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On a local holomorphic version of the fundamental theorem of projective geometry

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The fundamental theorem of projective geometry is the classical result that a map $\mathbb{R}P^n \to \mathbb{R}P^n$ ($\mathbb{C}P^n \to \mathbb{C}P^n$) that takes (complex) straight lines to straight lines is a projective transformation. In fact, the map here is only assumed to be injective. Starting from the 1990s, local versions of this result appeared, where a (biholomorphic, homeomorphic, or just injective) map is originally defined in a subdomain of a projective or a linear space (see, respectively, [1], [2], [3], and some other papers).

We consider here a local holomorphic version of the fundamental theorem of projective geometry in a 'relaxed' form, where we assume *a priori* that the map takes to straight lines open subsets not of all straight lines but only of those lines whose directions belong to a certain set. This setting has some interesting applications to multidimensional complex geometry, which we will treat elsewhere.

So let U be a bounded convex domain in $\mathbb{C}^n$, $n \geqslant 2$, and let $f\colon U \to \mathbb{C}^n$ be a biholomorphic map. Let $\gamma \subset \mathbb{C}P^{n-1}$ be the set of directions $p \in \mathbb{C}P^{n-1}$ such that for each (complex) affine straight line $l \subset \mathbb{C}^n$ that has direction p the set $f(l \cap U)$ lies on some straight line.

Lemma 1. *Let $p_0 \in \gamma$ be a limit point of γ. Then for each straight line $l_0 \subset \mathbb{C}^n$ in direction p_0 that intersects U the restriction $f\big|_{l_0}$ is a rational (vector-valued) function.*

Proof. Without loss of generality we can assume that $f(U)$ is also a convex domain in $\mathbb{C}^n$. Let λ_0 be the projective line in $\mathbb{C}P^n \supset \mathbb{C}^n$ that contains $f(l_0 \cap U)$, and let $p_1, p_2, \dots$ be a sequence of different points in γ that tends to p_0. For each $j \geqslant 1$ consider the family L_j of straight lines in $\mathbb{C}^n$ parallel to the direction p_j that have a common point with l_0 and intersect U. The straight lines in $\mathbb{C}P^n$ that contain the f-images of lines in L_j form a connected holomorphic family Λ_j; an open subset of this family consists of straight lines whose 'pre-images' intersect l_0 within U, so that these lines themselves clearly intersect λ_0 within $f(U)$. Hence all the lines in Λ_j intersect λ_0 (in $\mathbb{C}P^n$). The correspondence between the points in which the 'pre-images' of these lines intersect l_0 and the points in which the lines themselves intersect λ_0 produces an analytic (meromorphic) extension of f from the subdomain $l_0 \cap U$ of l_0 to some larger convex subdomain V_j of this line. Since the V_j are simply connected, such extensions agree pairwise and in combination give an extension of f to the whole of the straight line l_0 in $\mathbb{C}^n$.

Note that then the image of each point in $V_j \setminus V_k$, where $j > k \gg 1$ (which lies on some straight line λ_j in Λ_j, $\lambda_j \supset f(l_j \cap U)$) lies outside $f(U)$ (because the part of λ_j that lies in $f(U)$ is the f-image of $l_j \cap U$, while for large values of j and k the straight line l_j intersects l_0 at a point quite distant from U). Hence points in

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$\lambda_0 \cap f(U)$ are not among the limit values at infinity of the meromorphic map of l_0 that we have constructed. Thus, this map cannot have an essential singularity at infinity: it has a pole or a finite limit value there. That is, the restriction of f to l_0 is in fact a rational vector function. $\square$

By the theorem stating that holomorphic maps that are rational in each variable are rational (this goes back to Weierstrass and Hurwitz; for instance, see [4], § 5, Theorem 5), it follows from Lemma 1 that if γ contains n limit points not lying on one hyperplane, then f is a rational map.

On the other hand, since f is holomorphic, the set γ must in fact have a complex structure.

Lemma 2. *The set γ is a projective variety in $\mathbb{C}P^{n-1}$.*

Proof. In view of Chow's theorem it is sufficient to show that γ is an analytic subset of $\mathbb{C}P^{n-1}$. Fix points $p_0 \in \mathbb{C}P^{n-1}$ and $X \in U$. Let l be the line through X in the direction p_0. Taking a point $y_0 \neq X$ on $l \cap U$, we consider a hyperplane through this point that is transverse to l. Let Y be a small neighbourhood of y_0 on this hyperplane; then the directions of straight lines which are close to p_0 can be parametrized by points $y \in Y$. We denote the straight lines through X in these directions by l_y.

It is easy to see that f takes the set $l_y \cap U$ to a straight line if and only if for each scalar $a \in C$ close to 1 the vectors $f(y) - f(X)$ and $f(ay) - f(X)$ are linearly dependent. For fixed X and a such a linear dependence can be expressed as the vanishing of the $n-1$ determinants of 2×2 matrices formed from the coordinates of these vectors, that is, by holomorphic equations for y (or for the direction $p \in \mathbb{C}P^{n-1}$). It is clear that f takes each line in direction p to a straight line if and only if similar equations are satisfied for each scalar a in a small neighbourhood of 1 and each point in a small neighbourhood of X. That is, in a neighbourhood of p_0 the set γ is defined by a family of holomorphic equations (which has the cardinality of the continuum). This means that γ is an analytic set. $\square$

Note that we do not say that all distinct irreducible components of γ have the same dimension.

In view of Lemma 2 we can state the above consequence of Lemma 1 as follows.

Theorem 1. *Suppose that γ does not fully lie on a hyperplane with the exception of a finite set of points. Then f is a rational map.*

It is easy to show that in dimension $n = 3$ we can refine Lemma 2 and Theorem 1 as follows.

Corollary 1. *Let U be a domain in $\mathbb{C}^3$ and let $f\colon U \to \mathbb{C}^3$ be a biholomorphic map. Then one of the following cases holds*:

1) *the set γ is empty, finite, or a union of a line and a finite set of points*;

2) *γ is a union of a conic distinct from a line and a finite set of points, and f is a rational map*;

3) *$\gamma = \mathbb{C}P^2$, and f is a projective map.*

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