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An explicit local combinatorial formula for the first Pontryagin class

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The rational Pontryagin classes $p_i(M) \in H^{4i}(M;\mathbb{Q})$ are well defined for PL-manifolds M. A classical problem in algebraic topology is to compute $p_i(M)$ explicitly from the combinatorial data of a manifold triangulation.

In 2004 the first author [1] described all local combinatorial formulae for the first rational Pontryagin class of a combinatorial manifold M, that is, formulae of the form

$$f_\sharp(M) = \sum_{\sigma\in M,\ \operatorname{codim}\sigma=4} f(\operatorname{link}\sigma)\sigma, \tag{1}$$

where f denotes a function on the isomorphism classes of oriented simplicial 3-spheres which does not depend on the underlying manifold M, and $f_\sharp(M)$ is a cycle whose homology class is Poincaré dual to $p_1(M)$. Here we present a natural choice for the explicit function f, which was not done in [1].

A *bistellar move* is a transformation of combinatorial manifolds that replaces a full subcomplex of full dimension of the form $\sigma * \partial\tau$ with simplexes σ and τ by the subcomplex $\partial\sigma * \tau$. (The following conventions are used: $\partial\,\mathrm{pt} = \varnothing$, $\sigma * \varnothing = \sigma$.)

Define the *combinatorial curvature* of a simplicial 2-sphere L at a vertex v to be the number $W_L(v) = 1 - \frac{d_v}{6}$, where d_v is the degree of the vertex v. The sum of the combinatorial curvatures over all vertices is equal to 2, the Euler characteristic of the sphere.

To each bistellar move $\beta\colon L_1 \rightsquigarrow L_2$ of oriented simplicial 2-spheres we assign an oriented simplicial 3-sphere $L_\beta = \operatorname{cone} L_1 \cup \operatorname{cone} L_2 \cup (\sigma * \tau)$, where the cones are glued over the common subcomplex $L_1 \setminus \operatorname{Int}(\sigma * \partial\tau) = L_2 \setminus \operatorname{Int}(\partial\sigma * \tau)$. Denote the apex of $\operatorname{cone} L_i$ by a_i.

Further, we assign to β a set $\mathscr{H}$ of simplicial chains $\eta \in C_1(L_\beta;\mathbb{Q})$ with attributed weights $W(\eta)$ using the following rules.

1) For any vertex w that is present in both spheres L_1 and L_2 we include the chain $\eta = [a_1, w] + [w, a_2]$ in the set $\mathscr{H}$ and attribute to it the weight

$$W(\eta) = \begin{cases} W_{L_1}(w) = W_{L_2}(w) & \text{if } w \notin \sigma * \tau, \\ W_{L_2}(w)\ (\neq W_{L_1}(w)) & \text{if } w \in \sigma, \\ W_{L_1}(w)\ (\neq W_{L_2}(w)) & \text{if } w \in \tau. \end{cases}$$

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2) For any pair of vertices $w_1 \in \sigma$, $w_2 \in \tau$ we include the chain $\eta = [a_1, w_1] + [w_1, w_2] + [w_2, a_2]$ in the set $\mathscr{H}$ and attribute to it the weight $W(\eta) = -1/12$ if σ and τ are edges and $W(\eta) = 1/6$ if σ or τ is a vertex.

Proposition. *The chain $\xi(\beta) \in C_1(L_\beta; \mathbb{Q}) \otimes C_1(L_\beta; \mathbb{Q})$ defined by*

$$\xi(\beta) = \sum_{\eta_1, \eta_2 \in \mathscr{H}} W(\eta_1) W(\eta_2)\, \eta_1 \otimes \eta_2 - 2 \sum_{\eta \in \mathscr{H}} W(\eta)\, \eta \otimes \eta$$

is a cycle with respect to the differential $\partial \otimes 1 - 1 \otimes \partial$.

Let M be an oriented simplicial 3-sphere and let $Z_1(M; \mathbb{Q}) \subset C_1(M; \mathbb{Q})$ denote the kernel of the boundary operator. For cycles $\varphi, \psi \in Z_1(M; \mathbb{Q})$ with non-intersecting supports the classical linking number $\operatorname{lk}(\varphi, \psi) \in \mathbb{Q}$ is well defined. For arbitrary $\varphi, \psi \in Z_1(M; \mathbb{Q})$ we define their *generalized linking number* to be $\widetilde{\operatorname{lk}}(\varphi, \psi) = \operatorname{lk}(\varphi, \operatorname{Shift}(\psi))$, where $\operatorname{Shift}\colon C_1(M; \mathbb{Q}) \to C_1(M^*; \mathbb{Q})$ is an operator shifting a chain to the dual cellular complex M^*. The operator Shift is described explicitly and maps any edge to a chain whose support lies in the union of cells dual to the vertices of this edge. The map $\widetilde{\operatorname{lk}}$ is extended linearly to $Z_1(M; \mathbb{Q}) \otimes Z_1(M; \mathbb{Q})$.

According to a theorem of Pachner [2], any simplicial 3-sphere can be obtained from the simplex boundary $\partial\Delta^4$ using a sequence of bistellar moves.

Theorem. *Define a function f on the set of oriented simplicial 3-spheres inductively via the formulae $f(\partial\Delta^4) = 0$ and*

$$f(L_2) - f(L_1) = \sum \widetilde{\operatorname{lk}}(\xi(\beta_v))$$

if L_1 and L_2 are connected by the bistellar move β. Here the sum is taken over all vertices $v \in \sigma * \tau$ present in both spheres L_1 and L_2, and β_v denotes the bistellar move of 2-spheres induced by the move β in the link of the vertex v. Then f is well defined, and the corresponding cycle $f_\sharp(M)$ given by (1) represents the class $p_1(M)$.

Therefore, the computation of the first rational Pontryagin class of an n-dimenional combinatorial manifold M reduces to the following steps.

1) For any $(n-4)$-dimensional simplex σ find a sequence of bistellar moves $\partial\Delta^4 = L_0 \overset{\beta_1}{\rightsquigarrow} L_1 \overset{\beta_2}{\rightsquigarrow} \cdots \overset{\beta_k}{\rightsquigarrow} L_k = \operatorname{link}\sigma$.

2) For each $\beta_i\colon \sigma_i * \partial\tau_i \mapsto \partial\sigma_i * \tau_i$ take all the vertices $v \in \sigma_i * \tau_i$ such that bistellar moves $\beta_{i,v}$ are induced in their links. Next, for each of those moves compute the value $\widetilde{\operatorname{lk}}(\xi(\beta_{i,v}))$.

3) The cycle representing the homology class Poincaré dual to $p_1(M)$ has the form

$$f_\sharp(M) = \sum_{\sigma \in M,\ \operatorname{codim}\sigma = 4} \sum_{i=1}^{k} \sum_{v} \widetilde{\operatorname{lk}}(\xi(\beta_{i,v}))\sigma.$$

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