DOI: https://doi.org/10.1070/RM9915

Another view of the maximum principle for infinite-horizon optimal control problems in economics

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Abstract. The authors present their recently developed complete version of the Pontryagin maximum principle for a class of infinite-horizon optimal control problems arising in economics. The main distinguishing feature of the result is that the adjoint variable is explicitly specified by a formula analogous to the Cauchy formula for solutions of linear differential systems. In certain situations this formula implies the 'standard' transversality conditions at infinity. Moreover, it can serve as an alternative to them. Examples demonstrate the advantages of the proposed version of the maximum principle. In particular, its applications are considered to Halkin's example, to Ramsey's optimal economic growth model, and to a basic model for optimal extraction of a non-renewable resource. Also presented is an economic interpretation of the characterization obtained for the adjoint variable.

Bibliography: 62 titles.

Keywords: optimal control, Pontryagin maximum principle, adjoint variables, transversality conditions, Ramsey model, optimal extraction of a non-renewable resource.

Contents

The research of the first author was supported by the Russian Science Foundation under grant no. 19-11-00223. The research of the second author was supported by the Austrian Science Fund (FWF) under grant no. P31400-N32.

AMS 2010 *Mathematics Subject Classification*. Primary 49K15; Secondary 91B62.

1. Introduction

To the best of our knowledge, Pontryagin first announced his celebrated maximum principle for problems of optimal control in 1956 at a session of the USSR Academy of Sciences on Scientific Problems in Computer-Aided Manufacturing (see [47]). Then in 1961 Pontryagin and his collaborators published the seminal monograph *The mathematical theory of optimal processes*, which established the foundations of the new theory (see [48]). Starting from this time the theory of optimal control began its rapid development.

As indicated already in Pontryagin's lecture [47], the so-called *adjoint variable* plays a main role in the relations of the maximum principle. It enables one to determine the values of an optimal control via the maximum condition. The behaviour of the adjoint variable is governed by the adjoint system. However, the adjoint system has infinitely many solutions, and the particular solution of it that corresponds to the optimal control under consideration is usually determined by additional boundary conditions known as *transversality conditions*. This explains the role of the transversality conditions and suggests a standard way to complete the relations of the maximum principle.

In the last decades, the Pontryagin maximum principle has been extended to various classes of problems. One of the important classes of optimal control problems for which numerous attempts to develop the maximum principle have been made is the class of infinite-horizon problems arising in economics. Typically, the initial state is fixed and the terminal state (at infinity) is free in such problems, while the utility functional to be maximized is given by an improper integral on the time interval $[0,\infty)$. The last circumstance gives rise to specific mathematical features of the problems. More precisely, let $x_*(\cdot)$ be an optimal trajectory and let $(\psi^0,\psi(\cdot))$ be a pair of adjoint variables corresponding to $x_*(\cdot)$ according to the maximum principle. Although the state of the system at infinity is not constrained, such problems can be abnormal (that is, $\psi^0=0$), and the 'standard' transversality conditions at infinity, of the form

$$\lim_{t\to\infty}\psi(t)=0 \quad \text{or} \quad \lim_{t\to\infty}\langle\psi(t),x_*(t)\rangle=0,$$

may fail. The results justifying validity of these relations were obtained only under rather restrictive assumptions (see [22], [35], [42], [44], [51], [54], [62]) that make them inapplicable to many particular economic problems.

We remark that an additional characterization of the adjoint variable $\psi(\cdot)$ is critically important for the efficient use of the maximum principle, because without complementary conditions the set of extremals satisfying the maximum principle may be 'too broad' in the general case. Furthermore, a number of known examples (see [11], [26], [37], [44]) clearly demonstrate that complementary conditions for the adjoint variable which differ from the standard transversality conditions must be involved.

The aim of this paper is to present recent results of the authors which develop another view of the Pontryagin optimality conditions for infinite-horizon optimal control problems arising in economics, especially with regard to the correct determination of the adjoint variable $\psi(\cdot)$.

The main distinguishing feature of the version obtained for the maximum principle is that the adjoint variable is explicitly specified by a formula which resembles the Cauchy formula for solutions of linear differential systems. In certain situations this formula implies the standard transversality conditions at infinity. Moreover, it can serve as an alternative to them. Another important feature of the version obtained for the maximum principle is that it is proved under weak regularity assumptions. This makes it possible to directly apply it to some meaningful economic models. A third feature of the proposed approach is that it is also applicable to problems in which infinite objective values may appear. In this case the concept of *overtaking optimality* is employed, which is important for many economic considerations.

The paper is organized as follows. In § 2 we give a rigorous formulation of the problem, introduce the notion of optimality used in this paper, and formulate and discuss our main result — the normal-form version of the maximum principle with an explicitly specified solution of the adjoint equation. Here we also present two illustrative examples. The first is Halkin's classical example in which the standard transversality conditions at infinity are violated while all optimal controls in the problem are determined by the explicit representation obtained for the adjoint variable. The second example clarifies the role of our main growth assumption. Together, these two examples demonstrate the alternative character of the description obtained for the adjoint variable, compared with the standard transversality conditions at infinity.

In § 3 we specialize the main result for several classes of problems in terms of growth rates of the functions involved. This enables us to formulate conditions implying the standard transversality conditions at infinity in terms of the growth rate conditions.

Section 4 is devoted to applications of our main result in economics. Here the economic meaning of the Cauchy-type formula obtained is discussed in detail. In the case of autonomous problems with exponential discounting we establish a connection between the Cauchy-type formula and Michel's asymptotic condition for the Hamiltonian (see [44]). Then we apply our main result to two important economic problems: Ramsey's optimal economic growth model and a basic model of optimal extraction of a non-renewable resource. Ramsey's model is the most important theoretical construct in modern growth theory. The analysis of Ramsey's model presented in the economic literature is usually based on the assumption that the standard transversality condition holds as a necessary condition for optimality. However, as a rule this fact is not rigorously justified (see [19], § 2.6, for example). Here we present a rigorous analysis of the Ramsey model based on application of the version of the maximum principle developed. The basic model of optimal extraction of a non-renewable resource provides an example of an infinite-horizon optimal control problem in economics in which the standard transversality conditions at infinity fail while a correct characterization of the adjoint variable is provided by the Cauchy-type formula.

We conclude the paper with a brief bibliographical survey in § 5.

2. Statement of the problem and the main result

2.1. Statement of the problem. Let G be a non-empty open convex subset of $\mathbb{R}^n$ and let

$$f\colon [0,\infty)\times G\times\mathbb{R}^m\to\mathbb{R}^n \quad\text{and}\quad f^0\colon [0,\infty)\times G\times\mathbb{R}^m\to\mathbb{R}^1$$

be given functions.

Consider the following optimal control problem (P):

$$J(x(\cdot),u(\cdot))=\int_0^\infty f^0(t,x(t),u(t))\,dt\to\max, \tag{1}$$

$$\dot x(t)=f(t,x(t),u(t)),\qquad x(0)=x_0, \tag{2}$$

$$u(t)\in U(t). \tag{3}$$

Here $x(t)=(x^1(t),\dots,x^n(t))\in\mathbb{R}^n$ is a phase vector, $u(t)=(u^1(t),\dots,u^m(t))\in\mathbb{R}^m$ is a control vector at time $t\geqslant 0$, $x_0\in G$ is a fixed initial state, and $U\colon[0,\infty)\rightrightarrows\mathbb{R}^m$ is a multivalued map with non-empty values.

Infinite-horizon optimal control problems of type (P) arise in different areas of economics, in particular, in the theory of economic growth [19]. Typically, in economic applications the components of the vector $x(t)$ can be interpreted as values of various capital stocks, while the components of the vector $u(t)$ can be interpreted as values of different kinds of investments at the time $t\geqslant 0$.

As far as we know, it was Ramsey [49] who first considered (in the 1920s) the problem of optimization of economic growth as a variational problem of maximizing an integral functional on an infinite time horizon. This direction of research was continued by Cass [27], Koopmans [43], Shell [56], and Arrow and Kurz [3], and became the standard method for investigating optimal economic growth models. Nevertheless, the theory of first-order necessary optimality conditions for infinite-horizon problems is still less developed than that in the finite-horizon case.

It is well known that the proper choice of the present-value 'shadow prices function' (adjoint variable) $\psi(\cdot)$ along the optimal trajectory $x_*(\cdot)$ plays a crucial role in the identification of the corresponding optimal investment policy $u_*(\cdot)$ in the problem (P). Indeed, if such a function $\psi\colon[0,\infty)\to\mathbb{R}^n$ is known,[1] then the optimal investment policy $u_*(\cdot)$ can be determined by maximizing the instantaneous present net value utility on the time interval $[0,\infty)$:

$$\begin{aligned}&f^0(t,x_*(t),u_*(t))+\langle\psi(t),f(t,x_*(t),u_*(t))\rangle\\&\qquad\overset{\text{a.e.}}{=}\sup_{u\in U(t)}\bigl\{f^0(t,x_*(t),u)+\langle\psi(t),f(t,x_*(t),u)\rangle\bigr\}.\end{aligned} \tag{4}$$

Here the first term $f^0(t,x_*(t),u_*(t))$ on the left-hand side of (4) represents the present-value utility flow, while the second term $\langle\psi(t),f(t,x_*(t),u_*(t))\rangle$ is the present-value increment of the capital stock $x_*(t)$ at the time $t\geqslant 0$. Thus, one can say that (P) is, in fact, a problem of determining an appropriate shadow prices function $\psi(\cdot)$.

[1] Recall that in economics the 'shadow price' of capital is equal to the present discounted value of future marginal products (see [19], Chap. 2).

Note that for the finite-horizon counterpart of the problem (P) the Pontryagin maximum principle provides a unique function $\psi(\,\cdot\,)$ for which the *maximum condition* (4) holds.

Let us recall this classical result in optimal control theory [48]. Define the Hamilton–Pontryagin function $\mathscr{H}\colon [0,\infty)\times G\times\mathbb{R}^m\times\mathbb{R}^1\times\mathbb{R}^n\to\mathbb{R}^1$ for the problem (P) in the standard way:

$$\begin{gathered}\mathscr{H}(t,x,u,\psi^0,\psi)=\psi^0 f^0(t,x,u)+\langle\psi,f(t,x,u)\rangle,\\ t\in[0,\infty),\quad x\in G,\quad u\in\mathbb{R}^m,\quad \psi^0\in\mathbb{R}^1,\quad \psi\in\mathbb{R}^n.\end{gathered}$$

In the normal case, that is, when $\psi^0=1$, we will omit ψ^0 and write simply $\mathscr{H}(t,x,u,\psi)$ instead of $\mathscr{H}(t,x,u,1,\psi)$.

Now consider the following problem (P_T) on a fixed finite time interval $[0,T]$, $T>0$:

$$\begin{gathered}J_T(x(\,\cdot\,),u(\,\cdot\,))=\int_0^T f^0(t,x(t),u(t))\,dt\to\max,\\ \dot x(t)=f(t,x(t),u(t)),\qquad x(0)=x_0,\\ u(t)\in U(t).\end{gathered}$$

Here all the data in the problem (P_T) are the same as in (P). The only difference is that (P_T) is considered on the finite time interval $[0,T]$. Then the Pontryagin maximum principle asserts that, under suitable regularity assumptions, for any optimal admissible pair $(x_T(\,\cdot\,),u_T(\,\cdot\,))$ in the problem (P_T) there is an absolutely continuous function (adjoint variable) $\psi_T\colon[0,T]\to\mathbb{R}^n$ for which the maximum condition (4) is satisfied. This function $\psi_T(\,\cdot\,)$ is uniquely defined as the solution of the normal-form *adjoint system*

$$\dot\psi(t)=-\mathscr{H}_x(t,x_T(t),u_T(t),\psi(t)) \tag{5}$$

with boundary condition

$$\psi(T)=0. \tag{6}$$

The condition (6) is known in optimal control theory as the *transversality condition for free terminal state*. This condition identifies the function $\psi_T(\,\cdot\,)$ uniquely among all functions $\psi(\,\cdot\,)$ which together with the admissible pair $(x_T(\,\cdot\,),u_T(\,\cdot\,))$ satisfy the *core conditions* of the maximum principle: the adjoint system (5) and the maximum condition (4) (with the subscript $*$ replaced by T) .

This result motivated numerous attempts to extend the maximum principle for the problem (P_T) to the infinite-horizon problem (P) by involving 'natural' analogues of the transversality condition (6), in particular of the form

$$\lim_{t\to\infty}\psi(t)=0 \tag{7}$$

or

$$\lim_{t\to\infty}\langle\psi(t),x_*(t)\rangle=0. \tag{8}$$

Nevertheless, the positive results in this direction were obtained only under additional conditions that make them inapplicable to many particular problems (see the bibliographical comments in § 5). Moreover, as Halkin [37] pointed out by means of counterexamples, although the phase state at infinity is not constrained in the problem (P), such problems could be *abnormal* (that is, $\psi^0 = 0$), and complementary conditions of the form (7) or (8) may fail to be fulfilled for the 'right' adjoint function for which the core conditions of the maximum principle hold.

We should also mention another asymptotic condition of the form

$$\lim_{t\to\infty} H(t, x_*(t), \psi^0, \psi(t)) = 0 \tag{9}$$

on the adjoint variable. This was proved by Michel (see [44]) in the specific case when the problem (P) is autonomous with exponential discounting, that is,

$$f(t, x, u) \equiv f(x, u), \quad f^0(t, x, u) = e^{-\rho t} g(x, u),$$
$$x \in G, \quad u \in U(t) \equiv U, \quad t \geqslant 0,$$

the discount rate ρ is an arbitrary real number (not necessarily positive), and the optimal value of the functional is finite. Here

$$H(t, x_*(t), \psi^0, \psi(t)) = \sup_{u\in U} \mathscr{H}(t, x_*(t), u, \psi^0, \psi(t))$$

is the Hamiltonian. The condition (9) is similar to the *transversality in time condition*

$$H(T, x_T(T), \psi^0, \psi(T)) = 0,$$

which is well known for finite-horizon optimal control problems with free terminal time $T > 0$ (see [48]).

Let us return to the conditions (5) and (6) for the adjoint variable in the finite-horizon problem (P_T). It is easy to see that by the Cauchy formula for linear differential systems (see [38]), the adjoint system (5) and the transversality condition (6) give the representation

$$\psi(t) = Z_T(t) \int_t^T [Z_T(s)]^{-1} f_x^0(s, x_T(s), u_T(s))\, ds, \qquad t \in [0, T]. \tag{10}$$

Here $Z_T(\,\cdot\,)$ is the (normalized at $t = 0$) fundamental matrix solution on $[0, T]$ of the linear system

$$\dot z(t) = -[f_x(t, x_T(t), u_T(t))]^* z(t). \tag{11}$$

This means that the columns of the matrix function $Z_T(\,\cdot\,)$ are linearly independent solutions of (11) on $[0, T]$ and $Z_T(0) = I$, where I is the identity matrix.

The pointwise representation (10) suggests a 'natural' candidate for an appropriate adjoint function $\psi(\,\cdot\,)$ in the problem (P). Indeed, replacing $(x_T(\,\cdot\,), u_T(\,\cdot\,))$ by $(x_*(\,\cdot\,), u_*(\,\cdot\,))$ in (10) and formally passing to the limit as $T \to \infty$ (which can be justified under appropriate conditions that guarantee convergence of the integral in (12)), we obtain the expression

$$\psi(t) = Z_*(t) \int_t^\infty [Z_*(s)]^{-1} f_x^0(s, x_*(s), u_*(s))\, ds, \qquad t \geqslant 0, \tag{12}$$

where $Z_*(\cdot)$ is now the (normalized at $t = 0$) fundamental matrix solution of the linear system

$$\dot{z}(t) = -\big[f_x(t, x_*(t), u_*(t))\big]^* z(t), \qquad t \in [0, \infty).$$

Clearly, in the infinite-horizon case the formula (12) is a direct analogue of the Cauchy formula (10) for the adjoint function in the problem (P_T). However, (12) cannot be reduced to the asymptotic conditions (7) or (8). This explicit formula does not assume or imply the standard transversality conditions (7) or (8), which may be inconsistent with the core conditions of the maximum principle. Nevertheless, for particular classes of problems it may imply (7) and/or (8), as will be seen below. If the problem (P) is autonomous with exponential discounting and the optimal value of the functional is finite, then the explicit formula (12) can also imply the asymptotic condition (9). It turns out that the Cauchy-type formula (12) can be justified as part of necessary optimality conditions for the problem (P) under mild regularity and growth assumptions, and it can serve as an alternative to (7) and (8).

Let us now refine the formulation of the problem (P).

The following assumption applies throughout our paper and will not always be explicitly mentioned.

Assumption (A0). For almost every $t \in [0, \infty)$ the derivatives $f_x(t, x, u)$ and $f_x^0(t, x, u)$ exist for all $(x, u) \in G \times \mathbb{R}^m$, and the functions $f(\cdot, \cdot, \cdot)$, $f^0(\cdot, \cdot, \cdot)$, $f_x(\cdot, \cdot, \cdot)$, and $f_x^0(\cdot, \cdot, \cdot)$ are Lebesgue–Borel measurable (LB-measurable) with respect to (t, u) for every $x \in G$, and continuous with respect to x for almost every $t \in [0, \infty)$ and every $u \in \mathbb{R}^m$. The multivalued map $U(\cdot)$ is LB-measurable.

The LB-measurability with respect to (t, u) (see [31], Definition 6.33) means that the functions (and sets) with this property are measurable with respect to the σ-algebra generated by the Cartesian product of the Lebesgue σ-algebra on $[0, \infty)$ and the Borel σ-algebra on $\mathbb{R}^m$. It is important to note that for any LB-measurable function $g\colon [0, \infty) \times \mathbb{R}^m \to \mathbb{R}^n$, the superposition $t \mapsto g(t, u(t))$ with a Lebesgue measurable function $u\colon [0, \infty) \to \mathbb{R}^m$ is Lebesgue measurable (see [31], Proposition 6.34). The LB-measurability of the multivalued map $U(\cdot)$ means that the set

$$\operatorname{graph} U(\cdot) = \{(t, u) \in [0, \infty) \times \mathbb{R}^m \colon u \in U(t)\}$$

is an LB-measurable subset of $[0, \infty) \times \mathbb{R}^m$.

Remark 1. In some situations it is natural (and convenient) to consider problems (P) with functions $f(\cdot, \cdot, \cdot)$ and $f^0(\cdot, \cdot, \cdot)$ which are defined only for $(t, u) \in \operatorname{graph} U(\cdot)$, where $U(\cdot)$ is an LB-measurable multivalued map. In this case the LB-measurability of $f(\cdot, \cdot, \cdot)$ and $f^0(\cdot, \cdot, \cdot)$ with respect to (t, u) means that these functions are measurable with respect to the relative σ-algebra induced in $\operatorname{graph} U(\cdot)$ by the σ-algebra of all LB-measurable subsets of $[0, \infty) \times \mathbb{R}^m$. This is equivalent to the LB-measurability of the functions $f(\cdot, \cdot, \cdot)$ and $f^0(\cdot, \cdot, \cdot)$ extended as arbitrary constants from $\operatorname{graph} U(\cdot)$ to $[0, \infty) \times \mathbb{R}^m$ for all $x \in G$.

