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Pogorelov's problem on isometric transformations of a cylindrical surface

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In [2] the author found an error in Pogorelov's book [1] which could not be fixed. This led to the following problem.

Problem. Find at least one solution to Pogorelov's problem, or prove that the problem has no solution.

The present note analyses the substance of the problematic section in [1] and again confirms an error in it. Following [1], p. 194, we consider "developable surfaces with violation of regularity (two-fold differentiability) along particular curves."

Statement of the problem ([1], Chap. 8, § 4). Construct an isometric embedding in $\mathbb{R}^3$ of the surface of a right circular cylinder of finite height such that the image of the embedding is a piecewise smooth surface that is not congruent to the surface of the cylinder and satisfies the following four conditions: 1) both boundary components are circles; 2) these two circles lie in parallel planes; 3) one circle is projected onto the other circle orthogonally to these planes; 4) both planes are supporting for the embedded surface.

Pogorelov listed four types of embeddings which he considered to be solutions to this problem ([1], Figs. 47 and 48). We prove below that these are not solutions.

Searching for a solution [1] (the case of compression along the axis). Following [1], suppose that "the cylindrical surface undergoes geometric bending with the formation of a system of congruent dents along the whole length of the initial surface which are located regularly in the circumferential direction." The latter condition means that the dents are equivalent with respect to a finite group of rotations around the axis. Furthermore, each dent has two planes of mirror symmetry: one (vertical) plane passes through the axis, the other (horizontal) plane is orthogonal to the axis. Assuming that the geodesic line along which the dent intersects the vertical plane of mirror symmetry does not contain straight line segments (and omitting the ostensibly redundant case when the geodesic contains such segments; here we leave aside other flaws in the text), Pogorelov proved that "the dented part of the surface in question is cylindrical, with generatrices perpendicular to the [vertical] plane of symmetry." The part of the surface lying in a fundamental domain of the group generated by reflections in the two vertical planes of mirror symmetry is shown in Fig. 1, (a) and (b) (one symmetry plane "passes halfway between two dents" and the other cuts the dent in half). For the given group of rotations, Pogorelov presented these surfaces as solutions. However, he was wrong, as we show below.

Embedding a cylinder [1], that is, a cylindrical surface. Denote by Z the lateral surface of a right circular cylinder with boundary circles S_1 and S_2, and let $f\colon Z \to \mathbb{R}^3$ be an isometric embedding of Z in $\mathbb{R}^3$; see [2]. Assume that $f(Z)$ is

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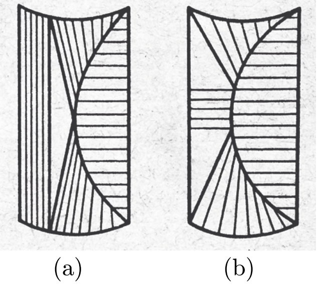
a piecewise smooth surface. Then each piece of it is homeomorphic to a disc and is part of a plane, a cylinder, a cone, or a torse. The images $f(S_1)$ and $f(S_2)$ are circles. Since $f(Z)$ consists of a finite number of smooth pieces, the circle $f(S)$, where $S = S_1$ or $S = S_2$, is subdivided into a finite number of arcs. Each arc is a geodesic in the smooth piece containing it. These pieces are parts of the surface of a circular cylinder, since the rectifying plane of an arc of the circle $f(S)$ is the tangent plane to the smooth piece containing the arc. The envelope of the family of rectifying planes of the arc is the only smooth developable surface for which the arc of $f(S)$ is a directrix, and its binormals are generatrices.

The border pieces [2]. The circle S is a geodesic on the surface Z. Therefore, an arc of the circle $f(S)$ is a geodesic on a smooth piece of $f(Z)$ containing this arc. Rectilinear generatrix lines of this piece are orthogonal to the plane of the circle $f(S)$ by Theorem 1 in [2].

Search for an error. Return to Fig. 1. Consider the two smooth pieces on opposite sides of a symmetric convex dent that adjoin it. A part of the surface including the upper part of the dent and the two adjoining pieces is shown in Fig. 2. Its boundary is formed by the generatrix AB of the dent, the generatrices AC and BD of the two pieces, and the arc $\widetilde{CD} \subset f(S)$. Suppose that the map f is the identity on *one* of the circles S (that is, $f(x) = x$ for any $x \in S$), and the surfaces $f(Z)$ and Z lie on the same side of the plane containing the circle $f(S) = S$. Then both the smooth pieces adjacent to the dent and adjoining the circle $f(S) = S$ belong not only to $f(Z)$, but also to Z, by Theorem 1 in [2]. On $f(Z)$, the line segment AB is the shortest path between A and B. On Z, the shortest path between A and B is the arc $\widetilde{AB}$ parallel to the arc $\widetilde{CD} \subset S$. Since the line segment AB is shorter than the arc $\widetilde{AB}$, the surface $f(Z)$ is not isometric to Z. Therefore, $f(Z)$ cannot be the surface shown in Fig. 1, (a) or (b). Thus we have found an error.



(a) (b)

Figure 1. Copy of Fig. 47 in [1]. The left half of the dent (with horizontal generatrices) is on the right.

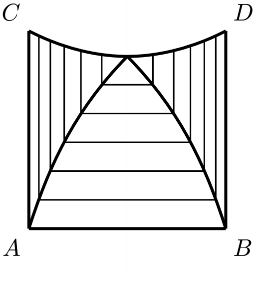


Figure 2. The upper part of the dent (with horizontal generatrices) and the two adjacent pieces (with vertical generatrices).

Unfortunately, Pogorelov did not notice the fact that generatrices of smooth pieces containing arcs of the circle $f(S)$ are *orthogonal to the plane containing* $f(S)$. Otherwise, he would have been aware of the incompatibility of the dent and the two smooth pieces adjacent to it, and then, judging by the text in [1] referring to Fig. 46, he would have rendered the contents of Remark 2 in [2] in one line.

On pages 213–214 of [1] he says: "We have considered an isometric transformation of a cylindrical surface under the condition of axial symmetry of the dents. $\langle\dots\rangle$ …under torsion, an isometric transformation having a periodic structure in the circumferential direction has centrally symmetric dents. $\langle\dots\rangle$ This result is shown in Fig. 48 and needs no explanation." However, the same mistake is present there.

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