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Characterization of the function spaces associated with weighted Sobolev spaces of the first order on the real line

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Abstract. A brief survey of results on the characterization of the spaces associated with given classes of function spaces is presented. It is shown that the situation differs in general for ideal and non-ideal spaces. In the second case the notion of associated space splits into two. In the main body of the text a complete description is given of the function spaces associated with weighted Sobolev spaces of the first order on the real line.

Bibliography: 54 titles.

Keywords: function spaces, associated spaces, duality principle, weighted Sobolev spaces, Oinarov–Otelbaev construction, Hardy–Steklov integral operators.

Contents

Introduction

Let X be a function space with norm $\|\cdot\|_X$. Alongside the notion of the dual (adjoint) space X^* of all bounded linear functionals on X one has the well-known notion of the space X' associated with X and also the problem of describing X'. In a number of classical cases these spaces are isometrically isomorphic.

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Banach function spaces. Let $(\mathcal{R}, m)$ be a space with a complete σ-finite measure, let $\mathfrak{M}$ be the set of measurable functions, and let $\mathfrak{M}^+ \subset \mathfrak{M}$ the subset of non-negative functions. We present several definitions and results known from [3], Chap. 1.

Definition 1. A Banach function norm is a map $\rho\colon \mathfrak{M}^+ \to [0, \infty]$ satisfying the following axioms:

(P1) $\rho(f) = 0$ if and only if $f = 0$ almost everywhere (a.e.), $\rho(af) = a\rho(f)$ for each $a \geqslant 0$, and $\rho(f+g) \leqslant \rho(f) + \rho(g)$;

(P2) if $0 \leqslant g \leqslant f$ a.e., then $\rho(g) \leqslant \rho(f)$;

(P3) if $0 \leqslant f_n \uparrow f$ a.e., then $\rho(f_n) \uparrow \rho(f)$ (the Fatou property);

(P4) if $m(E) < \infty$, then $\rho(\chi_E) < \infty$, where χ_E is the characteristic function of the set E;

(P5) if $m(E) < \infty$, then $\int_E f \leqslant C_E \rho(f)$.

Let ρ be a Banach function norm. Then

$$X := \{f \in \mathfrak{M}\colon \|f\|_X := \rho(|f|) < \infty\}$$

is called a Banach function space. In terms of the associated norm ρ' with

$$\rho'(g) := \sup\biggl\{\int fg\colon f \in \mathfrak{M}^+,\ \rho(f) \leqslant 1\biggr\}, \qquad g \in \mathfrak{M}^+,$$

one defines the associated space

$$X' := \{g \in \mathfrak{M}\colon \|g\|_{X'} := \rho'(|g|) < \infty\}.$$

Proposition 1. *If X is a Banach function space, then so also is X', both spaces are complete, $X = X''$, and Hölder's inequality $\int |fg| \leqslant \|f\|_X \|g\|_{X'}$ holds.*

Proposition 2. *If X is a Banach function space and X' is the associated space, then for each $g \in X'$*

$$\begin{aligned}\|g\|_{X'} &= \sup\biggl\{\int fg\colon f \in \mathfrak{M},\ \|f\|_X \leqslant 1\biggr\} \\ &= \sup\biggl\{\biggl|\int fg\biggr|\colon f \in \mathfrak{M},\ \|f\|_X \leqslant 1\biggr\} = \sup\biggl\{\int |fg|\colon f \in \mathfrak{M},\ \|f\|_X \leqslant 1\biggr\};\end{aligned}$$

and here X and X' can be interchanged.

Definition 2. A Banach function space X is said to have an absolutely continuous norm if

$$\|f\chi_{E_n}\|_X \to 0$$

for each $f \in X$ and each sequence of sets $\{E_n\} \subset \mathcal{R}$ such that $\chi_{E_n} \to 0$ a.e.

Proposition 3. *If X is a Banach function space with an absolutely continuous norm, then $X^* = X'$. In particular, if X' is also a space with an absolutely continuous norm, then X and X' are reflexive.*

There is a well-known problem of describing the space associated with a given Banach function space: this involves establishing an explicit functional equal or equivalent to the associated norm. A similar problem can be stated for a cone, a lattice, and so on.

We give a few examples.

Example 1. For $1 \leqslant p \leqslant \infty$ let

$$X = L_p := \left\{ f \in \mathfrak{M} \colon \|f\|_p := \left(\int |f|^p \right)^{1/p} < \infty \right\}.$$

Then

$$X' = L_{p'} \quad \text{and} \quad \|f\|_{X'} = \|f\|_{p'}, \quad \text{where } \frac{1}{p} + \frac{1}{p'} = 1.$$

Example 2. For $1 < p < \infty$ let $X = \mathrm{Ces}_p[0,\infty)$ be the Cesàro space

$$\|g\|_X := \left(\int_0^\infty \left(\frac{1}{x} \int_0^x |g| \right)^p \right)^{1/p}.$$

Then

$$\|f\|_{X'} \approx \|f^\downarrow\|_{p'}, \quad \text{where} \quad f^\downarrow(t) := \operatorname*{ess\,sup}_{s \geqslant t} |f(s)|$$

(see [24]).

Note that the norm $\|f^\downarrow\|_{p'}$ is not absolutely continuous, and therefore the Banach function space $\mathrm{Ces}_p[0,\infty)$ is not reflexive.

If $L_p^\downarrow$ is the cone of non-negative non-increasing functions on $[0,\infty)$ that are integrable to the power p, then by a theorem of Sawyer describing the space associated with the cone $L_p^\downarrow$ (Theorem 1 in [44]),

$$\mathrm{Ces}_p[0,\infty) = [L_{p'}^\downarrow]'$$

and

$$\|g\|_{[L_{p'}^\downarrow]'} := \sup_{f \in L_{p'}^\downarrow,\ f \neq 0} \frac{1}{\|f\|_{p'}} \int_0^\infty f|g| \approx \left(\int_0^\infty \left(\frac{1}{x} \int_0^x |g| \right)^p \right)^{1/p}.$$

Example 3. For $1 < p < \infty$ and $I = [0,1]$, let $X = L^{p)}(I)$ be the grand Lebesgue space [18] with

$$\|f\|_X := \sup_{0 < \varepsilon < p-1} \left(\varepsilon \int_I |f(x)|^{p-\varepsilon} \, dx \right)^{1/(p-\varepsilon)}.$$

Equivalent expressions for the norms of X and X' are found in [10] and [6]:

$$\|f\|_{L^{p)}(I)} \approx \sup_{0<t<1} (1 - \log t)^{-1/p} \left(\int_t^1 [f^*]^p \right)^{1/p},$$

$$\|g\|_{L^{p)'}(I)} \approx \int_0^1 (1 - \log t)^{-1/p'} \left(\int_0^t [g^*]^{p'} \right)^{1/p'} \frac{dt}{t},$$

where g^* is the decreasing rearrangement of g. These expressions were used in [19] and [20] to obtain a characterization of the space associated with the cone of non-increasing functions in $L^{p)}(I)$ in the form

$$\sup_{0\leqslant f\in\mathfrak{M}^{\downarrow}} \frac{1}{\|f\|_{L^{p)}(I)}} \int_I f|g| \approx \|A|g|\,\|_{L^{p)\prime}(I)} + \int_0^1 |g| \approx \|A'|g|\,\|_{L^{p)\prime}(I)},$$

where

$$Ag(x) := \frac{1}{x}\int_0^x g \quad \text{and} \quad A'g(x) := \int_x^1 \frac{g(y)\,dy}{y}\,.$$

It was also shown that the above operators are bounded on the grand Lebesgue space.

As we noted above, a description of the associated space is a topical problem in describing optimal function spaces containing a prescribed cone (for instance, see [16] and the references there). Under fairly general assumptions we can show that the associated space of the space associated with this cone is optimal.

Since the early 1990s there has been rapid development in the field of *Lorentz analysis*: questions involving the characterization of boundedness for classical operators acting in weighted Lorentz spaces, where a central role is played by the construction of the space associated with a prescribed cone of monotone functions (see [14] and [13]). We now give an example of spaces associated with the cone of quasi-concave functions; such spaces are closely connected with Lorentz gamma spaces.

Example 4. Let $\mathfrak{M}^{\downarrow} \subset \mathfrak{M}^{+}$ be the subset of non-increasing functions on $[0,\infty)$. For $0 < p \leqslant \infty$ and a weight $v \in \mathfrak{M}^{+}$ we define the weighted cone of monotone functions $X = \mathcal{L}_{p,v} \subset \mathfrak{M}^{\downarrow}$ of the form

$$\mathcal{L}_{p,v} := \biggl\{ f \in \mathfrak{M}^{\downarrow} \colon \|f\|_{p,v} := \biggl(\int_0^\infty \biggl(\frac{1}{t}\int_0^t f \biggr)^p v(t)\,dt \biggr)^{1/p} < \infty \biggr\}$$

for $0 < p < \infty$ and

$$\mathcal{L}_{\infty,v} := \biggl\{ f \in \mathfrak{M}^{\downarrow} \colon \|f\|_{\infty,v} := \operatorname*{ess\,sup}_{t\geqslant 0} \biggl(\frac{1}{t}\int_0^t f \biggr) v(t) < \infty \biggr\}$$

for $p = \infty$. Here we assume that the non-degeneracy conditions

$$0 < \int \frac{v(z)\,dz}{(z+t)^p} < \infty \quad \text{and} \quad \int_0^1 z^{-p} v(z)\,dz = \int_1^\infty v = \infty$$

hold; otherwise all results reduce to a few pathological situations. Let

$$Pv(t) := \frac{1}{t}\int_0^t v \quad \text{and} \quad Q_p v(t) := p t^{p-1} \int_t^\infty s^{-p} v(s)\,ds.$$

If

$$\|g\|_{X'} := \sup_{f\in\mathfrak{M}^{\downarrow}} \frac{\int_0^\infty f|g|}{\bigl(\int_0^\infty [Pf]^p v\bigr)^{1/p}}\,,$$

then [47]

$$\|g\|_{X'} \approx \left(\int_0^\infty \left(\sup_{t\geqslant x}\frac{1}{t}\int_0^t |g|\right)^{p'} \frac{Pv(x)Q_pv(x)}{(PQ_pv(x))^{p'+1}}\,dx\right)^{1/p'}, \qquad 1<p<\infty,$$

$$\|g\|_{X'} \approx \sup_{t\geqslant 0}\frac{t^{1/p'}P|g|(t)}{[PQ_pv(t)]^{1/p}}, \qquad 0<p<1,$$

$$\|g\|_{X'} = \sup_{t\geqslant 0}\frac{P|g|(t)}{PQ_1v(t)}, \qquad p=1,$$

and

$$\|g\|_{X'} \approx \int_{[0,\infty)} \overline{G}\,d\lambda, \qquad p=\infty,$$

where the measure $d\lambda$ is constructed from the weight function v, and $\overline{G}(x) := x\sup_{t\geqslant x} P|g|(t)$.

Spaces associated with non-ideal function spaces. A function space X is said to be *ideal* if

$$\forall\, f,g\in\mathfrak{M}\colon\ f\in X,\ |g|\leqslant|f| \quad\Longrightarrow\quad g\in X,\ \|g\|_X\leqslant\|f\|_X.$$

For ideal function spaces we have Proposition 2. However, for a non-ideal function space the definition of the associated space X' splits into two cases. Let

$$\mathfrak{D}_X := \left\{g\in\mathfrak{M}\colon\ \int|fg|<\infty\ \forall\, f\in X\right\}.$$

For $g\in\mathfrak{D}_X$ consider the functionals

$$J_X(g) := \sup_{f\in X\colon \|f\|_X\neq 0}\frac{1}{\|f\|_X}\left|\int fg\right| \quad\text{and}\quad \mathbf{J}_X(g) := \sup_{f\in X\colon \|f\|_X\neq 0}\frac{1}{\|f\|_X}\int|fg|,$$

which define norms on the linear space $\mathfrak{D}_X$ and thus determine the associated spaces X' and $\mathbf{X}'$ given by

$$X' := \{g\in\mathfrak{D}_X\colon \|g\|_{X'} := J_X(g)<\infty\}$$

and

$$\mathbf{X}' := \{g\in\mathfrak{D}_X\colon \|g\|_{\mathbf{X}'} := \mathbf{J}_X(g)<\infty\}.$$

Example 5. Let $X=C[0,1]$. Then by Riesz's classical theorem

$$J_X(g)=\mathbf{J}_X(g)=\|g\|_1.$$

Example 6. Let $X=H^1(\mathbb{R})$ be the Hardy space on the real line. Then by a theorem of Fefferman [8] (see also [9] and [46], Chap. IV, § 1.2)

$$J_X(g)\approx\|g\|_{\mathrm{BMO}} := \sup_{I\subset\mathbb{R}}\frac{1}{|I|}\int_I|g-g_I|,$$

where BMO denotes the space of functions of bounded mean oscillation, the supremum is taken over all intervals of the real line, and

$$g_I := \frac{1}{|I|} \int_I g.$$

However, using the atomic decomposition of functions in $H^1(\mathbb{R})$, we can show that

$$\mathbf{J}_X(g) = \|g\|_\infty.$$

Hence $\mathfrak{D}_X = L_\infty$, and $J_X(g) \neq \mathbf{J}_X(g)$ in general. Moreover, there exist an $f \in H^1(\mathbb{R})$ and a $g \in \mathrm{BMO}$ such that

$$\int_{\mathbb{R}} |fg| = \infty$$

([46], Chap. IV, § 6.2).

Thus, for all function spaces we have the embedding $\mathbf{X}' \subseteq X'$, but in general $\mathbf{X}' \neq X'$ for non-ideal spaces.

The aim of describing the associated space is to realize the *duality principle*, which is the equality

$$\|T\|_{X\to Y} = \|T'\|_{Y'\to X'}$$

of the norms of a linear operator T and the associated operator T' such that

$$\int_S gTf\,d\rho = \int_S fT'g\,d\rho \quad \text{for any } f \in X \quad \text{and} \quad g \in Y'.$$

In particular, if Y is an ideal space, then

$$\|T\|_{X\to Y} = \sup_{0\neq g\in Y'} \frac{J_X(T'g)}{\|g\|_{Y'}}.$$

Use of the duality principle is a key to investige the action of operators from classical analysis in new function spaces ([1], [2], [7], [15], [44], [45]).

In this paper we characterize the associated spaces X' and $\mathbf{X}'$ in the case when X is a weighted Sobolev space of the first order on the real line, with the same parameters of integration for the function and its derivative. We also describe these spaces completely (see the definitions and the history of the question in § 1). Our method is based on the Oinarov–Otelbaev constructions (see § 3) for the norm of a Sobolev space and on boundedness results for the Hardy–Steklov operators, which we present comprehensively here (see § 4).

Throughout the paper, products of the form $0\cdot\infty$ are set equal to 0. The relation $A \lesssim B$ means that $A \leqslant cB$ for a constant c which depends only on the parameter p; $A \approx B$ is equivalent to $A \lesssim B \lesssim A$. We use the signs := and =: to introduce new quantities. For $1 < p < \infty$ we set $p' := p/(p-1)$.

1. Weighted Sobolev spaces on the real line

Let $-\infty \leqslant a < b \leqslant \infty$, $I := (a,b)$, and $1 \leqslant p \leqslant \infty$, and let $\mathcal{L}^1$ be the Lebesgue measure on the real line. We let $\mathfrak{M}(I)$ denote the set of all Lebesgue-measurable functions on I, and let $\mathfrak{M}^+(I)$ denote the subset of non-negative functions in $\mathfrak{M}(I)$. Let $W^1_{1,\mathrm{loc}}(I)$ denote the space of functions $u \in L^1_{\mathrm{loc}}(I)$ that have a generalized derivative $Du \in L^1_{\mathrm{loc}}(I)$.

For functions $v_0, v_1 \in \mathfrak{M}^+(I)$ such that $1/v_1 \in L^{p'}_{\mathrm{loc}}(I)$, $v_0, v_1 \in L^p_{\mathrm{loc}}(I)$, and $\|v_0\|_{L^1(I)} > 0$ we define the weighted Sobolev space

$$W^1_p(I) := \{u \in W^1_{1,\mathrm{loc}}(I) : \|u\|_{W^1_p(I)} < \infty\},$$

where

$$\|u\|_{W^1_p(I)} := \|v_0 u\|_{L^p(I)} + \|v_1 Du\|_{L^p(I)},$$

and we define subspaces of this space,

$$\overset{\circ\circ}{W}{}^1_p(I) := \{f \in AC(I) \colon \operatorname{supp} f \text{ is a compact subset of } I,\\ \|v_0 f\|_{L^p(I)} + \|v_1 f'\|_{L^p(I)} < \infty\}$$

and $\mathring{W}^1_p(I) = \overline{\overset{\circ\circ}{W}{}^1_p(I)}$, the closure of $\overset{\circ\circ}{W}{}^1_p(I)$ in $W^1_p(I)$.

