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To the blessed memory of Vladimir Voevodsky

Cubillages of cyclic zonotopes

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Abstract. A survey is given of recent results on fine zonotopal tilings by cubes (briefly, cubillages) of cyclic zonotopes. The main interest of this theory is that it is interrelated with the theory of higher Bruhat orders, as well as with the parallel theory of triangulations of cyclic polytopes and Tamari–Stasheff posets, used in investigations of the Kadomtsev–Petviashvili equations and higher Auslander–Reiten algebras.

Bibliography: 35 titles.

Keywords: higher Bruhat orders, Tamari–Stasheff poset, polycategory, rhombus tiling, separated sets, purity.

Contents

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Introduction

Tessellations of figures by other figures are a favourite subject of geometers: recall crystalline lattices and parallelohedra, ball packings, Penrose tilings, and related rhombus tilings. One of the areas that have been developing intensively over the past thirty years concerns triangulations of so-called cyclic polytopes (see [27] and the survey [28] by Rambau and Reiner in [25]). We want to discuss here the largely parallel theory of cubillages of cyclic zonotopes, that is, tilings of zonotopes into (combinatorial) cubes.

Odd as it may seem, motivations to develop this theory were not only geometric. Though zonotopal (and fine zonotopal) tilings have long attracted the attention of geometers (see, for example, [29] or the historical sketch in [35]), two papers that are rather more algebraic or combinatorial have caused a revolution. One of them was the 1998 study [20] by Leclerc and Zelevinsky, which concerned the problem of quasi-commutativity of quantum minors and led to the study of rhombus tilings of zonogons (see [4] for a survey of some achievements in this area). Another was the 1986 paper [23] by Manin and Shekhtman (also motivated by quantum subjects), which was devoted to a generalization of the weak Bruhat order on the symmetric group S_n; the resulting ordered sets $B(n,d)$ have been called higher Bruhat orders. A little later, Voevodskii (Voevodsky) and Kapranov [33] gave an interpretation of these orders in terms of cubillages of cyclic zonotopes, of which the above rhombus tilings provide a particular (two-dimensional) case. Significant advances in this area were obtained by Ziegler [34], Galashin [13], and Galashin and Postnikov [14] (see also [1]).

We are going to discuss all this (including our own results) in this paper. We divide our presentation conditionally into two closely related parts: a geometric part and a combinatorial (or set-theoretic) part. In the first part, we discuss zonotopes and fine zonotopal tilings (cubillages) of zonotopes and introduce various useful objects in a cubillage: pies, tunnels, and so on. We also introduce there our main working tool, reductions and expansions, which are indispensable for inductive reasoning.

After introducing these general concepts, we will restrict ourselves exclusively to the case of cyclic (or, more correctly, totally positive) zonotopes $Z(n,d)$. Cubillages of just these zonotopes are related to higher Bruhat orders in the sense of Manin and Shekhtman. When we work inside a fixed cubillage, the structure of the natural order $\preceq$ on the set of its cubes plays an important role. When we switch to comparing different cubillages, the concept of a flip — a certain local reorganization of a cubillage — comes to the fore. Such a reorganization can most easily be explained by taking the simplest (after the cube) zonotope $Z(d+1,d)$, a so-called capsid, for example. It has only two cubillages, and replacing one by the other is called a flip. In the general case, a flip is replacement of a 'capsid' fragment of a cubillage by another ('flipped') fragment. The main fact concerning cubillages is that flips make it possible to obtain any cubillage of the zonotope $Z(n,d)$ from any other cubillage.

Another important concept is the concept of membranes, hypersurfaces of a special form in a zonogon. We show that any membrane can be embedded in a cubillage.

The combinatorial part of our paper relates cubillages to systems of subsets of the set $[n]=\{1,2,\dots,n\}$, which indexes the vectors generating a cyclic zonotope $Z(n,d)$. There are two methods to do this. The first method considers cubillages from the point of view of their *spectra*. Each cubillage $\mathscr{Q}$ of $Z(n,d)$ (d is the dimension of the zonotope, whereas n is the number of direction vectors) specifies a system $\operatorname{Sp}(\mathscr{Q})$ of subsets of $[n]=\{1,2,\dots,n\}$, that is, a subset of $2^{[n]}$. The set-system $\operatorname{Sp}(\mathscr{Q})$ uniquely determines the cubillage $\mathscr{Q}$. This result raises the question of which set-systems correspond to cubillages, since an answer makes it possible to replace geometry by combinatorics. An answer found by Galashin and Postnikov [14] is that set-systems of this kind have the so-called $(d-1)$-*separation* property and are maximal by size $\Bigl($which is $\dbinom{n}{\leqslant d}\Bigr)$. When maximality by size can be replaced by maximality by inclusion, we speak about the *purity* of the corresponding relation. There is a fundamental investigation of purity problems in [14]. However, that paper is difficult to read, because its authors worked with arbitrary (not necessarily cyclic) zonotopes and even with oriented matroids and therefore used the complicated matroid terminology and techniques. We deliberately confine ourselves to the cyclic case, which makes the presentation simpler and more accessible (in our opinion). In the cases $d=2$ and $d=3$, the property of purity was proved in [20] and [13], respectively. In the case $d\geqslant 4$ the answer is negative. To show this we discuss the situation for $n=d+2$ in detail and then show non-purity for $n-2\geqslant d>3$.

The map of types suggests another transition from geometry to combinatorics. With each cube of a cubillage we can associate its *type*, which is a subset of $[n]$ of size d. This defines a map (which is bijective, as is easy to show) of the cubillage to the set $\dbinom{[n]}{d}$ of subsets of size d in $[n]$. Using this bijection, one can transfer the natural order $\preceq_{\mathscr{Q}}$ from $\mathscr{Q}$ to $\dbinom{[n]}{d}$. It turns out that the transferred order also determines the cubillage $\mathscr{Q}$ uniquely. Thus, instead of cubillages, one can work with the class of so-called *admissible* orders on the discrete Grassmannian $\operatorname{Gr}([n],d)=\dbinom{[n]}{d}$. In this way we return to the origin of the theory, the definition of higher Bruhat orders as admissible orders on the Grassmannian $\dbinom{[n]}{d}$. In the geometric language, the higher Bruhat order $B(n,d)$ is an order on the set $\mathbf{Q}(n,d)$ of cubillages of the zonotope $Z(n,d)$, or equivalently, on the set $\mathbf{M}(n,d)$ of membranes in the zonotope $Z(n,d+1)$.

Cubillages have numerous interrelations with other objects in the world of mathematics. For example, there is a connection with the Kadomtsev–Petviashvili equations, which is so far mainly due to triangulations of cyclic polytopes. We have already noted that the development of the theory was motivated by Zamolodchikov's equations. Cubillages also provide a natural example of polycategories. Cubillages and triangulations are related to representations of quivers and finite-dimensional algebras (for example, see [26]). We will briefly discuss some such 'external' relations of cubillages in the appendices. There, as a rule, we will not give exact definitions and results, indicating only the general outlines of a relationship. There are four appendices in this paper. They concern the relationship with polycategories, the relationship with triangulations (and via them with the Kadomtsev–Petviashvili equations), and a discussion of weak membranes, which generalize the concept of membranes. A proof of the acyclicity theorem is also given in an appendix.

First, geometric part

Here we give the basic geometric concepts and facts concerning cubillages of zonotopes. A combinatorial (set-theoretic) standpoint will be used in the second part. We try to illustrate everything using two-dimensional cubillages (rhombus tilings), and sometimes give three-dimensional pictures.

1. Zonotopes

It is natural to begin by recalling facts about zonotopes[1] and introducing terminology. A cubillage is, roughly speaking, a regular filling (tiling) of a convex figure by 'cubes' or, more precisely, parallelohedra.

What is a zonotope? We fix a real vector space V (of dimension $d>0$) and a finite set $\mathbf{V}=(v_1,\dots,v_n)$ of n vectors in it. A *zonotope* $Z=Z(\mathbf{V})$ (generated by $\mathbf{V}$) is the Minkowski sum of the n line segments $[0,v_i]$, $i=1,\dots,n$. In other words, Z consists of points z of the form $\sum_i \alpha_i v_i$, where $0\leqslant\alpha_i\leqslant 1$ for any $i=1,\dots,n$. We can also say that a zonotope is the projection of the unit n-dimensional cube when the basis vectors of the space $\mathbb{R}^n$ are taken to the vectors v_i. This is a convex body symmetric with respect to its centre $\sum\frac{v_i}{2}$.

Any parallel translation of a zonotope is also called a zonotope. Replacing the vectors v_i by $-v_i$ if necessary, we can assume that all of them look in the same direction, say, upwards with respect to some horizontal hyperplane. Such an operation replaces the zonotope by a translated zonotope. Now our zonotope has the lower (root) vertex 0 and the upper vertex $v_1+\dots+v_n$.

The simplest example of a zonotope is a *cube* (more precisely, a parallelohedron, which for brevity we call a cube). In this case, the system $v_1,\dots,v_n$ forms a basis in the space V (and $n=d$). In what follows, we will deal with fillings of zonotopes by cubes.

[1] There is an extensive literature on zonotopes, including [2] and [35].

It is fairly easy to describe faces of a zonotope (which are also zonotopes of smaller dimension; see [35], Chap. 7, about coding of faces by signed vectors). We will make this problem even simpler. First, we assume that the system $\mathbf{V}$ is in *general position*, that is, that $n \geqslant d$ and any d vectors in the system $\mathbf{V}$ form a basis in V. From now on, this assumption always holds, because we are going to deal with cubillages of a zonotope Z. Second, we restrict ourselves to describing only the facets (faces of codimension 1) of Z. Clearly, facets of a general zonotope are cubes of dimension $d-1$.

To specify a facet, we fix $d-1$ arbitrary vectors $w_1, \dots, w_{d-1}$ (a subsystem $\mathbf{W}$) in the system $\mathbf{V}$. Let $H_{\mathbf{W}}$ be the hyperplane spanned by $\mathbf{W}$ in V. It divides the remaining vector points $\mathbf{V}$–$\mathbf{W}$ into two parts, say, $\mathbf{V}_+$ and $\mathbf{V}_-$. This yields two facets (opposite to each other) of the zonotope Z. One of them is formed by the zonotope (cube) $Z(\mathbf{W})$ rooted at the point $\sum \mathbf{V}_+$, and the other is formed by the same zonotope but rooted at the point $\sum \mathbf{V}_-$. All facets are obtained in this way. This shows that

$$\text{a general zonotope } Z \text{ has } 2\binom{n}{d-1} \text{ facets.} \tag{1.1}$$

We assume here implicitly that $d > 1$. The case of a one-dimensional zonotope is somewhat out of the general picture due to its triviality: it is just the line segment $[0, v_1 + \cdots + v_n]$.

We should also say something about vertices of a zonotope and their number. This is surely well known, but we have not been able to find a good reference. Therefore, we give a formula (and later discuss its proof):

$$\begin{gathered}\text{the number of vertices } v(n,d) \text{ of a zonotope } Z \text{ is}\\ 2\binom{n-1}{\leqslant (d-1)} = 2\left(\binom{n-1}{d-1} + \cdots + \binom{n-1}{0}\right).\end{gathered} \tag{1.2}$$

Another useful thing is to look at a zonotope along some direction. Assume that the direction is specified by some non-zero vector w in V and projects (by means of the map π_w) the space V (and the zonotope Z in it) onto the quotient space $V' = V/\mathbb{R}w$. For convenience, we again assume here that this vector w is in general position with respect to the system $\mathbf{V}$. In this case each facet F of Z is projected injectively to V'. The entire zonotope Z is projected onto the zonotope $Z' = \pi_w(Z) = Z(\pi_w(\mathbf{V}))$. The boundary $\partial Z'$ (of dimension $d-2$) is the bijective image of some (also $(d-2)$-dimensional) subcomplex of the boundary of Z, which can be called the *rim* (with respect to this projection π_w). The rim subdivides the boundary of Z (non-strictly) into two hemispheres: the upper hemisphere (we imagine that the projection π_w is vertical and upwardly directed) and the lower hemisphere.

It is convenient to introduce here the terms *visible* and *invisible* facets. Again, we take a general vector w (along which we look at the zonotope Z). Let F be a facet of Z. This facet F is said to be *visible* (in the direction w) if for a (relative interior) point p of this facet, the point $p - \varepsilon w$ does not belong to Z (for small

$\varepsilon > 0$). Then the point $p+\varepsilon w$ belongs to Z for small $\varepsilon > 0$. In other words, p is the first point in the zonotope when we move from $-\infty$ to $+\infty$ along the line $p+\mathbb{R}w$.

The visible facets cover the lower hemisphere of the zonotope, while the invisible facets cover the upper hemisphere. The projection of the upper hemisphere yields one cubillage of the zonotope Z', whereas the projection of the lower hemisphere yields another cubillage which is symmetric to the former. We will return to this subject in the next section.

One can project not only along general directions but also, for example, along the direction of the vector v_i, denoting this projection by π_i. The π_i-projection of the zonotope Z is again a zonotope $Z' = Z/v_i$, which is generated (in the space $V' = V/\mathbb{R}v_i$) by the images of all the vectors $v_1, \dots, v_n$ except for v_i. However, in this case the rim is not $(d-2)$-dimensional but $(d-1)$-dimensional (and it is called a *zone* or a *belt* of the zonotope). The zonotope Z is the sum of the zonotope $Z(\mathbf{V}-\{v_i\})$ and the line segment $[0, v_i]$.

After introducing these notions, we can proceed to a proof of (1.2). The zonotope $Z = Z(\mathbf{V})$ can be represented as the sum of the zonotope $\widetilde{Z} = Z(\mathbf{V}-\{v_n\})$ and the segment $[0, v_n]$. Its vertices are divided into visible and invisible vertices. However, the visible vertices (with respect to the direction of v_n) of $\widetilde{Z}$ are the same as the visible vertices of Z, and for invisible vertices (more precisely, vertices visible in the opposite direction) the situation is analogous. The only difference is that these two sets do not intersect in the first case, whereas they intersect in the second case exactly in the set of vertices of the rim of the zonotope $\widetilde{Z}$ in the direction v_n. Thus, the numbers of vertices of Z and $\widetilde{Z}$ differ by the number of vertices of this rim, that is, by the number of vertices of the zonotope Z' (being the projection of $\widetilde{Z}$ along v_n). This yields the relation

$$v(n,d) = v(n-1,d) + v(n-1,d-1).$$

In view of Pascal's rule, it remains to verify (1.2) in the cases $d = 1$ and $d = n$. In the first case, the right-hand side is equal to $2\binom{n-1}{0} = 2$, which is consistent with the fact that a line segment has two vertices. In the second case the right-hand side is equal to $2\binom{d-1}{\leqslant d-1} = 2(2^{d-1}) = 2^d$, which is consistent with the fact that a d-dimensional cube has 2^d vertices.

2. Cubillages

A cubillage (a fine zonotopal, hyperrhombus, or parallelohedral tiling) is a regular tesselation (a partitioning, a paving, or a tiling) of a zonotope by cubes. Specifically, a *cubillage* $\mathscr{Q}$ of a zonotope Z is a set of d-dimensional parallelohedra $Q_1, \dots, Q_N$ (called plates, tiles, or just *cubes*) covering Z so that the following two conditions hold:

(1) two cubes can intersect only in a common face;

(2) facets of the zonotope Z are facets of some cubes in $\mathscr{Q}$.

More precisely, this is the definition of a cubillage for zonotopes of dimension $d > 1$. By definition, a cubillage of a one-dimensional zonotope consists of n line segments congruent to the segments $[0, v_i]$, $i = 1, \dots, n$.

A *face* of a cubillage is a face of some cube in $\mathscr{Q}$.

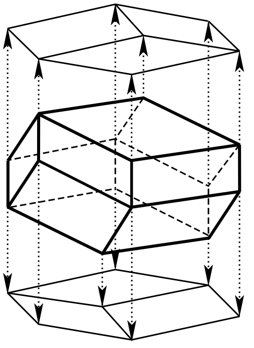

Figure 1. Top and bottom view of a rhombic dodecahedron.

Example 2.1. Let Z be a zonotope $Z(v_1, \dots, v_n)$, and let us project it along, say, the direction of v_n (denoting the projection by π; see Fig. 1). Then the projections of facets of the visible part of the boundary of Z form a cubillage of the zonotope $Z' = Z(\pi(v_1), \dots, \pi(v_{n-1}))$.

This construction is often viewed in the opposite direction. In this case the vector system $\mathbf{V} = \{v_1, \dots, v_n\}$ is called a one-element lifting of the system $\mathbf{V}' = \{\pi(v_1), \dots, \pi(v_{n-1})\}$ (using the vector v_n). A one-element lifting of the system $\mathbf{V}'$ yields a cubillage of the zonotope $Z' = Z(\mathbf{V}')$. The famous Bohne–Dress theorem (see [2] or [35]) states the converse result in a certain sense. We do not give an exact statement of their theorem, since this would draw us away from our purposes.

The following fact, which is often simply included in the definition of a cubillage, is useful.

Lemma 2.2. *Let Q be a cube of some cubillage $\mathscr{Q}$ of a zonotope $Z = Z(\mathbf{V})$. Then any edge of this cube is congruent to some line segment $[0, v_i]$.*

Proof. We can assume that $d > 1$. Let E be some edge of Q. When it is not on the boundary of the zonotope, there are many cubes around it whose edges are congruent to E. When going from one cube of this kind to another, we reach the boundary of the zonotope sooner or later. Then everything follows from the property (2) in the definition of a cubillage. □

In particular, any cube Q of a cubillage is congruent to the zonotope (cube) $Z(\mathbf{W})$, where $\mathbf{W}$ is a d-element subset of $\mathbf{V}$. The set of indices (or *colours*, which are understood as elements of the index set $[n] = \{1, \dots, n\}$) of this subset $\mathbf{W}$ is called the *type* of this cube. In other words, the type of a cube Q is a subset $\tau(Q) = \{i_1, \dots, i_d\}$ of $[n]$ such that Q is a shift of the zonotope (cube) $Z(v_{i_1}, \dots, v_{i_d})$.

The type of a face of a cubillage is understood similarly. In particular, the type of an edge of a cubillage is a one-element subset $\{i\}$ of $[n]$. We orient each edge of the cubillage along the direction of the vector v_i. A directed path from the lower vertex 0 to the upper vertex $v([n])$ along edges of the cubillage is called a *snake*. We will see later that an edge of colour i occurs exactly once in a snake. If $c(i)$ is the colour of the ith edge (arrow) of some snake, then we have a bijection of $[n]$ onto $[n]$, that is, a permutation of $[n]$. This permutation can also be understood as a linear order on $[n]$ in which the minimal element is the colour of the first edge of the snake.

Assigning to a cube Q of a cubillage $\mathscr{Q}$ its type $\tau(Q)$ results in a map $\tau = \tau_{\mathscr{Q}}\colon \mathscr{Q} \to \binom{\mathbf{V}}{d}$ to the set of d-element subsets of $\mathbf{V}$.

Proposition 2.3. *The map τ is a bijection. In particular, the number of cubes in any cubillage is $\binom{n}{d}$.*

We prove this in the next section. Here we give a simple corollary.

Corollary 2.4. *The number of vertices of any cubillage is $\binom{n}{\leqslant d}$.*

Indeed, we choose some general direction w for projecting (viewing). It can be seen from (1.2) that all but two vertices of each cube Q of a cubillage lie on the rim of this cube. There is one interior vertex on the visible half-boundary of Q, and similarly, there is one interior vertex on the invisible half-boundary. We denote the latter vertex of the cube, the farthest from us, by $h(Q)$. It is easy to see that the map h (from $\mathscr{Q}$ to the set of vertices of $\mathscr{Q}$) is injective. Its image contains all vertices except for the ones on the visible part $\partial_- Z$ of the boundary of Z. The $\pi = \pi_w$-projections of the facets of $\partial_- Z$ form a cubillage of the zonotope $Z' = Z(\pi(\mathbf{V}), d-1)$, which, by the induction assumption, has $\binom{n}{d-1} + \dots + \binom{n}{1} + \binom{n}{0}$ vertices.

3. Pies

Let $\mathscr{Q}$ be a cubillage of a zonotope $Z = Z(\mathbf{V})$, where $\mathbf{V} = \{v_1, \dots, v_n\}$. In what follows, we will consider various interesting subsets of $\mathscr{Q}$: pies, tunnels, garlands (tread-beads), stacks, capsids, and so on. We start with pies.

Definition 3.1. The *pie* of colour $i \in [n]$ in a cubillage $\mathscr{Q}$ is the set $\mathscr{P}_i$ of cubes of $\mathscr{Q}$ that contain the colour i in their types, that is,

$$\mathscr{P}_i = \{Q \in \mathscr{Q},\ i \in \tau(Q)\}.$$

(Some authors also call them *de Bruijn sections*.) The *body* of the pie $\mathscr{P}_i$ is the union of the cubes in $\mathscr{P}_i$ as subsets of the vector space V, and is a closed subset of the zonotope $Z = Z(\mathbf{V})$. The body of the pie $\mathscr{P}_i$ intersects the boundary of the zonotope in the belt of colour i (see § 1).

Main Lemma on pies 3.2. *The body of the pie $\mathscr{P}_i$ is a trivial fibration over the zonotope $Z' = \pi_i(Z)$ (where π_i is the projection along the vector v_i; see above) with fibre $[0, v_i]$.*

Proof. In each cube P of the pie $\mathscr{P} = \mathscr{P}_i$, we take the middle cross-section MP parallel to a facet of the cube whose type does not contain the colour i (that is, P is the (suitably translated) Minkowski sum of MP and the line segment $[-v_i/2, v_i/2]$). These cross-sections glue together into a $(d-1)$-dimensional complex $M\mathscr{P}$: the middle cross-section of the pie.

Let p be a point in $M\mathscr{P}$. For sufficiently small $\epsilon > 0$ we consider the ϵ-neighbourhood $\Omega(p, \epsilon)$ of p in Z. If (a) p lies in the interior of the pie $\mathscr{P}$, then $\Omega(p, \epsilon)$ is the d-ball with radius ϵ about p, which lies in the interior of the body of $\mathscr{P}$. If (b) p lies on the boundary of $\mathscr{P}$, then p lies on the middle cross-section of the belt of colour i in the boundary of Z, and $\Omega(p, \epsilon)$ looks (up to a deformation) like a half-ball containing a 'disk' in this belt.

Consider the section $S(p, \epsilon) = \Omega(p, \epsilon) \cap M\mathscr{P}_i$ and its π_i-image in Z'. In the case (a), the neighbourhood $\Omega(p, \epsilon)$ (with a small ϵ) either belongs to one cube of the pie or is divided into pieces of several cubes of the pie that contain a common line segment through p which is parallel to v_i. As a consequence, the cross-section $S(p, \epsilon)$ is homeomorphic to a $(d-1)$-dimensional disk with centre p, and the projection π_i is injective on $S(p, \epsilon)$ and maps it to a small neighbourhood of the point $\pi_i(p)$ in the interior of the zonotope Z'. Similarly, in the case (b) the projection π_i is injective on $S(p, \epsilon)$ and maps it onto a small neighbourhood of $\pi_i(p)$, which is now on the boundary of Z'. (Note that the boundary of $M\mathscr{P}_i$ is the middle cross-section of the belt of colour i of Z, and it is bijectively mapped to the boundary of Z' by π_i.)

Thus, π_i defines a local homeomorphism of a small neighbourhood of each point p of $M\mathscr{P}$ to a neighbourhood of the point $\pi_i(p)$ in Z', that is, it is an étale map of $M\mathscr{P}$ to Z'. In view of the fact that $M\mathscr{P}$ is compact, $\pi_i \colon M\mathscr{P} \to Z'$ is a finite non-ramified covering (for example, see [3], Chap. I, § 4, Corollary 2 to Theorem 1). Since the zonotope Z' is simply connected and $M\mathscr{P}$ is connected (which follows from the connectivity of the middle cross-section of the belt of colour i in Z), this covering has multiplicity one, that is, π_i is a global homeomorphism between $M\mathscr{P}_i$ and Z'. This implies the required assertion. $\square$

Remark 3.3. Pies (or even better, their middle cross-sections) resemble hyperplanes in their properties. Like the latter, they divide the zonotope into two domains (before and after). In addition, any d pies intersect in a single cube (their middle cross-sections intersect in one point). Thus, the system of all pies replaces an arrangement of hyperplanes in a certain sense. It is not without reason that Manin and Shekhtman began this theory with arrangements of hyperplanes.

The projection of the complex $M\mathscr{P}_i$ (or of the entire pie $\mathscr{P}_i$) yields a cubillage $\pi_i(\mathscr{P}_i)$ of the zonotope $Z' = \pi_i(Z) = Z/v_i$. This $(d-1)$-dimensional cubillage is denoted by $\mathscr{P}_i/v_i$ and called the *contraction* of the pie of colour i in the cubillage $\mathscr{Q}$.

