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MATHEMATICAL LIFE

On the 100th anniversary of the birth of Aleksei Vasil'evich Pogorelov

Aleksei Vasil'evich Pogorelov was born on 3 March 1919, in Korocha (now in Belgorod Oblast), in a farmer's family. Because of collectivization of family farms, his parents had to flee in 1931 from their town to Kharkov, where his father found work in the construction of the Kharkov Tractor Plant. In 1935 Pogorelov became a winner in a mathematical olympiad organized by Kharkov State University. After graduating from high school in 1937, he enrolled in the Department of Mathematics of the Faculty of Physics and Mathematics at Kharkov State University, where he was the best student in the department.

In 1941, when the war started, he was sent to 11-month courses at the Zhukovsky (Joukovski) Air Force Engineering Academy. After the defeat of the Nazis in the battle of Moscow this was extended to a full course of education. During this period, students of the academy were periodically deployed to the front as aircraft mechanics for several months at a time. For his part in World War II Pogorelov received the Order of the Patriotic War of the 2nd degree. After graduating from the academy, he was assigned to the Central Aerohydrodynamic Institute as a design engineer. However, his wish to complete his university education and to engage seriously in geometry brought him to Moscow State University. On the recommendation of I. G. Petrovsky, the then dean of the Faculty of Mechanics and Mathematics, and V. F. Kagan, a well-known geometer, Pogorelov was introduced to A. D. Aleksandrov, the founder of the theory of non-regular convex surfaces. There were many new problems in that theory, and Aleksandrov posed one of these for Pogorelov. A year later the problem was solved, and Pogorelov enrolled in the Faculty of Mechanics and Mathematics for distant postgraduate studies in the field developed by Aleksandrov, with N. V. Efimov as his advisor. After defending his Ph.D. thesis in 1947, Pogorelov was discharged from the army and moved to Kharkov, where he worked at the Research Institute of Mathematics and Mechanics of Kharkov

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State University and in the Department of Geometry of the university. In 1948 he defended his D.Sc. thesis, in 1951 he was elected a corresponding member of the Academy of Sciences of Ukraine, and in 1960 he was elected a corresponding member of the Academy of Sciences of the USSR, in the Department of Mathematics. From 1961 Pogorelov was an academician of the Academy of Sciences of Ukraine and from 1976 an academician of the Academy of Sciences of the USSR, in the Department of Mathematics. From 1950 to 1960 he was head of the Department of Geometry at Kharkov State University and from 1960 to 2000 head of the Department of Geometry at the Institute for Low Temperature Physics and Engineering of the Academy of Sciences of Ukraine. From 2000 he lived in Moscow as a researcher at the Steklov Mathematical Institute.

For his extraordinary research achievements Pogorelov was awarded:

- the Stalin Prize of the second degree (1950) for his investigations on the theory of convex surfaces published in his paper "The rigidity of convex surfaces" and in a series of papers in the journal *Doklady Akademii Nauk SSSR* in 1948–1949;
- the Lenin Prize (1962) for his investigations of geometry 'in the large';
- the N. I. Lobachevskii Prize (1959) for his book *Some questions in geometry in the large in a Riemannian space*;
- the N. M. Krylov Prize of the Academy of Sciences of Ukraine (1973);
- the State Prize of the Ukrainian Soviet Socialist Republic (1974) for his solution of the multidimensional Minkowski problem;
- the N. N. Bogolyubov Prize of the National Academy of Sciences of Ukraine (1998) for his works on mechanics;
- the State Prize of Ukraine (2005).

Pogorelov was the author of three brilliant university textbooks on analytic geometry, differential geometry, and the foundations of geometry. He was also the author of a high-school textbook on geometry, which is still in use in Russian schools and is republished each year.

1. Scientific interests

By the beginning of the 20th century mathematicians had developed methods for solving local problems for regular surfaces, and by the 1930s there were methods for solving geometry problems in the large, based mostly on the theory of partial differential equations. Mathematicians could do nothing in the case of non-regular surfaces (with conical points or ridge points) or when the intrinsic geometry is not determined by a regular positive-definite quadratic form but gives just a metric space of rather general type. It was the prominent geometer Aleksandrov who made a breakthrough in investigations of non-regular metrics and non-regular surfaces. He developed a theory of metric spaces with non-negative curvature (which covered also the intrinsic geometry of general convex surfaces, which are by definition domains on the boundary of an arbitrary convex body). Aleksandrov started investigating the connections between the intrinsic and extrinsic geometry of non-regular convex surfaces. He proved that any metric with non-negative curvature on a 2-sphere (for instance, non-regular metrics defined as metric spaces with intrinsic metrics) can be immersed isometrically in 3-dimensional Euclidean space as a closed convex surface. However, the answers to the following fundamental questions were unknown.

1. Is such an immersion unique up to motions?

2. If the metric on the sphere is regular and has positive Gaussian curvature, then is the convex surface realizing this metric regular?

3. G. Minkowski proved that there is a closed convex hypersurface with prescribed Gaussian curvature as a function of the normal vector under a certain natural condition on this function. However, the following problem remained open: if the function is regular on the sphere, then is the surface also regular?

Once these problems were solved, the theory developed by Aleksandrov would become a generally accepted part of mathematics and could be used also in the classical regular case. And Pogorelov was the person who answered all three questions in the affirmative. He used synthetic geometric methods and developed geometric methods for deriving *a priori* estimates for solutions of the Monge–Ampère equations. On the one hand, he used these solutions to solve geometric problems, and on the other hand, based on geometric considerations he constructed a generalized solution of the Monge–Ampère equation and then showed that for regular right-hand sides such solutions were regular. In fact, in these pioneering works Pogorelov laid the foundations of geometric analysis. On this path he obtain the following fundamental results.

1) Let F_1 and F_2 be two isometric closed convex surfaces in 3-dimensional Euclidean space or spherical space. Then they coincide up to a space motion (1951).
2) A closed convex surface in a space of constant curvature is rigid outside planar domains on it. This means that it admits only trivial infinitesimal deformations (1959).
3) If a convex surface K in a space of constant curvature c possesses a metric in the class C^k ($k \geqslant 2$) and if the Gaussian curvature of K is greater than c, then K is regular of class $C^{k-1,\alpha}$ (1949).

For subdomains of convex surfaces, the assertions 1) and 2) fail: the local and global properties of surfaces are essentially different. By proving 1) Pogorelov finished the solution of a problem which had been open for more than a century. The first result in this direction was obtained by Cauchy (1813) for the case of closed convex polyhedra. Recall that two surfaces are said to be isometric if there exists a map that takes one onto the other and preserves the lengths of curves.

Theorems proved by Pogorelov underlie his non-linear theory of thin shells. This theory considers elastic states of a shell which differ by significant changes in the original shape. Under such deformations the median surface of the thin shell undergoes bending with preservation of the metric. This makes it possible to investigate the loss of stability and the supercritical elastic state of convex shells under the action of a given load using Pogorelov's results for convex surfaces. Such shell structures are very common elements of contemporary buildings.

Pogorelov generalized the assertions 1) and 2) for regular surfaces in a Riemannian space. He also solved (1957) the Weyl problem for a Riemannian space by showing that *a regular metric on a 2-sphere with Gaussian curvature bounded below by a constant can be immersed isometrically as a regular surface in a complete simply connected 3-dimensional Riemannian space with curvature bounded above by* *a constant.* In analysing the methods used in the proof of this result, the Abel prize winner M. Gromov introduced pseudoholomorphic curves, which are now a major tool in symplectic geometry.