If X is one of the spaces $W^1_p(I)$, $\mathring{W}^1_p(I)$, and $\overset{\circ\circ}{W}{}^1_p(I)$, then we set

$$\mathfrak{D}_X := \left\{g \in L^1_{\mathrm{loc}}(I) \colon \int_I |f(x)g(x)|\,dx < \infty \text{ for all } f \in X\right\}.$$

Note that

$$\mathfrak{D}_{W^1_p(I)} \subset \mathfrak{D}_{\mathring{W}^1_p(I)} \subset \mathfrak{D}_{\overset{\circ\circ}{W}{}^1_p(I)} = L^1_{\mathrm{loc}}(I)$$

and the first two embeddings are proper, in general.

For a fixed Lebesgue-measurable function $g \in \mathfrak{D}_X$ consider the functionals

$$J_X(g) := \sup_{f \in X \colon \|f\|_{W^1_p(I)} \neq 0} \frac{1}{\|f\|_{W^1_p(I)}} \left|\int_I f(x)g(x)\,dx\right|$$

and

$$\mathbf{J}_X(g) := \sup_{f \in X \colon \|f\|_{W^1_p(I)} \neq 0} \frac{1}{\|f\|_{W^1_p(I)}} \int_I |f(x)g(x)|\,dx,$$

which define norms on $\mathfrak{D}_X$ that correspond to the associated spaces X' and $\mathbf{X}'$, respectively. Note that for $X \in \{\mathring{W}^1_p(I), W^1_p(I)\}$, $g \in \mathfrak{D}_X$ if and only if $J_X(g) < \infty$. This is a consequence of [42], Lemma 2.4 and Theorems 2.5 and 2.6.

For $X = W^1_p(I)$ or $X = \mathring{W}^1_p(I)$ the first two-sided bound for $\mathbf{J}_X(g)$ in terms of an integral functional was obtained by Oinarov [30] as part of a characterization of Hardy's inequality with three weights

$$\|gf\|_{L^q(I)} \leqslant C(\|v_0 f\|_{L^p(I)} + \|v_1 f'\|_{L^p(I)}), \qquad f \in W^1_p(I);$$

for $q = 1$ the best constant C here is equal to $\mathbf{J}_{W_p^1(I)}(g)$ (see also earlier alternative results in [34], [25], [27], [38], [5], and [26]). However, the functionals in [30] do not have the properties of a norm and cannot be used in a duality principle. This defect was subsequently corrected in [31], [32], and [7], but the duality principle based on the corresponding functionals could only be applied to positive operators. Furthermore, the results in [31], [32], and [7] were obtained under the additional assumption that $W_p^1(I) = \mathring{W}_p^1(I)$.

2. Properties of the functionals $J_X(g)$, $J_X(|g|)$, and $\mathbf{J}_X(g)$

In [40]–[42] we proved the set-theoretic equality $X' = \mathbf{X}'$ for a space $X \in \{\mathring{W}_p^1(I), W_p^1(I)\}$ and found sharp two-sided bounds for the norms of elements of the spaces X' and $\mathbf{X}'$ associated with an $X \in \{\overset{\circ\circ}{W}{}_p^1(I), \mathring{W}_p^1(I)\}$ in terms of integral functionals involving the weight functions v_0 and v_1, in the case when $\mathring{W}_p^1(I) = W_p^1(I)$ (for the model case where $I = (0,\infty)$); we also showed that for $X = \overset{\circ\circ}{W}{}_p^1(I)$ there is no equality $X' = \mathbf{X}'$ in general. We recall the main results in [40]–[42] involving the functionals $J_X(g)$, $J_X(|g|)$, and $\mathbf{J}_X(g)$.

Theorem 2.1. *Let $I \subseteq \mathbb{R}$, $1 \leqslant p \leqslant \infty$, $g \in \mathfrak{M}(I)$, $v_0, v_1 \in \mathfrak{M}^+(I)$, $1/v_1 \in L^{p'}_{\mathrm{loc}}(I)$, $v_0, v_1 \in L^p_{\mathrm{loc}}(I)$, and $\|v_0\|_{L^1(I)} > 0$. Then*

$$J_{W_p^1(I)}(g) < \infty \quad \Longleftrightarrow \quad \mathbf{J}_{W_p^1(I)}(g) < \infty,$$
$$J_{\mathring{W}_p^1(I)}(g) < \infty \quad \Longleftrightarrow \quad \mathbf{J}_{\mathring{W}_p^1(I)}(g) < \infty.$$

Corollary 2.1. *Let $I \subseteq \mathbb{R}$, $1 \leqslant p \leqslant \infty$, $g \in \mathfrak{M}(I)$, $v_0, v_1 \in \mathfrak{M}^+(I)$, $1/v_1 \in L^{p'}_{\mathrm{loc}}(I)$, $v_0, v_1 \in L^p_{\mathrm{loc}}(I)$, $\|v_0\|_{L^1(I)} > 0$, and*

$$X = W_p^1(I) \quad \text{or} \quad X = \mathring{W}_p^1(I).$$

Then $X' = \mathbf{X}'$ and for each $g \in X'$ there exists a constant C_g such that

$$C_g \|g\|_{\mathbf{X}'} \leqslant \|g\|_{X'} \leqslant \|g\|_{\mathbf{X}'}.$$

Remark 2.1. The following equalities hold:

$$J_{W_p^1(I)}(|g|) = \mathbf{J}_{W_p^1(I)}(g), \quad J_{\mathring{W}_p^1(I)}(|g|) = \mathbf{J}_{\mathring{W}_p^1(I)}(g), \quad \text{and} \quad J_{\overset{\circ\circ}{W}{}_p^1(I)}(|g|) = \mathbf{J}_{\overset{\circ\circ}{W}{}_p^1(I)}(g).$$

In this paper, which complements the results in [40], [43], [37], [41], and [42], we give a complete characterization of the functionals $J_X(g)$ and $\mathbf{J}_X(g)$ for

$$X = \overset{\circ\circ}{W}{}_p^1(I), \quad X = \mathring{W}_p^1(I), \quad \text{and} \quad X = W_p^1(I).$$

3. Oinarov–Otelbaev construction

The construction below is mainly borrowed from [30] (see also [7], [42], and the earlier papers [33] and [28]).

Let $I := (a,b) \subseteq \mathbb{R}$, $1 < p < \infty$, $v_0, v_1 \in \mathfrak{M}^+(I)$, $1/v_1 \in L^{p'}_{\mathrm{loc}}(I)$, $v_0, v_1 \in L^p_{\mathrm{loc}}(I)$, and $\|v_0\|_{L^1(I)} > 0$. Note that for each interval $I' \subset I$ with $\mathcal{L}^1(I') > 0$ we have

$$\int_{I'} v_1^{-p'} > 0.$$

First assume that $\int_I v_1^{-p'} < \infty$. Then there exists a $c \in (a,b)$ such that

$$\int_a^c v_1^{-p'} = \int_c^b v_1^{-p'}, \tag{3.1}$$

and there exist two non-negative strictly increasing functions $\tau_a(x)$ with $x \in (a,c]$ and $\delta_b(x)$ with $x \in [c,b)$ such that

$$\int_a^x v_1^{-p'} = \int_x^{x+\tau_a(x)} v_1^{-p'}, \qquad x \in (a,c],$$

and

$$\int_{x-\delta_b(x)}^x v_1^{-p'} = \int_x^b v_1^{-p'}, \qquad x \in [c,b).$$

For $(a,b) \subset \mathbb{R}$ let

$$\begin{aligned} D := \{(x,y) \in \mathbb{R}^2 \colon & x \in (a,c],\ a \leqslant y \leqslant x + \tau_a(x); \\ & x \in [c,b),\ x - \delta_b(x) \leqslant y \leqslant b\} \end{aligned}$$

and

$$\begin{aligned} D_0 := \{(x,y) \in \mathbb{R}^2 \colon & x \in (a,c],\ a - x \leqslant y \leqslant \tau_a(x); \\ & x \in [c,b),\ -\delta_b(x) \leqslant y \leqslant b - x\}, \end{aligned}$$

where for $a = -\infty$ and $b = +\infty$, or for $a = -\infty$ and $b = +\infty$ we make appropriate modifications. Also, let

$$D_0^+ := \{(x,y) \in D_0 \colon y \geqslant 0\}.$$

Note that

$$(x,y) \in D_0 \quad \Longleftrightarrow \quad (x, x+y) \in D,$$

we have the inclusion $D_0 \supset D_0^+$, and

$$(x,y) \in D_0^+ \quad \Longleftrightarrow \quad (x, x+y) \in D \cap \{y \geqslant 0\}.$$

Moreover, it follows from (3.1) that for $(x,y) \in D_0^+ \cap \{x \in [c,b)\}$ we have

$$\int_a^x v_1^{-p'} \geqslant \int_x^{x+y} v_1^{-p'}. \tag{3.2}$$

In fact, if $x \in [c,b)$ and $x + y \leqslant b$, then

$$\int_a^x v_1^{-p'} \geqslant \int_a^c v_1^{-p'} \geqslant \int_c^b v_1^{-p'} \geqslant \int_x^{x+y} v_1^{-p'}.$$

Similarly, if $(x,y) \in D_0^+ \cap \{x \in (a,c]\}$, then (3.2) holds. Indeed,

$$\int_a^x v_1^{-p'} = \int_x^{x+\tau_a(x)} v_1^{-p'} \geqslant \int_x^{x+y} v_1^{-p'}.$$

The converse is also true: if $x \in (a,c]$, $y \geqslant 0$, and (3.2) holds, then

$$\int_x^{x+y} v_1^{-p'} \leqslant \int_a^x v_1^{-p'} = \int_x^{x+\tau_a(x)} v_1^{-p'},$$

and therefore $0 \leqslant y \leqslant \tau_a(x)$, so that $(x,y) \in D_0^+ \cap \{x \in (a,c]\}$.

In the case when $\int_I v_1^{-p'} = \infty$ the constructions are similar. Leaving out the definitions of the sets D and D_0 in this case, we write down only the expressions for D_0^+. If there exists an $e \in (a,b)$ such that

$$\int_a^e v_1^{-p'} < \infty \quad \text{and} \quad \int_e^b v_1^{-p'} = \infty,$$

then $\tau_a(x)$ is defined for all $x \in (a,b)$ and

$$D_0^+ := \{(x,y)\colon x \in (a,b),\ 0 \leqslant y \leqslant \tau_a(x)\}.$$

But if for some $e \in (a,b)$ we have

$$\int_a^e v_1^{-p'} = \infty \quad \text{and} \quad \int_e^b v_1^{-p'} < \infty,$$

then

$$D_0^+ := \{(x,y)\colon x \in (a,b),\ 0 \leqslant b-x\}.$$

When there exists an $e \in (a,b)$ such that

$$\int_a^e v_1^{-p'} = \int_e^b v_1^{-p'} = \infty,$$

we have

$$D_0^+ := \{(x,y)\colon x \in (a,b),\ 0 \leqslant y\}.$$

In any case, if $(x,y) \in D_0^+$, then (3.2) holds, and since the integral is monotonic, for any $(x,y) \in D_0^+$ there exists a unique $\delta(x,y) > 0$ such that

$$\int_{x-\delta(x,y)}^x v_1^{-p'} = \int_x^{x+y} v_1^{-p'}. \tag{3.3}$$

In the original paper [30] (see also [31], [32], [7], and [42]) the quantity $\delta(x,y)$ was defined in a certain domain $\widetilde{D}$ by the equality

$$\delta(x,y) := \sup\biggl\{d > 0\colon \int_{x-d}^x v_1^{-p'} \leqslant \int_x^{x+y} v_1^{-p'},\ (x-d,x] \subset I\biggr\},$$

and inequality was allowed to hold in (3.3), but this case was then ruled out in the subsequent steps.

For $x \in I$ let

$$D_0^+(x) := \{y \geqslant 0 \colon (x,y) \in D_0^+\}$$

be the cross-section of D_0^+ by the line with abscissa $x \in I$, and let

$$d^+(x) := \sup\left\{ y \in D_0^+(x) \colon \left\| \frac{1}{v_1} \right\|_{L^{p'}((x-\delta(x,y),x+y))} \|v_0\|_{L^p((x-\delta(x,y),x+y))} \leqslant 1 \right\},$$
$$d^-(x) := \delta(x, d^+(x)), \quad \mu^-(x) := x - d^-(x), \quad \mu^+(x) := x + d^+(x),$$
$$\Delta^+(x) := [x, \mu^+(x)] \cap I, \quad \Delta^-(x) := [\mu^-(x), x] \cap I,$$
$$\Delta(x) := [\mu^-(x), \mu^+(x)] \cap I, \quad x^- := \mu^-(x), \quad x^+ := \mu^+(x).$$

We use the notation

$$a_0 := \inf\{x \in I \colon x - d^-(x) > a\}$$

and

$$b_0 := \sup\{x \in I \colon x + d^+(x) < b\}.$$

Let $e \in (a,b)$ be a point such that $\|v_0\|_{L^p((a,e))} > 0$ and $\|v_0\|_{L^p((e,b))} > 0$, and let

$$h_a := \left\| \frac{1}{v_1} \right\|_{L^{p'}((a,e))} \|v_0\|_{L^p((a,e))} \quad \text{and} \quad h_b := \left\| \frac{1}{v_1} \right\|_{L^{p'}((e,b))} \|v_0\|_{L^p((e,b))}.$$

Then

$$a \leqslant a_0 < b \quad (a < a_0 \ \Leftrightarrow \ h_a < \infty), \qquad a < b_0 \leqslant b \quad (b_0 < b \ \Leftrightarrow \ h_b < \infty),$$
$$h_a = h_b = \infty \quad \Longleftrightarrow \quad W_p^1(I) = \mathring{W}_p^1(I). \tag{3.4}$$

Moreover,

$$\int_{\Delta^-(x)} v_1^{-p'} = \int_{\Delta^+(x)} v_1^{-p'}, \qquad x \in (a,b), \tag{3.5}$$

$\mu^+(x) < b$ for any $x \in (a, b_0)$, $\mu^-(x) > a$ for any $x \in (a_0, b)$,

$$\left\| \frac{1}{v_1} \right\|_{L^{p'}(\Delta(x))} \|v_0\|_{L^p(\Delta(x))} \leqslant 1, \qquad x \in (a,b), \tag{3.6}$$

and

$$\left\| \frac{1}{v_1} \right\|_{L^{p'}(\Delta(x))} \|v_0\|_{L^p(\Delta(x))} = 1, \qquad x \in (a_0,b_0).$$

The equivalence (3.4) was established in [30], Lemma 1.6. Lemma 3.1 below was proved in Lemma 1.1 in [30], and our Lemma 3.2 is a consequence of Lemma 2.1 in [32].

Lemma 3.1. *Let $I := (a, b) \subseteq \mathbb{R}$, $1 < p < \infty$, $v_0, v_1 \in \mathfrak{M}^+(I)$, $1/v_1 \in L^{p'}_{\mathrm{loc}}(I)$, $v_0, v_1 \in L^p_{\mathrm{loc}}(I)$, and $\|v_0\|_{L^1(I)} > 0$. Then μ^- is a strictly increasing continuous function on $[a_0, b) \cap I$ that is continuous on (a_0, b), and μ^+ is a strictly increasing function on $(a, b_0] \cap I$ that is continous on (a, b_0). If $h_a < \infty$, then*

$$\mu^+(a+0) = a \quad \textit{and} \quad \mu^-(x) = a = \mu^-(a_0+0) \quad \textit{for all } x \in (a, a_0].$$

If $h_b < \infty$, then

$$\mu^-(b-0) = b \quad \textit{and} \quad \mu^+(x) = b = \mu^+(b_0-0) \quad \textit{for all } x \in [b_0, b).$$

If $h_a = \infty$, then

$$\mu^+(a+0) = a^* := \inf\left\{t \in I \colon \int_a^t v > 0\right\}.$$

If $h_b = \infty$, then

$$\mu^-(b-0) = b^* := \sup\left\{t \in I \colon \int_t^b v > 0\right\}.$$

Lemma 3.2. *Let $I := (a, b) \subseteq \mathbb{R}$, $1 < p < \infty$, $v_0, v_1 \in \mathfrak{M}^+(I)$, $1/v_1 \in L^{p'}_{\mathrm{loc}}(I)$, $v_0, v_1 \in L^p_{\mathrm{loc}}(I)$, and $\|v_0\|_{L^1(I)} > 0$. Then*

$$\mu^- \in AC_{\mathrm{loc}}((a_0, b)) \quad \textit{and} \quad \mu^+ \in AC_{\mathrm{loc}}((a, b_0)).$$

For $t \in I$ let

$$V_0(t) := \int_{\Delta(t)} v_0^p, \qquad V_1(t) := \int_{\Delta(t)} v_1^{-p'},$$
$$V_1^-(t) := \int_{\Delta^-(t)} v_1^{-p'}, \qquad V_1^+(t) := \int_{\Delta^+(t)} v_1^{-p'}.$$

Note that for any $t \in I$ we have $V_1(t) = 2V_1^-(t) = 2V_1^+(t)$ by (3.5). We also have $V_1(t) > 0$ since $\mu^-(t) < t < \mu^+(t)$.