The first consequence of the main lemma on pies is the above assertion about snakes. Recall that a *snake* is a directed path along edges (arrows) of a cubillage from the lower (root) vertex of the zonotope to the upper vertex. The colours of the edges of a snake are stated to be bijective to the set $[n]$, that is, each colour $i \in [n]$ occurs exactly once as the colour of an edge of a snake. Indeed, the pie $\mathscr{P}_i$ of colour i divides the zonotope into three parts: the part below $\mathscr{P}_i$, $\mathscr{P}_i$ itself, and the part above $\mathscr{P}_i$. Our snake must intersect the pie (as we move from the root to the top), and thus it must contain an arrow of colour i. Such an intersection is unique since after intersecting the pie the snake is in the upper part of the zonotope and can no longer leave it (all arrows of colour i in the pie point upwards).

This consequence makes it possible to introduce the important notion of the *spectrum* (the set of colours) of a vertex of a cubillage, which will play a central role in the combinatorial part of this study. Specifically, for a vertex v of a cubillage we consider an upwards directed path from 0 to v along edges (arrows) of the cubillage. Obviously, such a path exists (but may not be unique). This path is a starting piece of some snake, and therefore the colours of its edges form a subset of $[n]$. This subset does not depend on the choice of a path. We denote it by $\mathrm{sp}(v)$ and call it the *spectrum* of v. The independence of the choice of a path can be seen from another description of $\mathrm{sp}(v)$: a colour i belongs to $\mathrm{sp}(v)$ if and only if the pie $\mathscr{P}_i$ lies below the vertex v (so that it separates v from the root vertex 0). We also note that $\mathrm{sp}(v)$ does not depend on $\mathscr{Q}$. Indeed, we have

$$v = \sum_{i\in\mathrm{sp}(v)} v_i.$$

Along with pies, we can consider objects in a cubillage $\mathscr{Q}$ which are dual in a certain sense and are called *tunnels*. We fix a subset D of $[n]$ of size $d-1$ and include in a set $\mathscr{T}_D$ the cubes whose type contains D (the tunnel of type D). Each cube Q in $\mathscr{T}_D$ has two facets of type D. Let F be a facet of this kind; then either it is a facet of the zonotope Z or our cube Q is adjacent to another cube Q' in $\mathscr{T}_D$ across this facet. Repeating this procedure with Q', we obtain an adjacent cube Q'', and so on until we reach a facet of Z. (This construction was already used in the proof of Lemma 2.2.) Thus, a tunnel is a set of several thick paths (or cycles) inside a cubillage. In fact, we can easily see that a tunnel is a connected chain of cubes going from one (visible) part of the boundary of Z to another (invisible) part. This is indeed so, since each tunnel of type D intersects each pie $\mathscr{P}_i$ (where $i\notin D$) in exactly one cube (of type Di) and consists precisely of $n-d+1$ cubes.

Let us now turn to the proof of Proposition 2.3 (which is close to Shephard's reasoning in [29] on the number of cubes in a cubillage of a zonotope). We use induction on d. For $d=1$ the assertion is trivial. Now let $d>1$. We take some d-subset $D\subset[n]$ and show that there exists a unique cube Q of type D in our cubillage $\mathscr{Q}$. To do this we choose one of the colours in D (say, i) and consider the pie $\mathscr{P}_i$ of colour i. Let $\pi_i\colon Z\to Z'=\pi_i(Z)$ be the projection along the vector v_i. As we said above, the projection of $\mathscr{P}_i$ yields a cubillage $\pi_i(\mathscr{P}_i)$ of the zonotope Z'. By the induction assumption it contains a unique cube Q' (of dimension $d-1$) of type $D-i$. The cube Q in $\mathscr{P}_i$, projected onto Q', has type $(D-i)\cup\{i\}=D$. This proves the existence and the uniqueness of a cube of type D.

4. Reductions and expansions

We continue deducing consequences of the structure of pies. Again, let us assume provisionally (and for clarity) that the vector v_i (of arbitrary colour) is directed vertically upwards, so that we look at the zonotope Z from the bottom up. The pie

$\mathscr{P} = \mathscr{P}_i$ divides Z into three parts: the pie $\mathscr{P}$ proper; the part lying non-strictly below $\mathscr{P}$ (including the lower boundary of $\mathscr{P}$), which we denote by Z_-; and the upper part Z_+.

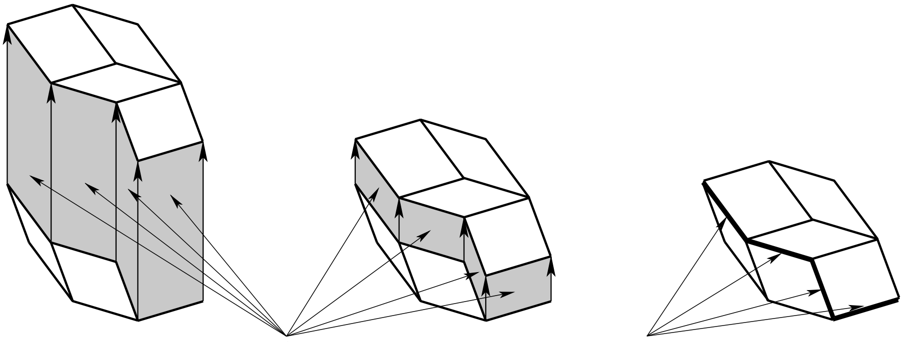

Figure 2. A pie and its gradual reduction.

The *reduction* of a cubillage $\mathscr{Q}$ by a colour i is the cubillage $\mathscr{Q}_{-i}$ of the zonotope $Z(\mathbf{V} - \{v_i\})$ obtained as follows: the lower part Z_- remains intact (along with all its cubes), while the upper part Z_+ is shifted downwards by the vector $-v_i$ (also along with all its cubes). Then the pie $\mathscr{P}$ disappears, or better to say, contracts to its lower boundary, thus turning into what we will later call a membrane. This membrane $\mathscr{M} = Z_- \cap |\mathscr{P}_i|$ is called the *seam* (or scar) remaining after the operation of reduction. See Fig. 2.

The operation of reduction is invertible: if the membrane (seam) $\mathscr{M}$ is widened by means of the line segment $[0, v_i]$ (by shifting the part of the reduced cubillage lying above the seam by the vector v_i), then we return to the original cubillage $\mathscr{Q}$. This inverse operation is called *expansion*.

Generally, if there are a cubillage $\widetilde{\mathscr{Q}}$ of the zonotope $\widetilde{Z} = Z(\mathbf{V} - \{v_i\})$ and a membrane $\mathscr{M}$ (that is, a $(d-1)$-dimensional subcomplex of $\mathscr{Q}$ which is projected bijectively onto $\widetilde{Z}' = \pi(\widetilde{Z})$ under the vertical projection) in it, then we can expand $\mathscr{M}$ by the vector v_i and obtain some cubillage $\mathscr{Q}$ of the zonotope Z. This operation is called the *expansion* of a membrane $\mathscr{M}$ in a cubillage $\widetilde{\mathscr{Q}}$ in the direction of v_i.

Corollary 4.1. *A cubillage of a zonotope $Z(\mathbf{V})$ is determined by the set of its vertices.*

Proof. Let $\mathscr{Q}$ and $\mathscr{R}$ be two cubillages of the zonotope $Z(\mathbf{V})$ which have the same set of vertices. We choose some colour, say, n, and perform the reduction of this colour in $\mathscr{Q}$ and $\mathscr{R}$. As a result, we obtain two new cubillages $\mathscr{Q}_{-n}$ and $\mathscr{R}_{-n}$ (of the zonotope $Z(\mathbf{V} - \{v_n\})$) with seams S and T. The seam S consists of the merged vertices v and v' such that $\mathrm{sp}(v') = \mathrm{sp}(v) \cup \{n\}$. For T the situation is analogous. Hence these two seams coincide, as do the sets of vertices of the cubillages $\mathscr{Q}_{-n}$ and $\mathscr{R}_{-n}$. We deduce by induction that the cubillages themselves coincide: $\mathscr{Q}_{-n} = \mathscr{R}_{-n}$. The original cubillages $\mathscr{Q}$ and $\mathscr{R}$ are obtained by the n-expansion of the common seam $S = T$. Thus, they also coincide. $\square$

Since we are considering vertices of a cubillage, the question arises as to which points (or collections of points) *in the interior of a zonotope* $Z = Z(\mathbf{V})$ *can be vertices of some cubillage of this zonotope* (can be embedded in a cubillage). One requirement is evident (see above for the definition of the spectrum): these points must be of the form $v(S) = \sum_{i\in S} v_i$ for some subset $S \subset [n]$. Points of this form are called *integer* points, and this condition is assumed to hold in what follows. We will return to this problem in the second part of our paper, but for now, we just give a simple result in this direction. Specifically, we claim that any single integer point can be embedded in some cubillage. In fact, even more is true: any subzonotope can be embedded in some cubillage. More precisely, the set of vertices of any subzonotope $T \subset Z$ can be embedded in some cubillage of the zonotope Z. We assume here that T is the shift of the zonotope $Z(\mathbf{W})$ by an integer point $v(S)$, where $\mathbf{W} \subset \mathbf{V}$ and S does not intersect the index set $\mathbf{W}$.

Proposition 4.2. *Any subzonotope* T *of a zonotope* $Z = Z(\mathbf{V})$ (*and any cubillage of* T) *can be embedded in some cubillage of* Z.

Proof. We use induction on n (or on the size of $\mathbf{V}-\mathbf{W}$). Let $t(T)$ be the upper vertex of the subzonotope T. We assume first that this vertex is not the upper vertex of the full zonotope Z. In this case, the set $\mathrm{sp}(t(T))$ differs from $[n]$. Let i be an arbitrary colour not belonging to $\mathrm{sp}(t(T))$. Then T lies in the zonotope $Z' = Z(\mathbf{V} - \{i\})$ and, by the induction assumption, can be embedded in some cubillage $\mathscr{Q}'$ of the zonotope $Z' = Z(\mathbf{V} - \{i\})$. It remains to perform an expansion by the colour i. See Fig. 3.

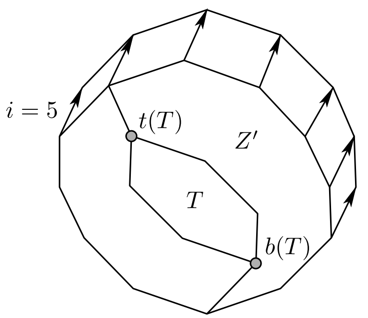

Figure 3. Subzonogon T in the zonogon $Z(7, 2)$.

Hence, we can assume that the upper vertices of T and Z coincide. Arguing symmetrically, we can assume that the lower (root) vertices also coincide. Then T and Z coincide, and the assertion holds trivially. □

Expansion of the membrane $\mathscr{M}$ can be performed not only in the direction of the vector v_i but also in any close direction v_i'. More precisely, it is not important that these directions are close but only that v_i and v_i' point in the same direction with respect to all plates ($(d-1)$-dimensional faces) of $\mathscr{M}$. Expanding the membrane $\mathscr{M}$ in the direction of v_i', we obtain another cubillage of another zonotope, though in a certain sense it resembles the old one, is similar to it. Let us briefly elaborate on this.

When dealing with a zonotope $Z(\mathbf{V})$ and its cubillages, we fixed some vector configuration $\mathbf{V}$. However, it is intuitively clear that small perturbations of $\mathbf{V}$ have no effect on the structure of cubillages. More precisely: assume that we have two configurations of vectors $\mathbf{V} = \{v_1, \dots, v_n\}$ and $\mathbf{W} = \{w_1, \dots, w_n\}$ (in the same d-dimensional space V and with the same number n of vectors). Then $\mathbf{V}$ and $\mathbf{W}$ are said to be *similar* if for any subset $B \subset [n]$ of size d, the determinants $\det(v_B)$ and $\det(w_B)$ have the same sign (or the corresponding bases v_B and w_B have the same orientation).

Provided that the configurations $\mathbf{V}$ and $\mathbf{W}$ are similar, for any cubillage $\mathscr{Q}$ of $Z(\mathbf{V})$ we can naturally construct a similar cubillage $\mathscr{Q}'$ of $Z(\mathbf{W})$ that has the same combinatorial structure as $\mathscr{Q}$. To be precise, let Q be a cube of a cubillage $\mathscr{Q}$; it has a root vertex v and a type $\tau(Q) \subset [n]$. There is a path P from the starting vertex 0 of the zonotope $Z(\mathbf{V})$ to the vertex v along arrows of the cubillage of the zonotope $Z(\mathbf{V})$. We take a corresponding path P' in $Z(\mathbf{W})$ and regard its endpoint v' as the image of the vertex v. Clearly, this construction does not depend on the choice of the path P. The image of Q is a cube of the same type as that of Q, which is rooted at v'. We thus obtain a new set $\mathscr{Q}'$ of cubes in $Z(\mathbf{W})$, and we only need to verify that these cubes do not overlap. It suffices to do this for adjacent cubes, that is, if cubes Q and R of $\mathscr{Q}$ are adjacent (across a common facet F), then their images Q' and R' (which obviously have a common facet F') must be separated by F'.

Assume that the facet F (that is, a cube of dimension $d-1$) has type $\tau(F)$. The cubes Q and R are obtained by supplementing the vectors $v_{\tau(F)}$ with some vectors v_i and v_j. Since they are separated by F, the orientations of the bases $v_{\tau(F)i}$ and $v_{\tau(F)j}$ are opposite. Then the orientations of the bases $w_{\tau(F)i}$ and $w_{\tau(F)j}$ are also opposite, which means that the cubes Q' and R' are separated by F'.

This fact shows that the structure (and variety) of cubillages of $Z(\mathbf{V})$ depends only on the oriented matroid generated by the vector configuration $\mathbf{V}$. We will actively use this simple observation when we consider totally positive configurations.

5. Cyclic (totally positive) vector configurations

So far, configurations $\mathbf{V}$ of vectors in the space V have actually been arbitrary (apart from the condition of general position). From now on, we restrict ourselves to more special configurations commonly said to be cyclic or totally positive. Here and below, we assume that $V = \mathbb{R}^d$ (with basis $e_1, \dots, e_d$). Moreover, the order in which the vectors v_i are indexed will now play an important role (that is, the set of indices (colours) $[n]$ will be endowed with the natural order $1 < 2 < \dots < n$).

Definition 5.1. A vector configuration $\mathbf{V} = (v_1, \dots, v_n)$ is said to be *totally positive* (or, briefly, *cyclic*) if for any (increasing) collection of d colours $i_1 < i_2 < \dots < i_d$ in $[n]$ the determinant of the matrix $\mathrm{Mat}(v_{i_1}, \dots, v_{i_d})$ formed by the column vectors v_{i_j} is positive.

Cyclic configurations are similar to each other and are determined by the two numbers d and $n \geqslant d$. For this reason, in what follows the cyclic zonotope $Z(\mathbf{V})$ will be denoted simply by $Z(n,d)$, while the set of its cubillages will be denoted by $\mathbf{Q}(n,d)$.

A convenient representative for such a configuration can be obtained as follows. We map the line $\mathbb{R}$ to $\mathbb{R}^d$ by the map

$$\mathbf{v}_d(t) = (1, t, t^2, \dots, t^{d-1}).$$

(When d is clear from the context, we omit the index d and just speak of $\mathbf{v}$. The image of the line $\mathbb{R}$ under this map is called the *Veronese curve* or the *moment curve*.) When we take a finite subset $\{t_1 < \dots < t_n\}$ of $\mathbb{R}$, we obtain a configuration of vectors $\mathbf{V}$ consisting of the vectors $v_i = \mathbf{v}(t_i)$, $i = 1, \dots, n$. This configuration is totally positive, as can be seen from the formula for Vandermonde determinants.

For cyclic configurations we note two properties which are important in what follows. When some element k is excluded from $[n]$, the configuration remains cyclic. Thus, we can speak about the projections $\pi_k \colon Z(n,d) \to Z([n]-\{k\},d)$ and the corresponding reductions of cubillages (see above). The second property (applicable to Veronese configurations) consists in the following: when we use the canonical projection π of $\mathbb{R}^d$ onto $\mathbb{R}^{d-1}$ (the projection along the last basis vector e_d, or the 'forgetting' of the dth coordinate), a cyclic configuration is taken to a cyclic configuration. In particular, we obtain the projection $\pi \colon Z(n,d) \to Z(n,d-1)$, which is useful for induction on d. We also note that this projection π (along the dth coordinate axis) can be understood as the projection along the direction of $\mathbf{v}(+\infty)$, since the direction of the vector $\mathbf{v}(t)$ tends to the direction of e_d for large t. The interest in cyclic configurations is explained by the fact that cubillages of cyclic zonotopes are closely related to higher Manin–Shekhtman orders, which we will discuss in the combinatorial part of this paper.

We have already discussed the reduction of colours for general zonotopes. In the case of cyclic zonotopes, the reduction of the highest colour n plays a particularly important role. The thing is that the seam remaining after such a reduction is a membrane with respect to the projection π along the dth coordinate axis. We will go more into this in the next section.

6. Membranes

We recall the definition of a membrane in a cubillage.

Definition 6.1. Let $\mathscr{Q}$ be a cubillage of the cyclic zonotope $Z = Z(n,k)$. A *membrane* of $\mathscr{Q}$ or a membrane *embedded* in $\mathscr{Q}$ is a $(d-1)$-dimensional subcomplex $\mathscr{M}$ of $\mathscr{Q}$ that is projected bijectively onto the zonotope $Z(n,d-1)$ (under the projection π along the last, dth, coordinate).

The set of membranes of a cubillage $\mathscr{Q}$ is denoted by $\mathbf{M}(\mathscr{Q})$.

Therefore, a membrane is an $(d-1)$-dimensional film in Z (formed by faces of $\mathscr{Q}$) which is homeomorphic to the $(d-1)$-dimensional disk whose border is exactly the rim of the zonotope Z with respect to the projection π. A membrane subdivides the zonotope into two parts: the part located non-strictly before the membrane

$(Z_-(\mathscr{M}))$ and the part located non-strictly after it $(Z_+(\mathscr{M}))$. The projection of cells of $\mathscr{M}$ (its $(d-1)$-dimensional cubes called *plates*) yields a cubillage of the zonotope $Z' = Z(n, d-1) = \pi(Z)$, which we denote by $\pi(\mathscr{M})$.

It is not *a priori* clear whether there even exist membranes or how many of them there are. We will see later that there are a lot of them. For the present, we give an example of two important membranes.

Example 6.2. We have already discussed this in § 2. Let us go back in the cyclic case. We consider the visible part of the boundary of Z (in the direction of the dth coordinate), which we denote by $\partial_- Z$. Obviously, it is a membrane (of any cubillage of the zonotope Z), and its projection $\pi(\partial_- Z)$ yields a cubillage of the zonotope Z' which we say is *standard* (the corresponding rhombus tilings of a two-dimensional zonogon were so called in [4]). The invisible part $\partial_+ Z$ of the boundary yields an *antistandard* cubillage of Z'.

A universal way to obtain membranes is reduction of the highest colour. Let $\mathscr{Q}$ be a cubillage of the zonotope $Z = Z(n, d)$. Let $\widetilde{\mathscr{Q}} = \mathscr{Q}_{-n}$ be the cubillage of the zonotope $\widetilde{Z} = Z(n-1, d)$ obtained by reduction of the colour n. Then the seam $\mathscr{S}$ from this reduction is a membrane. In fact, replacing the vector v_n by the dth basis vector $e_d = (0, \dots, 0, 1)$ in $\mathbb{R}^d$ leads to a similar configuration of vectors. Therefore (see § 4), we can expand the seam $\mathscr{S}$ in the direction of e_d and obtain a cubillage $\mathscr{Q}'$ similar to $\mathscr{Q}$. This means (see the main theorem in § 3) that $\mathscr{S}$ is projected bijectively onto the zonotope $\pi(Z) = Z(n, d-1)$.

Conversely, given a membrane $\mathscr{M}$ in a cubillage $\widetilde{\mathscr{Q}}$ of the zonotope $\widetilde{Z} = Z(n-1, d)$, we can expand this membrane in the direction of a new vector $v_n = \mathbf{v}(t_n)$, where $t_n > t_{n-1}$. The thing is that the direction specified by v_n is also transversal to $\mathscr{M}$ (as is also the direction of $\mathbf{v}(+\infty)$). We discussed that in § 4. As a result, we obtain a new cubillage $\mathscr{Q}$ of $Z = Z(n, d)$. The 'widened' membrane $\mathscr{M} + [0, v_n]$ becomes the pie of the highest colour n in $\mathscr{Q}$. Reducing this colour, we return to the original cubillage $\widetilde{\mathscr{Q}}$.

Thus, the pair $(\widetilde{\mathscr{Q}}, \mathscr{M})$, where $\widetilde{\mathscr{Q}}$ is a cubillage of the zonotope $\widetilde{Z} = Z(n-1, d)$ and $\mathscr{M}$ is a membrane in $\widetilde{\mathscr{Q}}$, determines a cubillage $\mathscr{Q}$ of the zonotope $Z = Z(n, d)$. All cubillages of Z can be obtained in this way. Indeed, let $\mathscr{Q}$ be an arbitrary cubillage of Z. We take the pie $\mathscr{P}$ of colour n in it and reduce it. The seam $\mathscr{S}$ from this reduction, as already mentioned, is a membrane for the reduced cubillage $\widetilde{\mathscr{Q}} = \mathscr{Q}_{-n}$ of the reduced zonotope $Z(n-1, d)$. Expansion of this membrane $\mathscr{S}$ takes us back to the original cubillage $\mathscr{Q}$.

This discussion can be summarized as follows.

Proposition 6.3. *Specifying a cubillage of the zonotope $Z(n, d)$ is the same as specifying the pair $(\widetilde{\mathscr{Q}}, \mathscr{M})$, where $\widetilde{\mathscr{Q}}$ is a cubillage of the zonotope $Z(n-1, d)$ and $\mathscr{M}$ is a membrane for $\widetilde{\mathscr{Q}}$.*

In other words, there exists a natural surjective map

$$\mathbf{Q}(n, d) \to \mathbf{Q}(n-1, d).$$

Its fibre over any point $\mathscr{Q} \in \mathbf{Q}(n-1, d)$ coincides with the set $\mathbf{M}(\mathscr{Q})$ of membranes for $\mathscr{Q}$.

Example 6.4. The standard cubillage of the zonotope $Z(n,d)$ can be obtained from the standard cubillage of the zonotope $Z'=Z(n-1,d)$ using the rear membrane $\partial_+ Z'$ (the invisible part of the boundary of Z'). See Fig. 7, where $d=2$.

Indeed, we have defined the standard cubillage of $Z(n,d)$ to be the image of the front (visible) membrane $\partial_- Z(n,d+1)$ of the zonotope $Z(n,d+1)$. But what does the latter look like? The zonotope $Z(n,d+1)$ can be obtained from $Z(n-1,d+1)$ by adding the line segment $[0,\mathbf{v}_{d+1}(n)]$. Correspondingly, its visible boundary consists of the visible boundary ∂_- of $Z(n-1,d+1)$ plus facets of the form $F+[0,\mathbf{v}_{d+1}(n)]$, where F runs through the set of invisible facets (in the direction of the coordinate d) of this visible boundary ∂_-. When we project along the coordinate $d+1$, the first part turns into the standard cubillage of the zonotope $Z'=Z(n-1,d)$, while the facets F turn into invisible facets of this Z', thus forming together the invisible boundary of Z'. Therefore, the second part is projected exactly onto the expansion (by the colour n) of the rear (invisible) membrane of Z'.

Similarly, the antistandard cubillage can be obtained by the expansion of the front (visible) membrane $\partial_- Z'$ of the antistandard cubillage of the zonotope $Z'=Z(n-1,d)$ by the colour n.

7. Cubes and flops

We consider more closely the case of the cube, that is, the zonotope $C=Z(d,d)$. It has only two membranes: the visible (front) membrane $\partial_- C$ and the invisible (rear) membrane $\partial_+ C$, which yield the standard and antistandard cubillages of the zonotope $Z(d,d-1)$ after the projection π. It can be seen from counting the number of vertices of the rim (see (1.2)) that there is exactly one interior vertex t on the front membrane (we call it the *tail*) and one interior vertex h on the rear membrane (we call it the *head*).