A closed convex hypersurface is not only uniquely determined by its metric but also by its Gaussian curvature as a function of the normal vector. In the second case it is determined up to a parallel translation, as shown by Minkowski. However, is it regular in the case when the Gaussian curvature $k(n)$ is a regular function of the normal? Pogorelov proved (in 1952 for 2-dimensional surfaces and in 1971 in the multidimensional case) that if $k(n)$ is a positive function in the class C^k ($k \geqslant 3$), then the support function is $C^{k+1,\nu}$-regular with $0 < \nu < 1$.

The hardest part of the proof was to find *a priori* estimates for the derivatives of the support function up to third order. Subsequently, S. T. Yau used Pogorelov's method to get *a priori* estimates of solutions of the complex Monge–Ampère equation. This was the main step in the proof of the existence of Calabi–Yau manifolds, which are important for theoretical physics. The Monge–Ampére equation has the form

$$|z_{ij}| = f(x_1, \dots, x_n, z, z_1, \dots, z_n).$$

A priori estimates in the Minkowski problem are estimates for the solution of the Monge–Ampère equation with

$$f = \frac{1}{k(1 + x_1^2 + \dots + x_n^2)^{n/2+1}}\,.$$

No approaches to this fully non-linear equation were known at that time. Pogorelov developed a theory of the Monge–Ampère equation using geometric methods. First, he began with polytopes and proved that generalized solutions exist under natural assumptions on the right-hand side. Next, for regular solutions he found *a priori* estimates for the derivatives up to third order. Finally, using these *a priori* estimates he showed that strictly convex solutions are regular, and proved the solvability of the Dirichlet problem and its regularity.

The Monge–Ampère equation is a significant part of the Monge–Kantorovich transportation problem; it is used in conformal, affine, and Kähler geometry, in meteorology and financial mathematics. Pogorelov once said of the Monge–Ampère equation: "*This is a great equation with which I have had the honour to work.*"

Among the most conceptual works by Pogorelov was a cycle of papers on smooth surfaces with bounded extrinsic curvature. Aleksandrov developed the theory of general metric spaces, which are a natural generalization of Riemannian manifolds. In particular, he introduced the class of 2-manifolds with bounded curvature. They exhaust the class of 2-manifolds with metric that can be uniformly approximated, in a neighbourhood of each point, by Riemannian manifolds with absolute integral curvatures (integrals of the absolute value of the Gaussian curvature) bounded by a common constant.

Of course, there was the natural question of the class of surfaces in 3-dimensional Euclidean space that carry such a metric with preservation of the connection between the metric and the extrinsic geometry of the surface. As a partial answer to this question, Pogorelov introduced the class of C^1-surfaces satisfying the condition that their spherical image has finite area (where the multiplicity of the covering in a neighbourhood of each point is taken into account). Such surfaces are called surfaces of bounded curvature. For such surfaces there is a very close connection between the intrinsic geometry of the surface and its external shape: a complete surface with bounded extrinsic curvature and non-negative (non-zero) intrinsic curvature is either a closed convex surface or an infinite convex surface; a complete surface with zero intrinsic curvature and bounded extrinsic curvature is a cylinder.

Pogorelov's first paper on surfaces with bounded extrinsic curvature appeared in 1953. However, in 1954 J. Nash published a paper on C^1-isometric immersion, the results in which were then improved by N. Kuiper in 1955. Their work showed that, under rather general assumptions, a Riemannian metric on a 2-manifold can be realized on a C^1-smooth surface in 3-dimensional Euclidean space. Moreover, such a realization can be implemented as freely as a topological immersion of the manifold with metric in the ambient space. From this it is clear that for C^1-surfaces there can be no such connection between the extrinsic and intrinsic curvatures, even with a nice intrinsic metric. And even when a C^1-surface carries a regular metric with positive Gaussian curvature, this does not mean that it is locally convex. All this underscores that Pogorelov's class of surfaces with bounded extrinsic curvature ([1], [2]) is a natural class.

2. Hilbert's fourth problem

Pogorelov solved Hilbert's fourth problem, posed by Hilbert in 1900 at the 2nd International Congress of Mathematicians in Paris.

Statement of the problem [3]: *find, up to isomorphism, all geometries that have the axiomatic systems of the classical geometries (Euclidean, hyperbolic, and elliptic), without the axioms of congruence involving the notion of angle and with the triangle inequality as an added axiom.*

In the planar case if we assume also the continuity axiom, then we arrive at a problem posed by G. Darboux: *find all variational problems in the plane whose solutions are the plane straight lines* [4].

For Desarguesian spaces G. Hamel proved that every solution of the Hilbert problem can be represented in real projective space $\mathbb{R}P^n$ or in a convex domain in $\mathbb{R}P^n$ if the congruence of line intervals is defined to be the equality of their lengths in a certain special metric for which the straight lines in the projective space are geodesics. Such metrics are said to be *flat* or *projective*. Thus, the solution of the Hilbert problem was reduced to the problem of a constructive definition of all complete flat metrics.

Hamel solved this problem under the assumption that the metric is sufficiently regular [5]: he showed that a regular Finsler metric

$$F(x,y) = F(x_1,\dots,x_n,y_1,\dots,y_n)$$

is flat if and only if it satisfies the conditions

$$\frac{\partial^2 F^2}{\partial x^i\,\partial y^j} = \frac{\partial^2 F^2}{\partial x^j\,\partial y^i}\,, \qquad i,j=1,\dots,n.$$

However, simple examples show that regular flat metrics by no means exhaust all the flat metrics. The axioms of the geometries under consideration ensure only that the metrics are continuous. Therefore, a complete solution of the Hilbert problem presupposes a constructive definition of all continuous flat metrics.

Consider the set of oriented straight lines in the plane. A straight line is determined by the parameters ρ and φ, where ρ is the distance from the origin to the line and φ is the angle between the line and the x-axis. Then the set of oriented lines is homeomorphic to a circular cylinder of unit radius with area element $dS = d\rho\, d\varphi$. Let γ be a rectifiable plane curve. Then as M. Crofton showed in 1870, its length can be defined as

$$L = \frac{1}{4} \iint_{\Omega} n(\rho, \varphi)\, dp\, d\varphi,$$

where Ω is the set of straight lines crossing the curve and $n(p, \varphi)$ is the number of points of intersection of the straight line and the curve.

A similar result also holds in the projective space.

On the set of straight lines in the projective plane $\mathbb{R}P^2$ H. Busemann introduced a completely additive non-negative measure σ with the following properties:

1) $\sigma(\tau P) = 0$, where τP is the set of straight lines through the point P;

2) $\sigma(\tau X) > 0$, where τX is the set of straight lines intersecting a set X containing a line segment;

3) $\sigma(\mathbb{R}P^n)$ is finite.

When we consider a σ-measure in an arbitrary convex domain Ω in the projective plane $\mathbb{R}P^2$, the condition 3) must be replaced by the condition that $\sigma(\pi H) < \infty$ for each set H in Ω such that the closure of H is disjoint from the boundary of Ω [6].

Using such a measure, we define a σ-metric in $\mathbb{R}P^2$:

$$|x, y| = \sigma(\tau[x, y]), \tag{1}$$

where $\tau[x, y]$ is the set of straight lines intersecting the line segment $[x, y]$. For this metric the triangle inequality is a consequence of Pasch's theorem.

