\mathrm{d}X_s = (C - \kappa X_s)\,\mathrm{d}s + \sigma \mathrm{d}W_s, \qquad \mathrm{d}t = \mathrm{d}U_s, \tag{1.1}$$

where $\{W_s : s > 0\}$ is a standard Brownian motion. The evolution of the marginal probability density function of the resultant process is described by the fractional Fokker–Planck equation [19, 23]

$$\frac{\partial}{\partial t}p(x,t) = {}_0D_t^{1-\alpha}\left[-\frac{\partial}{\partial x}(C - \kappa x) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\right] p(x,t), \tag{1.2}$$

with $p(x, 0) = \delta_0(x)$, where ${}_0D_t^{1-\alpha}$ is the Riemann–Liouville fractional derivative of order $1 - \alpha$. For a comprehensive review of the relationship between inverse stable subordinators and fractional derivatives, we refer the reader to [17]. Generalisation of the above fractional Fokker–Planck equation and its corresponding Langevin equations to the case with jump coefficient and the general subordinator can be found in [16]. A special case of interest is when the outer process is simply a driftless Brownian motion, in which case $C = \kappa = 0$ and the solution to equation (1.2) yields

$$p(x,t) = \int_0^{+\infty} \frac{1}{\sigma\sqrt{s}}\phi\left(\frac{x}{\sigma\sqrt{s}}\right)\frac{\partial}{\partial s}\mathbb{P}(S_t \leqslant s)\,\mathrm{d}s,$$

with the variance proportional to t^α [8], where ϕ is the standard normal probability density function.

The analysis of distribution and statistical inference of increments starting from zero time $X_{S_\Delta} - X_{S_0}$ have been extensively studied in the literature, for instance, [6, 7, 12] to mention a few. The general increment $X_{S_{t+\Delta}} - X_{S_t}$ for $t \geqslant 0$ is considered to be more nuanced than the case for $t = 0$. Increments starting from $t = 0$ will almost surely observe movement, while any increment starting from a nonzero time has a positive probability of remaining trapped throughout its observation window, as we will shortly show. This leads to a source of technical inconvenience; the increment $X_{S_{t+\Delta}} - X_{S_t}$ is not identical in law to $X_{S_\Delta} - X_{S_0}$ for $t > 0$ even when the outer process has stationary increments. In what follows, we hope to shed more light onto the inference and relevance of such subdiffusion processes by investigating general increments in spite of the aforementioned technical complications.

This paper is organised as follows. In section 2, we present the density and probability of degeneracy for increments of the inverse stable subordinator. We then present an integral representation for the probability of degeneracy for increments of general inverse subordinators, with asymptotics provided for a particular class of inverse subordinators. In section 3, surprising relations to subdiffusive dynamics are discussed, along with implications of asymptotic degeneracy on optimal inference. We also demonstrate that asymptotic degeneracy is an essential yet overlooked criterion for determining modelling relevance of such processes towards real-world phenomena. We summarise our investigation in section 4. All derivations and some technical details are deferred to the appendix to avoid detracting from the discourse.

2. Results

We begin our investigation by considering the probability distribution of increments of the inverse stable subordinator. Fix $t \geqslant 0$ and $\Delta > 0$. Via the selfsimilarity of the marginal density of the stable subordinator $g_\alpha(r;t)$, we present the density of the increment $S_{t+\Delta} - S_t$ as

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{P}(S_{t+\Delta} - S_t \leqslant s) = -\frac{1}{\alpha\Gamma(\alpha)}\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{s}\int_{0+}^{\Delta}\frac{u}{(t+\Delta-u)^{1-\alpha}}g_\alpha\left(u;s\right)\,\mathrm{d}u\right], \tag{2.1}$$

for every $s > 0$. Since $S_0 = 0$ almost surely, the $t = 0$ case corresponds to the probability density function of S_Δ, given by [1, 2]

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{P}(S_\Delta \leqslant s) = \frac{\Delta}{\alpha s}g_\alpha\left(\Delta;s\right).$$

Integrating the density (2.1) over $s > 0$, we find that the probability of a period of immobilisation engulfing the fixed-length increment is

$$\mathbb{P}(S_{t+\Delta} - S_t = 0) = \frac{\sin(\alpha\pi)}{\pi}B\left(\frac{t}{t+\Delta};\alpha,1-\alpha\right), \tag{2.2}$$

where $B(z;a,b) := \int_0^z u^{a-1}(1-u)^{b-1}\,\mathrm{d}u$ is the incomplete beta function with $z \in [0,1]$, $\mathfrak{Re}(a) > 0$ and $\mathfrak{Re}(b) > 0$. Note that (2.2) is nonzero for every $t > 0$, so the probability distribution of the increment $S_{t+\Delta} - S_t$ contains a point mass at the origin, and thus the density function (2.1) is not a *probability* density function for $t > 0$. From this, we see that $X_{S_{t+\Delta}} - X_{S_t}$ is not equal in distribution to $X_{S_\Delta} - X_{S_0}$. Remarkably, the point mass (2.2) implies that every fixed-length increment is asymptotically degenerate. Specifically, as $t \to +\infty$ it holds that

$$\mathbb{P}(S_{t+\Delta} - S_t = 0) \sim 1 - \frac{\sin(\alpha\pi)}{(1-\alpha)\pi}\left(\frac{\Delta}{t+\Delta}\right)^{1-\alpha},$$

that is, the inverse stable subordinator is asymptotically degenerate. The convergence of the probability (2.2) to one as t increases is faster with smaller α. This is quite natural, since decreasing α corresponds to increasingly heavy tails of the waiting time distribution. Interestingly, the incomplete beta function also appears in the second moment

$$\mathbb{E}\left[(S_{t+\Delta} - S_t)^2\right] = \frac{2(t+\Delta)^{2\alpha}}{\alpha(\Gamma(\alpha))^2}B\left(\frac{\Delta}{t+\Delta};\alpha+1,\alpha\right), \tag{2.3}$$

which can be derived from the density (2.1). From this, we can straightforwardly recover the correlation structure of the inverse stable subordinator [15].

Example 2.1. To provide an explicit example of the above results, consider the special case $\alpha = 1/2$ where the density is available in the closed form

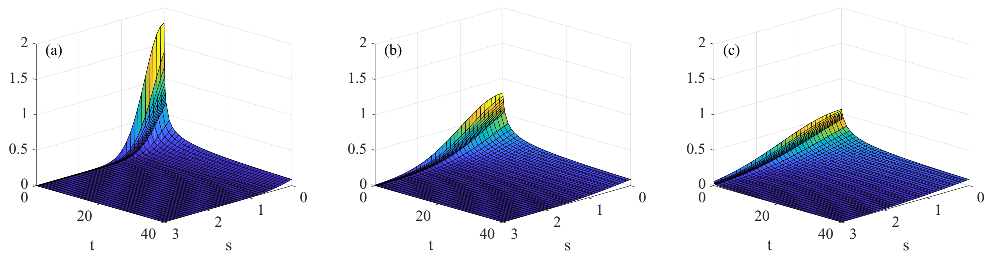

Figure 1. Plots of the density (2.1) for $\alpha = 1/2$ with (a) $\Delta = 0.1$, (b) $\Delta = 0.5$ and (c) $\Delta = 1.0$.

$$g(r;s) = \frac{s}{\sqrt{4\pi}r^{3/2}} \exp\left[-\frac{s^2}{4r}\right], \quad (r,s) \in (0,+\infty)^2.$$

Fix $\Delta > 0$. For the density of $S_{t+\Delta} - S_t$ for $t > 0$, we substitute the above into (2.1) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{P}(S_{t+\Delta} - S_t \leqslant s) = -\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}s}\int_{0+}^{\Delta} \frac{1}{\sqrt{r(t+\Delta-r)}} \exp\left[-\frac{s^2}{4r}\right] \mathrm{d}r.$$

Examples of this density are provided in figure 1.