Lemma 3.3. *Let $I := (a, b) \subseteq \mathbb{R}$, $1 < p < \infty$, $v_0, v_1 \in \mathfrak{M}^+(I)$, $1/v_1 \in L^{p'}_{\mathrm{loc}}(I)$, $v_0, v_1 \in L^p_{\mathrm{loc}}(I)$, and $\|v_0\|_{L^1(I)} > 0$. Then*

$$V_1 \in AC_{\mathrm{loc}}(I).$$

Proof. 1. Let $h_a = h_b = \infty$. Then $a_0 = a$, $b_0 = b$, and

$$\mu^-, \mu^+ \in AC_{\mathrm{loc}}(I)$$

by Lemma 3.2. Fix $[\alpha, \beta] \subset (a, b)$ and note that $\mu^-(\alpha), \mu^+(\beta) \in I$ and $V_1(x) = G(\mu^+(x)) - G(\mu^-(x))$, where

$$G(t) := \int_{\mu^-(\alpha)}^t v_1^{-p'}, \qquad t \in [\mu^-(\alpha), \mu^+(\beta)].$$

Since μ^- and μ^+ belong to $AC([\alpha,\beta])$ and are increasing functions and since $G \in AC([\mu^-(\alpha),\mu^+(\beta)])$, it follows that $V_1 \in AC([\alpha,\beta])$.

2. Let $h_a < \infty$. Then for each $t \in (a,b)$ the integral

$$\int_a^t v_1^{-p'} =: G(t)$$

exists. Note that $V_1(x) = 2(G(x) - G(\mu^-(x)))$ for $x \in I$ and $G \in AC_{\rm loc}(I)$. We show that

$$G \circ \mu^- \in AC_{\rm loc}(I).$$

Fix a closed interval $[\alpha,\beta] \subset (a,b)$. If $\beta \leqslant a_0$, then $G \circ \mu^- = 0$ on $[\alpha,\beta]$, so that $G \circ \mu^- \in AC([\alpha,\beta])$. Suppose that $\beta > a_0$. Then fix $\varepsilon > 0$. By Lemma 3.1 there exists a $t_0 \in (a_0,\beta)$ such that $\int_a^{\mu^-(t_0)} v_1^{-p'} < \dfrac{\varepsilon}{2}$. Since

$$G \in AC\biggl(\biggl[\mu^-\biggl(\frac{a_0+t_0}{2}\biggr),\ \mu^-(\beta)\biggr]\biggr)$$

and the function μ^- belongs to $AC\biggl(\biggl[\dfrac{a_0+t_0}{2}\,,\beta\biggr]\biggr)$ and is increasing, we have

$$G \circ \mu^- \in AC\biggl(\biggl[\frac{a_0+t_0}{2}\,,\beta\biggr]\biggr).$$

Hence, there exists a $\delta_0 > 0$ such that for any system of non-overlapping intervals $\{(\alpha_j,\beta_j)\}$ such that

$$[\alpha_j,\beta_j] \subset \biggl[\frac{a_0+t_0}{2}\,,\beta\biggr] \quad \text{and} \quad \sum_j |\beta_j - \alpha_j| < \delta_0$$

we have

$$\sum_j |(G\circ\mu^-)(\beta_j) - (G\circ\mu^-)(\alpha_j)| < \frac{\varepsilon}{2}\,.$$

Take $\delta \in \biggl(0, \min\biggl\{\delta_0, \dfrac{t_0-a_0}{3}\biggr\}\biggr)$. Then for any system of non-overlapping intervals $\{(\alpha_j,\beta_j)\}$ such that

$$[\alpha_j,\beta_j] \subset [\alpha,\beta] \quad \text{and} \quad \sum_j |\beta_j - \alpha_j| < \delta,$$

we have

$$\begin{aligned}
&\sum_j |(G\circ\mu^-)(\beta_j) - (G\circ\mu^-)(\alpha_j)| \\
&\qquad \leqslant \biggl[\sum_{j\colon [\alpha_j,\beta_j]\subset(a,t_0]} + \sum_{j:[\alpha_j,\beta_j]\subset[(a_0+t_0)/2,\beta]}\biggr] |(G\circ\mu^-)(\beta_j) - (G\circ\mu^-)(\alpha_j)| \\
&\qquad \leqslant \int_a^{\mu^-(t_0)} v_1^{-p'} + \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

3. The case $h_b < \infty$ is treated similarly. $\square$

By Lemma 1.3 in [30], for $f \in AC_{\mathrm{loc}}(I)$ such that $f(a+0)=0$ for $h_a<\infty$ and $f(b-0)=0$ for $h_b<\infty$ we have

$$\sup_{s\in\Delta(x)} |f(s)| \leqslant 2^{1/p'} \left\| \frac{1}{v_1} \right\|_{L^{p'}(\Delta^{\pm}(x))} \|f\|_{W^1_p(\Delta(x))}, \qquad x \in I. \tag{3.7}$$

4. Hardy–Steklov operators

Let $-\infty \leqslant a_i < b_i \leqslant \infty$ and $I_i := (a_i, b_i)$, $i=1,2$, and let

$$\vartheta \in \mathfrak{M}^+(I_1) \quad \text{and} \quad \omega \in \mathfrak{M}^+(I_2)$$

be weight functions. For $q_1>1$ and $q_2>0$ we consider a Hardy–Steklov operator

$$\mathcal{H}f(x) := \omega(x) \int_{\phi(x)}^{\psi(x)} f(y)\vartheta(y)\,dy, \tag{4.1}$$

from $L^{q_1}(I_1)$ to $L^{q_2}(I_2)$, where the boundary functions ϕ and ψ satisfy the inequalities

$$-\infty \leqslant a_1 \leqslant \phi(x) \leqslant \psi(x) \leqslant b_1 \leqslant \infty \quad \text{on } I_2$$

and the following conditions:

(i) ϕ and ψ are locally absolutely continuous and strictly increasing on I_2;

(ii) $\phi(a_2)=\psi(a_2)=a_1,\ \phi(x)<\psi(x)$ for $x\in I_2,$ and $\phi(b_2)=\psi(b_2)=b_1.$ (4.2)

The integral operators $\mathcal{H}$ with variable limits of integration arise in certain problems concerned with the properties of operators acting in function spaces ([7], [30], [31]). The two limiting cases of operators $\mathcal{H}$ when $\phi(x)=a_1$ or $\psi(x)=b_1$ are very well understood. In particular, much is known about the Hardy integral operator

$$Hf(x) := \omega(x) \int_a^x f(y)\vartheta(y)\,dy, \tag{4.3}$$

acting from $L^{q_1}(I)$ to $L^{q_2}(I)$ [22], [23]. In the literature (for instance, see [11] and [36]) the reader can find precise characteristics of boundedness of two basic types of operator $H\colon L^{q_1}(I) \to L^{q_2}(I)$. They have the form of functionals (or boundedness constants) equivalent to the norm $\|H\|_{L^{q_1}(I)\to L^{q_2}(I)}$ and depending only on the fixed parameters of the problem: the weight functions, the boundary functions, and the integration parameters q_i, $i=1,2$. The Muckenhoupt functional A_{M} and the Maz'ya–Rozin functional B_{MR},

$$A_{\mathrm{M}} = \sup_{t\in I} \biggl(\int_t^b \omega^{q_2}\biggr)^{1/q_2} \biggl(\int_a^t \vartheta^{q_1'}\biggr)^{1/q_1'} \qquad (1<q_1\leqslant q_2<\infty)$$

and

$$B_{\mathrm{MR}} = \biggl(\int_a^b \biggl[\int_t^b \omega^{q_2}\biggr]^{r/q_1} \biggl[\int_a^t \vartheta^{q_1'}\biggr]^{r/q_1'} \omega^{q_2}(t)\,dt\biggr)^{1/r} \qquad (0<q_2<q_1<\infty,\ q_1>1),$$

where $q_i' = q_i/(q_i-1)$, $i=1,2$, and $1/r = 1/q_2 - 1/q_1$, and also the dual constants $(A_{\mathrm{M}})^* = A_{\mathrm{M}}$ and

$$(B_{\mathrm{MR}})^* = \biggl(\int_a^b \biggl[\int_t^b \omega^{q_2}\biggr]^{r/q_2} \biggl[\int_a^t \vartheta^{q_1'}\biggr]^{r/q_2'} \vartheta^{q_1'}(t)\,dt\biggr)^{1/r}$$

with respect to them, relate to the first type of boundedness characteristics for $H\colon L^{q_1}(I) \to L^{q_2}(I)$. The second type, characteristics alternative to A_{M} and B_{MR}, includes the Tomaselli functional A_{T} and the Persson–Stepanov functional B_{PS}, of the form

$$A_{\mathrm{T}} = \sup_{t\in I}\biggl(\int_a^t \biggl[\int_a^x \vartheta^{q_1'}\biggr]^{q_2} \omega^{q_2}(x)\,dx\biggr)^{1/q_2} \biggl(\int_a^t \vartheta^{q_1'}\biggr)^{-1/q_1} \qquad (1 < q_1 \leqslant q_2 < \infty)$$

and

$$\begin{aligned} B_{\mathrm{PS}} = \biggl(\int_a^b \biggl[\int_a^t \biggl\{\int_a^x \vartheta^{q_1'}\biggr\}^{q_2} \omega^{q_2}(x)\,dx\biggr]^{r/q_1} \\ \times \biggl[\int_a^t \vartheta^{q_1'}\biggr]^{q_2 - r/q_1} \omega^{q_2}(t)\,dt\biggr)^{1/r} \qquad (0 < q_2 < q_1 < \infty,\ q_1 > 1), \end{aligned}$$

respectively, and also the boundedness constants dual to these functionals (see [36]). The functionals A_{M} and B_{MR} are basic boundedness characteristics for the operator H acting from $L^{q_1}(I)$ into $L^{q_2}(I)$, and they are usually used in the further analysis of the properties of H and in applications of these properties. The alternative boundedness constants A_{T} and B_{PS} have proved to be indispensable, in particular, in investigations of the non-linear geometric mean operator

$$Gf(x) = \exp\biggl(\frac{1}{x}\int_a^x \log f(y)\,dy\biggr)$$

(see [35]) and also in certain other problems [52].

The first steps in devising a scale of boundedness characteristics for the Hardy–Steklov operator $\mathcal{H}$ that is similar to the above scale for H were made in [12], [17], and [4]. The most noticeable progress in this direction was achieved when the notion of the fairway function [50] was introduced into the problem of characterizing a bounded operator (4.1) acting from $L^{q_1}(I_1)$ to $L^{q_2}(I_2)$, with boundary functions ϕ and ψ satisfying (4.2).

Definition 4.1 ([48], [50]). Given boundary functions ϕ and ψ satisfying (4.2), a number $q_1 \in (1,\infty)$, and a weight function ϑ such that

$$0 < \vartheta(y) < \infty \quad \text{for almost all } y \in I_1 \quad \text{and} \quad \vartheta \in L^{q_1'}_{\mathrm{loc}}(I_1),$$

the *fairway function* σ is a function such that $\phi(x) < \sigma(x) < \psi(x)$ for $x \in I_2$ and

$$\int_{\phi(x)}^{\sigma(x)} \vartheta^{q_1'}(y)\,dy = \int_{\sigma(x)}^{\psi(x)} \vartheta^{q_1'}(y)\,dy. \tag{4.4}$$

Slightly later, in [51], the dual fairway ρ was introduced, which also opened up new ways of characterizing the $(L^{q_1} - L^{q_2})$-boundedness of $\mathcal{H}$.

Definition 4.2 ([29], [51], [53]). For given boundary functions ϕ and ψ satisfying (4.2), a positive number q_2, and a weight function ω such that

$$0 < \omega < \infty \quad \text{a.e. on } I_2 \quad \text{and} \quad \omega \in L^{q_2}_{\mathrm{loc}}(I_2),$$

a *dual fairway function* ρ must satisfy the inequalities $\psi^{-1}(y) < \rho(y) < \phi^{-1}(y)$ on I_1 and the equality

$$\int_{\psi^{-1}(y)}^{\rho(y)} \omega^{q_2}(x)\,dx = \int_{\rho(y)}^{\phi^{-1}(y)} \omega^{q_2}(x)\,dx \quad \text{for all } y \in I_1. \tag{4.5}$$

Here ϕ^{-1} and ψ^{-1} are the inverse functions of ϕ and ψ, respectively.

Definitions 4.1 and 4.2 show that both the fairways σ and ρ are strictly increasing, almost-everywhere differentiable functions on I_2 and I_1, respectively ([41], § 2.2.1).

The first result in [48] was the determination of boundedness characteristics of the Muckenhoupt ($\mathcal{A}_{\mathrm{M}}$) and Maz'ya–Rozin ($\mathcal{B}_{\mathrm{MR}}$) type for the operator $\mathcal{H}\colon L^{q_1}(I_1) \to L^{q_2}(I_2)$ for all q_1 and q_2 satisfying $1 < q_1 < \infty$ and $0 < q_2 < \infty$, in terms of the fairway σ:

$$\begin{aligned}
\mathcal{A}_{\mathrm{M}} &=: \mathcal{A}_\sigma = \sup_{t\in I_2}\biggl(\int_{\psi^{-1}(\sigma(t))}^{\phi^{-1}(\sigma(t))} \omega^{q_2}\biggr)^{1/q_2}\biggl(\int_{\phi(t)}^{\psi(t)} \vartheta^{q_1'}\biggr)^{1/q_1'} \quad (1 < q_1 \leqslant q_2 < \infty),\\
\mathcal{B}_{\mathrm{MR}} &=: \mathcal{B}_\sigma = \biggl(\int_{I_2}\biggl[\int_{\psi^{-1}(\sigma(t))}^{\phi^{-1}(\sigma(t))} \omega^{q_2}\biggr]^{r/q_1}\\
&\qquad\qquad \times\biggl[\int_{\phi(t)}^{\psi(t)} \vartheta^{q_1'}\biggr]^{r/q_1'} \omega^{q_2}(t)\,dt\biggr)^{1/r} \quad (0 < q_2 < q_1 < \infty,\ q_1 > 1).
\end{aligned} \tag{4.6}$$

From duality, but only for $q_i > 1$, $i = 1,2$, the dual functionals $(\mathcal{A}_\rho)^*$ and $(\mathcal{B}_\rho)^*$ with respect to $\mathcal{A}_{\mathrm{M}}$ and $\mathcal{B}_{\mathrm{MR}}$ appeared in [50] and [51], expressed in terms of the fairway ρ:

$$\begin{aligned}
(\mathcal{A}_\rho)^* &= \sup_{t\in I_1}\biggl(\int_{\psi^{-1}(t)}^{\phi^{-1}(t)} \omega^{q_2}\biggr)^{1/q_2}\biggl(\int_{\phi(\rho(t))}^{\psi(\rho(t))} \vartheta^{q_1'}\biggr)^{1/q_1'} \quad (1 < q_1 \leqslant q_2 < \infty),\\
(\mathcal{B}_\rho)^* &= \biggl(\int_{I_1}\biggl[\int_{\psi^{-1}(t)}^{\phi^{-1}(t)} \omega^{q_2}\biggr]^{r/q_2}\biggl[\int_{\phi(\rho(t))}^{\psi(\rho(t))} \vartheta^{q_1'}\biggr]^{r/q_2'} \vartheta^{q_1'}(t)\,dt\biggr)^{1/r} \quad (1 < q_2 < q_1 < \infty).
\end{aligned}$$

A full scale for the boundedness characteristics of Muckenhoupt and Maz'ya–Rozin type for operators $\mathcal{H}\colon L^{q_1}(I_1) \to L^{q_2}(I_2)$, expressed in terms of both the direct fairway σ and the dual fairway ρ, was presented in [29] (see also [53]) for all $q_1 > 1$ and $q_2 > 0$. In particular, in [29] the validity of the boundedness constant $(\mathcal{B}_\rho)^*$ was extended to the case $0 < q_2 < q_1 < \infty$, $q_1 > 1$, and two further pairs of functionals $(\mathcal{A}_\sigma)^*$, $(\mathcal{B}_\sigma)^*$ and $\mathcal{A}_\rho$, $\mathcal{B}_\rho$ were found, which, like $\mathcal{A}_\sigma$, $\mathcal{B}_\sigma$ and $(\mathcal{A}_\rho)^*$, $(\mathcal{B}_\rho)^*$, are mutually dual for $q_i > 1$, $i = 1,2$ (see details in [29] or [53]).