Any σ-metric in $\mathbb{R}P^2$ is flat, that is, geodesics in this metric are straight lines in the projective plane.

However, Busemann was far from the opinion that σ-metrics exhaust all flat metrics. He wrote: "*The freedom in the choice of a metric with given geodesics is for non-Riemannian metrics so great that it may be doubted whether there really exists a convincing characterization of all Desarguesian spaces ...*" [6].

The following theorem, established by Pogorelov in 1973 [7], [8], solved Hilbert's fourth problem.

Theorem 2.1. *Each flat 2-dimensional complete continuous metric is a σ-metric.*

In 1976, R. V. Ambartsumyan [9] gave another proof for Hilbert's fourth problem with $n = 2$.

In the case $n = 3$ Pogorelov proved the following theorem.

Theorem 2.2. *Each flat 3-dimensional complete regular continuous metric is a σ-metric.*

However, in the 3-dimensional case σ-measures may be signed. Pogorelov showed that in the 3-dimensional case each flat complete continuous metric is a limit of regular σ-metrics in the topology of uniform convergence on compact subdomains of its domain of definition. He called such metrics generalized σ-metrics.

In his review [10] of Pogorelov's book "Hilbert's fourth problem" Busemann wrote: "*In the spirit of the time Hilbert restricted himself to $n = 2, 3$ and so does Pogorelov. However, ... the real difference is between $n = 2$ and $n > 2$. Pogorelov's method works for $n > 3$, but requires greater technicalities.*"

The multidimensional case of Hilbert's fourth problem was studied by Z. I. Szabo. In 1986, he proved a generalized Pogorelov theorem [11].

Theorem 2.3. *Each n-dimensional Desarguesian space in the class C^{n+2}, $n > 2$, is generated by the Blaschke–Busemann construction. A σ-measure that generates a flat metric has the following properties*:

1) *the σ-measure of the hyperplanes passing through any fixed point is zero*;

2) *the σ-measure of the set of hyperplanes intersecting two line segments $[x, y]$ and $[y, z]$ with the points x, y, and z not collinear is positive.*

In the same paper Szabo gave an example of a flat metric not generated by the Blaschke–Busemann construction. Subsequently, R. Alexander [12] showed that the flat metric in the Minkowski space with the norm

$$\|x\| = \max\{|x_1|, |x_2|, |x_3|\}$$

is not generated by the Blaschke–Busemann construction.

Szabo also described all flat continuous metrics in terms of generalized functions [11].

The circle of topics related to Hilbert's fourth problem is still under development. A correspondence was discovered between flat n-dimensional Finsler metrics and certain special symplectic forms on the Grassmannian manifold $G(n+1, 2)$ in E^{n+1} [13]. Also, the statement of the fourth problem was extended to the case of symmetric spaces [14].

Some problems in this theory are still open:

1) Hilbert's fourth problem has not been solved for asymmetric distances;

2) the generalization of Hilbert's fourth problem for rank-1 symmetric spaces has not been established;

3) the metrics on $\mathbb{R}P^n$ such that the k-planes minimize the k-area (Busemann; see [15]) have not been described.

3. The Dirichlet problem for the Monge–Ampère equation

We can state the Dirichlet problem for the Monge–Ampère equation in a bounded domain $\Omega \in \mathbb{R}^n$:

$$T_n[u] := \det u_{xx} = \varphi(x) > 0, \quad u\big|_\partial = \psi, \tag{2}$$

where u_{xx} is the Hessian matrix of the function u. This problem has long attracted the attention of experts in partial differential equations, in particular, because it is a model problem in the theory of fully non-linear (that is, non-linear in the second derivatives) second-order elliptic differential equations. It is well known that the operator T_n is elliptic on the set of convex functions on Ω, and one of the central problems was to describe conditions on the data in (2) that ensure solvability in the classical sense, that is, in the cone of convex functions in the space $C^k(\overline{\Omega})$, $k > 2$.

By 1953 this problem had been fairly thoroughly investigated in the 2-dimensional case [16]. However, in the multidimensional case with $\Omega \in \mathbb{R}^n$, $n \geqslant 3$, it remained open up to 1970, and even its well-posedness in the space $C^k(\Omega)\cap C^0(\overline{\Omega})$, $k>2$, was in question for $n>2$.

In proving the existence of classical solutions of the Dirichlet problem for elliptic equations, authors usually used continuation with respect to a parameter (the method of continuity) or the Leray–Schauder theorem, and both require a set of fairly strong *a priori* estimates. For (2) this must be an *a priori* estimate of a convex solution in the space $C^{2+\alpha}(\overline{\Omega})$. More precisely, in contrast to the linear theory it was necessary

(i) to find an *a priori* estimate for the second derivatives of the solution u in $C(\overline{\Omega})$;

(ii) to find an *a priori* estimate of $\|u_{xx}\|_{C^\alpha(\overline{\Omega})}$, $\alpha>0$.

Methods for implementing the programme (i), (ii) for all values $n\geqslant 2$ were developed only in the early 1980s, after Pogorelov ([17], [18]) introduced the notion of a regular solution and worked out the theory of regular solutions of (2).

In 1958, Aleksandrov introduced the concept of a generalized solution of (2) in the framework of the geometric theory of multidimensional convex surfaces, and he proved by geometric methods that such a solution exists and is unique [19]. However, purely geometric methods are not enough to solve the problem of C^k-regularity of generalized solutions of (2) for $k\geqslant 2$, and it remained open till 1971, when Pogorelov published the papers [20] and [21] (see also [17]).

In his book [17] (see § 5.2) Pogorelov presented a completely transparent definition of a generalized solution of (2) and a proof of the generalized solvability of (2) (Theorem 1 in § 5.2).

Theorem 3.1. *For each strictly convex domain Ω, each positive continuous function φ on Ω, and each continuous function ψ on the boundary of Ω there exists a unique generalized solution of the problem* (2).

However, the main aim was to prove that for sufficiently smooth data in the problem the generalized solution is regular. We remark that Pogorelov defined a solution u of (2) to be regular if $u\in C^k(\Omega)\cap C^0(\overline{\Omega})$ for some $k\geqslant 2$. As a first step, he solved the problem of *a priori* regularity in $C^2(\Omega)\cap C^0(\overline{\Omega})$ ([17], § 5.3).

Theorem 3.2. *Let $u(x)$ be a regular convex solution of* (2) *in Ω that satisfies the boundary condition $u=0$. Then the second derivatives of the solution interior to Ω have an estimate depending on the maximum modulus of the solution and its first-order derivatives, the function f and its first- and second-order derivatives, and the distance from the point at which the estimate is considered to the boundary of Ω.*

For the proof, Pogorelov introduced the auxiliary function

$$w=-ue^{u_\alpha^2/2}u_{\alpha\alpha}, \tag{3}$$

where α is a fixed direction ([17], § 5.3, (2)). Since the solution $u(x)$ vanishes on the boundary, w attains its maximum value at an interior point of the domain. Differentiating (2) twice and analysing the result at a maximum point of w ([17], § 5.3) yields the required estimate for the second derivatives of u. Thus, the *a priori* estimate in Theorem 3.2 is obtained by classical methods in the theory of second-order elliptic equations. We note that in Pogorelov's argument the estimate in Theorem 3.2 was an adequate substitute for an estimate in (i).

The following theorem was central in Pogorelov's theory ([17], § 2.3, Theorem 3).

