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Junction conditions in scalar–tensor theories

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Abstract

We analyze junction conditions at a null or non-null hypersurface Σ in a large class of scalar–tensor theories in arbitrary $n(\geqslant 3)$ dimensions. After showing that the metric and a scalar field must be continuous at Σ as the first junction conditions, we derive the second junctions conditions from the Einstein equations and the equation of motion for the scalar field. Subsequently, we study C^1 regular matching conditions as well as vacuum conditions at Σ both in the Jordan and Einstein frames. Our result suggests that the following configurations may be possible; (i) a vacuum thin-shell at null Σ in the Einstein frame, (ii) a vacuum thin-shell at null and non-null Σ in the Jordan frame, and (iii) a non-vacuum C^1 regular matching at null Σ in the Jordan frame. Lastly, we clarify the relations between the conditions for C^1 regularity and also for vacuum Σ in the Jordan and Einstein frames.

Keywords: junction conditions, scalar tensor theories, regular matching conditions, Jordan frame, non-minimally coupled scalar fields

Contents

1. Introduction

For given two spacetimes, can one attach them at a hypersurface Σ? If so, what kind of configurations of Σ is possible? How smooth is the spacetime at Σ? These are well-defined problems in gravitation physics and have a variety of applications. The basic equations to answer these problems are called the *junction conditions* which are obtained from the field equations and describe the relation between the discontinuity of the metric and the matter field on the junction hypersurface Σ embedded in a bulk spacetime.

In general relativity, a manifestly covariant formalism of the junction conditions has been formulated in the sixties by Israel for non-null (namely, timelike or spacelike) Σ, which relates the jump of the extrinsic curvature of Σ to the energy-momentum tensor for a matter field on

Σ [1]. By the Israel junction conditions, it is shown that the spacetime is C^1 (continuously differentiable) and hence regular at Σ if and only if there is no matter field on Σ. If the spacetime is C^0 and hence there is a jump of the extrinsic curvature at Σ, the matching hypersurface Σ is refered to as a *thin-shell* or a *singular hypersurface*. In general relativity, a matter field is required on Σ for this C^0 matching and then Σ is refered to as a *massive thin-shell*[4].

Obviously, Israel's formulation does not work for null hypersurfaces because the extrinsic curvature is necessarily continuous when Σ is null. (See section 3.11.3 in [3].) Indeed, it took more than twenty years until the extension of Israel's formalism for null hypersurfaces was developed by Barrabès and Israel [4]. After being applied in several contexts [5–10], this extension has been reformulated by Poisson [11]. Poisson's new formulation makes systematic use of the null generators of the hypersurface and provides a simple characterization of the thin-shell energy-momentum tensor in terms of the jump of the transverse curvature at Σ. (See [12] for recent developments in the research of junction conditions.)

Alternatively, the junction conditions can also be obtained from the variational principle. This method relies on the action principle under Dirichlet boundary conditions for a composite manifold made out of two submanifolds joined at a non-null hypersurface Σ [13]. The action contains surface terms and its extremum yields not only the field equations in the bulk spacetime but also the junction conditions at Σ. In contrast, derivation of the junction conditions in this method is still unknown in the case where Σ is null. This is because a general well-defined action principle has not been established on null hypersurfaces. (See, for instance [14–16].)

The junction conditions have been studied also in scalar–tensor theories, which are natural generalizations of general relativity and contain a non-minimally coupled scalar field to gravity. Extensions of Israel's formalism for non-null Σ have been presented in a class of scalar–tensor theories [17–20]. However, these analyses did not consider the case where Σ is null. Although the junction conditions have been studied both for null and non-null Σ in a class of four-dimensional scalar–tensor theories in [21, 22], the analyses were performed only in the Einstein frame and therefore non-minimal couplings for the scalar field were not taken into account. As far as the authors know, a study of the junction conditions for null Σ in the Jordan frame is absent in the literature in spite of their potential importance for future applications. One of the purposes of the present paper is to fill this gap.

In this article, we study junction conditions at a null or non-null hypersurface Σ in a large class of scalar–tensor theories in arbitrary $n(\geqslant 3)$ dimensions, in which a real scalar field with self-interaction potential is non-minimally coupled to gravity. The article is organized as follows. In the next section, we will present the action and the field equations of the system both in the Jordan and Einstein frames. In section 3, we will derive the junction conditions in the case where the matching hypersurface Σ is non-null and study the C^1 regular matching conditions and the vacuum conditions at Σ in both frames. In section 4, we will perform the same analysis as in section 3, but in the case where Σ is null. For this purpose, we adopt the formalism presented in [11]. In section 5, we will clarify the relations between the conditions for C^1 regularity and also for vacuum Σ in the Jordan and Einstein frames and apply the result to two different exact solutions. Our results are summerized in the final section. Some technical details are presented in two appendices.

[4] In contrast, a *vacuum* thin-shell is possible in a class of quadratic curvature gravity called Einstein–Gauss–Bonnet gravity [2].

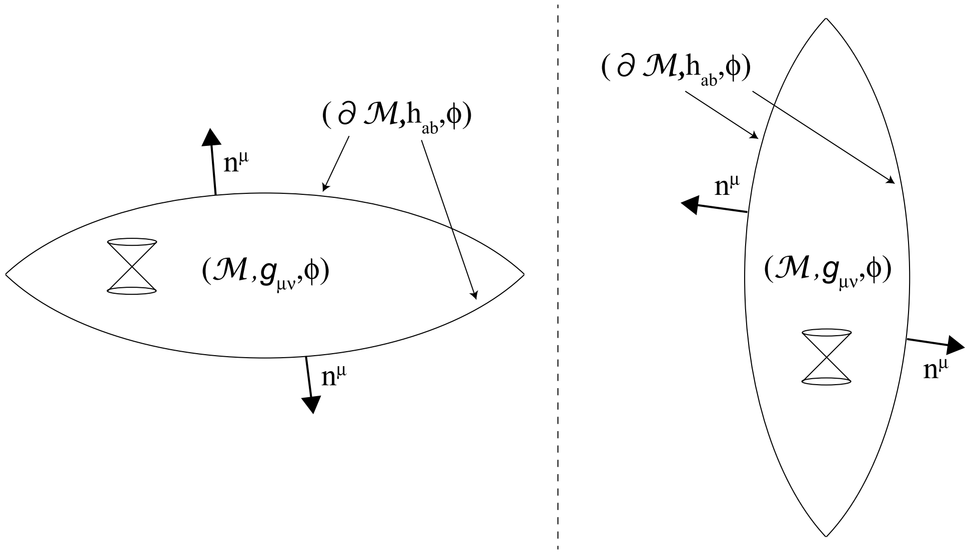

Figure 1. A schematic figure of a spacetime $\mathcal{M}$ with a spacelike boundary (left) or a timelike boundary (right), denoted by $\partial\mathcal{M}$.

2. Action and field equations in scalar–tensor theories

2.1. Preliminaries

Our basic notations follow [3] and [23]. We use the conventions for the curvature tensors such that $[\nabla_\rho, \nabla_\sigma]V^\mu = R^\mu{}_{\nu\rho\sigma}V^\nu$ and $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$. The Minkowski metric has the signature $(-, +, \cdots, +)$ and Greek indices run over all spacetime indices. We adopt the units such that $c = 1$ and κ_n denotes the n-dimensional gravitational constant.