As a *control* we take any Lebesgue measurable function $u\colon [0, \infty) \to \mathbb{R}^m$ satisfying (3) for all $t \geqslant 0$. If $u(\cdot)$ is a control, then the corresponding *trajectory* $x(\cdot)$

is a locally absolutely continuous solution of (2) which (if it exists) is defined with values in G on some (maximal) finite or infinite time interval $[0,\tau)$, $\tau > 0$. The local absolute continuity of $x(\cdot)$ means that $x(\cdot)$ is absolutely continuous on any compact subinterval $[0,T]$ of its domain of definition $[0,\tau)$.

By definition, a pair $(x(\cdot),u(\cdot))$, where $u(\cdot)$ is a control and $x(\cdot)$ is the corresponding trajectory, is an *admissible pair* in the problem (P) if the trajectory $x(\cdot)$ is defined on the whole time interval $[0,\infty)$ and the function $t \mapsto f^0(t,x(t),u(t))$ is locally integrable on $[0,\infty)$ (that is, it is integrable on any finite time interval $[0,T]$, $T>0$). Thus, for any admissible pair $(x(\cdot),u(\cdot))$ and any $T>0$ the integral

$$J_T(x(\cdot),u(\cdot)) := \int_0^T f^0(t,x(t),u(t))\,dt$$

is well defined. If $(x(\cdot),u(\cdot))$ is an admissible pair, then we refer to $u(\cdot)$ as an *admissible control* and to $x(\cdot)$ as the corresponding *admissible trajectory*.

We now recall two basic concepts of optimality used in the literature (see [26], for instance).

In the first, the integral in (1) is understood in the improper sense, that is, for an arbitrary admissible pair $(x(\cdot),u(\cdot))$, by definition,

$$J(x(\cdot),u(\cdot)) = \lim_{T\to\infty} \int_0^T f^0(t,x(t),u(t))\,dt,$$

if this limit exists.

Definition 2. An admissible pair $(x_*(\cdot),u_*(\cdot))$ is said to be *strongly optimal* in the problem (P) if the corresponding integral in (1) converges (to a finite number), and for any other admissible pair $(x(\cdot),u(\cdot))$

$$J(x_*(\cdot),u_*(\cdot)) \geqslant \limsup_{T\to\infty} \int_0^T f^0(t,x(t),u(t))\,dt.$$

In the second definition, the integral in (1) is not necessarily convergent.

Definition 3. An admissible pair $(x_*(\cdot),u_*(\cdot))$ is said to be *finitely optimal* in the problem (P) if for any $T>0$ this pair (restricted to $[0,T]$) is optimal in the following optimal control problem (Q_T) with fixed initial and final states:

$$\begin{gathered} J_T(x(\cdot),u(\cdot)) = \int_0^T f^0(t,x(t),u(t))\,dt \to \max, \\ \dot{x}(t) = f(t,x(t),u(t)), \qquad x(0)=x_0, \quad x(T)=x_*(T), \\ u(t)\in U(t). \end{gathered}$$

It is easy to see that strong optimality implies finite optimality.

The following weak regularity assumption plays a key role for the validity of the general version of the Pontryagin maximum principle for a finitely optimal pair $(x_*(\cdot),u_*(\cdot))$ in the problem (P) (one can find similar assumptions for problems with finite time-horizons in [30], Chap. 5, and [31], Hypothesis 22.25).

Assumption (A1). There exist a continuous function $\gamma\colon [0,\infty) \to (0,\infty)$ and a locally integrable function $\varphi\colon [0,\infty) \to \mathbb{R}^1$, such that

$$\{x\colon \|x - x_*(t)\| \leqslant \gamma(t)\} \subset G \quad \text{for all } t \geqslant 0$$

and for almost every $t \in [0,\infty)$

$$\max_{\{x\colon \|x-x_*(t)\|\leqslant\gamma(t)\}} \bigl\{\|f_x(t,x,u_*(t))\| + \|f_x^0(t,x,u_*(t))\|\bigr\} \leqslant \varphi(t). \tag{13}$$

Note that if $(x_*(\,\cdot\,), u_*(\,\cdot\,))$ is an admissible pair and Assumption (A1) holds, then $x_*(\,\cdot\,)$ is the unique trajectory corresponding to $u_*(\,\cdot\,)$ (see [36], Chap. 1, Theorem 2).

Remark 4. Assumption (A1) holds automatically under the standard regularity conditions that $u_*(\,\cdot\,) \in L^\infty_{\mathrm{loc}}[0,\infty)$, $U(t) \equiv U$ for $t \geqslant 0$, and the functions $f_x(\,\cdot\,,\,\cdot\,,\cdot\,)$ and $f_x^0(\,\cdot\,,\,\cdot\,,\cdot\,)$ are measurable with respect to t, continuous with respect to (x,u), and locally bounded. Here local boundedness of these functions of t, x, and u (take $\phi(\,\cdot\,,\cdot\,,\cdot\,)$ to represent them) means that for every $T > 0$, every compact set $D \subset G$, and every bounded set $V \subset U$ there exists a number M such that $\|\phi(t,x,u)\| \leqslant M$ for almost all $t \in [0,T]$ and all $x \in D$ and $u \in V$.

If Assumption (A1) holds, then any finitely optimal admissible pair $(x_*(\,\cdot\,), u_*(\,\cdot\,))$ satisfies the following general version of the maximum principle, which is proved in [37] under the standard regularity conditions.

Theorem 5. *Let $(x_*(\,\cdot\,), u_*(\,\cdot\,))$ be a finitely optimal admissible pair in the problem (P) and let Assumption (A1) be valid. Then there is a non-vanishing pair of adjoint variables $(\psi^0, \psi(\,\cdot\,))$, with $\psi^0 \geqslant 0$ and a locally absolutely continuous function $\psi(\,\cdot\,)\colon [0,\infty) \to \mathbb{R}^n$, such that the core conditions of the maximum principle hold, that is,*

(i) *$\psi(\,\cdot\,)$ is a solution of the adjoint system*

$$\dot\psi(t) = -\mathscr{H}_x(t, x_*(t), u_*(t), \psi^0, \psi(t)),$$

(ii) *the maximum condition is satisfied,*

$$\mathscr{H}(t, x_*(t), u_*(t), \psi^0, \psi(t)) \overset{a.e.}{=} \sup_{u\in U(t)} \mathscr{H}(t, x_*(t), u, \psi^0, \psi(t)).$$

The main points in the proof of this theorem are essentially the same as in Halkin's original result (see [37], Theorem 4.2). The necessary changes are given in [16].

Theorem 5 provides the most general known version of the maximum principle for the problem (P). Nevertheless, this theorem establishes only the core conditions (i) and (ii) of the maximum principle without any additional characterizations of the adjoint variables $(\psi^0, \psi(\,\cdot\,))$. Due to this circumstance, the relations of Theorem 5 are incomplete, and, as a rule, application of the theorem to particular problems is ineffective (see the discussion in Example 9 below). To complete the relations of Theorem 5 we need to employ a stronger concept of optimality and to use an additional growth assumption.

The following concept of optimality appears to be the most useful among the numerous alternative definitions proposed in the context of economics. It takes an intermediate place between finite optimality and strong optimality (see [26]).

Definition 6. The admissible pair $(x_*(\cdot), u_*(\cdot))$ is said to be *weakly overtaking optimal* if for arbitrary $\varepsilon > 0$ and $T > 0$ and any other admissible pair $(x(\cdot), u(\cdot))$ there is a $T' > T$ such that

$$\int_0^{T'} f^0(t, x_*(t), u_*(t))\, dt \geqslant \int_0^{T'} f^0(t, x(t), u(t))\, dt - \varepsilon.$$

The following growth assumption for an admissible pair $(x_*(\cdot), u_*(\cdot))$ was introduced in [15] as an extension of the so-called *dominating discount condition* (see [7], [9]–[11], [14], [17]).

Assumption (A2). There exist a number $\beta > 0$ together with an integrable function $\lambda\colon [0, \infty) \to \mathbb{R}^1$ such that for every $\zeta \in G$ with $\|\zeta - x_0\| < \beta$ the equation (2) with $u(\cdot) = u_*(\cdot)$ and initial condition $x(0) = \zeta$ (instead of $x(0) = x_0$) has a solution $x(\zeta; \cdot)$ with values in G on $[0, \infty)$, and

$$\max_{x \in [x(\zeta;t), x_*(t)]} \bigl|\langle f_x^0(t, x, u_*(t)), x(\zeta; t) - x_*(t)\rangle\bigr| \overset{\text{a.e.}}{\leqslant} \|\zeta - x_0\| \lambda(t).$$

Here $[x(\zeta; t), x_*(t)]$ is the line segment between the points $x(\zeta; t)$ and $x_*(t)$.

By the Lipschitz dependence of the solution $x(\zeta; t)$ on the initial condition ζ, the inequality $\|x(\zeta; t) - x_*(t)\| \leqslant l(t)\|\zeta - x_0\|$ always holds, with $l(\cdot)$ independent of ζ. The function $\lambda(\cdot)$ incorporates the growth of this Lipschitz constant and the growth of the derivative of the objective integrand in a neighbourhood of the pair under consideration. The real assumption here is actually that $\lambda(\cdot)$ is integrable.

Note that the constant $\beta > 0$ and the integrable function $\lambda(\cdot)$ may depend on the admissible pair $(x_*(\cdot), u_*(\cdot))$ in (A2). In some cases Assumption (A2) can be *a priori* justified for all optimal (or even for all admissible) pairs $(x_*(\cdot), u_*(\cdot))$ in (P), taken together with their own constants β and functions $\lambda(\cdot)$ (see the examples in § 4).

The following auxiliary result (see [16], Lemma 3.2) implies that the integral in (12) is finite.

Lemma 7. *Let (A1) and (A2) be satisfied. Then the following estimation holds*:

$$\bigl\|[Z_*(t)]^{-1} f_x^0(t, x_*(t), u_*(t))\bigr\| \leqslant \sqrt{n}\, \lambda(t) \quad \textit{for a.e. } t \geqslant 0. \tag{14}$$

By Lemma 7 and the integrability of $\lambda(\cdot)$, the function $\psi\colon [0, \infty) \to \mathbb{R}^n$ defined by (12) is locally absolutely continuous. By direct differentiation we verify that the so-defined function $\psi(\cdot)$ satisfies on $[0, \infty)$ the adjoint system

$$\dot\psi(t) = -\mathscr{H}_x(t, x_*(t), u_*(t), \psi(t)).$$

2.2. Main result. The following version of the Pontryagin maximum principle for the infinite-horizon problem (P) is the main result in this paper.

Theorem 8. *Let* $(x_*(\cdot), u_*(\cdot))$ *be a weakly overtaking optimal pair in the problem* (P). *Assume that the regularity Assumption* (A1) *and the growth Assumption* (A2) *are satisfied. Then the vector function* $\psi\colon [0,\infty) \to \mathbb{R}^n$ *defined by* (12) *is* (*locally*) *absolutely continuous and satisfies the core conditions of the normal-form maximum principle*, *that is*,

(i) $\psi(\cdot)$ *is a solution of the adjoint system*

$$\dot{\psi}(t) = -\mathscr{H}_x(t, x_*(t), u_*(t), \psi(t)), \tag{15}$$

(ii) *the maximum condition is satisfied*,

$$\mathscr{H}(t, x_*(t), u_*(t), \psi(t)) \overset{a.e.}{=} \sup_{u \in U(t)} \mathscr{H}(t, x_*(t), u, \psi(t)). \tag{16}$$

The rather technical proof of Theorem 8 is presented in detail in [16].

An important feature of Theorem 8 is that it is proved under weak regularity assumptions. This makes it possible to apply it directly to some meaningful economic models. Here the admissible controls $u(\cdot)$ are not necessarily bounded (even in a local sense), and the functions $f(\cdot, \cdot, \cdot)$ and $f^0(\cdot, \cdot, \cdot)$ are not necessarily continuous with respect to the variable u. Instead, we assume that the functions $f(\cdot, \cdot, \cdot)$, $f^0(\cdot, \cdot, \cdot)$, and the multivalued map $U(\cdot)$ are LB-measurable with respect to the variables (t, u), while their partial derivatives $f_x(\cdot, \cdot, u_*(\cdot))$ and $f^0_x(\cdot, \cdot, u_*(\cdot))$ are locally integrally bounded in some (t, x)-tube in a neighbourhood of the graph of the optimal trajectory $x_*(\cdot)$ under consideration. From the practical point of view (especially in economics), considering LB-measurable functions $f(\cdot, \cdot, \cdot)$ and $f^0(\cdot, \cdot, \cdot)$ is important, since it allows for their discontinuity with respect to the control. Such a discontinuity appears, for example, when the cost of a positive control (say, maintenance) is fixed, and by a jump goes to zero if a zero control is applied. The unboundedness of admissible controls $u(\cdot)$ allows one to treat some economic problems (such as problems of optimal exploitation of renewable or non-renewable resources) in their most natural settings when the rate of extraction of the resource satisfies only an integral constraint in an L-space.

We remark that the maximum principle ('extended maximum principle') for finite horizon problems with LB-measurable data $f(\cdot, \cdot, \cdot)$ and $f^0(\cdot, \cdot, \cdot)$ and not necessarily bounded admissible controls $u(\cdot)$ was established by Clarke (see [30] and [31]) by methods of non-smooth analysis. Our proof of Theorem 8 employs a modification of the classical needle variations technique and makes essential use of the Yankov–von Neumann–Aumann selection theorem (see [40], Theorem 2.14). The use of simple needle variations enables us to treat the case of the unbounded time interval $[0, \infty)$.

Another useful feature of our main result is that it applies to problems with infinite objective integral (1), where the notion of *overtaking optimality* is adopted.

It is easy to see that under the assumptions of Theorem 8 together with the additional assumption that $\|Z_*(t)\| \leqslant c$ for some constant $c \geqslant 0$ and all sufficiently large t, the formula (12) immediately implies the 'standard' asymptotic condition (7). In § 3 we consider some other situations when the formula (12) implies the 'standard' asymptotic conditions (7) and (8). In § 4 we establish a link between (12) and the condition (9).

The convergence of the integral in (12) immediately implies that the adjoint function $\psi(\,\cdot\,)$ defined in (12) satisfies the asymptotic relation

$$\lim_{t\to\infty}[Z_*(t)]^{-1}\psi(t)=0. \tag{17}$$

Even more, it is straightforward to prove that under the assumptions of Theorem 8 the function $\psi(\,\cdot\,)$ defined by (12) is the unique solution of the adjoint equation (15) that satisfies (17). Indeed, if $\psi(\,\cdot\,)$ and $\widetilde{\psi}(\,\cdot\,)$ are two solutions of (15) satisfying (17), then

$$\frac{d}{dt}(\psi(t)-\widetilde{\psi}(t))=-[f_x(t,x_*(t),u_*(t))]^*(\psi(t)-\widetilde{\psi}(t)).$$

From this, for any $t\geqslant 0$ we get that

$$\psi(0)-\widetilde{\psi}(0)=[Z_*(t)]^{-1}(\psi(t)-\widetilde{\psi}(t)).$$

Since the right-hand side converges to zero as $t\to\infty$, we have $\psi(0)-\widetilde{\psi}(0)=0$, which implies that $\psi(\,\cdot\,)=\widetilde{\psi}(\,\cdot\,)$.

Another direct corollary of the convergence of the integral in (12) is the equality

$$\psi(0)=\int_0^\infty [Z_*(s)]^{-1}f_x^0(s,x_*(s),u_*(s))\,ds. \tag{18}$$

Then since $\psi(\,\cdot\,)$ is a solution of the linear differential system (15), we have

$$\psi(t)=Z_*(t)\psi(0)-Z_*(t)\int_0^t [Z_*(s)]^{-1}f_x^0(s,x_*(s),u_*(s))\,ds, \qquad t\geqslant 0. \tag{19}$$

Obviously, (18) and (19) imply (12). Thus, under the assumptions of Theorem 8 the function $\psi(\,\cdot\,)$ defined by (12) is the only solution of the adjoint equation (15) that satisfies the initial condition (18).

2.3. Two illustrative examples. The first example is Halkin's original example (see [37], § 5). It demonstrates the completeness of the conditions of Theorem 8 and the advantage of (12) in comparison with the asymptotic conditions (7), (8), and (9). The second example clarifies the role of Assumption (A2) in Theorem 8. Together, these examples illustrate the alternative character of the Cauchy-type formula (12) in comparison with the standard transversality conditions (7) and (8).

Example 9 (Halkin's example). Consider the following problem (P1):

$$J(x(\,\cdot\,),u(\,\cdot\,))=\int_0^\infty (1-x(t))u(t)\,dt\to\max, \tag{20}$$
$$\dot{x}(t)=(1-x(t))u(t), \qquad x(0)=0,$$
$$u(t)\in[0,1].$$

This example is interesting in that it shows that the standard asymptotic conditions (7) and (8) are inconsistent with the core conditions (i) and (ii) of the maximum principle (see Theorem 5), while the asymptotic condition (9) does not bear any substantial information in this case. Let us clarify this statement.

For any $T>0$ and for an arbitrary admissible pair $(x(\,\cdot\,),u(\,\cdot\,))$ we have

$$J_T(x(\,\cdot\,),u(\,\cdot\,)) = \int_0^T \dot{x}(t)\,dt = x(T) = 1-\exp\biggl\{-\int_0^T u(t)\,dt\biggr\}. \tag{21}$$

This implies that an admissible pair $(x_*(\,\cdot\,),u_*(\,\cdot\,))$ is weakly overtaking optimal (also strongly optimal)[2] if and only if $\int_0^\infty u_*(t)\,dt=\infty$. Also, note that $x_*(t)\to 1$ as $t\to\infty$.

According to Theorem 5, any optimal admissible pair $(x_*(\,\cdot\,),u_*(\,\cdot\,))$ satisfies, together with the corresponding adjoint function $\psi(\,\cdot\,)$, the adjoint equation in (i) and the maximum condition in (ii), which in this particular case take the forms

$$\dot{\psi}(t) = (\psi(t)+\psi^0)u_*(t), \tag{22}$$

$$(1-x_*(t))(\psi(t)+\psi^0)u_*(t) \overset{\text{a.e.}}{=} \max_{u\in[0,1]}\{(1-x_*(t))(\psi(t)+\psi^0)u\} \tag{23}$$

for some $\psi^0\geqslant 0$. From the adjoint equation (22) we obtain

$$\psi(t) = (\psi(0)+\psi^0)\exp\biggl\{\int_0^t u_*(s)\,ds\biggr\}-\psi^0.$$

Thus, for all $t\geqslant 0$ and $u\in[0,1]$ we have

$$\mathscr{H}(t,x_*(t),u,\psi^0,\psi(t)) = (1-x_*(t))(\psi(t)+\psi^0)u = (\psi^0+\psi(0))u.$$

Since (23) implies that

$$(\psi^0+\psi(0))u_*(t) \overset{\text{a.e.}}{=} \max_{u\in[0,1]}(\psi^0+\psi(0))u$$

and any strongly optimal control $u_*(\,\cdot\,)$ cannot be identically zero, we must have $\psi^0+\psi(0)\geqslant 0$. If $\psi^0+\psi(0)=0$, then without loss of generality we can set $\psi^0=1$ and $\psi(0)=-1$. Then both (7) and (8) are obviously violated. If $\psi^0+\psi(0)>0$, then $\psi(t)\to+\infty$ as $t\to+\infty$, and again both (7) and (8) are violated. Thus, for an arbitrary optimal admissible pair $(x_*(\,\cdot\,),u_*(\,\cdot\,))$ both the conditions (7) and (8) are inconsistent with the core conditions of the maximum principle.