Theorem 3.3. *At each interior point of Ω the third derivatives of the convex solution of (2) have an estimate which depends only on the second derivatives of the solution, the derivatives of φ up to third order, and the distance from the point to the boundary of Ω.*

In his proof of this result Pogorelov used differential-geometric constructions due to E. Calabi [22] to construct a certain auxiliary function, and then he derived the desired estimate by analysing this function at a maximum point ([17], § 3.3).

The $C^{k+1,\alpha}$-regularity of solutions of (2) with $k \geqslant 3$ and $0 < \alpha < 1$ for $\varphi \in C^k(\Omega)$ is a simple consequence of Theorem 3.3, because this theorem opens the way to applications of the machinery of Schauder *a priori* estimates from the linear theory of elliptic equations. Indeed, let $u \in C^3(\Omega)$ be a convex solution of (2), and let $\varphi \in C^k(\Omega)$, with $k \geqslant 3$. Differentiating (2) in an arbitrary direction l and letting $v = u_l$, we have

$$A^{ij}[u]v_{ij} = \varphi_l, \quad \text{where} \quad A^{ij} = \frac{\partial T_n}{\partial u_{ij}}, \qquad i, j = 1, \dots, n. \tag{4}$$

It follows from Theorem 3.1 that equation (4) is uniformly elliptic in any interior subdomain of Ω, while Theorem 3.2 ensures that

$$A[u] = (A^{ij}(u_{xx})) \in C^{0,1}(\Omega).$$

Furthermore, it follows from Schauder's theory that $v \in C^{2,\alpha}(\Omega)$ for any $0 < \alpha < 1$. Since l is arbitrary, the last inclusion is equivalent to $u \in C^{3,\alpha}(\Omega)$. But then $A[u] \in C^{1,\alpha}(\Omega)$, so that $u \in C^{4,\alpha}(\Omega)$, and so on.

Theorems 3.2 and 3.3 provide a functional-geometric basis for proving regularity of the generalized solution in Theorem 3.1 in the case of a linear boundary condition ([17], § 5.4, Theorem 4). In summary, [17] contains a complete proof of the existence theorem for regular solutions of the Dirichlet problem (Theorem 7 in § 5.6), the first such result (originally published in 1971 in [21]) in the theory of the multidimensional Monge–Ampère equation.

Theorem 3.4. *The Dirichlet problem for equation (2) with a regular right-hand side $\varphi(x)$ in a strictly convex domain Ω always has a regular solution if the boundary of Ω and the function ψ prescribed as the boundary values of the solution are in the class C^2. The solution is in the class $C^{k+1,\alpha}$ with $k \geqslant 3$ and $0 < \alpha < 1$ if φ is in the class C^k. The solution is analytic if φ is.*

It was a natural extension of Pogorelov's theory to investigate regular solvability of the problem (2) for a wider class of functions φ. In 1977, sufficient conditions were found in [23] for the existence of a regular solution of the problem

$$\det u_{xx} = f(x, u, u_x) > 0, \quad u\big|_{\partial\Omega} = 0, \quad x \in \Omega \subset \mathbb{R}^n, \quad n \geqslant 2. \tag{5}$$

The proof was based on a system of *a priori* estimates deduced using the Pogorelov–Calabi method (see [24], Chap. 17, comments).

In 1982 the problem (5) was treated in [25] as a problem in weighted spaces. As a result, in addition to Theorem 3.4 the Leray–Schauder principle could also be used to prove solvability of (5). Investigations along these lines are still going on.

The proof of Theorem 3.2 (see [17], § 5.4) follows the classical scheme for constructing *a priori* estimates in the theory of second-order elliptic equations, and taking the function (3) as the auxiliary function allows one to extend the result of Theorem 3.2 to a wide class of fully non-linear second-order equations.

The situation with Theorem 3.3 is different. Its proof involves a differential geometric machinery developed by Pogorelov to construct a regular solution of the Minkowski problem, and this theorem was a challenge to experts in elliptic differential equations. In fact, without an analytic proof that the problem (2) has a regular solution, this result could not be extended to a wider class of fully non-linear equations. However, it seemed incredible that, among all fully non-linear equations in the multidimensional case, only for the Monge–Ampère equation could the Dirichlet problem have regular solutions (see Theorem 3.4).

New analytic methods for local analysis of the smoothness of second derivatives of solutions of a wide class of fully non-linear second-order elliptic equations were found by L. C. Evans [26] and N. V. Krylov [27] in the early 1980s. The most concise statement and proof of this remarkable result were presented in [24]. For applications to geometric problems we formulate a version of this theorem in other terms.

Let $\mathrm{Sym}(n)$ denote the space of symmetric $n \times n$ matrices, let $D \subset \mathrm{Sym}(n)$ be a connected domain, and let $F \in C^2(D)$.

Definition 3.5. We call a set $D_F \subset D$ an F-centre if D_F is convex in $\mathrm{Sym}(n)$, F is (downwards) concave in D_F, and for each $S \in D_F$

$$0 < \lambda(S) \leqslant F^{ij}(S)\xi_i\xi_j \leqslant \Lambda(S) \quad \text{for } |\xi| = 1, \quad \text{where } F^{ij}(S) = \frac{\partial F(S)}{\partial s_{ij}}. \tag{6}$$

We call D_F a maximal F-centre if for each $S \notin \overline{D}_F$ at least one of the above conditions fails to hold.

A simplest example here is the linear function $F(S; A) = (A, S)$, where A is a constant symmetric matrix. If $A > \mathbf{0}$, then $D_F = \mathrm{Sym}(n)$ is a maximal centre. Otherwise, $D_F = \varnothing$, that is, the function $F(S; A)$ has no centre. It is also well known that in the case of the function $F_n(S) = \det^{1/n} S$ the maximal F_n-centre is equal to the cone of positive-definite matrices. We denote it by K_n.

With the function F and an $n \times n$ matrix $\tau(p)$ with $p \in \mathbb{R}^n$ and $\det \tau \neq 0$ we associate the operator $F[u] = F(S[u])$, $u \in C^2(\Omega)$, where

$$S[u] = u_{(xx)} \ \text{ and } \ u_{(xx)} = \tau^{\mathrm{T}}(u_x) u_{xx} \tau(u_x), \ \ (u_{(ij)} = (\tau_i^k u_{kl} \tau_j^l), \ \ i, j = 1, \dots, n). \tag{7}$$

Theorem 3.6. *Let F, τ, and f be C^2-smooth parameters of the equation*

$$F(u_{(xx)}) = f(x, u, u_x), \qquad x \in \Omega, \tag{8}$$

and let $u \in C^4(\Omega)$ be a solution of it. Assume that $u_{(xx)}(x) \in D_F$ for any $x \in \Omega' \subset \Omega$. Then for any $\beta \in (0; 1)$,

$$\|u_{xx}\|_{C^{1,\beta}(\Omega')} \leqslant C\bigl(\beta, \|u\|_{C^2(\Omega')}, \|F, f\|_{C^2}, \mathrm{dist}(\Omega'; \partial\Omega)\bigr). \tag{9}$$

Let us turn to the equation (2). The next result is a consequence of Theorem 3.6.