We consider an $n(\geqslant 3)$-dimensional Lorentzian (bulk) spacetime $\mathcal{M}$, of which line element is written as

$$\mathrm{d}s_n^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu. \tag{2.1}$$

Let $\partial\mathcal{M}$ be an $(n-1)$-dimensional non-null hypersurface as a boundary of $\mathcal{M}$, defined by $\Phi(x)$ =constant and let y^a be a set of coordinates on $\partial\mathcal{M}$. Since the location of $\partial\mathcal{M}$ in $\mathcal{M}$ is described by $x^\mu = x^\mu(y)$, the line element on $\partial\mathcal{M}$ is given by

$$\mathrm{d}s_{n-1}^2 = h_{ab}(y)\mathrm{d}y^a\mathrm{d}y^b, \tag{2.2}$$

where

$$h_{ab}(y) := g_{\mu\nu}e^\mu_a e^\nu_b, \qquad e^\mu_a := \frac{\partial x^\mu}{\partial y^a}. \tag{2.3}$$

While $g_{\mu\nu}$ and $g^{\mu\nu}$ are respectively used to raise or lower Greek indices, the induced metric h_{ab} and its inverse h^{ab} are used to raise or lower Latin indices, respectively. For a given vector v_μ, its components on $\partial\mathcal{M}$ in the coordinates y^a are given by $v_a := e^\mu_a v_\mu$. Covariant derivative of $v_a(:= e^\mu_a v_\mu)$ on $\partial\mathcal{M}$ is given by $D_a v_b \equiv e^\mu_a e^\nu_b(\nabla_\mu v_\nu)$.

A unit normal vector n^μ of $\partial\mathcal{M}$ is given by

$$n_\mu := \frac{\varepsilon \nabla_\mu \Phi}{(\varepsilon g^{\rho\sigma}\nabla_\rho\Phi\nabla_\sigma\Phi)^{1/2}}, \tag{2.4}$$

which satisfies $n^\mu n_\mu = \varepsilon$, where $\varepsilon = 1\,(-1)$ corresponds to the case where $\partial\mathcal{M}$ is a timelike (spacelike) hypersurface. (See figure 1.) Because Φ is constant on $\partial\mathcal{M}$ and hence independent of y^a, $n_\mu e^\mu_a = 0$ is satisfied. The Stokes' theorem for a vector field v^μ in $\mathcal{M}$ is expressed as

$$\int_{\mathcal{M}} \mathrm{d}^n x\sqrt{-g}\nabla_\mu v^\mu = \varepsilon\int_{\partial\mathcal{M}} \mathrm{d}^{n-1}x\sqrt{|h|}n_\mu v^\mu. \tag{2.5}$$

A projection tensor defined by $h_{\mu\nu} := g_{\mu\nu} - \varepsilon n_\mu n_\nu$ satisfies $h_{\mu\nu}n^\nu = 0$ and $h_{ab} = h_{\mu\nu}e^\mu_a e^\nu_b$ (and therefore $h_{\mu\nu} = h_{ab}e^a_\mu e^b_\nu$). The extrinsic curvature (or the second fundamental form) $K_{\mu\nu}$ of $\partial\mathcal{M}$ and its trace are defined by

$$K_{\mu\nu} := h^\rho_\mu h^\sigma_\nu \nabla_\rho n_\sigma \left(\equiv \frac{1}{2}\mathcal{L}_n h_{\mu\nu}\right), \tag{2.6}$$

$$K := g^{\mu\nu}K_{\mu\nu} = \nabla_\mu n^\mu. \tag{2.7}$$

If a symmetric tensor $A_{\mu\nu}$ is tangent to $\partial\mathcal{M}$, i.e. $A_{\mu\nu}n^\nu \equiv 0$, it admits a decomposition on $\partial\mathcal{M}$ such that

$$A^{\mu\nu} = A^{ab}e^\mu_a e^\nu_b, \tag{2.8}$$

where $A_{ab}(y) = A_{\mu\nu}(x)e^\mu_a e^\nu_b$ is an $(n-1)$-dimensional tensor on $\partial\mathcal{M}$. Since $K_{\mu\nu}$ is symmetric and tangent to $\partial\mathcal{M}$ as $h_{\mu\nu}$, we can write

$$K^{\mu\nu} = K^{ab}e^\mu_a e^\nu_b \quad\Leftrightarrow\quad K_{ab} = K_{\mu\nu}e^\mu_a e^\nu_b, \tag{2.9}$$

which show $K = g^{\mu\nu}K_{\mu\nu} = h^{ab}K_{ab}$.

2.2. Jordan frame

In this work we deal with a class of scalar–tensor theories in $n(\geqslant 3)$ dimensions characterized by a non-minimally coupled real scalar field ϕ endowed with a self-interaction potential $V(\phi)$. Our system is described in the Jordan frame by the following action:

$$\begin{aligned} I_{\mathrm{J}} = &\int_{\mathcal{M}} \mathrm{d}^n x\sqrt{-g}\left(f(\phi)R - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right) + \int_{\mathcal{M}} \mathrm{d}^n x\sqrt{-g}\mathcal{L}^{(m)}_{\mathcal{M}} \\ &+ 2\varepsilon\int_{\partial\mathcal{M}} \mathrm{d}^{n-1}y\sqrt{|h|}f(\phi)K, \end{aligned} \tag{2.10}$$

where $(\nabla\phi)^2 := g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi)$ and $\sqrt{-g}\mathcal{L}^{(m)}_{\mathcal{M}}$ is the Lagrangian density for matter fields other than ϕ. The last term in equation (2.10) is a boundary term leading a well-defined action principle under Dirichlet boundary conditions, $\delta g_{\mu\nu}|_{\partial\mathcal{M}} = 0 = \delta\phi|_{\partial\mathcal{M}}$. This term will be used to provide an alternative derivation of the junction conditions for non-null hypersurfaces in section 3.5.1. For simplicity, here we do not consider the case where the boundary $\partial\mathcal{M}$ consists of several spacelike and timelike portions.

The action (2.10) provides the following field equations in the Jordan frame:

$$\begin{aligned} &2f(\phi)G_{\mu\nu} + g_{\mu\nu}\left(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right) \\ &\quad - (\nabla_\mu\phi)(\nabla_\nu\phi) - 2\nabla_\mu\nabla_\nu f(\phi) + 2g_{\mu\nu}\Box f(\phi) = T_{\mu\nu}, \end{aligned} \tag{2.11}$$

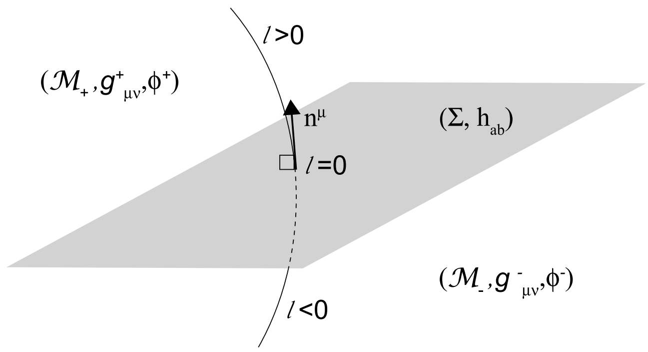

Figure 2. A non-null hypersurface Σ partitions a spacetime into two regions $\mathcal{M}_+$ and $\mathcal{M}_-$.

$$\Box\phi + f'(\phi)R - V'(\phi) = 0, \tag{2.12}$$

where a prime denotes derivative with respect to the argument and the energy-momentum tensor $T_{\mu\nu}$ for other matter fields is defined by

$$T_{\mu\nu} := -2\frac{\partial\mathcal{L}^{(m)}_{\mathcal{M}}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}^{(m)}_{\mathcal{M}}. \tag{2.13}$$

We assume that: (a) the Lagrangian density for matter fields $\sqrt{-g}\mathcal{L}^{(m)}_{\mathcal{M}}$ does not depend on the scalar field ϕ, and (b) the matter fields are minimally coupled to gravity. In consequence, the energy-momentum tensor (2.13) does not contribute to the energy-momentum tensor on the matching hypersurface Σ.