We will speak more specifically about the spectra of these vertices. For this purpose, we return to § 1, where we described the facets of a zonotope. We can do the same here to describe the vertices t and h. Recall that the vectors $v_1,\dots,v_d$ were specified by numbers $t_1<\dots<t_d$, and let $v_i=\mathbf{v}(t_i)$. Take some numbers $s_1,\dots,s_{d-1}$ alternating with the t_j, that is, $t_1<s_1<t_2<\dots<t_{d-1}<s_{d-1}<t_d$. As a linear functional on the space $\mathbb{R}^d$, take $\det(\mathbf{v}(s_1),\dots,\mathbf{v}(s_{d-1}),\cdot)$. Note that this functional is positive on the vectors v_d and e_d. It is also positive on the vectors v_i when i has the same parity as d, and it is negative on v_i when the parity of i is opposite to that of d. It follows that the head vertex h is the sum of vectors v_i such that $d-i$ is even, while the tail vertex t is the sum of vectors v_i such that $d-i$ is odd. Thus, $\mathrm{sp}(h)=\{d,d-2,\dots\}$, whereas $\mathrm{sp}(t)=\{d-1,d-3,\dots\}$. For example, in the case $d=3$ (see Fig. 4) we have $\mathrm{sp}(t)=2$ and $\mathrm{sp}(h)=13$ (here and below, 13 denotes the set $\{1,3\}$). Obviously, the union of these sets (spectra) yields the entire set $[d]$.

The $t\to h$ arrow is called the *chord* of the cube C. In the case when d is even, the chord is horizontal, whereas it raises by 1 for odd d, as can be seen from Fig. 4. Here the difference between even and odd d manifests itself for the first time.

After considering an isolated cube, we pass to the general situation. Let $\mathscr{Q}$ be a cubillage of the zonotope $Z=Z(n,d)$ and let Q be some cube of this cubillage,

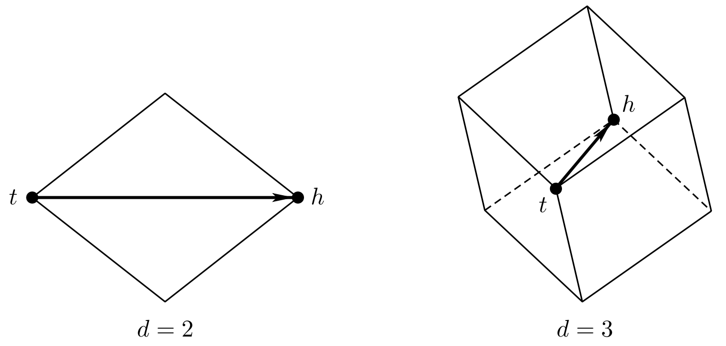

Figure 4. The tail and the head.

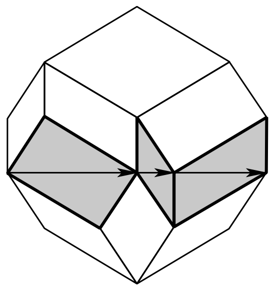

Figure 5. Garland in the zonogon $Z(5, 2)$.

with tail t_Q, head h_Q, and chord $t_Q \to h_Q$. When we do this with each cube of the cubillage, the chords give the structure of an oriented graph on the set of vertices of $\mathscr{Q}$. This graph consists of the paths (threads) connecting (interior) vertices of the front membrane $\partial_- Z$ with (interior) vertices of the rear membrane $\partial_+ Z$. Each cube is 'strung' on its chord, and the cubes strung on paths look like garlands (see Fig. 5). Assuming that the vertices of the rim are connected with themselves by paths of length 0, we obtain an interesting (and little studied) garland bijection of the set of vertices of the front membrane to the set of vertices of the rear membrane of Z. In fact, we obtain more: something like a braid going from the vertices of the front membrane to the vertices of the rear membrane.

Of course, we cannot limit ourselves to boundary membranes. If $\mathscr{M}$ and $\mathscr{M}'$ are two arbitrary membranes (in the same cubillage), then each garland pierces each membrane exactly once. Thus, we obtain a garland bijection between the set of vertices of $\mathscr{M}$ and the set of vertices of $\mathscr{M}'$. This shows again that the number of vertices of each membrane is the same as that of the standard membrane. In fact, we have more: all membranes in the zonotope $Z(n, d)$ with even d have the same number of vertices at each height. Or: each cubillage of the zonotope $Z(n, d)$ with odd d has the same number of vertices on each level.

Example 7.1. We illustrate the garland bijection using the standard cubillage of the zonotope $Z(4, 3)$ as an example. As already noted in the previous section, this cubillage is obtained from the cube $C = Z(3,3)$ using the expansion by the colour 4 of the rear side of this cube. The rear side $\partial_+ C$ consists of three plates (rhombi), and hence the standard cubillage of $Z(4,3)$ consists of four cubes: the original cube C and the three cubes expanded from these plates. All this is shown in Fig. 6, where the first cube C is pulled out and slightly moved away for clarity.

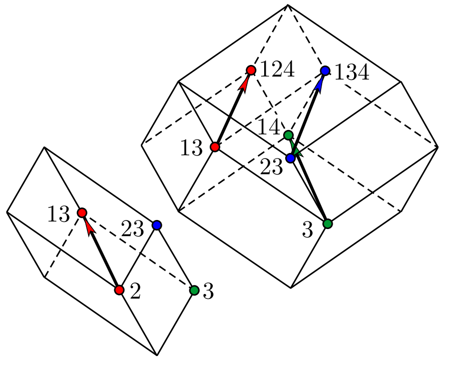

Figure 6. Garland map for the standard cubillage of the zonotope $Z(4,3)$.

The front side (or the membrane) of $Z(4,3)$ contains three vertices with the spectra 2, 3, and 23, in addition to the peripheral vertices (lying on the rim). Since the garland map γ_S is trivial on the rim, we need to see where these three vertices 2, 3, and 23 are mapped to by γ_S. It can be seen from Fig. 6 that 2 is first mapped to the point 13, which is then mapped to the point 124. The point 3 is sent to 14, while the point 23 is taken to 134. Finally, we have

$$\gamma_S\colon \begin{cases} 2 \mapsto 124, \\ 3 \mapsto 14, \\ 23 \mapsto 134. \end{cases}$$

Reasoning similarly, one can verify that the garland map γ_A (for the antistandard cubillage of $Z(4,3)$) is structured as follows:

$$\gamma_A\colon \begin{cases} 2 \mapsto 14, \\ 3 \mapsto 134, \\ 23 \mapsto 124. \end{cases}$$

However, this can also be seen from the following general considerations. An arbitrary zonotope Z is symmetric with respect to its centre. We denote this symmetry by α. For any cubillage $\mathscr{Q}$ we can consider the symmetric cubillage $\alpha\mathscr{Q}$ of the same zonotope Z. Then the standard cubillage is transformed into the antistandard cubillage, and vice versa. We let S denote the vertices of the zonogon on the front side of Z, and we let A denote the vertices on the rear side. The involution α gives a bijection of S to A, and a subset X is mapped to its complement $[n] - X$.

Proposition 7.2. *If $\gamma_{\mathscr{Q}}\colon S \to A$ is a garland bijection associated with a cubillage $\mathscr{Q}$, then $\gamma_{\alpha\mathscr{Q}} = \alpha^{-1}\gamma_{\mathscr{Q}}\alpha$.*

Proof. Each chord of a cube Q is mapped to the oppositely directed chord of the cube αQ by the involution α. □

Let $\mathscr{M}$ be a membrane in some cubillage $\mathscr{Q}$. Assume that some cube Q of $\mathscr{Q}$ touches the membrane along the entire visible boundary $\partial_- Q$. (We will see below that such a situation is not exceptional and is constantly encountered.) In this case we can form a new membrane $\mathscr{M}'$ coinciding with $\mathscr{M}$ everywhere except that this fragment $\partial_- Q$ is replaced by the new fragment $\partial_+ Q$. In other words, if the membrane $\mathscr{M}$ was positioned ahead of the cube C, then it is now behind C. We say that $\mathscr{M}'$ is obtained from $\mathscr{M}$ by a *raising flop* (while $\mathscr{M}$ is obtained from $\mathscr{M}'$ by a *lowering flop*).

This can be expressed slightly differently. We let $\mathscr{Q}_-(\mathscr{M})$ denote the collection of cubes of $\mathscr{Q}$ located before the membrane $\mathscr{M}$. Then $\mathscr{Q}_-(\mathscr{M}')$ is obtained by adding exactly one cube Q to $\mathscr{Q}_-(\mathscr{M})$. Thus, the membrane is moved away from us by raising flops, capturing cubes of the cubillage one by one in turn, until it reaches the rear boundary of the zonotope.

8. Capsids and flips

After thoroughly treating the case of the cube, we naturally pass to the zonotope $Z = Z(d+1, d)$ next in complexity. For brevity we will call this zonotope (or a subzonotope of a larger zonotope $Z(n, d)$) a *capsid.*[2]

The capsid can be represented as a projection of the $(d+1)$-dimensional cube $\widehat{Z} = Z(d+1, d+1)$. The projection of the front membrane of this cube yields the standard cubillage of the capsid Z. The projection of the tail $t_{\widehat{Z}}$ yields the only central vertex (not lying on the rim) of the standard cubillage, which we denote by c^{st}. Its spectrum $\mathrm{sp}(c^{\mathrm{st}})$ is $\{d, d-2, \dots\}$. This vertex is surrounded by cubes of the standard cubillage of the capsid on all sides, and it is a vertex of each cube of this cubillage. Symmetrically, the invisible membrane contains the central vertex c^{an} of the antistandard cubillage of the capsid, with the spectrum $\{d+1, d-1, \dots\}$. For example, for $d = 2$ we have

$$\mathrm{sp}(c^{\mathrm{st}}) = 2 \quad \text{and} \quad \mathrm{sp}(c^{\mathrm{an}}) = 13,$$

while for $d = 3$ we have

$$\mathrm{sp}(c^{\mathrm{st}}) = 13 \quad \text{and} \quad \mathrm{sp}(c^{\mathrm{an}}) = 24.$$

From Corollary 4.1 we conclude that the capsid has only the two cubillages described above. Let us summarize the above.

Proposition 8.1. *The capsid $Z(d+1, d)$ admits only two cubillages: the standard and antistandard cubillages. The standard cubillage is characterized by the fact that it contains the vertex c^{st} with the spectrum $\{d, d-2, \dots\}$ in addition to the vertices*

[2] The term was borrowed from virology. It is the name of the protein shell of a virus, which is usually polyhedral (like a rhombic dodecahedron, that is, $Z(5, 3)$).

of the rim. Symmetrically, the antistandard cubillage is characterized by the fact that it contains the vertex c^{an} *with the spectrum* $\{d+1, d-1, \dots\}$ *in addition to the vertices of the rim.*

The assertion about the two cubillages of the capsid can also be deduced from Proposition 6.3, which says that specifying a cubillage of the capsid $Z(d+1,d)$ is the same as specifying a membrane in the cube $C = Z(d,d)$. However, the cube has only two membranes: the visible part of the boundary $\partial_- C$ and the invisible part of the boundary $\partial_+ C$. Correspondingly, the capsid $Z(d+1,d)$ has two cubillages. One of them is obtained by the $(d+1)$-expansion of the rear membrane $\partial_+ C$; this is the standard cubillage. The other (antistandard) cubillage is obtained by the expansion of the front membrane $\partial_- C$. See Fig. 7.

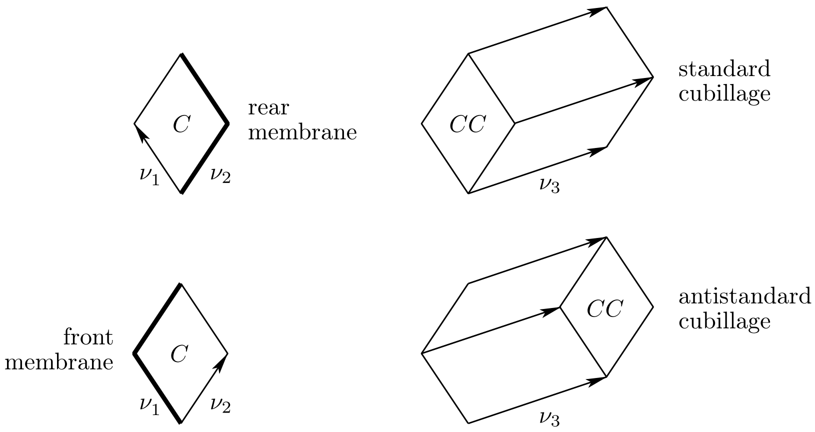

Figure 7. Expansion of the membranes of the cube.

We now assume that there is a cubillage $\mathscr{Q}$ of an arbitrary zonotope $Z(n,d)$ and a fragment in it of the form $Z(d+1,d)$ (that is, the boundary of this capsid is a subcomplex of the complex $\mathscr{Q}$). The restriction of $\mathscr{Q}$ to this capsid $Z(d+1,d)$ is the standard or antistandard cubillage. Replacing the cubillage of this fragment by the opposite cubillage is called a *flip* (a *raising* flip if the standard fragment is replaced by the antistandard one, and a *lowering* flip otherwise). Flips transform $\mathscr{Q}$ into some new cubillage $\mathscr{Q}'$ which differs from $\mathscr{Q}$ only locally, inside the capsid.

The connection of flips with the flops considered in the previous section is obvious. Assume that a cubillage $\mathscr{Q}$ of the zonotope $Z(n,d)$ is obtained as the projection of a membrane $\mathscr{M}$ in a cubillage $\widehat{\mathscr{Q}}$ of the larger zonotope $\widehat{Z} = Z(n,d+1)$, and perform a (say, raising) flop with this membrane, thus replacing it by a membrane $\mathscr{M}'$. Then the projection $\mathscr{Q}'$ of the new membrane $\mathscr{M}'$ is obtained by applying a (raising) flip to the cubillage $\mathscr{Q}'$.

We will see below that there are many flips, at least in the sense that by using lowering flips one can obtain the standard cubillage $\mathscr{Q}_{\mathrm{st}}$ from any cubillage $\mathscr{Q}$. However, regardless of this fact, we can introduce a notion of *order* on the set $\mathbf{Q}(n,d)$. More precisely, $\mathscr{Q} \leqslant \mathscr{Q}'$ if the cubillage $\mathscr{Q}'$ can be obtained from the cubillage $\mathscr{Q}$ using a series of raising flips. To show that there are many flips, it is

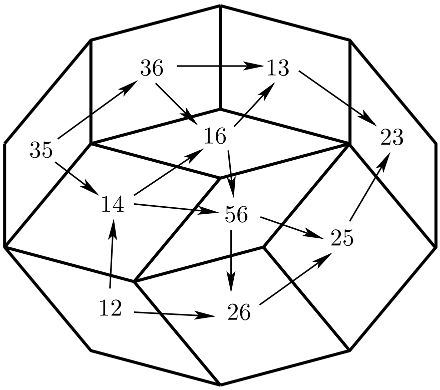

Figure 8. The relation $\prec$ on a cubillage of the zonogon $Z(5, 2)$.

useful to introduce the important notion of the natural order on the set of cubes of an arbitrary cubillage.

9. Ordering cubes in a cubillage

Let $\mathscr{Q}$ be a cubillage of the (cyclic) zonotope $Z(n, d) \subset \mathbb{R}^d$. It turns out that the cubes of this cubillage can be partially ordered in a natural way. Roughly speaking, this is the ascending order of the dth coordinate. We can formulate this more correctly: the order is based on whether one cube is positioned after another when viewed in the direction of the dth coordinate.

Let Q be a cube of $\mathscr{Q}$ and let F be a facet of Q. Then this facet is said to be *visible* (or illuminated) if there exists a line (in $\mathbb{R}^d$) that is parallel to the coordinate vector e_d and that first intersects F and only then enters Q. The opposite (to F in Q) facet is said to be *invisible* or shaded. Obviously, half of the $2d$ facets of a cube are visible and the rest are shaded.

Now let Q and Q' be two cubes of $\mathscr{Q}$. Then Q is said to *immediately precede* Q' (which we denote by $Q \prec Q'$) if Q and Q' are adjacent across a facet F which is invisible for Q and visible for Q'. Thus, a binary relation $\prec$ (or more precisely, $\prec_{\mathscr{Q}}$) is defined on $\mathscr{Q}$.

Fig. 8 suggests the following generalization.

Proposition 9.1. *The relation $\prec$ is acyclic.*

This assertion (as well as the following lemma) is proved in Appendix D. The lemma applies to the following situation. Let $\mathscr{P} = \mathscr{P}_n$ be the pie of the highest colour n in a cubillage $\mathscr{Q}$ of the zonotope $Z = Z(n, d)$. Reduction of this pie yields a cubillage $\widetilde{\mathscr{Q}}$ of the zonotope $\widetilde{Z} = Z(n-1, d)$ and a seam (membrane) $\mathscr{M}$ in $\widetilde{\mathscr{Q}}$. By projecting $\mathscr{M}$ along the dth coordinate axis e_d, we obtain the cubillage $\mathscr{Q}' = \pi(\mathscr{M})$ of the zonotope $Z' = Z(n, d-1)$.

Now let Q and R be two cubes of $\mathscr{P}$ related by the immediate precedence relation $R \prec Q$. Reduction transforms the cubes Q and R into plates (facets) $\gamma(Q)$ and $\gamma(R)$ of the membrane $\mathscr{M}$. Their images under the projection π are cubes Q' and R'

of the cubillage $\mathscr{Q}'$. It is obvious that Q' and R' are also adjacent in $\mathscr{Q}'$ and can be compared by the immediate precedence relation $\prec'$ in $\mathscr{Q}'$.

Lemma 9.2 (on the reverse). *If R and Q are cubes in the pie $\mathscr{P}$ and $R \prec Q$, then $Q' \prec' R'$.*

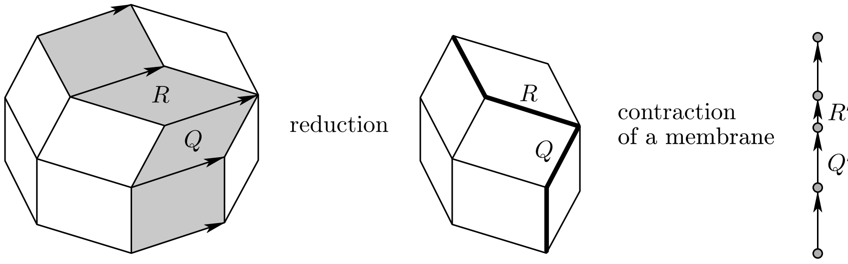

Figure 9. On the left, the cubes R and Q in a pie satisfy the relation $R \prec Q$. On the right, the cubillage $\mathscr{Q}'$ obtained by contracting the pie is shown. The reverse relation $Q' \prec R'$ holds for the corresponding cubes R' and Q'.

A direct corollary of the lemma on the reverse is that the restriction of the relation $\prec$ to the pie $\mathscr{P}$ (of colour n) is the reverse of the relation $\prec'$ for the cubillage $\mathscr{Q}'$ (here we identify the set $\mathscr{P}$ with $\mathscr{Q}'$ via the maps $P \mapsto \gamma(P) \mapsto P'$). This is shown for the zonotope $Z(5,2)$ in Fig. 9.

Definition 9.3. The *natural order* on a cubillage $\mathscr{Q}$ is the reflexive transitive closure of the relation $\prec$; we denote it by $\preceq$ or $\preceq_{\mathscr{Q}}$.

By definition, the relation $\preceq$ is a pre-order. The non-trivial part is that it is an *order*, that is, it is antisymmetric ($Q \preceq Q'$ and $Q' \preceq Q$ imply that $Q = Q'$). It is this fact that yields Proposition 9.1.

Example 9.4. Let Q and Q' be two cubes in the same tunnel and let Q' be positioned after Q as we go from the visible boundary of the zonotope toward the invisible boundary. Then $Q \preceq Q'$.

Example 9.5. Let the cube Q partially shade Q', that is, assume that there is a line parallel to the coordinate vector e_n that pierces (intersects in an interior point) Q prior to Q'. Then $Q \preceq Q'$. Indeed, we need to consider a chain of cubes $Q_1, \dots, Q_k$ crossed by this line on the way from Q to Q'. Then we have $Q \prec Q_1 \prec \dots \prec Q_k \prec Q'$.

Note that we could start from this stronger shading relation and obtain (upon transitive closure) the same relation $\preceq$.

Example 9.6. Let the cube Q' be located immediately after a cube Q in some garland (that is, the head of Q coincides with the tail of Q'; see § 7). Then Q shades Q', and in view of the previous example $Q \preceq Q'$.

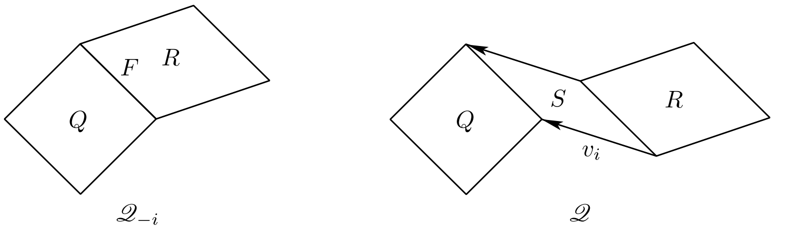

Figure 10

Let $\mathscr{Q}$ be a cubillage and let $\mathscr{Q}_{-i}$ be the reduction of $\mathscr{Q}$ by the colour i. As a set of cubes, $\mathscr{Q}_{-i}$ is identified with $\mathscr{Q}-\mathscr{P}_i$, and thus one can regard $\mathscr{Q}_{-i}$ as a subset of $\mathscr{Q}$.

Proposition 9.7. *The restriction of the order relation $\preceq$ to $\mathscr{Q}_{-i}$ is not weaker than the order relation $\preceq_{-i}$ on $\mathscr{Q}_{-i}$.*

In other words, if $Q \preceq_{-i} R$ for cubes Q and R in the reduced cubillage $\mathscr{Q}_{-i}$, then $Q \preceq R$.

To prove this fact we need to verify that if Q immediately precedes R in $\mathscr{Q}_{-i}$, then $Q \preceq R$. The relation $Q \prec_{-i} R$ means that the cubes Q and R are adjacent in $\mathscr{Q}_{-i}$ across a facet F, and Q is positioned before R. Then two cases are possible. The first is where F is not a plate of the reduction seam $\mathscr{S}$. Then Q and R are adjacent even in $\mathscr{Q}$, and everything is obvious. The second case is where F belongs to the seam. Then it has been obtained by the reduction of the colour i in the cube $S = F + [0, v_i]$. Obviously, we have $Q \prec S \prec R$ (in the cubillage $\mathscr{Q}$; see Fig. 10).

10. The order on cubillage of a capsid

Proposition 9.7 works most simply for the standard cubillage. The thing is that the reduction $\widetilde{\mathscr{Q}} = \mathscr{Q}_{-n}$ of the standard cubillage $\mathscr{Q} = \mathscr{Q}_{\mathrm{st}}$ of the zonotope $Z(n,d)$ by the colour n is the standard cubillage (of the zonotope $Z(n-1,d)$). Moreover, contraction of the pie $\mathscr{P} = \mathscr{P}_n$ of the colour n in $\mathscr{Q}$ yields the antistandard cubillage of the zonotope $Z' = Z(n-1, d-1)$. We can use these considerations to completely describe the natural order on the standard cubillage. We will do this in the case $n = d+1$, that is, for the capsid.

Let $\mathscr{Q}$ be the standard cubillage of the capsid $Z = Z(d+1, d)$. The reduced cubillage $\widetilde{\mathscr{Q}}$ is the trivial cubillage of the cube $\widetilde{Z} = Z(d,d)$. The pie $\mathscr{P}$ (of the colour $n = d+1$) adjoins the invisible (rear) part $\partial_-\widetilde{Z}$ of the boundary of $\widetilde{Z}$. More precisely, it is obtained as the sum of this rear part of the boundary and the segment $[0, v_n]$. Therefore, cubes in the pie have the form $F + [0, v_n]$, where F runs through the invisible facets of the cube $\widetilde{Z}$ (see Figs. 6 and 7). The relation $Q \succ \widetilde{Z}$ holds for any cube Q of this kind. It remains to clarify the situation with the ordering of the cubes of the pie $\mathscr{P}$. By Lemma 9.2 on the reverse, this order is the reverse of the order of the cubillage obtained by the projection π of the rear membrane $\partial_+\widetilde{Z}$, that is, of the antistandard cubillage of the smaller capsid $Z' = Z(d, d-1)$. Hence, the order on the cubes of $\mathscr{P}$ is exactly the order on the standard cubillage of the capsid $Z(d, d-1)$. Using induction on d, we deduce the following proposition.

Proposition 10.1. *The natural order on the standard cubillage of the capsid $Z(d+1,d)$ is total (or linear).*

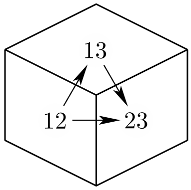

Figure 11. The order on the standard tiling of the zonogon $Z(3,2)$.