Theorem 3.7. *Let Ω be a bounded domain in $\mathbb{R}^n$, let $\Omega' \subset \Omega$, and assume that $\varphi(x) \geqslant \nu > 0$ and $\varphi \in C^2(\Omega)$. Let $u \in C^4(\Omega)$ be a convex solution of (2). Then for $\beta \in (0;1)$*

$$\|u\|_{C^{3,\beta}(\Omega')} \leqslant c\big(\nu, \beta, \|u\|_{C^2(\Omega')}, \|\varphi\|_{C^2(\Omega)}, \operatorname{dist}(\Omega'; \partial\Omega)\big). \tag{10}$$

In fact, in the cone of convex functions equation (2) is equivalent to $F_n[u] = \varphi^{1/n}$. The latter equation is covered by Theorem 3.6 with $F = F_n$, $\tau = \mathrm{Id}$, and $f = \varphi^{1/n}(x)$, and (10) is a special case of the inequality (9).

Formally, Theorem 3.3 is a consequence of Theorem 3.7, but it was published more than ten years before [26] and [27]. There is no doubt that it was this result of Pogorelov that prompted the investigations by Krylov and Evans, which now form a basis for the modern theory of fully non-linear second-order differential equations.

In Hölder spaces it is convenient to use the method of continuity to prove existence theorems for classical solutions. However, for this method we need *a priori* estimates for solutions in closed domains, and for the problem (2) we need a bound in $C^{2,\alpha}(\overline{\Omega})$ for some $\alpha > 0$. Sufficient conditions close to necessary conditions for the *a priori* boundedness of solutions of (2) in $C^2(\overline{\Omega})$ were found in 1980 in the note [28]. As for the local estimates of Krylov and Evans, in 1983 M. V. Safonov developed a method for extending them to closed domains, provided that the boundary and the Dirichlet condition are at least C^4-smooth [29]. All this led to existence theorems for classical solutions of the Dirichlet problem for a wide class of fully non-linear second-order elliptic equations (for instance, see [30]).

To formulate one contemporary version of the existence theorem for the problem (2), let $\mathbf{k}_N[\Gamma]$ denote the Gaussian curvature of a hypersurface $\Gamma \subset \mathbb{R}^{N+1}$.

Theorem 3.8. *Let Ω be a bounded domain in $\mathbb{R}^n$, $k \geqslant 2$, and $0 < \alpha \leqslant 1$. Assume that the following conditions are satisfied*:

(i) $\partial\Omega \in C^{k+2,\alpha}$, $\mathbf{k}_{n-1}[\partial\Omega] > 0$, *and* $\psi \in C^{k+2,\alpha}(\partial\Omega)$;

(ii) $\varphi \in C^{k,\alpha}(\overline{\Omega})$ *and* $\varphi(x) > 0$ *for* $x \in \overline{\Omega}$.

Then the problem (2) has a unique solution u in the cone of convex functions, and it belongs to the space $C^{k+2,\alpha}(\overline{\Omega})$. But if $\psi = 0$, then the solution u is unique in $C^2(\overline{\Omega})$.

Let us compare the hypotheses of Theorems 3.4 and 3.8. On the one hand, the inequality (10) is a valid replacement for Theorem 3.3 in the proof of Theorem 3.4, so the smoothness assumption on φ can be replaced by the weaker condition $\varphi \in C^{k-1,\alpha}$ with some $0 < \alpha \leqslant 1$.

On the other hand, Theorem 3.6 does not work when $\partial\Omega$ and ψ are C^2 (see (i)), whereas in Theorem 3.4 a C^2-smooth boundary ensures that a solution exists whose regularity depends on that of $\varphi(x)$.

We see that Theorems 3.4 and 3.8 are qualitatively different, but overlap. For example, the following result holds.

Theorem 3.9. *Let $\partial\Omega \in C^{4,\alpha}$, $\mathbf{k}_{n-1}[\partial\Omega] > 0$, and $\psi \in C^{4,\alpha}(\partial\Omega)$ with some $0 < \alpha \leqslant 1$. Then for any function $\varphi(x) > 0$ in $\overline{\Omega}$ such that $\varphi \in C^{2,\alpha}(\overline{\Omega}) \cap C^{k,\alpha}(\Omega)$, and $k \geqslant 2$, the cone of convex functions contains a unique solution u of the problem (2), and moreover, $u \in C^{k+2,\alpha}(\Omega) \cap C^{4,\alpha}(\overline{\Omega})$. On the other hand, if $\psi = 0$, then the solution u is unique in $C^2(\overline{\Omega})$.*

4. A generalization of the Minkowski problem

The main result obtained by Pogorelov in the first half of the 1970s was a regular solution of the multidimensional Minkowski problem (see [17], § 3, Theorem 1).

Theorem 4.1. *Let $K(\xi)$ be a positive C^m-regular function with $m \geqslant 3$ on the unit hypersphere Ω such that*

$$\int_\Omega \frac{\xi\,d\omega}{K(\xi)} = 0. \tag{11}$$

Then up to a parallel translation there exists a unique $C^{m+1,\alpha}$-regular $(\alpha > 0)$ convex hypersurface with Gaussian curvature $K(\xi)$. If $K(\xi)$ is an analytic function, then the hypersurface is also analytic.

Let $\varkappa_i[\Gamma](M)$, $i = 1,\dots,n$, denote the principal curvatures of a hypersurface $\Gamma \in \mathbb{R}^{n+1}$ at the point M, and let $\mathbf{n}[\Gamma](M)$ be the normal to the hypersurface pointing into the domain Ω, where $\partial\Omega = \Gamma$ and $|\Omega| < \infty$. Theorem 4.1 states that the condition (11) ensures that a regular convex solution $\Gamma \in \mathbb{R}^{n+1}$ of the equation

$$\mathbf{k}_n[\Gamma] = \prod_{i=1}^{n} \varkappa_i = K(\mathbf{n}[\Gamma]), \qquad M \in \Gamma, \tag{12}$$

exists and is unique if the function $K(\xi)$ on the hypersphere $|\xi| = 1$ is sufficiently regular. In the theory of second-order differential equations there are methods for proving existence theorems for classical solutions, based on constructing certain systems of *a priori* estimates. However, these methods cannot be applied to equations in the form (12).

Pogorelov found an equivalent formulation of (12) which permitted an analytical investigation of the multidimensional Minkowski problem, and he then developed new methods for deriving *a priori* estimates suitable for the use of results in functional analysis. More precisely, he reformulated the problem in terms of the support function of the convex hypersurface, and for this function (12) becomes an equation of Monge–Ampère type. The geometric analogue of the latter has the form

$$\prod_{i=1}^{n} R^i[\Gamma] = \varphi(\xi), \quad \text{where } R^i = \frac{1}{\varkappa_i} \quad \text{and} \quad \varphi(\xi) = \frac{1}{K(\xi)}\,, \quad |\xi| = 1. \tag{13}$$

Let $S_m(R)$, $1 \leqslant m \leqslant n$, denote the mth elementary symmetric function of the radii of curvature $R^1,\dots,R^n$ of Γ. In his note [31] Pogorelov called $S_m(R)$ the curvature function of order m of Γ and stated the problem

$$S_k(R[\Gamma]) = \varphi_k(\xi), \qquad \int_\Omega \xi\varphi_k(\xi)\,d\omega = 0, \quad |\xi| = 1, \tag{14}$$

which he regarded as a generalization of the Minkowski problem. It was shown in § 4.4 of [17] that the problem (14) has a regular solution under a certain additional condition on φ_k which led in § 2 to *a priori* bounds for the Hölder constants of the second derivatives of the support function of the hypersurface. It was also noted that for $k < n$ the positivity of the curvature function does not ensure the positivity of the principal radii of curvature. Now, after more than 45 years, it is clear that these difficulties are not essential: the problem with the Hölder constants is solved by Theorem 3.6 without any additional assumptions about φ_k. To deal with the second observation, we first make a slight algebraic digression.