We note that by [44] the asymptotic condition for the Hamiltonian (9) is a necessary optimality condition in Halkin's example. However, this condition does not give us any useful information in this case.

Indeed, in view of our above analysis this condition holds only in the case when $\psi^0+\psi(0)=0$ or, equivalently, $\psi^0=1$ and $\psi(0)=-1$. But in this case the core conditions (i) and (ii) of the maximum principle and the condition (9) hold trivially along any admissible pair, and hence they do not provide any useful information.

[2] We mention that in this example every admissible control is obviously finitely optimal. Thus, the concept of finite optimality is too weak here.

Now let us apply Theorem 8 with $G = \mathbb{R}^1$. Fix an arbitrary admissible pair $(x_*(\cdot), u_*(\cdot))$. Assumptions (A0) and (A1) are obviously fulfilled (see Remark 4). In order to check Assumption (A2), we note that

$$x(\zeta;t) = 1 - (1-\zeta)\exp\left\{-\int_0^t u_*(s)\,ds\right\} \quad \text{and} \quad f_x^0(t,x,u_*(t)) = -u_*(t)$$

for all $t \geqslant 0$. Hence,

$$\max_{x\in[x(\zeta;t),x_*(t)]} |\langle f_x^0(t,x,u_*(t)), x(\zeta;t) - x_*(t)\rangle| \overset{\text{a.e.}}{=} |\zeta - x_0|\lambda(t),$$

where

$$\lambda(t) = u_*(t)\exp\left\{-\int_0^t u_*(s)\,ds\right\} \quad \text{for all } t \geqslant 0.$$

The function $\lambda(\cdot)$ is integrable on $[0,\infty)$, so the condition (A2) is also satisfied.

We recall that in view of the explicit formula $x_*(t) = 1 - \exp\{-\int_0^t u_*(s)\,ds\}$, $t \geqslant 0$, the maximum condition (23) in the normal case $\psi^0 = 1$ has the form

$$(1+\psi(0))u_*(t) \overset{\text{a.e.}}{=} \max_{u\in[0,1]}\{(1+\psi(0))u\}. \tag{24}$$

The formula (12) for the adjoint variable gives us that

$$\begin{aligned}\psi(t) &= \exp\left\{\int_0^t u_*(s)\,ds\right\}\int_t^\infty \exp\left\{-\int_0^s u_*(\tau)\,d\tau\right\}(-u_*(s))\,ds\\ &= \exp\left\{\int_0^t u_*(s)\,ds\right\}\left[\lim_{T\to\infty}\exp\left\{-\int_0^T u_*(s)\,ds\right\}\right.\\ &\qquad \left.-\exp\left\{-\int_0^t u_*(s)\,ds\right\}\right], \qquad t \geqslant 0.\end{aligned}$$

We consider two cases. First, if $\int_0^\infty u_*(t)\,dt = \infty$ (that is, $u_*(\cdot)$ is optimal), then $\psi(t) = -1$ for all $t \geqslant 0$, and thus the maximum condition (24) is obviously satisfied. Second, if $\int_0^\infty u_*(t)\,dt$ is finite (that is, $u_*(\cdot)$ is not optimal), then $\psi(t) > -1$ for all $t \geqslant 0$, and (24) implies that $u_*(t) = 1$ for almost every $t \geqslant 0$, which contradicts the assumption that $\int_0^\infty u_*(t)\,dt < \infty$.

Summarizing, Theorem 8 provides a complete characterization of all optimal controls in the problem (P1), while the core conditions of the maximum principle are inconsistent with the standard asymptotic conditions (7) and (8), while the asymptotic condition (9) is uninformative (satisfied by any admissible control).

As can be easily seen, if for the admissible pair $(x_*(\cdot), u_*(\cdot))$ Assumption (A1) holds and the integral

$$I_*(t) = \int_t^\infty Z_*^{-1}(s) f_x^0(s, x_*(s), u_*(s))\,ds, \qquad t \geqslant 0, \tag{25}$$

converges, then the function $\psi(\cdot)$ (see (12)) is defined. If, moreover, the pair $(x_*(\cdot), u_*(\cdot))$ is weakly overtaking optimal and the stronger condition (A2) holds (see Lemma 7), then all the assumptions of Theorem 8 are satisfied, and thus the normal-form maximum principle holds with the adjoint variable $\psi(\,\cdot\,)$ given by (12). Since convergence of the integral (25) is sufficient for defining the function $\psi(\,\cdot\,)$, it is natural to ask whether Assumption (A2) in Theorem 8 could be relaxed to convergence of the improper integral (25).

The analysis of the problem below shows that convergence of the integral (25) (together with (A1)) is not enough for validity of Theorem 8, although the function $\psi(\,\cdot\,)$ defined by (12) is locally absolutely continuous and satisfies the adjoint equation (15) and the transversality conditions (7) and (8) in this case.

Example 10. Consider the following problem (P2):

$$J(x(\,\cdot\,),u(\,\cdot\,)) = \int_0^\infty e^{-t}[u(t) - 5x(t)^2]\,dt \to \max,$$
$$\dot{x}(t) = [u(t) + x(t)]\phi(x(t)), \qquad x(0) = 0,$$
$$u(t) \in [0,1].$$

Here $\phi\colon \mathbb{R}^1 \to [0,1]$ is a $C^\infty(\mathbb{R}^1)$-function such that

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \leqslant 1, \\ 0 & \text{if } |x| \geqslant 2. \end{cases}$$

Let $G = (-\infty,\infty)$. Obviously, (P2) is a particular case of the problem (P), and the condition (A1) holds for any admissible pair $(x_*(\,\cdot\,),u_*(\,\cdot\,))$ in (P2) (see Remark 4).

We show that the pair $(x_*(\,\cdot\,),u_*(\,\cdot\,))$ with $x_*(t) \equiv 0$ and $u_*(t) \overset{\text{a.e.}}{=} 0$ for $t \geqslant 0$ is the unique optimal pair in (P2). Indeed, by Theorem 3.6 in [18] there is an optimal admissible pair $(x_*(\,\cdot\,),u_*(\,\cdot\,))$ in (P2). Assume that $u_*(\,\cdot\,)$ is non-vanishing on a set of positive measure. Then for the corresponding optimal trajectory $x_*(\,\cdot\,)$ there is a unique time $\tau > 0$ such that $x_*(\tau) = 1$.

Consider the following auxiliary problem (P2$_\tau$):

$$J_\tau(x(\,\cdot\,),u(\,\cdot\,)) = \int_0^\tau e^{-t}[u(t) - 5x(t)^2]\,dt \to \max,$$
$$\dot{x}(t) = [u(t) + x(t)]\phi(x(t)), \qquad x(0) = 0, \quad x(\tau) = 1,$$
$$u(t) \in [0,1].$$

Here all the data in (P2$_\tau$) are the same as in (P2), and the only difference is that (P2$_\tau$) is considered on the fixed time interval $[0,\tau]$ with the terminal condition $x(\tau) = 1$.

As can be easily seen, the pair $(x_*(\,\cdot\,),u_*(\,\cdot\,))$ that is optimal in (P2), is also optimal in the problem $(P2_\tau)$. Hence, according to the classical Pontryagin maximum principle [48], for problems on finite time intervals with fixed endpoints there are adjoint variables $\psi^0 \geqslant 0$ and $\psi(\,\cdot\,)$ not vanishing simultaneously such that the absolutely continuous function $\psi(\,\cdot\,)$ is a solution on $[0,\tau]$ of the adjoint system

$$\dot{\psi}(t) = 10\psi^0 e^{-t} x_*(t) - \psi(t) \tag{26}$$

and for almost every $t \in [0, \tau]$ the maximum condition holds:

$$u_*(t)\big(\psi^0 e^{-t} + \psi(t)\big) = \max\{0, \psi^0 e^{-t} + \psi(t)\}. \tag{27}$$

Here we have used the fact that $x_*(t) < 1$ for all $t < \tau$, and hence $\phi_x(x_*(t)) \equiv 0$ for all $t \in [0, \tau]$.

If $\psi^0 = 0$, then $\psi(t) = \psi(0)e^{-t}$ for $t \geqslant 0$ by (26). Due to the maximum condition (27), this implies that either $u_*(t) \overset{\text{a.e.}}{=} 0$ (if $\psi(0) < 0$) or $u_*(t) \overset{\text{a.e.}}{=} 1$ (if $\psi(0) > 0$). By assumption $u_*(\,\cdot\,)$ is non-vanishing on a set of positive measure. Hence $u_*(t) \overset{\text{a.e.}}{=} 1$ for $t \in [0, \tau]$.

Substituting $u_*(t) \overset{\text{a.e.}}{=} 1$ into the control system, we get that $x_*(t) = e^t - 1$ for $t \in [0, \tau]$. This implies that $\tau = \log 2$. Further, by direct calculation we get that

$$\begin{aligned} J_\tau(x_*(\,\cdot\,), u_*(\,\cdot\,)) &= \int_0^{\log 2} e^{-t}[1 - 5(e^{2t} - 2e^t + 1)]\,dt \\ &= -4\int_0^{\log 2} e^{-t}\,dt - 5\int_0^{\log 2} e^t\,dt + 10\int_0^{\log 2} dt = 10\log 2 - 7 < 0. \end{aligned} \tag{28}$$

Since $x_*(\log 2) = 1$ and $x_*(t) \geqslant 1$ for $t \geqslant \log 2$, we find that

$$\int_{\log 2}^{\infty} e^{-t}[u_*(t) - 5x_*(t)^2]\,dt < 0.$$

Hence

$$J(x_*(\,\cdot\,), u_*(\,\cdot\,)) < J_\tau(x_*(\,\cdot\,), u_*(\,\cdot\,)) < 0,$$

which contradicts the optimality of the pair $(x_*(\,\cdot\,), u_*(\,\cdot\,))$ in the problem (P2). Thus, either $\psi^0 > 0$ or $u_*(t) \overset{\text{a.e.}}{=} 0$, $t \geqslant 0$.

Consider the case $\psi^0 > 0$. In this case we can assume without loss of generality that $\psi^0 = 1/10$. By (26),

$$\psi(t) = e^{-t}\left[\psi(0) + \int_0^t x_*(s)\,ds\right], \qquad t \in [0, \tau].$$

This implies that

$$\psi^0 e^{-t} + \psi(t) = e^{-t}\left[\frac{1}{10} + \psi(0) + \int_0^t x_*(s)\,ds\right], \qquad t \in [0, \tau].$$

If $\psi(0) > -1/10$, then by the maximum condition (27) we get that $u_*(t) \overset{\text{a.e.}}{=} 1$, $t \in [0, \tau]$. But, as we showed above, $J(x_*(\,\cdot\,), u_*(\,\cdot\,)) < 0$ in this case, which contradicts the optimality of the pair $(x_*(\,\cdot\,), u_*(\,\cdot\,))$.

If $\psi(0) \leqslant -1/10$, then by the maximum condition (27) we get that the control $u_*(\,\cdot\,)$ is vanishing on some interval $[0, \tau_1]$ with $\tau_1 < \tau$, and then $u_*(t) = 1$ for almost every $t \in [\tau_1, \tau]$. In this case $x_*(t) \equiv 0$ on the time interval $[0, \tau_1]$, and $x_*(t) = e^{t-\tau_1} - 1$ for all $t \in [\tau_1, \tau]$. This implies that $\tau = \tau_1 + \log 2$. Hence, in this case we find that

$$J_\tau(x_*(\cdot), u_*(\cdot)) = \int_{\tau_1}^{\tau_1+\log 2} e^{-t}[1 - 5(e^{2(t-\tau_1)} - 2e^{t-\tau_1} + 1)]\,dt$$
$$= e^{-\tau_1}\int_0^{\log 2} e^{-t}[1 - 5(e^{2t} - 2e^t + 1)]\,dt < 0$$

(see (28)). But this again contradicts the optimality of the pair $(x_*(\cdot), u_*(\cdot))$ in the problem (P2).

Thus, we have proved that $(x_*(\cdot), u_*(\cdot))$ with $x_*(t) \equiv 0$ and $u_*(t) \stackrel{\text{a.e.}}{=} 0$ for $t \geqslant 0$ is the unique strongly optimal pair in (P2).

Along the pair $(x_*(\cdot), u_*(\cdot))$ we have

$$f_x^0(t, x_*(t), u_*(t)) = -10x_*(t)e^{-t} \equiv 0, \qquad t \geqslant 0.$$

Thus, for any $t \geqslant 0$ the integral (25) converges absolutely, $I_*(t) \equiv 0$ for $t \geqslant 0$, and the adjoint function $\psi(\cdot)$ defined by (12) is also vanishing: $\psi(t) \equiv 0$ for $t \geqslant 0$. However, the maximum condition (16) (that is, (27) with $\psi^0 = 1$ in the present example) does not hold for $u_*(t) \equiv 0$, $t \geqslant 0$, with the adjoint variable $\psi(t) \equiv 0$, $t \geqslant 0$. Thus, the assertion of Theorem 8 fails in the case of the problem (P2). The reason for this phenomenon is the violation of the growth condition (A2) for the pair $x_*(t) \equiv 0$, $u_*(t) \stackrel{\text{a.e.}}{=} 0$, $t \geqslant 0$.

Nevertheless, all the assumptions of the general maximum principle (see Theorem 5) are satisfied for the problem (P2). In particular, the pair $(x_*(\cdot), u_*(\cdot))$ with $x_*(t) \equiv 0$ and $u_*(t) \stackrel{\text{a.e.}}{=} 0$ for $t \geqslant 0$, which is strongly optimal in (P2), satisfies the conditions (15) and (16) of the general maximum principle with the adjoint variables $\psi^0 = 1$ and $\psi(t) = -e^{-t}$, $t \geqslant 0$. Obviously, this adjoint variable satisfies both the asymptotic conditions (7) and (8). Thus, the normal form of the maximum principle holds, although the correct adjoint function is not provided by (12). The explanation is that Assumption (A2) is not only used to ensure the convergence of the integral in (25) but is also essential in the proof of Theorem 8.

Example 10 also shows that the formula (12) is not implied by the asymptotic conditions (7) or (8).

3. Problems with dominating discount

As Example 10 shows, Assumption (A2) plays an essential role in Theorem 8. In this section we consider a class of infinite-horizon problems (P) for which conditions ensuring (A2) can be expressed in terms of the growth rates of the functions involved. Use of the growth rates allows us also to describe some situations when the explicit formula (12) implies the asymptotic conditions (7) and (8).

In addition to (A1) (see also Remark 4) we make the following assumptions.

Assumption (B1). There exist numbers $\mu \geqslant 0$, $r \geqslant 0$, $\beta > 0$, $\rho \in \mathbb{R}^1$, $\nu \in \mathbb{R}^1$, and $c \geqslant 0$ such that

(B1.i) $\|x_*(t)\| \leqslant ce^{\mu t}$ for all $t \geqslant 0$;

(B1.ii) for any $\zeta \in G$ with $\|\zeta - x_0\| < \beta$ the equation (2) with $u(\cdot) = u_*(\cdot)$ and initial condition $x(0) = \zeta$ (instead of $x(0) = x_0$) has a solution $x(\zeta; \cdot)$ with values in G on $[0, \infty)$, and moreover,

$$\|x(\zeta; t) - x_*(t)\| \leqslant c\|\zeta - x_0\| e^{\nu t}, \qquad t \geqslant 0, \tag{29}$$

and

$$\|f_x^0(t, y, u_*(t))\| \leqslant c(1 + \|y\|^r) e^{-\rho t} \quad \text{for any } t \geqslant 0 \text{ and } y \in [x(\zeta; t), x_*(t)].$$

Some comments about the above assumptions follow. The first inequality in Assumption (B1.ii) specifies the known fact that the solution of a system of ordinary differential equations has a Lipschitz dependence on the initial condition, and it is required in addition that the Lipschitz constant depend exponentially (with rate ν) on the time horizon. Note that the number ν can be negative. The factor $e^{-\rho t}$ in the second inequality in Assumption (B1.ii) indicates that the objective integrand may contain a 'discount' factor with the (possibly negative) discount rate ρ. Assumption (B1.i) requires *a priori* information about the exponential growth rate of the optimal trajectory, which can often be obtained in economic contexts.

While Assumption (B1) is needed mainly to define the constants ρ, r, μ, and ν, the next assumption imposes a certain relationship among them, which is called the *dominating discount condition* (see [7], [9]–[11], [14], [17]).

Assumption (B2).

$$\rho > \nu + r \max\{\mu, \nu\}.$$

In Lemma 5.1 of [15] it was proved that Assumptions (B1) and (B2) imply (A2). Thus, the following corollary of Theorem 8 holds.

Corollary 11. *The assertions of Theorem 8 hold under Assumptions (A1), (B1), and (B2).*

We mention that although the dominating discount condition (B2) may be easier to check than (A2), its satisfaction depends on the time scale chosen (see [45] or [15], Part 3, § 5). In contrast, Assumption (A2) is invariant with respect to any diffeomorphic change of the time variable. Indeed, if the time variable is changed as $t = \xi(s)$ for $s \geqslant 0$ (where $\xi(\,\cdot\,)$ maps $[0, \infty)$ to itself diffeomorphically), then it can be directly checked that in the resulting problem Assumption (A2) is fulfilled with the function $\widetilde{\lambda}(s) = \lambda(\xi(s))\dot{\xi}(s)$, $s \geqslant 0$, which is integrable if and only if $\lambda(\,\cdot\,)$ is.

Below, in §§ 3.1 and 3.2, we consider two more specific classes of problems where the dominating discount condition can be verified in a more convenient way: problems for one-sided Lipschitz systems and problems for systems with regular linearization.

3.1. Problems with one-sided Lipschitz dynamics. Assumption (B1), and hence also (B2), can be verified in a more convenient way for systems with one-sided Lipschitz right-hand sides. The result below essentially extends the result obtained previously in [14], § 4, and therefore we present it in somewhat greater detail. First, we recall the following definition.