This completed the determination of a base scale of boundedness characteristics of the first type for Hardy–Steklov operators (4.1) acting from $L^{q_1}(I_1)$ to $L^{q_2}(I_2)$. The above functionals were used in [29] to characterize the embeddability of a certain class of AC-functions in a fractional Sobolev space, and also in [7], [40], and [42] to find two-sided bounds (of alternative forms to the bounds presented here) for the norms of the spaces associated with weighted Sobolev spaces.

The first alternative boundedness characteristics of type A_{T} and B_{PS} for the Hardy–Steklov operator $\mathcal{H}$ were found in [49], where it was shown that the norm $\|\mathcal{H}\|_{L^{q_1}(I_1)\to L^{q_2}(I_2)}$ is equivalent to the functional

$$\mathcal{A}_{\mathrm{T}} =: \mathbb{A}_\sigma = \sup_{t\in I_2}\biggl(\int_{\sigma^{-1}(\phi(t))}^{\sigma^{-1}(\psi(t))}\biggl[\int_{\phi(x)}^{\psi(x)}\vartheta^{q_1'}\biggr]^{q_2}\omega^{q_2}(x)\,dx\biggr)^{1/q_2}\biggl(\int_{\phi(t)}^{\psi(t)}\vartheta^{q_1'}\biggr)^{-1/q_1} \tag{4.7}$$

when $1<q_1\leqslant q_2<\infty$, and to the functional

$$\begin{aligned}\mathcal{B}_{\mathrm{PS}} =: \mathbb{B}_\sigma = \biggl(\int_{I_2}\biggl[\int_{\sigma^{-1}(\phi(t))}^{\sigma^{-1}(\psi(t))}\biggl\{\int_{\psi(x)}^{\psi(x)}\vartheta^{q_1'}\biggr\}^{q_2}\omega^{q_2}(x)\,dx\biggr]^{r/q_1}&\\ \times\biggl[\int_{\phi(t)}^{\psi(t)}\vartheta^{q_1'}\biggr]^{r_2-r/q_1}\omega^{q_2}(t)\,dt\biggr)^{1/r}&\end{aligned} \tag{4.8}$$

when $0<q_2<q_1<\infty$, $q_1>1$ (see also [50], Theorem 4.2). In [49] the functionals $\mathbb{A}_\sigma$ and $\mathbb{B}_\sigma$ were successfully applied to the problem of characterizing the boundedness of the geometric mean operator of Hardy–Steklov type which has the form

$$\mathcal{G}f(x)=\exp\biggl(\frac{1}{\psi(x)-\phi(x)}\int_{\phi(x)}^{\psi(x)}\log f\biggr).$$

By duality the following two boundedness characteristics for $\mathcal{H}\colon L^{q_1}(I_1)\to L^{q_2}(I_2)$ are consequences of the relations

$$\|\mathcal{H}\|_{L^{q_1}(I_1)\to L^{q_2}(I_2)}\approx\mathbb{A}_\sigma \quad\text{and}\quad \|\mathcal{H}\|_{L^{q_1}(I_1)\to L^{q_2}(I_2)}\approx\mathbb{B}_\sigma$$

for $q_i>1$, $i=1,2$:

$$(\mathbb{A}_\rho)^* = \sup_{t\in I_1}\biggl(\int_{\rho^{-1}(\psi^{-1}(t))}^{\rho^{-1}(\phi^{-1}(t))}\biggl[\int_{\psi^{-1}(y)}^{\phi^{-1}(y)}\omega^{q_2}\biggr]^{q_1'}\vartheta^{q_1'}(y)\,dy\biggr)^{1/q_1'}\biggl(\int_{\psi^{-1}(t)}^{\phi^{-1}(t)}\omega^{q_2}\biggr)^{-1/q_2'}, \tag{4.9}$$

$$\begin{aligned}(\mathcal{B}_{\mathrm{PS}})^* = \biggl(\int_{I_1}\biggl[\int_{\rho^{-1}(\psi^{-1}(t))}^{\rho^{-1}(\phi^{-1}(t))}\biggl\{\int_{\psi^{-1}(y)}^{\phi^{-1}(y)}\omega^{q_2}\biggr\}^{q_1'}\vartheta^{q_1'}(y)\,dy\biggr]^{r/q_2'}&\\ \times\biggl[\int_{\psi^{-1}(t)}^{\phi^{-1}(t)}\omega^{q_2}\biggr]^{q_1'-r/q_2'}\vartheta^{q_1'}(t)\,dt\biggr)^{1/r}.&\end{aligned} \tag{4.10}$$

Together with $\mathbb{A}_\sigma$ and $\mathbb{B}_\sigma$, the functionals $(\mathbb{A}_\rho)^*$ and $(\mathcal{B}_{\mathrm{PS}})^*$ form part of the corresponding system of alternative boundedness characteristics of the second type for an operator $\mathcal{H}$ from $L^{q_1}(I_1)$ to $L^{q_2}(I_2)$. A full scale for such boundedness constants for $\mathcal{H}\colon L^{q_1}(I_1) \to L^{q_2}(I_2)$ was found for all $q_1 > 1$ and $q_2 > 0$ in [54], where all possible forms of such characteristics, expressed in terms of both the direct fairway σ and the dual fairway ρ, were found. A full system of boundedness constants of the second type, apart from $\mathbb{A}_\sigma$, $\mathbb{B}_\sigma$, $(\mathbb{A}_\rho)^*$, and $(\mathbb{B}_{\mathrm{PS}})^*$, also includes the functionals

$$(\mathbb{A}_\sigma)^* = \sup_{t\in I_2}\biggl(\int_{\sigma^{-1}(\phi(\sigma^{-1}(t)))}^{\sigma^{-1}(\psi(\sigma^{-1}(t)))} \Theta^{q_2}\omega^{q_2}\biggr)^{1/q_2}\biggl(\int_{\phi(\sigma^{-1}(t))}^{\psi(\sigma^{-1}(t))} \vartheta^{q_1'}\biggr)^{-1/q_1},$$

$$(\mathbb{B}_\sigma)^* = \biggl(\int_{I_1}\biggl[\int_{\sigma^{-1}(\phi(\sigma^{-1}(t)))}^{\sigma^{-1}(\psi(\sigma^{-1}(t)))} \Theta^{q_2}\omega^{q_2}\biggr]^{r/q_2}\biggl[\int_{\phi(\sigma^{-1}(t))}^{\phi(\sigma^{-1}(t))} \vartheta^{q_1'}\biggr]^{-r/q_2}\vartheta^{q_1'}(t)\,dt\biggr)^{1/r},$$

$$\mathbb{A}_\rho = \sup_{t\in I_2}\biggl(\int_{\rho^{-1}(\psi^{-1}(\rho^{-1}(t)))}^{\rho^{-1}(\phi^{-1}(\rho^{-1}(t)))} \Omega^{q_1'}\vartheta^{q_1'}\biggr)^{1/q_1'}\biggl(\int_{\psi^{-1}(\rho^{-1}(t))}^{\phi^{-1}(\rho^{-1}(t))} \omega^{q_2}\biggr)^{-1/q_2'},$$

$$\mathbb{B}_\rho = \begin{cases} \mathbb{B}_{q_2>1} \ll \mathbb{B}_{q_2<1}, & q>1,\\ \mathbb{B}_{q_2<1} \ll \mathbb{B}_{q_2>1}, & q<1,\end{cases}$$

$$\mathbb{B}_{q_2>1} = \biggl(\int_{I_2}\biggl[\int_{\rho^{-1}(\psi^{-1}(\rho^{-1}(t)))}^{\rho^{-1}(\phi^{-1}(\rho^{-1}(t)))} \Omega^{q_1'}\vartheta^{q_1'}\biggr]^{r/q_1'}\biggl[\int_{\psi^{-1}(\rho^{-1}(t))}^{\phi^{-1}(\rho^{-1}(t))} \omega^{q_2}\biggr]^{-r/q_1'}\omega^{q_2}(t)\,dt\biggr)^{1/r},$$

and

$$\mathbb{B}_{q_2<1} = \biggl(\int_{I_2}\biggl[\int_{\phi(t)}^{\psi(t)} \Omega^{q_1'}\vartheta^{q_1'}\biggr]^{r/q_1'}\biggl[\int_{\psi^{-1}(\rho^{-1}(t))}^{\phi^{-1}(\rho^{-1}(t))} \omega^{q_2}\biggr]^{-r/q_1'}\omega^{q_2}(t)\,dt\biggr)^{1/r},$$

where $\Theta(x) := \int_{\phi(x)}^{\psi(x)} \vartheta^{q_1'}$ and $\Omega(y) := \int_{\psi^{-1}(y)}^{\phi^{-1}(y)} \omega^{q_2}$, and also the functionals

$$(\mathbb{B}_\rho)^* = \begin{cases} (\mathbb{B}^-_{q_2>1})^* + (\mathbb{B}^+_{q_2>1})^*, & q_2>1,\\ (\mathbb{B}^-_{q_2>1})^* + (\mathbb{B}^+_{q_2>1})^* + (\mathbb{B}^-_{q_2<1})^* + (\mathbb{B}^+_{q_2<1})^*, & q_2<1,\end{cases}$$

$$(\mathbb{B}^-_{q_2>1})^* = \biggl(\int_{I_1}\biggl[\int_{\rho^{-1}(\psi^{-1}(t))}^{t} \omega^{q_1'}\vartheta^{q_1'}\biggr]^{r/q_2'}[\Omega(t)]^{q_1'-r/q_2'}\vartheta^{q_1'}(t)\,dt\biggr)^{1/r},$$

$$(\mathbb{B}^+_{q_2>1})^* = \biggl(\int_{I_1}\biggl[\int_{t}^{\rho^{-1}(\phi^{-1}(t))} \Omega^{q_1'}\vartheta^{q_1'}\biggr]^{r/q_2'}[\Omega(t)]^{q_1'-r/q_2'}\vartheta^{q_1'}(t)\,dt\biggr)^{1/r},$$

$$(\mathbb{B}^-_{q_2<1})^* = \biggl(\int_{I_1}\biggl[\int_{\phi(\rho(t))}^{t} \Omega^{q_1'}\vartheta^{q_1'}\biggr]^{r/q_2'}[\Omega(t)]^{q_1'-r/q_2'}\vartheta^{q_1'}(t)\,dt\biggr)^{1/r},$$

and

$$(\mathbb{B}^+_{q_2<1})^* = \biggl(\int_{I_1}\biggl[\int_{t}^{\psi(\rho(t))} \Omega^{q_1'}\vartheta^{q_1'}\biggr]^{r/q_2'}[\Omega(t)]^{q_1'-r/q_2'}\vartheta^{q_1'}(t)\,dt\biggr)^{1/r},$$

where $(\mathbb{B}_\rho)^* \approx (\mathcal{B}_{\mathrm{PS}})^*$ for $q_2 > 1$. Constants of type A are usually valid for $1 < q_1 \leqslant q_2 < \infty$, while characteristics of type B are equivalent to the norm $\|\mathcal{H}\|_{L^{q_1}(I_1)\to L^{q_2}(I_2)}$ for $0 < q_2 < q_1 < \infty$ and $q_1 > 1$ (see [54] for details).

To conclude this section note that for $0 < q_1 < 1$ and all $0 < q_2 < \infty$ the operator $\mathcal{H}$ is bounded from $L^{q_1}(I_1)$ to $L^{q_2}(I_2)$ only in the trivial case [39]. For $q_1 = 1,\infty$ or $q_2 = 1,\infty$ a precise equality for the norm $\|\mathcal{H}\|_{L^{q_1}(I_1)\to L^{q_2}(I_2)}$ can be derived from a general theorem ([21], Chap. XI, § 1.5, Theorem 4).

5. Two-sided bounds for the functionals $J_{\overset{\circ\circ}{W}{}^1_p(I)}(g)$

Let

$$\nu^+(y) := \begin{cases} a_0, & y = a, \\ (\mu^-)^{-1}(\{y\}), & y \in (a, \mu^-(b-0)); \end{cases}$$

$$\nu^-(y) := \begin{cases} b_0, & y = b, \\ (\mu^+)^{-1}(\{y\}), & y \in (\mu^+(a+0), b). \end{cases}$$

Note that $\nu^+ \colon [a, \mu^-(b-0)) \to [a_0, b)$ is a strictly increasing function on $[a, \mu^-(b-0))$, while $\nu^- \colon (\mu^+(a+0), b] \to (a, b_0]$ is strictly increasing on $(\mu^+(a+0), b]$. Consider the functionals

$$\mathfrak{G}_{I,\nu^+}(g) := \left(\int_I v_1^{-p'}(t) \left| \int_t^{\nu^+(t)} g(x)\, dx \right|^{p'} dt \right)^{1/p'},$$

and

$$\mathbb{G}_{I,\mu^-}(g) := \left(\int_I \frac{v_1^{-p'}(t)}{V_1^{p'}(t)} \left| \int_{\mu^-(t)}^t v_1^{-p'}(y) \int_t^{\nu^+(y)} g(x)\, dx\, dy \right|^{p'} dt \right)^{1/p'}.$$

Theorem 5.1. *Let $I := (a,b) \subseteq \mathbb{R}$, $1 < p < \infty$, and $g \in L^1_{\mathrm{loc}}(I)$. Assume that $v_0, v_1 \in \mathfrak{M}^+(I)$, $1/v_1 \in L^{p'}_{\mathrm{loc}}(I)$, $v_0, v_1 \in L^p_{\mathrm{loc}}(I)$, $\|v_0\|_{L^1(I)} > 0$, and let $\mu^-(b-0) = b$ and $\mu^+(a+0) = a$. Then*

$$J_{\overset{\circ\circ}{W}{}^1_p(I)}(g) \approx \mathbb{G}_{I,\mu^-}(g) + \mathfrak{G}_{I,\nu^+}(g),$$

where the constants of equivalence can be taken to depend only on p.

