Russian Math. Surveys **74**:6 1117–1119 *Uspekhi Mat. Nauk* **74**:6 159–160
DOI: https://doi.org/10.1070/RM9918

BRIEF COMMUNICATIONS

Variational principle for multidimensional conservation laws and pressureless media

A. I. Aptekarev and Yu. G. Rykov

Multidimensional systems of n *conservation laws* often have a hyperbolic character, and in the process of evolution of their solutions this can result in the appearance of singularities (shock waves, fusion of masses). This is a motivation for seeking universal ways of defining *generalized solutions*. In [1] (for $n = 1, 2$) the author proposed a family of functionals which are connected with the original system of conservation laws so that the set of their extremal points determines a generalized solution of the system. In the degenerate case of the *system of equations of pressureless gas dynamics* this method can be implemented in the form of a particular *variational representation* for solutions, which can be used to describe the process of concentration of the substance; see [2] (for $n = 2, 3$). This process corresponds to conservation laws for the mass and the momentum, in contrast with models not taking account of the conservation of momentum (see [3]), so it is of interest for applications to astrophysics and, in particular, for investigations of the distribution of the hypothetical *dark matter* in the Universe. In this note we present this variational principle in its multidimensional version.

Consider a quasi-linear system of conservation laws

$$\frac{\partial}{\partial t}\mathbf{U}(t,\mathbf{x}) + \sum_{j=1}^{m} \frac{\partial}{\partial x_j}(\mathbf{F}_j(\mathbf{U}(t,\mathbf{x}))) = 0, \tag{1}$$

where $\mathbf{F}_j := (f_{1j}, \dots, f_{nj})$ are sufficiently smooth (at least continuously differentiable) functions of the variables $(u_1, \dots, u_n)$, $(t, \mathbf{x}) := (t, x_1, \dots, x_m) \in \mathbb{R}_+ \times \mathbb{R}^m$, and the *vector function* $\mathbf{U}(t, \mathbf{x}) := (u_1, \dots, u_n)\colon \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^n$ belongs to some Banach space B (for instance, to BV, L_1, $L_{1,\mathrm{loc}}$, L^∞, and so on).

In the $(t, \mathbf{x})$-space we look at the set Σ of continuously differentiable hypersurfaces S (of codimension 1) parametrized as $t = \tau$, $\mathbf{x} = \boldsymbol{\chi}(\tau, \mathbf{s})$, where $\mathbf{s} = (s_1, \dots, s_{m-1})$ and $\boldsymbol{\chi} = (\chi_1, \dots, \chi_m)$. We set $dx := dx_1 \cdots dx_m$ (the usual volume element), $d\mathbf{x} := dx_1 \wedge \cdots \wedge dx_m$ (the oriented volume element), and $\bar{d}\mathbf{x}_j := dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_m$ (an oriented projection of the element of 'hypersurface area' leaving out the factor dx_j). Now consider a functional $\mathbf{J}\colon \Sigma \to \mathbb{R}^n$ given by

$$\mathbf{J} := \int \cdots \int \biggl(\mathbf{U}\, dx + \sum_{j=1}^{m} (-1)^j \mathbf{F}_j(\mathbf{U})\, dt \wedge \bar{d}\mathbf{x}_j\biggr), \tag{2}$$

where the integral is taken over S.

This research was supported by the Russian Science Foundation under grant no. 19-71-30004.
AMS 2010 *Mathematics Subject Classification*. Primary 35L45, 35L60, 35L65.

Definition 1. Let $\Gamma_{\mathbf{U}} \subset \Sigma$ be the set of hypersurfaces (which depends on the function $\mathbf{U}$ under consideration) such that the integral in (2) is well defined, and let $\Delta_{\mathbf{U}} \subset \Gamma_{\mathbf{U}}$ be the set of hypersurfaces such that the variation $\delta\mathbf{J}$ of the functional (2) is well defined. A function $\mathbf{U}(t,\mathbf{x}) \in B$ is called a generalized solution of the system (1) if each hypersurface $S \in \Delta_{\mathbf{U}}$ is critical for $\mathbf{J}$, that is, $\delta\mathbf{J} = 0$.

Interpreting Definition 1 for a degenerate system of conservation laws, for the system of equations of pressureless gas dynamics

$$\begin{cases} \rho_t + \operatorname{div}_x(\rho\mathbf{U}) = 0, \\ (\rho\mathbf{U})_t + \operatorname{div}_x(\rho\mathbf{U}\otimes\mathbf{U}) = 0, \end{cases} \qquad \rho(0,\mathbf{x}) = \rho_0(\mathbf{x}), \quad \mathbf{U}(0,\mathbf{x}) = \mathbf{U}_0(\mathbf{x}), \tag{3}$$

where $\rho(t,\mathbf{x})$ is the density of the medium, $\mathbf{U}(t,\mathbf{x})$ is the velocity vector, and $\mathbf{U}\otimes\mathbf{U}$ denotes the tensor product, we arrive at the following result.