The action with a typical non-minimally coupled scalar field is realized with the following form of $f(\phi)$:

$$f(\phi) = \frac{1}{2\kappa_n} - \frac{1}{2}\xi\phi^2, \tag{2.14}$$

where ξ is the non-minimal coupling parameter. However, the analysis throughout the text is done for *an arbitrary C^1 function* $f(\phi)$, namely $f(\phi)$ and its first derivative are both continuous (and hence finite). In order to simplify the descriptions in the following analysis, we define

$$\begin{aligned} E_{\mu\nu} :=\; & 2f(\phi)G_{\mu\nu} + g_{\mu\nu}\left(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right) \\ & - (\nabla_\mu\phi)(\nabla_\nu\phi) - 2\nabla_\mu\nabla_\nu f(\phi) + 2g_{\mu\nu}\Box f(\phi), \end{aligned} \tag{2.15}$$

$$\Pi := \Box\phi + f'(\phi)R - V'(\phi), \tag{2.16}$$

so that the field equations (2.11) and (2.12) are described as $E_{\mu\nu} = T_{\mu\nu}$ and $\Pi = 0$, respectively.

2.3. Einstein frame

The scalar–tensor theory in the Jordan frame (2.10) is often compared with the following theory:

$$\begin{aligned} I_{\mathrm{E}} = & \int_{\mathcal{M}} \mathrm{d}^n x\sqrt{-\bar{g}}\left(\frac{1}{2\kappa_n}\bar{R} - \frac{1}{2}(\bar{\nabla}\psi)^2 - \bar{V}(\psi)\right) + \int_{\mathcal{M}} \mathrm{d}^n x\sqrt{-\bar{g}}\bar{\mathcal{L}}^{(m)}_{\mathcal{M}} \\ & + \frac{\varepsilon}{\kappa_n}\int_{\partial\mathcal{M}} \mathrm{d}^{n-1}y\sqrt{|\bar{h}|}\bar{K}, \end{aligned} \tag{2.17}$$

which is called the Einstein frame of the theory. As adopted in equation (2.17), we will describe geometric quantities in the Einstein frame with bars.

As in the Jordan frame, under an assumption that $\bar{\mathcal{L}}^{(m)}_{\mathcal{M}}$ does not depend on ψ, the action (2.17) provides the following field equations:

$$\bar{G}_{\mu\nu} - \kappa_n \left\{ (\bar{\nabla}_\mu \psi)(\bar{\nabla}_\nu \psi) - \bar{g}_{\mu\nu} \left(\frac{1}{2} (\bar{\nabla}\psi)^2 + \bar{V}(\psi) \right) \right\} = \kappa_n \bar{T}_{\mu\nu}, \tag{2.18}$$

$$\bar{\Box}\psi = \bar{V}'(\psi), \tag{2.19}$$

where $\bar{T}_{\mu\nu}$ is defined by

$$\bar{T}_{\mu\nu} := -2 \frac{\partial \bar{\mathcal{L}}^{(m)}_{\mathcal{M}}}{\partial \bar{g}^{\mu\nu}} + \bar{g}_{\mu\nu} \bar{\mathcal{L}}^{(m)}_{\mathcal{M}}. \tag{2.20}$$

As in the Jordan frame, we assume (a) the Lagrangian density for matter fields $\sqrt{-\bar{g}}\bar{\mathcal{L}}^{(m)}_{\mathcal{M}}$ does not depend on the scalar field ψ and (b) the matter fields are minimally coupled to gravity, so that the energy-momentum tensor (2.20) does not contribute to the energy-momentum tensor on the matching hypersurface Σ.

2.4. Proper mapping between the Jordan and Einstein frames

By a conformal transformation and a redefinition of the scalar field such that

$$\bar{g}_{\mu\nu} = (2\kappa_n f(\phi))^{2/(n-2)} g_{\mu\nu}, \tag{2.21}$$

$$\psi(\phi) := \pm \int \sqrt{\frac{2(n-1)f'(\phi)^2 + (n-2)f(\phi)}{2(n-2)\kappa_n f(\phi)^2}} \mathrm{d}\phi, \tag{2.22}$$

the action in the Jordan frame (2.10) is mapped to the action (2.17), where

$$\bar{V}(\psi) := (2\kappa_n f(\phi(\psi)))^{-n/(n-2)} V(\phi(\psi)), \tag{2.23}$$

$$\bar{\mathcal{L}}^{(m)}_{\mathcal{M}} := (2\kappa_n f(\phi(\psi)))^{-n/(n-2)} \mathcal{L}^{(m)}_{\mathcal{M}}. \tag{2.24}$$

(See appendix A for details.)

Here it should be emphasized that matter fields other than the scalar field may violate a proper mapping between the Jordan and Einstein frames [24]. In general, under the assumption that $\mathcal{L}^{(m)}_{\mathcal{M}}$ is independent of ϕ, required to give the field equations (2.15) and (2.16) in the Jordan frame, the conformally transformed action (2.17) in the Einstein frame does *not* give the equation of motion (2.19) for ψ. This is because $\bar{\mathcal{L}}^{(m)}_{\mathcal{M}}$ may depend on ψ, as seen in equation (2.24). Then, not only $\bar{T}_{\mu\nu}$ depends on ψ, but also there appear additional terms in the equation of motion (2.19) for ψ. An exception is the case where $\mathcal{L}^{(m)}_{\mathcal{M}}$ is for a conformally invariant matter field such as an electromagnetic field in four dimensions. In such a case, $\sqrt{-g}\mathcal{L}^{(m)}_{\mathcal{M}} = \sqrt{-\bar{g}}\bar{\mathcal{L}}^{(m)}_{\mathcal{M}}$ holds and then the equation of motion (2.19) for ψ is obtained in the Einstein frame.

Also, independent of the extra matter fields, a proper mapping between two frames is violated for the following non-minimal coupling

$$f(\phi) = -\frac{n-2}{8(n-1)}(\phi - \phi_0)^2, \tag{2.25}$$

where ϕ_0 is a constant[5]. With this form of $f(\phi)$, the integrand in equation (2.22) is identically zero. As a result, ψ is constant and there is no inverse transformation $\phi = \phi(\psi)$ even locally.

These observations are summarized in the following lemma, where the assumption (ii) includes the vacuum case, $\mathcal{L}^{(m)}_{\mathcal{M}} = \bar{\mathcal{L}}^{(m)}_{\mathcal{M}} \equiv 0$.

Lemma 1. *Suppose that*

(i) $f(\phi)$ is a C^1 function and not in the exceptional form (2.25), and
(ii) $\sqrt{-g}\mathcal{L}^{(m)}_{\mathcal{M}} = \sqrt{-\bar{g}}\bar{\mathcal{L}}^{(m)}_{\mathcal{M}}$ holds.

Then, there is a proper mapping between the Einstein frame (2.17) and the Jordan frame (2.10) by a conformal transformation (2.21) and redefinitions (2.22)–(2.24).

3. Junction conditions for non-null hypersurfaces

3.1. Setup

We consider a non-null hypersurface Σ which partitions a spacetime into two regions $\mathcal{M}_+$ and $\mathcal{M}_-$. (See figure 2.) Hence Σ is a part of both $\partial\mathcal{M}_+$ and $\partial\mathcal{M}_-$. In $\mathcal{M}_+$, the metric and the scalar field are $g^+_{\mu\nu}$ and ϕ^+, respectively, which are functions of the coordinates x^μ_+. In $\mathcal{M}_-$, the metric and the scalar field are $g^-_{\mu\nu}$ and ϕ^-, respectively, which are expressed in coordinates x^μ_-. We set the same coordinates y^a on both sides of Σ, and we choose n^μ, the unit normal to Σ, to point from $\mathcal{M}_-$ to $\mathcal{M}_+$.