Proof. We start by proving the inequality

$$J_{\overset{\circ\circ}{W}{}^1_p(I)}(g) \lesssim \mathbb{G}_{I,\mu^-}(g) + \mathfrak{G}_{I,\nu^+}(g).$$

Fix $f \in \overset{\circ\circ}{W}{}^1_p(I)$ and note that

$$f/V_1^- \in AC_{\mathrm{loc}}(I),$$

because $f, V_1^- \in AC_{\mathrm{loc}}(I)$ and $V_1^- > 0$ on I. Furthermore, the integral

$$A := \int_{\mu^-(\inf\{\operatorname{supp} f\})}^{\sup\{\operatorname{supp} f\}} v_1^{-p'}$$

is finite: for $a = a_0$ this is because $v_1^{-p'} \in L^1_{\rm loc}(I)$, and for $a < a_0$ it follows from the relations $v_1^{-p'} \in L^1_{\rm loc}(I)$ and $h_a < \infty$. We have

$$\begin{aligned}\int_I f(x)g(x)\,dx &= \int_I V_1^-(x)\frac{f(x)}{V_1^-(x)}g(x)\,dx \\ &= \int_I \frac{f(x)}{V_1^-(x)}g(x)\Big(\int_I v_1^{-p'}(y)\chi_{\Delta^-(x)}(y)\,dy\Big)\,dx \\ &= \int_I v_1^{-p'}(y)\Big(\int_I \frac{f(x)}{V_1^-(x)}g(x)\chi_{\Delta^-(x)}(y)\,dx\Big)\,dy \\ &= \int_I v_1^{-p'}(y)\int_y^{\nu^+(y)} g(x)\frac{f(x)}{V_1^-(x)}\,dx\,dy.\end{aligned}$$

Here, the integrals (in the third equality) can be interchanged by Fubini's theorem, since f has a compact support, f/V_1^- is bounded on

$$[\inf\{\operatorname{supp} f\}, \sup\{\operatorname{supp} f\}],$$

and $g, v_1^{-p'} \in L^1_{\rm loc}(I)$, because

$$\begin{aligned}&\int_I \frac{|f(x)|}{V_1^-(x)}\,|g(x)|\,dx\int_I v_1^{-p'}(y)\chi_{\Delta^-(x)}(y)\,dy \\ &\qquad\leqslant \int_{\inf\{\operatorname{supp} f\}}^{\sup\{\operatorname{supp} f\}}\frac{|fg|}{V_1^-}\cdot\int_{\mu^-(\inf\{\operatorname{supp} f\})}^{\sup\{\operatorname{supp} f\}} v_1^{-p'} < \infty.\end{aligned}$$

Continuing the chain of equalities, we write

$$\int_I f(x)g(x)\,dx = \int_I v_1^{-p'}(y)\int_y^{\nu^+(y)} g(x)\left(\int_y^x\left[\frac{f(t)}{V_1^-(t)}\right]'dt + \frac{f(y)}{V_1^-(y)}\right)dx\,dy. \tag{5.1}$$

Since for $y \in I$ we have

$$[y, \nu^+(y)] \subset I \quad \text{and} \quad g, \left(\frac{f}{V_1^-}\right)' \in L^1_{\rm loc}(I),$$

it follows that

$$\int_y^{\nu^+(y)} g(x)\,dx\int_y^x\left[\frac{f(t)}{V_1^-(t)}\right]'dt = \int_y^{\nu^+(y)}\left[\frac{f(t)}{V_1^-(t)}\right]'dt\int_t^{\nu^+(y)} g(x)\,dx.$$

Moreover,

$$\begin{aligned}&\int_I v_1^{-p'}(y)\,dy\int_y^{\nu^+(y)}\left|\left[\frac{f(t)}{V_1^-(t)}\right]'dt\int_t^{\nu^+(y)} g(x)\,dx\right| \\ &\qquad\leqslant \int_{\mu^-(\inf\{\operatorname{supp} f\})}^{\sup\{\operatorname{supp} f\}} v_1^{-p'}\cdot\int_{\inf\{\operatorname{supp} f\}}^{\sup\{\operatorname{supp} f\}}\left|\left[\frac{f}{V_1^-}\right]'\right|\cdot\int_{\inf\{\operatorname{supp} f\}}^{\nu^+(\sup\{\operatorname{supp} f\})}|g| < \infty.\end{aligned}$$

Hence

$$\int_I v_1^{-p'}(y)\,dy \int_y^{\nu^+(y)} g(x)\,dx \int_y^x \left[\frac{f(t)}{V_1^-(t)}\right]' dt$$
$$= \int_I \left[\frac{f(t)}{V_1^-(t)}\right]' dt \int_{\mu^-(t)}^t v_1^{-p'}(y) \left[\int_t^{\nu^+(y)} g\right] dy.$$

Setting

$$G_{\mu^-}(t) := \int_{\mu^-(t)}^t v_1^{-p'}(y) \left[\int_t^{\nu^+(y)} g\right] dy, \quad G_{\nu^+}(t) := \int_t^{\nu^+(t)} g(x)\,dx, \qquad t \in I,$$

in place of (5.1) we can write down the decomposition

$$\int_I f(x)g(x)\,dx = \int_I \frac{f'(t)}{V_1^-(t)}\, G_{\mu^-}(t)\,dt - \int_I \frac{f(t)(V_1^-)'(t)}{[V_1^-(t)]^2}\, G_{\mu^-}(t)\,dt$$
$$+ \int_I \frac{f(y)\, v_1^{-p'}(y)}{V_1^-(y)}\, G_{\nu^+}(y)\,dy =: \mathrm{I} - \mathrm{II} + \mathrm{III}.$$

From Hölder's inequality with exponents p and p' and the equality $V_1 = 2V_1^-$ we get that

$$|\mathrm{I}| \leqslant 2 \int_I \frac{|f'(t)|}{V_1(t)}\, |G_{\mu^-}(t)|\,dt \lesssim \mathbb{G}_{I,\mu^-}(g) \|f'v_1\|_{L^p(I)}. \tag{5.2}$$

Note that for $x \in (a, a_0)$ we have $V_1^-(x) = \int_a^x v_1^{-p'}$. Consequently,

$$(V_1^-)'(x) = v_1^{-p'}(x) \quad \text{for } \mathcal{L}^1\text{-almost all } x \in (a, a_0).$$

Furthermore, from (3.5) we get that for $x \in (b_0, b)$

$$V_1^-(x) = V_1^+(x) = \int_x^b v_1^{-p'}$$

so that

$$(V_1^-)'(x) = -v_1^{-p'}(x) \quad \text{for } \mathcal{L}^1\text{-almost all } x \in (b_0, b).$$

Now let $c \in (a_0, b_0)$ or let $c = a_0$ if $a_0 = b_0$, and let

$$F(x) := \int_c^x v_1^{-p'}, \qquad x \in I.$$

Then $F \in AC_{\mathrm{loc}}(I)$. The functions $\mu^-\colon (a_0, b) \to I$ and $\mu^+\colon (a, b_0) \to I$ are increasing. Therefore,

$$(F \circ \mu^-)'(x) = v_1^{-p'}(\mu^-(x))\,(\mu^-)'(x) \geqslant 0 \quad \text{for } \mathcal{L}^1\text{-almost all } x \in (a_0, b), \tag{5.3}$$
$$(F \circ \mu^+)'(x) = v_1^{-p'}(\mu^+(x))\,(\mu^+)'(x) \geqslant 0 \quad \text{for } \mathcal{L}^1\text{-almost all } x \in (a, b_0).$$

Differentiating (3.5), we get that for $\mathcal{L}^1$-almost all $x \in (a_0, b_0)$

$$2\, v_1^{-p'}(x) = (F \circ \mu^-)'(x) + (F \circ \mu^+)'(x),$$

which yields

$$(F \circ \mu^-)'(x) \leqslant 2 v_1^{-p'}(x) \quad \text{for } \mathcal{L}^1\text{-almost all } \ x \in (a_0, b_0). \tag{5.4}$$

Since

$$V_1^-(x) = F(x) - (F \circ \mu^-)(x),$$

for $\mathcal{L}^1$-almost all $x \in (a_0, b_0)$ we have

$$|(V_1^-)'(x)| = |v_1^{-p'}(x) - (F \circ \mu^-)'(x)| \leqslant 3 v_1^{-p'}(x).$$

Note also that for $\mathcal{L}^1$-almost all $x \in (b_0, b)$

$$v_1^{-p'}(x) - (F \circ \mu^-)'(x) = (V_1^-)'(x) = -v_1^{-p'}(x).$$

Hence

$$(F \circ \mu^-)'(x) = 2 v_1^{-p'}(x) \quad \text{for } \mathcal{L}^1\text{-almost all } x \in (b_0, b). \tag{5.5}$$

Summarizing, we obtain

$$|(V_1^-)'(x)| \leqslant 3 v_1^{-p'}(x) \quad \text{for } \mathcal{L}^1\text{-almost all } x \in I. \tag{5.6}$$

To estimate the quantities |II| and |III|, we define a sequence $\{\xi_k\}_{k \in \overline{\mathbb{Z}} \subset \mathbb{Z}}$ of points in I. If $h_a = h_b = \infty$, then we set $\xi_0 = c \in I$ and define the ξ_k for all $k \in \mathbb{Z}$ so that

$$\xi_{k+1} = \nu^+(\mu^+(\xi_k)), \tag{5.7}$$

$$\xi_{k-1} = \nu^-(\mu^-(\xi_k)). \tag{5.8}$$

If $h_a < \infty$ and $h_b = \infty$, then we set $\xi_0 = a_0$ and for $k \in \mathbb{N}$ we choose ξ_k in accordance with (5.7). Similarly, if $h_a = \infty$ and $h_b < \infty$, then we set $\xi_0 = b_0$, and for $-k \in \mathbb{N}$ we choose ξ_k in accordance with (5.8). Finally, if $h_a < \infty$ and $h_b < \infty$, then we define the string $\{\xi_k\}_{k=0}^K$, where $0 \leqslant K < \infty$, as follows. Let $\xi_0 = a_0$ and $\xi_K = b_0$. Here if $a_0 = b_0$, then $K = 0$, and if $\nu^+(\mu^+(a_0)) \geqslant b_0$, then $K = 1$. Otherwise $K > 1$, and for $1 \leqslant k < K - 1$ we choose the ξ_k in accordance with (5.7) so that $\xi_{K-1} < \xi_K$. Then $I \subset \bigcup_{k \in \overline{\mathbb{Z}}} \Delta(\xi_k)$. By (5.6),

$$\int_{\Delta^-(\xi_k)} \left| \frac{f(t)(V_1^-)'(t)}{[V_1^-(t)]^2} G_{\mu^-}(t) \right| dt \lesssim \int_{\Delta^-(\xi_k)} |f(t) - f(\xi_k^-)| \frac{v_1^{-p'}(t)}{[V_1^-(t)]^2} |G_{\mu^-}(t)|\, dt$$

$$+ |f(\xi_k^-)| \int_{\Delta^-(\xi_k)} \frac{v_1^{-p'}(t)}{[V_1^-(t)]^2} |G_{\mu^-}(t)|\, dt =: \mathrm{II}_{k,1} + \mathrm{II}_{k,2}, \tag{5.9}$$

where $f(\xi_0^-) := 0$ for $h_a < \infty$ (in this case $\xi_0^- = a$). Using Hölder's inequality again and taking into account the inclusion $[y, \nu^+(y)] \supset [y, \xi_k]$ for $y \in \Delta^-(\xi_k)$, we obtain

$$\begin{aligned}
\mathrm{II}_{k,1} &= \int_{\Delta^-(\xi_k)} \left| \int_{\xi_k^-}^{t} f'(y)\, dy \right| \frac{v_1^{-p'}(t)}{[V_1^-(t)]^2} \, |G_{\mu^-}(t)| \, dt \\
&\leqslant \int_{\Delta^-(\xi_k)} |f'(y)| \int_y^{\xi_k} \frac{v_1^{-p'}(t)}{[V_1^-(t)]^2} \, |G_{\mu^-}(t)| \, dt \\
&\leqslant \left(\int_{\Delta^-(\xi_k)} v_1^{-p'}(y) \left(\int_y^{\xi_k} \frac{v_1^{-p'}(t)}{[V_1^-(t)]^2} \, |G_{\mu^-}(t)| \, dt \right)^{p'} dy \right)^{1/p'} \|f' v_1\|_{L^p(\Delta^-(\xi_k))} \\
&\leqslant \left(\int_{\Delta^-(\xi_k)} v_1^{-p'}(y) \left(\int_y^{\nu^+(y)} \frac{v_1^{-p'}(t)}{[V_1^-(t)]^2} \, |G_{\mu^-}(t)| \, dt \right)^{p'} dy \right)^{1/p'} \|f' v_1\|_{L^p(\Delta^-(\xi_k))}.
\end{aligned} \tag{5.10}$$

To estimate $\mathrm{II}_{k,2}$, in view of (3.7) we write

$$|f(\xi_k^-)| \lesssim \left(\int_{\Delta^-(\xi_k^-)} v_1^{-p'} \right)^{1/p'} \|f\|_{W_p^1(\Delta(\xi_k^-))}.$$

In combination with (3.5) this yields

$$\mathrm{II}_{k,2} \lesssim \left(\int_{\Delta^+(\xi_k^-)} v_1^{-p'}(y)\, dy \right)^{1/p'} \left(\int_{\Delta^-(\xi_k)} \frac{v_1^{-p'}(t)}{[V_1^-(t)]^2} \, |G_{\mu^-}(t)| \, dt \right) \|f\|_{W_p^1(\Delta(\xi_k^-))}.$$

For $y \in \Delta^+(\xi_k^-)$ we have the inclusions

$$[\nu^-(y), \nu^+(y)] \supset [\nu^-(y), \xi_k] \supset \Delta^-(\xi_k),$$

which show that

$$\begin{aligned}
\mathrm{II}_{k,2} &\lesssim \left(\int_{\Delta^+(\xi_k^-)} v_1^{-p'}(y) \left(\int_{\nu^-(y)}^{\nu^+(y)} \frac{v_1^{-p'}(t)}{[V_1^-(t)]^2} \, |G_{\mu^-}(t)| \, dt \right)^{p'} dy \right)^{1/p'} \|f\|_{W_p^1(\Delta(\xi_k^-))} \\
&\leqslant \left(\int_{\Delta(\xi_k)} v_1^{-p'}(y) \left(\int_{\nu^-(y)}^{\nu^+(y)} \frac{v_1^{-p'}(t)}{[V_1^-(t)]^2} \, |G_{\mu^-}(t)| \, dt \right)^{p'} dy \right)^{1/p'} \|f\|_{W_p^1(\Delta(\xi_k^-))} \\
&=: \mathbf{G}_{\mu^-,k}(g) \|f\|_{W_p^1(\Delta(\xi_k^-))}.
\end{aligned} \tag{5.11}$$

From (5.9), (5.10), and (5.11) we obtain

$$\int_{\Delta^-(\xi_k)} \left| \frac{f(t)(V_1^-)'(t)}{[V_1^-(t)]^2} \, G_{\mu^-}(t) \right| dt \lesssim \mathbf{G}_{\mu^-,k}(g) \|f\|_{W_p^1(\Delta^-(\xi_k^-) \cup \Delta(\xi_k))}.$$

Similar calculations with Δ^+ in place of Δ^- in (5.9) give an upper bound for the same integral over Δ^+:

$$\int_{\Delta^+(\xi_k)} \left| \frac{f(t)(V_1^-)'(t)}{[V_1^-(t)]^2} \, G_{\mu^-}(t) \right| dt \lesssim \mathbf{G}_{\mu^-,k}(g) \|f\|_{W_p^1(\Delta(\xi_k) \cup \Delta^+(\xi_k^+))}.$$

Since $\Delta^-(\xi_k^-)\cup\Delta(\xi_k)$ and $\Delta(\xi_k)\cup\Delta^+(\xi_k^+)$ lie in the interior of

$$\Gamma_k := \Delta(\xi_{k-1})\cup\Delta(\xi_k)\cup\Delta(\xi_{k+1}),$$

it follows that

$$\int_{\Delta(\xi_k)}\left|\frac{f(t)(V_1^-)'(t)}{[V_1^-(t)]^2}\,G_{\mu^-}(t)\right|dt \lesssim \mathbf{G}_{\mu^-,k}(g)\|f\|_{W_p^1(\Gamma_k)}.$$

Using Hölder's inequality and bearing in mind that each point in I belongs to at most two intervals $\Delta(\xi_k)$ and therefore to at most four intervals Γ_k, we obtain

$$\begin{aligned}
|\mathrm{II}| &\leqslant \sum_k\int_{\Delta(\xi_k)}\left|\frac{f(t)(V_1^-)'(t)}{[V_1^-(t)]^2}\,G_{\mu^-}(t)\right|dt \lesssim \sum_k \mathbf{G}_{\mu^-,k}(g)\|f\|_{W_p^1(\Gamma_k)}\\
&\leqslant \Big(\sum_k \mathbf{G}_{\mu^-,k}(g)^{p'}\Big)^{1/p'}\Big(\sum_k\|f\|^p_{W_p^1(\Gamma_k)}\Big)^{1/p}\\
&\lesssim \bigg(\int_I v_1^{-p'}(y)\Big(\int_{\nu^-(y)}^{\nu^+(y)}\frac{v_1^{-p'}(t)}{[V_1^-(t)]^2}\,|G_{\mu^-}(t)|\,dt\Big)^{p'}dy\bigg)^{1/p'}\|f\|_{W_p^1(I)}\\
&=: \mathbf{G}_{\mu^-}(g)\|f\|_{W_p^1(I)}.
\end{aligned}\tag{5.12}$$

Similarly, we find that

$$\begin{aligned}
|\mathrm{III}| &\leqslant \sum_k\int_{\Delta(\xi_k)}\left|\frac{f(t)v_1^{-p'}(t)}{V_1^-(t)}\,G_{\nu^+}(t)\right|dt\\
&\lesssim \bigg(\int_I v_1^{-p'}(y)\Big(\int_{\nu^-(y)}^{\nu^+(y)}\frac{v_1^{-p'}(t)}{V_1^-(t)}\,|G_{\nu^+}(t)|\,dt\Big)^{p'}dy\bigg)^{1/p'}\|f\|_{W_p^1(I)}\\
&=: \mathbf{G}_{\nu^+}(g)\|f\|_{W_p^1(I)}.
\end{aligned}\tag{5.13}$$