Theorem 1. *For the system* (3) *the functional* $\mathbf{J}$ (*see* (2)) *can be expressed by* $\mathbf{J} := \mathbf{J}(A) := \displaystyle\int\cdots\int\left[\mathbf{U}_0(\mathbf{a}) - \frac{\mathbf{x}-\mathbf{a}}{t}\right]\rho_0(\mathbf{a})\,d\mathbf{a}$, *where the integral is taken over the compact set* $A \Subset \mathbb{R}^m$ *and the variation of* $\mathbf{J}$ *is understood to be its variation with respect to* A.

From Theorem 1 we obtain the following test for the appearance of a singularity in (3). Fix a point $(t,\mathbf{x})$ and some parametrization of the Lagrangian variables $\mathbf{a}$, that is, fix a function $\mathbf{a} = \mathbf{a}(\tau,\mathbf{s})$ (recall that $\mathbf{s} = (s_1,\dots,s_{m-1})$).

Theorem 2. *If there exist a parametrization* $\bar{\mathbf{a}}(\tau,\mathbf{s})$ *and a value* $\mathbf{s} = \bar{\mathbf{s}}$ *such that the function* $\mathbf{\Phi}(\tau,\bar{\mathbf{s}}) := \displaystyle\int_0^\tau\left[\mathbf{U}_0(\bar{\mathbf{a}}(\tau,\bar{\mathbf{s}})) - \frac{\mathbf{x}-\bar{\mathbf{a}}(\tau,\bar{\mathbf{s}})}{t}\right]\rho_0(\bar{\mathbf{a}}(\tau,\bar{\mathbf{s}}))\frac{\partial\bar{\mathbf{a}}(\tau,\bar{\mathbf{s}})}{\partial(\tau,\bar{\mathbf{s}})}\,d\tau$ *has more than one global minimum with respect to* τ, *then generalized solutions of* (3) *acquire a singularity in the form of a concentration of masses* (*on a manifold of one dimension or another*).

Theorem 2 generalizes to several variables the dynamics of fusion described in [4] for the one-dimensional case.

For another hyperbolic system (the so-called *continuous analogue of the Toda lattice*) a variational principle was proposed in [5] for a certain functional $\mathbf{J}(A)$ defined on compact sets $A \Subset \mathbb{R}^m$ in the case $m = 1$; the case $m > 1$ was considered in [6]–[8].

Bibliography

[1] Ю. Г. Рыков, Тр. МИАН, **301**, 2018, с. 225–240; English transl., Yu. G. Rykov, *Proc. Steklov Inst. Math.* **301** (2018), 213–227.

[2] А. И. Аптекарев, Ю. Г. Рыков, *Докл. РАН* **484**:6 (2019), 655–658; English transl., A. I. Aptekarev and Yu. G. Rykov, *Dokl. Math.* **99**:1 (2019), 79–82.

[3] С. Н. Гурбатов, А. И. Саичев, С. Ф. Шандарин, *УФН* **182**:3 (2012), 233–261; English transl., S. N. Gurbatov, A. I. Saichev, and S. F. Shandarin, *Phys. Usp.* **55**:3 (2012), 223–249.

[4] И. Вейнан, Ю. Г. Рыков, Я. Г. Синай, *УМН* **50**:1(301) (1995), 193–194; English transl., I. Veinan (W. E), Yu. G. Rykov, and Ya. G. Sinai, *Russian Math. Surveys* **50**:1 (1995), 220–222.

[5] A. I. Aptekarev and Yu. G. Rykov, *Russ. J. Math. Phys.* **13**:1 (2006), 4–12.

[6] A. I. Aptekarev, *Modern trends in constructive function theory*, Contemp. Math., vol. 661, Amer. Math. Soc., Providence, RI 2016, pp. 167–186.

[7] А. И. Аптекарев, Г. Лопес Лагомасино, А. Мартинес-Финкельштейн, *УМН* **72**:3(435) (2017), 3–64; English transl., A. I. Aptekarev, G. López Lagomasino, and A. Martínez-Finkelshtein, *Russian Math. Surveys* **72**:3 (2017), 389–449.

[8] A. I. Aptekarev, M. A. Lapik, and V. G. Lysov, *Anal. Math. Phys.* **9**:3 (2019), 919–935.

Alexander I. Aptekarev
Keldysh Institute of Applied Mathematics
of Russian Academy of Sciences
E-mail: aptekaa@keldysh.ru

Yuri G. Rykov
Keldysh Institute of Applied Mathematics
of Russian Academy of Sciences
E-mail: rykov@keldysh.ru

Presented by A. G. Sergeev
Accepted 14/OCT/19
Translated by N. KRUZHILIN