In the late 1950s the prominent Swedish mathematician L. Gårding (born four days after Pogorelov) developed the theory of a-hyperbolic polynomials, which he defined to be homogeneous polynomials $P_m(x)$, $x \in \mathbb{R}^N$, $0 < m \leqslant N$, such that the polynomial $p_m(t;a) = P_m(x+at)$, $t \in \mathbb{R}^1$, has only real zeros. The central point in Gårding's theory is the concept of the cone $C(P_m;a) \subset \mathbb{R}^N$. By definition, x belongs to $C(P_m;a)$ if all the zeros of $p_m(t;a)$ are negative, that is, $P_m(x+at) \neq 0$ for $t \geqslant 0$ ([32], p. 960). The following theorem is one of the core results in Gårding's theory.

Theorem 4.2. *The cones $C(P_m;a)$ are convex. If $b \in C(P_m;a)$, then the polynomial P_m is b-hyperbolic, and moreover,*

$$C(P_m;b) = C(P_m;a).$$

Setting $\mathbb{R}^N = \mathrm{Sym}(n)$, $N = n(n+1)/2$, and $a = \mathrm{Id}$, we can add to the set of examples illustrating Definition 3.5. By the m-trace of a matrix S we mean the sum of all principal minors of order m of S, and we denote the m-trace by $T_m(S)$, $T_0(S) \equiv 1$. All the polynomials T_m with $m = 1,\dots,n$ are Id-hyperbolic; let K_m, $m = 1,\dots,n$, be the corresponding Gårding cones. The following result is a consequence of Theorem 4.2.

Corollary 4.3. *The cone K_m is the connected component of the domain of positivity of T_m that contains the identity matrix.*

In [33], K_m was defined constructively:

$$K_m = \{S \in \mathrm{Sym}(n),\ T_i(S) > 0,\ i = 1,\dots,m\}, \tag{15}$$

which implies the strict inclusion $K_j \subset K_i$ for $0 \leqslant i < j \leqslant n$. Let

$$F_m = T_m^{1/m}(S), \qquad S \in K_m.$$

It is known that the functions F_m are (downwards) concave on K_m, $m = 1,\dots,n$, and satisfy the inequality (6) (for instance, see [34]). Hence K_m is an F_m-centre. Moreover, we can show that K_m is the maximal F_m-centre.

Most authors prefer to formulate the above for diagonal matrices, that is, in terms of the elementary symmetric functions $S_m(\lambda)$, $\lambda \in \mathbb{R}^n$. In particular, Pogorelov introduced the curvature function of order m for a convex hypersurface as an elementary symmetric function of the radii of curvature ([17], § 4). However, the inclusion $K^n \subset K^m$ for $0 \leqslant m < n$ does not guarantee that all the R^i, $i = 1,\dots,n$, are positive, and the *a priori* assumption that the hypersurface is strictly convex must be justified by confirming an *a priori* lower bound for the radii of curvature, whereas only an upper bound was given in [17]. Nevertheless, for regular closed hypersurfaces the fact that the curvature function of any order is positive does guarantee the strict convexity of the surface in an informal way. For a formal verification of this, we present another series of examples illustrating Definition 3.5.

In 1990, N. S. Trudinger [35] introduced the symmetric fractions

$$\sigma_{m,l}(\lambda) = \frac{\sigma_m}{\sigma_l}(\lambda), \qquad 0 \leqslant l < m \leqslant n, \tag{16}$$

and noted that the cone K_m plays the same role for these fractions as for the elementary symmetric function $S_m(\lambda)$ (see [35], p. 157). In particular, if $m = n$, $D \subset \mathbb{R}^n$ is a connected component of the positivity domain of $S_{n,l}$, and there exists an element $\lambda \in D$ that has positive components, then $D = K_n$, and moreover,

$$\sigma_{m,l}(\lambda) < \sigma_m^{(m-l)/m}(\lambda), \qquad \lambda \in K_m. \tag{17}$$

We denote the ratio (16) with $\lambda = \varkappa[\Gamma](M)$, where $\varkappa = (\varkappa_1, \dots, \varkappa_n)$, by $\mathbf{k}_{m,l}[\Gamma](M)$ and call it the (m,l)-curvature of the hypersurface Γ at the point M. We remark that the geometric invariant $\mathbf{k}_m[\Gamma] := \mathbf{k}_{m,0}[\Gamma]$ was originally introduced in [30].

The following lemma is a consequence of Corollary 4.3 and the inequality (17) for $m = n$.

Lemma 4.4. *Let Ω be a bounded connected domain in $\mathbb{R}^{n+1}$ and let $\Gamma = \partial\Omega$ be a regular hypersurface. Assume that there exist numbers $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ with $\boldsymbol{\mu} > \boldsymbol{\nu} > 0$ such that*

$$\boldsymbol{\nu} \leqslant \mathbf{k}_{n,l}[\Gamma](M) \leqslant \boldsymbol{\mu}, \qquad M \in \Gamma. \tag{18}$$

Then Γ is strictly convex, with Gaussian curvature positive at every point.

Let us rewrite (14) in the equivalent form

$$\mathbf{k}_{n,n-k}[\Gamma] = K_{n,n-k}(\mathbf{n}[\Gamma]), \qquad \int_{|\xi|=1} \frac{\xi\, d\omega}{K_{n,n-k}(\xi)} = 0, \tag{19}$$

where $K_{n,n-k}$ is a prescribed regular function. Assume that the right-hand side of (14) satisfies the inequalities $0 < \nu^k \leqslant \varphi_k(\xi) \leqslant \mu^k$. Then every regular solution of (19) satisfies (18) with $l = n - k$, $\boldsymbol{\nu} = 1/\mu^k$, and $\boldsymbol{\mu} = 1/\nu^k$ and the principal curvatures of Γ are positive by Lemma 4.4. In this sense the fact that the Pogorelov curvature functions $S_k(R)[\Gamma]$ are positive ensures the following inequalities for $1 \leqslant k \leqslant n$:

$$R^i[\Gamma] > 0, \qquad i = 1, \dots, n.$$

However, in contrast to the relationship between (12) and (13), in replacing (19) by (14) we lose information. Namely, there is no analogue of Lemma 4.4 for solutions of (14) for $k < n$, so we cannot estimate the radii of curvature of Γ from below using (14).

Let us turn to (19). To implement Pogorelov's pioneering idea that the maximum principle in the theory of second-order elliptic differential equations can be used in the construction of *a priori* estimates, we must represent the operator $\mathbf{k}_{n,n-k}[\Gamma]$ in a form inheriting the regularity of the hypersurface Γ. The key notion here is the curvature matrix $\mathscr{K}[\Gamma]$ introduced in [36], [34], [37]. Using the terminology from the preprint [37], we can describe it as follows.