Definition 12. A function $f(\cdot,\cdot,\cdot)$ with values $f(t,x,u)$ in $\mathbb{R}^n$ and defined for $x\in G$ and $(t,u)\in \operatorname{graph} U(\cdot)$ is said to be *one-sided Lipschitz* with respect to x (uniformly with respect to $(t,u)\in \operatorname{graph} U(\cdot)$) if there exists a $\nu\in\mathbb{R}^1$ such that

$$\langle f(t,x,u)-f(t,y,u),x-y\rangle \leqslant \nu\|x-y\|^2$$

for all $x,y\in G$ and $(t,u)\in \operatorname{graph} U(\cdot)$.

Note that the constant ν can be negative.

The following property is an important well-known property of one-sided Lipschitz systems.

Lemma 13. *For any control $u(\cdot)$ and any two solutions $x_1(\cdot)$ and $x_2(\cdot)$ of the equation $\dot{x}(t)=f(t,x(t),u(t))$ with values in G and defined on an interval $[\tau,T]$,*

$$\|x_1(t)-x_2(t)\|\leqslant e^{\nu(t-\tau)}\|x_1(\tau)-x_2(\tau)\| \quad \textit{for any } t\in[\tau,T].$$

This property allows us to prove the following lemma.

Lemma 14. *If $f(\cdot,\cdot,\cdot)$ is one-sided Lipschitz, then*

$$\|Z_*(\tau)[Z_*(s)]^{-1}\|\leqslant \sqrt{n}\, e^{\nu(s-\tau)} \quad \textit{for all } \tau,s\in[0,\infty),\ \tau\leqslant s.$$

Proof. Let us fix an arbitrary τ and s as in the formulation of the lemma. Let $x_i(\cdot)$ be the solution of the equation $\dot{x}(t)=f(t,x(t),u_*(t))$ with $x_i(\tau)=x_*(\tau)+\alpha e_i$, where e_i is the ith canonical unit vector in $\mathbb{R}^n$ and α is a positive scalar. Clearly, $x_i(\cdot)$ exists in G on $[\tau,s]$ for all sufficient small $\alpha>0$.

It is a known (see [2], Chap. 2.5.6, for example) that under our standing assumptions

$$x_i(t)=x_*(t)+\alpha y_i(t)+o(\alpha,t), \qquad t\in[\tau,s],$$

where $\|o(\alpha,t)\|/\alpha\to 0$ as $\alpha\to 0$ uniformly with respect to $t\in[\tau,s]$, and $y_i(\cdot)$ is the solution of the equation $\dot{y}(t)=f_x(t,x_*(t),u_*(t))y(t)$ with $y(\tau)=e_i$. This solution, however, has the form $y_i(t)=[Z_*(\tau)^*]^{-1}Z_*(t)^*e_i$. That is, $y_i(t)$ is the ith row of the matrix $Z_*(\tau)[Z_*(t)]^{-1}$. Hence

$$\|Z_*(\tau)[Z_*(s)]^{-1}\|=\biggl(\sum_{i=1}^n\|y_i(s)\|^2\biggr)^{1/2}=\biggl(\sum_{i=1}^n\biggl(\frac{\|x_i(s)-x^*(s)-o(\alpha,t)\|}{\alpha}\biggr)^2\biggr)^{1/2}.$$

Using Lemma 13 and taking the limit as $\alpha\to 0$, we get the desired inequality. □

Using Lemma 14 and Assumption (B2), we can estimate the norm of the adjoint vector $\psi(t)$, $t\geqslant 0$, defined by (12) as follows:

$$\begin{aligned}\|\psi(t)\|&\leqslant \int_t^\infty \|Z_*(t)[Z_*(s)]^{-1}\|\,\|f_x^0(s,x_*(s),u_*(s))\|\,ds\\&\leqslant \int_t^\infty \sqrt{n}\,e^{\nu(s-t)}\kappa(1+c_1^r e^{\mu r s})e^{-\rho s}\,ds\leqslant c_3 e^{-(\rho-r\mu)t},\end{aligned} \tag{30}$$

where $c_3 \geqslant 0$ is a suitable constant. This estimation leads to the next corollary of Theorem 8. In its formulation we use the weighted space $L_\infty(e^{\gamma t}; [0,\infty))$ consisting of all measurable functions $\psi\colon [0,\infty) \to \mathbb{R}^n$ for which the norm

$$\|\psi(\,\cdot\,)\|_{\infty,\gamma} := \operatorname*{ess\,sup}_{t\in[0,\infty)} e^{\gamma t}\|\psi(t)\|$$

is finite.

Corollary 15. *Assume that the function $f(\,\cdot\,,\,\cdot\,,\cdot)$ is one-sided Lipschitz in the sense of Definition 12. Let $(x_*(\,\cdot\,), u_*(\,\cdot\,))$ be a weakly overtaking optimal pair in the problem (P), and let Assumptions (A1) and (B1) hold without requiring the inequality (29). Assume also that (B2) holds with the number ν in Definition 12. Then the function $\psi\colon [0,\infty) \to \mathbb{R}^n$ defined by (12) is locally absolutely continuous, and the conditions (i) and (ii) in Theorem 8 are satisfied. Moreover, the function $\psi(\,\cdot\,)$ is the unique solution of the adjoint equation (15) belonging to the weighted space $L_\infty(e^{(\rho-r\mu)t}; [0,\infty))$.*

Proof. The inequality in Lemma 13, applied with $x_1(\,\cdot\,) = x_*(\,\cdot\,)$, $x_2(\,\cdot\,) = x(\zeta;\,\cdot\,)$, and $\tau = 0$, implies (29) in (B1). Then Assumption (A2) holds by Corollary 11. Thus, the first part of the corollary follows from Theorem 8.

The inequality (30) establishes that $\psi(\,\cdot\,) \in L_\infty(e^{(\rho-r\mu)t}; [0,\infty))$. Let $\widetilde{\psi}(\,\cdot\,)$ be another solution of (15) which belongs to $L_\infty(e^{(\rho-r\mu)t}; [0,\infty))$. Then for any $t \geqslant 0$ we have

$$\psi(0) - \widetilde{\psi}(0) = [Z_*(t)]^{-1}(\psi(t) - \widetilde{\psi}(t)).$$

Hence,

$$\begin{aligned}\|\psi(0) - \widetilde{\psi}(0)\| &\leqslant \|[Z_*(t)]^{-1}\|(\|\psi(t)\| + \|\widetilde{\psi}(t)\|)\\ &\leqslant \sqrt{n}\, e^{\nu t} e^{-(\rho-r\mu)t}(\|\psi(\,\cdot\,)\|_{\infty,\rho-r\mu} + \|\widetilde{\psi}(\,\cdot\,)\|_{\infty,\rho-r\mu})\\ &\leqslant c_4 e^{-(\rho-\nu-r\mu)t}, \qquad t \geqslant 0,\end{aligned}$$

for an appropriate constant $c_4 \geqslant 0$ (which may depend on $\widetilde{\psi}(\,\cdot\,)$). Since the right-hand side goes to zero as $t \to \infty$, we get that $\|\psi(0) - \widetilde{\psi}(0)\| = 0$. $\square$

The next corollary connects the relation

$$\psi(\,\cdot\,) \in L_\infty(e^{(\rho-r\mu)t}; [0,\infty))$$

provided by Corollary 15 with the asymptotic conditions (7) and (8).

Corollary 16. *If the assumptions of Corollary 15 hold and also $\rho > r\mu$, then (7) is valid. Moreover, if in addition to the assumptions of Corollary 15 the stronger inequality $\rho > (r+1)\mu$ holds, then both asymptotic conditions (7) and (8) are valid.*

Proof. Note first that since $\mu \geqslant 0$ and $r \geqslant 0$ (see (B1)), both $\rho > r\mu$ and $\rho > (r+1)\mu$ imply that $\rho > 0$. Under the assumptions of Corollary 15 we have $\psi(\,\cdot\,) \in L_\infty(e^{(\rho-r\mu)t}; [0,\infty))$. This means that there is a constant $c_3 \geqslant 0$ such that the inequality (30) holds. Hence, the inequality $\rho > r\mu$ implies the asymptotic condition (7) in this case. Further, due to the condition (B1.i) we have $\|x_*(t)\| \leqslant ce^{\mu t}$ for $t \geqslant 0$. Therefore, the stronger inequality $\rho > (r+1)\mu$ implies both asymptotic conditions (7) and (8) in this case. $\square$

3.2. Systems with regular linearization. Here we consider another special case where Assumption (B1) takes a more explicit form.

First we recall a few facts from the stability theory of linear systems (for more details see [28] and [33], for example). Consider a linear differential system

$$\dot{y}(t) = A(t)y(t), \tag{31}$$

where $t \in [0, \infty)$, $y \in \mathbb{R}^n$, and all the components of the real $n \times n$ matrix function $A(\,\cdot\,)$ are bounded measurable functions.

Let $y(\,\cdot\,)$ be a non-zero solution of the system (31). Then

$$\widetilde{\lambda} = \limsup_{t\to\infty} \frac{1}{t} \log \|y(t)\|$$

is called the *characteristic Lyapunov exponent* or, briefly, the *characteristic exponent* of the solution $y(\,\cdot\,)$. The characteristic exponent $\widetilde{\lambda}$ of any non-zero solution $y(\,\cdot\,)$ of (31) is finite. The set of characteristic exponents corresponding to all non-zero solutions of (31) is called the *spectrum* of the system. The spectrum always consists of at most n different numbers.

The solutions of (31) form a finite-dimensional linear space of dimension n. Any basis of this space (any set of n linearly independent solutions $y_1(\,\cdot\,), \dots, y_n(\,\cdot\,)$) is called a *fundamental system* of solutions of (31). A fundamental system of solutions $y_1(\,\cdot\,), \dots, y_n(\,\cdot\,)$ is said to be *normal* if the sum of the characteristic exponents of these solutions $y_1(\,\cdot\,), \dots, y_n(\,\cdot\,)$ is minimal among all fundamental systems of solutions of (31).

It turns out that a normal fundamental system of solutions of (31) always exists. If $y_1(\,\cdot\,), \dots, y_n(\,\cdot\,)$ is a normal fundamental system of solutions, then its characteristic exponents cover the whole spectrum of (31). This means that for any number $\widetilde{\lambda}$ in the spectrum $\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_l$ of the system (31) there exists a solution in the set $y_1(\,\cdot\,), \dots, y_n(\,\cdot\,)$ that has this number as its characteristic exponent. Note that different members $y_j(\,\cdot\,)$ and $y_k(\,\cdot\,)$ of the fundamental system $y_1(\,\cdot\,), \dots, y_n(\,\cdot\,)$ may have the same characteristic exponent. Denote by n_s the multiplicity of the characteristic exponent $\widetilde{\lambda}_s$, $s = 1, \dots, l$, in the spectrum of (31). Any normal fundamental system contains the same number n_s of solutions of (31) with characteristic number $\widetilde{\lambda}_s$, $1 \leqslant s \leqslant l$, in the Lyapunov spectrum of (31).

Let

$$\sigma = \sum_{s=1}^{l} n_s \widetilde{\lambda}_s.$$

The linear system (31) is said to be *regular* if

$$\sigma = \liminf_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{trace} A(s)\, ds,$$

where $\operatorname{trace} A(s)$ is the sum of all the elements of $A(s)$ on the principal diagonal.

Note that a differential system (31) with constant matrix $A(t) \equiv A$, $t \geqslant 0$, is always regular. In this case the maximal element $\overline{\lambda}$ of the spectrum of (31) equals the maximal real part of the eigenvalues of A. Another important class of regular differential systems consists of systems (31) with periodic components (with the same period) of the matrix $A(\,\cdot\,)$.

It is known (for example, see [33]), that if the system (31) is regular, then for any $\varepsilon > 0$

$$\|Z_*(\tau)[Z_*(s)]^{-1}\| \leqslant c(\varepsilon)e^{\overline{\lambda}(s-\tau)+\varepsilon s} \quad \text{for any } \tau, s \in [0,\infty),\ \tau \leqslant s. \tag{32}$$

The inequality (32) is similar to the inequality in the assertion of Lemma 14 above. As in the case of Lemma 14, (32) leads to the following corollary of Theorem 8.

Corollary 17. *Let $(x_*(\,\cdot\,), u_*(\,\cdot\,))$ be a weakly overtaking optimal pair in the problem (P), and let Assumptions (A1) and (B1) hold for this pair, except without requiring (29). Let the linearized system*

$$\dot{y}(t) = f_x(t, x_*(t), u_*(t))y \tag{33}$$

be regular. Assume also that (B2) holds with ν greater than or equal to the maximal element $\overline{\lambda}$ of the spectrum of (33). Then the function $\psi\colon [0,\infty) \to \mathbb{R}^n$ defined by (12) is locally absolutely continuous and the conditions (15) and (16) in Theorem 8 are satisfied. Moreover, $\psi(\,\cdot\,)$ is the unique solution in $L_\infty(e^{(\rho-r\mu)t}; [0,\infty))$ of the adjoint system (15).

Essentially, the proof repeats the argument in the proof of Lemma 14 above (see also [10], § 5, and [14], Corollary 2).

Like Corollary 16, the following result connects the relation $\psi(\,\cdot\,) \in L_\infty(e^{(\rho-r\mu)t}; [0,\infty))$ provided by Corollary 17 with the asymptotic conditions (7) and (8).

Corollary 18. *If the assumptions of Corollary 17 hold and also $\rho > r\mu$, then the asymptotic condition (7) is valid. Moreover, if in addition to the assumptions of Corollary 17 the stronger inequality $\rho > (r+1)\mu$ holds, then both asymptotic conditions (7) and (8) are valid.*

4. Applications in economics

In this section we discuss economic interpretations of the adjoint variable $\psi(\,\cdot\,)$ given by (12) in view of Theorem 8 and in comparison with the dynamic programming principle [20]. Then we present applications of Theorem 8 to two basic optimal economic growth models.

4.1. Economic interpretations. First, we note that the traditional interpretation of the components of the adjoint vector $\psi(t)$, $t \geqslant 0$, as the present-value *shadow prices* of the corresponding components (types) of the optimal capital stock $x_*(t)$ is based on the identification of the present net value of the capital stock vector $x_*(t)$ with the value $V(t, x_*(t))$ of the optimal value function $V(\,\cdot\,,\cdot\,)$, and on subsequent use of the dynamic programming method (see [34] and [1], Chap. 7). Let us recall these standard constructions in optimal control theory.

Consider the following family $\{(\mathrm{P}(\tau,\zeta))\}_{\tau\geqslant 0,\zeta\in G}$ of optimal control problems:

$$J_\tau(x(\,\cdot\,),u(\,\cdot\,)) = \int_\tau^\infty f^0(s,x(s),u(s))\,ds \to \max,$$
$$\dot{x}(t) = f(t,x(t),u(t)), \quad x(\tau)=\zeta, \qquad u(t)\in U(t).$$

Here, the initial time $\tau \geqslant 0$ and the initial state $\zeta \in G$ are regarded as parameters. Admissible pairs $(x(\,\cdot\,),u(\,\cdot\,))$ in the problem $(\mathrm{P}(\tau,\zeta))$ are defined as in the problem (P), but with the initial data (τ,ζ) instead of $(0,x_0)$. Thus, $P(0,x_0)$ is identical to (P).

Assume now that the problem $(\mathrm{P}(\tau,\zeta))$ has a strongly optimal solution for any $(\tau,\zeta)\in[0,\infty)\times G$. Then we can define the corresponding optimal value function $V(\,\cdot\,,\cdot\,)$ of the variables $\tau\in[0,\infty]$ and $\zeta\in G$ as follows:

$$V(\tau,\zeta) = \max_{(x(\,\cdot\,),u(\,\cdot\,))} J_\tau(x(\,\cdot\,),u(\,\cdot\,)). \tag{34}$$

Here the maximum is taken over all admissible pairs $(x(\,\cdot\,),u(\,\cdot\,))$ in the problem $(\mathrm{P}(\tau,\zeta))$.

Let $(x_*(\,\cdot\,),u_*(\,\cdot\,))$ be a strongly optimal pair in (P). If $V(\,\cdot\,,\cdot\,)$ is a twice continuously differentiable function in some open neighbourhood of graph $x_*(\,\cdot\,)$, then by applying the dynamic programming approach, it is not difficult to show that all the conditions of the maximum principle (Theorem 5) hold in the normal form ($\psi^0=1$) with the adjoint variable $\psi(\,\cdot\,)$ defined along the optimal trajectory $x_*(\,\cdot\,)$ by

$$\psi(t) = \frac{\partial V(t,x_*(t))}{\partial x}, \qquad t\geqslant 0. \tag{35}$$

By the definition of the value function $V(\,\cdot\,,\cdot\,)$, one can identify the present value of the capital vector $\zeta\in G$ at time $\tau\geqslant 0$ with $V(\tau,\zeta)$. Then by (35), at each time $t\geqslant 0$ the components of $\psi(t)$ can be interpreted as the present-value *marginal prices* (also called *shadow prices*) of the corresponding components of the capital vector $x_*(t)$. This observation gives an economic meaning to the relations of the maximum principle.

Note that the optimal value function $V(\,\cdot\,,\cdot\,)$ is not necessarily differentiable. However, the differentiability of $V(t,\,\cdot\,)$ at the point $x_*(t)$, $t\geqslant 0$, is of critical importance for the interpretation of the vector $\psi(t)$ that appears in the maximum principle relations (Theorem 5) as the vector of marginal prices. Indeed, $\psi(t)$ being the marginal price vector at $x_*(t)$ means that

$$V(t,x_*(t)+\Delta x) = V(t,x_*(t)) + \langle\psi(t),\Delta x\rangle + o(\|\Delta x\|)$$

for any increment vector Δx, where $o(\|\Delta x\|)/\|\Delta x\|\to 0$ as $\Delta x\to 0$. This implies the (Fréchet) differentiability of $V(t,\,\cdot\,)$ at $x_*(t)$.

It turns out that under the assumptions of Theorem 8 the adjoint variable $\psi(\,\cdot\,)$ defined by (12) can be interpreted as the function of *integrated intertemporal prices*, without any *a priori* assumptions about the optimal value function $V(\,\cdot\,,\cdot\,)$. We explain this interpretation in the next paragraphs.