Now we assume that continuous *canonical coordinates* x^μ, which are different from $x^\mu_\pm$, can be introduced in an open region containing both sides of Σ. Actually, the metric and scalar field in $\mathcal{M}_\pm$ are not described as $g^\pm_{\mu\nu}$ and $\phi^\pm$ in terms of x^μ. Nevertheless, hereafter in this section, we keep using the same expressions in the canonical coordinates for simplicity as long as there is no risk of confusion.

Here we use distributions[6] to derive the junction conditions. The hypersurface Σ is considered to be pierced by a congruence of geodesics that intersect it orthogonally. The proper distance (or proper time) along the geodesics is denoted by l, and the parametrization is adjusted so that $l = 0$ when the geodesics cross Σ. Our convention is that l is negative in $\mathcal{M}_-$ and positive in $\mathcal{M}_+$. Now we introduce the Heaviside distribution $\Theta(l)$, equal to $+1$ if $l > 0$, 0 if $l < 0$, and indeterminate if $l = 0$. The distribution $\Theta(l)$ satisfies

$$\Theta(l)^2 = \Theta(l), \quad \Theta(l)\Theta(-l) = 0, \quad \frac{\mathrm{d}\Theta}{\mathrm{d}l} = \delta(l), \tag{3.1}$$

where $\delta(l)$ is the Dirac distribution, which verifies $\delta(l) = \delta(-l)$. It is important to remark that $\Theta(l)\delta(l)$ is not defined as a distribution. The metric $g_{\mu\nu}$ and the scalar field ϕ are expressed in the canonical coordinates x^μ as

$$g_{\mu\nu} = \Theta(l)g^+_{\mu\nu} + \Theta(-l)g^-_{\mu\nu}, \tag{3.2}$$

[5] The coupling (2.25) with $\phi_0 = 0$ makes the sector $\sqrt{-g}\{f(\phi)R - (\nabla\phi)^2/2\}$ in the action (2.10) conformal invariant.

[6] We follow the classical textbooks [25] and [26] on distributions. Due to its nonlinear nature, there are technical subtleties and problems in the use of distributions in general relativity. (See [27] for a review.)

$$\phi = \Theta(l)\phi^{+} + \Theta(-l)\phi^{-}, \tag{3.3}$$

which are distribution-valued tensors.

Since the following equations hold along the geodesics,

$$\varepsilon \mathrm{d}l^2 = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}, \qquad \varepsilon\frac{\partial l}{\partial x^{\mu}}\mathrm{d}l = g_{\mu\nu}\mathrm{d}x^{\nu}, \tag{3.4}$$

where $\varepsilon = 1\ (-1)$ if Σ is timelike (spacelike), a displacement away from Σ along one of the geodesics is described by $\mathrm{d}x^{\mu} = n^{\mu}\mathrm{d}l$, where n_{μ} is given by

$$n_{\mu} = \varepsilon\partial_{\mu}l \tag{3.5}$$

and $n^{\mu}n_{\mu} = \varepsilon$ holds. The factor ε in equation (3.5) is in order for n^{μ} to point from $\mathcal{M}_{-}$ to $\mathcal{M}_{+}$. In the canonical coordinates x^{μ}, the following relations hold:

$$[n^{\mu}] = [e^{\mu}_{a}] = 0, \tag{3.6}$$

where e^{μ}_{a} is defined by equation (2.3). Here $[X]$ is defined by

$$[X] := X^{+} - X^{-}, \tag{3.7}$$

where $X^{\pm}$ are X's evaluated either on the $+$ or $-$ side of Σ. The first of equation (3.6) follows from the relation $\mathrm{d}x^{\mu} = n^{\mu}\mathrm{d}l$ and the continuity of both l and x^{μ} across Σ, while the second follows from the fact that the coordinates y^{a} are the same on both sides of Σ.

3.2. Continuity of $g_{\mu\nu}$ and ϕ: first junction conditions

The metric $g_{\mu\nu}$ and the scalar field ϕ are expressed as equations (3.2) and (3.3) in canonical coordinates x^{μ}, respectively. Differentiating them, we obtain

$$\partial_{\rho}g_{\mu\nu} = \Theta(l)\partial_{\rho}g^{+}_{\mu\nu} + \Theta(-l)\partial_{\rho}g^{-}_{\mu\nu} + \varepsilon\delta(l)[g_{\mu\nu}]n_{\rho}, \tag{3.8}$$

$$\partial_{\mu}\phi = \Theta(l)\partial_{\mu}\phi^{+} + \Theta(-l)\partial_{\mu}\phi^{-} + \varepsilon\delta(l)[\phi]n_{\rho}. \tag{3.9}$$

Thus, to removed the last terms in the right-hand sides which generate terms proportional to $\Theta(l)\delta(l)$ in the Einstein equation (2.15) and the equation of motion (2.16) for ϕ, we impose continuity of the metric $g_{\mu\nu}$ and the scalar field ϕ across Σ:

$$[g_{\mu\nu}] = [\phi] = 0. \tag{3.10}$$

This set of conditions is dubbed as the first junction conditions. By equation (2.3), $[g_{\mu\nu}] = 0$ is equivalent to $[h_{ab}] = 0$, which means that the induced metric on Σ is the same on both sides of Σ. The difference of the numbers of equations $[g_{\mu\nu}] = 0$ and $[h_{ab}] = 0$ is n, which coincides with the number of the coordinate conditions $[x^{\mu}] = 0$.

Hereafter, we impose the conditions (3.10) and the derivatives (3.8) and (3.9) then become

$$\partial_{\rho}g_{\mu\nu} = \Theta(l)\partial_{\rho}g^{+}_{\mu\nu} + \Theta(-l)\partial_{\rho}g^{-}_{\mu\nu}, \tag{3.11}$$

$$\partial_{\mu}\phi = \Theta(l)\partial_{\mu}\phi^{+} + \Theta(-l)\partial_{\mu}\phi^{-}. \tag{3.12}$$

Since the metric and the scalar field are continuous across Σ in the canonical coordinates x^{μ}, the tangential derivatives of the metric and scalar field are also continuous. Thus, if $\partial_{\rho}g_{\mu\nu}$ and $\partial_{\rho}\phi$ are to be discontinuous, the discontinuity must be directed along the normal vector n^{μ}. Therefore, there must exist a tensor field $\omega_{\mu\nu}$ and a scalar field M such that

$$[\partial_{\mu}g_{\alpha\beta}] = n_{\mu}\omega_{\alpha\beta}, \qquad [\partial_{\mu}\phi] = n_{\mu}M. \tag{3.13}$$

Namely, $\omega_{\mu\nu}$ and M are defined by

$$\omega_{\alpha\beta} := \varepsilon n^{\mu}[\partial_{\mu} g_{\alpha\beta}], \qquad M := \varepsilon n^{\mu}[\partial_{\mu}\phi], \tag{3.14}$$

respectively.