Consider a regular hypersurface $\Gamma \subset \mathbb{R}^{n+1}$, take a point $M \in \Gamma$, and consider a parametrization $X[\Gamma] = (x^1(\theta), \dots, x^n(\theta))$ of Γ in a neighbourhood of M, where

θ ranges in a connected domain Θ in $\mathbb{R}^n$. Here we assume that $X \in C^k(\Theta)$, $k > 2$. We introduce the following notation:

$$(\,\cdot\,)_i = \frac{\partial}{\partial \theta^i}\,, \qquad (\,\cdot\,)_{(i)} = (\,\cdot\,)_k \tau_i^k, \qquad (\,\cdot\,)_{(\theta)} = \bigl((\,\cdot\,)_{(1)}, \dots, (\,\cdot\,)_{(n)}\bigr). \tag{20}$$

Here the $n \times n$ matrices $\tau = \tau[\Gamma]$ are defined so that

$$g^{-1}[\Gamma] = \tau\tau^{\mathrm{T}},$$

where $g[\Gamma] = ((X_i, X_j))_{i,j=1}^n(\theta)$ is the metric tensor of Γ. Namely, each matrix of the form

$$\tau = \tau_0 B, \qquad \tau_0 = \sqrt{g^{-1}}\,, \quad B \in O(n),$$

is admissible. The derivatives $(\,\cdot\,)_{(i)}$ in (20) do not depend on the parametrization, and therefore they were called invariant derivatives in [34].

Let $\mathbf{n}[\Gamma](M)$ denote the normal to Γ at M. If Γ is the boundary of a connected domain, then $\mathbf{n}[\Gamma]$ is always the inward normal.

Definition 4.5. We call the matrix

$$\mathscr{K}[\Gamma] = (X_{(\theta)(\theta)}, \mathbf{n})(M) = (X_{(\theta\theta)}, \mathbf{n})(M), \qquad M \in \Gamma, \tag{21}$$

the curvature matrix of Γ, and we call the functions

$$\mathbf{k}_p[\Gamma] = T_p(\mathscr{K})[\Gamma], \quad \mathbf{k}_{p,q}[\Gamma] = \frac{\mathbf{k}_p}{\mathbf{k}_q}[\Gamma], \qquad p = 1, \dots, n, \quad q \geqslant 0, \tag{22}$$

the p- and (p,q)-curvatures of Γ, respectively.

The characteristics (21) and (22) are absolute geometric invariants, and they are C^{k-2}-smooth if the regular hypersurface Γ is C^k-smooth, where $k > 2$. More precisely, (21) defines the family of symmetric matrices $\{B^{\mathrm{T}}\mathscr{K}_0 B\}$, $B \in O(n)$, where, for example, $\mathscr{K}_0$ corresponds to $\tau_0 = \sqrt{g^{-1}}$. The eigenvalues of all the curvature matrices coincide, up to order, with the principal curvatures of the hypersurface. On the other hand, our freedom in the choice of the matrix τ allows us to reduce the curvature matrix at a fixed point $M \in \Gamma$ to diagonal form, and therefore by the orthogonal invariance of the trace of a symmetric matrix the curvatures (22) are well defined and depend only on the hypersurface Γ.

It follows from the above that each of the equations (12), (14), and (19) is equivalent to a fully non-linear second-order elliptic differential equation

$$T_{m,l}(\mathscr{K}[\Gamma]) = f(M) > 0, \qquad M \in \Gamma, \quad 0 \leqslant l, \tag{23}$$

for a suitable choice of the subscripts and for $f(M) = f(\mathbf{n}[\Gamma](M))$. The core feature of the geometric problems (23) is that we can investigate them in the framework of the modern theory of second-order elliptic partial differential equations.

We see that Pogorelov's generalization of the multidimensional Minkowski problem and Gårding's theory of hyperbolic polynomials have brought about a new field in both differential geometry and the theory of second-order partial differential equations. The reader can find examples of investigations in this field in [38] and [39].

Pogorelov's methods for deriving *a priori* bounds have been used in the theory of the optimal transportation problem and in geometric optics [40], [41].

5. Right-angled polyhedra in Lobachevskii space

Pogorelov's paper "A regular partition of Lobachevskian space" [42] was published in the first issue of the journal *Matematicheskie Zametki*.[1] He started with the following observation, which we cite here: "Let P be a closed convex polyhedron with right dihedral angles in Lobachevskii space. By a mirror reflection of P at each of its faces we obtain polyhedra that are then reflected at the faces not adjacent to P, and so on. By continuing this operation indefinitely we fill the whole of space with polyhedra equal to P. Such a partition of Lobachevskii space into equal polyhedra can be effected by a system of planes. Any two planes of this system either are disjoint or intersect at right angles. $\langle \dots \rangle$ On the Lobachevskii plane it is possible to construct a regular pentagon with any interior angles smaller than $3\pi/5$. In particular, there exists a regular pentagon with right interior angles. From such plane pentagons one can obviously construct a regular dodecahedron in Lobachevskii space. All its dihedral angles are right angles, and hence the whole of Lobachevskii space can be covered by such dodecahedra in the indicated manner"([42], p. 3).

The object of that paper was to find necessary and sufficient conditions for the existence of a closed convex polyhedron of given structure with right dihedral angles in Lobachevskii space. Here Pogorelov meant that two polyhedra P and P' have the *same structure* or are *analogous* if a correspondence can be established between their faces, edges, and vertices that preserves the incidence relation. He proved the following result.

Theorem 5.1 [42]**.** *In order that a closed convex polyhedron which has right dihedral angles and is analogous to a given polyhedron P exist in Lobachevskii space, it is necessary and sufficient that the following conditions be satisfied:*

1) *precisely three edges meet at each vertex of P;*

2) *each face of P has at least five sides;*

3) *every simple closed contour on the surface of P that separates any two of its faces intersects at least five edges;*

4) *in Lobachevskii space there exists a polyhedron P' that is analogous to P and has acute dihedral angles.*

A polyhedron $\overline{P}$ that is analogous to P and has right dihedral angles is uniquely defined up to a motion and a mirror reflection.

Conditions 1) and 2) in Theorem 5.1 are combinatorial and easy to verify on the basis of the 1-skeleton of the polyhedron (the graph formed by its vertices and edges).

In Fig. 1 we present the Schlegel diagram of a polyhedron satisfying the conditions 1) and 2), but not 3). The closed dashed curve is a simple closed contour on the surface of the polyhedron that separates two of its facets (for example, the central hexagonal facet and the 'external' hexagonal facet) but intersects only four edges of it.

We remark that verifying the condition 4) is quite difficult for polyhedra with a complicated combinatorial structure. Several years later E. M. Andreev [43] overcame this difficulty by establishing necessary and sufficient conditions for a bounded acute-angled polyhedron that is combinatorially analogous to a fixed polyhedron

[1] translated as *Mathematical Notes*.

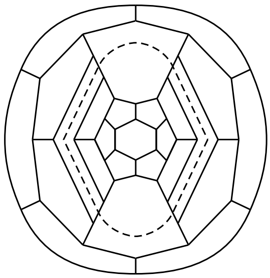

Figure 1. Schlegel diagram of a polyhedron not satisfying the condition 3)

to exist in Lobachevskii space. Because of the importance of his result we present its statement. We recall the following definition. Let $\Gamma_{i_1}, \dots, \Gamma_{i_s}$ be facets of some 3-dimensional polyhedron such that each facet is adjacent to the next in the list, the last is adjacent to the first, no other two facets are adjacent, and no three intersect in one point. Then we say that $\Gamma_{i_1}, \dots, \Gamma_{i_s}$ form an *s-angled prismatic element*.