The plots indicate that the distribution decreases monotonically with increasing t, which affirms the increase in probability of degeneracy (2.2). Specifically, for the probability of observing movement in the time interval $(t, t+\Delta]$, we also apply the general expression (2.2) to obtain

$$\mathbb{P}(S_{t+\Delta} - S_t > 0) = \frac{1}{\pi}\int_{0+}^{\frac{\Delta}{t+\Delta}} \frac{1}{\sqrt{u(1-u)}}\,\mathrm{d}u = \frac{1}{\pi}\int_{0+}^{\Delta} \frac{1}{\sqrt{r(t+\Delta-r)}}\,\mathrm{d}r = \frac{2}{\pi}\arctan\sqrt{\Delta/t},$$

where the second equality follows from the substitution $u = r/(t+\Delta)$. That is,

$$\mathbb{P}(S_{t+\Delta} - S_t = 0) = 1 - \frac{2}{\pi}\arctan\sqrt{\Delta/t} = \frac{2}{\pi}\arctan\sqrt{t/\Delta}.$$

As a function of t, the zero increment probability is plotted in figure 2. $\square$

At the core of asymptotic degeneracy is the property that any increment of fixed length becomes more likely to be degenerate as we shift it forward in time. A related concept from renewal theory is the well-known inspection paradox, in which the flat interval containing the fixed time $t > 0$ is stochastically larger than the initial interval from $t = 0$. In the language of subordinators, let $\{U_s : s \geqslant 0\}$ be any subordinator and define Z_t to be the size of the jump which crosses a fixed level $t \geqslant 0$. In the same vein, we can show that $\mathbb{P}(Z_t > x) \geqslant \mathbb{P}(Z_0 > x)$ for every $t > 0$, with severity at least nondecreasing in t. By applying the inspection paradox to the size of the flat period including the fixed time t, we derive the following representation of $\mathbb{P}(S_{t+\Delta} - S_t = 0)$ when $\{S_t : t \geqslant 0\}$ is a general inverse subordinator. To this end, let $\nu(\mathrm{d}z)$ be the Lévy measure of the subordinator $\{U_s : s \geqslant 0\}$ and denote the largest jump size as $c := \inf\{s > 0 : \nu((s,+\infty)) = 0\}$, which is infinite if the support of $\nu(\mathrm{d}z)$ is unbounded. For every $t \geqslant 0$ and $\Delta \geqslant 0$, it holds that

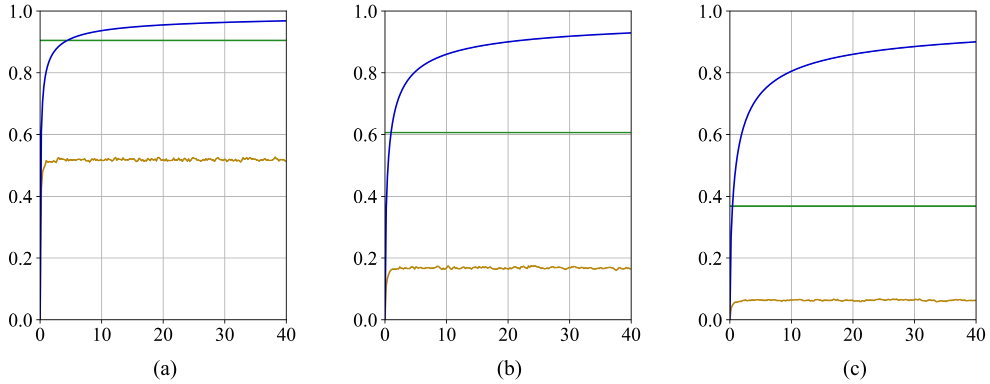

Figure 2. Plots of $\mathbb{P}(S_{t+\Delta} - S_t = 0)$ against t for inverse subordinators corresponding to Lévy measures of the form (2.5). The case $(\alpha, \beta) = (1/2, 0)$ is represented by the dominating curve (2.2), the case $(\alpha, \beta) = (-1, 1)$ is represented by the straight line while the bottom represents the Monte Carlo approximation for $(\alpha, \beta) = (1/2, 1)$. See appendix B for details.

$$\mathbb{P}\left(S_{t+\Delta} - S_t = 0\right) = \int_{\max\{t+\Delta-c,0\}}^{t} \frac{\nu((t+\Delta-u,+\infty))}{\nu((t-u,+\infty))} \mathbb{P}\left(U_{\tau_t-} \in \mathrm{d}u\right), \tag{2.4}$$

where $\tau_t := \inf\{s > 0 : U_s > t\}$ with $\tau_0 = 0+$ and $\tau_0 - = 0$. In particular, consider Lévy measures of the form

$$\nu(\mathrm{d}z) = \frac{\mathrm{e}^{-\beta z}}{z^{\alpha+1}} \mathrm{d}z, \qquad z > 0, \tag{2.5}$$

with parameters $\alpha \geqslant -1$ and $\beta \geqslant 0$ but $(\alpha, \beta) \neq (-1, 0)$, so $c = +\infty$. Applying (2.4) (see appendix A.5), we can show that for every $\Delta > 0$

$$\begin{cases} \lim_{t\to+\infty} \mathbb{P}(S_{t+\Delta} - S_t = 0) = 1, & (\alpha,\beta) \in (0,1) \times \{0\}, \\ \mathbb{P}(S_{t+\Delta} - S_t = 0) = \mathrm{e}^{-\beta\Delta}, & (\alpha,\beta) \in \{-1\} \times (0,+\infty), \\ \limsup_{t\to+\infty} \mathbb{P}(S_{t+\Delta} - S_t = 0) < \mathrm{e}^{-\beta\Delta}, & (\alpha,\beta) \in (-1,+\infty) \times (0,+\infty), \end{cases} \tag{2.6}$$

where the second case holds for every $t > 0$. Recall that (2.5) generalises the Lévy measures of gamma and tempered stable subordinators [12, 14], which fall under the inequality case in (2.6), so neither inverse subordinator exhibits asymptotic degeneracy. See figure 2 for a comparison between the cases in (2.6). To interpret this, recall the Lévy measure corresponding to that of a stable subordinator is of the form (2.5) with $(\alpha, \beta) \in (0, 1) \times \{0\}$. Since the tail of this Lévy measure is heavier than that of the tempered stable or gamma cases, the waiting time distribution of the inverse stable subordinator tends to produce much longer flat periods and is sufficiently extreme to induce asymptotically degenerate increments. Therefore, asymptotic degeneracy requires the tail of the Lévy measure to be at least heavier than exponential. It should also be noted that the sensitivity of the probability of degeneracy for an increment to the length of the increment depends on the subordinator. For example, for t fixed, in the case of the inverse stable subordinator it holds that as $\Delta \to +\infty$,

$$\mathbb{P}(S_{t+\Delta} - S_t = 0) \sim \frac{\sin(\alpha\pi)}{\alpha\pi} \left(\frac{t}{t+\Delta}\right)^{\alpha}, \quad t > 0. \tag{2.7}$$

In contrast to this power-law sensitivity towards Δ which becomes more subdued for larger t, the subordinator with Lévy measure given by (2.5) with $(\alpha, \beta) \in \{-1\} \times (0, +\infty)$ is statically and exponentially sensitive to Δ, as $\mathbb{P}(S_{t+\Delta} - S_t = 0) = \mathrm{e}^{-\beta\Delta}$. This sharp contrast in sensitivity can be seen in figure 2, as varying Δ leads to large impacts for the straight line, whereas the dominating curve (2.2) experiences only very mild displacement for large t. This increasing insensitivity of the increment length is intrinsically linked to asymptotic degeneracy of fixed-length increments.