3.3. Discontinuity of geometric and physical quantities

From equations (3.2) and (3.11), we obtain

$$\Gamma^{\mu}_{\nu\rho} = \Theta(l)\Gamma^{+\mu}{}_{\nu\rho} + \Theta(-l)\Gamma^{-\mu}{}_{\nu\rho}, \tag{3.15}$$

where $\Gamma^{\pm\mu}{}_{\nu\rho}$ is the Christoffel symbols constructed from $g^{\pm}_{\mu\nu}$. Then, a straightforward calculation with equations (3.5) and (3.13) reveals

$$\partial_{\sigma}\Gamma^{\mu}_{\nu\rho} = \Theta(l)\partial_{\sigma}\Gamma^{+\mu}{}_{\nu\rho} + \Theta(-l)\partial_{\sigma}\Gamma^{-\mu}{}_{\nu\rho} + \varepsilon\delta(l)[\Gamma^{\mu}_{\nu\rho}]n_{\sigma}, \tag{3.16}$$

where $[\Gamma^{\rho}_{\sigma\mu}]$ is given by

$$[\Gamma^{\rho}_{\sigma\mu}] = \frac{1}{2}(\omega^{\rho}{}_{\sigma}n_{\mu} + \omega^{\rho}{}_{\mu}n_{\sigma} - \omega_{\sigma\mu}n^{\rho}). \tag{3.17}$$

By equations (3.6) and (3.17), we obtain

$$\begin{aligned}[\nabla_{\mu}n_{\nu}] &= -[\Gamma^{\sigma}_{\mu\nu}]n_{\sigma} \\ &= \frac{1}{2}(\varepsilon\omega_{\mu\nu} - \omega_{\sigma\mu}n_{\nu}n^{\sigma} - \omega_{\sigma\nu}n_{\mu}n^{\sigma}),\end{aligned} \tag{3.18}$$

and hence the jump of the extrinsic curvature (2.6) and its trace are given by

$$\begin{aligned}[K_{\mu\nu}] &= h_{\mu}^{\ \rho}h_{\nu}^{\ \sigma}[\nabla_{\rho}n_{\sigma}] \\ &= \frac{1}{2}(\varepsilon\omega_{\mu\nu} - \omega_{\mu\sigma}n^{\sigma}n_{\nu} - \omega_{\nu\rho}n^{\rho}n_{\mu} + \varepsilon\omega_{\rho\sigma}n^{\rho}n^{\sigma}n_{\mu}n_{\nu}),\end{aligned} \tag{3.19}$$

$$[K] = g^{\mu\nu}[K_{\mu\nu}] = \frac{1}{2}(\varepsilon\omega_{\mu}^{\ \mu} - \omega_{\mu\nu}n^{\mu}n^{\nu}). \tag{3.20}$$

By equation (2.9), we obtain

$$[K_{ab}] = [K_{\mu\nu}]e^{\mu}_{a}e^{\nu}_{b} = \frac{1}{2}\varepsilon\omega_{\mu\nu}e^{\mu}_{a}e^{\nu}_{b}. \tag{3.21}$$

On the other hand, using equation (3.17), we obtain the Riemann tensor as

$$R^{\rho}{}_{\sigma\mu\nu} = \Theta(l)R^{+\rho}{}_{\sigma\mu\nu} + \Theta(-l)R^{-\rho}{}_{\sigma\mu\nu} + \delta(l)\tilde{R}^{\rho}{}_{\sigma\mu\nu}, \tag{3.22}$$

where the δ-function part of the Riemann tensor is given by

$$\begin{aligned}\tilde{R}^{\rho}{}_{\sigma\mu\nu} &:= \varepsilon([\Gamma^{\rho}_{\sigma\nu}]n_{\mu} - [\Gamma^{\rho}_{\sigma\mu}]n_{\nu}) \\ &= \frac{1}{2}\varepsilon(\omega^{\rho}{}_{\nu}n_{\sigma}n_{\mu} - \omega^{\rho}{}_{\mu}n_{\sigma}n_{\nu} - \omega_{\sigma\nu}n^{\rho}n_{\mu} + \omega_{\sigma\mu}n^{\rho}n_{\nu}).\end{aligned} \tag{3.23}$$

Equation (3.23) shows that the δ-function parts of the Ricci tensor and the Ricci scalar are expressed as

$$\begin{aligned}\tilde{R}_{\sigma\nu} = \tilde{R}^{\mu}{}_{\sigma\mu\nu} &= \frac{1}{2}\varepsilon(\omega_{\nu\mu}n^{\mu}n_{\sigma} + \omega_{\sigma\mu}n^{\mu}n_{\nu} - \omega_{\mu}^{\ \mu}n_{\sigma}n_{\nu} - \varepsilon\omega_{\sigma\nu}) \\ &= -\varepsilon[K_{\sigma\nu}] - [K]n_{\sigma}n_{\nu},\end{aligned} \tag{3.24}$$

$$\tilde{R} = g^{\sigma\nu}\tilde{R}_{\sigma\nu} = -2\varepsilon[K]. \tag{3.25}$$

From equations (3.24) and (3.25), the δ-function part of the Einstein tensor $\tilde{G}_{\mu\nu}$ is given as

$$\tilde{G}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tilde{R} = -\varepsilon\left([K_{\mu\nu}] - h_{\mu\nu}[K]\right). \tag{3.26}$$

A C^1 regular matching of two spacetimes $\mathcal{M}_+$ and $\mathcal{M}_-$ at Σ is defined by $[g_{\alpha\beta}] = [\partial_\mu g_{\alpha\beta}] = 0$. The following lemma provides several different expressions of a C^1 regular matching, among which the condition (i) means that the full Riemann tensor is certainly non-singular at Σ.

Lemma 2. *If* $[g_{\alpha\beta}] = 0$ *holds, the following six conditions are equivalent: (i)* $\tilde{R}^\rho{}_{\sigma\mu\nu} = 0$, *(ii)* $[K_{\mu\nu}] = 0$, *(iii)* $[K_{ab}] = 0$, *(iv)* $\omega_{\mu\nu} = 0$, *(v)* $[\partial_\mu g_{\alpha\beta}] = 0$, *and (vi)* $[\Gamma^\rho_{\sigma\mu}] = 0$.

Proof. Equation (3.21) shows that the conditions (ii), (iii) and (iv) are equivalent. By equations (3.13) and (3.14), the conditions (iv) and (v) are equivalent. Next we show that the conditions (i) and (ii) are equivalent. By equation (3.19), $[K_{\mu\nu}] = 0$ implies

$$\omega_{\mu\nu} = \varepsilon\omega_{\mu\sigma}n^\sigma n_\nu + \varepsilon\omega_{\nu\rho}n^\rho n_\mu - \omega_{\rho\sigma}n^\rho n^\sigma n_\mu n_\nu. \tag{3.27}$$

Substituting this into equation (3.23), we obtain $\tilde{R}^\rho{}_{\sigma\mu\nu} = 0$. On the other hand, if $\tilde{R}^\rho{}_{\sigma\mu\nu} = 0$ holds, we have $\tilde{R}_{\sigma\nu} = \tilde{R} = 0$ and then equations (3.24) and (3.25) show $[K_{\mu\nu}] = 0$. Since we have shown that the conditions (i)–(v) are equivalent, we complete the proof by showing that the conditions (iv) and (vi) are equivalent. The condition (iv) implies the condition (vi) by equation (3.17). The condition (vi) implies the condition (i) by equation (3.23), which is equivalent to the condition (iv). ■

In the following subsections, we will derive the junction conditions from the equation of motion (2.12) for ϕ and the Einstein equation (2.11). For this purpose, differentiating equation (3.12), we obtain

$$\partial_\mu\partial_\nu\phi = \Theta(l)\partial_\mu\partial_\nu\phi^+ + \Theta(-l)\partial_\mu\partial_\nu\phi^- + \varepsilon\delta(l)Mn_\mu n_\nu, \tag{3.28}$$

where we used equations (3.5) and (3.13). From the above expression, we obtain

$$\nabla_\mu\nabla_\nu\phi = \Theta(l)\nabla_\mu\nabla_\nu\phi^+ + \Theta(-l)\nabla_\mu\nabla_\nu\phi^- + \varepsilon\delta(l)Mn_\mu n_\nu, \tag{3.29}$$

$$\Box\phi = \Theta(l)\Box\phi^+ + \Theta(-l)\Box\phi^- + \delta(l)M. \tag{3.30}$$