Theorem 5.2 [43]**.** *Let P be a bounded convex polyhedron with dihedral angles $\alpha_{ij}(M)$ at most $\pi/2$ in the 3-dimensional Lobachevskii space $\mathbb{H}^3$. Then the dihedral angles satisfy the following system of inequalities* (*depending only on the combinatorial type*):

0) $0 < \alpha_{ij} \leqslant \pi/2$;

1) *if Γ_{ijk} is a vertex of P*, *then*

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} > \pi;$$

2) *if facets Γ_i, Γ_j, and Γ_k form a three-angled prismatic element*, *then*

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} < \pi;$$

3) *if four facets Γ_i, Γ_j, Γ_k, and Γ_l form a four-angled prismatic element*, *then*

$$\alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} < 2\pi;$$

4) *if Γ_s is a facet with edges Γ_{is}, Γ_{js}, Γ_{ks}, and Γ_{ls} ordered successively*, *then*

$$\alpha_{is} + \alpha_{ks} + \alpha_{ij} + \alpha_{il} + \alpha_{jk} + \alpha_{kl} < 3\pi,$$
$$\alpha_{js} + \alpha_{ls} + \alpha_{ij} + \alpha_{il} + \alpha_{jk} + \alpha_{kl} < 3\pi.$$

We remark also that Andreev proved the following uniqueness result.

Theorem 5.3 [43]**.** *Let P and P' be two bounded convex polyhedra in the n-dimensional Lobachevskii space $\mathbb{H}^n$, $n \geqslant 3$. Let $P \sim P'$ and suppose that the corresponding dihedral angles of these polyhedra are equal and do not exceed $\pi/2$. Then P and P' are congruent.*

Polyhedra satisfying the combinatorial conditions in Theorem 5.1 have recently come to be called *Pogorelov polyhedra*. They are connected in a natural way with interesting constructions in 3-dimensional hyperbolic geometry and toric topology (for instance, see the surveys [44] and [45]).

We recall that in answering affirmatively the question of the existence of closed oriented hyperbolic 3-manifolds, Löbell constructed in 1931 the first example of such a manifold by using eight copies of a right-angled 14-hedron [47] (see Fig. 2). In [48] A. Yu. Vesnin proposed a method for constructing closed oriented hyperbolic 3-manifolds by using four-colourings of the facets of right-angled polyhedra, and he showed that Löbell's example can also be constructed using that method. Namely, a four-colouring of the facets of a Pogorelov polyhedron determines an epimorphism of the reflection group of the polyhedron onto the eight-element Abelian group $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ and leads to the construction of a hyperbolic 3-manifold. The fundamental group of this manifold is a torsion-free index-8 subgroup of the corresponding right-angled Coxeter group. In [48] this method was used to construct an infinite family of hyperbolic manifolds generalizing Löbell's example, in [49] their volumes were calculated, and in [50] it was shown that if the 1-skeleton of a Pogorelov polyhedron contains a Hamiltonian cycle, then the corresponding manifold can be represented as a multiplicity-2 branched cover of the 3-sphere, and a method for describing the branch locus was indicated.

In [51] M. Davis and T. Januszkiewicz proposed a construction of manifolds with a torus action that correspond to simple (on their vertices) polyhedra. Surprisingly, such manifolds are cohomologically rigid precisely when the polyhedron in question is a Pogorelov polyhedron [44]. This enabled V. M. Buchstaber and T. E. Panov [52] to find necessary and sufficient conditions for four-colourings of the facets of Pogorelov polyhedra to determine isometric hyperbolic 3-manifolds.

We see from Theorem 5.1 that Pogorelov polyhedra can have a rather complicated combinatorial structure. The simplest example is a dodecahedron. The example next in simplicity is a 14-hedron whose top and bottom bases are hexagons. We present the Schlegel diagrams of these polyhedra in Fig. 2.

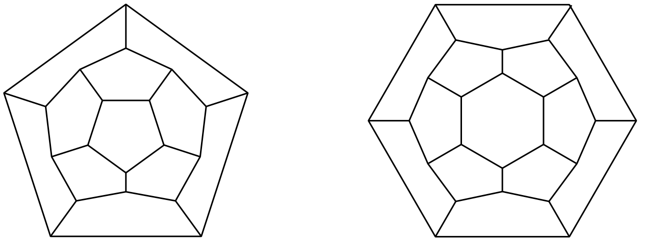

Figure 2. Schlegel diagrams of a dodecahedron and a 14-hedron

The class of Pogorelov polyhedra contains the interesting class of objects called fullerenes. Recall that *fullerenes* are chemical compounds whose molecular structure is a 1-skeleton of a polyhedron each of whose vertices is incident to three edges and each of whose faces is a pentagon or a hexagon. Investigations of molecules of fullerenes have been closely connected with mathematics from the outset. Geometry, topology, graph theory, number theory, and other areas of mathematics are instrumental to a significant degree for classifying the structures of fullerenes and predicting their unique physical and chemical properties. For more information on the mathematical theory of fullerenes the reader is advised to consult the collection of papers [53] and *An atlas of fullerenes* [54].

By Theorem 5.3 a bounded right-angled hyperbolic polyhedron is completely determined by its combinatorics. Moreover, it turns out that some of its geometric parameters can be estimated in terms of just the number of vertices. Namely, C. K. Atkinson [55] found the following two-sided bounds for volumes.

Theorem 5.4 [55]**.** *Let P be a bounded right-angled hyperbolic polyhedron with N vertices. Then*

$$(N-2)\frac{v_8}{32} \leqslant \operatorname{vol}(P) < (N-10)\frac{5v_3}{8},$$

where v_8 is the maximum volume of a hyperbolic octahedron, and v_3 is the maximum volume of a hyperbolic tetrahedron. Furthermore, there exists a sequence of bounded right-angled polyhedra P_i with N_i vertices such that the ratio $\operatorname{vol}(P_i)/N_i$ tends to $5v_3/8$ as $i \to \infty$.

The constants v_3 and v_8 in the theorem have the numerical values

$$v_3 = 3\Lambda\left(\frac{\pi}{3}\right) = 1.0149416064096535\dots, \tag{24}$$

$$v_8 = 8\Lambda\left(\frac{\pi}{4}\right) = 3.663862376708876\dots. \tag{25}$$

In fact, v_3 is the volume of a regular ideal hyperbolic tetrahedron with dihedral angles $\pi/3$, and v_8 is the volume of a regular ideal hyperbolic octahedron with dihedral angles $\pi/2$. Here $\Lambda(\theta)$ denotes the Lobachevskii function

$$\Lambda(\theta) = -\int_0^\theta \log|2\sin t|\,dt.$$

In view of Theorem 5.3 it seems natural to list the Pogorelov polyhedra in order of increasing volumes of their right-angled realizations in Lobachevskii space. T. Inoue [56], [57] listed the first 825 bounded right-angled hyperbolic polyhedra with their volumes. In particular, the last, 825th polyhedron in the list has volume $13.4204\dots$. Inoue also presented the Schlegel diagrams of the first 100 Pogorelov polyhedra.

We underscore that the study of conditions for the existence of bounded right-angled polyhedra in Lobachevskii space that Pogorelov [42] initiated in 1967 has developed in recent years into intensive investigations of their geometric properties and has led to interesting constructions in topology. It is rather exciting that, after several decades, the class of combinatorial polyhedra that he described has turned out to be of interest not only from a mathematical point of view, but also because it contains a subclass of polyhedra important in structural chemistry.

A. A. Borisenko, A. Yu. Vesnin, and N. M. Ivochkina

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