In other words, this ordering is a chain of $n = d+1$ elements. Moreover, we see that even the relation $\prec$ is transitive in this case. (See Fig. 11 for an example in the case $n = 3$.) The cube $\widetilde{Z}$ is the minimal element with respect to this order. Note that its type is $[d] = [n] - n$. The same induction as above shows that in terms of cube types the order on the standard cubillage of the capsid $Z(d+1,d)$ coincides with the lexicographic order $\prec_{\text{lex}}$ on the set $\mathrm{Gr}([d+1],d)$ of subsets of $[d+1]$ of size d. For example, for $d = 6$ this order is as follows:

$$12345 \prec 12346 \prec 12356 \prec 12456 \prec 13456 \prec 23456.$$

For the antistandard cubillage of the capsid the order $\prec$ is also linear, but it coincides with the reverse, antilexicographic, order $\prec_{\text{alex}}$ when expressed in terms of types.

We leave as an exercise an analysis of the case next in complexity: the standard cubillage of the zonotope $Z(d+2,d)$. An answer for the standard cubillage of the zonotope $Z(6,4)$ is shown in Fig. 12.[3]

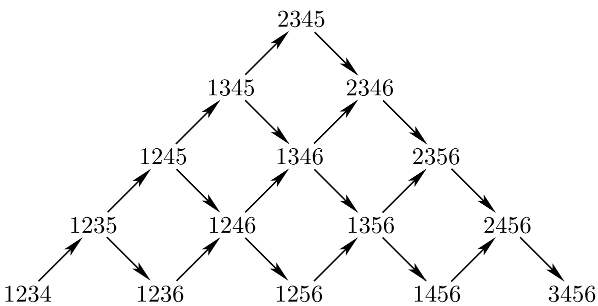

Figure 12. The standard cubillage of the zonotope $Z(6,4)$.

The following considerations play an important role in studying the natural order on a general cubillage. Each capsid (that is, each subzonotope $Z(d+1,d)$ in

[3]We note an analogy with pictures of quivers in [26], Table 1. We also note that, in the general case, the natural poset (to be precise, its Hasse diagram) of the standard cubillage $\mathscr{Q}_{\text{st}}(d+r,d)$ of the zonotope $Z(d+r,d)$ coincides with the graph of the $\mathbb{A}_d$-crystal corresponding to the rth multiplicity of the last fundamental weight (see [8], (7.7) c).

a cubillage $\mathscr{Q}$) has its type K, which is a subset of $[n]$ of cardinality $d+1$. In turn, let $K \subset [n]$ be an arbitrary subset of cardinality $d+1$. We consider the set $\mathscr{F}(K)$ in $\mathscr{Q}$ that consists of the cubes $Q \in \mathscr{Q}$ whose types are contained in K. The number of such cubes is $d+1$, and they are parameterized by their types $K-i$, where i runs through K. The union of these cubes is something like a 'dispersed' capsid (an actual capsid consists of cubes that are closely pressed together). By reducing the colours not contained in K, we press these cubes together. More precisely, let $\mathscr{Q}_K$ denote the cubillage obtained from $\mathscr{Q}$ by the reduction of all colours not contained in K. As a set of cubes, $\mathscr{Q}_K$ is identified with $\mathscr{F}(K)$. The cubillage $\mathscr{Q}_K$ is a cubillage of the capsid $Z(K,d)$, and therefore it is either standard or antistandard. Since the natural order $\preceq$ on $\mathscr{F}(K)$ coincides with the order on $\mathscr{Q}_K$ (see Example 9.4), it is, first, linear (that is, $\mathscr{F}(K)$ is a chain for $\preceq$) and, second, either lexicographic or antilexicographic in terms of types. It is easy to see which of the latter is the case. There is a maximal element, say, k, in the set K as a subset of $[n]$. Thus, everything is determined by whether the cube Q of type $K-k$ is located before the pie $\mathscr{P}_k$ of colour k in $\mathscr{Q}$ (and then Q is the minimal element in $\mathscr{F}(K)$ and the order is lexicographic) or after it (and then Q is maximal and the order $\preceq$ on $\mathscr{F}(K)$ is antilexicographic).

Thus, the restrictions of $\preceq$ to such capsid-like systems $\mathscr{F}(K)$ in $\mathscr{Q}$ are linear orders (chains). In turn, these chains determine $\preceq$ uniquely. More precisely, the following proposition holds.

Proposition 10.2. *The natural order $\preceq$ on a cubillage $\mathscr{Q}$ is the transitive closure of the chains $(\mathscr{F}(K), \preceq_{\mathscr{F}(K)})$ over all $K \subset [n]$ of size $d+1$.*

To prove this fact it suffices to verify that if $Q \prec R$ for cubes Q and R of $\mathscr{Q}$, then Q and R occur in some 'dispersed' capsid $\mathscr{F}(K)$. Indeed, we take K to be the union of the types of Q and R. Since these cubes are adjacent, the cardinality of K is $d+1$.

Thus, the natural order $\preceq$ on a cubillage $\mathscr{Q}$ could be defined in terms of the chains $\mathscr{F}(K)$ of cardinality $d+1$ instead of in terms of the immediate precedence relation $\prec$. We will return to this observation in the combinatorial part of the paper.

11. Stacks and membranes

Definition 11.1. A set $\mathscr{S}$ of cubes in $\mathscr{Q}$ is called a *stack* if, together with each cube Q in it, $\mathscr{S}$ contains the smaller cubes (with respect to $\prec$ or $\preceq$). In other words, it is an order ideal in the poset $(\mathscr{Q}, \preceq)$. The *body* of $\mathscr{S}$ is the subset of $Z = Z(n,k)$ that is the union of the cubes in $\mathscr{S}$ plus the set of points in the visible boundary of Z.

The interest in stacks is due to the fact that they determine membranes in $\mathscr{Q}$. Indeed, when any line parallel to the dth coordinate axis goes in the zonotope Z, it first crosses the body of $\mathscr{S}$ and then intersects the boundary of this stack at some point and never returns back to $\mathscr{S}$. The set of these 'terminal' points of the stack forms a membrane $\mathscr{M} = \mathscr{M}(\mathscr{S})$ in the cubillage $\mathscr{Q}$. Indeed, it consists of facets of $\mathscr{Q}$, and its projection onto $Z' = \pi(Z)$ is bijective. Conversely, let $\mathscr{M}$

be a membrane in $\mathscr{Q}$. When we take the union of all the cubes lying before the membrane, we obtain a stack $\mathscr{S}(\mathscr{M})$. Obviously, these operations are mutually inverse, and we obtain the following proposition.

Proposition 11.2. *The set* $\mathbf{S}(\mathscr{Q})$ *of stacks in a cubillage* $\mathscr{Q}$ *is connected with the set* $\mathbf{M}(\mathscr{Q})$ *of membranes by a natural bijection.*

Corollary 11.3. *The zonotope* $Z(d+2,d)$ *has* $2(d+2)$ *distinct cubillages.*

Proof. According to Proposition 6.3, specifying a cubillage of the zonotope $Z(d+2,d)$ is the same as specifying a membrane in some cubillage of the capsid $Z(d+1,d)$. The latter has only two cubillages, the standard and antistandard cubillages. The natural order on each of them is a chain (of length $d+1$). Correspondingly, there are $d+2$ stacks. By the previous proposition, there are $d+2$ membranes. $\square$

It is easy to see that the poset $\mathbf{Q}(d+2,d)$ consists of two branches of a ring of size $2(d+2)$, as in Fig. 13.

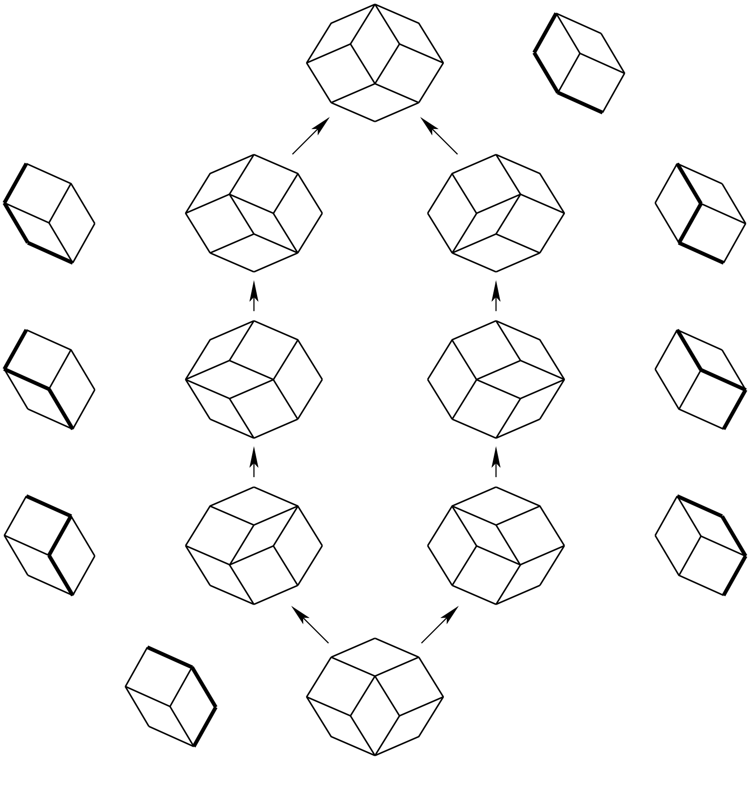

Figure 13. A ring. Eight cubillages of the zonogon $Z(4,2)$ are shown. Arrows denote raising flips. Alongside each cubillage, there is a cubillage of $Z(3,2)$ and a membrane in it specifying the corresponding cubillage of $Z(4,2)$ (see Proposition 6.3).

The membranes in a cubillage $\mathscr{Q}$ are naturally ordered: a membrane $\mathscr{M}$ in $\mathscr{Q}$ is said to be encountered *earlier* than a membrane $\mathscr{M}'$ if any line parallel to e_d intersects $\mathscr{M}$ no later than $\mathscr{M}'$. In terms of stacks, this is precisely the inclusion relation for the corresponding subsets of $\mathscr{Q}$. We deduce the following assertions from this.

Proposition 11.4. (a) *The set* $\mathbf{M}(\mathscr{Q})$ *of membranes of a cubillage* $\mathscr{Q}$ *is a distributive lattice.*

(b) *This lattice is graded* (*like any distributive lattice*; *see* [30], § 3.4). *The rank of a membrane* $\mathscr{M}$ *is equal to the cardinality of the stack* $\mathscr{S}(\mathscr{M})$.

(c) *For any membrane* $\mathscr{M}$ *that is not maximal*, *there exists a cube* $Q \in \mathscr{Q}$ *such that all its visible facets are contained in* $\mathscr{M}$. *Adding this cube leads to a membrane* $\mathscr{M}'$ *that differs from* $\mathscr{M}$ *by a raising flop.*

Thus, one can see from part (c) that any membrane $\mathscr{M}$ in $\mathscr{Q}$ can be connected with the minimal membrane (the visible part of the boundary of the zonotope) by lowering flops (and with the maximal membrane by raising flops). This also means that the cubillage $\pi(\mathscr{M})$ of the zonotope $Z' = Z(n, d-1)$ can be transformed into the standard cubillage of Z' by a series of lowering flips.

Corollary 11.5. *Let* $\mathscr{Q}$ *be a cubillage of the zonotope* $Z = Z(n,d)$ *and let* v *be some vertex of this cubillage. Then there exists a membrane* $\mathscr{M}$ *of* $\mathscr{Q}$ *which passes through* v.

This is obvious when v is on the visible (front) boundary of Z. Otherwise, let Q be the only cube in $\mathscr{Q}$ whose head is the vertex v: $v = h(Q)$. It remains to take the stack in $\mathscr{Q}$ generated by Q. The corresponding membrane passes through v.

As for two vertices of a cubillage, they do not necessarily lie on the same membrane. The main obstacle has to do with the separation relation and is discussed in § 16. Here we limit ourselves to the following observations. Let $\mathscr{M}_{\min}(v)$ be the minimal membrane through a vertex v in a cubillage (we constructed this membrane in the proof of Corollary 11.5), and let $\mathscr{M}_{\max}(v)$ be the maximal membrane through v. The other membranes through v lie between these two. A vertex w of a cubillage $\mathscr{Q}$ is said to be *similar* to a vertex v if w lies between the membranes $\mathscr{M}_{\min}(v)$ and $\mathscr{M}_{\max}(v)$ (between meaning with respect to the last coordinate x_d).

Corollary 11.6. *A vertex* w *is similar to a vertex* v *if and only if there exists a membrane through* w *and* v.

This is obvious in one direction. We show that if w lies between the membranes $\mathscr{M}_{\min}(v)$ and $\mathscr{M}_{\max}(v)$, then there exists a membrane through v and w. Let $\mathscr{M}$ be an arbitrary membrane through w. For the required membrane we take the median among $\mathscr{M}_{\min}(v)$, $\mathscr{M}_{\max}(v)$, and $\mathscr{M}$.

The membranes $\mathscr{M}_{\min}(v)$ and $\mathscr{M}_{\max}(v)$ divide the zonotope Z into three parts: before $\mathscr{M}_{\min}(v)$, after $\mathscr{M}_{\max}(v)$, and the middle part between them. A vertex w is said to lie *before* (respectively, *after*) a vertex v if it is positioned strictly before the membrane $\mathscr{M}_{\min}(v)$ (respectively, strictly after $\mathscr{M}_{\max}(v)$). We can denote this by $w <_{\mathrm{Vert}(\mathscr{Q})} v$. We thus obtain a binary relation $<_{\mathrm{Vert}(\mathscr{Q})}$ on the set $\mathrm{Vert}(\mathscr{Q})$ of vertices of $\mathscr{Q}$, which is easily seen to be transitive and irreflexive. One can show (using the reduction of the colour n) that if $w <_{\mathrm{Vert}(\mathscr{Q})} v$, then the spectrum of the vertex v is lexicographically larger than the spectrum of w.

12. Existence of flips

The notion of a flip was already introduced in § 8 as the replacement of a cubillage of the capsid $Z(d+1,d)$ in Z by the other cubillage. Here we prove the existence of flips.

Theorem 12.1. *Let $\mathscr{Q}$ be a cubillage of the zonotope $Z = Z(n,k)$. If $\mathscr{Q}$ is not the standard cubillage, then a lowering flip can be performed in it.*

In other words, if it is impossible to perform a lowering flip in a cubillage, then this cubillage is standard. Of course, the converse is also true: the standard cubillage does not admit lowering flips. This can be seen, for example, from the fact that the standard cubillage is realized as the visible (or front) membrane in any cubillage of the zonotope $\hat{Z} = Z(n,d+1)$, and there is nowhere to go back from this membrane.

Proof. Consider the pie $\mathscr{P}$ of colour n in $\mathscr{Q}$. If it adjoins the invisible side of Z, we just eliminate it from both $\mathscr{Q}$ and Z, thus obtaining a cubillage $\mathscr{Q} - \mathscr{P}$ of the zonotope $Z(n-1,d)$. If $\mathscr{Q} - \mathscr{P}$ is the standard cubillage, then $\mathscr{Q}$ is also standard, contrary to the assumption. Consequently, the cubillage $\mathscr{Q} - \mathscr{P}$ of $Z(n-1,d)$ is not standard, and by induction we can perform a lowering flip in it (and therefore in Z).

Hence, we can assume that the pie $\mathscr{P}$ does not adjoin the invisible part of the boundary of Z, and there are actually cubes behind this pie. Among these cubes (strictly behind $\mathscr{P}$) we take the minimal cube Q with respect to the natural order $\preceq$ on $\mathscr{Q}$. This cube adjoins $\mathscr{P}$ along the whole of its visible side. Adding to Q all the cubes in $\mathscr{P}$ that are adjacent to Q, we obtain the needed subzonotope (capsid) Z_0. Indeed, these cubes have the structure of products (sums) of the line segment $[-v_n, 0]$ by visible facets of Q. Hence, we obtain a subzonotope of type Bn, where B is the type of Q. This is a collection of cubes with which a lowering flip can be performed (see § 8). □

As a result of such a lowering flip with the participation of the colour n, we move the pie of the colour n by one cube toward the invisible boundary of the zonotope Z. Proceeding in this way step by step, we move the pie right up to the invisible boundary of Z. However, we could do this directly, in one 'big' step. With this aim in view, we let $\mathscr{Q}_+$ denote the set of cubes of our cubillage $\mathscr{Q}$ that lie behind the pie $\mathscr{P} = \mathscr{P}_n$. Then we shift this whole collection of cubes $\mathscr{Q}_+$ by the vector $-v_n$. As a result, this set is shifted right up to the front (visible) boundary of $\mathscr{P}$ and forms a cubillage of the zonotope $Z(n-1,d)$. Adding to $Z(n-1,d)$ the product (sum) of the invisible boundary of this zonotope by the line segment $[0, v_n]$ (in other words, expanding this cubillage along the rear membrane of the zonotope $Z(n-1,k)$ by the colour n), we obtain the original zonotope Z with a new cubillage of it, in which the new pie $\mathscr{P}'$ of the colour n adjoins the invisible side of Z. Such a significant restructuring of a cubillage can be called a large flip or an *avalanche* (see Fig. 14).

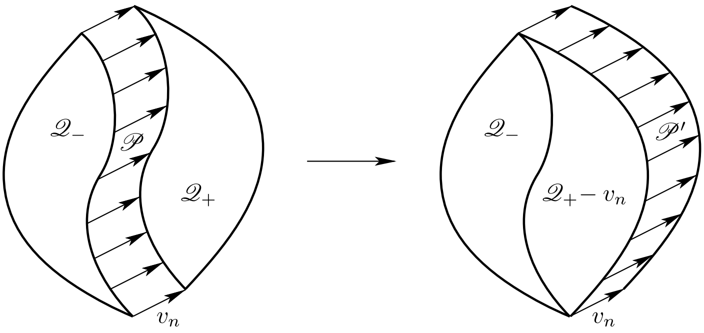

Figure 14. An avalanche.

As a result of such an avalanche, we obtain a cubillage $\widetilde{\mathscr{Q}}$ of the zonotope $\widetilde{Z} = Z(n-1,d)$. One can apply to it the same operation of lifting the pie of colour $n-1$, and so on. This sequence of n avalanches can be called the *standardization* of the cubillage $\mathscr{Q}$, since it becomes increasingly more standard after each step (avalanche) and is completely standard at the end of the process. The main advantage of standardization is that it is canonical: whereas lowering flips can be performed in different places and in different sequences, avalanches are uniquely determined.

Symmetrically, we can perform the antistandardization of a cubillage by pressing the pie $\mathscr{P}$ to the visible boundary of the zonotope $Z(n,d)$ (in the first step) and then continuing in this way. As a result, we obtain the antistandard cubillage in n steps.

We have already indicated how one can introduce an order $\leqslant$ on the set $\mathbf{Q}(n,d)$ of cubillages of the zonotope $Z(n,d)$ using flips. Namely, we say that $Q \leqslant Q'$ if Q' is reachable from Q using a sequence of raising flips. There can be no cycles in this process, and thus this relation $\leqslant$ is a real partial order relation on the set $\mathbf{Q}(n,d)$. In fact, it was this poset that Manin and Shekhtman called the *higher Bruhat order* and denoted by $B(n,d)$. Theorem 12.1 states that this poset contains a minimal element (the standard cubillage) and a maximal element (the antistandard cubillage). Posets of this kind often turn out to be lattices. And indeed, the first Bruhat poset $B(n,1)$ (the so-called *weak Bruhat order*) is a lattice. For small n, the poset $B(n,2)$ is also a lattice (see [12] for a picture of the poset $B(5,2)$). However, $B(6,2)$ is not a lattice [34]. Moreover, this poset is in general not a poset of vertices of a polyhedron [12]. Nevertheless, it is a graded (rank) poset.

13. Membranes in zonotopes

So far, we have been dealing with membranes in cubillages. Now we introduce the more general notion of a *membrane in the zonotope* $Z = Z(n,k)$. It is again a $(d-1)$-dimensional cube complex $\mathscr{M}$ in $Z(n,k)$ such that the following two properties (borrowed from membranes in cubillages) hold:

(1) $\mathscr{M}$ (more precisely, the body of $\mathscr{M}$) is projected bijectively onto $\pi(Z) = Z' = Z(n,d-1)$;

(2) the edges of $\mathscr{M}$ are congruent to vectors v_i (however, for $d > 2$ this can be deduced from the property (1) roughly as in Lemma 2.2).

The projections of cubes in $\mathscr{M}$ yield a $(d-1)$-dimensional cubillage of Z' which we again denote by $\mathscr{Q}' = \pi(\mathscr{M})$. Conversely, let $\mathscr{Q}'$ be a cubillage of Z'. Then we can construct a (unique) membrane $\mathscr{M}$ such that $\mathscr{Q}' = \pi(\mathscr{M})$. This can be done in the following way. For any vertex v' of $\mathscr{Q}'$ we specify a vertex v as $\sum_{i\in \mathrm{sp}(v')} v_i$. Then we arrange these vertices into a cubillage by repeating the arrangement of the vertices v' into cubes in $\mathscr{Q}'$. This can also be done in a slightly different way: we construct the membrane $\mathscr{M}$ (more precisely, its body) as a certain natural (piecewise linear) section over the zonotope Z' (a section with respect to the projection π). As we already noted, each point in the cubillage $\mathscr{Q}'$ can be expressed uniquely in terms of the vectors v'_i. It remains to rewrite this in terms of the v_i. Thus,

> *the set* $\mathbf{M}(n, d)$ *of membranes in the zonotope* $Z = Z(n, d)$ *is identified with the set* $\mathbf{Q}(n, d-1)$ *of cubillages of the zonotope* $Z(n, d-1)$.

We illustrate this by an example with $d = 3$. A cubillage of the zonogon $Z' = Z(n, 2)$ is a rhombus tiling, which is a two-dimensional shape. A transition to a membrane in $Z = Z(n, 3)$ makes it more salient: this is like the transition from a plane drawing to a three-dimensional model. Flips, which look like artificial reshuffles of rhombi, turn into more vivid flexures of films (membranes). However, our membranes are still 'hanging in the air'. The following statement corrects this defect.

Theorem 13.1. *For any membrane $\mathscr{M}$ in $Z = Z(n, d)$ there exists a cubillage $\mathscr{Q}$ of this zonotope for which it is a membrane.*

In other words, $\mathscr{M}$ can be embedded in some cubillage $\mathscr{Q}$.

Proof. We let $Z_-(\mathscr{M})$ denote the domain in Z before the membrane $\mathscr{M}$. We will use Theorem 12.1. By means of lowering flops of $\mathscr{M}$ (more precisely, we perform flips of the cubillage $\pi(\mathscr{M})$ and then lift them to membranes) we obtain a series of membranes going from $\mathscr{M}$ to the front membrane, that is, to the visible part of the boundary of Z. Thus, we obtain a cubillage of $Z_-(\mathscr{M})$.

We can do the same with the domain $Z_+(\mathscr{M})$ behind the membrane. In combination, they yield a cubillage including the membrane $\mathscr{M}$. $\square$

Moreover, when we make the canonical standardization of the cubillage $\pi(\mathscr{M})$, we obtain the canonical (or standard) cubillage of the domain $Z_-(\mathscr{M})$ in the zonotope Z. We see that this cubillage is standard from the fact that it is impossible to perform a lowering flip in it.

We can proceed symmetrically with the domain $Z_+(\mathscr{M})$ lying in Z behind the membrane $\mathscr{M}$. We obtain the canonical (antistandard) cubillage of this domain. Upon uniting these two cubillages we obtain a cubillage $\mathscr{Q}$ of the entire zonotope Z, in which $\mathscr{M}$ is a membrane. This standard-antistandard cubillage of the zonotope Z is called the *canonical extension* of the membrane $\mathscr{M}$ (or the *canonical lift* of the cubillage $\pi(\mathscr{M})$). (See Fig. 15 for the case of the zonotope $Z(6, 2)$.)

Remark 13.2. Arguing as in the proof of Theorem 12.1, we can show that if a cubillage of $Z_-(\mathscr{M})$ is non-standard (see above), then a lowering flip can be performed

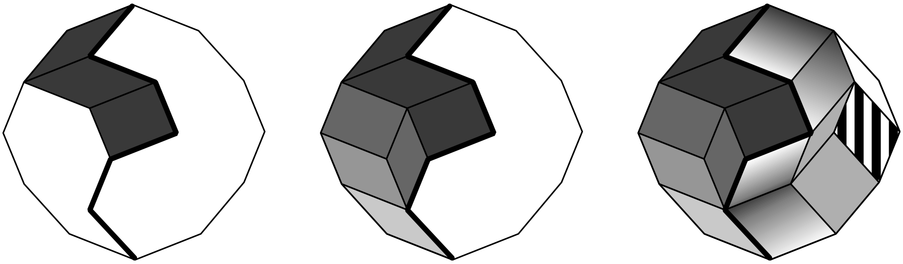

Figure 15. The stages of a canonization. A rhombus tiling and a membrane (snake) $\mathscr{M}$ in it are shown. The figure on the left shows the first step of the standardization for the domain before the membrane: constructing a part of the pie of colour 6. Constructing all pies in the domain before the membrane is completed in the central figure, that is, the standardization of this domain is shown. The figure on the right shows the (anti)standardization of the domain behind the membrane.

in it. This fact shows that cubillages of $Z_-(\mathscr{M})$ (as well as cubillages of the entire zonotope) are related by flips. A similar observation is true for a cubillage of $Z_+(\mathscr{M})$.