Next, consider the scenario where we completely truncate the tail of the Lévy measure, so the largest waiting time is finite ($c < +\infty$). Since all observation windows with length exceeding the largest waiting time almost surely observe movement, asymptotic degeneracy is absent. However, rather than considering large increments, one may be interested in high frequency sampling schemes in which the length of increments are kept short. We demonstrate that convergence is not guaranteed for such cases. Fix $\Delta > 0$ and consider $\nu(\mathrm{d}z) = \delta_{\{2\Delta\}}(\mathrm{d}z)$, where every jump is of size 2Δ (so $c = 2\Delta$). In other words, $\{U_s : s \geqslant 0\}$ is a standard Poisson process with jumps of fixed height 2Δ. By the representation (2.4), it holds that for every $t \geqslant 0$,

$$\mathbb{P}(S_{t+\Delta} - S_t = 0) = \int_{\max\{t-\Delta,0\}}^{t} \frac{\nu((t+\Delta-u,+\infty))}{\nu((t-u,+\infty))} \mathbb{P}(U_{\tau_t-} \in \mathrm{d}u) = \mathbb{P}(U_{\tau_t-} \in (\max\{t-\Delta,0\},t]).$$

Since $\{U_s : s \geqslant 0\}$ takes values in $\{0, 2\Delta, 4\Delta, \dots\}$, we have only two possibilities:

1. If $(t, t+\Delta] \subseteq (2k\Delta, 2(k+1)\Delta)$ for some $k \in \{0, 1, 2, \dots\}$, then $\mathbb{P}(S_{t+\Delta} - S_t = 0) = 1$.
2. If $2k\Delta \in (t, t+\Delta]$ for some $k \in \{0, 1, 2, \dots\}$, then $\mathbb{P}(S_{t+\Delta} - S_t = 0) = 0$.

Thus, as we shift the observation window $(t, t+\Delta]$ forward in time, the probability of the increment becoming degenerate oscillates between zero and one, so the probability does not converge as $t \to +\infty$. The increment is therefore not asymptotically degenerate. So in the case of a high frequency sampling scheme with a bounded waiting time distribution, caution is advised when working with fixed-length increments in long-time as convergence is not guaranteed.

3. Discussion

So far, we have derived the density of increments of the inverse stable subordinator (2.1) as well as the corresponding degenerate increment probability (2.2). In doing so, we derived the asymptotically degenerate increment structure of the inverse subordinator. Moreover, we have represented the degenerate increment probability for general inverse subordinators as a ratio of its corresponding Lévy measure (2.4), and applied the result to show that a class of inverse subordinators do not exhibit asymptotic degeneracy (2.6). In what follows, we discuss various practical and overlooked implications of asymptotic degeneracy towards time-changed processes.

3.1. Asymptotic degeneracy for time-changed processes

We now establish the carrying over of asymptotic degeneracy into time-changed processes. Let $\{S_t : t \geqslant 0\}$ be a nondecreasing process independent of another process $\{X_s : s \geqslant 0\}$. Since the outer process is dependent on the internal time variable, it holds that for every $t \geqslant 0$ and $\Delta \geqslant 0$

$$\mathbb{P}\left(X_{S_{t+\Delta}} - X_{S_t} = 0\right) \geqslant \mathbb{P}\left(S_{t+\Delta} - S_t = 0\right), \tag{3.1}$$

with equality when increments of positive length of $\{X_s : s \geqslant 0\}$ are almost surely nonzero. That is, if the time-changing process exhibits asymptotic degeneracy of increments, then so must the time-changed process $\{X_{S_t} : t \geqslant 0\}$. It may be tempting to conjecture that any time-changed process with asymptotically degenerate increments exhibits eventual rest (that is, $\lim_{t\to+\infty} X_{S_t}$ exists), as any arbitrarily large increment converges to zero in probability as we shift it towards infinity. However, this is not generally true. Consider the time-changed Brownian motion $\{W_{S_t} : t \geqslant 0\}$ where $\{S_t : t \geqslant 0\}$ is an inverse stable subordinator, so the zero increment probability (3.1) is given explicitly by (2.2) and asymptotic degeneracy of increments is guaranteed. For any fixed $t \geqslant 0$, the probability (3.1) decreases to zero at the rate $\mathcal{O}(\Delta^{-\alpha})$ as Δ increases, as seen in (2.7). That is, this stable subdiffusion process will continue to move, albeit very slowly. Hence, we see that asymptotic degeneracy generalises the concept of eventual rest.

Given the previous discussion, a natural question to ask is whether the act of time-changing by a temporal process with asymptotically degenerate increments always slows diffusivity. This appears to be reasonable, as introducing asymptotically degenerate increments via time-changing always impedes the movement of the outer process with increasing severity along time. Consider the case where the outer process is a Brownian motion. Recall that the tempered stable and gamma inverse subordinators do not exhibit asymptotic degeneracy, and that timing-changing the Brownian motion with either leads to linear diffusion in the long-run, whereas time-changing by the inverse stable subordinator leads to lifetime subdiffusion [6]. On the whole, there is a necessary and sufficient correspondence. By the self-similarity of the Brownian motion and its independence with the inner temporal process, it holds that $\mathrm{Var}(W_{S_t}) = \mathbb{E}[S_t]$. The inner process exhibits asymptotic degeneracy if and only if $\mathbb{E}[S_{t+\Delta}] - \mathbb{E}[S_t] \to 0$ as $t \to +\infty$ for every $\Delta \geqslant 0$. That is, $\mathbb{E}[S_t]$ is sublinear with respect to t. Hence, asymptotic degeneracy of the inner temporal process is necessary and sufficient for lifetime subdiffusion of the time-changed Brownian motion. Interestingly, this relationship between asymptotic degeneracy and lifetime subdiffusion breaks down in the presence of an external force. Consider processes of the form $\{\mu S_t + \sigma W_{S_t} : t \geqslant 0\}$ with $\mu \neq 0$, that is, the advection of the particle is disrupted while it is trapped. Where $\{S_t : t \geqslant 0\}$ is an inverse stable subordinator, the probability of a degenerate increment is given by (2.2) and so asymptotic degeneracy is present. It is known [6] that the process diffuses at a rate of $\mathcal{O}(t^{\alpha})$ as $t \to 0$ and $\mathcal{O}(t^{2\alpha})$ as $t \to +\infty$. This leads to a rather counterintuitive result: in the presence of the external force, the stable drift-*sub*diffusion process with $\alpha \in (1/2, 1)$ exhibits *super*diffusion in the long-run along with asymptotically degenerate increments. So surprisingly, time-changing by an inner process with asymptotic degeneracy does not necessarily guarantee slower lifetime diffusivity. One should also note that the converse does not hold either, as the fractional Brownian motion with Hurst parameter $H \in (0, 1/2)$ is an example of a lifetime subdiffusive process without asymptotically degenerate increments.

We mention an application of degenerate increment probabilities (2.2) and (3.1) to the context of subdiffusion in the presence of a general space-dependent external binding potential. The scenario is modelled with the fractional Fokker–Planck equation (1.2) such that a general force function $F(x)/(k_BT)$ replaces $C - \kappa x$, where k_BT is thermal energy. Combined with the findings in [4], the uncentred covariance is of the form

$$\mathbb{E}[X_{S_{t+\Delta}} X_{S_t}] \sim K_\alpha \sigma_B^2 \mathbb{P}\left(X_{S_{t+\Delta}} - X_{S_t} = 0\right) + \mu_B, \tag{3.2}$$

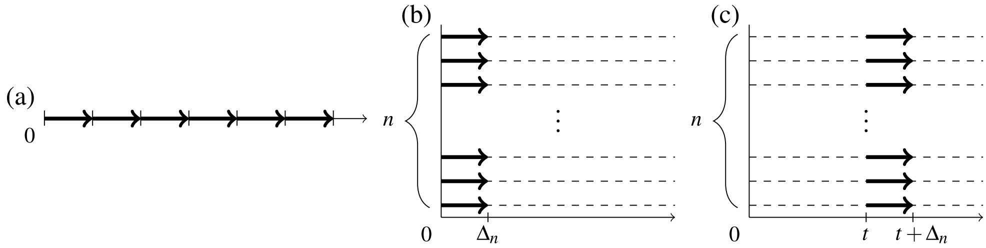

Figure 3. Sampling schemes: (a) discretisation of single particle tracking, and n iid observations on (b) $(0, \Delta_n]$ and (c) $(t, t+\Delta_n]$.

for every $\Delta \geqslant 0$ and t sufficiently large, where $K_\alpha \geqslant 4$ is constant depending on $\alpha \in (0,1)$, and μ_B and σ_B^2 are the mean and variance of the corresponding Boltzmann distribution respectively. This leads to a natural interpretation: as t increases, the locations of the process at the times t and $t+\Delta$ become more correlated due to the growing likelihood of both becoming stuck within the same period of immobility. Therefore, it is very intuitive that the covariance (3.2) asymptotically depends on the zero increment probability.