To estimate the functionals $\mathbf{G}_{\mu^-}(g)$ and $\mathbf{G}_{\nu^+}(g)$ on the right-hand sides of (5.12) and (5.13), consider the inequality

$$\Big(\int_I|H^*h(y)|^{p'}\,dy\Big)^{1/p'}\leqslant C\Big(\int_I|h(x)|^{p'}\,dx\Big)^{1/p'}\tag{5.14}$$

for the operator

$$H^*h(t) := u(t)\int_{\nu^-(t)}^{\nu^+(t)}h(x)w(x)\,dx,$$

acting from $L^{p'}(I)$ to $L^{p'}(I)$. The transformation adjoint to H^* has the form

$$\begin{aligned}
Hh(x) &:= w(x)\int_{\mu^-(x)}^{\mu^+(x)}h(t)u(t)\,dt = H_ah(x)+\mathscr{H}h(x)+H_bh(x)\\
&:= w(x)\bigg[\chi_{(a,a_0]}(x)\int_a^{\mu^+(x)}+\chi_{(a_0,b_0)}(x)\int_{\mu^-(x)}^{\mu^+(x)}+\chi_{[b_0,b)}(x)\int_{\mu^-(x)}^{b}\bigg]h(t)u(t)\,dt
\end{aligned}$$

and acts on $L^p(I)$. Here

$$H_a\colon L^p((a,\mu^+(a_0)]) \to L^p((a,a_0]) \quad \text{and} \quad H_b\colon L^p([\mu^-(b_0),b)) \to L^p([b_0,b))$$

are Hardy operators with a variable (upper or lower) limit of integration, and

$$\mathscr{H}\colon L^p((a,b)) \to L^p((a_0,b_0))$$

is a Hardy–Steklov operator with two variable limits of integration. By Definition 1 in [48], for μ^-, μ^+, and u the fairway is the function $\sigma_u\colon I \to \mathbb{R}$ such that

$$\int_{\mu^-(x)}^{\sigma_u(x)} u^{p'} = \int_{\sigma_u(x)}^{\mu^+(x)} u^{p'} \quad \text{for all } x \in I.$$

Let $u = 1/v_1$. Then $u^{p'} = v_1^{-p'}$ and in view of (3.5), $\sigma_u(x) = x$. By analogy with the proof of Theorem 1 in [48] we can show that

$$\begin{aligned} \|\mathscr{H}\| &:= \|\mathscr{H}\|_{L^p((a,b))\to L^p((a_0,b_0))} \\ &\lesssim \sup_{t\in(a_0,b_0)} \biggl(\int_{\nu^-(t)}^{\nu^+(t)} \chi_{(a_0,b_0)} w^p\biggr)^{1/p} \biggl(\int_{\mu^-(t)}^{\mu^+(t)} v_1^{-p'}\biggr)^{1/p'}. \end{aligned} \tag{5.15}$$

For the norms of H_a and H_b we have the estimates

$$\|H_a\| := \|H_a\|_{L^p((a,\mu^+(a_0)])\to L^p((a,a_0])} \lesssim \sup_{t\in(a,a_0]} \biggl(\int_{\nu^-(t)}^{a_0} w^p\biggr)^{1/p} \biggl(\int_{a}^{\mu^+(t)} v_1^{-p'}\biggr)^{1/p'} \tag{5.16}$$

and

$$\|H_b\| := \|H_b\|_{L^p([\mu^-(b_0),b))\to L^p([b_0,b))} \lesssim \sup_{t\in[b_0,b)} \biggl(\int_{b_0}^{\nu^+(t)} w^p\biggr)^{1/p} \biggl(\int_{\mu^-(t)}^{b} v_1^{-p'}\biggr)^{1/p'} \tag{5.17}$$

(for instance, see [41], Lemmas 1.10 and 1.11, or [50], Lemmas 2.1 and 2.2).

Let $w = v_1^{1-p'}/V_1$. Then it is known that for each $\alpha > 0$

$$\int_{\nu^-(t)}^{\nu^+(t)} v_1^{-p'} V_1^{-\alpha-1} \lesssim [V_1(t)]^{-\alpha} \tag{5.18}$$

(as concerns the proof, see [42], (4.10), or [41], (3.2.18)). For the w under consideration this yields the inequality

$$\int_{\nu^-(t)}^{\nu^+(t)} \chi_E w^p = \int_{\nu^-(t)}^{\nu^+(t)} \chi_E \cdot v_1^{-p'} V_1^{-p} \lesssim [V_1(t)]^{1-p}$$

for any measurable $E \subset I$. Therefore,

$$\|H_a\| \lesssim 1, \quad \|\mathscr{H}\| \lesssim 1, \quad \text{and} \quad \|H_b\| \lesssim 1.$$

Hence, applying the inequality (5.14) for

$$h = h_{\mu^-} := \frac{|G_{\mu^-}|}{v_1\, V_1} \quad \text{and} \quad h = h_{\nu^+} := \frac{|G_{\nu^+}|}{v_1}$$

to (5.12) and (5.13), respectively, we obtain the required estimates

$$|\mathrm{II}| \lesssim \mathbb{G}_{I,\mu^-}(g)\|f\|_{W_p^1(I)} \quad \text{and} \quad |\mathrm{III}| \lesssim \mathfrak{G}_{I,\nu^+}(g)\|f\|_{W_p^1(I)}.$$

In combination with (5.2) this completes the proof of this part of the theorem.

To prove that

$$J_{\overset{\circ\circ}{W}{}_p^1(I)}(g) \gtrsim \mathbb{G}_{I,\mu^-}(g) + \mathfrak{G}_{I,\nu^+}(g),$$

consider sequences $\{a_N\}_1^\infty$ and $\{b_N\}_1^\infty$ such that $a_N \downarrow a$ and $b_N \uparrow b$, $a_1 < b_1$, and let $I_N := [a_N, b_N]$. For $x \in I$ let

$$\mathbb{F}_N(x) := \int_{\mu^-(x)}^x v_1^{-p'}(y)\left[\int_y^x \chi_{I_N}(t)v_1^{-p'}(t)[V_1(t)]^{-p'}[\operatorname{sgn} G_{\mu^-}(t)]\,|G_{\mu^-}(t)|^{p'-1}\,dt\right]dy$$

and

$$\mathfrak{F}_N(x) := \int_{\mu^-(x)}^x \chi_{I_N}(t)v_1^{-p'}(t)[\operatorname{sgn} G_{\nu^+}(t)]\,|G_{\nu^+}(t)|^{p'-1}\,dt.$$

Note that

$$\mathbb{F}_N(x) = \left[\int_{\mu^-(x)}^{b_N} v_1^{-p'}\right]\int_{\mu^-(x)}^x f_1(t)\,dt - \int_{\mu^-(x)}^x f_2(t)\,dt,$$

where

$$f_1(t) := \chi_{I_N}(t)v_1^{-p'}(t)[V_1(t)]^{-p'}[\operatorname{sgn} G_{\mu^-}(t)]\,|G_{\mu^-}(t)|^{p'-1}$$

and

$$f_2(t) := f_1(t)\left[\int_t^{b_N} v_1^{-p'}\right].$$

Here we have

$$\int_I |f_1| \lesssim \{\min_{x\in[a_N,b_N]} V_1(x)\}^{-1}\left[\int_{a_N}^{\nu^+(b_N)}|g|\right]^{p'-1}\left[\int_{a_N}^{b_N} v_1^{-p'}\right] < \infty$$

and

$$\int_I |f_2| < \infty.$$

Hence, $\mathbb{F}_N \in AC_{\mathrm{loc}}(I)$ (see the proof of Lemma 3.3). Furthermore,

$$\operatorname{supp}\mathbb{F}_N \subset [a_N, \nu^+(b_N)] \subset I.$$

We can show in a similar way that $\mathfrak{F}_N \in AC_{\mathrm{loc}}(I)$ and the support of $\mathfrak{F}_N$ is a compact subset of I.

Now by Fubini's theorem,

$$\int_I g(x)\mathbb{F}_N(x)\,dx = \int_{a_N}^{b_N} v_1^{-p'}(t)[V_1(t)]^{-p'}\,|G_{\mu^-}(t)|^{p'}\,dt =: \mathbb{G}_N(g) < \infty \tag{5.19}$$

and

$$\int_I g(x)\mathfrak{F}_N(x)\,dx = \int_{a_N}^{b_N} v_1^{-p'}(t)\,|G_{\nu^+}(t)|^{p'}\,dt =: \mathfrak{G}_N(g) < \infty. \tag{5.20}$$

For estimates of $\|\mathbb{F}_N v_0\|_{L^p(I)}$ and $\|\mathfrak{F}_N v_0\|_{L^p(I)}$, consider the inequality

$$\biggl(\int_I v_0^p(x)\biggl|\int_{\mu^-(x)}^{\mu^+(x)} h(t)[v_1(t)]^{-1}\,dt\biggr|^p\,dx\biggr)^{1/p} \leqslant C\biggl(\int_I |h(y)|^p\,dy\biggr)^{1/p}, \tag{5.21}$$

which is dual to (5.14) for $w = v_0$ and $u = 1/v_1$; the best constant $C \approx \|H_a\| + \|\mathscr{H}\| + \|H_b\|$ in this inequality has the form of the sum of the right-hand sides of (5.16), (5.15), and (5.17). Since

$$V_1^{\pm}(t) \leqslant V_1(\nu^{\pm}(t))$$

and for measurable subsets E of I we have

$$\int_{\nu^-(t)}^{\nu^+(t)} \chi_E\cdot v_0^p = \int_{\nu^-(t)}^{t} \chi_E\cdot v_0^p + \int_t^{\nu^+(t)} \chi_E\cdot v_0^p \leqslant V_0(\nu^-(t)) + V_0(\nu^+(t)),$$

it follows from (3.6) for $x = \nu^{\pm}(t)$ that $C \lesssim 1$. Applying (5.21) to the function

$$h(t) = h_{\mathbb{F}_N}(t) := \chi_{I_N}(t)v_1^{1-p'}(t)[V_1(t)]^{1-p'}|G_{\mu^-}(t)|^{p'-1},$$

we arrive at the estimate

$$\|\mathbb{F}_N v_0\|_{L^p(I)} \lesssim [\mathbb{G}_N(g)]^{1/p}. \tag{5.22}$$

Similarly, if in (5.21) we set

$$h(t) = h_{\mathfrak{F}_N}(t) := \chi_{I_N}(t)v_1^{1-p'}(t)|G_{\nu^+}(t)|^{p'-1},$$

then we obtain

$$\|\mathfrak{F}_N v_0\|_{L^p(I)} \lesssim [\mathfrak{G}_N(g)]^{1/p}. \tag{5.23}$$

For estimates of $\|\mathbb{F}'_N v_1\|_{L^p(I)}$ and $\|\mathfrak{F}'_N v_1\|_{L^p(I)}$, we find from the definitions of $\mathbb{F}_N$ and $\mathfrak{F}_N$ that

$$\mathbb{F}'_N(x) = \biggl[\int_a^{b_N} v_1^{-p'}\biggr] f_1(x) - f_2(x) = f_1(x)V_1^-(x) =: \mathbb{F}_{N,1}(x)$$

and

$$\mathfrak{F}'_N(x) = \chi_{I_N}(x)v_1^{-p'}(x)[\operatorname{sgn} G_{\nu^+}(x)]\,|G_{\nu^+}(x)|^{p'-1} =: \mathfrak{F}_{N,1}(x)$$

for $\mathcal{L}^1$-almost all $x \in (a, a_0)$. Moreover,

$$\mathbb{F}'_N(x) = \mathbb{F}_{N,1}(x) - \chi_{(a_0,b)}(x) v_1^{-p'}(\mu^-(x))(\mu^-)'(x) \int_{\mu^-(x)}^x f_1 =: \mathbb{F}_{N,1}(x) - \mathbb{F}_{N,2}(x)$$

and

$$\mathfrak{F}'_N(x) = \mathfrak{F}_{N,1}(x) - \chi_{(a_0,b)}(x)\mathfrak{F}_{N,1}(\mu^-(x))(\mu^-)'(x) =: \mathfrak{F}_{N,1}(x) + \mathfrak{F}_{N,2}(x)$$

for $\mathcal{L}^1$-almost all $x \in (a_0, b)$. We have

$$\|\mathbb{F}_{N,1} v_1\|^p_{L^p(I)} = 2^{-p}\mathbb{G}_N(g) \quad \text{and} \quad \|\mathfrak{F}_{N,1} v_1\|^p_{L^p(I)} = \mathfrak{G}_N(g). \tag{5.24}$$

For an estimate of $\|\mathbb{F}_{N,2} v_1\|_{L^p(I)}$ let

$$\begin{aligned} E_0 &:= \{t \in (a_0, b) \colon (\mu^-)'(t) > 0\}, \\ E_1 &:= \{t \in (a_0, b) \colon v_1(\mu^-(t)) < \infty\}, \\ E_2 &:= \{t \in (a_0, b) \colon v_1(t) > 0\}, \end{aligned}$$

and

$$E_3 := \{t \in (a_0, b) \colon (F \circ \mu^-)'(t) = v_1^{-p'}(\mu^-(t))(\mu^-)'(t)\}.$$

Then $\mathcal{L}^1((a_0, b) \setminus E_3) = 0$ by (5.3), we have

$$v_1(x) v_1^{-p'}(\mu^-(x))(\mu^-)'(x) = 0$$

for $x \in (a_0, b) \setminus \bigcap_{i=0}^2 E_i$, and in view of (5.4), (5.5), and (5.3) the estimate

$$\begin{aligned} v_1^p(x) &\leqslant 2^{p/p'}[(F \circ \mu^-)'(x)]^{-p/p'} = 2^{p/p'}[v_1^{-p'}(\mu^-(x))\,(\mu^-)'(x)]^{-p/p'} \\ &= 2^{p/p'}[(\mu^-)'(x)]^{1-p} v_1^p(\mu^-(x)) \end{aligned}$$

holds for $\mathcal{L}^1$-almost all $x \in \bigcap_{i=0}^3 E_i$. Hence, for $\mathcal{L}^1$-almost all $x \in (a_0, b)$ we have

$$[v_1(x) v_1^{-p'}(\mu^-(x))(\mu^-)'(x)]^p \leqslant 2^{p/p'} v_1^{-p'}(\mu^-(x))(\mu^-)'(x). \tag{5.25}$$

Using the above inequality and making the substitution $\mu^-(x) = y$, we arrive at the following estimate for the norm $\|\mathbb{F}_{N,2} v_1\|_{L^p(I)}$:

$$\|\mathbb{F}_{N,2} v_1\|_{L^p(I)} \lesssim \left(\int_I v_1^{-p'}(y) \left|\int_y^{\nu^+(y)} h_{\mathbb{F}_N}(x)[v_1(x)V_1(x)]^{-1}\,dx\right|^p dy\right)^{1/p}. \tag{5.26}$$

Let us turn to the inequality (5.14) (with p in place of p') for the operator H^* with weights $w = 1/(v_1 V_1)$ and $u = v_1^{1-p'}$ (so that $u^p = v_1^{-p'}$ and $\sigma_u(x) = x$) and also to the estimates (5.15)–(5.17). It follows from (5.18) that for measurable $E \subset I$,

$$\int_{\nu^-(t)}^{\nu^+(t)} \chi_E w^{p'} = \int_{\nu^-(t)}^{\nu^+(t)} \chi_E \cdot v_1^{-p'} V_1^{-p'} \lesssim [V_1(t)]^{1-p'}.$$

Therefore,

$$\|H_a\| \lesssim 1, \quad \|\mathscr{H}\| \lesssim 1, \quad \text{and} \quad \|H_b\| \lesssim 1,$$

that is, $\|H^*\| \lesssim 1$. Applying (5.14) with $h = h_{\mathbb{F}_N}$ to the right-hand side of (5.26), we obtain

$$\left(\int_I v_1^{-p'}(y) \left| \int_{\nu^-(y)}^{\nu^+(y)} h_{\mathbb{F}_N}(x)[v_1(x)V_1(x)]^{-1}\, dx \right|^p dy \right)^{1/p} \lesssim \|h_{\mathbb{F}_N}\|_{L^p(I)} = [\mathbb{G}_N(g)]^{1/p}.$$

Hence

$$\|\mathbb{F}_{N,2} v_1\|_{L^p(I)} \lesssim [\mathbb{G}_N(g)]^{1/p}.$$

Taking (5.24), the equality $\mathfrak{F}_{N,2}(x) = \mathfrak{F}_{N,1}(\mu^-(x))(\mu^-)'(x)$, and (5.25) into account, we arrive at the inequalities

$$\|\mathbb{F}_N' v_1\|_{L^p(I)} \lesssim [\mathbb{G}_N(g)]^{1/p} \quad \text{and} \quad \|\mathfrak{F}_N' v_1\|_{L^p(I)} \lesssim [\mathfrak{G}_N(g)]^{1/p}.$$

In combination with (5.22) and (5.23) this implies that

$$\|\mathbb{F}_N\|_{W_p^1(I)} \lesssim [\mathbb{G}_N(g)]^{1/p} \quad \text{and} \quad \|\mathfrak{F}_N\|_{W_p^1(I)} \lesssim [\mathfrak{G}_N(g)]^{1/p}.$$