Membranes serve as a kind of means of connecting cubillages having successive dimensions. A cubillage $\mathscr{Q}$ can be represented as a network of membranes and thus as a network of cubillages of one dimension lower. Recall that $\mathbf{Q}(n,d)$ is the set of cubillages of the zonotope $Z(n,d)$. Cubillages $\mathscr{Q}$ in $\mathbf{Q}(n,d)$ and $\mathscr{Q}'$ in $\mathbf{Q}(n,d-1)$ are said to be *consistent* if $\mathscr{Q}'$ can be lifted to a membrane $\mathscr{M}$ in $\mathscr{Q}$ (so that $\mathscr{Q}'=\pi(\mathscr{M})$). The consistency relation can be considered as a correspondence c from $\mathbf{Q}(n,d)$ to $\mathbf{Q}(n,d-1)$. The above canonical lifting of cubillages produces a section can: $\mathbf{Q}(n,d-1)\to\mathbf{Q}(n,d)$ of this correspondence c.

The abundance of relationships among cubillages motivates us to look at them from a categorical standpoint. We will discuss this in Appendix A on polycategories.

Second, combinatorial part

The combinatorial approach to cubillages is understood as the study of systems of subsets of the base set $[n]$ that are generated by cubillages. We already dealt with this when we considered types of cubes of a cubillage. However, when only types of cubes are taken into account, any cubillage yields the entire set $\binom{[n]}{d}=\mathrm{Gr}([n],d)$, called the Grassmannian. To reflect the specifics of a particular cubillage $\mathscr{Q}$, we need to transfer the natural order $\preceq_{\mathscr{Q}}$ to $\mathrm{Gr}([n],d)$. This throws a bridge between the geometric approach and the approach by Manin and Shekhtman [23], [24] and Ziegler [34]. We recall that Manin and Shekhtman, the creators of higher Bruhat orders, worked precisely in terms of orders on $\mathrm{Gr}([n],d)$.

The notion of a spectrum opens up another way to combinatorics. By combining the spectra of all vertices of a cubillage $\mathscr{Q}$ into one system $\mathrm{Sp}(\mathscr{Q})$, we obtain an interesting system of subsets of $[n]$. The main property of systems of this kind is the property of separation, which was first discovered by Leclerc and Zelevinsky [20] in the case $d = 2$ and then was generalized and studied by Galashin and Postnikov [13], [14].

In our paper we will constantly have to deal with subsets and systems of subsets of $[n]$. The base set $[n]$ itself can be understood as an analogue of an n-dimensional vector space (over an arbitrary field or the one-element 'field' $\mathbf{F}_1$). From this standpoint, a subset of $[n]$ of cardinality k should be understood as an analogue of a vector subspace of dimension k. Because of this, we call the set of all subsets of cardinality k the discrete Grassmannian and denote it by $\operatorname{Gr}([n],k)$ $\bigl($instead of the more common notation $\binom{[n]}{k}\bigr)$. An analogue of a complete flag of subspaces is an unrefinable chain of subsets of $[n]$, which is in fact a linear order on $[n]$. Recall that we regard $[n]$ as a set (chain) with the natural order $(1 < 2 < \cdots < n)$, which looks as if we have fixed some complete flag.

For convenience of identification, elements of the base set $[n]$ are called *colours* and usually denoted by $i, j, \dots$, and subsets of $[n]$ are denoted by $X, Y, \dots$ and often simply called sets. For brevity, a set of the form $X \cup \{i\}$ is usually denoted by Xi. Subsets of $2^{[n]}$ are called *set systems* and are denoted by handwritten letters (like $\mathscr{S}$ or $\mathscr{X}$).

14. Admissible orders

Let Q be a cube of some cubillage $\mathscr{Q}$ of the zonotope $Z(n,d)$. It determines two sets (and is determined by them): the spectrum $\operatorname{sp}(b(Q)) \subset [n]$ of the lower (root) vertex $b(Q)$ and the type $\tau(Q) \subset [n]$. These two subsets do not intersect, and the second has cardinality d. In fact, this is the above-mentioned coding by signed vectors, $Q \mapsto \dfrac{\tau(Q)}{\operatorname{sp}(b(Q))}$. By Proposition 2.3 the type map τ yields a bijection from the set $\mathscr{Q}$ of all cubes of the cubillage $\mathscr{Q}$ onto the Grassmannian $\operatorname{Gr}([n],d)$. Transferring the natural order $\preceq$ on $\mathscr{Q}$ by means of this bijection τ (see § 9), we obtain some order $\preceq_{\mathscr{Q}}$ on the Grassmannian $\operatorname{Gr}([n],d)$.

Proposition 14.1. *The order $\preceq_{\mathscr{Q}}$ determines the cubillage $\mathscr{Q}$.*

In other words, the order $\preceq_{\mathscr{Q}}$ on $\operatorname{Gr}([n],d)$ makes it possible to uniquely reconstruct the cubillage $\mathscr{Q}$.

Proof. Assume that there are two cubillages $\mathscr{Q}_1$ and $\mathscr{Q}_2$ of the zonotope $Z = Z(n,d)$ producing the same order on $\operatorname{Gr}([n],d)$. Let $\mathscr{S}$ be an (order) ideal in $\operatorname{Gr}([n],d)$. We will show, using induction on the cardinality of $\mathscr{S}$, that the cubes (in $\mathscr{Q}_1$ and $\mathscr{Q}_2$) corresponding to $\mathscr{S}$ (that is, $\tau^{-1}(\mathscr{S})$) are identically placed in Z. Let $K \in \operatorname{Gr}([n],d)$ be the maximal element in $\mathscr{S}$ and let $\mathscr{S}' = \mathscr{S} - \{K\}$. By the induction assumption, the cubes corresponding to $\mathscr{S}'$ are identically placed in Z. They lie in the front part with respect to some membrane $\mathscr{M}'$. Then the cube Q_1 (in $\mathscr{Q}_1$ of type K) adjoins some part (capsid) of $\mathscr{M}'$ by its whole visible boundary, in exactly the same way as the cube Q_2 (in $\mathscr{Q}_2$ of type K). Thus, they adjoin the same part of $\mathscr{M}'$ and therefore coincide. $\square$

We can explain this in a slightly different way. Let Q_1 and Q_2 be cubes in the cubillages $\mathscr{Q}_1$ and $\mathscr{Q}_2$, respectively, with the same type (say, K). Let $b_1 = b(Q_1)$ and $b_2 = b(Q_2)$ be their root vertices. It suffices to show that $b_1 = b_2$. We take any visible facet F_1 of Q_1 (with type $K - i$ for some $i \in K$) and consider the part of the tunnel of type $K - i$ passing from F_1 to the front boundary of the zonotope Z. We do not know how this tunnel goes, but we know that it consists of cubes Q which (a) lie in the ideal $\mathscr{S}$ and (b) contain $K - i$ in $\tau(Q)$. This tunnel reaches the front boundary of Z by the unique facet $\widetilde{F}$ of type $K - i$. We denote its root vertex by $\widetilde{b}$. Then we have $b_1 = \widetilde{b} + \sum_j (\pm) v_j$, where the sum is taken over the j such that $K - i + j$ belongs to $\mathscr{S}$, and the signs are the signs of the determinant $\det(\dots v_k \dots v_j)$, where k runs through $K - i$. Since this expression depends on $K - i$ and $\mathscr{S}$ but not on $\mathscr{Q}$, we have $b_2 = b_1$.

This assertion raises the question of what conditions distinguish orders of the form $\preceq_{\mathscr{Q}}$. Proposition 10.2 suggests one such condition. A subset K of $[n]$ of cardinality $d+1$ is called a *parent*, while the system $\mathscr{F}(K)$ of subsets of K of cardinality d is called its *family*. (In other words, $\mathscr{F}(K)$ is the image of $\mathrm{Gr}(K, d)$ in $\mathrm{Gr}([n], d)$ under the natural embedding. Manin and Shekhtman called this a *packet*.) A family has two distinguished linear orders, namely, the lexicographic order $\preceq_{\mathrm{lex}}$ and the antilexicographic order $\preceq_{\mathrm{alex}}$ (the reverse of the former order). If $K = \{i_1 < i_2 < \dots < i_{d+1}\}$, then the lexicographic order $\preceq_{\mathrm{lex}}$ on $\mathscr{M}(k)$ has the form

$$K - i_{d+1} < K - i_d < \dots < K - i_1,$$

while the antilexicographic order has the reverse form.

Definition 14.2. A partial order $\preceq$ on $\mathrm{Gr}([n], d)$ is said to be *admissible* if its restriction to any family $\mathscr{F}(K)$ (where K is an arbitrary set of cardinality $d + 1$) is the lexicographic or antilexicographic order.

Of course, any strengthening of an admissible order (in particular, any linear extension) is also admissible. Manin and Shekhtman dealt just with linear admissible orders.

In these terms, Proposition 10.2 states that an order on $\mathrm{Gr}([n], d)$ induced by some cubillage is admissible. It turns out that this condition is not only necessary but also sufficient in a certain sense.

Theorem 14.3. *Let $\preceq$ be an admissible order on* $\mathrm{Gr}([n], d)$*. Then there exists a* (*unique*: *see Proposition* 14.1) *cubillage $\mathscr{Q}$ such that $\preceq$ is a strengthening of $\preceq_{\mathscr{Q}}$.*

To clarify the idea of the proof, we imagine that the order $\preceq$ has already arisen from a cubillage $\mathscr{Q}$. Let $\mathscr{P}$ be the pie of colour n. We reduce the colour n. As a result, the pie is contracted to a membrane $\mathscr{M}$, while the cubillage $\mathscr{Q}$ is reduced to the cubillage $\mathscr{Q}' = \mathscr{Q}_{-n}$ of the zonotope $Z' = Z(n-1, d)$. The cubillage $\mathscr{Q}'$ as a set of cubes is naturally embeddable in $\mathscr{Q}$. The restriction of the natural order of $\mathscr{Q}$ to $\mathscr{Q}'$ is stronger than the natural order on $\mathscr{Q}'$ (Proposition 9.7). Thus, the restriction of $\preceq$ to $\mathrm{Gr}([n-1], d)$ is an admissible order. By induction we can recover $\mathscr{Q}'$. It remains to reconstruct also the membrane $\mathscr{M}$. But it is specified by the set of (types of) cubes in $\mathscr{Q}'$ located before the membrane, or, which is the same, by the set of cubes in $\mathscr{Q}$ located before the pie. Let Q be such a cube and let $T \subset [n-1]$ be its type. We consider the parent $K = Tn$ and its family as a set of cubes in $\mathscr{Q}$. This family starts with Q, and then go cubes in the pie $\mathscr{P}$, since their types contain the colour n. It follows that Q is the minimal member of this family. Conversely, if Q is behind the pie, then Q is the maximal member of the family. Thus, the cubes (in $\mathscr{Q}'$) before the membrane $\mathscr{M}$ are the ones whose type T is minimal in the family with parent Tn. (This explanation can be regarded as a third proof of Proposition 14.1.)

Let us now proceed to the proof, which is the inversion of the previous arguments. We start with an admissible order $\preceq$ on $\mathrm{Gr}([n], d)$ and restrict it to $\mathrm{Gr}([n-1], d) \subset \mathrm{Gr}([n], d)$. Clearly, this is again an admissible order on $\mathrm{Gr}([n-1], d)$. By induction, it arises from some cubillage $\mathscr{Q}'$ of the zonotope $Z' = Z(n-1, d)$. We now form the subset $\mathscr{T}$ of $\mathrm{Gr}([n-1], d)$ consisting of those T such that the restriction of $\preceq$ to the family $\mathscr{F}(Tn)$ of the parent Tn is the lexicographic order.

We claim that $\mathscr{T}$ is an ideal with respect to the natural order $\preceq'$ on $\mathscr{Q}'$. Indeed, let $Q \in \mathscr{T}$ and $R \prec Q$, so that R immediately precedes the cube Q in the cubillage $\mathscr{Q}'$ (see § 9). We need to show that R also belongs to $\mathscr{T}$. Assume the converse: the restriction of $\preceq$ to the family of the parent $\tau(R)n$ is the antilexicographic order. In particular, $\tau(R)$ is the maximal element in the family $\mathscr{F}(\tau(R)n)$. In turn, $\tau(Q)$ is the minimal element in $\mathscr{F}(\tau(Q)n)$. The cubes R and Q have a common facet F. Thus, its type $\tau(F)$ is contained in both $\tau(R)$ and $\tau(Q)$. Therefore, $\tau(F)n$ belongs to both the family $\mathscr{F}(\tau(R)n)$ and the family $\mathscr{F}(\tau(Q)n)$. This implies that $\tau(Q) \preceq \tau(F)n \preceq \tau(R)$. By transitivity, $\tau(Q) \preceq \tau(R)$. However, this contradicts the fact that $R \prec Q$, and therefore $\tau(R) \prec \tau(Q)$.

Thus, $\mathscr{T}$ is an ideal in $\mathscr{Q}'$. This ideal specifies a membrane $\mathscr{M}$ in $\mathscr{Q}'$. By expanding the colour n in this membrane, we obtain a cubillage $\mathscr{Q}$ (see Proposition 6.3). It remains to verify that the original relation $\preceq$ is not weaker than the relation $\preceq_{\mathscr{Q}}$, that is, if a cube R immediately precedes a cube S in $\mathscr{Q}$ ($R \prec S$), then $\tau(R) \preceq \tau(S)$.

This is evident when R and S are in $\mathscr{Q}'$. It is almost evident when one of the cubes is in the pie $\mathscr{P}$ (obtained by expansion of the membrane $\mathscr{M}$) and the other is not. Assume, for example, that S is in the pie and R is not. Then R is located before the pie. By the construction of $\mathscr{T}$, the family $\mathscr{F}(\tau(R) \cup \tau(S))$ is 'lexicographic' and $\tau(R)$ is the minimal member of this family, which implies that $\tau(R) \preceq \tau(S)$.

It remains to consider the case when R and S are in the pie. Upon reducing all the colours not contained in $\tau(R) \cup \tau(S)$, we can assume that our cubillage $\mathscr{Q}$ is one of the two cubillages of the capsid, for which the assertion is obvious due to results in § 10. This completes the proof of Theorem 14.3.

Theorem 14.3 provides a bridge between the geometric approach (cubillages) and the Manin–Shekhtman approach based on admissible orders on the Grassmannians $\mathrm{Gr}([n], d)$. Such a relationship was announced by Voevodskii and Kapranov in [33] and was described in greater detail in [32], Theorem 2.1.

A certain inconvenience in describing cubillages in terms of admissible orders is that distinct orders can lead to the same cubillage. We have already noted that if $\preceq$ is an admissible order, then any strengthening $\preceq'$ of it is also an admissible order, and produces the same cubillage as does $\preceq$. To restore uniqueness, we can work with minimal admissible orders. Clearly, these are precisely the orders generated (in the sense of transitive closure) by restrictions to all kinds of families in $\mathrm{Gr}([n], d)$. These minimal admissible orders are determined by indicating whether the family $\mathscr{F}(K)$ of each parent (a set K of size $d+1$) is ordered lexicographically or antilexicographically. In other words, they are specified by the map $\sigma\colon \mathrm{Gr}([n], d+1) \to \{+,-\}$, where $+$ corresponds to 'lexicography'. Of course, this assignment σ cannot be arbitrary: it is necessary that an acyclic relation be obtained as a result (cf. Proposition 9.1). The problem of which assignments yield acyclicity (and therefore lead to cubillages) will be discussed in the next section.

15. Inversions

Section 13 makes it possible to consider cubillages of the zonotope $Z(n, d-1)$ as membranes in the zonotope $Z(n, d)$ of dimension larger by one. This gives another way to specify cubillages.

Let $\mathscr{Q}'$ be a cubillage of the zonotope $Z' = Z(n, d-1)$. We realize it as a membrane $\mathscr{M} = \mathscr{M}(\mathscr{Q}')$ in $Z = Z(n, d)$. According to Theorem 13.1, there exists a cubillage (as a rule, more than one) $\mathscr{Q}$ of Z which includes $\mathscr{M}$. This membrane divides the cubes of $\mathscr{Q}$ into the ones positioned before and after $\mathscr{M}$. We combine the cubes before $\mathscr{M}$ into a set $\mathscr{Q}_-(\mathscr{M})$. Of course, this set depends on the cubillage $\mathscr{Q}$ containing $\mathscr{M}$. However, the set of types of cubes in $\mathscr{Q}_-(\mathscr{M})$ depends only on $\mathscr{M}$ and not on $\mathscr{Q}$. This can be seen from the following lemma.

Lemma 15.1. *Let Q be a cube of a cubillage $\mathscr{Q}$ located before* (*respectively*, *after*) *a membrane $\mathscr{M}$, and let $K = \{k_1 < k_2 < \cdots < k_d\}$ be the type of Q. Then there exists a vertex v of $\mathscr{M}$ such that $K \cap \mathrm{sp}(v) = \{k_d, k_{d-2}, \dots\}$* (*respectively*, *$K \cap \mathrm{sp}(v) = \{k_{d-1}, k_{d-3}, \dots\}$*).

Proof. We reduce all the colours distinct from the ones of K. As a result, we obtain a zonotope $Z' = Z(K, d)$ (consisting of only the cube $Q = Z'$) and a membrane $\mathscr{M}'$ in Z'. Assume that Q was before the membrane $\mathscr{M}$. Then the cube $Q = Z'$ is also before $\mathscr{M}'$, that is, the membrane $\mathscr{M}'$ is the invisible part of the boundary of Q. Let v' be the head $h(Q)$ of Q (see § 7). As we know, its spectrum is $\{k_d, k_{d-2}, \dots\}$. Let v be the pre-image of v' in $\mathscr{M}$, that is, the vertex of $\mathscr{M}$ that is reduced to v'. Clearly, $\mathrm{sp}(v)$ differs from $\mathrm{sp}(v')$ only by colours not contained in K, and this implies that $\mathrm{sp}(v) \cap K = \{k_d, k_{d-2}, \dots\}$.

We reason similarly in the case when K is located after the membrane. $\square$

Remark 15.2. A less technical argument is as follows. Cubillages of the domain in $Z(n, d)$ located before a membrane $\mathscr{M}$ can be obtained one from another using flips (see Remark 13.2). And flips do not change the types of cubes but only permute them.

Let $\mathscr{Q}$ be a cubillage of the zonotope $Z(n, d)$, let $\mathscr{M}$ be the corresponding membrane in $Z(n, d+1)$ ($\mathscr{Q} = \pi(\mathscr{M})$), and let $\mathscr{Q}'$ be some cubillage of $Z(n, d+1)$ containing $\mathscr{M}$ (which exists by Theorem 13.1).

Definition 15.3. The *system of inversions* $\operatorname{Inv}(\mathscr{Q})$ of a cubillage $\mathscr{Q}$ is the system $\tau(\mathscr{Q}'_{-}(\mathscr{M}))$ in $\operatorname{Gr}([n],d+1)$ (which is independent of the choice of $\mathscr{Q}'$ by Lemma 15.1). The system $\operatorname{Inv}(\mathscr{Q})$ is also denoted by $\operatorname{Inv}(\mathscr{M})$.

The same arguments as in the proof of Lemma 15.1 (with the reduction of all colours not contained in the type of Q) give another description of the system of inversions.

Proposition 15.4. *A parent $T\in\operatorname{Gr}([n],d+1)$ belongs to $\operatorname{Inv}(\mathscr{Q})$ if and only if the family of this parent in $\operatorname{Gr}([n],d)$ is antilexicographically ordered by the relation $\preceq_{\mathscr{Q}}$.*

In other words,

$$\operatorname{Inv}(\mathscr{Q})=\{T\in\operatorname{Gr}([n],d+1)\text{ such that the restriction of }\preceq_{\mathscr{Q}}\text{ to }\operatorname{Gr}(T,d)\\ \text{is the antilexicographic order}\}.$$

For example, the system Inv for the standard cubillage is empty. On the contrary, Inv for the antistandard cubillage is the whole of $\operatorname{Gr}([n],d)$. When we perform a raising flip in $\mathscr{Q}$, we add one more cube to the pre-membrane cubes. Thus, the system $\operatorname{Inv}(\mathscr{Q})$ is enlarged by one element in this case.

An important property of the system of inversions $\operatorname{Inv}(\mathscr{Q})$ is that this system makes it possible to uniquely reconstruct $\mathscr{Q}$.

Proposition 15.5. *The map* $\operatorname{Inv}\colon \mathbf{Q}(n,d)\to 2^{\operatorname{Gr}([n],d+1)}$ *is injective.*

Proof. It follows from Proposition 14.1 that the order $\preceq_{\mathscr{Q}}$ determines the cubillage $\mathscr{Q}$. The order $\preceq_{\mathscr{Q}}$ is generated by orders on families, whereas the system $\operatorname{Inv}(\mathscr{Q})$ shows which family is ordered antilexicographically. □

As in the previous section, this raises the problem of what systems in $\operatorname{Gr}([n],d)$ are inversive, that is, arise (using Inv) from cubillages of the zonotope $Z(n,d-1)$. This problem was solved by Ziegler [34] (though the foundation for the solution was laid earlier by Las Vergnas [19]). We introduce the following definition.

Definition 15.6. A set $\mathscr{S}$ in $\operatorname{Gr}([n],d)$ is said to be *consistent* if the family $\mathscr{F}(T)$ of any parent $T\subset[n]$ of cardinality $d+1$ intersects $\mathscr{S}$ in a starting or ending segment of $\mathscr{F}(T)$ (where $\mathscr{F}(T)$ is structured in the lexicographic order).

Lemma 15.7. *For any membrane $\mathscr{M}$ in $Z(n,d)$ the system $\operatorname{Inv}(\mathscr{M})$ forms a consistent subset of $\operatorname{Gr}([n],d)$.*

Proof. We embed $\mathscr{M}$ in a cubillage $\mathscr{Q}$ of the zonotope $Z=Z(n,d)$. Let $\preceq$ be the natural order on $\mathscr{Q}$ and let $\preceq_{\mathscr{Q}}$ be the corresponding (admissible) order on the Grassmannian $\operatorname{Gr}([n],d)$. We know from § 11 that $\operatorname{Inv}(\mathscr{M})$ is an ideal with respect to $\preceq_{\mathscr{Q}}$. On the other hand, for any family $\mathscr{F}(T)$ the order $\preceq_{\mathscr{Q}}$ induces the lexicographic or antilexicographic order. In the first case, the intersection of $\operatorname{Inv}(\mathscr{M})$ with $\mathscr{F}(T)$ is an ideal with respect to the lexicographic order, that is, a starting segment. In the second case, it is an ideal with respect to the reverse order, that is, an ending segment. □

Theorem 15.8. *Let $\mathscr{S}$ be a consistent subset of* $\mathrm{Gr}([n],d)$. *Then there exists a (unique, according to Proposition 15.5) membrane $\mathscr{M}$ in the zonotope $Z=Z(n,d)$ such that $\mathscr{S}=\mathrm{Inv}(\mathscr{M})$.*

Proof. We consider the subsystem $\mathscr{S}'$ of $\mathscr{S}$ that consists of the sets $S\in\mathscr{S}$ not containing the colour n. The system $\mathscr{S}'$ lies in $\mathrm{Gr}([n-1],d)$ and is also consistent. Therefore, by induction $\mathscr{S}'$ is realized by some membrane $\mathscr{M}'$ in $Z'=Z([n-1],d)$: $\mathscr{S}'=\mathrm{Inv}(\mathscr{M}')$. We complete this membrane to form a cubillage $\mathscr{Q}'$ of the zonotope Z' (Theorem 13.1) and then we perform the expansion by the colour n of $\mathscr{M}'$ in $\mathscr{Q}'$. We obtain a cubillage $\mathscr{Q}$ of the zonotope Z, whose reduction $\mathscr{Q}_{-n}$ is $\mathscr{Q}'$. In particular, the membrane $\mathscr{M}'$ is thickened to the pie $\mathscr{P}$ of colour n in $\mathscr{Q}$. The cubes of $\mathscr{Q}$ whose types belong to $\mathscr{S}-\mathscr{S}'$ lie in $\mathscr{P}$.