Finally, using the following expression;

$$\nabla_\mu\nabla_\nu f(\phi) = f'(\phi)\nabla_\mu\nabla_\nu\phi + f''(\phi)(\nabla_\mu\phi)(\nabla_\nu\phi), \tag{3.31}$$

we obtain

$$\nabla_\mu\nabla_\nu f(\phi) = \Theta(l)\nabla_\mu\nabla_\nu f(\phi^+) + \Theta(-l)\nabla_\mu\nabla_\nu f(\phi^-) + \varepsilon f'(\phi)\delta(l)Mn_\mu n_\nu, \tag{3.32}$$

$$\Box f(\phi) = \Theta(l)\Box f(\phi^+) + \Theta(-l)\Box f(\phi^-) + f'(\phi)\delta(l)M. \tag{3.33}$$

3.4. Second junction conditions

3.4.1. Equation of motion for a scalar field. Here we derive the junction condition from the equation of motion (2.12), namely $\Pi = 0$, where Π is defined by equation (2.16). Using equations (3.3), (3.25) and (3.29), we write down Π as

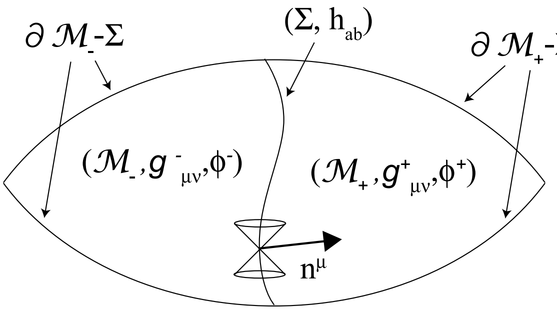

Figure 3. A schematic figure of a spacetime consisting of two portions $\mathcal{M}_+$ and $\mathcal{M}_-$ which are separated by a hypersurface Σ, of which sides are denoted by $\Sigma_\pm$. This figure shows the case of $\epsilon_+ = \epsilon_- = -1$ $((\partial\mathcal{M}_- - \Sigma_-) \cup (\partial\mathcal{M}_+ - \Sigma_+)$ is spacelike) and $\varepsilon = 1$ (Σ is timelike), but there are other possible configurations.

$$\Pi = \Theta(l)\Pi^+ + \Theta(-l)\Pi^- + \delta(l)\tilde{\Pi}, \tag{3.34}$$

where the δ-function part $\tilde{\Pi}$ is given by

$$\tilde{\Pi} := M - 2\varepsilon f'(\phi)[K]. \tag{3.35}$$

The equation of motion (2.16) on Σ gives $\tilde{\Pi} = 0$, namely

$$M = 2\varepsilon f'(\phi)[K]. \tag{3.36}$$

We shall refer to this condition as the junction condition from the equation of motion for a scalar field. This junction condition is a constraint between the metric and scalar field on Σ. For a minimally coupled scalar field, namely $f(\phi) = 1/(2\kappa_n)$, this condition is simply $M = 0$, which means continuity of $n^\mu\partial_\mu\phi$ at Σ.

3.4.2. Einstein equations. Next let us derive the junction conditions from the Einstein equation (2.11), namely $E_{\mu\nu} = T_{\mu\nu}$, where $E_{\mu\nu}$ is defined by equation (2.15). Using equations (3.26), (3.32) and (3.33), we write down $E_{\mu\nu}$ as

$$E_{\mu\nu} = \Theta(l)E^+_{\mu\nu} + \Theta(-l)E^-_{\mu\nu} + \delta(l)\tilde{E}_{\mu\nu}, \tag{3.37}$$

where the δ-function part $\tilde{E}_{\mu\nu}$ is given by

$$\tilde{E}_{\mu\nu} = -2\varepsilon f(\phi)\left([K_{\mu\nu}] - h_{\mu\nu}[K]\right) + 2Mf'(\phi)h_{\mu\nu}. \tag{3.38}$$

Since the bulk matter fields do not contribute to the energy-momentum tensor on Σ under the assumptions (a) and (b) in section 2.2, we can write $T_{\mu\nu}$ as

$$T_{\mu\nu} = \Theta(l)T^+_{\mu\nu} + \Theta(-l)T^-_{\mu\nu}. \tag{3.39}$$

By equations (3.37) and (3.39), the Einstein equations $E_{\mu\nu} = T_{\mu\nu}$ on Σ give $\tilde{E}_{\mu\nu} = 0$, namely

$$\varepsilon f(\phi)\left([K_{\mu\nu}] - h_{\mu\nu}[K]\right) = Mf'(\phi)h_{\mu\nu}. \tag{3.40}$$

We shall refer to equation (3.40) as the junction conditions from the Einstein equations, which are other constraints between the metric and scalar field on Σ. Under the conditions (3.40), there is no matter field on Σ other than ϕ. We shall describe this situation as 'Σ is vacuum' throughout this paper. For a minimally coupled scalar field, namely $f(\phi) = 1/(2\kappa_n)$, equation (3.40) reduces to $[K_{\mu\nu}] = h_{\mu\nu}[K]$, which is the same as the general relativistic case.

For embedding configurations of Σ with $\tilde{E}_{\mu\nu} \neq 0$, the Einstein equations require an additional matter field on Σ for consistency. Then, Σ is no more vacuum and the Einstein equations on Σ become

$$\tilde{E}_{\mu\nu} = t_{\mu\nu}, \tag{3.41}$$

where $t_{\mu\nu}$ is the energy-momentum tensor of the matter field on Σ.

We have now derived the junction conditions (3.36) and (3.41) in the scalar–tensor theories (2.10) in the Jordan frame, namely

$$-2\varepsilon f(\phi)\left([K_{\mu\nu}] - h_{\mu\nu}[K]\right) + 2Mf'(\phi)h_{\mu\nu} = t_{\mu\nu}, \tag{3.42}$$

$$M = 2\varepsilon f'(\phi)[K]. \tag{3.43}$$

Since $h_{\mu\nu}$, $K_{\mu\nu}$, and $t_{\mu\nu}$ are symmetric and tangent to Σ, we can write equation (3.42) in terms of intrinsic coordinates y^a on Σ such that

$$-2\varepsilon f(\phi)\left([K_{ab}] - h_{ab}[K]\right) + 2Mf'(\phi)h_{ab} = t_{ab}, \tag{3.44}$$

where h_{ab} and K_{ab} are defined by equations (2.3) and (2.9), respectively, and $t_{ab} := t_{\mu\nu}e^{\mu}_{a}e^{\nu}_{b}$. The trace of equations (3.42) with (3.43) gives

$$2\varepsilon\left\{(n-2)f(\phi) + 2(n-1)f'(\phi)^2\right\}[K] = t. \tag{3.45}$$