Hence it follows from (5.19) and (5.20) that

$$J_{\overset{\circ\circ}{W}{}_p^1(I)}(g) \gtrsim [\mathbb{G}_N(g)]^{1/p'} + [\mathfrak{G}_N(g)]^{1/p'}.$$

Letting $N \to \infty$, we obtain the required inequality

$$J_{\overset{\circ\circ}{W}{}_p^1(I)}(g) \gtrsim \mathbb{G}_{I,\mu^-}(g) + \mathfrak{G}_{I,\nu^+}(g),$$

which now shows that $J_{\overset{\circ\circ}{W}{}_p^1(I)}(g) \approx \mathbb{G}_{I,\mu^-}(g) + \mathfrak{G}_{I,\nu^+}(g)$. Theorem 5.1 is proved. $\square$

Remark 5.1. Similarly, for $I := (a, b) \subseteq \mathbb{R}$, $1 < p < \infty$, and $g \in L^1_{\mathrm{loc}}(I)$, the following estimate holds under the assumptions that $v_0, v_1 \in \mathfrak{M}^+(I)$, $1/v_1 \in L^{p'}_{\mathrm{loc}}(I)$, $v_0, v_1 \in L^p_{\mathrm{loc}}(I)$, $\|v_0\|_{L^1(I)} > 0$, $\mu^-(b-0) = b$, and $\mu^+(a+0) = a$:

$$J_{\overset{\circ\circ}{W}{}_p^1(I)}(g) \approx \mathbb{G}_{I,\mu^+}(g) + \mathfrak{G}_{I,\nu^-}(g),$$

with equivalence constants depending only on p, where

$$\mathfrak{G}_{I,\nu^-}(g) := \left(\int_I v_1^{-p'}(t) \left| \int_{\nu^-(t)}^t g(x)\, dx \right|^{p'} dt \right)^{1/p'}$$

and

$$\mathbb{G}_{I,\mu^+}(g) := \left(\int_I \frac{v_1^{-p'}(t)}{V_1^{p'}(t)} \left| \int_t^{\mu^+(t)} v_1^{-p'}(y) \int_{\nu^-(y)}^t g(x)\, dx\, dy \right|^{p'} dt \right)^{1/p'}.$$

6. Estimates for the norms of elements of the spaces associated with $[\mathring{W}_p^1(I)]$ and $[W_p^1(I)]$

Corollary 6.1. *Let $I := (a,b) \subseteq \mathbb{R}$, $1 < p < \infty$, and $g \in L^1_{\mathrm{loc}}(I)$. Assume that $v_0, v_1 \in \mathfrak{M}^+(I)$, $1/v_1 \in L^{p'}_{\mathrm{loc}}(I)$, $v_0, v_1 \in L^p_{\mathrm{loc}}(I)$, and $\|v_0\|_{L^1(I)} > 0$, and let $\mu^-(b-0) = b$ and $\mu^+(a+0) = a$. Then*

$$\mathbf{J}_{\mathring{W}_p^1(I)}(g) = \mathbf{J}_{\overset{\circ\circ}{W}{}_p^1(I)}(g) = J_{\overset{\circ\circ}{W}{}_p^1(I)}(|g|) \approx \mathbb{G}_{I,\mu^-}(|g|) + \mathfrak{G}_{I,\nu^+}(|g|) \approx \mathfrak{G}_{I,\nu^+}(|g|).$$

Proof. It is clear that

$$\mathbf{J}_{\mathring{W}_p^1(I)}(g) \geqslant \mathbf{J}_{\overset{\circ\circ}{W}{}_p^1(I)}(g).$$

To prove the reverse inequality, fix an $f \in \mathring{W}_p^1(I)$. If $\mathbf{J}_{\overset{\circ\circ}{W}{}_p^1(I)}(g) = \infty$, then the reverse inequality holds, and therefore we assume that $\mathbf{J}_{\overset{\circ\circ}{W}{}_p^1(I)}(g) < \infty$. Since $\overset{\circ\circ}{W}{}_p^1(I)$ is dense in $\mathring{W}_p^1(I)$, there is a sequence $\{f_n\} \subset \overset{\circ\circ}{W}{}_p^1(I)$ such that $\|f - f_n\|_{W_p^1(I)} \to 0$ as $n \to \infty$. Because $f \in \mathring{W}_p^1(I)$, there exists an equivalent function $\bar{f} \in AC_{\mathrm{loc}}(I)$, and we have $\bar{f}(a+0) = 0$ for $h_a < \infty$ and $\bar{f}(b-0) = 0$ for $h_b < \infty$. We also have $I \subset \bigcup_{k\in\mathbb{Z}} \Delta(\xi_k)$ (see the proof of Theorem 5.1). By (3.7), $f_n(x) \to \bar{f}(x)$ on each $\Delta(\xi_k)$. Then $f_n(x) \to \bar{f}(x)$ on I, and by Fatou's lemma

$$\begin{aligned}\int_I |fg| = \int_I |\bar{f}g| &\leqslant \liminf_{n\to\infty} \int_I |f_n g| \leqslant \mathbf{J}_{\overset{\circ\circ}{W}{}_p^1(I)}(g) \liminf_{n\to\infty} \|f_n\|_{W_p^1(I)} \\ &= \mathbf{J}_{\overset{\circ\circ}{W}{}_p^1(I)}(g) \|f\|_{W_p^1(I)}.\end{aligned}$$

It follows, in particular, that if $\mathbf{J}_{\overset{\circ\circ}{W}{}_p^1(I)}(g) < \infty$, then $g \in \mathfrak{D}_{\mathring{W}_p^1(I)}$. Now we observe that for any function $f \in \overset{\circ\circ}{W}{}_p^1(I)$ its absolute value $|f|$ belongs to $\overset{\circ\circ}{W}{}_p^1(I)$ and $\|f\|_{W_p^1(I)} = \|\,|f|\,\|_{W_p^1(I)}$. Hence

$$\mathbf{J}_{\overset{\circ\circ}{W}{}_p^1(I)}(g) = J_{\overset{\circ\circ}{W}{}_p^1(I)}(|g|).$$

Now Theorem 5.1 and the estimate $\mathbb{G}_{I,\mu^-}(|g|) \lesssim \mathfrak{G}_{I,\nu^+}(|g|)$ yield the required result. □

Corollary 6.2. *Let $I := (a,b) \subseteq \mathbb{R}$, $1 < p < \infty$ and $g \in L^1_{\mathrm{loc}}(I)$. Assume that $v_0, v_1 \in \mathfrak{M}^+(I)$, $1/v_1 \in L^{p'}_{\mathrm{loc}}(I)$, $v_0, v_1 \in L^p_{\mathrm{loc}}(I)$, $\|v_0\|_{L^1(I)} > 0$, and let $\mu^-(b-0) = b$ and $\mu^+(a+0) = a$. Then*

$$J_{\mathring{W}_p^1(I)}(g) < \infty \quad \Longleftrightarrow \quad \mathbb{G}_{I,\mu^-}(|g|) + \mathfrak{G}_{I,\nu^+}(|g|) < \infty.$$

Furthermore,

$$J_{\mathring{W}_p^1(I)}(g) \approx \mathbb{G}_{I,\mu^-}(g) + \mathfrak{G}_{I,\nu^+}(g).$$

The proof follows the scheme of the proof of Theorem 3.7 in [41].

In order to estimate $\mathbf{J}_{W_p^1(I)}(g)$ and $J_{W_p^1(I)}(g)$ we need the following lemma.

Lemma 6.1. *Let $I := (a,b) \subseteq \mathbb{R}$, $1 < p < \infty$, $v_0, v_1 \in \mathfrak{M}^+(I)$, $1/v_1 \in L^{p'}_{\mathrm{loc}}(I)$, $v_0, v_1 \in L^p_{\mathrm{loc}}(I)$, $\|v_0\|_{L^1(I)} > 0$, and $a_0 < b_0$. Assume that $f \in W^1_p(I)$, and let $\bar{f} \in AC_{\mathrm{loc}}(I)$ be a representative of f. Let*

$$\omega(x) := \begin{cases} \left(\int_a^{a_0} v_1^{-p'}\right)^{-1} \int_a^x v_1^{-p'}, & x \in (a, a_0), \\ 1, & x \in [a_0, b_0] \cap I, \\ \left(\int_{b_0}^{b} v_1^{-p'}\right)^{-1} \int_x^b v_1^{-p'}, & x \in (b_0, b), \end{cases}$$

and let $f_0 := \omega \bar{f}$. Then $f_0 \in \mathring{W}^1_p(I)$, $\|f_0\|_{W^1_p(I)} \leqslant C\|f\|_{W^1_p(I)}$, and $\bar{f} = f_0$ on $[a_0, b_0] \cap I$.

Proof. Let $\widetilde{I}$ be some interval in I and let $\phi \in AC_{\mathrm{loc}}(I)$. For $t, x \in \widetilde{I}$ we have

$$|\phi(x)| \leqslant \left|\int_t^x |\phi'|\right| + |\phi(t)| \leqslant \left\|\frac{1}{v_1}\right\|_{L^{p'}(\widetilde{I})} \|v_1\phi'\|_{L^p(\widetilde{I})} + |\phi(t)|. \tag{6.1}$$

Multiplying (6.1) by $v_0(t)$, raising to the power p, and integrating over $\widetilde{I}$ with respect to t, we obtain

$$|\phi(x)|\, \|v_0\|_{L^p(\widetilde{I})} \leqslant \left\|\frac{1}{v_1}\right\|_{L^{p'}(\widetilde{I})} \|v_1\phi'\|_{L^p(\widetilde{I})} \|v_0\|_{L^p(\widetilde{I})} + \|v_0\phi\|_{L^p(\widetilde{I})}, \qquad x \in \widetilde{I}. \tag{6.2}$$

Assume that $0 < \|v_0\|_{L^p(\widetilde{I})} < \infty$ and let $\left\|\frac{1}{v_1}\right\|_{L^{p'}(\widetilde{I})} \|v_0\|_{L^p(\widetilde{I})} = 1$. It follows from (6.2) that

$$\sup_{x \in \widetilde{I}} |\phi(x)| \leqslant \left\|\frac{1}{v_1}\right\|_{L^{p'}(\widetilde{I})} \|\phi\|_{W^1_p(\widetilde{I})}$$

so that

$$\|v_1^{1-p'}\phi\|_{L^p(\widetilde{I})} \leqslant \left\|\frac{1}{v_1}\right\|^{p'-1}_{L^{p'}(\widetilde{I})} \sup_{x \in \widetilde{I}} |\phi(x)| \leqslant \left\|\frac{1}{v_1}\right\|^{p'}_{L^{p'}(\widetilde{I})} \|\phi\|_{W^1_p(\widetilde{I})}. \tag{6.3}$$

Since $\bar{f} \in AC_{\mathrm{loc}}(I)$, for any $x, t \in \widetilde{I}$ we have

$$|\bar{f}(t) - \bar{f}(x)| = \left|\int_x^t \bar{f}'\right| \leqslant \|v_1 f'\|_{L^p(\widetilde{I})} \left\|\frac{1}{v_1}\right\|_{L^{p'}(\widetilde{I})} \leqslant \|f\|_{W^1_p(I)} \left\|\frac{1}{v_1}\right\|_{L^{p'}(\widetilde{I})}. \tag{6.4}$$

Assume that $h_a < \infty$. Then for each $\alpha \in (a,b)$ we have

$$\left\|\frac{1}{v_1}\right\|_{L^{p'}((a,\alpha))} < \infty.$$

Hence, taking account of (6.4) with $\widetilde{I} := (a, \alpha)$, we see by the Cauchy criterion that the limit $\bar{f}(a+0)$ exists. Similarly, we prove that $\bar{f}(b-0)$ exists for $h_b < \infty$.

Hence, $f_0(a+0)=0$ for $h_a<\infty$ and $f_0(b-0)=0$ for $h_b<\infty$. Since $|f_0|\leqslant|\bar f|$, it follows that $\|v_0f_0\|_{L^p(I)}\leqslant\|v_0f\|_{L^p(I)}$. Moreover, $f_0\in AC_{\rm loc}(I)$, $f_0'=\omega'\bar f+\omega\bar f'$, and $\|v_1\omega\bar f'\|_{L^p(I)}\leqslant\|v_1\bar f'\|_{L^p(I)}$. Furthermore, for $\mathcal{L}^1$-almost all $x\in I$

$$\omega'(x)=\begin{cases}\left(\displaystyle\int_a^{a_0}v_1^{-p'}\right)^{-1}v_1^{-p'}(x), & x\in(a,a_0),\\ 0, & x\in(a_0,b_0),\\ -\left(\displaystyle\int_{b_0}^{b}v_1^{-p'}\right)^{-1}v_1^{-p'}(x), & x\in(b_0,b).\end{cases}$$

Let

$$K(x):=\left\|\frac{1}{v_1}\right\|_{L^{p'}(\Delta(x))}\|v_0\|_{L^p(\Delta(x))},\qquad x\in I.$$

Take $c\in(a_0,b_0)$. Then $a<c^-$ and $c^+<b$. If $h_a<\infty$, then $1/v_1\in L^{p'}((a,c^+))$ and $v_0\in L^p((a,c^+))$, so that K is continuous on (a,c). Bearing in mind that $K(x)=1$ for $x\in(a_0,b_0)$, we get that $K(a_0)=1$. In particular,

$$\left\|\frac{1}{v_1}\right\|_{L^{p'}(\Delta(a_0))},\ \|v_0\|_{L^p(\Delta(a_0))}\in(0,\infty).$$

Similarly, for $h_b<\infty$ we have $K(b_0)=1$ and

$$\left\|\frac{1}{v_1}\right\|_{L^{p'}(\Delta(b_0))},\ \|v_0\|_{L^p(\Delta(b_0))}\in(0,\infty).$$

Applying (6.3) to $\phi=\bar f$, $\widetilde I=\Delta(a_0)$, and $\widetilde I=\Delta(b_0)$, we find that

$$\|v_1^{1-p'}\bar f\|_{L^p(\Delta(a_0))}\leqslant\left\|\frac{1}{v_1}\right\|^{p'}_{L^{p'}(\Delta(a_0))}\|f\|_{W_p^1(\Delta(a_0))}$$

and

$$\|v_1^{1-p'}\bar f\|_{L^p(\Delta(b_0))}\leqslant\left\|\frac{1}{v_1}\right\|^{p'}_{L^{p'}(\Delta(b_0))}\|f\|_{W_p^1(\Delta(b_0))}.$$

It follows that

$$\|v_1\omega'\bar f\|_{L^p(I)}\leqslant\|f\|_{W_p^1(\Delta(a_0))}+\|f\|_{W_p^1(\Delta(b_0))}\leqslant2\|f\|_{W_p^1(I)}.$$

Thus, $\|f_0\|_{W_p^1(I)}\leqslant3\|f\|_{W_p^1(I)}$. Lemma 1.6 in [30] implies that $f_0\in\overset{\circ}{W}{}_p^1(I)$. $\square$

Let $g\in L^1_{\rm loc}(I)$. Setting

$$A_1(g):=\left(\int_a^{a_0}v_1^{-p'}(t)\left|\int_a^t g\right|^{p'}dt\right)^{1/p'},\qquad A_2(g):=\left(\int_{b_0}^{b}v_1^{-p'}(t)\left|\int_t^b g\right|^{p'}dt\right)^{1/p'},$$
$$\text{and}\quad A_3(g):=\left(\int_a^b v_0^p\right)^{-1/p}\left|\int_a^b g\right|$$

we observe that $\mathbf{J}_{W_p^1(I)}(g)=J_{W_p^1(I)}(|g|)$ and $\mathbf{J}_{\overset{\circ}{W}{}_p^1(I)}(g)=J_{\overset{\circ}{W}{}_p^1(I)}(|g|)$ (see Remark 2.1).

Theorem 6.1. *Let $I := (a,b) \subseteq \mathbb{R}$, $1 < p < \infty$, $v_0, v_1 \in \mathfrak{M}^+(I)$, $1/v_1 \in L^{p'}_{\mathrm{loc}}(I)$, $v_0, v_1 \in L^p_{\mathrm{loc}}(I)$, $\|v_0\|_{L^1(I)} > 0$, $a_0 < b_0$, and $g \in L^1_{\mathrm{loc}}(I)$. Then*

$$\mathbf{J}_{W^1_p(I)}(g) \approx A_1(|g|) + A_2(|g|) + \mathbf{J}_{\mathring{W}^1_p(I)}(g). \tag{6.5}$$

Proof. If $a = a_0$ and $b = b_0$, then $W^1_p(I) = \mathring{W}^1_p(I)$, and (6.5) holds. Assume that $a < a_0$ or $b_0 < b$.