3.2. Optimal inference in light of asymptotic degeneracy

Having established asymptotic degeneracy of fixed-length increments of a large class of subdiffusion processes, we consider its practical implications for optimal inference. Statistical inference on subdiffusion models are possible by modern developments in single particle tracking technology (figure 3(a)). The non-ergodicity brought on by time-changing via the inverse stable subordinator is intrinsically linked to asymptotically degenerate increments, as the irreproducibility of time-averaged observables across independent sample paths can be attributed to the frustration of convergence of time averages caused by the *quasi*-eventual rest of asymptotic degeneracy. When quantified, this non-ergodicity can still yield a viable method of parameter inference [3, 7, 18, 24]. However, due to the demanding amount of data required for accurate inference via non-ergodicity methods based on single-particle tracking, classical inferential techniques based on incremental data of many independent and identically distributed trajectories (figure 3(b)) may be more favourable [7]. From a practical perspective, it is often the case that particles have a history of movement well before their tracking (figure 3(c)). The only realistic choice of increment starting time for statistical inference in those cases is arbitrarily large time. In the case of subdiffusion processes with asymptotic degeneracy, the zero increment probability (3.1) converges to one. Any increment data interpreted to be of arbitrarily far starting time is nonsensical for inference, thus generally rendering these processes as inappropriate for such modelling scenarios.

Interestingly, asymptotic degeneracy of increments sheds light on existing results on Fisher information. We will only consider the stable subdiffusion process $\{\sigma W_{S_t} : t \geqslant 0\}$ for which the Fisher information matrix is known asymptotically. Consider sampling increments over the observation window $(0, \Delta_n]$ (figure 3(b)), where the fixed length Δ_n is indexed by the number of particles n. As n increases, it has been shown that in both long-time ($\Delta_n \to +\infty$) and short-time ($\Delta_n \to 0$) sampling schemes the Fisher information matrix for the parameters (α, σ) is given by

$$\mathscr{I}(\alpha,\sigma) = \frac{M_\alpha}{4\sigma^2}\begin{bmatrix} \sigma^2 & \pm 2\sigma \\ \pm 2\sigma & 4 \end{bmatrix},$$

where $M_\alpha > 0$ depends on α and the off-diagonal terms are positive in long-time and negative in short-time [7]. The Fisher information for either α or σ is finite and nonzero if the other is known *a priori*, thus we are able to obtain simple optimality criteria for inference on either parameter based on increments starting from zero time. Surprisingly, short-time increments appear to be no less optimal than long-time increments and hence should be preferred to reduce experimental effort. We conjecture that asymptotic degeneracy of increments plays a key role here; increments of the process with asymptotically far starting times contain no information on either parameter, so the inclusion of infinite time in long-time increments does not seem to provide any additional information for optimal inference over using short-time increments. By the zero increment probability (2.2), we know that increments starting from zero time almost surely observe movement, so we expect optimal inference for α to use short-time increments starting after, but not far away from zero time. However, preferred choices of increments may differ depending on the inverse subordinator. For instance, due to asymptotically linear diffusion of the gamma and tempered stable cases [6, 12, 14], the effect of the inverse subordinator in the long-run is isolated to the coefficient for linear diffusion, and so long-time increments may harbour advantages for inference on the parameter σ.

3.3. Asymptotic degeneracy as a modelling criterion

In the context of modelling, asymptotic degeneracy of increments can be pivotal. We first address an important practical question: are there real-world phenomena which exhibit subdiffusion driven by sporadic trapping and asymptotic degeneracy but not eventual rest? In the literature, stable subdiffusion processes have been used to describe financial data [10, 11]. However to our knowledge, there has not yet been a modelling scenario where asymptotic degeneracy is required or even considered. To rectify this, we propose that short-term rates maintained by open market operations under an inflation targeting regime can be modelled by a mean-reverting process time-changed by an inverse stable subordinator, such as the process described by the Langevin equations (1.1) or the more general subordinated stable Ornstein–Uhlenbeck process considered in [10]. As the central bank periodically adjusts the short-term rate based on its inflation target, over time the inflation rate converges and fluctuations are alleviated. Consequently, not only does the short-term rate demonstrate mean-reverting behaviour away from the initial value, but also flat periods which increase in size with time as the policy objective is approached. Short-term rates have known starting times, so sampling from the asymptotic increment distribution is inappropriate for statistical inference. Moreover, asymptotic degeneracy of increments is desired but not eventual rest; the short-term rate will tend towards rest due to the stabilising of inflation rate, but it will not experience eventual rest due to the existence of sudden shocks and secondary policy objectives. For example, from March to June of 2008, the Reserve Bank of Australia's cash rate was constant at 7.25%. However, by April 2009, the cash rate had steadily lowered to 3.00% in the wake of the global financial crisis before reaching another flat period [20]. Similar characteristics can be observed in the short-term rates which influence the European interbank rates examined in [10]. Therefore, the dynamics described by a mean-reverting process time-changed by an inverse stable subordinator captures many stylised facts of short-term rates determined by open market operations under an inflation targeting regime. It is crucial to note that in this modelling scenario, tempered stable and gamma subdiffusion models are inadequate precisely because of their lack of asymptotic degeneracy. For real-world physical examples of asymptotic degeneracy exhibited in a subdiffusion phenomenon, we suggest turning to biophysics. It has been observed experimentally that Kv2.1 potassium channels in cells exhibit both sporadic episodes of immobilisation as well as tendency towards rest [26]. While each channel may remain immobile for extended periods of time, like many physical processes in living cells, eventual rest is never realised in practice. The lifetime of the channel may be terminated, for example, by absorption for recycling. Thus, a stable subdiffusion process with a finite lifetime may be useful in capturing increasing waiting times on average over its lifetime.

We next address another important question: What should we do if we would like lifetime subdiffusion with trapping dynamics but without asymptotic degeneracy? Firstly, to guarantee sporadic trapping without asymptotic degeneracy, we need to choose the inverse of a subordinator with a sufficiently light-tailed Lévy measure. By the result (2.6), we have seen two such examples, namely the tempered stable and gamma inverse subordinators [6, 12, 14]. In addition, another example is that of a truncated stable Lévy measure, which has been used to model the trapping of magnetic bright points on the photosphere [5]. This leaves us with the task of selecting an appropriate outer process so as to guarantee lifetime subdiffusion. A natural choice is the fractional Brownian motion $\{B_t^H : t \geqslant 0\}$ with Hurst parameter $H \in (0, 1/2)$, which is subdiffusive for all time. For example, time-changing by the inverse tempered stable subordinator, we have that $\mathrm{Var}(B_{S_t}^H) = \mathcal{O}(t^{2H\alpha})$ in short-time and $\mathrm{Var}(B_{S_t}^H) = \mathcal{O}(t^{2H})$ in long-time [14], as we desired.