We assert that $\mathscr{S}$ is a stack (an ideal) in the cubillage $\mathscr{Q}$. Then $\mathscr{M}$ is the membrane corresponding to $\mathscr{S}$. For a proof we need to verify that if $R\prec S$ are two cubes of the pie $\mathscr{P}$ and S belongs to $\mathscr{S}$, then R also belongs to $\mathscr{S}$. The cubes R and S are adjacent, and therefore the union of their types $T=\tau(R)\cup\tau(S)$ has cardinality $d+1$. We consider the family $\mathscr{F}$ of this parent T, which is ordered lexicographically. The minimal element (starting member) of this family is $T-n$. Let Q be a cube of type $T-n$ in $\mathscr{Q}$. This cube does not belong to the pie $\mathscr{P}$, and consequently it lies before or after the pie.

Consider the case when Q lies before the pie. Then it is the minimal element with respect to the natural order $\preceq$ on $\mathscr{Q}$, and the family $\mathscr{F}$ is ordered lexicographically. Since $\mathscr{S}$ is consistent, the whole interval between $\tau(Q)=T-n$ and $\tau(S)$ (and, in particular, the set $\tau(R)$) belongs to $\mathscr{S}$. Hence R belongs to $\mathscr{S}$.

The case where Q lies after the pie can be considered similarly. In this case Q is the maximal element (with respect to the restriction of $\preceq$ to the family $\mathscr{F}$) and $\tau(Q)$ does not belong to $\mathscr{S}$. Hence, the natural order $\preceq$ on $\mathscr{F}$ is antilexicographic: the cube R goes after S, and, again by consistency, R belongs to $\mathscr{S}$. $\square$

We see that there are two equivalent (better to say, cryptomorphic) descriptions of cubillages of the zonotope $Z(n,d)$: as (minimal) admissible orders on $\mathrm{Gr}([n],d)$ (Manin and Shekhtman) and as consistent subsets of $\mathrm{Gr}([n],d+1)$ (Ziegler). Given these descriptions, we can reformulate various notions and constructions concerning cubillages in the relevant language. For example, a raising flip corresponds to the enlargement of a consistent set by one element. Thus, the relation $\leqslant$ for cubillages (see § 8) is in agreement with the inclusion relation for consistent sets: if $\mathscr{Q}\leqslant\mathscr{Q}'$, then $\mathrm{Inv}(\mathscr{Q})\subset\mathrm{Inv}(\mathscr{Q}')$. The converse is true if $d=1$ (a classical property of the weak Bruhat order) or $d=2$ [11], [10], but fails for large d (Ziegler gave a counterexample for $d=3$ in [34]). In the next section we give another description (initiated by Leclerc and Zelevinsky [20] and developed by Galashin and Postnikov [14]) in terms of systems $\mathrm{Sp}(\mathscr{Q})$ of the spectra of vertices of the cubillage.

16. Separation relation

Recall that each vertex v of a cubillage $\mathscr{Q}$ of the zonotope $Z(n,d)$ is associated with the subset $\mathrm{sp}(v)$ of $[n]$, the spectrum of this vertex (see § 3). When v runs through all vertices of $\mathscr{Q}$, we obtain the system $\mathrm{Sp}(\mathscr{Q})$, which is a subset of $2^{[n]}$.

By Corollary 4.1 the system $\mathrm{Sp}(\mathscr{Q})$ uniquely determines $\mathscr{Q}$. Therefore, there arises the natural question of the properties of such systems.

The first property of such a system is its size. Corollary 2.4 states that the cardinality of $\mathrm{Sp}(\mathscr{Q})$ is $\binom{n}{\leqslant d}$. This is surely not enough to characterize such systems. The second important structure property is that any two sets in $\mathrm{Sp}(\mathscr{Q})$ are $(d-1)$-separated.

Definition 16.1. Let r be a positive integer. Two subsets X and Y of $[n]$ are said to be *r-separated* if there is no increasing chain $i_0 < i_1 < \cdots < i_{r+1}$ (of size $r+2$) such that the elements with even indices belong to one of the set differences $X-Y$ and $Y-X$, while the elements with odd indices belong to the other. We denote this symmetric and reflexive relation by S_r. A system of subsets is said to be *r-separated* if any two members of it are r-separated.

In other words, X and Y are r-separated if the set $[n]$ can be divided into $r+1$ successive intervals $I_0, \dots, I_r$ (some of which may be empty) so that $X-Y$ is in one part of these intervals and $Y-X$ is in the other part.

It is convenient to specify these intervals by dividers (separators), that is, by r points $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_r$ on the real line so that the interval I_i consists of the integer points x satisfying $c_i < x < c_{i+1}$ (we assume that $c_0 = 0$ and $c_{r+1} = n+1$).

The relations S_r become weaker with increasing r: $S_0 \subset S_1 \subset \cdots \subset S_n$. We consider several first terms of this series of relations.

Example 16.2. The relation XS_0Y means that the sets X and Y are comparable by inclusion: $X \subset Y$ or $Y \subset X$. We can also say that the relation S_{-1} is the equality relation $=$.

Example 16.3. S_1 is the strong separation relation introduced by Leclerc and Zelevinsky in [20]. It means that when we eliminate the common part, either X lies to the left of Y ($X < Y$) or Y lies to the left of X.

Example 16.4. S_2 is the chord separation relation studied in [13].

The separation relation S_r is directly connected with the spectra of cubillages, as shown by the next proposition.

Proposition 16.5. *Let $\mathscr{Q}$ be a cubillage of the zonotope $Z(n,d)$. Then for any two vertices v and w of $\mathscr{Q}$, the sets $\mathrm{sp}(v)$ and $\mathrm{sp}(w)$ are $(d-1)$-separated.*

In other words, the system $\mathrm{Sp}(\mathscr{Q})$ is $(d-1)$-separated. For example, the system $\mathrm{Sp}(\mathscr{T})$ is 1-separated for any rhombus tiling $\mathscr{T}$ (this was noted in [20]). There is another, special case. As in § 7, we consider the cube $Z(d,d)$. It has two distinguished points, the tail t and the head h. The spectrum of the head is $\{d, d-2, \dots\}$, while the spectrum of the tail is $\{d-1, d-3, \dots\}$. Taken together, their elements alternate and cover the whole of $[n]$. Therefore, the sets $\mathrm{sp}(h)$ and $\mathrm{sp}(t)$ are $(d-1)$-separated (but not $(d-2)$-separated). When we flatten (project) this cube onto the capsid $Z(d,d-1)$, t turns into an interior (central) point of the standard cubillage of the zonotope $Z(d,d-1)$ and it is easy to verify that it is $(d-2)$-separated from all other vertices of this cubillage. However, it is not $(d-2)$-separated from the projection of the point h, which is the central point of another cubillage, the antistandard one.

Proof of Proposition 16.5. Assume that $X = \mathrm{sp}(v)$ and $Y = \mathrm{sp}(w)$ are not $(d-1)$-separated. Then there exist elements (colours) $i_0 < i_1 < \cdots < i_d$ such that $i_0, i_2, \ldots$ belong to, say, $X - Y$, while $i_1, i_3, \ldots$ belong to $Y - X$. Performing the reduction of all colours distinct from $i_0, \ldots, i_d$, we obtain a cubillage of the capsid $Z(d+1, d)$. There are two points v' and w' in this cubillage (the images of v and w under the reduction) with the spectra $i_0, i_2, \ldots$ and $i_1, i_3, \ldots$. However, as we have seen above, these points belong to different cubillages: one of them (v' for odd d and w' for even d) is in the standard cubillage, while the other is in the antistandard cubillage. □

The following proposition gives a description of the spectrum of the standard (and the antistandard) cubillage of $Z(n, d)$.

Proposition 16.6. *Let v be a vertex of the standard cubillage of the zonotope $Z(n, d)$. Then there exists a decomposition $[n] = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_d$ into $d+1$ successive intervals such that $\mathrm{sp}(v) = I_{d-1} \sqcup I_{d-3} \sqcup \cdots$. Conversely, any set of this form is the spectrum of some vertex of the standard tiling.*

For the antistandard cubillage, the spectra of vertices have the form $I_d \sqcup I_{d-2} \sqcup \cdots$.

Proof. The standard cubillage of $Z(n, d)$ can be obtained from the standard cubillage of $Z(n-1, d)$ by expanding (adding) the colour n along the rear (invisible) membrane (see Example 6.2). Therefore, the spectrum of the standard cubillage consists of two parts: the first part includes sets of the form $I_{d-1} \cup I_{d-3} \cup \cdots$, where the intervals $I_d, I_{d-1}, \ldots, I_0$ are from $[n-1]$, while the second part is obtained from the spectra of vertices of the rear membrane (the invisible part of the boundary of $Z(n-1, d)$) by adding the colour n. However, the rear membrane is actually the antistandard cubillage of the zonotope $Z(n-1, d-1)$. The spectra of its vertices have the form $J_{d-1} \cup J_{d-3} \cup \cdots$, where the intervals $J_{d-1}, \ldots, J_0$ are again from $[n-1]$. In the first case we need to correct the interval I_d by replacing it by $I_d \cup \{n\}$ (note that I_d either contains the colour $n-1$ or is empty). In the second case we add the new interval $J_d = \{n\}$. □

For example, in the case $d = 1$ the spectrum of the standard cubillage of $Z(n, 1)$ consists of the $(n+1)$ intervals $[k]$, $k = 0, 1, \ldots, n$. Correspondingly, the spectrum of the antistandard tiling consists of the complementary intervals $[n], [n-1 \ldots n], \ldots, [n-k \ldots n], \ldots, \varnothing$.

In the case $d = 2$ the spectrum of the standard rhombus tiling consists of arbitrary intervals in $[n]$, whereas the spectrum of the antistandard tiling consists of co-intervals (the complements of intervals).

In the case $d = 3$ the spectrum of the standard cubillage consists of the so-called sesquilateral intervals, that is, sets of the form $[1, i] \cup [j, k]$, where $0 \leqslant i \leqslant j \leqslant k-1 \leqslant n$.

As a consequence of Proposition 16.6, we obtain a description of the spectra of *peripheral* vertices, that is, zonotope vertices. These are precisely vertices that belong simultaneously to the standard and antistandard cubillages. It is convenient to introduce the notion of the *interval rank* of a subset $X \subset [n]$ as the smallest integer r such that X is representable as the union of intervals $I_1 \sqcup \cdots \sqcup I_r$.

Corollary 16.7. *A set X is the spectrum of a peripheral vertex of the zonotope $Z(n,d)$ if and only if the sum of the interval ranks of X and the complement $[n]-X$ is at most d.*

In particular, if the size of some set X is at most $(d-1)/2$ (that is, $2|X| \leqslant d-1$), then the corresponding point $v(X)$ is peripheral, that is, a vertex of $Z(n,d)$. In other words, interior (not lying on the boundary of the zonotope) integer points appear only at height $h \geqslant d/2$. This fact is closely related to a phenomenon which is usually mentioned in discussions of cyclic polytopes (for example, see [35], Example 0.6): any set of vertices of cardinality at most $(d-1)/2$ of a cyclic polytope belongs to some face of this polytope.

Corollary 16.8. *The spectrum of a peripheral vertex is $(d-1)$-separated from any subset of $[n]$.*

This correlates with Proposition 6.3, which states, in particular, that any integer point can be embedded in some cubillage.

17. Separation and cubillages

We have already deduced that for a cubillage $\mathscr{Q}$ of the zonotope $Z(n,d)$ the system $\mathrm{Sp}(\mathscr{Q})$ is $(d-1)$-separated and has cardinality $\binom{n}{\leqslant d}$. We now proceed in the reverse direction and show that a system with these properties can be obtained as the spectrum of some cubillage. To begin with, we show that $\binom{n}{\leqslant d}$ is the upper bound for the cardinality of a $(d-1)$-separated system in $[n]$.

Proposition 17.1. *Let $\mathscr{S}$ be a $(d-1)$-separated system of subsets of $[n]$. Then it has cardinality at most $\binom{n}{\leqslant d}$.*

Proof. We divide $\mathscr{S}$ into two parts: the part $\mathscr{S}_0$ whose elements do not contain the colour n and the part $\mathscr{S}_1$ whose elements contain the colour n. We let $\mathscr{S}_2$ denote the system of sets of the form $X-n$, where X runs through $\mathscr{S}_1$. It is obvious that $\mathscr{S}_2$ has the same cardinality as $\mathscr{S}_1$. It is easy to see that $\mathscr{S}_0 \cup \mathscr{S}_2$ is a $(d-1)$-separated system in $[n-1]$, and therefore by induction its cardinality is at most $\binom{n-1}{\leqslant d}$.

Now consider the intersection $\mathscr{T} = \mathscr{S}_0 \cap \mathscr{S}_2$. It consists of sets $X \subset [n-1]$ belonging to $\mathscr{S}$ such that Xn also belongs to $\mathscr{S}$. We claim that the system $\mathscr{T}$ is $(d-2)$-separated. Indeed, assume that there are sets X and Y in $\mathscr{T}$ that are not $(d-2)$-separated. Hence, there is a sequence $i_0 < i_1 < \cdots < i_{d-1}$ such that its elements with even indices belong to (say) $X-Y$, while the elements with odd indices belong to $Y-X$. But then the sequence $i_0 < i_1 < \cdots < i_{d-1} < i_d = n$ has the same property either for $\widetilde{X} = Xn$ and Y or for X and $\widetilde{Y} = Yn$. This contradicts the assumption that $\mathscr{S}$ is $(d-1)$-separated.

Again by the induction assumption, $\mathscr{T}$ has cardinality at most $\binom{n-1}{\leqslant d-1}$. Therefore, the cardinality of $\mathscr{S}$, which is equal to the sum of the cardinalities of $\mathscr{S}_0$ and $\mathscr{S}_2$, that is, the sum of the cardinalities of $\mathscr{S}_0 \cup \mathscr{S}_2$ and $\mathscr{S}_0 \cap \mathscr{S}_2$, does not exceed

$$\binom{n-1}{\leqslant d} + \binom{n-1}{\leqslant d-1} = \binom{n}{\leqslant d}. \qquad \square$$

In view of Proposition 17.1, $(d-1)$-separated systems in $[n]$ of cardinality $\binom{n}{\leqslant d}$ are said to be *maximal by cardinality*, as opposed to systems *maximal by inclusion*. The next result was established by Galashin and Postnikov [14] in a slightly more general case (they did not restrict themselves to cubillages of cyclic zonotopes).

Theorem 17.2. *Let $\mathscr{S}$ be a $(d-1)$-separated system of cardinality $\binom{n}{\leqslant d}$ in $[n]$. Then it has the form* $\operatorname{Sp}(\mathscr{Q})$ *for some (unique) cubillage of the zonotope $Z(n,d)$.*

Proof. This is close to the proof of Proposition 17.1. We again divide $\mathscr{S}$ into $\mathscr{S}_0$ and $\mathscr{S}_1$ and define $\mathscr{S}_2$ as above. It follows from the proof of Proposition 17.1 that $\mathscr{S}_0 \cup \mathscr{S}_2$ has cardinality $\binom{n-1}{\leqslant d}$, while the intersection $\mathscr{S}_0 \cap \mathscr{S}_2$ has cardinality $\binom{n-1}{\leqslant d-1}$. We can assume by induction that the system $\mathscr{S}_0 \cup \mathscr{S}_2$ is realized by some cubillage $\mathscr{Q}'$ of $Z(n-1,d)$, while the intersection $\mathscr{S}_0 \cap \mathscr{S}_2$ is realized by a cubillage $\mathscr{Q}''$ of $Z(n-1,d-1)$. Obviously, $\mathscr{Q}''$ is realized by a membrane $\mathscr{M}$ in the cubillage $\mathscr{Q}'$. It remains to take $\mathscr{Q}$ to be the expansion by the colour n of the cubillage $\mathscr{Q}'$ along this membrane. At the level of spectra, this reduces to replacing all the sets X in $\mathscr{S}_2$ by Xn. Then $\operatorname{Sp}(\mathscr{Q}) = \mathscr{S}_0 \cup \mathscr{S}_1 = \mathscr{S}$. $\square$

Thus, we obtain yet another cryptomorphic implementation of cubillages. It is again possible to reformulate various constructions in these new terms.

We give another interesting example of separated systems.

Proposition 17.3. *Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two membranes in the zonotope $Z(n,d)$ such that* $\operatorname{Inv}(\mathscr{M}_1) \subset \operatorname{Inv}(\mathscr{M}_2)$. *Then the system* $\operatorname{Sp}(\mathscr{M}_1) \cup \operatorname{Sp}(\mathscr{M}_2)$ *is $(d-1)$-separated.*

Proof. Assume that this is not the case, and some sets $X \in \operatorname{Sp}(\mathscr{M}_1)$ and $Y \in \operatorname{Sp}(\mathscr{M}_2)$ are not $(d-1)$-separated. Then there exists a sequence of elements $i_1 < i_2 < \cdots < i_{d+1}$ of $[n]$ that alternately lie in $X-Y$ and $Y-X$. We consider the case where the elements $i_{d+1}, i_{d-1}, \dots$ are in $X-Y$ (while the elements $i_d, i_{d-2}, \dots$ are in $Y-X$) and form the d-element set $K = \{i_1, \dots, i_d\}$. Applying Lemma 15.1 to X, we deduce that the set K is inversive for the membrane $\mathscr{M}_1$ and thus also for $\mathscr{M}_2$. However, according to Lemma 15.1 the set X is not inversive for $\mathscr{M}_2$.

One can similarly analyze the case when the elements $i_{d+1}, i_{d-1}, \dots$ are in $Y-X$. But now they form a d-element set $K' = \{i_2, \dots, i_{d+1}\}$. By Lemma 15.1 it is non-inversive for the membrane $\mathscr{M}_2$ and thus also for $\mathscr{M}_1$. However, according to the same lemma, the set K' is inversive for $\mathscr{M}_1$. $\square$

18. Purity and extendability

The following question arises from Proposition 17.1: when is a $(d-1)$-separated system $\mathscr{S}$ of subsets of $[n]$ extendable to a system of maximal cardinality, that is, to a $(d-1)$-separated system of cardinality $\binom{n}{\leqslant d}$? In terms of § 4 this question can be reformulated as follows: When can the corresponding set of integer points of the zonotope $Z(n,d)$ be embedded in some cubillage of this zonotope? When this is the case, the system $\mathscr{S}$ is said to be *extendable* (or, more precisely, (n,d)-*extendable*). If each $(d-1)$-separated system in $[n]$ is extendable, then the property of *purity* is said to hold for the pair (n,d).

For example, we have seen in § 4 that any one-element system (that is, any subset of $[n]$) is extendable. According to Theorem 13.1, any membrane can be embedded in a cubillage. We now show that any two-element system is extendable.

Proposition 18.1. *Let X and Y be two $(d-1)$-separated subsets of $[n]$. Then the two-element system $\{X,Y\}$ is extendable.*

Proof. We prove this proposition using induction on n. Assume that the colour i does not lie in X and Y. Then these sets can be regarded as subsets of $[n]-i$. By induction, X and Y can be embedded in a cubillage $\mathscr{Q}$ of the zonotope $Z([n]-i,d)$. It remains to perform the expansion by the colour i (see § 4) with respect to the invisible part (in the direction of v_i) of the boundary of this zonotope.

We reason similarly in the case when the colour i lies in both X and Y. But now we need to perform the expansion by the colour i with respect to the visible part (in the direction of v_i) of the boundary of the zonotope.

Hence, everything reduces to the case when X and Y are the complements of each other in $[n]$. Using the interval representation, we see that X and Y are formed by alternating intervals in some decomposition $I_0 \sqcup I_1 \sqcup \dots \sqcup I_r$ of the chain $[n]$ into successive non-empty intervals. By the $(d-1)$-separation of X and Y we have $r<d$. This means that the sets X and Y (more precisely, the points $v(X)$ and $v(Y)$ realizing them) belong to the periphery of the zonotope $Z(n,d)$ and can thus be embedded in any cubillage. □

As we shall see in the next section, this assertion already fails for three sets. This means that purity may not hold for every pair (n,d). For which pairs does it hold?

It is trivial that purity holds for $d=1$. For $d=2$ purity was established by Leclerc and Zelevinsky [20] (see also our survey [4]). For $d=3$ purity was established by Galashin [13]. In addition, purity holds for $n=d+1$ (capsids) for trivial reasons. These cases exhaust purity, as was shown in [14]. In the next section we consider the case $n=d+2$ for $d\geqslant 4$ in detail and give simple explicit examples of non-extendable systems. To extend these results to the case $n>d+2$ the following two facts are useful.

Proposition 18.2. *A system $\mathscr{S}\subset 2^{[n]}$ is (n,d)-extendable if and only if (considered as a system of subsets of $[n+1]$) it is $(n+1,d)$-extendable.*

Proof. Let $\mathscr{S}$ be (n,d)-extendable. Then it is $(d-1)$-separated and is realized as a subset of vertices of some cubillage $\mathscr{Q}$ of $Z=Z(n,d)$. Assume that a cubillage

$\mathscr{Q}'$ of $Z(n+1,d)$ is obtained from $\mathscr{Q}$ by means of the expansion by the colour $n+1$ with respect to the invisible (rear) membrane of Z. Since $\mathscr{Q} \subset \mathscr{Q}'$, $\mathscr{S}$ can be embedded in $\mathscr{Q}'$.

Conversely, assume that $\mathscr{S}$ can be embedded in a cubillage $\mathscr{Q}'$ of $Z(n+1,d)$. Let $\mathscr{P} = \mathscr{P}_{n+1}$ be the pie of colour $n+1$ in $\mathscr{Q}'$. Since the sets in $\mathscr{S}$ do not contain the colour $n+1$, this system of vertices lies (non-strictly) before the pie $\mathscr{P}$. Therefore, by reducing $\mathscr{P}$ we obtain a cubillage $\mathscr{Q}$ of Z in which $\mathscr{S}$ can be embedded. $\square$

Proposition 18.3. *Let $\mathscr{S}$ be a $(d-1)$-separated system of subsets in $[n]$ and let $n' = n+1$. We form a system $\mathscr{S}' \subset 2^{[n+1]}$ as $\mathscr{S}' = \{Xn', X \in \mathscr{S}\}$. The system $\mathscr{S}$ is (n,d)-extendable if and only if the system $\mathscr{S} \cup \mathscr{S}'$ is $(n+1,d+1)$-extendable.*

Proof. Assume that $\mathscr{S}$ is an (n,d)-extendable system, that is, can be embedded in a cubillage $\mathscr{Q}$ of the zonotope $Z(n,d)$. We realize $\mathscr{Q}$ as a membrane $\mathscr{M}$ in some cubillage $\mathscr{Q}'$ of the zonotope $Z(n,d+1)$; thus, $\mathscr{S}$ forms a part of the vertices of $\mathscr{M}$. Finally, let $\mathscr{Q}''$ be the expansion of $\mathscr{Q}'$ by the colour n' with respect to this membrane $\mathscr{M}$. It is a cubillage of the zonotope $Z(n+1,d+1)$. In this case $\mathscr{M}$ extends to the pie $\mathscr{P}$ of colour n' in $\mathscr{Q}''$. The system $\mathscr{S}$ is on one (visible) side of this pie, whereas $\mathscr{S}'$ is on the other (invisible) side of $\mathscr{P}$. It follows that $\mathscr{S} \cup \mathscr{S}'$ can be embedded in $\mathscr{Q}''$.

Conversely, assume that the system $\mathscr{S} \cup \mathscr{S}'$ (as a set of integer points in the zonotope $Z'' = Z(n+1,d+1)$) can be embedded in a cubillage $\mathscr{Q}''$ of the zonotope Z''. We consider the pie $\mathscr{P}$ of colour n' in $\mathscr{Q}''$. It is clear that $\mathscr{S}$ is located on one side of this pie, whereas $\mathscr{S}'$ is located on the other side (and is obtained from $\mathscr{S}$ by a shift by the vector $v_{n'}$). The reduction of this pie yields a cubillage $\mathscr{Q}'$ of $Z' = Z(n,d+1)$. The pie $\mathscr{P}$ is contracted to a membrane $\mathscr{M}$, while $\mathscr{S}$ and $\mathscr{S}'$ merge into one system $\mathscr{S}$ of points on this membrane. It remains to project this membrane and obtain a cubillage $\mathscr{Q}$ of $Z(n,d)$ containing $\mathscr{S}$ as a subset of vertices. $\square$

Corollary 18.4. *Let $\mathscr{S}$ be a non-extendable system in the zonotope $Z(d+r,d)$ and let $r \leqslant r'$ and $d \leqslant d'$. Then there exists a non-extendable system in the zonotope $Z(d'+r',d')$ as well.*

We noted above that any membrane can be embedded in a cubillage. Now let us show that not every pair of $(d-1)$-separated membranes can be embedded in a cubillage. We use an example constructed by Ziegler in [34]. Namely, he constructed two membranes $\mathscr{M}_1$ and $\mathscr{M}_2$ (specified as consistent systems) in the zonotope $Z(8,4)$ which have the following two properties:

(1) $\mathrm{Inv}(\mathscr{M}_1) \subset \mathrm{Inv}(\mathscr{M}_2)$;

(2) these two membranes cannot be embedded in the same cubillage of $Z(8,4)$.