Let $(x(\cdot), u(\cdot))$ be an admissible (not necessarily optimal) pair in (P) for which Assumption (A1) holds (with $(x(\cdot), u(\cdot))$ instead of $(x_*(\cdot), u_*(\cdot))$; see also Remark 4). Fix an arbitrary $s > 0$. By the theorems on continuous dependence and differentiability of solutions of the Cauchy problem with respect to the initial conditions (see [2], § 2.5.5 and § 2.5.6), for any $\tau \in [0, s)$ there is an open neighbourhood $\mathscr{V}_s(\tau) \subset G$ of $x(\tau)$ such that for any $\zeta \in \mathscr{V}(\tau)$ the solution $x(\tau, \zeta; \cdot)$ of the Cauchy problem

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(\tau) = \zeta,$$

exists on $[\tau, s]$ and has values in G, and the function $x(\tau, \cdot\,; s)\colon \mathscr{V}(\tau) \to \mathbb{R}^n$ is continuously (Fréchet) differentiable. Moreover, the following equality holds:

$$x_\zeta(\tau, x(\tau); s) = [Z(\tau)[Z(s)]^{-1}]^*, \tag{36}$$

where (consistently with our previous notation) $Z(t)$ is the fundamental matrix solution of the linear differential equation

$$\dot{z}(t) = -[f_x(t, x(t), u(t))]^* z(t)$$

normalized at $t = 0$, so that $[Z(\tau)[Z(s)]^{-1}]^*$ is the state transition (Cauchy) matrix of the linearized system

$$\dot{y}(t) = f_x(t, x(t), u(t))y.$$

Now we define the *intertemporal instantaneous utility function* $\pi(\tau, \cdot\,; s)$ on $\mathscr{V}(\tau)$ by

$$\pi(\tau, \zeta, s) = f^0(s, x(\tau, \zeta; s), u(s)), \qquad \zeta \in \mathscr{V}(\tau). \tag{37}$$

In essence, $\pi(\tau, \zeta, s)$ is the *instantaneous utility* gained at the time s by the capital stock ζ at the time τ after transition of the system from the state ζ to the state $x(\tau, \zeta; s)$ via the control $u(\cdot)$ given on the time interval $[\tau, s]$. Hence, one can interpret the vector $\pi_\zeta(\tau, x(\tau), s)$ as the vector of *intertemporal prices* corresponding to the capital stock $x(\tau)$.

Due to the properties of the functions $f^0(\cdot\,, \cdot\,, \cdot)$ and $x(\tau, \cdot\,; s)$, the function $\pi(\tau, \cdot, s)$ defined by (37) is differentiable at $x(\tau)$. Using the chain rule and taking into account the equality $x(\tau, x(\tau), s) = x(s)$ and (36), we get that

$$\begin{aligned}
\pi_\zeta(\tau, x(\tau), s) &= \bigl[[f^0_x(s, x(s), u(s))]^* x_\zeta(\tau, x(\tau); s)\bigr]^* \\
&= \bigl[[f^0_x(s, x(s), u(s))]^* [Z(\tau)[Z(s)]^{-1}]^*\bigr]^* \\
&= Z(\tau)[Z(s)]^{-1} f^0_x(s, x(s), u(s)).
\end{aligned} \tag{38}$$

Note that $s > 0$ was arbitrarily chosen, and thus the function $(t, s) \mapsto \pi_\zeta(t, x(t), s)$ is defined for all $s > 0$ and $t \in [0, s)$. Moreover, the representation (38) implies that this function is Lebesgue measurable. Thus, we can define the function

$$\mu(t) = \int_t^\infty \pi_\zeta(t, x(t), s)\, ds, \qquad t \geqslant 0, \tag{39}$$

provided that the integral converges for any $t \geqslant 0$. Therefore, the *integrated intertemporal prices function* $\mu(\cdot)$ is defined by (39) along any (not necessarily optimal) admissible trajectory $x(\cdot)$ in the problem (P). Note that only Assumption (A1) and the convergence of the improper integral in (39) are needed to define the integrated intertemporal prices function $\mu(\cdot)$. We require no smoothness, Lipschitzness, continuity, nor even finiteness assumptions on the corresponding optimal value function $V(\cdot,\cdot)$ in a neighbourhood of the admissible trajectory $x(\cdot)$ under consideration.

Now let $(x_*(\cdot), u_*(\cdot))$ be a weakly overtaking optimal admissible pair in (P) and let Assumption (A1) be satisfied for this pair. The matrix function $Z(\cdot)$ and the function $\mu(\cdot)$ associated with the pair $(x_*(\cdot), u_*(\cdot))$ will be denoted by $Z_*(\cdot)$ and $\mu_*(\cdot)$, respectively. From (38) and (39) we have

$$\mu_*(t) = Z_*(t) \int_t^\infty [Z_*(s)]^{-1} f_x^0(s, x_*(s), u_*(s))\, ds, \qquad t \geqslant 0. \tag{40}$$

If the above integral is finite for every $t \geqslant 0$, then $\mu_*(\cdot)$ coincides with the function $\psi(\cdot)$ defined in (12) and appearing in the formulation of Theorem 8. Observe that if Assumption (A2) also holds for $(x_*(\cdot), u_*(\cdot))$, then by Lemma 7 the improper integral in (40) converges for any $t \geqslant 0$, and thus $\mu_*(\cdot) = \psi(\cdot)$ is well defined on $[0,\infty)$. Hence, under Assumptions (A1) and (A2) the adjoint variable $\psi(\cdot)$ in Theorem 8 coincides with the integrated intertemporal prices function $\mu_*(\cdot)$.

Assumption (A2) is sufficient but not necessary for the finiteness of the integral in (40) for all $t \geqslant 0$. Given also that the function $\mu_*(\cdot)$ has the economic meaning of the integrated intertemporal prices function, it is natural to ask whether Assumption (A2) in Theorem 8 can be relaxed to the condition of convergence of the improper integral in (12) or (40). The answer to this question is negative, as Example 10 shows. It can happen (if (A2) fails) that for a unique strongly optimal admissible pair $(x_*(\cdot), u_*(\cdot))$ in the problem (P) Assumption (A1) is satisfied, the corresponding improper integral in (12) converges absolutely, and the general maximum principle (Theorem 5) holds in the normal form with an adjoint variable $\psi(\cdot)$ which is not equal to the integrated intertemporal prices function $\mu(\cdot)$, although $\mu(\cdot)$ is well defined by (40). Thus, in general the adjoint variable $\psi(\cdot)$ that appears in the normal-form conditions of the general maximum principle (Theorem 5) can be something different from the integrated intertemporal prices function $\mu(\cdot)$, while under the conditions of Theorem 8 these functions coincide. Assumption (A2) is not only needed to ensure the finiteness of $\mu_*(\cdot)$ via Lemma 7; it is also essential for the proof of Theorem 8.

Consider now a weakly overtaking optimal admissible pair $(x_*(\cdot), u_*(\cdot))$ in (P) for which the assumptions of Theorem 8 (that is, Assumptions (A1) and (A2)) are satisfied, and in addition, $J(x_*(\cdot), u_*(\cdot))$ in (1) is finite. In this case the assertion of Theorem 8 can be strengthened.

By (A2), there is an open neighbourhood Ω of the set $\operatorname{graph} x_*(\cdot)$ such that the integral below converges for any $(\tau,\zeta) \in \Omega$, and hence the following *conditional value* function $W(\cdot,\cdot)\colon \Omega \to \mathbb{R}^1$ is well defined:

$$W(\tau,\zeta) = \int_\tau^\infty \pi(\tau,\zeta;s)\, ds, \qquad (\tau,\zeta) \in \Omega.$$

Note that the optimal value function $V(\,\cdot\,,\cdot\,)$ (see (34)) is not necessarily defined in this case. In essence, the quantity $W(\tau,\zeta)$, $(\tau,\zeta)\in\Omega$, has the economic meaning of the integrated intertemporal value of the capital vector ζ at the time τ (under the condition that the given investment plan $u_*(\,\cdot\,)$ is realized for the initial capital vector ζ at the initial time τ on the whole infinite time interval $[\tau,\infty)$).

The following result strengthens the assertion of Theorem 8 under the additional assumption of convergence of the improper integral in (1).

Theorem 19. *Let $(x_*(\,\cdot\,),u_*(\,\cdot\,))$ be a locally weakly overtaking optimal pair in the problem (P) for which Assumptions (A1) and (A2) are satisfied, and suppose that the integral in (1) converges to the finite value $J(x_*(\,\cdot\,),u_*(\,\cdot\,))$. Then the following assertions hold.*

(i) *For any $t\geqslant 0$ the partial (Fréchet) derivative $W_x(t,x_*(t))$ exists. Moreover, the vector function $\psi(\,\cdot\,)\colon[0,\infty)\to\mathbb{R}^n$ defined by*

$$\psi(t)=W_x(t,x_*(t)),\qquad t\geqslant 0,$$

is locally absolutely continuous and satisfies the core conditions (15) and (16) of the maximum principle in the normal form for the problem (P).

(ii) *The partial derivative $W_t(t,x_*(t))$ exists for almost every $t\geqslant 0$, and*

$$W_t(t,x_*(t))+\sup_{u\in U(t)}\{\langle W_x(t,x_*(t)),f(t,x_*(t),u)\rangle+f^0(t,x_*(t),u)\}\overset{a.e.}{=}0.$$

The proof in [5] (§ 2) of Theorem 19 is based on the theorem on differentiability of solutions of the Cauchy problem with respect to the initial conditions, Theorem 8, and the fact that under the conditions of Theorem 19 we have $W_x(t,x_*(t))\equiv\mu(t)$, $t\geqslant 0$ (see (39)) and the equality (40).

In essence, the assertion (i) of the theorem is a reformulation of Theorem 8 in the economic terms of the function $W(\,\cdot\,,\cdot\,)$ under the additional assumption that the integral $J(x_*(\,\cdot\,),u_*(\,\cdot\,))$ converges. However, the assertion (ii) is a complementary fact. In particular, it lets us connect the adjoint variable $\psi(\,\cdot\,)$ in Theorem 19 with Michel's asymptotic condition (9).

Corollary 20. *Assume that the conditions of Theorem 19 are satisfied and that the problem (P) is autonomous with exponential discounting, that is,*

$$f(t,x,u)\equiv f(x,u),\quad f^0(t,x,u)\equiv e^{-\rho t}g(x,u),\quad \text{and}\quad U(t)\equiv U$$

for all $t\geqslant 0$, $x\in G$, and $u\in\mathbb{R}^m$, where $\rho\in\mathbb{R}^1$ is not necessarily positive. Then the following stationarity condition holds:

$$\mathscr{H}(t,x_*(t),u_*(t),\psi(t))\overset{a.e.}{=}\rho\int_t^\infty e^{-\rho s}g(x_*(s),u_*(s))\,ds,\qquad t\geqslant 0.\tag{41}$$

Proof. Indeed, for all $t\geqslant 0$ we have

$$W(t,x_*(t))=e^{-\rho t}\int_t^\infty e^{-\rho(s-t)}g(x_*(s),u_*(s))\,ds=e^{-\rho t}W(0,x_*(t)).$$

Hence

$$W_t(t,x_*(t)) = -\rho e^{-\rho t} W(0,x_*(t)) = -\rho \int_t^\infty e^{-\rho s} g(x_*(s),u_*(s))\,ds, \qquad t \geqslant 0.$$

By virtue of assertions (i) and (ii) of Theorem 19, this implies (41). $\square$

Finally, note that if the problem (P) is autonomous with discounting, and the usual regularity assumptions hold for the weakly overtaking optimal control $u_*(\,\cdot\,)$ and the functions $f(\,\cdot\,,\cdot\,)$ and $g(\,\cdot\,,\cdot\,)$ (see Remark 4), then the core conditions (15) and (16) of the normal-form maximum principle imply that the function

$$h(\,\cdot\,)\colon h(t) = H(t,x_*(t),\psi(t)) = \sup_{u\in U} \mathscr{H}(t,x_*(t),u,\psi(t)), \qquad t \geqslant 0,$$

is locally absolutely continuous and

$$\dot h(t) \stackrel{\text{a.e.}}{=} \frac{\partial H(t,x_*(t),\psi(t))}{\partial t} = -\rho g(x_*(t),u_*(t)), \qquad t \geqslant 0$$

(see [48], Chap. 2). Since the functional $J(x_*(\,\cdot\,),u_*(\,\cdot\,))$ converges in Theorem 19, the conditions (9) and (41) are equivalent in this case.

4.2. Two economic examples.

Example 21 (Ramsey's model). This example demonstrates the applicability of Theorem 8 to the Ramsey model of optimal economic growth (see [19], Chap. 2). This model is the most important construct in the modern theory of economic growth. It was first presented by Ramsey [49] in 1928 and then developed by Cass [27] and Koopmans [43] in the 1960s. It is also known in the literature as the Ramsey–Cass–Koopmans model. Here for simplicity of presentation we restrict our consideration to the canonical setting of the model with the Cobb–Douglas production function and the iso-elastic instantaneous utility function. For the case of the general neoclassical production function, see [23].

Below we present a rigorous analysis of the Ramsey model based on Theorem 8. We show that Theorem 8 is applicable, and hence the core conditions of the normal-form maximum principle hold with the adjoint variable $\psi(\,\cdot\,)$ specified by the formula (12). In this case (12) directly implies the asymptotic conditions (7) and (8).

Consider a closed aggregated economy that at each moment of time $t \geqslant 0$ produces a single homogeneous product in the quantity $Y(t) > 0$ in accordance with the Cobb–Douglas production function (see [19], Chap. 1):

$$Y(t) = AK(t)^\alpha L(t)^{1-\alpha}. \tag{42}$$

Here $A > 0$ is a technological coefficient, $0 < \alpha < 1$ is the output elasticity of capital, and $K(t) > 0$ and $L(t) > 0$ are the capital stock and the labour force available at the time $t \geqslant 0$, respectively.

In a closed economy, at each moment of time $t \geqslant 0$ a part $I(t) = u(t)Y(t)$, $u(t) \in [0,1)$, of the product produced is invested, while the remaining (non-vanishing) part

$C(t) = (1-u(t))Y(t)$ is consumed. Therefore, the capital dynamics can be described by the following differential equation:

$$\dot{K}(t) = u(t)Y(t) - \widetilde{\delta}K, \qquad K(0) = K_0 > 0, \tag{43}$$

where $\widetilde{\delta} > 0$ is the capital depreciation rate.

Assume that the labor resource $L(\,\cdot\,)$ grows exponentially, that is,

$$\dot{L}(t) = \mu L(t), \qquad L(0) = L_0 > 0, \tag{44}$$

where $\mu \geqslant 0$ is a constant. Assume also that the instantaneous utility function $g\colon (0,\infty) \to \mathbb{R}^1$ is iso-elastic (see [19], Chap. 2). In this case

$$g(c) = \begin{cases} \dfrac{c^{1-\sigma}-1}{1-\sigma}\,, & \sigma > 0,\ \sigma \neq 1, \\ \log c, & \sigma = 1, \end{cases} \tag{45}$$

where $c > 0$ is the per-capita consumption. Then with the new (capital-labour ratio) variable $x(t) = K(t)/L(t)$, $t \geqslant 0$, we arrive at the following optimal control problem (P3) in view of (42)–(45) and the homogeneity of the Cobb–Douglas production function (42):

$$J(x(\,\cdot\,), u(\,\cdot\,)) = \int_0^\infty e^{-\rho t} g((1-u(t))Ax(t)^\alpha)\,dt \to \max, \tag{46}$$

$$\dot{x}(t) = u(t)Ax(t)^\alpha - \delta x(t), \qquad x(0) = x_0 = \frac{K_0}{L_0}\,, \tag{47}$$

$$u(t) \in [0,1). \tag{48}$$

Here $\rho > 0$ is the social discount rate,

$$(1-u(t))Ax(t)^\alpha = \frac{C(t)}{L(t)}$$

is the per-capita consumption at the time $t \geqslant 0$, and $\delta = \widetilde{\delta} + \mu > 0$ is the adjusted depreciation rate.

Let $G = (0,\infty)$. Then any measurable function $u\colon [0,\infty) \to [0,1)$ is an admissible control in the problem (P3). Indeed, by (47) the trajectory $x(\,\cdot\,)$ corresponding to $u(\,\cdot\,)$ is defined on $[0,\infty)$ with values in G, and the integrand in (46), that is, the function $t \mapsto e^{-\rho t} g((1-u(t))Ax(t)^\alpha)$, is locally integrable on $[0,\infty)$. Thus, the trajectory $x(\,\cdot\,)$ is admissible. Moreover, by (45), (47), and (48) the integrand in (46) is bounded above by an exponentially decreasing function (uniformly with respect to all admissible pairs $(x(\,\cdot\,), u(\,\cdot\,))$). Hence, there is a decreasing non-negative function $\omega\colon [0,\infty) \to \mathbb{R}^1$ with $\lim_{t\to\infty} \omega(t) = 0$ such that for any $0 \leqslant T < T'$ the following inequality holds:

$$\int_T^{T'} e^{-\rho t} g((1-u(t))Ax(t)^\alpha)\,dt \leqslant \omega(T)$$

(see [7], § 2, Assumption (A3)). This implies that for any admissible pair $(x(\,\cdot\,), u(\,\cdot\,))$ the improper integral in (46) either converges to a finite number or diverges to $-\infty$, and $J(x(\cdot),u(\cdot)) \leqslant \omega(0)$ (see [7], § 2). Hence, in the case of the problem (P3) the concepts of strong optimality and weak overtaking optimality coincide. So everywhere below in this example we understand optimality of an admissible pair $(x_*(\cdot),u_*(\cdot))$ in the problem (P3) in the strong sense. In particular, if an optimal admissible pair $(x_*(\cdot),u_*(\cdot))$ exists, then $J(x_*(\cdot),u_*(\cdot))$ is finite.

We define an auxiliary state variable $y(\cdot)$ via the Bernoulli transformation:

$$y(t) = x(t)^{1-\alpha}, \qquad t \geqslant 0.$$

Then it can be easily seen that in terms of the state variable $y(\cdot)$ the problem (P3) takes the following equivalent form $(\widetilde{\mathrm{P3}})$:

$$\widetilde{J}(y(\cdot),u(\cdot)) = \int_0^\infty e^{-\rho t} g((1-u(t))Ay(t)^{\alpha/(1-\alpha)})\,dt \to \max,$$

$$\dot{y}(t) = (1-\alpha)Au(t) - (1-\alpha)\delta y(t), \qquad y(0) = y_0 = x_0^{1-\alpha}, \tag{49}$$

$$u(t) \in [0,1). \tag{50}$$

For the problem $(\widetilde{\mathrm{P3}})$ we again let $G = (0,\infty)$. Since

$$f(t,y,u) := (1-\alpha)Au - (1-\alpha)\delta y \quad \text{and} \quad f^0(t,y,u) := e^{-\rho t} g((1-u)Ay^{\alpha/(1-\alpha)}),$$
$$t \geqslant 0, \quad y \in G, \quad u \in [0,1),$$

in this problem, Assumption (A0) holds for these functions and the multivalued map $U(\cdot)\colon U(t) \equiv [0,1)$, $t \geqslant 0$ (see Remark 1). Obviously, an arbitrary measurable function $u\colon [0,\infty) \to \mathbb{R}^1$ satisfying the pointwise constraint (50) is an admissible control in $(\widetilde{\mathrm{P3}})$. Thus, $(\widetilde{\mathrm{P3}})$ is a particular case of the problem (P).