4. Concluding remarks

We have investigated asymptotic degeneracy of fixed-length increments pertaining to a large class of time-changed processes. While asymptotic degeneracy slows diffusivity in some cases, the long-run diffusivity of a process is generally attributed to the interaction between various anomalous diffusive components. We discussed implications for statistical inference, such as the inadequacy of processes with asymptotically degenerate increments for modelling contexts which require asymptotic increment sampling. Using the example of short-term rates maintained by open market operations under an inflation targeting regime, we showed that asymptotic degeneracy can arise naturally in real-world contexts and deserves serious consideration in modelling scenarios as a criterion for model selection. We also demonstrated that lifetime subdiffusive processes exist with sporadic trapping but without asymptotically degenerate increments. The consideration of asymptotic degeneracy of fixed-length increments not only sheds light on properties of many subdiffusion processes, but also leads to various practical implications valuable to modelling applications.

Acknowledgments

The authors are grateful to Sean Carnaffan for his valuable feedback that helped improve the quality of this manuscript.

Appendix A. Derivations

The derivation of mathematical results in the main paper are presented in the following sections.

A.1. Density function of an increment of the inverse stable subordinator

We begin with the derivation of the density function (2.1). Fix $t > 0$. Denote the marginal probability density of the subordinator U_t by $g(r; t) := \frac{\partial}{\partial r}\mathbb{P}(U_t \leqslant r)$ for every $r \geqslant 0$. We begin by considering the density of $S_{t+\Delta} - S_t$ without making assumptions on the subordinator $\{U_t : t \geqslant 0\}$. By the definitions of the cumulative distribution function and the density function, we obtain for every $s > 0$,

$$\begin{aligned}\mathbb{P}(S_{t+\Delta} - S_t \leqslant s) &= \mathbb{P}(S_t \geqslant S_{t+\Delta} - s)\\ &= \int_0^{+\infty} \int_{\max\{x_2 - s, 0\}}^{x_2} \frac{\partial}{\partial x_1} \left(\frac{\partial}{\partial x_2} \mathbb{P}(\{S_t \leqslant x_1\} \cap \{S_{t+\Delta} \leqslant x_2\}) \right) \mathrm{d}x_1 \, \mathrm{d}x_2\\ &= \int_0^{+\infty} \left[\frac{\partial}{\partial x_2} \mathbb{P}(\{S_t \leqslant x_1\} \cap \{S_{t+\Delta} \leqslant x_2\}) \right]_{x_1 = \max\{x_2 - s, 0\}}^{x_1 = x_2} \mathrm{d}x_2.\end{aligned}$$

Hence, we wish to compute its derivative

$$\frac{\mathrm{d}}{\mathrm{d}s} \mathbb{P}(S_{t+\Delta} - S_t \leqslant s) = -\frac{\mathrm{d}}{\mathrm{d}s} \int_0^{+\infty} \left(\frac{\partial}{\partial x_2} \mathbb{P}(\{S_t \leqslant x_1\} \cap \{S_{t+\Delta} \leqslant x_2\}) \right) \Big|_{x_1 = \max\{x_2 - s, 0\}} \mathrm{d}x_2.$$

By the independence of the two increments $U_{x_2} - U_{x_1}$ and $U_{x_1} - U_0$, for each nonnegative constants $x_1 < x_2$ we have that

$$\begin{aligned}\mathbb{P}(\{S_t \leqslant x_1\} \cap \{S_{t+\Delta} \leqslant x_2\}) &= \mathbb{P}(\{U_{x_1} \geqslant t\} \cap \{U_{x_2} \geqslant t + \Delta\})\\ &= \mathbb{P}(\{U_{x_1} \geqslant t\} \cap \{(U_{x_2} - U_{x_1}) + U_{x_1} \geqslant t + \Delta\})\\ &= \int_t^{+\infty} \mathbb{P}((U_{x_2} - U_{x_1}) + r \geqslant t + \Delta) g(r; x_1) \mathrm{d}r\\ &= \int_t^{(t+\Delta)-} \mathbb{P}(U_{x_2} - U_{x_1} \geqslant t + \Delta - r) g(r; x_1) \mathrm{d}r + \int_{t+\Delta}^{+\infty} g(r; x_1) \mathrm{d}r\\ &= \int_{0+}^{\Delta} \mathbb{P}(U_{x_2 - x_1} \geqslant u) g(t + \Delta - u; x_1) \mathrm{d}u + \int_{t+\Delta}^{+\infty} g(r; x_1) \mathrm{d}r,\end{aligned}$$

where the last equality follows from the substitution $r = t + \Delta - u$ and stationarity of increments. Taking the partial derivative with respect to x_2, we have that

$$\frac{\partial}{\partial x_2} \mathbb{P}(\{S_t \leqslant x_1\} \cap \{S_{t+\Delta} \leqslant x_2\}) = -\int_{0+}^{\Delta} \frac{\partial}{\partial x_2} \mathbb{P}(U_{x_2 - x_1} \leqslant u) g(t + \Delta - u; x_1) \, \mathrm{d}u,$$

where the interchanging of integration and differentiation holds by the Leibniz integral rule.

We now consider the case when $\{U_t : t \geqslant 0\}$ is a stable subordinator with fixed $\alpha \in (0, 1)$. We proceed with denoting the stable marginal probability density by $g(r; t)$ without the subscript as α is fixed hereafter. The probability density $g(r; t)$ satisfies $g(0+; t) = 0$ and the selfsimilarity property

$$g(r; t) = t^{-1/\alpha} g(t^{-1/\alpha} r; 1), \quad (r, t) \in (0, +\infty)^2. \tag{A.1}$$

Thus, we obtain

$$\mathbb{P}(\{S_t \leqslant x_1\} \cap \{S_{t+\Delta} \leqslant x_2\}) = \int_{0+}^{\Delta} \mathbb{P}\left(U_1 \geqslant \frac{u}{(x_2 - x_1)^{1/\alpha}} \right) g(t + \Delta - u; x_1) \mathrm{d}u + \int_{t+\Delta}^{+\infty} g(r; x_1) \mathrm{d}r.$$

Partially differentiating with respect to x_2, we have that

$$\begin{aligned}\frac{\partial}{\partial x_2} \mathbb{P}(\{S_t \leqslant x_1\} \cap \{S_{t+\Delta} \leqslant x_2\}) &= \int_{0+}^{\Delta} \frac{u}{\alpha (x_2 - x_1)^{1/\alpha + 1}} g\left(\frac{u}{(x_2 - x_1)^{1/\alpha}}; 1 \right) g(t + \Delta - u; x_1) \, \mathrm{d}u\\ &= \int_{0+}^{\Delta} \frac{u}{\alpha (x_2 - x_1)} g(u; x_2 - x_1) g(t + \Delta - u; x_1) \, \mathrm{d}u.\end{aligned}$$

By setting $x_1 = \max\{x_2 - s, 0\}$, we obtain

$$\left(\frac{\partial}{\partial x_2}\mathbb{P}(\{S_t \leqslant x_1\} \cap \{S_{t+\Delta} \leqslant x_2\})\right)\Big|_{x_1=\max\{x_2-s,0\}}$$
$$= \int_{0+}^{\Delta} \frac{u}{\alpha \min\{x_2, s\}} g\left(u; \min\{x_2, s\}\right) g(t+\Delta-u; \max\{x_2-s, 0\})\,\mathrm{d}u.$$