Thus, if $(n-2)f(\phi) + 2(n-1)f'(\phi)^2 \neq 0$ holds at Σ, we can rewrite equations (3.42) and (3.43) in the following alternative forms:

$$2\varepsilon f(\phi)[K_{\mu\nu}] = -t_{\mu\nu} + \frac{f(\phi) + 2f'(\phi)^2}{(n-2)f(\phi) + 2(n-1)f'(\phi)^2}h_{\mu\nu}t, \tag{3.46}$$

$$M = \frac{f'(\phi)}{(n-2)f(\phi) + 2(n-1)f'(\phi)^2}t. \tag{3.47}$$

3.5. Some notes on junction conditions

3.5.1. Derivation by the variational principle. Actually, the junction conditions (3.42) and (3.43) can be derived also by the variational principle. Now the spacetime consists of two parts $\mathcal{M}_+$ and $\mathcal{M}_-$ separated by a non-null hypersurface Σ such as figure 3. In such a spacetime, the action is given by

$$\begin{aligned}
I_{\rm J} = &\int_{\mathcal{M}_+} \mathrm{d}^n x_+ \sqrt{-g^+}\left(f(\phi^+)R^+ - \frac{1}{2}(\nabla\phi^+)^2 - V(\phi^+) + \mathcal{L}^{(m)}_{\mathcal{M}_+}\right)\\
&+ \int_{\mathcal{M}_-} \mathrm{d}^n x_- \sqrt{-g^-}\left(f(\phi^-)R^- - \frac{1}{2}(\nabla\phi^-)^2 - V(\phi^-) + \mathcal{L}^{(m)}_{\mathcal{M}_-}\right)\\
&+ 2\epsilon_+ \int_{\partial\mathcal{M}_+ - \Sigma_+} \mathrm{d}^{n-1}z_+ \sqrt{|\zeta^+|}f(\phi^+)K^+ + 2\epsilon_- \int_{\partial\mathcal{M}_- - \Sigma_-} \mathrm{d}^{n-1}z_- \sqrt{|\zeta^-|}f(\phi^-)K^-\\
&+ 2\varepsilon \int_{\Sigma_+} \mathrm{d}^{n-1}y\sqrt{|h|}f(\phi)K^+ + 2\varepsilon \int_{\Sigma_-} \mathrm{d}^{n-1}y\sqrt{|h|}f(\phi)K^- + \int_{\Sigma} \mathrm{d}^{n-1}y\sqrt{|h|}\mathcal{L}^{(m)}_{\Sigma},
\end{aligned} \tag{3.48}$$

where $\Sigma_{+(-)}$ denotes a side of Σ in $\mathcal{M}_{+(-)}$. (See appendix B for details.) ϵ_+, ϵ_-, and ε independently take their values ± 1 and $\phi^{\pm}|_{\Sigma} = \phi$. Here $\sqrt{|h|}\mathcal{L}^{(m)}_{\Sigma}$ is the Lagrangian density for the matter field other than ϕ on Σ and we used $z^i_\pm$ and $\zeta^\pm_{ij}$ for the coordinates and induced metric on the boundary $\partial\mathcal{M}_\pm - \Sigma_\pm$, respectively.

Under the assumptions that $\mathcal{L}^{(m)}_{\mathcal{M}_\pm}$ and $\mathcal{L}^{(m)}_\Sigma$ do not depend on $\phi^\pm$ and ϕ, respectively, variation of the above action, with the boundary condition $\delta g^\pm_{\mu\nu}|_{\partial\mathcal{M}_\pm-\Sigma_\pm} = 0 = \delta\phi^\pm|_{\partial\mathcal{M}_\pm-\Sigma_\pm}$, provides the Einstein equation (2.11) and the equation of motion for the scalar field (2.12) in the bulk spacetimes $\mathcal{M}_+$ and $\mathcal{M}_-$, as well as, the junction conditions (3.42) and (3.43) on Σ, where $t_{\mu\nu} = t_{ab}e^a_\mu e^b_\mu$ is given by the energy-momentum tensor t_{ab} for other matter fields on Σ defined by

$$t_{ab} := -2\frac{\partial\mathcal{L}^{(m)}_\Sigma}{\partial h^{ab}} + h_{ab}\mathcal{L}^{(m)}_\Sigma. \tag{3.49}$$

The details of derivation are presented in appendix B.

3.5.2. Comments on the matter field on Σ. Here we should comment on the energy-momentum tensor t_{ab} on Σ. We have seen in equation (3.49) that t_{ab} is obtained from its Lagrangian density in the variational approach. In such a case, $D^b t_{ab} = 0$ holds and hence the energy-momentum conservation is satisfied on Σ under the assumptions in the following lemma[7]. (See appendix E in [23].)

Lemma 3. *Let $\sqrt{|h|}\mathcal{L}^{(m)}_\Sigma$ be a matter Lagrangian density for a matter field Ψ (not necessary to be a scalar field) on a non-null hypersurface Σ. If the bulk action does not contain Ψ, then the energy-momentum tensor t_{ab} defined by equation (3.49) satisfies $D^a t_{ab} = 0$.*

Proof. The matter action is given by

$$I^{(m)}_\Sigma := \int_\Sigma \mathrm{d}^{n-1}y\sqrt{|h|}\mathcal{L}^{(m)}_\Sigma. \tag{3.50}$$

The variation of the action (3.50) on a non-null hypersurface on Σ results in the following form:

$$\delta I^{(m)}_\Sigma = \int_\Sigma \mathrm{d}^{n-1}y\sqrt{|h|}\left(-\frac{1}{2}t_{ab}\delta h^{ab} + \tilde{\mathcal{E}}_{(\Psi)}\delta\Psi\right) + \int_{\partial\Sigma}\mathrm{d}^{n-2}z\sqrt{|\tilde{h}|}\left(\tilde{\mathcal{F}}_{ab}\delta h^{ab} + \tilde{\mathcal{F}}_{(\Psi)}\delta\Psi\right), \tag{3.51}$$

where z^i and $|\tilde{h}|$ are the coordinates and the determinant of the induced metric at the boundary of Σ, respectively. By assumptions, variation of the bulk action does not generate any term proportional to $\delta\Psi$ on Σ. Thus, the action principle on Σ with the boundary conditions $\delta h^{ab} = \delta\Psi = 0$ at $\partial\Sigma$ gives $\tilde{\mathcal{E}}_{(\Psi)} = 0$ as an equation of motion for Ψ on Σ.

Using the equation of motion $\tilde{\mathcal{E}}_{(\Psi)} = 0$ and the boundary conditions $\delta h^{ab} = \delta\Psi = 0$ at $\partial\Sigma$, we can rewrite the variation (3.51) as

$$\delta I^{(m)}_\Sigma = -\frac{1}{2}\int_\Sigma \mathrm{d}^{n-1}y\sqrt{|h|}t_{ab}\delta h^{ab}. \tag{3.52}$$

Now we use the fact that the action is diffeomorphism invariant on Σ, namely the coordinate invariant, and therefore $\delta I^{(m)}_\Sigma = 0$ holds for such variations. If the diffeomorphism is

[7] In contrast, the energy-momentum tensor introduced in (3.41) has no such a requirement.

generated by an infinitesimal vector field w^a on Σ, we have $\delta h^{ab} = \mathcal{L}_w h^{ab} = 2D^{(a}w^{b)}$, where $\mathcal{L}_w$ is the Lie derivative along w^a. Then, from equation (3.52), $\delta I_\Sigma^{(m)} = 0$ implies

$$\begin{aligned}
0 &= -\frac{1}{2}\int_\Sigma \mathrm{d}^{n-1}y\sqrt{|h|}t_{ab}D^{(a}w^{b)} = -\int_\Sigma \mathrm{d}^{n-1}y\sqrt{|h|}t_{ab}D^a w^b \\
&= -\int_\Sigma \mathrm{d}^{n-1}y\sqrt{|h|}\left(D^a(t_{ab}w^b) - (D^a t_{ab})w^b\right) \\
&= -\varepsilon\int_{\partial\Sigma} \mathrm{d}^{n-2}z\sqrt{|\tilde{h}|}n^a t_{ab}w^b + \int_\Sigma \mathrm{d}^{n-1}y\sqrt{|h|}(D^a t_{ab})w^b \\
&= \int_\Sigma \mathrm{d}^{n-1}y\sqrt{|h|}(D^a t_{ab})w^b,
\end{aligned} \tag{3.53}$$

where we used the Stokes' theorem (2.5) and the boundary condition $w^a = 0$ at $\partial\Sigma$. Since the above equation is satisfied for an arbitrary generator w^a, $D^a t_{ab} = 0$ is concluded. ■