Further, in $(\widetilde{\mathrm{P3}})$ for all $t \geqslant 0$, $y > 0$, and $u \in [0,1)$ we have $f_y(t,y,u) \equiv -(1-\alpha)\delta$, and moreover

$$\begin{aligned} f^0_y(t,y,u) &= e^{-\rho t} \frac{dg((1-u)Ay^{\alpha/(1-\alpha)})}{dc} \frac{(1-u)A\alpha y^{\alpha/(1-\alpha)-1}}{1-\alpha} \\ &= \frac{(1-u)Ae^{-\rho t}\alpha y^{\alpha/(1-\alpha)-1}}{1-\alpha}[(1-u)Ay^{\alpha/(1-\alpha)}]^{-\sigma} \\ &= \frac{\alpha e^{-\rho t}}{(1-\alpha)y}[(1-u)Ay^{\alpha/(1-\alpha)}]^{1-\sigma} \end{aligned}$$

if $\sigma > 0$ and $\sigma \neq 1$, and

$$f^0_y(t,y,u) = \frac{\alpha e^{-\rho t}}{(1-\alpha)y}$$

if $\sigma = 1$.

Thus, for any $\sigma > 0$ and all $t \geqslant 0$, $y > 0$, and $u \in [0,1)$ we have

$$f^0_y(t,y,u) = \frac{\alpha e^{-\rho t}}{(1-\alpha)y}[(1-u)Ay^{\alpha/(1-\alpha)}]^{1-\sigma}. \tag{51}$$

Hence, Assumption (A1) is satisfied for any admissible pair $(y_*(\cdot),u_*(\cdot))$ in $(\widetilde{\mathrm{P3}})$.

Note that the control system (49) in $(\widetilde{P3})$ is linear. Hence, for an arbitrary admissible control $u_*(\cdot)$ and any initial state $y(0) = \zeta > 0$ the corresponding admissible trajectory $y(\zeta;\cdot)$ is given by the Cauchy formula

$$y(\zeta;t) = e^{-(1-\alpha)\delta t}\zeta + (1-\alpha)Ae^{-(1-\alpha)\delta t}\int_0^t e^{(1-\alpha)\delta s}u_*(s)\,ds, \qquad t \geqslant 0. \tag{52}$$

Let us show that Assumption (A2) is also satisfied for any optimal admissible pair $(y_*(\cdot), u_*(\cdot))$ in $(\widetilde{\mathrm{P3}})$ (if such a pair exists).

Take an arbitrary admissible pair $(y_*(\cdot), u_*(\cdot))$ such that $\widetilde{J}(y_*(\cdot), u_*(\cdot)) > -\infty$, and let $\beta = y_0/2$. Then by (51) and (52), for any ζ such that $|\zeta - y_0| < \beta$ and all $t \geqslant 0$ we get that

$$\begin{aligned}
&\max_{y\in[y(\zeta;t),y_*(t)]} |f^0_y(t,y,u_*(t))(y(\zeta;t)-y_*(t))| \\
&\quad= \frac{\alpha}{1-\alpha}\max_{y\in[y(\zeta;t),y_*(t)]}\frac{e^{-\rho t}e^{-(1-\alpha)\delta t}|\zeta-y_0|[(1-u_*(t))Ay^{\alpha/(1-\alpha)}]^{1-\sigma}}{y} \\
&\quad\leqslant \frac{\alpha|\zeta-y_0|}{1-\alpha}\max_{y\in[y(\zeta;t),y_*(t)]}\frac{e^{-\rho t}[(1-u_*(t))Ay^{\alpha/(1-\alpha)}]^{1-\sigma}}{y_0/2+(1-\alpha)A\int_0^t e^{(1-\alpha)\delta s}u_*(s)\,ds} \\
&\quad\leqslant \frac{2\alpha|\zeta-y_0|}{y_0(1-\alpha)}\max_{y\in[y(\zeta;t),y_*(t)]}\{e^{-\rho t}[(1-u_*(t))Ay^{\alpha/(1-\alpha)}]^{1-\sigma}\} = |\zeta-y_0|\lambda(t),
\end{aligned}$$

where

$$\lambda(t) = \frac{2\alpha}{y_0(1-\alpha)}\max_{y\in[y(\zeta;t),y_*(t)]}\{e^{-\rho t}[(1-u_*(t))Ay^{\alpha/(1-\alpha)}]^{1-\sigma}\}. \tag{53}$$

Note that for any $t \geqslant 0$ and $\zeta \in [y_0-\beta, y_0+\beta]$ the formula (52) (where the integral term is non-negative) implies the chain of inequalities

$$\frac{1}{2}y_*(t) \leqslant y\left(\frac{1}{2}y_0;t\right) \leqslant y(\zeta;t) \leqslant y\left(\frac{3}{2}y_0;t\right) \leqslant \frac{3}{2}y_*(t). \tag{54}$$

Due to the choice of β we have $\zeta \in [y_0/2, 3y_0/2]$. The monotonicity of the function $\zeta \mapsto y(\zeta;t)$ implies that $y(\zeta;t) \in [y(y_0/2;t), y(3y_0/2;t)]$, which together with (54) gives $[y(\zeta;t), y_*(t)] \subset [y_*(t)/2, 3y_*(t)/2]$. Thus,

$$\lambda(t) \leqslant \frac{2\alpha}{y_0(1-\alpha)}\max_{y\in[y_*(t)/2,3y_*(t)/2]}\{e^{-\rho t}[(1-u_*(t))Ay^{\alpha/(1-\alpha)}]^{1-\sigma}\}, \qquad t \geqslant 0.$$

In view of the monotonicity of the function in the braces with respect to y (it is non-increasing for $\sigma \in (0,1]$ and non-decreasing for $\sigma \geqslant 1$) we have

$$\begin{aligned}
0 \leqslant \lambda(t) \leqslant \frac{2\alpha}{y_0(1-\alpha)}e^{-\rho t}\max\left\{\left(\frac{1}{2}\right)^{\alpha(1-\sigma)/(1-\alpha)}, \left(\frac{3}{2}\right)^{\alpha(1-\sigma)/(1-\alpha)}\right\} \\
\times [(1-u_*(t))Ay_*(t)^{\alpha/(1-\alpha)}]^{1-\sigma}.
\end{aligned} \tag{55}$$

Since $\widetilde{J}(y_*(\cdot), u_*(\cdot)) > -\infty$, the function

$$t \mapsto e^{-\rho t}g\big((1-u_*(t))Ay_*(t)^{\alpha/(1-\alpha)}\big)$$

is integrable on $[0,\infty)$. Then the function

$$t \mapsto e^{-\rho t}[(1-u_*(t))Ay_*(t)^{\alpha/(1-\alpha)}]^{1-\sigma} = e^{-\rho t}\big[1+(1-\sigma)g\big((1-u_*(t))Ay_*(t)^{\alpha/(1-\alpha)}\big)\big]$$

is also integrable. By (55) this implies that the function $\lambda(\,\cdot\,)$ defined in (53) is integrable on $[0,\infty)$. Thus, Assumption (A2) is satisfied for arbitrary $\sigma > 0$ and all admissible pairs $(y_*(\,\cdot\,), u_*(\,\cdot\,))$ with $\widetilde{J}(y_*(\,\cdot\,), u_*(\,\cdot\,)) > -\infty$.

Therefore, for arbitrary $\sigma > 0$ and any optimal admissible pair $(y_*(\,\cdot\,), u_*(\,\cdot\,))$ in $(\widetilde{\text{P3}})$ all the assumptions of Theorem 8 are satisfied. Hence, for any optimal admissible pair $(y_*(\,\cdot\,), u_*(\,\cdot\,))$ in $(\widetilde{\text{P3}})$ the core conditions (15) and (16) of the normal-form maximum principle hold with the adjoint variable $\psi(\,\cdot\,)$ specified by (12) (see (51) and (52)):

$$\begin{aligned}\psi(t) &= \frac{\alpha e^{(1-\alpha)\delta t}}{1-\alpha}\int_t^\infty \frac{e^{-(1-\alpha)\delta s}e^{-\rho s}[(1-u_*(s))Ay_*(s)^{\alpha/(1-\alpha)}]^{1-\sigma}}{y_*(s)}\,ds\\ &= \frac{\alpha e^{(1-\alpha)\delta t}}{1-\alpha}\int_t^\infty \frac{e^{-\rho s}[(1-u_*(s))Ay_*(s)^{\alpha/(1-\alpha)}]^{1-\sigma}}{y_0+(1-\alpha)A\int_0^s e^{(1-\alpha)\delta\tau}u_*(\tau)\,d\tau}\,ds, \qquad t \geqslant 0.\end{aligned}$$

Replacing $\int_0^s$ on the right-hand side with $\int_0^t$ (which is not larger) and using (52), we obtain the following relations:

$$\begin{aligned}0 < \psi(t)y_*(t) &\leqslant \frac{\alpha}{(1-\alpha)}\int_t^\infty e^{-\rho s}[(1-u_*(s))Ay_*(s)^{\alpha/(1-\alpha)}]^{1-\sigma}\,ds\\ &= \frac{\alpha}{(1-\alpha)}\int_t^\infty e^{-\rho s}\big[1+(1-\sigma)g\big((1-u_*(s))Ay_*(s)^{\alpha/(1-\alpha)}\big)\big]\,ds\\ &= \frac{\alpha e^{-\rho t}}{(1-\alpha)\rho} + \frac{\alpha(1-\sigma)}{1-\alpha}\int_t^\infty e^{-\rho s}g\big((1-u_*(s))Ay_*(s)^{\alpha/(1-\alpha)}\big)\,ds. \end{aligned}\tag{56}$$

Note that (56) is stronger than the asymptotic condition (8).

Introducing the current-value adjoint variable $p(\,\cdot\,)$, $p(t) = e^{\rho t}\psi(t)$, $t \geqslant 0$, we now arrive at the current-value adjoint system

$$\dot{p}(t) = \big((1-\alpha)\delta+\rho\big)p(t) - \frac{\alpha}{(1-\alpha)y(t)}[(1-u_*(t))Ay(t)^{\alpha/(1-\alpha)}]^{1-\sigma} \tag{57}$$

(see (i) in Theorem 8 and (51)) and the current-value maximum condition

$$\begin{aligned}&(1-\alpha)Au_*(t)p(t) + g\big((1-u_*(t))Ay_*(t)^{\alpha/(1-\alpha)}\big)\\ &\quad\stackrel{\text{a.e.}}{=} \max_{u\in[0,1)}\big\{(1-\alpha)Aup(t) + g\big((1-u)Ay_*(t)^{\alpha/(1-\alpha)}\big)\big\}\end{aligned}\tag{58}$$

(see (ii) in Theorem 8).

Since for any $\sigma > 0$ the iso-elastic function $g(\,\cdot\,)$ is strictly concave (see (45)), the current-value maximum condition (58) implies that $u_*(t) \stackrel{\text{a.e.}}{=} u_*(y_*(t), p(t))$, where for any $y > 0$ and $p > 0$ the feedback $u_*(y,p)$ is defined via the unique solution of the equation

$$(1-\alpha)Ap + \frac{d}{du}g\big((1-u)Ay^{\alpha/(1-\alpha)}\big) = (1-\alpha)Ap - \frac{(Ay^{\alpha/(1-\alpha)})^{1-\sigma}}{(1-u)^\sigma} = 0,$$

that is,

$$u_*(y,p)=\begin{cases}1-\dfrac{y^{\alpha(1-\sigma)/(\sigma(1-\alpha))}}{A(1-\alpha)^{1/\sigma}p^{1/\sigma}} & \text{if } p>\dfrac{1}{A(1-\alpha)}y^{\alpha(1-\sigma)/(1-\alpha)},\\ 0 & \text{if } p\leqslant\dfrac{1}{A(1-\alpha)}y^{\alpha(1-\sigma)/(1-\alpha)}.\end{cases}\tag{59}$$

Substituting $u_*(y(t),p(t))$ defined in (59) into the control system (49) and into the adjoint system (57) in place of $u_*(t)$, we arrive at the following normal-form current-value Hamiltonian system of the maximum principle:

$$\dot y(t)=(1-\alpha)Au_*(y(t),p(t))-(1-\alpha)\delta y(t),\tag{60}$$

$$\dot p(t)=\big((1-\alpha)\delta+\rho\big)p(t)-\frac{\alpha}{(1-\alpha)y(t)}[(1-u_*(y(t),p(t)))Ay(t)^{\alpha/(1-\alpha)}]^{1-\sigma}.\tag{61}$$

By Theorem 8 an optimal admissible trajectory $y_*(\cdot)$ (if there is any) together with the corresponding current-value adjoint variable $p(\cdot)$ must satisfy the system (60), (61), as well as the initial condition $y(0)=y_0=x_0^{1-\alpha}$ and the estimate (56).

By the linearity of the equation (49) and the concavity of the iso-elastic function $g(\cdot)$ for any $\sigma>0$ (see (45)), the Hamiltonian in the problem $(\widetilde{P3})$ is a concave function of the state variable $y>0$. This fact, together with (56), implies that all the conditions in Arrow's theorem on sufficient conditions for optimality (see [53], Theorem 10) are satisfied. Thus, any solution $(y_*(\cdot),p(\cdot))$ of the system (60), (61) on $[0,\infty)$ which satisfies the initial condition $y(0)=y_0=x_0^{1-\alpha}$ and the estimate (56) corresponds to the optimal admissible pair $(y_*(\cdot),u_*(\cdot))$, where $u_*(t)=u_*(y_*(t),p(t))$, $t\geqslant 0$. Thus, the assertion of Theorem 8 is necessary and sufficient (a criterion) for the optimality of an admissible pair $(y_*(\cdot),u_*(\cdot))$ in $(\widetilde{\mathrm{P3}})$.

A direct analysis (which we omit here) shows that for any $\sigma>0$ and an arbitrary initial condition $y_0>0$ there is a unique solution $(y_*(\cdot),p(\cdot))$ of the system (60), (61) which satisfies both the initial condition $y(0)=y_0$ and the estimate (56). Hence, for any initial state $y_0>0$ there is a unique optimal admissible pair $(y_*(\cdot),u_*(\cdot))$ in $(\widetilde{\mathrm{P3}})$. It can also be shown that the solution $(y_*(\cdot),p(\cdot))$ tends asymptotically to the unique equilibrium $(\hat y,\hat p)$ (of saddle type) of the system (60), (61). Since $p(t)\to\hat p$ and $y_*(t)\to\hat y$ as $t\to\infty$, it is obvious that both standard asymptotic conditions (7) and (8) hold in this example.

Finally, returning to the initial state variable $x_*(t)=y_*(t)^{1/(1-\alpha)}$, $t\geqslant 0$, we get a unique optimal admissible pair $(x_*(\cdot),u_*(\cdot))$ in the problem (P3).

Example 22 (Model of optimal extraction of a non-renewable resource). Here we apply Theorem 8 to the basic model of optimal extraction of a non-renewable resource. We remark that the issue of the optimal use of an exhaustible resource was first raised by Hotelling [39] in 1931. A model involving both man-made capital and an exhaustible resource (now commonly called the Dasgupta–Heal–Solow–Stiglitz (DHSS) model) was subsequently developed in a series of papers (see [32], [59], [60]). A complete analysis of the DHSS model in the case of constant return to scale and no capital depreciation was presented in [21]. An application of Theorem 8 to the DHSS model with logarithmic instantaneous utility function, arbitrary return to scale, and capital depreciation can be found in [6]. Here we consider the case of a non-renewable (not necessarily completely extractable) resource. For simplicity we do not consider any man-made capital.

Consider the following problem (P4):

$$J(x(\cdot),u(\cdot)) = \int_0^\infty e^{-\rho t} g(u(t)(x(t)-a))\,dt \to \max, \tag{62}$$

$$\dot{x}(t) = -u(t)(x(t)-a), \qquad x(0) = x_0 > a, \tag{63}$$

$$u(t) \in (0,\infty). \tag{64}$$

Here $x(t)$ is the stock of a non-renewable resource at the time $t \geqslant 0$, and $a \geqslant 0$ is the non-extractable part of the stock. In the case $a = 0$ the resource can be asymptotically exhausted, while in the case $a > 0$ it can be depleted only up to the given minimal level $a > 0$ (for technological (or some other) reasons). Further, $u(t)$ is the (non-vanishing) rate of extraction of the part $x(t) - a$ of the total stock $x(t)$ of resource available for exploitation at the time $t \geqslant 0$. All the extracted amount $u(t)(x(t) - a)$ of the resource at each moment of time $t \geqslant 0$ is consumed. Thus, $c(t) = u(t)(x(t) - a)$, $t \geqslant 0$, is the corresponding amount of consumption. As in Example 21, we assume that $\rho > 0$ is the social discount rate, and the instantaneous utility function $g(\cdot)$ is iso-elastic (see (45)).

Let $G = (a,\infty)$. Obviously, Assumption (A0) holds for the corresponding functions $f(\cdot,\cdot,\cdot)$ and $f^0(\cdot,\cdot,\cdot)$ in the problem (P4),

$$f(t,x,u) = -u(x-a) \quad \text{and} \quad f^0(t,x,u) = e^{-\rho t} g(u(x-a)),$$
$$t \geqslant 0, \quad x > a, \quad u \in (0,\infty),$$

and the multivalued map $U(\cdot)$: $U(t) \equiv (0,\infty)$, $t \geqslant 0$ (see Remark 1). Thus, (P4) is a particular case of the problem (P).

By (63), for any locally integrable function $u\colon [0,\infty) \to (0,\infty)$ and an arbitrary initial state $\zeta > a$ the corresponding solution $x(\zeta,\cdot)$ of the Cauchy problem (63) is defined by the formula

$$x(\zeta,t) = (\zeta - a)\exp\biggl\{-\int_0^t u(s)\,ds\biggr\} + a, \qquad t \geqslant 0. \tag{65}$$

Hence, any locally integrable function $u(\cdot)$ satisfying the pointwise constraint (64) is an admissible control in (P4).

Since for any $\sigma > 0$ and all $t \geqslant 0$, $x > a$,and $u \in (0,\infty)$ in (P4) we have

$$f_x(t,x,u) \equiv -u, \qquad f_x^0(t,x,u) = e^{-\rho t} u^{1-\sigma}(x-a)^{-\sigma},$$

Assumption (A1) is also satisfied in (P4) for any admissible pair $(x_*(\cdot),u_*(\cdot))$.