To be fair, it must be said that Ziegler formulated the property (2) in slightly different terms. By Proposition 17.3, the union of the spectra of $\mathscr{M}_1$ and $\mathscr{M}_2$ is a 3-separated system in $\mathrm{Gr}([8],4)$, but this system is not extendable to a system maximal by cardinality. This is also an example of non-purity.

19. The case of $Z(6,4)$

In this section we study the question of extendable/non-extendable systems of points in $Z(d+2,d)$. In § 11 we discussed cubillages of such zonotopes and showed that there are $2n$ of them, where $n=d+2$. Let us consider the case of $Z(6,4)$ more closely (in the case of $Z(5,3)$ all 2-separated subsystems are extendable). This simple example shows almost all the effects of the general case $Z(d+2,d)$.

For small $n-d$ almost all integer points are on the boundary of the zonotope $Z(n,d)$. For example, for $n=d+1$, only two points are not on the boundary. There are $2n$ non-peripheral points in the case $n=d+2$. Indeed, by (1.2) the number of peripheral points is

$$2\left(\binom{d+1}{d-1}+\cdots+\binom{d+1}{0}\right)=2(2^{d+1}-1-(d+1))=2^n-2n.$$

Since all peripheral points are $(d-1)$-separated from all points, we need to analyze the separation of these $2n$ non-peripheral points (sets). It is here that we start to assume that $n=6$ and $d=4$.

In this case, there are $52=64-12$ peripheral points (sets). We write 12 non-peripheral points (sets) explicitly. As Corollary 16.7 shows, peripheral points are given by decomposing the segment $[n]$ into d successive intervals. Then we need to take the union of either the even intervals or the odd intervals. In the case where $d=4$ and $n=6$, the non-peripheral points (sets) are the ones that cannot be included (alternately) in four intervals but have to engage five or six (non-empty) intervals. Clearly, either these are one-element intervals (and then there are six of them) or almost all of them are one-element intervals with a single two-element interval (and then there are five intervals). We write these 12 sets as rows of the following table:

1		3		5	
1		3		5	6
1		3			6
1		3	4		6
1			4		6
1	2		4		6
	2		4		6
	2		4		
	2		4	5	
	2			5	
	2	3		5	
		3		5	

It is convenient to arrange these 12 sets in a circle, as on a clock face, by placing 135 at the 6 o'clock position and its complement 246 at the 12 o'clock position (see Fig. 16).

Which sets among these 12 sets are 3-separated? We can verify that a set and its complement and also the two neighbours of this complement are non-separated. That is, a set and any of the three sets 'opposite' to it form non-separated pairs. This can also be seen from the fact that five successive sets on a clock face can

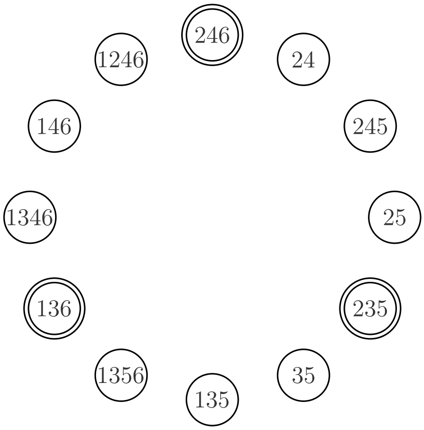

Figure 16. Clock face.

be embedded in the same cubillage (and yield all the 12 cubillages of the zonotope $Z(6,4)$). For example, the numbers from 1 o'clock to 5 o'clock (namely, the sets 24, 245, 25, 235, and 35) can be embedded in the standard cubillage. It can be seen from this description of 3-non-separation that three sets (at the 12 o'clock, 4 o'clock, and 8 o'clock positions, that is, the sets 246, 235, and 136) form (along with the periphery) a maximal-by-inclusion 3-separated system of $52+3=55$ sets, whereas the maximum by cardinality is $52+5=57$. This shows that there is a system of cardinality 55 in $Z(6,4)$ which is not (6,4)-extendable (and is maximal by inclusion).

A similar argument shows that in $Z(d+2,d)$ there is a $(d+2,d)$-non-extendable system (for $d \geqslant 4$).[4] It is interesting to note that in the case $d=8$ we can indicate five numbers (0, 4, 8, 12, and 16) on a clock face with 20 'hours' that yield a maximal-by-inclusion 7-separated subsystem of this 20-element non-peripheral system. In combination with the periphery, this gives a 1009-element maximal-by-inclusion 7-separated system of subsets of the set [10], whereas the maximum by cardinality is $1004+9=1013$. The difference is $1013-1009=4=d/2$, which suggests that the cardinality of a maximal-by-inclusion $(d-1)$-separated system differs from the maximum by cardinality at most by $d/2$.

Along with Corollary 18.4, this yields the Galashin–Postnikov theorem [14]: if $d \geqslant 4$ and $n \geqslant d+2$, then there exists an (n,d)-non-extendable system of subsets of $[n]$, that is, there is no purity for such (n,d).

20. Weak separation

In addition to the concept of strong separation (1-separation in our terms), Leclerc and Zelevinsky proposed and developed the concept of weak separation. Below we generalize this concept for arbitrary k. Weak k-separation can be understood as both a weakening of k-separation and a strengthening of $(k+1)$-separation.

[4] Non-purity for $d=7$ was actually stated in [26], Proposition 8.1.

We recall that two subsets A and B of $[n]$ are said to be $(k+1)$-separated if there exists a decomposition $[n] = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_{k+1}$ of $[n]$ into $k+2$ successive intervals such that the set $A - B$ lies in the union $I_0 \sqcup I_2 \sqcup \cdots$ of the intervals with even indices, while $B - A$ lies in the union $I_1 \sqcup I_3 \sqcup \cdots$ of the intervals with odd indices (or vice versa). We also say that A *surrounds* B if $\max(A-B) > \max(B-A)$; in other words, A is lexicographically larger than B.

Definition 20.1. Two sets A and B are said to be *weakly k-separated* if one of the following two properties holds:

(1) A and B are k-separated;

(2) A and B are $(k+1)$-separated (but not k-separated), and if A surrounds B, then the cardinality of A does not exceed that of B.

For example, let $k = 0$. Then weak 0-separation of sets A and B means that either one of them is contained in the other one, or $A-B < B-A$ and the cardinality of A is not less than that of B. For example, the set 123 is weakly 0-separated from 45 and 456 but not from 4567. The main justification of the concept we have introduced is that for $k = 1$ it coincides with weak Leclerc–Zelevinsky separation. Indeed, what does (2) mean in this case? It means that A and B can be included in three intervals and if A surrounds B, then A surrounds B and the cardinality of A does not exceed that of B.

It is obvious that k-separated sets are weakly k-separated and weakly k-separated sets are $(k+1)$-separated. If A and B have the same cardinality, then weak k-separation is equivalent to $(k+1)$-separation.

It is worth noting here that weak separation properties depend on whether k is odd or even. As we show in Proposition 20.2, the cardinality of a weakly k-separated system with odd k does not exceed $\binom{n}{\leqslant k+1}$ (the same estimate as for k-separated systems; see Proposition 17.1). If k is even, then there is no estimate of this type. For example, for $k = 0$ and $n = 5$ we have the following weakly 0-separated system of 12 sets:

$$\{\varnothing, 5, 4, 45, 34, 345, 234, 134, 124, 123, 1234, 12345\},$$

whereas $\binom{5}{\leqslant 1} = 5 + 1 = 6$. The weak 0-separation of this system can be verified directly. A rather general method to construct weakly separated systems using weak membranes will be proposed in Appendix D.

Proposition 20.2. *Let k be an odd integer. Then the cardinality of a weakly k-separated system of subsets of $[n]$ does not exceed* $\binom{n}{\leqslant k+1}$.

Proof. We prove this proposition using induction on k. For $k = -1$ the assertion is evident, since strong 0-separation is just equality. Therefore, the size of such a system is $1 = \binom{n}{0}$. For $k \geqslant 1$ we start arguing as in the proof of Proposition 17.1. Let $\mathscr{W}$ be a weakly k-separated system in $[n]$. We divide it into two parts: the part $\mathscr{W}_0$ whose elements do not contain the colour n and the part $\mathscr{W}_1$ whose elements contain the colour n. Let $\mathscr{W}_2$ consist of the sets of the form $X - n$, where X runs through $\mathscr{W}_1$. It is clear that $\mathscr{W}_0 \cup \mathscr{W}_2$ is again a weakly k-separated system (now in $[n-1]$). By induction its size does not exceed $\binom{n-1}{\leqslant k+1}$.

Let $\mathscr{D}$ denote the system of sets X such that X is in $\mathscr{W}_0$, and its 'counterpart' Xn is in $\mathscr{W}_1$. We assert that the cardinality of $\mathscr{D}$ does not exceed $\binom{n-1}{\leqslant k}$. As in the case of Proposition 17.1, Proposition 20.2 follows from this estimate. The following three lemmas prove the indicated estimate.

Lemma 20.3. *Let two subsets A and B of $\mathscr{D}$ have the same cardinality. Then they are $(k-1)$-separated.*

Proof. The property of $(k-1)$-separation means that $(A-B)$ and $(B-A)$ cannot be alternately included into a decomposition of $[n-1]$ into k intervals, that is, that $k+1$ intervals must be used for this. Assume that such a arrangement has the scheme $ABAB\dots AB$ (a word with $k+1$ letters). We compare B and An in this case. These two sets can be included in $k+2$ intervals. However, the cardinality of the surrounding set An exceeds that of the surrounded set B, which contradicts the fact that An and B are weakly k-separated. The case of the scheme $BA\dots BA$, when we compare the sets A and Bn, can be analyzed similarly. $\square$

Let $\mathscr{D}_i$ denote the subsystem of sets of cardinality i in $\mathscr{D}$. We will supplement it with two other systems of subsets of $[n-1]$. To define them, we consider two auxiliary systems $\mathscr{S}$ and $\mathscr{A}$. To describe these, consider a decomposition of $[n-1]$ into k intervals: $[n-1] = I_0 \sqcup \dots \sqcup I_{k-1}$. The union of the odd intervals $(I_1 \sqcup \dots \sqcup I_{k-2})$ in this decomposition yields an element of $\mathscr{S}$, while the union of the even intervals yields an element of $\mathscr{A}$. Let $\mathscr{S}_i$ denote the subsystem of $\mathscr{S}$ formed by sets of size i, and define $\mathscr{A}_i$ similarly.

Lemma 20.4. *The union of systems $\mathscr{S}_{n-1} \sqcup \dots \sqcup \mathscr{S}_{i+1} \sqcup \mathscr{D}_i \sqcup \mathscr{A}_{i-1} \sqcup \dots \sqcup \mathscr{A}_0$ forms a weakly $(k-2)$-separated system of subsets of the set $[n-1]$.*

Proof. The proof is a consequence of the following three simple observations. The first is that a set in $\mathscr{S}_l$ is weakly $(k-2)$-separated from any set whose cardinality is not larger. In fact, it has the form $I_1 \sqcup \dots \sqcup I_{k-2}$ for some decomposition $[n-1] = I_0 \sqcup \dots \sqcup I_{k-1}$ into k intervals. The condition on the cardinality yields strong $(k-1)$-separation, that is, weak $(k-2)$-separation. The second observation is that a set in $\mathscr{A}_l$ is weakly $(k-2)$-separated from any set whose cardinality is not smaller, for the same reasons. The third observation is that sets in $\mathscr{D}_i$ are weakly $(k-2)$-separated in view of Lemma 20.3 (they are $(k-1)$-separated and have the same cardinality). $\square$

By the induction assumption the system $\mathscr{S}_{n-1} \sqcup \dots \sqcup \mathscr{S}_{i+1} \sqcup \mathscr{D}_i \sqcup \mathscr{A}_{i-1} \sqcup \dots \sqcup \mathscr{A}_0$ has cardinality at most $\binom{n-1}{\leqslant k-1}$. On the other hand, the system $\mathscr{S}$ (as the spectrum of the standard cubillage of the zonotope $Z(n-1, k-1)$) has cardinality $\binom{n-1}{\leqslant k-1}$. Thus, the cardinality of $\mathscr{D}_i$ does not exceed the difference between the cardinality of $\mathscr{S}_0 \sqcup \mathscr{S}_1 \sqcup \dots \sqcup \mathscr{S}_i$ and the cardinality of $\mathscr{A}_0 \sqcup \dots \sqcup \mathscr{A}_{i-1}$. It remains to find this difference.

Lemma 20.5. *The indicated difference is equal to the number of vertices of the standard cubillage (in fact, of any cubillage) of the zonotope $Z(n-1,k)$ at height i.*

Given this equality, we can quickly complete the proof of Proposition 20.2. Indeed, the size of $\mathscr{D}_i$ does not exceed the number of vertices of the standard cubillage at height i. Therefore, the size of the whole system $\mathscr{D}$ does not exceed the number of vertices of the standard cubillage of $Z(n-1,k)$, that is, $\dbinom{n-1}{\leqslant k}$. But this is precisely what we needed to prove.

Proof of Lemma 20.5. Here geometry comes into play. We consider the zonotope $Z = Z(n-1,k)$. Elements of the system $\mathscr{S}$ are realized as the spectra of vertices of the visible part of the boundary of Z, while elements of the system $\mathscr{A}$ are realized as the spectra of vertices of the invisible part of the boundary. We truncate Z at height i. Then elements of $\mathscr{S}_0 \sqcup \mathscr{S}_1 \sqcup \cdots \sqcup \mathscr{S}_i$ are realized as vertices of the visible part of this truncated piece (cup), while elements of $\mathscr{A}_0 \sqcup \cdots \sqcup \mathscr{A}_{i-1}$ are realized as vertices of the invisible part (except for those located at height i).

We now take the standard cubillage of Z and consider garlands in it (see § 7), more precisely, those beginning at points in the visible part of the cup boundary. Each garland intersects somewhere another part of the cup boundary, which is ether the invisible boundary of the cup or its upper horizontal base (ceiling). Any vertex of the cubillage of this invisible part of the cup boundary is obtained in this way. It follows that the difference in question is exactly equal to the number of vertices of the cubillage at height i. Thus, Lemma 20.5 is proved, and with it Proposition 20.2. $\square$

Finally, we can pose the question of the purity of the weak separation relation. For $k = 1$ a positive answer was obtained in [4]. However, the answer is negative already for $k = 3$. To give a counterexample, we turn again to § 19, that is, to the system of subsets of the set $[6] = \{1,\dots,6\}$. In this case, there are 52 peripheral points (sets) that are 3-separated from any subsets of [6]. We assert the following.

(a) The three sets 25, 1356, and 1246 are weakly 3-separated. They are even 3-separated (see § 19).

(b) Taken together with the peripheral points (sets), these three sets form a non-extendable (maximal-by-inclusion) weakly 3-separated system of size $52+3 = 55$, whereas $\dbinom{6}{\leqslant 4} = 57$. Non-extendability can be verified directly. For example, the set 25 is 4-separated from the sets 136, 1346, and 146, but it is not weakly 3-separated, for reasons of size.

On the other hand, note that the non-extendable 3-separated system $\{246, 235, 136\}$, regarded as weakly 3-separated, admits an extension by adding the two sets 146 and 245.

Appendix A. Polycategorical view of cubillages

We will try to give some idea of this rather sophisticated construction invented by Manin, Shekhtman, Kapranov, and Voevodsky ([24], [33], [17]).

It is based on the idea that cubillages can be interpreted as morphisms. However, between what objects do these morphisms act? They act between cubillages of smaller dimension. These latter, in turn, can be regarded as morphisms. In this way, we arrive at not just a category but rather at a polycategory. Roughly speaking, a polycategory is a system of categories such that the set $\operatorname{Hom}(a, b)$ for any objects a and b of the same category is in turn the set of objects of another, higher-level category.

To tune in to this polycategorical standpoint, we return to rhombus tilings. Assume that there are two membranes in a tiling, that is, two snakes in the zonogon $Z(n, 2)$ which pass from the lower vertex $\varnothing$ to the upper vertex $[n]$. We also assume that the second membrane $\mathscr{M}_2$ goes to the right of the first membrane $\mathscr{M}_1$. Then we can say that $\mathscr{M}_1 \leqslant \mathscr{M}_2$ if a (partial) rhombus tiling can be embedded between these membranes (or, which is equivalent, both membranes are embedded in some rhombus tiling). Interpreting membranes (snakes) as linear orders on (or permutations of) the set $[n]$, we obtain a (weak) Bruhat order.

However, we can act more subtly and say that we are interested not only in the existence of tilings connecting $\mathscr{M}_1$ with $\mathscr{M}_2$ but also in the connecting tilings themselves. In other words, a morphism from $\mathscr{M}_1$ to $\mathscr{M}_2$ is understood as an arbitrary (partial) tiling $\mathscr{T}$ between $\mathscr{M}_1$ and $\mathscr{M}_2$. Thus, instead of a Bruhat poset, we obtain a Bruhat category such that permutations (or linear orders on $[n]$) are objects of it, the tilings connecting them are morphisms, and composition of morphisms is obtained by taking the union of the tilings. In fact, we obtain more: two tilings $\mathscr{T}_1$ and $\mathscr{T}_2$ between $\mathscr{M}_1$ and $\mathscr{M}_2$ can also be compared by raising flips. So $\operatorname{Hom}(\mathscr{M}_1, \mathscr{M}_2)$ is not just a (rather rich) set but also a poset! And composition of morphisms (tilings) is consistent with these posets.

Once we have embarked on this path, it would be a sin to stop. We can say not just that tilings $\mathscr{T}_1$ and $\mathscr{T}_2$ are linked by raising flips, but that such a link is a morphism (now of the next, second level). A link is a certain (partial) cubillage now in the zonotope $Z(n, 3)$. And this construction can be continued by raising the dimension further and further until we obtain the zonotope (cube) $Z(n, n)$, which has only the one trivial cubillage.

This is an outline of this polycategorical picture.

In the following example, we give only the most interesting fragment of the polycategorical canvas, and in a very simple case. Namely, we take $n = 3$ and describe only the first, second, and third levels of the resulting 'pagoda'.

Objects of the first level are vertices of the cube $Z(3, 3)$, that is, subsets of the base set $[3] = \{1, 2, 3\}$. But what can be regarded as the morphisms? It seems natural to assume that an arrow goes from X to Y if $X \subset Y$. However, we obtain only the poset $2^{[3]}$ in this way. The cubillage point of view says that we need to look at a morphism as a path from X to Y, a partial snake from X to Y. The composition of snakes is obvious: given a snake from X to Y and a snake from Y to Z, we must just attach them one to the other, thus obtaining a snake from X to Z.

Objects of the second level are partial snakes. When there are two snakes a and b with the same startpoint and the same endpoint, the question arises of whether it is possible to tessellate the domain between them by rhombi. Morphisms on the

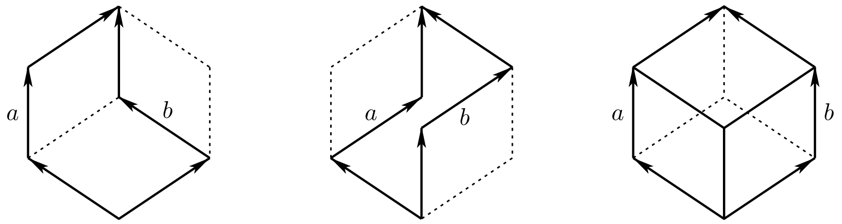

Figure 17. Snakes in the zonogon $Z(3, 2)$.

second level (between a and b) are precisely rhombus tilings of the domain between a and b (we need to assume here that b lies to the right of a). In our case with $n = 3$, there can be either one tiling of this kind (as on the left in Fig. 17) or no such tilings (as in the middle in the same figure); here snakes go from $\varnothing$ to $[3]$.

However, there is a unique exception (in our simple case), when two pavings or tilings (the standard and antistandard ones) are possible between the snakes 123 and 321 (here we understand the abbreviation 123 as a linear order on the set $\{1, 2, 3\}$). Thus, drawing the part of the second level containing only complete snakes (from $\varnothing$ to $[3]$, that is, the six linear orders on the set $[3]$), we obtain the picture shown in Fig. 18.

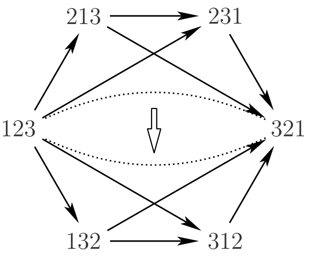

Figure 18. Partial polycategory picture in the case $n = 3$.

We would like to draw attention to the fact that there are two morphisms (the standard and antistandard tilings) from the snake 123 to the snake 321. These two morphisms as objects of the third level are also connected by a (third-level) morphism, which we show by a double arrow $\Rightarrow$. Of course, there are also many trivial (isolated) objects on the third level.

We complete the story of the polycategorical view of cubillages with this example. Details of the formalism were given by Manin and Shekhtman [24] and by Kapranov and Voevodskii [33], [17]. The notion of a polycategory was discussed briefly in [22] and in more detail in [21]. We also note that the Zamolodchikov equation was treated in the polycategorical language in [16].

Appendix B. Interrelation with triangulations and the Tamari–Stasheff poset

Almost simultaneously with the introduction of cubillages as a form of higher Bruhat poset, Tamari–Stasheff posets appeared (in [17] by Kapranov and Voevodskii), which are largely parallel to Bruhat posets and closely related to them. Several studies have been devoted to this topic (see the survey [28]).

To define these posets, we again consider a cyclic configuration of vectors $C(n, d)$. The endpoints of these vectors are points on the hyperplane in $\mathbb{R}^d$ given by the equation $x_1 = 1$, that is, on the hyperplane at height 1. The convex hull of these points $v_1, \dots, v_n$ is called a *cyclic polytope* and is denoted by $P(n, d)$. It is the intersection of the zonotope $Z(n, d)$ with the above hyperplane and has dimension $d - 1$. For $d > 2$ the points v_i are vertices of this polytope.

Instead of cubillages of $Z(n, d)$, we now consider triangulations of the polytope $P(n, d)$, that is, decompositions of it into $(d-1)$-dimensional simplices whose vertices are in the set $\{v_1, \dots, v_n\}$. For $d = 2$, this is a partition of the line segment $[v_1, v_n]$ into subsegments. For $d = 3$, it is a partition of a convex n-gon into $n - 2$ triangles, and so on. We denote the set of triangulations by $\mathbf{TS}(n, d)$.

Triangulations are similar to cubillages in many respects. For example, there are standard and antistandard triangulations (or lower and upper triangulations in the terminology of [28]) among them, which are obtained as restrictions of the corresponding cubillages. In particular, these triangulations exhaust the sets $\mathbf{TS}(d+1, d)$. Replacement of the standard triangulation by the antistandard triangulation in $\mathbf{TS}(d+1, d)$ is called a raising flip. A similar thing (a raising flip) can be done for any n if we manage to find a fragment of the form of the standard triangulation in $\mathbf{TS}(d+1, d)$. We say that $\mathscr{T} \leqslant \mathscr{T}'$ if $\mathscr{T}'$ is reachable from $\mathscr{T}$ by a series of raising flips. This gives the structure of a poset on the set $\mathbf{TS}(n, d)$, which is known as the *Tamari–Stasheff poset*.

Example B.1. Consider the case $d = 2$. Specifying a triangulation here is specifying a partition of the segment $[t_1, t_n]$ into smaller segments so that the endpoints of these segments are some of the points t_i. Any subset of $\{t_2, \dots, t_{n-1}\}$ defines such a partition into segments. A typical raising flip is the replacement of two successive segments $[t_i, t_j]$ and $[t_j, t_k]$ $(t_i < t_j < t_k)$ by $[t_i, t_k]$, that is, it is actually the removal of t_j from the subset. We see that the Tamari–Stasheff poset in this case is (anti)isomorphic to the Boolean lattice of subsets of $\{2, \dots, n-1\}$.

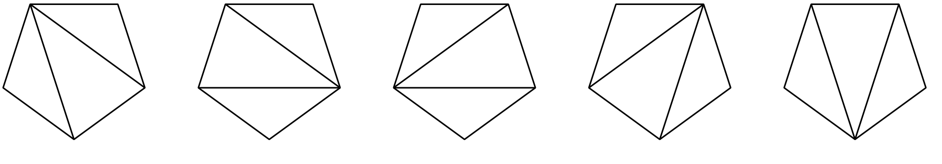

Figure 19. Five triangulations of the pentagon $P(5, 2)$.

