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Moments of particle numbers in a branching random walk with heavy tails

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We consider a continuous-time branching random walk (BRW) on $\mathbb{Z}^d$, $d \in \mathbb{N}$, in which a random walk of particles is defined in terms of a matrix $A = (a(x,y))_{x,y\in\mathbb{Z}^d}$ of transition intensities satisfying for all x the condition $\sum_{y\neq x} a(x,y) = |a(x,x)| < \infty$, where $a(x,y) \geqslant 0$ for $y \neq x$ and $a(x,x) < 0$. It is assumed that a particle at a point x moves to a point $y \neq x$ in a small time h with probability $p(h,x,y) = a(x,y)h + o(h)$ and that it remains at x with probability $p(h,x,x) = 1 + a(x,x)h + o(h)$. Let the branching at a source $x_0 \in \mathbb{Z}^d$ be determined by the infinitesimal generating function $f(u) = \sum_{k=0}^{\infty} b_k u^k$ for $0 \leqslant u \leqslant 1$, where $b_k \geqslant 0$ for $k \neq 1$, $b_1 < 0$ and $\sum_{k\neq 1} b_k = |b_1| < \infty$. Also, let $f^{(r)}(1) < \infty$ for all $r \in \mathbb{N}$. Thus, if a particle is at the source x_0, then in a small time h either it moves with probability $p(h,x_0,y) = a(x_0,y)h + o(h)$ to some point $y \neq x_0$, or it does not execute any such transition and either produces $k \neq 1$ descendant particles with probability $p_*(h,x_0,k) = b_k h + o(h)$ which stay at x_0 (we assume that the particle itself is counted here; in the case $k = 0$ we say that it dies), or undergoes no changes with probability $1 - \sum_{y\neq x_0} a(x_0,y)h - \sum_{k\neq 1} b_k h + o(h)$. The value $\beta := f'(1) = \sum_k kb_k$ is called the intensity of the branching source.

We assume that the intensities $a(x,y)$ are symmetric and spatially homogeneous; that is, $a(x,y) = a(y,x) = a(0,y-x)$. For brevity, we set $a(z) := a(0,z)$. We also assume that a random walk is irreducible, that is, for each $z \in \mathbb{Z}^d$ there exist $z_1,\dots,z_k \in \mathbb{Z}^d$ such that $z = \sum_{i=1}^{k} z_i$ and $a(z_i) \neq 0$ for $i = 1,\dots,k$. For a BRW under consideration, the condition $\sum_z a(z)|z|^2 = \infty$ is usually interpreted as the presence of *heavy tails* for the corresponding random walk. The presence of heavy tails, which cause the random walk to have infinite variance of the jumps, takes place, for example, for

$$a(z) \sim \frac{H(z/|z|)}{|z|^{d+\alpha}}, \qquad |z| \to \infty, \tag{1}$$

where $\alpha \in (0,2)$ and $H(\,\cdot\,)$ is a positive continuous symmetric function on the sphere $\mathbb{S}^{d-1} := \{z \in \mathbb{R}^d \colon |z| = 1\}$. According to [3], under the condition (1) the transition probabilities $p(t,x,y)$ behave as follows for fixed x and y as $t \to \infty$: $p(t,x,y) \sim \gamma_{d,\alpha} t^{-d/\alpha}$, where $\gamma_{d,\alpha} > 0$. Let $G_\lambda(x,y) := \displaystyle\int_0^\infty e^{-\lambda t} p(t,x,y)\,dt$. If $G_0(0,0) < \infty$, then the random walk is said to be transient [5]. From (1) it follows that $G_0(0,0) < \infty$ for $d = 1$ and $\alpha \in (0,1)$, and also for $d \geqslant 2$. Note that for $\sum_z a(z)|z|^2 < \infty$ the situation is different: $p(t,x,y) \sim \gamma_d t^{-d/2}$ as $t \to \infty$, where $\gamma_d > 0$, and correspondingly, $G_0(0,0) < \infty$ for $d \geqslant 3$ (see, for example, [4]).

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By $m_n(t,x,y)$ and $m_n(t,y)$, $n \geqslant 1$, we denote the moments of particle numbers at a point $y \in \mathbb{Z}^d$ and on the entire lattice $\mathbb{Z}^d$ at time t, respectively, provided that initially, at $t=0$, there is only one particle at $x \in \mathbb{Z}^d$. An approach to the analysis of the limit structure of the field of particles in a BRW with the help of a study of the moments of the particle numbers at an arbitrary point of $\mathbb{Z}^d$, $d \in \mathbb{N}$, was used in [1] and [4] in the case when $\sum_z a(z)|z|^2 < \infty$. Equations for the moments of the particle numbers were derived, for example, in [2] and [4] without any restrictions on the variance of the jumps of a random walk. In the present note, our aim is to give a classification of the moments of particle numbers. Under the condition (1) this classification exhausts all possible combinations of the parameters α, β, and d.