Let $(u(\cdot),x(\cdot))$ be an arbitrary admissible pair. In the case $\sigma = 1$, by (63) we have $g(u(t)(x(t)-a)) = \log(-\dot{x}(t)) \leqslant -\dot{x}(t)$ for almost every $t \geqslant 0$, and hence for any $0 \leqslant T < T'$

$$\int_T^{T'} e^{-\rho t}\log(u(t)(x(t)-a))\,dt \leqslant -\int_T^{T'} e^{-\rho t}\dot{x}(t)\,dt \leqslant (x_0 - a)e^{-\rho T}. \tag{66}$$

In the case $\sigma < 1$ we have

$$g(c) = \frac{c^{1-\sigma} - 1}{1-\sigma} \leqslant \frac{c}{1-\sigma}, \qquad c > 0,$$

and hence for any $0 \leqslant T < T'$

$$\int_T^{T'} e^{-\rho t} \frac{(u(t)(x(t)-a))^{1-\sigma} - 1}{1-\sigma}\, dt \leqslant \int_T^{T'} e^{-\rho t} \frac{-\dot{x}(t)}{1-\sigma}\, dt \leqslant \frac{x_0 - a}{1-\sigma} e^{-\rho T}. \tag{67}$$

In the case $\sigma > 1$ we have $g(u(t)(x(t)-a)) \leqslant 1/(\sigma-1)$ for almost every $t \geqslant 0$, and hence for every $0 \leqslant T < T'$

$$\int_T^{T'} e^{-\rho t} \frac{(u(t)(x(t)-a))^{1-\sigma} - 1}{1-\sigma}\, dt \leqslant \frac{1}{\sigma - 1} e^{-\rho T}. \tag{68}$$

By (66), (67), and (68), for any $\sigma > 0$ there is a decreasing non-negative function $\omega\colon [0,\infty) \to \mathbb{R}^1$ with $\lim_{t\to\infty} \omega(t) = 0$ such that for an arbitrary admissible pair $(x(\cdot), u(\cdot))$ the following estimate holds:

$$\int_T^{T'} e^{-\rho t} g(u(t)x(t))\, dt \leqslant \omega(T), \qquad 0 \leqslant T < T'. \tag{69}$$

As in Example 21 above, the estimate (69) implies that for any admissible pair $(x(\cdot), u(\cdot))$ the improper integral in (62) either converges to a finite number or diverges to $-\infty$, and $J(x(\cdot), u(\cdot)) \leqslant \omega(0)$. Hence, in the problem (P4) for any $\sigma > 0$ the concepts of strong optimality and weak overtaking optimality coincide. Therefore, everywhere below in this example we understand optimality of an admissible pair $(x_*(\cdot), u_*(\cdot))$ in the problem (P4) in the strong sense. In particular, if an optimal admissible pair $(x_*(\cdot), u_*(\cdot))$ exists, then $J(x_*(\cdot), u_*(\cdot))$ is finite. This fact will be used later.

Below we focus our analysis on the case $\sigma \neq 1$, since the case of a logarithmic instantaneous utility ($\sigma = 1$) was considered in [5], Example 3, and [6], § 5.

Fix an optimal admissible pair $(x_*(\cdot), u_*(\cdot))$ in (P4) (if such a pair exists). We shall show that Assumption (A2) is satisfied for this pair. Let $\beta = (x_0 - a)/2$. Taking into account that the function $\zeta \mapsto x(\zeta; t)$ is monotone increasing and the function $x \mapsto (x-a)^{-\sigma}$ is monotone decreasing, we get that for $\zeta \in [x_0 - \beta, x_0 + \beta] = [(x_0 + a)/2, (3x_0 - a)/2]$

$$\begin{aligned}
&\max_{x \in [x(\zeta;t), x_*(t)]} |f_x^0(t, x, u_*(t))(x(\zeta;t) - x_*(t))| \\
&\qquad\leqslant |\zeta - x_0| \exp\biggl\{-\int_0^t u_*(s)\, ds\biggr\} e^{-\rho t} u_*(t)^{1-\sigma} \max_{x \in [x(\zeta;t), x_*(t)]} (x-a)^{-\sigma} \\
&\qquad\leqslant |\zeta - x_0| \exp\biggl\{-\int_0^t u_*(s)\, ds\biggr\} e^{-\rho t} u_*(t)^{1-\sigma} \biggl(x\biggl(\frac{x_0 + a}{2}; t\biggr) - a\biggr)^{-\sigma} \\
&\qquad= |\zeta - x_0| \biggl(\frac{x_0 + a}{2}\biggr)^{-\sigma} e^{-\rho t} \biggl[u_*(t) \exp\biggl\{-\int_0^t u_*(s)\, ds\biggr\}\biggr]^{1-\sigma} \\
&\qquad\leqslant |\zeta - x_0| \biggl(\frac{x_0 + a}{2}\biggr)^{-\sigma} e^{-\rho t} \biggl(\frac{-\dot{x}_*(t)}{x_0 - a}\biggr)^{1-\sigma} = |\zeta - x_0| \lambda(t), \qquad t \geqslant 0,
\end{aligned}$$

where the function

$$\lambda(t) = \left(\frac{x_0 + a}{2}\right)^{-\sigma} e^{-\rho t} \left(\frac{-\dot{x}_*(t)}{x_0 - a}\right)^{1-\sigma}, \qquad t \geqslant 0,$$

is integrable on $[0, \infty)$ because of the integrability of the function

$$t \mapsto e^{-\rho t}(-\dot{x}_*(t))^{1-\sigma} = e^{-\rho t}\bigl(1 + (1-\sigma)g(u_*(t)(x_*(t) - a))\bigr).$$

Thus, Assumption (A2) holds with $\beta = (x_0 - a)/2$ and the function $\lambda(\,\cdot\,)$ defined above.

Then by Theorem 8, for any optimal admissible pair $(x_*(\,\cdot\,), u_*(\,\cdot\,))$ (if there is one) the core conditions (15) and (16) of the normal-form maximum principle hold with an adjoint variable $\psi(\,\cdot\,)$ of the form

$$\begin{aligned} \psi(t) = \exp\biggl\{\int_0^t u_*(s)\,ds\biggr\} \int_t^\infty & e^{-\rho s} u_*(s)^{1-\sigma} (x_0 - a)^{-\sigma} \\ & \times \exp\biggl\{-(1-\sigma)\int_0^s u_*(\xi)\,d\xi\biggr\}\,ds, \qquad t \geqslant 0. \end{aligned} \tag{70}$$

If an optimal control $u_*(\,\cdot\,)$ does exist, then according to (16) we have

$$\mathscr{H}_u(t, x_*(t), u_*(t), \psi(t)) \overset{\text{a.e.}}{=} 0$$

on $[0, \infty)$, where the Hamilton–Pontryagin function has the form

$$\mathscr{H}(t, x, u, \psi) = \frac{e^{-\rho t}(u^{1-\sigma}(x-a)^{1-\sigma} - 1)}{1-\sigma} - \psi u(x - a),$$
$$t \geqslant 0, \quad x > a, \quad u > 0.$$

Differentiating with respect to u, we get that

$$e^{-\rho t} u_*(t)^{-\sigma} (x_*(t) - a)^{-\sigma} - \psi(t) \overset{\text{a.e.}}{=} 0, \qquad t \geqslant 0.$$

Substituting the expressions (65) (with $\zeta = x_0$) and (70) for $x_*(t)$ and $\psi(t)$, we get that

$$\begin{aligned} u_*(t)^{-\sigma} \overset{\text{a.e.}}{=} e^{\rho t} \exp\biggl\{(1-\sigma)\int_0^t u_*(s)\,ds\biggr\} & \\ \times \int_t^\infty e^{-\rho s} u_*(s)^{1-\sigma} \exp\biggl\{-(1-\sigma)\int_0^s u_*(\xi)\,d\xi\biggr\}\,ds, & \qquad t \geqslant 0. \end{aligned} \tag{71}$$

In view of the absolute convergence of the integral in (71), the last expression implies that $u_*(\,\cdot\,)$ is (equivalent to) a locally absolutely continuous function on $[0, \infty)$.

Further, by (65) the equality (71) implies that

$$\begin{aligned} u_*(t)^{-\sigma} &= \frac{e^{\rho t} \exp\bigl\{(1-\sigma)\int_0^t u_*(s)\,ds\bigr\}}{(x_0 - a)^{1-\sigma}} \int_t^\infty e^{-\rho s}(u_*(s)(x_*(s) - a))^{1-\sigma}\,ds \\ &= \frac{e^{\rho t}}{(x_*(t) - a)^{1-\sigma}} \int_t^\infty e^{-\rho s}(u_*(s)(x_*(s) - a))^{1-\sigma}\,ds, \qquad t \geqslant 0. \end{aligned}$$

Hence

$$u_*(t) = \frac{e^{-\rho t}(u_*(t)(x_*(s)-a))^{1-\sigma}}{\int_t^\infty e^{-\rho s}(u_*(s)(x_*(s)-a))^{1-\sigma}\,ds} = -\dot{z}(t), \qquad t \geqslant 0, \tag{72}$$

where the locally absolutely continuous function $z(\,\cdot\,)$ is defined by

$$z(t) = \log \int_t^\infty e^{-\rho s}\big(u_*(s)(x_*(s)-a)\big)^{1-\sigma}\,ds, \qquad t \geqslant 0.$$

Integrating (72) on an arbitrary time interval $[0,T]$, $T>0$, we get that

$$\begin{aligned}\int_0^T u_*(s)\,ds = \log \int_0^\infty e^{-\rho s}\big(u_*(s)(x_*(s)-a)\big)^{1-\sigma}\,ds \\ - \log \int_T^\infty e^{-\rho s}\big(u_*(s)(x_*(s)-a)\big)^{1-\sigma}\,ds.\end{aligned}$$

Since $(x_*(\,\cdot\,), u_*(\,\cdot\,))$ is an optimal admissible pair, the first term on the right-hand side is finite, while the second converges to $-\infty$. Thus, we have $\int_0^\infty u_*(s)\,ds = \infty$.

Differentiating (71) with respect to t and utilizing the same expression for $u_*^{-\sigma}(t)$, we conclude that for almost every $t \geqslant 0$ the function $u_*(\,\cdot\,)$ satisfies the equality

$$-\sigma u_*(t)^{-\sigma-1}\dot{u}_*(t) = \rho u_*(t)^{-\sigma} - u_*(t)^{1-\sigma} + (1-\sigma)u_*(t)^{1-\sigma}.$$

Dividing by $-\sigma u_*(t)^{-\sigma-1}$, we get that $u_*(\,\cdot\,)$ is a locally absolutely continuous solution of the differential equation

$$\dot{u}(t) = u(t)^2 - \frac{\rho}{\sigma}u(t).$$

The general solution of this simple Riccati equation is

$$u_*(t) = e^{-\rho t/\sigma}\left[c - \frac{\sigma}{\rho}(1 - e^{-\rho t/\sigma})\right]^{-1}, \qquad t \geqslant 0,$$

where c is a constant (equal to $u_*(0)^{-1}$). Since $u_*(\,\cdot\,)$ takes only positive values, this expression for $u_*(\,\cdot\,)$ implies that $c \geqslant \sigma/\rho$, and due to the equality $\int_0^\infty u_*(s)\,ds = \infty$ we get finally that $c = \sigma/\rho$. Thus, we conclude that application of Theorem 8 determines a unique admissible pair $(x_*(\,\cdot\,), u_*(\,\cdot\,))$ which is 'suspectable' for optimality in the problem (P4) (see (65)):

$$x_*(t) = (x_0 - a)e^{-\rho t/\sigma} + a, \quad u_*(t) \equiv \frac{\rho}{\sigma}, \qquad t \geqslant 0. \tag{73}$$

Note that the explicit formula (70) for the corresponding adjoint variable $\psi(\,\cdot\,)$ gives us that

$$\psi(t) \equiv \left(\frac{\rho(x_0-a)}{\sigma}\right)^{-\sigma}, \qquad t \geqslant 0. \tag{74}$$

We show that the admissible pair $(x_*(\,\cdot\,), u_*(\,\cdot\,))$ defined in (73) is indeed optimal in (P4). To do this consider the function $\Phi\colon [0,\infty)\times(0,\infty) \to \mathbb{R}^1$ defined by

$$\Phi(t,y) = e^{-\rho t}g(y) - \psi(t)y, \qquad t \geqslant 0, \quad y > 0,$$

where $\psi(\,\cdot\,)$ is given by (74). It is easy to see that for any $t \geqslant 0$ the strict concavity of the iso-elastic function $g(\,\cdot\,)$ (see (45)) implies that the function $\Phi(t,\,\cdot\,)$ has a unique point $y_*(t)$ of global maximum on $(0,\infty)$. For any $t \geqslant 0$ the point $y_*(t)$ is the unique solution of the equation $e^{-\rho t} g_y(y) = \psi(t)$. Since $g_y(y) = y^{-\sigma}$, $y > 0$ (see (45)), we get by solving this equation that

$$y_*(t) = \bigl(e^{\rho t}\psi(t)\bigr)^{-1/\sigma} = \frac{\rho(x_0 - a)e^{-\rho t/\sigma}}{\sigma}, \qquad t \geqslant 0$$

(see (74)). However, by (73),

$$(x_*(t) - a)u_*(t) = \frac{\rho(x_0 - a)e^{-\rho t/\sigma}}{\sigma} = y_*(t), \qquad t \geqslant 0.$$

Thus, we have proved the global inequality

$$\begin{aligned} &e^{-\rho t} g(u_*(t)(x_*(t) - a)) - \psi(t)u_*(t)(x_*(t) - a) \\ &\quad \geqslant e^{-\rho t} g(u(x - a)) - \psi(t)u(x - a), \qquad t \geqslant 0, \quad x > 0, \quad u > 0. \end{aligned} \tag{75}$$

Now let $(x(\,\cdot\,), u(\,\cdot\,))$ be another arbitrary admissible pair. Then by (75) we have

$$\begin{aligned} &e^{-\rho t} g(u_*(t)(x_*(t) - a)) - e^{-\rho t} g(u(t)(x(t) - a)) \\ &\quad \geqslant \left(\frac{\rho(x_0 - a)}{\sigma}\right)^{-\sigma} [u_*(t)(x_*(t) - a) - u(t)(x(t) - a)], \qquad t \geqslant 0 \end{aligned}$$

(see (74)). Integrating the last inequality on an arbitrary time interval $[0, T]$, $T > 0$, we get that

$$\begin{aligned} &\int_0^T e^{-\rho t} g(u_*(t)(x_*(t) - a))\,dt - \int_0^T e^{-\rho t} g(u(t)(x(t) - a))\,dt \\ &\quad \geqslant \left(\frac{\rho(x_0 - a)}{\sigma}\right)^{-\sigma} \int_0^T [u_*(t)(x_*(t) - a) - u(t)(x(t) - a)]\,dt \\ &\quad = \left(\frac{\rho(x_0 - a)}{\sigma}\right)^{-\sigma} \bigl(x(T) - x_*(T)\bigr), \qquad T \geqslant 0. \end{aligned}$$

Passing to the limit as $T \to \infty$ and taking into account that $\lim_{T\to\infty} x_*(T) = a$ and $x(T) \geqslant a$, we find that

$$\int_0^\infty e^{-\rho t} g(u_*(t)(x_*(t) - a))\,dt \geqslant \limsup_{T\to\infty} \int_0^T e^{-\rho t} g(u(t)(x(t) - a))\,dt.$$

Hence, $(x_*(\,\cdot\,), u_*(\,\cdot\,))$ is indeed the (unique) optimal admissible pair in the problem (P4).

Thus, we have proved that for any $\sigma > 0$ with $\sigma \neq 1$, the asymptotic condition (7) always fails in this example in view of (73) and (74). Moreover, since $x_*(t) \to a$ as $t \to \infty$ (see (73)), the asymptotic condition (8) also fails if $a > 0$, while it holds if $a = 0$. Therefore, if $\sigma > 0$, $\sigma \neq 1$, and $a > 0$, then the explicit formula (12) plays the role of an alternative to the asymptotic conditions (7) and (8), which are both inconsistent with the core conditions (15) and (16) of the maximum principle. The same phenomenon, that is, the simultaneous violation of both the standard asymptotic conditions (7) and (8), can be observed also in the case when $\sigma = 1$ and $a > 0$ (see [5], Example 3, and [6], § 5). In the last case the optimal extraction rate $u_*(\cdot)$ coincides with the classical Hotelling rule (see [39]), that is, $u_*(t) \equiv \rho$, $t \geqslant 0$. In the case of $\sigma > 0$ with $\sigma \neq 1$ considered above, the optimal extraction rate $u_*(\cdot)$ is given by the adjusted Hotelling rule (corresponding to the value of σ), that is, $u_*(t) \equiv \rho/\sigma$, $t \geqslant 0$ (see (73)).

5. Bibliographical comments

To the best of our knowledge, optimal control problems with infinite time horizon were first considered in Chap. 4 of the fundamental monograph [48]. The problem considered there is completely autonomous, involves no discounting, satisfies the usual regularity assumptions (see Remark 4), and contains the additional asymptotic terminal condition $\lim_{t\to\infty} x(t) = x_1$, where x_1 is a given asymptotic terminal state in $\mathbb{R}^n$. The approach proposed in [48] is potentially applicable to a broad scope of infinite-horizon optimal control problems, in particular, to the problem (P) with free terminal state in the focus of the present paper. This approach is based on the construction of an 'initial cone' K_{t_0} at the initial time t_0 instead of the 'limiting cone' K_{t_1} at the terminal time t_1 (which does not exist in the infinite-horizon case). The initial cone K_{t_0} is constructed in the same way as the limiting cone K_{t_1} at the terminal time t_1 in the case of a finite-horizon problem on the time interval $[t_0, t_1]$, $t_0 < t_1$. This construction is based on the classical needle variations technique (see [48]). The only difference from the finite-horizon case is that the increment of the principal linear part of the varied trajectory is transmitted (by solving a system of variational equations) to the initial moment t_0 rather than to the terminal time t_1 (which does not exist). All other points of this construction are essentially the same as in the finite-horizon case. When the initial cone K_{t_0} has been constructed, the subsequent application of a topological result and the separation theorem provides a corresponding version of the maximum principle (see [48] for details). Note that this construction employs only the property of finite optimality of the optimal control $u_*(\cdot)$ under consideration. Therefore, when applied to the problem (P), this construction leads to exactly the same result as the general version of the maximum principle for the problem (P) that was developed later by Halkin (see [37]).

Halkin's paper [37] considers the problem (P) with free terminal state at infinity under the usual regularity assumptions (see Remark 4). The integral functional (1) is not assumed there to be finite. The approach in [37] is based on consideration of the family in Definition 3 of auxiliary optimal control problems (Q_T) on the finite time intervals $[0, T]$, $T > 0$. The finite optimality of the admissible pairs $(x_*(\cdot), u_*(\cdot))$ in the problem (P) implies that on any finite time interval $[0, T]$, $T > 0$, the core conditions of the Pontryagin maximum principle for the pair $(x_*(\cdot), u_*(\cdot))$ hold with a corresponding non-vanishing pair of adjoint variables $\psi_T^0 \geqslant 0$, $\psi_T(\cdot)$. This implies the validity of the core conditions of the infinite-horizon maximum principle after taking the limit as $T \to \infty$ in the conditions of the maximum principle for these auxiliary problems (Q_T) (see details in [26] and [37]). No additional characterizations of the adjoint variables ψ^0 and $\psi(\,\cdot\,)$ such as normality of the problem and/or some boundary conditions at infinity are provided in [37]. Moreover, [37] presents two counterexamples demonstrating possible pathologies in the relations of the general version of the maximum principle for the problem (P), namely, the possible abnormality of the problem ($\psi^0=0$ in this case) and the possible violation of the standard asymptotic conditions (7) and (8).

Apparently, [56] and [37] were the first papers in which the authors demonstrated by means of counterexamples that abnormality is possible, that the 'natural' asymptotic conditions (7) and (8) may be violated in the case of infinite-horizon problems with free terminal state at infinity. Since the discount rate ρ is equal to zero in these counterexamples, for a long time the opinion was common in the economic literature that such pathologies were possible only in problems without time discounting (for example, see [19], § A.3.9, and [29], Chap. 9). However, many 'pathological' examples with positive discount rate are known nowadays (for example, see [11], Chap. 1, § 6, and [44], § 2), including models developed quite recently and having clear economic interpretations (see [16], § 4, [5], Example 3, [6], § 2.2, and Example 22 in § 4).