Integrating with respect to x_2 yields

$$\int_0^{+\infty} \left(\frac{\partial}{\partial x_2}\mathbb{P}(\{S_t \leqslant x_1\} \cap \{S_{t+\Delta} \leqslant x_2\})\right)\Big|_{x_1=\max\{x_2-s,0\}} \mathrm{d}x_2$$
$$= \int_0^s \int_{0+}^{\Delta} \frac{u}{\alpha x_2} g\left(u; x_2\right)\ \delta_{\{t+\Delta\}}(\mathrm{d}u)\,\mathrm{d}x_2$$
$$+ \int_{s+}^{+\infty} \int_{0+}^{\Delta} \frac{u}{\alpha s} g\left(u; s\right) g\left(t+\Delta-u; x_2-s\right)\ \mathrm{d}u\,\mathrm{d}x_2,$$

where $\delta_{\{t+\Delta\}}(\mathrm{d}u)$ denotes the Dirac measure concentrated at $u = t + \Delta$ and we have applied selfsimilarity (A.1). The first term vanishes due to $t > 0$. For the second term, we apply Tonelli's theorem by the nonnegativity of the integrand to obtain

$$\int_{s+}^{+\infty} \int_{0+}^{\Delta} \frac{u}{\alpha s} g\left(u; s\right) g\left(t+\Delta-u; x_2-s\right)\ \mathrm{d}u\,\mathrm{d}x_2$$
$$= \int_{0+}^{\Delta} \frac{u}{\alpha s} g(u; s) \left[\int_{s+}^{+\infty} g(t+\Delta-u; x_2-s)\,\mathrm{d}x_2\right] \mathrm{d}u$$
$$= \frac{1}{\alpha\Gamma(\alpha)s} \int_{0+}^{\Delta} \frac{u}{(t+\Delta-u)^{1-\alpha}} g(u; s)\,\mathrm{d}u,$$

where the second equality holds by the potential density formula [22]

$$\int_{s+}^{+\infty} g(t+\Delta-u; x_2-s)\,\mathrm{d}x_2 = \int_{0+}^{+\infty} g(t+\Delta-u; x_2)\,\mathrm{d}x_2 = \frac{1}{\Gamma(\alpha)(t+\Delta-u)^{1-\alpha}}. \tag{A.2}$$

The density function for $s > 0$ is therefore given by

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{P}(S_{t+\Delta} - S_t \leqslant s) = -\frac{1}{\alpha\Gamma(\alpha)} \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{s}\int_{0+}^{\Delta} \frac{u}{(t+\Delta-u)^{1-\alpha}} g\left(u; s\right)\ \mathrm{d}u\right],$$

which yields (2.1) as desired.

A.2. Probability of degeneracy for increments of the inverse stable subordinator

In what follows, we derive the probability of degenerate increment (2.2) for the inverse stable subordinator. Let $t > 0$ and denote

$$h(u, s) := \frac{1}{s} \frac{u}{(t+\Delta-u)^{1-\alpha}} g(u; s) = \frac{1}{s^{1+1/\alpha}} \frac{u}{(t+\Delta-u)^{1-\alpha}} g\left(s^{-1/\alpha}u; 1\right), \quad (u, s) \in (0, \Delta] \times (0, +\infty), \tag{A.3}$$

where the second representation follows from selfsimilarity (A.1). Consider the integral of the density of the random variable $S_{t+\Delta} - S_t$ over $(0, +\infty)$:

$$\begin{aligned}\mathbb{P}\left(S_{t+\Delta} - S_t > 0\right) &= \int_{0+}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}s}\mathbb{P}(S_{t+\Delta} - S_t \leqslant s)\,\mathrm{d}s \\ &= -\frac{1}{\alpha\Gamma(\alpha)}\int_{0+}^{+\infty}\frac{\mathrm{d}}{\mathrm{d}s}\left[\int_{0+}^{\Delta} h(u,s)\,\mathrm{d}u\right]\mathrm{d}s \\ &= \frac{1}{\alpha\Gamma(\alpha)}\left[\lim_{s\to 0}\int_{0+}^{\Delta} h(u,s)\,\mathrm{d}u - \lim_{s\to+\infty}\int_{0+}^{\Delta} h(u,s)\,\mathrm{d}u\right].\end{aligned}$$

To justify that the second term vanishes, note that the probability density $g(u,s)$ is uniformly bounded for $s \geqslant 1$, thus $h(u,s)$ is also uniformly bounded over $(u,s) \in (0,\Delta] \times [1,+\infty)$. Moreover, we have that $\lim_{s\to+\infty} h(u,s) = 0$ for every $u \in (0,\Delta]$, so we obtain the result through an application of the bounded convergence theorem. It remains for us to compute the first term. Firstly, we justify the passage of the limit into the integrand. For every $u \in (0,\Delta]$, by the asymptotics of the marginal stable density it holds that [22, 14.37]

$$\lim_{s\to 0} h(u,s) = \frac{1}{\pi}\Gamma(\alpha+1)\sin(\alpha\pi)\frac{1}{(t+\Delta-u)^{1-\alpha}u^{\alpha}} =: h(u,0).$$

As $h(\cdot,s)$ is bounded for every $s \in (0,1]$, we have that $h(\cdot,s) \in L^1((0,\Delta])$. Moreover, $h(\cdot,0)$ is bounded on $[\epsilon,\Delta]$ for every $\epsilon \in (0,\Delta)$, so $M_\epsilon := \sup_{(u,s)\in[\epsilon,\Delta]\times(0,1]} h(u,s) < +\infty$. It remains to dominate $h(\cdot,s)$ about the origin. For every fixed $s \in (0,1]$ we have $\lim_{u\to 0} h(u,s) = 0$, yet at the limit we have $\lim_{u\to 0} h(u,0) = +\infty$, so by continuity there exists $\epsilon > 0$ sufficiently small such that $h(u,s_2) \leqslant h(u,s_1)$ for every $0 < s_1 \leqslant s_2 \leqslant 1$ and every $u \in (0,\epsilon)$. Thus, there exists a dominating function $h(u,s) \leqslant M_\epsilon + Cu^{-\alpha} \in L^1((0,\Delta])$, where C is some positive constant. We apply the dominated convergence theorem to obtain

$$\begin{aligned}&\frac{1}{\alpha\Gamma(\alpha)}\lim_{s\to 0}\int_{0+}^{+\infty} h(u,s)\,\mathrm{d}u \\ &= \frac{\sin(\alpha\pi)}{\pi}\int_{0+}^{\Delta}\frac{1}{(t+\Delta-u)^{1-\alpha}u^{\alpha}}\,\mathrm{d}u \\ &= \frac{\sin(\alpha\pi)}{\pi}\int_{0+}^{\frac{\Delta}{t+\Delta}} y^{-\alpha}(1-y)^{\alpha-1}\,\mathrm{d}y = \frac{\sin(\alpha\pi)}{\pi}B\left(\frac{\Delta}{t+\Delta};1-\alpha,\alpha\right),\end{aligned}$$

where the second equality follows from the substitution $y = u/(t+\Delta)$. Thus, it holds that

$$\mathbb{P}(S_{t+\Delta} - S_t > 0) = \frac{1}{\alpha\Gamma(\alpha)}\lim_{s\to 0}\int_{0+}^{\Delta} h(u,s)\,\mathrm{d}u = \frac{\sin(\alpha\pi)}{\pi}B\left(\frac{\Delta}{t+\Delta};1-\alpha,\alpha\right).$$

Therefore, we obtain

$$\mathbb{P}(S_{t+\Delta} - S_t = 0) = 1 - \mathbb{P}(S_{t+\Delta} - S_t > 0) = \frac{\sin(\alpha\pi)}{\pi}B\left(\frac{t}{t+\Delta};\alpha,1-\alpha\right),$$

which yields (2.2) as desired. Note that this formula also holds for $t = 0$; for every $\alpha \in (0,1)$ and $\Delta > 0$, it holds that

$$\mathbb{P}(S_\Delta - S_0 = 0) = \frac{\sin(\alpha\pi)}{\pi} B\left(\frac{0}{0+\Delta}; \alpha, 1-\alpha\right) = 0.$$