Theorem. *Let* $\beta_c = G_0^{-1}(0,0)$ *and let* (1) *hold. Then*

$$m_n(t,x,y) \sim C_n(x,y)u_n(t) \quad \textit{and} \quad m_n(t,x) \sim C_n(x)v_n(t) \quad \textit{as } t \to \infty,$$

where

$u_n(t) = e^{n\lambda_0 t}$,	$v_n(t) = e^{n\lambda_0 t}$	*for* $\beta > \beta_c$, $d/\alpha \in (1/2, \infty)$;
$u_n(t) = t^{-1/\alpha}$,	$v_n(t) = t^{(1-1/\alpha)(n-1)}$	*for* $\beta = \beta_c$, $d/\alpha \in (1/2, 1)$;
$u_n(t) = t^{-1}$,	$v_n(t) = (\log t)^{n-1}$	*for* $\beta = \beta_c$, $d/\alpha = 1$;
$u_n(t) = t^{d/\alpha-2}$,	$v_n(t) = t^{(d/\alpha-1)(2n-1)}$	*for* $\beta = \beta_c$, $d/\alpha \in (1, 3/2)$;
$u_n(t) = t^{-1/2}(\log t)^{n-1}$,	$v_n(t) = t^{n-1/2}$	*for* $\beta = \beta_c$, $d/\alpha = 3/2$;
$u_n(t) = t^{(d/\alpha-2)(2n-1)+n-1}$,	$v_n(t) = t^{(d/\alpha-1)(2n-1)}$	*for* $\beta = \beta_c$, $d/\alpha \in (3/2, 2)$;
$u_n(t) = t^{n-1}(\log t)^{1-2n}$,	$v_n(t) = t^{2n-1}(\log t)^{1-2n}$	*for* $\beta = \beta_c$, $d/\alpha = 2$;
$u_n(t) = t^{n-1}$,	$v_n(t) = t^{2n-1}$	*for* $\beta = \beta_c$, $d/\alpha \in (2, \infty)$;
$u_n(t) = t^{1/\alpha-2}$,	$v_n(t) = t^{1/\alpha-1}$	*for* $\beta < \beta_c$, $d/\alpha \in (1/2, 1)$;
$u_n(t) = t^{-1}\log^{-2} t$,	$v_n(t) = \log^{-1} t$	*for* $\beta < \beta_c$, $d/\alpha = 1$;
$u_n(t) = t^{-d/\alpha}$,	$v_n(t) = 1$	*for* $\beta < \beta_c$, $d/\alpha \in (1, \infty)$

and λ_0, $C_n(x,y)$, *and* $C_n(x)$ *are some positive constants.*

The classification of the asymptotic behaviour of the moments of the particle numbers in a BRW with heavy tails is more varied than in the case of finite variance of the jumps [4]. From the above theorem it follows that, under the condition (1) and for α close to 2, the properties of the BRW are close to those of a BRW with finite variance of the jumps (for small α these properties are substantially different). Thus, if a branching process is supercritical ($\beta > 0$) at a source on $\mathbb{Z}^1$, then in the case of finite variance of the jumps the BRW is also supercritical for $\beta > 0$ [2], [4] (for such a BRW the number of particles grows exponentially as $t \to \infty$), and in the case (1) the BRW is supercritical for all $\beta > 0$ and $\alpha \in [1,2)$ (as long as the random walk preserves the recurrence property [2]). For $\alpha \in (0,1)$ the random walks ceases to be recurrent, and so a 'stronger source of particle generation' (that is, the condition $\beta > \beta_c > 0$) is required to achieve a supercritical regime. For $0 < \beta \leqslant \beta_c$ (a supercritical branching process at a source), already on $\mathbb{Z}^1$ the function $m_1(t,x,y)$ for $\beta = \beta_c$ and $\alpha \in (0,1/2)$ tends to some positive constant as $t \to \infty$ (for $\sum_z a(z)|z|^2 < \infty$ the function $m_1(t,x,y)$ can behave in this way only if $d \geqslant 5$), while for $\beta = \beta_c$ and $\alpha \in [1/2,1)$, and also for $0 < \beta < \beta_c$ and $\alpha \in (0,1)$ the function $m_1(t,x,y)$ tends to zero, which is impossible for $\beta > 0$ for a BRW with finite variance of the jumps on $\mathbb{Z}^1$ and $\mathbb{Z}^2$. Note that for $\beta = \beta_c$ seven different cases of asymptotic behaviour as $t \to \infty$ are possible for the pairs $m_1(t,x,y)$ and

$m_1(t,x)$, as classified by the values of d/α, whereas for a BRW with finite variance of the jumps there are only five cases [4].

In the proof of the theorem we use methods from the spectral theory of operators, the Laplace transform method, Tauberian theorems, and some approaches from [2]–[6].

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