After the publication of [37] many authors attempted to develop normal-form versions of the maximum principle for the problem (P) and to characterize, under various additional assumptions, the asymptotic behaviour of the adjoint variable for which the maximum condition (4) is satisfied. The first positive results in this direction were obtained in [17] and [22].

In [17], a particular case of the problem (P) is investigated in which the control system is linear and autonomous:

$$\dot{x}(t)=Fx(t)+u(t), \qquad x(0)=x_0.$$

The constraining set $U\subset\mathbb{R}^n$ is convex and compact, and the instantaneous utility function has the form

$$f^0(t,x,u)=e^{-\rho t}g(x,u), \qquad t\geqslant 0, \quad x\in\mathbb{R}^n, \quad u\in U,$$

with a positive discount rate ρ and a function $g(\,\cdot\,,\cdot\,)$ which is locally Lipschitz with respect to both variables x and u.[3] The authors assume that the following dominating discount condition holds:

$$\rho>(r+1)\lambda_F. \tag{76}$$

Here λ_F is the greatest real part of the eigenvalues of the $n\times n$ matrix F, and r is a non-negative number that characterizes the growth of the function $g(\,\cdot\,,\cdot\,)$ in terms of its generalized gradient $\partial g(\,\cdot\,,\cdot\,)$ (in the sense of Clarke [30]; see [17] for more details):

$$\|\zeta\|\leqslant\kappa(1+\|(x,u)\|^r) \quad \text{for any } \zeta\in\partial g(x,u),\ x\in G,\ u\in U.$$

[3]Here and below, for uniformity we use notation sometimes slightly different from that used in the original papers under discussion.

Note that the generalized gradient is taken here with respect to both variables, x and u.

Since $\rho > 0$, it is easy to see that the condition (76) guarantees the convergence of the functional $J(x(\,\cdot\,), u(\,\cdot\,))$ for any admissible pair $(x(\,\cdot\,), u(\,\cdot\,))$. Accordingly, the concept of strong optimality is employed in [17].

In the case of $r > 0$, the authors of [17] proposed a normal-form version of the Pontryagin maximum principle which contains a characterization of the behaviour of the adjoint variable $\psi(\,\cdot\,)$ in terms of convergence of the improper integral:

$$\int_0^\infty e^{(q-1)\rho t} \|\psi(t)\|^q \, dt < \infty. \tag{77}$$

Here the constant $q > 1$ is defined by the equality $1/q + 1/(r + 1) = 1$. As pointed out in [17], (77) implies that the asymptotic condition (7) holds.

Later the result obtained in [17] was generalized and strengthened using different methods (which also differ from the method in [17]) in the series of papers [7], [9]–[11], [14], [23]. In these papers a few different (non-equivalent) extensions of the condition (76) to the case of non-linear problems (P) were proposed, and various normal-form versions of the maximum principle with adjoint variable $\psi(\,\cdot\,)$ specified explicitly by (12) were developed. Here we mention only that in the linear case (considered in [17]) the dominating discount condition (76) implies the validity of Assumption (B2) in § 3, and hence of (A2). In this case, by Theorem 8 the core conditions (15) and (16) of the normal-form maximum principle hold with the adjoint variable $\psi(\,\cdot\,)$ specified by the formula (12), and this directly implies the estimate (77) (see [11], § 16, and [12]). Moreover, since $\rho > 0$, (12) implies both asymptotic conditions (7) and (8) in this case (see [11], § 12, and Corollary 18).

In [22] a version of first-order necessary optimality conditions containing the asymptotic condition at infinity (8) was obtained for the infinite-horizon dynamic optimization problem of the form

$$J(x(\,\cdot\,)) = \int_0^\infty f^0(t, x(t), \dot{x}(t)) \, dt \to \max, \tag{78}$$

$$(x(t), \dot{x}(t)) \in K, \qquad x(0) = x_0. \tag{79}$$

Here the set $K \subset \mathbb{R}^{2n}$ is assumed to be convex and closed with non-empty interior, the function $f^0 \colon [0, \infty) \times K \to \mathbb{R}^1$ is jointly concave in the variables x, $\dot{x}$ for all $t \geqslant 0$, and the optimal trajectory $x_*(\,\cdot\,)$ is assumed to take values in the interior of the set $\operatorname{dom} V(t, \,\cdot\,)$ for all $t \geqslant 0$, where

$$\operatorname{dom} V(t, \,\cdot\,) = \{x_0 \in \mathbb{R}^n \colon V(t, x_0) < \infty\}$$

is the effective set of the optimal value function $V(t, \,\cdot\,)$:

$$V(t, x_0) = \sup\biggl\{ \int_t^\infty f^0(s, x(s), \dot{x}(s)) \, ds \colon (x(s), \dot{x}(s)) \in K \text{ for } s \geqslant t; \ x(t) = x_0 \biggr\}.$$

Under the assumptions made, the problem (78), (79) is 'completely convex'. In particular, the optimal value function $V(\,\cdot\,, \cdot\,)$ in it is concave with respect to the variable x_0 for all $t \geqslant 0$, and the set of all admissible trajectories is convex in the space $C([0,T],\mathbb{R}^n)$ for any $T>0$.

The main result of [22] states that there exists an adjoint variable $\psi(\,\cdot\,)$ corresponding to the optimal trajectory $x_*(\,\cdot\,)$ such that

$$\psi(t) \in \partial_x V(t, x_*(t)), \qquad t \geqslant 0. \tag{80}$$

Here $\partial_x V(t,x_*(t))$ is the partial subdifferential (in the sense of convex analysis) of the concave function $V(t,\,\cdot\,)$ at the point $x_*(t)$ for fixed t. Further, a certain generalized Euler equation and the asymptotic condition (8) were derived from (80) in [22] under certain additional assumptions. In particular, it was assumed that the phase vector x is non-negative and the function $f^0(\,\cdot\,,\,\cdot\,,\cdot)$ is monotone in the variable $\dot{x}$ (see [22] for more details).

The question of whether asymptotic conditions of the form (8) hold for the problem (78), (79) was considered in [42] without convexity assumptions in the situation when the optimal trajectory $x_*(\,\cdot\,)$ is regular and interior and the control system satisfies a homogeneity condition.

The next step in developing complementary necessary conditions characterizing the asymptotic behaviour of the adjoint variable $\psi(\,\cdot\,)$ was due to Michel in 1982 (see [44]). In the special case when the problem (P) is autonomous with exponential discounting (that is, $f(t,x,u)\equiv f(x,u)$, $f^0(t,x,u)\equiv e^{-\rho t}g(x,u)$, and $U(t)\equiv U$, where $\rho\in\mathbb{R}^1$ is not necessarily positive) and under the assumption that the optimal value $J(x_*(\,\cdot\,),u_*(\,\cdot\,))$ is finite, he established the validity of the asymptotic condition (9) for any strongly optimal admissible trajectory $x_*(\,\cdot\,)$. This asymptotic condition is analogous to the transversality condition with respect to time in problems with free final time [48]. Since the standard regularity conditions are employed (see Remark 4), in this case (9) is equivalent to the stationarity condition

$$H(t,x_*(t),\psi^0,\psi(t)) = \psi^0\rho\int_t^\infty e^{-\rho s} g(x_*(s),u_*(s))\,ds, \qquad t\geqslant 0. \tag{81}$$

Note that the adjoint variable ψ^0 can be equal to zero here, and an example of an autonomous problem (P) with positive discount rate ($\rho>0$) in which the equality $\psi^0=0$ necessarily holds was presented in [44], § 2. The complementary character of the condition (9) was demonstrated in [11], Example 6.6. A generalization of Michel's result to the case when the instantaneous utility $f^0(\,\cdot\,,\,\cdot\,,\cdot)$ depends on the variable t in a more general way was developed in [55], using a slightly modified argument. A normal-form version of the maximum principle with adjoint variable $\psi(\,\cdot\,)$ having all positive components and including the condition (41) was also obtained in [13] under certain assumptions of monotonicity type.

In some cases, in particular, when the function $g(\,\cdot\,,\cdot)$ is non-negative and there exists a neighbourhood $\mathscr{V}$ of 0 in $\mathbb{R}^n$ such that $\mathscr{V}\subset f(x_*(t),U)$ for all large enough times t, the asymptotic condition (9) implies (7). Nevertheless, being one-dimensional, the condition (9) (as well as (81)) cannot provide a full set of complementary conditions for the adjoint variable $\psi(\,\cdot\,)$ in the general multidimensional case.

The relationship between the explicit formula (12) and the asymptotic condition (9) was discussed in Corollary 20 (see also [5] and [4]).

In [62] Ye obtained the stationarity condition (81) in the case of a non-smooth problem (P) with discounting (provided that the autonomous functions $f(\,\cdot\,,\cdot\,)$ and $g(\,\cdot\,,\cdot\,)$ are Lipschitz with respect to the phase variable x uniformly with respect to u and Borel measurable with respect to u, and the function $g(\,\cdot\,,\cdot\,)$ is bounded).

The main result in [62] provided a version of the maximum principle with the asymptotic conditions (7) and (9) under the additional assumption that

$$\rho > \max\left\{0, \sup_{x,y\in G,\ u\in U} \frac{\langle x-y, f(x,u)-f(y,u)\rangle}{\|x-y\|^2}\right\}. \tag{82}$$

This assumption means that ρ is positive and $\rho > \nu$, where ν is the one-sided Lipschitz constant of the vector function $f(\,\cdot\,,\cdot\,)$ with respect to the phase variable x (see Definition 12). It is easy to see that if the functions $f(\,\cdot\,,\cdot\,)$, $f_x(\,\cdot\,,\cdot\,)$, $g(\,\cdot\,,\cdot\,)$, and $g_x(\,\cdot\,,\cdot\,)$ are continuous with respect to x, the function $g(\,\cdot\,,\cdot\,)$ is Lipschitz with respect to x uniformly with respect to u (this implies that $r=0$ in (B1)), and (82) holds, then by Theorem 8 the core conditions (15) and (16) of the normal-form maximum principle hold together with the adjoint variable $\psi(\,\cdot\,)$ specified via (12) (see Lemma 15). In this case the formula (12) implies the asymptotic condition (7).

In [58], Smirnov characterized the asymptotic behaviour of the adjoint variable $\psi(\,\cdot\,)$ in terms of Lyapunov exponents (see [28], [33], and § 3.2 for the relevant definitions from stability theory). The main assumption of [58] is that the system of variational equations considered along a particular optimal pair $(x_*(\,\cdot\,),u_*(\,\cdot\,))$ is regular. In this case, under certain additional assumptions[4] it was proved in [58] that the characteristic Lyapunov exponent of the adjoint variable $\psi(\,\cdot\,)$ corresponding to the optimal pair $(x_*(\,\cdot\,),u_*(\,\cdot\,))$ under consideration is non-positive (see [58], Theorem 3.1). However, this result does not guarantee the normality of the optimal control problem nor the satisfaction of the asymptotic conditions (7) or (8). As pointed out in [58] by means of a counterexample, regularity of the system of variational equations is essential here.

In [51] Seierstad considered (as a minimum problem) an infinite-horizon optimal control problem that is more general than (P). The statement of this problem includes a non-autonomous non-smooth control system

$$\dot x(t) = f(t,x(t),u(t)), \qquad u(t)\in U,$$

an initial condition $x(0)=x_0\in\mathbb{R}^n$, and equality- and inequality-type terminal boundary constraints at infinity that are imposed on some of the phase coordinates $x^i(\infty)$, $i=1,\dots,m$ (it is assumed that $x(t)\in\mathbb{R}^n$, $m<n$, and that the corresponding limits $x^i(\infty)=\lim_{t\to\infty}x^i(t)$, $i=1,\dots,m$, exist). The problem considered in [51] consists in minimization of a terminal functional of the form

$$J(\pi x(\infty)) = \sum_{i=1}^m \nu_i x^i(\infty).$$

[4]We remark that in [58] an important condition used in the proof is missing from the formulation of the main result. Namely, the gradient of the integrand must be bounded: $\|b(t)\|\leqslant K$ for almost every $t\geqslant 0$ (see [58], Theorem 3.1). Example 10.4 in [51] shows that this condition is essential.

Here π is the $m \times n$ matrix of the operator of projection onto the subspace of the first m coordinates in $\mathbb{R}^n$, and ν_i, $i = 1, \dots, m$, are real numbers.

In [51] a version of the maximum principle that contains a full set of asymptotic conditions at infinity was presented, though under rather restrictive growth conditions and some other assumptions. In application to the problem (P) with free right-hand endpoint, this result implies the normality of the maximum principle and the validity of the asymptotic condition (7). For a discussion of the growth conditions in [51] see [11], § 16.

We note also that a version of the maximum principle containing a full set of asymptotic conditions at infinity was obtained in [52] for a smooth infinite-horizon optimal control problem with inequality-type state constraints and with terminal conditions on the states at infinity. However, the growth conditions in [52] are very restrictive.

For an autonomous problem (P) with exponential discounting, an approach based on approximations of it by a specially constructed sequence $\{(\mathrm{P}_k)\}_{k=1}^{\infty}$ of finite-horizon problems on time intervals $[0, T_k]$ with $T_k > 0$ and $\lim_{k\to\infty} T_k = \infty$ was developed in [9]–[11] and [13]. In this case on any finite time interval $[0, T]$ with $T > 0$, the sequence of optimal controls $\{u_k(\,\cdot\,)\}$ (which do exist) in the approximating problems (P_k), $k = 1, 2, \dots$, converges weakly in $L_1([0, T], \mathbb{R}^m)$ (or in another suitable sense) to the optimal control $u_*(\,\cdot\,)$ under consideration in the problem (P). The necessary optimality conditions for (P) are obtained by passing to the limit as $k \to \infty$ in the relations of the Pontryagin maximum principle for the approximating problems (P_k). It was proved in [9]–[11] that the maximum principle holds in normal form with the adjoint variable $\psi(\,\cdot\,)$ specified by (12) under some conditions of dominating discount type. A similar characterization of the adjoint variable $\psi(\,\cdot\,)$ was obtained by means of this technique also in the so-called 'monotone case' (see [13] and [11] for details). The main constructions and results in [9]–[11] were extended in [7] and [23].

Although the method of finite-horizon approximations enables us to develop different versions of the normal-form maximum principle that contain full sets of necessary conditions for the problem (P), there are inherent limitations for the applicability of this approach. In particular, application of this approximation technique assumes conditions guaranteeing the existence of solutions in the corresponding finite-horizon approximating problems. Moreover, it is required that the improper integral in the functional (1) converge uniformly with respect to all admissible pairs (for example, see the condition (A3) in [11]). In many cases of interest, assumptions of this type either fail or cannot easily be verified *a priori*. For instance, in problems without discounting and in certain models of endogenous economic growth (especially with declining discount rates) the corresponding integral utility functional may diverge to infinity.

For deriving first-order necessary optimality conditions for infinite-horizon optimal control problems, an approach based on methods in the general theory of extremal problems (see [41]) was recently obtained by Pickenhain in [45] and [46] (in the linear-quadratic case) and by Tauchnitz [61] (in the general non-linear case). The key idea of this approach is to introduce certain weighted Sobolev spaces as state spaces and certain weighted Lebesgue spaces as control spaces. The value of the functional is assumed to be finite and the optimality of an admissible control $u_*(\cdot) \in L_\infty([0,\infty), \mathbb{R}^m)$ is understood in the strong sense. The general version of the maximum principle obtained using this approach (see [61], Theorem 4.1) is not necessarily normal (the case $\psi^0 = 0$ is not excluded). It involves an adjoint variable $\psi(\cdot)$ belonging to the corresponding weighted function space (that is, satisfying a certain exponential growth condition). In this sense the result extends a result in [17]. Both the asymptotic conditions (7) and (8) follow from the fact that $\psi(\cdot)$ is in the corresponding weighted function space. However, this characterization does not necessarily uniquely determine an adjoint function satisfying the maximum principle.

In the linear-quadratic case the corresponding maximum principle holds in normal form (see [46], Theorem 5). The normal-form version of the maximum principle with adjoint variable specified by the Cauchy-type formula (12) was obtained in [61], Theorem 6.1, under an additional 'stability'-type condition (see the condition (A3) in [61]). It was also shown in [61], Example 6.2, that all the assumptions of the last result can be satisfied for an optimal admissible pair $(x_*(\cdot), u_*(\cdot))$ in this example, while the assumptions of Theorem 8 fail. However, the weighted Sobolev and Lebesgue spaces in this example are constructed using an *a priori* known optimal pair $(x_*(\cdot), u_*(\cdot))$ in this case.

Methods from the general theory of extremal problems were also used in the earlier paper [24] by Brodskii to develop a variant of the maximum principle for an infinite-horizon problem with terminal and mixed control-state constraints. The result there was obtained under very restrictive growth assumptions, and it does not imply the normality of the problem. In the case of a free terminal state at infinity, the result involves the asymptotic condition (7).

The relationship between the maximum principle with infinite-horizon and the dynamic programming method was studied in [25] and [50]. In the case when the optimal value function $V(\cdot,\cdot)$ is locally Lipschitz with respect to the state variable x, some normal versions of the maximum principle together with sensitivity-type relations involving generalized gradients of $V(\cdot,\cdot)$ were obtained.

The normal-form version of the Pontryagin maximum principle for the problem (P) with adjoint variable specified by the Cauchy-type formula (12) in our paper (see Theorem 8) was developed recently by the authors in [14]–[16]. The approach in these papers is based on the classical needle variations technique and the Yankov–von Neumann–Aumann selection theorem (see [40], Theorem 2.14). The main results obtained using this approach have been presented here, including economic applications (§ 4). The advantage of our approach is that it can be applied under less restrictive regularity and growth assumptions than approaches like the approximations-based technique or the methods of the general theory of extremal problems. In particular, this method can justify the Cauchy-type formula as a part of the Pontryagin necessary conditions for optimality even in the case when the optimal value of the functional is infinite. The notion of weakly overtaking optimality (see [26]) can be used in this case.

The importance of the Cauchy-type formula (12) is determined not only by the fact that in some cases it can imply the standard asymptotic conditions (7), (8), and (9), or provide even more complete information about the adjoint variable

$\psi(\,\cdot\,)$, but also by the fact that in cases when the asymptotic conditions (7) and (8) are inconsistent with the core conditions (15) and (16) of the maximum principle, the formula (12) can serve as an alternative to them. As was shown in [4], [5], and our § 4, the formula (12) also provides the possibility of interpreting the adjoint variable $\psi(\,\cdot\,)$ as the integrated intertemporal price function.

We mention that the same approach has also proved to be productive for distributed control systems, as shown in [57] for a class of age-structured optimal control problems, and for discrete-time problems with infinite horizons, as shown in [8].

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Received 4/APR/19

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