A.3. Second moment of increments of the inverse stable subordinator

Using the density (2.1), we derive the second moment of increments of the inverse stable subordinator (2.3) directly via integration by parts to obtain

$$\begin{aligned}
\mathbb{E}\left[(S_{t+\Delta} - S_t)^2\right] &= 0^2\mathbb{P}(S_{t+\Delta} - S_t = 0) + \int_{0+}^{+\infty} s^2 \frac{\mathrm{d}}{\mathrm{d}s}\mathbb{P}(S_{t+\Delta} - S_t \leqslant s)\,\mathrm{d}s \\
&= \frac{2}{\alpha\Gamma(\alpha)} \int_{0+}^{+\infty}\int_{0+}^{\Delta} \frac{u}{(t+\Delta-u)^{1-\alpha}} g(u;s)\,\mathrm{d}u\,\mathrm{d}s \\
&\quad - \frac{1}{\alpha\Gamma(\alpha)} \int_{0+}^{+\infty} \left(\frac{\mathrm{d}}{\mathrm{d}s}\int_{0+}^{\Delta} \frac{u}{(t+\Delta-u)^{1-\alpha}} s g(u;s)\,\mathrm{d}u\right)\mathrm{d}s \\
&= \frac{2}{\alpha\Gamma(\alpha)} \int_{0+}^{\Delta} \frac{u}{(t+\Delta-u)^{1-\alpha}} \left[\int_{0+}^{+\infty} g(u;s)\,\mathrm{d}s\right]\mathrm{d}u \\
&= \frac{2}{\alpha(\Gamma(\alpha))^2} \int_{0+}^{\Delta} u^\alpha (t+\Delta-u)^{\alpha-1}\,\mathrm{d}u \\
&= \frac{2(t+\Delta)^{2\alpha}}{\alpha(\Gamma(\alpha))^2} B\left(\frac{\Delta}{t+\Delta}; \alpha+1, \alpha\right),
\end{aligned}$$

where the third equality holds by the dominated convergence theorem, the fourth equality follows from (A.2) and the last equality follows by the substitution $y = u/(t+\Delta)$.

A.4. Probability of degeneracy of an increment for the general subordinator

Before presenting the derivation of (2.4), we recall the inspection paradox to provide some intuition. Define $Z_t := U_{\tau_t} - U_{\tau_t-}$ as the size of the jump which crosses a fixed level $t > 0$, with Z_0 the size of the first jump. For any fixed $x > 0$, we condition on U_{τ_t-} to obtain

$$\begin{aligned}
\mathbb{P}(Z_t > x) &= \int_0^t \mathbb{P}(Z_t > x | U_{\tau_t-} = u)\mathbb{P}(U_{\tau_t-} \in \mathrm{d}u) \\
&= \int_0^t \mathbb{P}(Z_0 > x | Z_0 > t-u)\mathbb{P}(U_{\tau_t-} \in \mathrm{d}u) \\
&= \int_0^t \frac{\mathbb{P}(Z_0 > \max\{x, t-u\})}{\mathbb{P}(Z_0 > t-u)}\mathbb{P}(U_{\tau_t-} \in \mathrm{d}u) \\
&= \int_0^t \min\left\{\frac{\mathbb{P}(Z_0 > x)}{\mathbb{P}(Z_0 > t-u)}, 1\right\}\mathbb{P}(U_{\tau_t-} \in \mathrm{d}u) \\
&\geqslant \int_0^t \min\left\{\mathbb{P}(Z_0 > x), 1\right\}\mathbb{P}(U_{\tau_t-} \in \mathrm{d}u) \\
&= \mathbb{P}(Z_0 > x).
\end{aligned}$$

Note that the denominator of the integrand $\mathbb{P}(Z_0 > t - u)$ is nonincreasing in t, so the severity of the stochastic dominance $\mathbb{P}(Z_t > x) \geqslant \mathbb{P}(Z_0 > x)$ is at least nondecreasing in t.

We now provide the derivation of (2.4). Fix $t > 0$ and $\Delta > 0$. Recall that $\{U_{\tau_t} - t > \Delta\} = \{S_{t+\Delta} - S_t = 0\}$. We proceed in a fashion similar to the inspection paradox but with respect to the size of the flat period containing the fixed time t instead. Once again, by conditioning on $U_{\tau_t -}$, we can express the probability of observing no movement over the time window $(t, t+\Delta]$ as

$$\begin{aligned}
\mathbb{P}(S_{t+\Delta} - S_t = 0) &= \mathbb{P}(U_{\tau_t} - t > \Delta) \\
&= \int_{\max\{t+\Delta-c,0\}}^{t} \mathbb{P}(U_{\tau_t} - t > \Delta | U_{\tau_t -} = u)\mathbb{P}(U_{\tau_t -} \in \mathrm{d}u) \\
&= \int_{\max\{t+\Delta-c,0\}}^{t} \frac{\mathbb{P}(Z_0 > t+\Delta-u)}{\mathbb{P}(Z_0 > t-u)}\mathbb{P}(U_{\tau_t -} \in \mathrm{d}u) \\
&= \int_{\max\{t+\Delta-c,0\}}^{t} \frac{\nu((t+\Delta-u, +\infty))}{\nu((t-u, +\infty))}\mathbb{P}(U_{\tau_t -} \in \mathrm{d}u).
\end{aligned}$$

For the case of $t = 0$, note that $\mathbb{P}(U_{\tau_0 -} \in \mathrm{d}u) = \delta_{\{0\}}(\mathrm{d}u)$. By the inspection paradox it immediately holds that for every $\Delta > 0$,

$$\mathbb{P}(S_\Delta - S_0 = 0) = \mathbb{P}(U_{\tau_0} - 0 > \Delta) = \mathbb{P}(Z_0 > \Delta) = \int_{\max\{\Delta-c,0\}}^{0} \frac{\nu((\Delta-u, +\infty))}{\nu((0, +\infty))}\mathbb{P}(U_{\tau_0 -} \in \mathrm{d}u),$$

as required.

A.5. Asymptotics of zero increment probability for a class of Lévy measures

For the derivation of (2.6), let $\alpha > -1$. Define $f(s) := \mathrm{e}^{-\beta\Delta}\nu((s, +\infty)) - \nu((s+\Delta, +\infty))$ for every $s > 0$, so $\lim_{s\to+\infty} f(s) = 0$. It is easy to check that $\mathrm{d}f(s)/\mathrm{d}s < 0$ for every $s > 0$, so f is strictly decreasing and thus strictly positive, that is, for every $s > 0$ it holds that $\nu((s+\Delta, +\infty))/\nu((s, +\infty)) < \mathrm{e}^{-\beta\Delta}$. Thus, for every $t \geqslant 0$ and $\Delta > 0$ we have that

$$\mathbb{P}(S_{t+\Delta} - S_t = 0) < \mathrm{e}^{-\beta\Delta}\int_0^t \mathbb{P}(U_{\tau_t -} \in \mathrm{d}u) = \mathrm{e}^{-\beta\Delta}.$$

For $(\alpha, \beta) \in \{-1\} \times (0, +\infty)$, we directly apply the formula (2.4) to obtain

$$\mathbb{P}(S_{t+\Delta} - S_t = 0) = \int_0^t \frac{\mathrm{e}^{-\beta(t+\Delta-u)}/\beta}{\mathrm{e}^{-\beta(t-u)}/\beta}\mathbb{P}(U_{\tau_t -} \in \mathrm{d}u) = \mathrm{e}^{-\beta\Delta}, \quad t > 0,$$

which corresponds to the straight line in figure 2.

Appendix B. Sample path generation for tempered stable subordinator

In figure 2, we have explicit formulas for $\mathbb{P}(S_{t+\Delta} - S_t = 0)$ in the cases $(\alpha, \beta) = (1/2, 0)$ (given by (2.2)) and $(\alpha, \beta) = (-1, 1)$ (see previous section). However, numerical approximation is required in the case $(\alpha, \beta) = (1/2, 1)$, which corresponds to an inverse tempered stable subordinator. In the following, we describe our strategy for numerically approximating $\mathbb{P}(S_{t+\Delta} - S_t = 0)$ in the tempered stable case.