Example B.2. The case $d = 3$ is more interesting. We take n points on the parabola $y = x^2$ sorted in ascending order of x. The convex hull of these points

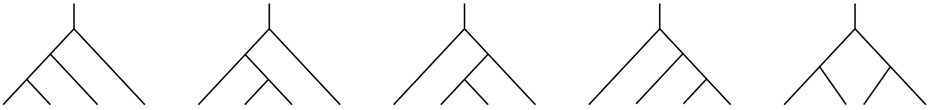

Figure 20. Five binary trees corresponding to the triangulations in Fig. 19.

is the cyclic polygon $P = P(n,3)$. We take some triangulation of it and enter the triangle with the side $[v_1, v_n]$ (from above). Then we exit from it across either of the other sides. If we leave P in this case, then the path ends. If we find ourselves in another triangle of the triangulation, then the process continues: we again exit through one of the two other sides of this triangle; and so on. As a result, we obtain what is called a *plane binary tree* (with $n-2$ non-root vertices corresponding to triangles of the triangulation); see Figs. 19 and 20. In [31] (Exercise 6.19), 65 other ways to specify the set of such trees were described, among which we should distinguish the method of placing the parentheses 'correctly' in a row of $n-1$ letters. Tamari used this method when he defined his poset $\mathbf{TS}(n,3)$.

To speak not just about the set $\mathbf{TS}(n,3)$ but also about a partial order on it, we need to define raising flips more precisely. They are structured as follows: we take a triangulation fragment shown on the left in Fig. 21 and replace the diagonal (ik) of the tetragon by the other diagonal (jl).

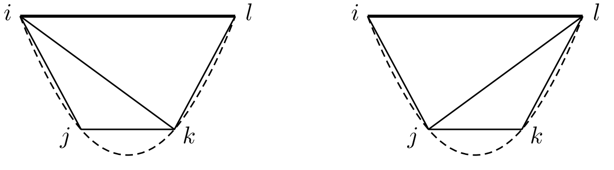

Figure 21. Triangulation flip.

Figs. 19 and 20 show five triangulations of the pentagon and the corresponding binary trees.

In addition to the obvious analogy with cubillages, there is a more clear-cut relationship between triangulations and cubillages. Let $\mathscr{Q}$ be a cubillage of the zonotope $Z(n,d)$. When this zonotope is intersected by the hyperplane $x_1 = 1$, we obtain precisely the polytope $P(n,d)$. The sections of the cubes by the same hyperplane yield a triangulation of $P(n,d)$, and this gives a natural map

$$\mathrm{sec}\colon \mathbf{Q}(n,d) \to \mathbf{TS}(n,d).$$

The triangulation $\mathscr{T} = \mathrm{sec}(\mathscr{Q})$ gives a full idea of the cubes at the bottom of the cubillage $\mathscr{Q}$ (that is, the ones rooted at 0) but says little about how the cubes above are placed. Therefore, there is no reason to hope that the map sec is injective. Even very simple examples show that there is no injection. At the same time, there is good reason to hope that the following conjecture is true.

Conjecture B.3. The map $\mathrm{sec}\colon \mathbf{Q}(n,d) \to \mathbf{TS}(n,d)$ is surjective.

In any case, this is true for $d \leqslant 3$. In the case $d = 2$ this is a simple exercise. The case $d = 3$ can be derived from [6], § 7, or from the results in [13].

We remark that the relationship between the Tamari and Bruhat posets was also mentioned in [28] (see § 8.3 there), but it was another map that was discussed there. As for the map sec, it was discussed in [9], where it was said to be surjective. However, the Tamari poset was understood there in a slightly different way, more 'combinatorial'. The whole of [9] was devoted to application of Tamari posets to soliton solutions of the Kadomtsev–Petviashvili equation. There are several other papers concerning the relationship between the Kadomtsev–Petviashvili equation and cubillages (triangulations), among which we note [18], the most recent one.

We also note that the map $\operatorname{sec}\colon \mathbf{Q}(n,d) \to \mathbf{TS}(n,d)$ is consistent with flips. When we perform a flip in a cubillage $\mathscr{Q}$, it can induce a flip in the triangulation $\operatorname{sec}(\mathscr{Q})$. More precisely, when we perform a flip in some capsid rooted at 0, we obtain a flip of the corresponding triangulation. If the capsid lies 'high', then the flip of the cubillage has no effect on the triangulation. In any case, this means that the map $\operatorname{sec}\colon \mathbf{Q}(n,d) \to \mathbf{TS}(n,d)$ is consistent with the poset structures.

Appendix C. Weak membranes

This appendix is a direct continuation of the material in §§ 9–11. On the other hand, it borders on the subjects concerning triangulations of cyclic polytopes in Appendix B.

The concept of a membrane in a cubillage was introduced in § 6. It is a $(d-1)$-dimensional subcomplex of a cubillage that is bijectively projected onto the π-projection of the whole zonotope $Z(n,d)$ along the dth coordinate vector e_d. A weak membrane is understood similarly. It is also some $(d-1)$-dimensional film in $Z(n,d)$ that is bijectively projected onto the projection of the whole zonotope. However, there are two distinctions. The first is that this latter film is no longer a subcomplex of the cubillage $\mathscr{Q}$: it is a subcomplex of some refinement of this cubillage. The second is that we project not along the vector (direction) e_d but along the vector $e_d + \varepsilon e_1$, where ε is a small positive number, that is, we look at the zonotope almost in the direction of e_d but also a little bit from the bottom up. In other words, the projection π_ε along $e_d + \varepsilon e_1$ maps the point $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ to the point

$$\pi_\varepsilon(x) = (x_1 - \varepsilon x_d, x_2, \dots, x_{d-1}) \in \mathbb{R}^{d-1}.$$

More precisely, the cyclic zonotope $Z = Z(n,d)$ grows from the point 0 'upwards' to height n, and all its vertices, as well as the vertices of the cubillage $\mathscr{Q}$, are at integer heights. We dissect Z and all the cubes of $\mathscr{Q}$ by the horizontal hyperplanes H_k given by the equations $x_1 = k$, where k runs through the integers from 1 to $n-1$. As a result, each cube Q is cut into d parts called *fragments*. Each fragment is a hypersimplex (in the terminology of [15]). Fig. 22 shows a dissection of a three-dimensional cube into three fragments: a lower tetrahedron, an octahedron, and an upper tetrahedron.

We obtain a partition of Z, though no longer into cubes, but rather into finer fragments (hypersimplices), and we denote it by $\mathscr{Q}\!\equiv$ and call it the *refinement* (or *fragmentation*) of the cubillage $\mathscr{Q}$.

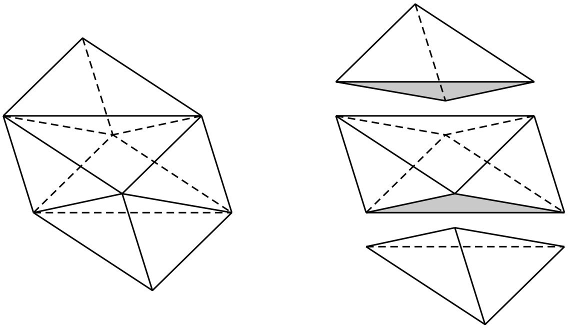

Figure 22. Dissection of the three-dimensional cube into three fragments.

The refined cubillage $\mathscr{Q}\equiv$ has the same vertices as $\mathscr{Q}$. However, we are more interested in facets of $\mathscr{Q}\equiv$, or facets of its fragments. They are divided into vertical facets (fragments of facets of cubes of $\mathscr{Q}$) and horizontal facets, which lie in some hyperplanes H_k. (Both are hypersimplices, though now of dimension $d-1$.) Most facets of a fragment P are vertical, and only two are horizontal (or one, if the fragment is a simplex).

A *weak membrane* (or *w-membrane*) in a cubillage $\mathscr{Q}$ is a subcomplex $\mathscr{W}$ of the refinement of $\mathscr{Q}$ that is bijectively (homeomorphically) projected onto the zonotope $\pi_\varepsilon(Z)$ (of smaller dimension) by the projection π_ε along the vector $e_d+\varepsilon e_1$. Ordinary membranes (from § 6; they can be called strong membranes) can also be understood as weak membranes. Like strong membranes, weak membranes are homeomorphic to the $(d-1)$-dimensional disk whose border is the rim of the zonotope Z (with respect to π or π_ε). They also divide Z into two parts: the part $(Z_-(\mathscr{W}))$ before the membrane and the part $(Z_+(\mathscr{W}))$ after the membrane; each of these parts is also fragmented. The projections (by π_ε) of cells of a membrane yield some partition of the zonotope $\pi_\varepsilon(Z)$ into plates (hypersimplices), which can be called *hypercombi*, by analogy with the concept of a combi in dimension 2 (see [5], [6]). This is an interesting object, but we will not deal with it yet.

The main distinction between weak and strong membranes is that the former can have horizontal parts (ledges or balconies); see Figs. 23 and 24.

The simplest examples of w-membranes (we call these *principal* membranes) are obtained as follows. Let $\mathscr{Q}$ be a cubillage of the zonotope $Z=Z(n,d)$. We go (from top down) first along the front (visible) part of the boundary of Z to level k, $0\leqslant k\leqslant n$. We then go horizontally along this level k to the invisible (rear) part of the boundary. Finally, we go down to 0 along the rear boundary. The left-hand picture in Fig. 24 shows the principal w-membrane (with $k=2$) for a tiling of the zonogon $Z(5,2)$, while the right-hand picture shows the one for a cubillage of the zonotope $Z(5,3)$ (this is a front view and a slightly bottom-up view; the cubillage itself is not shown).

In these cases there are ledges only at height 2. In the general case ledges can occur at any height. All membranes are principal for the trivial cubillage of the cube $Z(d,d)$.

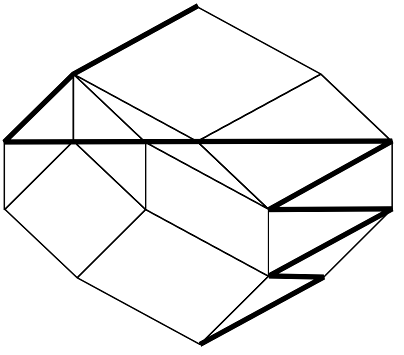

Figure 23. Weak membrane in the rhombus tiling; three balconies can be seen.

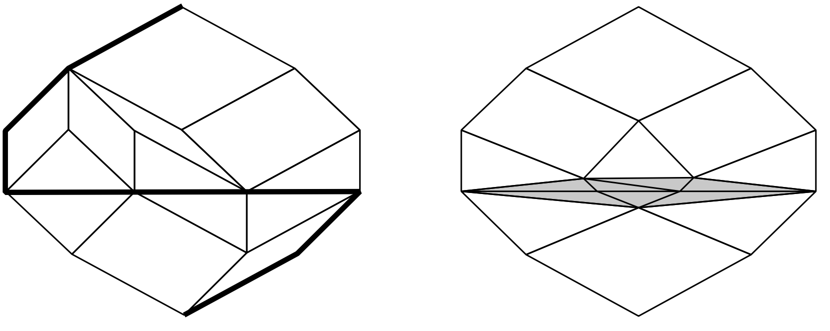

Figure 24. Examples of principal membranes.

Note that triangulations of the cyclic polytope in Appendix B can be understood as principal weak membranes (for $k=1$).

The main thing we want to discuss here is an analogue of the natural order on the set of fragments of the refined cubillage $\mathscr{Q}\equiv$. We proceed exactly as in § 9. Namely, we introduce the concept of *immediate precedence* $\prec$ for fragments. However, we project using not the map π but the slightly modified map π_ε (and correspondingly we consider the visible and invisible facets of fragments). Here we note the following. Assume that a fragment P (in a cube Q) immediately precedes a fragment P' (in a cube Q'): $P \prec P'$. Then the following two cases are possible:

(1) P and P' lie on the same level (between the hyperplanes H_k and H_{k+1}), and then $Q \prec Q'$;

(2) P and P' are separated by a hyperplane H_k, and then $Q = Q'$ and P lies under P'.

This remark helps us prove the following analogue of Proposition 9.1.

Assertion C.1. *The relation $\prec$ on $\mathscr{Q}\equiv$ is acyclic.*

Indeed, let

$$P_1 \prec \cdots \prec P_m$$

be a directed path in $\mathscr{Q}\equiv$. Let Q_l be the cube of the cubillage $\mathscr{Q}$ containing the fragment P_l. Then we obtain a directed path $Q_1, \dots, Q_m$ in $\mathscr{Q}$ such that adjacent cubes either are connected by the relation $\prec_{\mathscr{Q}}$ or coincide. By Proposition 9.1, if the relation $\prec_{\mathscr{Q}}$ occurs at least once on this path, then Q_1 is distinct from Q_m and thus P_1 is distinct from P_m. Therefore, we can assume that all the Q_l are the same cube Q. Then all the P_l are fragments of Q. Now the assertion is obvious, since each next fragment is higher than the previous fragment by one level.

Based on this assertion, we can close the relation $\prec$ transitively and obtain an order relation $\preceq$ on $\mathscr{Q}\equiv$, which again is said to be *natural*. Exactly as in § 11, we can introduce the concept of a stack (or an ideal) for this order and can identify weak membranes (for $\mathscr{Q}$) with stacks. This simple observation is useful because a stack can be disassembled by removing some maximal element (fragment) (with respect to the order $\preceq$). In terms of weak membranes, we obtain the concept of a weak (lowering) flop. An important consequence is that the minimal (and now strong) membrane is reachable from any weak membrane by a series of such weak lowering flops. Moreover, the set of weak membranes (in $\mathscr{Q}$) is a distributive lattice (as in Proposition 11.4).

However, with this the analogies come to an end in part, and the situation starts to depend on whether d is even or odd. We return to the simplest cubillage of the cube $Z(d, d)$. The refinement of this cube has d fragments. And the vertices of almost all fragments (except for the middle ones) all lie on the rim of the cube. As we know from § 7, only two vertices, denoted by t and h (the tail and the head), are not on the rim. Both these vertices belong to either one middle fragment (when d is odd) or one middle section (when d is even; see Fig. 4). This fact affects the number of vertices of a weak membrane. When we remove a non-central fragment from the stack $\mathscr{S}(\mathscr{W})$ of pre-membrane fragments (by an appropriate lowering flop), the number of vertices of the weak membrane does not change. However, when we remove central fragments, the situation becomes more complicated. Removing the above-positioned fragment (with even d) results in an increase in the number of membrane vertices by 1, removing the below-positioned fragment results in its decrease by 1, and removing the only central fragment (with odd d) does not change the number of membrane vertices.

Conclusion C.2. *For an odd d all weak membranes have the same number of vertices, equal to $\binom{n}{\leqslant d-1}$. For an even d the number of vertices of weak membranes can vary (in what ranges?).*

For example, Fig. 23 shows a tiling of the zonogon $Z(5, 2)$ with a weak membrane having 12 vertices (instead of the 'normal', six).

We now recall Proposition 20.2, which asserts that (for odd d) the cardinality of a weakly $(d-2)$-separated system in $[n]$ does not exceed $\binom{n}{\leqslant d-1}$. This fact suggests the following two conjectures.

Conjecture C.3 (cf. Proposition 16.5). The spectrum of any weak membrane in $Z(n, d)$ is a weakly $(d-2)$-separated system.

Conjecture C.4 (cf. Theorem 17.2). Let d be odd and let $\mathscr{W}$ be a weakly $(d-2)$-separated system of size $\binom{n}{\leqslant d-1}$. Then it is realized as the spectrum of some weak membrane in the zonotope $Z(n,d)$.

In any case, both conjectures are true for $d \leqslant 3$ (see [6]). Conjecture C.3 was proved in [7] for odd d, along with other assertions concerning weakly separated systems.

Appendix D. Proof of the acyclicity

In this appendix we prove Proposition 9.1, which asserts that the relation $\prec$ on the set of cubes of a cubillage $\mathscr{Q}$ is acyclic. In fact, we establish a stronger result: the relation $\prec$ is acyclic not only on the set of cubes of a fixed cubillage $\mathscr{Q}$ but also on the set $\mathscr{C}$ of all cubes of all cubillages. An (abstract) *cube* is an arbitrary d-dimensional cube C in the zonotope $Z(n,d)$ spanned by some set of vectors in $\mathbf{V}$ and growing from an integer point. Such a cube is specified by indicating its root $v(X)$ ($X \subset [n]$) and its type $T \subset [n]$; T has cardinality d and does not intersect X. As shown in Proposition 6.3, such a cube can be embedded in a cubillage. The set of all cubes in $Z(n,d)$ is denoted by $\mathscr{C}(n,d)$. It consists of all cubes of all cubillages of $Z(n,d)$.

We introduce a binary relation $\prec$ on $\mathscr{C}(n,d)$ in the same way as in § 9. Recall that $Q \prec Q'$ if these cubes Q and Q' are adjacent across a common facet F that is invisible in Q and visible in Q'. The following stronger assertion implies Proposition 9.1.

Theorem D.1. *The relation $\prec$ on the set $\mathscr{C}(n,d)$ is acyclic.*

To prove this theorem, we express the relation $\prec$ combinatorially. Assume that a cube Q is (X,T) and a cube Q' is (X',T') and that $Q \prec Q'$. We let F denote the facet across which these cubes are adjacent. Combinatorially, this facet is (S,J), where S is the root of the facet and J is its type. Obviously, $J = T \cap T'$ and J has cardinality $d-1$. Let $T = Ji$ and $T' = Jk$ for some colours i and k. The root of F can be either the head of the vector v_i or its tail. In the first case we have $S = Xi$, and in the second case $X = S$. Similarly for k. Thus, one of the four cases shown in Fig. 25 is possible.

Now we need to express the fact that F is invisible in Q. To do this we assume provisionally that the facet F is rooted at zero. Let

$$J = \{j_1 < j_2 < \cdots < j_{d-1}\}.$$

Then the linear equation $\det(v_{j_1}, \dots, v_{j_{d-1}}, \cdot) = 0$ defines a hyperplane containing F. The fact that this facet is invisible in Q means that the vector v_i lies in the positive half-space when the root of F is its head, and lies in the negative half-space when the root of F is its tail. In other words, $\det(v_{j_1}, \dots, v_{j_{d-1}}, v_i) < 0$ in cases I and II and > 0 in cases III and IV.

Symmetrically, the facet F is visible in Q' if the vector v_k is in the positive half-space in cases II and IV and if it is in the negative half-space in cases I and III,

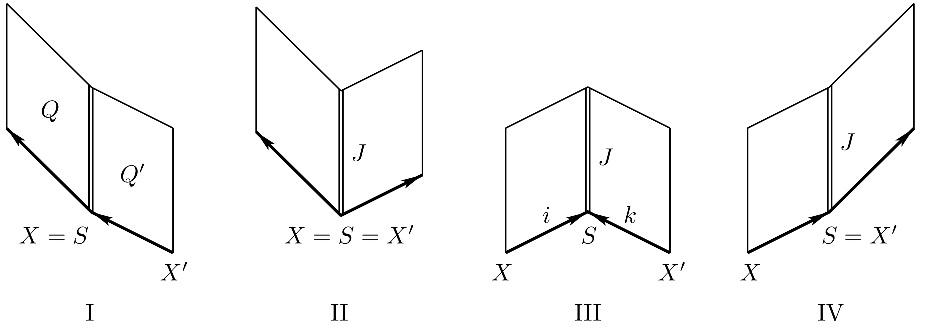

Figure 25. Four variants of cubes adjacent across a facet F of type J.

that is, the determinant $\det(v_{j_1}, \dots, v_{j_{d-1}}, v_k)$ must have the sign $+$ or the sign $-$, respectively.

This takes us back to the problem of signs of determinants, which we discussed in § 5. Namely, points $j_1 < j_2 < \dots < j_{d-1}$ on the segment $[n]$ subdivide this segment into d intervals, which are conveniently numbered from right to left. The zero interval consists of the points (integers) lying strictly above j_{d-1}; the first interval consists of the points between j_{d-2} and j_{d-1}, and so on; the last interval consists of the points lying strictly below j_1. If an integer j lies in an interval with even index, then the determinant $\det(v_{j_1}, \dots, v_{j_{d-1}}, v_j)$ is positive, and if this interval has an odd index, then the determinant is negative. We can say that j is *even* with respect to J if j lies in an interval with even index. Otherwise, this integer is said to be *odd* with respect to J.

We summarize. Assume that we have a facet F given as a pair (S, J) and a cube $Q = (X, T)$ such that F is a facet of Q. Thus, $T = Ji$ for some i not belonging to J. This facet F is invisible in Q if and only one of the following two conditions holds:

(a) $X = S$ and i is odd with respect to J;

(b) $X = S - i$ and i is even with respect to J.

Symmetrically, assume that F is a facet of a cube $Q' = (X', T')$, so that $T' = Jk$. This facet F is visible in Q if and only if one of the following two conditions holds:

(a′) $X' = S$ and k is even with respect to J;

(b′) $X' = S - k$ and k is odd with respect to J.

Hence, $Q \prec Q'$ if and only if the following combinations take place: (a) and (a′) (case II in Fig. 25), (a) and (b′) (case I), (b) and (a′) (case IV), or (b) and (b′) (case III).

Accordingly, we have expressed the relation $\prec$ combinatorially. We now divide the whole set $\mathscr{C}$ of cubes into three groups, or levels: $\mathscr{C}_0$, $\mathscr{C}_1$, and $\mathscr{C}_2$. The cubes (X, T) such that colour n is not contained in either X or T belong to the zero level $\mathscr{C}_0$; the first level (the pie of colour n) $\mathscr{C}_1$ consists of the cubes such that n belongs to the type T; the second level $\mathscr{C}_2$ includes the cubes such that $n \in X$. The following assertion is basic.

Lemma D.2. *The level is monotonically non-decreasing with respect to $\prec$.*

Proof. Let $Q \prec Q'$. We let l and l' denote the levels of Q and Q', respectively. We need to verify two things. First: if $l' = 0$, then $l = 0$. Second: if $l = 2$, then $l' = 2$.

We start with the second assertion. Assume that $l = 2$ (that is, n belongs to X) but n does not belong to X'. Since X' contains X in case (a$'$), this case is not realizable and (b$'$) takes place. But then $k = n$, and n is always even, so we arrive at a contradiction.

Let us now verify the first implication. Assume that $l' = 0$, while $l > 0$. This means that n belongs to neither X' nor T' and at the same time belongs to either X or T. The colour n cannot belong to X, as we have just shown. Thus, $n \in T = Si = (T' - k) \cup \{i\}$. Since n does not belong to T', we have $i = n$. But the colour n is always even, hence the case (a) holds when $X = S - i = S - n$. Consequently, n belongs to S and thus to $X' = Sk$. This is a contradiction. $\square$

We now proceed to a proof of the theorem using induction on n. The assertion of the theorem is true for $n = d$ since there is only one cube. Let $d < n$.

Assume that there is a cyclic monotonic path

$$Q_0 \prec Q_1 \prec \cdots \prec Q_N = Q_0.$$

By the previous lemma this path lies completely on one of the levels, namely, the zeroth, first, or second level.

If the cycle lies on the zeroth level, then it is in the zonotope $Z(n-1, d)$, which contradicts the induction assumption.

If the cycle lies on the second level, then we replace each root X_i by $X_i - n$. We again obtain a cyclic path in $Z(n-1, d)$, which is impossible in view of the induction assumption.

It remains to consider the case when the cyclic path lies on the first level, that is, if $Q_i = (X_i, T_i)$ for $i = 0, \dots, N$, then $n \in T_i$ for all i. In this case, replacing each set T_i by the reduced set $\widetilde{T}_i = T_i - n$, we obtain a path $\widetilde{Q}_0, \dots, \widetilde{Q}_N$ in $Z(n-1, d-1)$, where $\widetilde{Q}_i = (X_i, \widetilde{T}_i)$. The main observation is that we again obtain a cycle, although it has the opposite orientation, which contradicts the induction assumption. This observation follows from the next lemma, on the reverse.

Lemma D.3. *Assume that cubes $Q = (X, T)$ and $Q' = (X', T')$ in $\mathscr{C}(n, d)$ are such that*

$$Q \prec Q'.$$

Let $n \in T, T'$ and let $\widetilde{T} = T - n$ and $\widetilde{T}' = T' - n$. Then the cubes $\widetilde{Q} = (X, \widetilde{T})$ and $\widetilde{Q}' = (X', \widetilde{T}')$ in the zonotope $\widetilde{Z} = Z(n-1, d-1)$ satisfy the reverse relation

$$\widetilde{Q}' \mathrel{\widetilde{\prec}} \widetilde{Q},$$

where $\widetilde{\prec}$ is the corresponding relation on the set $\mathscr{C}(n-1, d-1)$.

Proof. Let $J = T \cap T'$. Clearly, J also contains n. We set $\widetilde{J} = J - n$. The reason for the reverse relation is that, since $J = \widetilde{J}n$, the parity of any colour i with respect to the set $\widetilde{J}$ is opposite to the parity of this colour with respect to J. $\square$

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