Upper estimate. Suppose that $A_1(|g|)+A_2(|g|)+\mathbf{J}_{\mathring{W}^1_p(I)}(g) < \infty$. Since $A_1(|g|)+A_2(|g|)$ is finite and $g \in L^1_{\mathrm{loc}}(I)$, it follows that

$$\int_{(a,a_0)\cup(b_0,b)} |g| < \infty.$$

Fix some $f \in W^1_p(I)$. Let $f_0 \in \mathring{W}^1_p(I)$ be the function constructed from f as in Lemma 6.1. Then we have

$$\begin{aligned}
\int_a^b f|g| &= \int_a^{a_0} f|g| + \int_{a_0}^{b_0} f|g| + \int_{b_0}^b f|g| \\
&= \int_a^{a_0} \Big(-\int_x^{a_0} \bar{f}' + \bar{f}(a_0) \Big) |g(x)|\, dx + \int_{a_0}^{b_0} f_0|g| \\
&\quad + \int_{b_0}^b \Big(\int_{b_0}^x \bar{f}' + \bar{f}(b_0) \Big) |g(x)|\, dx \\
&= -\int_a^{a_0} \bar{f}'(t) \Big(\int_a^t |g| \Big)\, dt + \bar{f}(a_0) \int_a^{a_0} |g| + \int_{a_0}^{b_0} f_0|g| \\
&\quad + \int_{b_0}^b \bar{f}'(t) \Big(\int_t^b |g| \Big)\, dt + \bar{f}(b_0) \int_{b_0}^b |g|,
\end{aligned}$$

where the integrals can be interchanged because

$$\int_a^{a_0} |\bar{f}'(t)| \Big(\int_a^t |g| \Big)\, dt \leqslant A_1(|g|) \|v_1 \bar{f}'\|_{L^p((a,a_0))} < \infty$$

and

$$\int_{b_0}^b |\bar{f}'(t)| \Big(\int_t^b |g| \Big)\, dt \leqslant A_2(|g|) \|v_1 \bar{f}'\|_{L^p((b_0,b))} < \infty.$$

We have

$$\begin{aligned}
\bar{f}(a_0) \int_a^{a_0} |g| &= \int_a^{a_0} f_0' \cdot \int_a^{a_0} |g| = \int_a^{a_0} f_0'(t) \Big(\int_a^t |g| \Big)\, dt + \int_a^{a_0} f_0'(t) \Big(\int_t^{a_0} |g| \Big)\, dt \\
&= \int_a^{a_0} f_0'(t) \Big(\int_a^t |g| \Big)\, dt + \int_a^{a_0} |g(x)| \Big(\int_a^x f_0' \Big)\, dx \\
&= \int_a^{a_0} f_0'(t) \Big(\int_a^t |g| \Big)\, dt + \int_a^{a_0} f_0|g|.
\end{aligned}$$

Similarly,

$$\bar{f}(b_0)\int_{b_0}^{b}|g| = -\int_{b_0}^{b} f_0'(t)\left(\int_t^b |g|\right)dt + \int_{b_0}^{b} f_0|g|.$$

Thus,

$$\int_a^b f|g| = \int_a^{a_0}(f_0'(t)-\bar{f}'(t))\left(\int_a^t |g|\right)dt + \int_{b_0}^{b}(\bar{f}'(t)-f_0'(t))\left(\int_t^b |g|\right)dt + \int_a^b f_0|g|,$$

so that

$$\begin{aligned}\left|\int_a^b f|g|\right| &\leqslant A_1(|g|)\|v_1(f_0'-\bar{f}')\|_{L^p((a,a_0))} + A_2(|g|)\|v_1(\bar{f}'-f_0')\|_{L^p((b_0,b))}\\ &\quad + J_{\mathring{W}_p^1(I)}(|g|)\|f_0\|_{W_p^1(I)}\\ &\lesssim \left[A_1(|g|)+A_2(|g|)+J_{\mathring{W}_p^1(I)}(|g|)\right]\|f\|_{W_p^1(I)}.\end{aligned}$$

Lower estimate. Assume that $\mathbf{J}_{W_p^1(I)}(g) < \infty$. By Lemma 1.6 in [30], for $a < a_0$ the space $W_p^1(I)$ contains a function $\eta_a \in C^1(I)$ equal to one in a neighbourhood of a and to zero in a neighbourhood of b. Therefore, the finiteness of $\mathbf{J}_{W_p^1(I)}(g)$ implies that $g \in L^1((a,a_0))$. Similarly, $g \in L^1((b_0,b))$ for $b_0 < b$.

Using test functions $F_1, F_2 \in W_p^1(I)$ of the form

$$F_1(x) := \chi_{(a,a_0)}(x)\int_x^{a_0} v_1^{-p'}(t)\left(\int_a^t |g|\right)^{p'-1}dt$$

and

$$F_2(x) := \chi_{(b_0,b)}(x)\int_{b_0}^{x} v_1^{-p'}(t)\left(\int_t^b |g|\right)^{p'-1}dt,$$

we obtain

$$\int_a^b F_1|g| = \int_a^{a_0} v_1^{-p'}(t)\left(\int_a^t |g|\right)^{p'}dt = A_1(|g|)^{p'}$$

and

$$\int_a^b F_2|g| = \int_{b_0}^{b} v_1^{-p'}(t)\left(\int_t^b |g|\right)^{p'}dt = A_2(|g|)^{p'}.$$

Furthermore,

$$\|F_1'v_1\|_{L^p(I)}^p = \int_a^{a_0} v_1^p(x)\left[v_1^{-p'}(x)\left|\int_a^x |g|\right|^{p'-1}\right]^p dx = A_1(|g|)^{p'},$$

and by Hölder's inequality

$$\begin{aligned}\|F_1 v_0\|^p_{L^p(I)} &= \int_a^{a_0} v_0^p(x)\bigg(\int_x^{a_0} v_1^{-p'}(t)\bigg|\int_a^t |g|\bigg|^{p'-1} dt\bigg)^p dx\\ &\leqslant \int_a^{a_0} v_0^p(x)\bigg[\int_x^{a_0} v_1^{-p'}\bigg]^{p/p'}\bigg[\int_x^{a_0} v_1^{-p'}(t)\bigg|\int_a^t |g|\bigg|^{p'} dt\bigg] dx\\ &= \int_a^{a_0} v_1^{-p'}(t)\bigg|\int_a^t |g|\bigg|^{p'}\bigg(\int_a^t v_0^p(x)\bigg[\int_x^{a_0} v_1^{-p'}\bigg]^{p/p'} dx\bigg) dt\\ &\leqslant A_1(|g|)^{p'}\bigg(\|v_0\|_{L^p(\Delta(a_0))}\bigg\|\frac{1}{v_1}\bigg\|_{L^{p'}(\Delta(a_0))}\bigg)^p = A_1(|g|)^{p'}.\end{aligned}$$

Similarly, $\|F_2' v_1\|^p_{L^p(I)} = A_2(|g|)^{p'}$ and $\|F_2 v_0\|^p_{L^p(I)} \leqslant A_2(|g|)^{p'}$.

Combining all the estimates, we arrive at (6.5). □

Theorem 6.2. *Let $I := (a,b) \subseteq \mathbb{R}$, $1 < p < \infty$, $v_0, v_1 \in \mathfrak{M}^+(I)$, $1/v_1 \in L^{p'}_{\mathrm{loc}}(I)$, $v_0, v_1 \in L^p_{\mathrm{loc}}(I)$, $\|v_0\|_{L^1(I)} > 0$, $a_0 < b_0$, and $g \in L^1_{\mathrm{loc}}(I)$. Then*

$$J_{W^1_p(I)}(g) < \infty \quad \iff \quad \mathbf{J}_{W^1_p(I)}(g) < \infty. \tag{6.6}$$

In addition,

$$J_{W^1_p(I)}(g) \approx A_1(g) + A_2(g) + J_{\mathring{W}^1_p(I)}(g).$$

Proof. The assertion (6.6) was proved in [42], Theorem 2.5. Let $\mathbf{J}_{W^1_p(I)}(g) < \infty$. Then by Theorem 6.1, $A_1(|g|) + A_2(|g|) < \infty$ and $\int_{(a,a_0)\cup(b_0,b)} |g| < \infty$. The upper estimate is established by the same argument as in the proof of Theorem 6.1, with $|g|$ replaced by g. To prove the lower estimate we use test functions $F_{1,0}, F_{2,0} \in W^1_p(I)$ of the form

$$F_{1,0}(x) := \chi_{(a,a_0)}(x)\int_x^{a_0} v_1^{-p'}(t)\bigg|\int_a^t g\bigg|^{p'-1}\operatorname{sgn}\bigg[\int_a^t g\bigg]\, dt$$

and

$$F_{2,0}(x) := \chi_{(b_0,b)}(x)\int_{b_0}^x v_1^{-p'}(t)\bigg|\int_t^b g\bigg|^{p'-1}\operatorname{sgn}\bigg[\int_t^b g\bigg]\, dt. \qquad \square$$

Theorem 6.3. *Let $I := (a,b) \subseteq \mathbb{R}$, $1 < p < \infty$, $v_0, v_1 \in \mathfrak{M}^+(I)$, $1/v_1 \in L^{p'}_{\mathrm{loc}}(I)$, $v_0, v_1 \in L^p_{\mathrm{loc}}(I)$, $\|v_0\|_{L^1(I)} > 0$, $a_0 = b_0$, and $g \in L^1_{\mathrm{loc}}(I)$. Then*

$$\mathbf{J}_{W^1_p(I)}(g) \approx A_1(|g|) + A_2(|g|) + A_3(|g|).$$

Proof. Upper estimate. Assume that $A_1(|g|) + A_2(|g|) + A_3(|g|) < \infty$. Fix an arbitrary $f \in W^1_p(I)$. Then we have

$$\int_a^b f|g| = -\int_a^{a_0} \bar{f}'(t)\bigg(\int_a^t |g|\bigg) dt + \bar{f}(a_0)\int_a^b |g| + \int_{b_0}^b \bar{f}'(t)\bigg(\int_t^b |g|\bigg) dt.$$

Note that since $a = \mu^-(a_0)$, $b = \mu^+(b_0)$, and $a_0 = b_0$, it follows that $\Delta(a_0) = I$, and therefore $\|1/v_1\|_{L^{p'}(I)}\|v_0\|_{L^p(I)} \leqslant 1$. Using (6.2) for $\widetilde{I} = I$ and $x = a_0$, we get that

$$\left|\bar f(a_0)\int_a^b |g|\right| \leqslant A_3(|g|)|\bar f(a_0)|\,\|v_0\|_{L^p(I)} \leqslant A_3(|g|)\|f\|_{W_p^1(I)}.$$

Hence

$$\left|\int_a^b f|g|\right| \leqslant [A_1(|g|) + A_2(|g|) + A_3(|g|)]\|f\|_{W_p^1(I)}.$$

For the lower estimate use the test functions F_1, F_2, and $F_3 := \chi_{(a,b)}$. □

Theorem 6.4. *Let $I := (a,b) \subseteq \mathbb{R}$, $1 < p < \infty$, $v_0, v_1 \in \mathfrak{M}^+(I)$, $1/v_1 \in L^{p'}_{\mathrm{loc}}(I)$, $v_0, v_1 \in L^p_{\mathrm{loc}}(I)$, $\|v_0\|_{L^1(I)} > 0$, $a_0 = b_0$, and $g \in L^1_{\mathrm{loc}}(I)$. Then*

$$J_{W_p^1(I)}(g) < \infty \quad\Longleftrightarrow\quad \mathbf{J}_{W_p^1(I)}(g) < \infty. \tag{6.7}$$

Moreover,

$$J_{W_p^1(I)}(g) \approx A_1(g) + A_2(g) + A_3(g).$$

Proof. The assertion (6.7) was proved in [42], Theorem 2.5. Since $\mathbf{J}_{W_p^1(I)}(g) < \infty$, it follows that $A_1(|g|) + A_2(|g|) + A_3(|g|) < \infty$. The upper estimate is proved by the same argument as in the proof of Theorem 6.3, with $|g|$ replaced by g. For the lower estimate use the test functions $F_{1,0}$, $F_{2,0}$, and F_3. □

7. Examples

Example 7.1. Let $I = (0,\infty)$, $1 < p < \infty$, $v_0(t) = t^{-1}$, $v_1(t) = 1$, and

$$\|f\|_{W_p^1(I)} := \left(\int_0^\infty \frac{|f(x)|^p\,dx}{x^p}\right)^{1/p} + \left(\int_0^\infty |f'(x)|^p\,dx\right)^{1/p}.$$

Then $a_0 = 0$, $b_0 = \infty$, and $\mu^-(t) = (1-\alpha)t$, where $\alpha \in (0,1)$; furthermore, the equivalence (3.4) ensures that $W_p^1(I) = \mathring{W}_p^1(I)$. Thus, by Theorem 5.1

$$J_{\overset{\circ\circ}{W}{}_p^1(I)}(g) \approx \left(\int_0^\infty t^{-p'}\left|\int_{(1-\alpha)t}^t\left(\int_t^{y/(1-\alpha)} g\right)dy\right|^{p'} dt\right)^{1/p'} + \left(\int_0^\infty \left|\int_t^{t/(1-\alpha)} g\right|^{p'} dt\right)^{1/p'}.$$

It follows from Corollary 6.1 that

$$\mathbf{J}_{W_p^1(I)}(g) = \mathbf{J}_{\mathring{W}_p^1(I)}(g) = \mathbf{J}_{\overset{\circ\circ}{W}{}_p^1(I)}(g) \approx \left(\int_0^\infty\left(\int_t^{t/(1-\alpha)} |g|\right)^{p'} dt\right)^{1/p'}.$$

However, using the method from [42], we can show that

$$\mathfrak{D}_{\overset{\circ\circ}{W}{}_p^1(I)} = L^1_{\mathrm{loc}}(I) \quad\text{and}\quad \mathfrak{D}_{\mathring{W}_p^1(I)} = \mathfrak{L}^{p'}(I),$$

where

$$\mathfrak{L}^{p'}(I) := \left\{ g \in L^1_{\mathrm{loc}}(I) : \|g\|_{\mathfrak{L}^{p'}(I)} := \left(\int_0^\infty \left(\int_t^\infty |g| \right)^{p'} dt \right)^{1/p'} < \infty \right\},$$

and

$$\mathbf{J}_{W_p^1(I)}(g) = \mathbf{J}_{\mathring{W}_p^1(I)}(g) = \mathbf{J}_{\overset{\circ\circ}{W}{}_p^1(I)}(g) \approx \|g\|_{\mathfrak{L}^{p'}(I)}.$$

We can also show that

$$J_{\mathring{W}_p^1(I)}(g) \approx \left(\int_0^\infty \left| \int_t^\infty g \right|^{p'} dt \right)^{1/p'}, \qquad g \in \mathfrak{L}^{p'}(I).$$

Example 7.2. Let $I = (0, \infty)$, $v_0(t) = 1$, $v_1(t) = 1$, and

$$\|f\|_{W_p^1(I)} := \left(\int_0^\infty |f(x)|^p \, dx \right)^{1/p} + \left(\int_0^\infty |f'(x)|^p \, dx \right)^{1/p}.$$

Then $a_0 = 1/2$, $b_0 = \infty$, and

$$\mu^-(t) = \begin{cases} 0, & 0 < t < \dfrac{1}{2}, \\ t - \dfrac{1}{2}, & \dfrac{1}{2} \leqslant t, \end{cases} \qquad \nu^+(t) = t + \frac{1}{2}.$$

Hence

$$\begin{aligned} J_{\overset{\circ\circ}{W}{}_p^1(I)}(g) \approx \left(\int_0^\infty \left| \int_t^{t+1/2} g \right|^{p'} dt \right)^{1/p'} + \Biggl(\int_0^{1/2} t^{-p'} \left| \int_0^t \left(\int_t^{y+1/2} g \right) dy \right|^{p'} dt \\ + \int_{1/2}^\infty \left| \int_{t-1/2}^t \left(\int_t^{y+1/2} g \right) dy \right|^{p'} dt \Biggr)^{1/p'} \end{aligned}$$

by Theorem 5.1.

Example 7.3. Let $I = (0, 1)$, $v_0(t) = 1$, $v_1(t) = 1$, and

$$\|f\|_{W_p^1(I)} := \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p} + \left(\int_0^1 |f'(x)|^p \, dx \right)^{1/p}.$$

Then $a_0 = b_0 = 1/2$ and

$$J_{W_p^1(I)}(g) \approx \left(\int_0^{1/2} \left| \int_0^t g \right|^{p'} dt \right)^{1/p'} + \left(\int_{1/2}^1 \left| \int_t^1 g \right|^{p'} dt \right)^{1/p'} + \left| \int_0^1 g \right|$$

by Theorem 6.4.

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