Let $\{U_s : s \geqslant 0\}$ be a tempered stable subordinator corresponding to the Lévy measure $\nu(\mathrm{d}z) = \mathrm{e}^{-\beta z} z^{-\alpha-1} \mathrm{d}z$ with $\alpha \in (0, 1)$ and $\beta > 0$. Evaluation of the event $\{U_{\tau_t} - t > \Delta\}$ requires observing jumps crossing the fixed level t, so generation of increments is insufficient for our purpose. We estimated probabilities in the latter case by generating 20 000 sample paths of the inverse tempered stable subordinator. To generate sample paths of $\{U_s : s \in [0, T]\}$, we take advantage of the series representation [13, 21]

$$\{U_s : s \in [0, T]\} \stackrel{\mathscr{L}}{=} \left\{ \sum_{k=1}^{+\infty} \min \left\{ \left(\frac{\alpha \Gamma_k}{T} \right)^{-1/\alpha}, \frac{V_k U_k^{1/\alpha}}{\beta} \right\} \mathbb{1}(T_k \in [0, s]) : s \in [0, T] \right\}, \tag{B.1}$$

where

- $\{\Gamma_k\}_{k \in \mathbb{N}}$ is a sequence of arrival times of a standard Poisson process,
- $\{V_k\}_{k \in \mathbb{N}}$ is a sequence of iid standard exponential random variables,
- $\{U_k\}_{k \in \mathbb{N}}$ is a sequence of iid standard uniform random variables, and
- $\{T_k\}_{k \in \mathbb{N}}$ is a sequence of independent random variables distributed uniformly on $[0, T]$,

such that $\{\Gamma_k\}_{k \in \mathbb{N}}$, $\{V_k\}_{k \in \mathbb{N}}$, $\{U_k\}_{k \in \mathbb{N}}$ and $\{T_k\}_{k \in \mathbb{N}}$ are mutually independent. An approximate sample path of $\{U_s : s \in [0, T]\}$ can be obtained by summation of the summands in (B.1) over the index set $\{k \in \mathbb{N} : \Gamma_k / T \leqslant n\}$, where $n > 0$ is a truncation parameter. Our numerical results for figure 2 correspond to $\alpha = 1/2$, $\beta = 1$ and $n = 20$. To justify our choice of $n = 20$, note that for every $k \in \mathbb{N}$ such that $\Gamma_k / T \leqslant 20$ we have $(\Gamma_k / (2T))^{-2} \geqslant 0.01$, that is, all jumps of sizes greater or equal to 0.01 are simulated exactly. This implies that all jumps which can cross any observation window $(t, t + \Delta]$ for $\Delta \in \{0.1, 0.5\}$ are generated, as required. It is also important to note that since the process is increasing, the truncation of jumps leads to approximate sample paths which are below the true sample path and hence the estimated probability has a positive bias. However, this bias becomes largely eliminated as t increases due to the convergence of $\mathbb{P}(S_{t+\Delta} - S_t = 0)$, and we refer the reader to [9] for details regarding the error analysis of the truncation method for various series representations.

ORCID iDs

Sida Yuan https://orcid.org/0000-0003-1848-988X
Reiichiro Kawai https://orcid.org/0000-0002-3845-015X

References

[1] Barkai E 2001 Fractional Fokker–Planck equation, solution, and application *Phys. Rev.* E **63** 046118
[2] Baule A and Friedrich R 2005 Joint probability distributions for a class of non-Markovian processes *Phys. Rev.* E **71** 026101
[3] Burov S, Jeon J H, Metzler R and Barkai E 2011 Single particle tracking in systems showing anomalous diffusion: the role of weak ergodicity breaking *Phys. Chem. Chem. Phys.* **13** 1800–12
[4] Burov S, Metzler R and Barkai E 2010 Aging and nonergodicity beyond the Khinchin theorem *Proc. Natl Acad. Sci.* **107** 13228–33
[5] Cadavid A, Lawrence J and Ruzmaikin A 1999 Anomalous diffusion of solar magnetic elements *Astrophys. J.* **521** 844
[6] Carnaffan S and Kawai R 2017 Cusping, transport and variance of solutions to generalized Fokker–Planck equations *J. Phys. A: Math. Theor.* **50** 245001
[7] Carnaffan S and Kawai R 2019 Optimal statistical inference for subdiffusion processes *J. Phys. A: Math. Theor.* **52** 135001
[8] Fogedby H C 1994 Langevin equations for continuous time Lévy flights *Phys. Rev.* E **50** 1657

[9] Imai J and Kawai R 2011 On finite truncation of infinite shot noise series representation of tempered stable laws *Physica* A **390** 4411–25
[10] Janczura J, Orzeł S and Wyłomańska A 2011 Subordinated α-stable Ornstein–Uhlenbeck process as a tool for financial data description *Physica* A **390** 4379–87
[11] Janczura J and Wyłomańska A 2009 Subdynamics of financial data from fractional Fokker–Planck equation *Acta Phys. Pol.* B **40** 1341–51
[12] Janczura J and Wyłomańska A 2012 Anomalous diffusion models: different types of subordinator distribution *Acta Phys. Pol.* B **43** 1000–16
[13] Kawai R and Masuda H 2011 Exact discrete sampling of finite variation tempered stable Ornstein–Uhlenbeck processes *Monte Carlo Methods Appl.* **17** 279–300
[14] Kumar A and Vellaisamy P 2015 Inverse tempered stable subordinators *Stat. Probab. Lett.* **103** 134–41
[15] Leonenko N N, Meerschaert M M, Schilling R L and Sikorskii A 2014 Correlation structure of time-changed Lévy processes *Commun. Appl. Ind. Math.* **6** e–483
[16] Magdziarz M and Zorawik T 2016 Stochastic representation of a fractional subdiffusion equation. The case of infinitely divisible waiting times, Lévy noise and space-time-dependent coefficients *Proc. Am. Math. Soc.* **144** 1767–78
[17] Meerschaert M M and Straka P 2013 Inverse stable subordinators *Math. Modelling Nat. Phenom.* **8** 1–16
[18] Metzler R, Jeon J H, Cherstvy A G and Barkai E 2014 Anomalous diffusion models and their properties: non-stationarity, non-ergodicity, and ageing at the centenary of single particle tracking *Phys. Chem. Chem. Phys.* **16** 24128–64
[19] Plastino A and Plastino A 1995 Non-extensive statistical mechanics and generalized Fokker–Planck equation *Physica* A **222** 347–54
[20] Reserve Bank of Australia 2008 Statement by Glenn Stevens, Governor: Monetary Policy | Media Releases (www.rba.gov.au/media-releases/2008/mr-08-20.html) last accessed on 17 January 2020
[21] Rosiński J 2007 Tempering stable processes *Stoch. Process. Appl.* **117** 677–707
[22] Sato K 1999 *Lévy Processes and Infinitely Divisible Distributions* (Cambridge: Cambridge University Press)
[23] Schertzer D, Larchevêque M, Duan J, Yanovsky V and Lovejoy S 2001 Fractional Fokker–Planck equation for nonlinear stochastic differential equations driven by non-Gaussian Lévy stable noises *J. Math. Phys.* **42** 200–12
[24] Schwarzl M, Godec A and Metzler R 2017 Quantifying non-ergodicity of anomalous diffusion with higher order moments *Sci. Rep.* **7** 3878
[25] Stanislavsky A, Weron K and Weron A 2008 Diffusion and relaxation controlled by tempered α-stable processes *Phys. Rev.* E **78** 051106
[26] Weigel A V, Simon B, Tamkun M M and Krapf D 2011 Ergodic and nonergodic processes coexist in the plasma membrane as observed by single-molecule tracking *Proc. Natl Acad. Sci.* **108** 6438–43