The junction conditions from the Einstein equation (3.41) can be written as $\tilde{E}_{ab} = t_{ab}$, where

$$\begin{aligned}
\tilde{E}_{ab} :&= \tilde{E}_{\mu\nu}e^\mu_a e^\nu_b \\
&= -2\varepsilon f(\phi)\left([K_{ab}] - h_{ab}[K]\right) + 2Mf'(\phi)h_{ab}.
\end{aligned} \tag{3.54}$$

Divergence of $\tilde{E}_{ab}$ is written as

$$D^a\tilde{E}_{ab} = -2\varepsilon f(\phi)[R_{\nu\sigma}]e^\nu{}_b n^\sigma - 2\varepsilon f'(\phi)(D^a\phi)\left([K_{ab}] - h_{ab}[K]\right) + 2D_b(Mf'(\phi)), \tag{3.55}$$

where we used the Codazzi equation:

$$R_{\mu\nu\rho\sigma}e^\mu{}_a e^\nu{}_b e^\rho{}_c n^\sigma = D_a K_{bc} - D_b K_{ac} \;\Rightarrow\; R_{\nu\sigma}e^\nu{}_b n^\sigma = D^c K_{bc} - D_b K. \tag{3.56}$$

We note that, if $[\partial_\rho g_{\mu\nu}]$ is non-vanishing at Σ, we have $D^a\tilde{E}_{ab} \neq 0$ in general (even in general relativity). Therefore, by lemma 3, the junction conditions $\tilde{E}_{ab} = t_{ab}$ require (i) an embedding configuration satisfying $D^a\tilde{E}_{ab} \equiv 0$ or (ii) violation of the assumption in lemma 3, which means that $\mathcal{L}_\Sigma^{(m)}$ depends on ϕ.

While to achieve the case (i) is rather difficult in the Jordan frame, there is a simple example of such configurations of Σ in the Einstein frame. In the Einstein frame, where $f(\phi) = 1/2\kappa_n$ holds, equation (3.55) reduces to

$$D^a\tilde{E}_{ab} = -\frac{\varepsilon}{\kappa_n}[R_{\nu\sigma}]e^\nu{}_b n^\sigma \tag{3.57}$$

and hence $[R_{\nu\sigma}]e^\nu_b n^\sigma = 0$ is required for $D^a\tilde{E}_{ab} = D^a t_{ab} = 0$. This condition is accomplished for any non-null Σ embedded in an Einstein space because $R_{\mu\nu} \propto g_{\mu\nu}$ implies $[R_{\nu\sigma}]e^\nu_b n^\sigma \propto [g_{\nu\sigma}]e^\nu_b n^\sigma = 0$. The condition $[R_{\nu\sigma}]e^\nu_b n^\sigma = 0$ is also satisfied for the following spacetime:

$$\mathrm{d}s^2 = -F(r)\mathrm{d}t^2 + F(r)^{-1}\mathrm{d}r^2 + r^2\gamma_{ij}\mathrm{d}z^i\mathrm{d}z^j, \tag{3.58}$$

if the shell is described by $t = t(\tau)$ and $r = r(\tau)$, where τ is a parameter and $\gamma_{ij}\mathrm{d}z^i\mathrm{d}z^j$ is the line element on an $(n-2)$-dimensional Einstein space.

Next let us consider the case (ii), namely the case where $\mathcal{L}_\Sigma^{(m)}$ depends on $\phi(= \phi^\pm|_\Sigma)$. As an example, we consider $\mathcal{L}_\Sigma^{(\phi)} (\in \mathcal{L}_\Sigma^{(m)})$ for ϕ on Σ as in the bulk:

$$\mathcal{L}_\Sigma^{(\phi)} := f(\phi)\mathcal{R} - \frac{1}{2}(D\phi)^2 - V(\phi), \tag{3.59}$$

where $(D\phi)^2 := h^{ab}(D_a\phi)(D_b\phi)$ and $\mathcal{R}$ is the Ricci scalar constructed from h_{ab}. Then, the equation of motion for ϕ on Σ becomes

$$D_aD^a\phi + f'(\phi)\mathcal{R} - V'(\phi) = -\tilde{\Pi}, \tag{3.60}$$

where $\tilde{\Pi}$ is defined by equation (3.35). In this case, the energy-momentum tensor $t^{(\phi)}_{ab}$ for ϕ on Σ, defined by

$$t^{(\phi)}_{ab} := -2\frac{\partial \mathcal{L}^{(\phi)}_{\Sigma}}{\partial h^{ab}} + h_{ab}\mathcal{L}^{(\phi)}_{\Sigma}, \tag{3.61}$$

satisfies $D^a t^{(\phi)}_{ab} \neq 0$ if $\tilde{\Pi} \neq 0$. However in general, it is highly nontrivial whether there exists a configuration of Σ with this t_{ab}. There is even a possibility that the junction conditions $\tilde{E}_{ab} = t_{ab}$ and the equation of motion (3.60) do not allow any solution.

In summary, when the energy-momentum tensor of a matter field on Σ is assumed to come from a Lagrangian density, the junction conditions (3.41) in the Jordan frame severely constrain the configuration of Σ with non-vanishing $[\partial_\rho g_{\mu\nu}]$.

3.6. Conditions for C^1 matching and vacuum Σ

3.6.1. Jordan frame. Now let us study the conditions for a C^1 matching and also for vacuum Σ. The following proposition shows that $M = t_{ab} = 0$ is a necessary and sufficient condition for a C^1 matching at Σ in the Jordan frame if $f(\phi) \neq 0$ holds there.

Proposition 1 (J-regularity at non-null Σ). *Suppose in the Jordan frame that*

(i) $f(\phi)$ is a C^1 function,
(ii) $[g_{\mu\nu}] = [\phi] = 0$ holds at a non-null hypersurface Σ, and
(iii) the second junction conditions at Σ are given by equations (3.43) and (3.44).

Then, the C^1 regularity at Σ implies $M = t_{ab} = 0$. Moreover, $M = t_{ab} = 0$ and $f(\phi) \neq 0$ at Σ implies the C^1 regularity at Σ.

Proof. By lemma 2, the C^1 regularity at Σ is equivalent to $[K_{ab}] = 0$. Then, the proposition follows from equations (3.43) and (3.44). ■

In the special case where $f(\phi) = 0$ holds at Σ, $M = t_{ab} = 0$ is just a necessary condition for a C^1 regular matching. Actually, $M = t_{ab} = 0$ only implies $f'(\phi)[K] = 0$ in this case.

The following proposition shows the conditions for vacuum Σ ($t_{ab} \equiv 0$).

Proposition 2 (J-vacuum at non-null Σ). *Let ϕ_Σ be the value of ϕ at a non-null hypersurface Σ. Then, under the assumptions (i)–(iii) in proposition 1, $t_{\mu\nu} \equiv 0$ is realized at Σ only in one of the following four cases:*

(I) $[K_{\mu\nu}] = M = 0$,
(II) $f(\phi_\Sigma) = [K] = M = 0$,
(III) $f'(\phi_\Sigma) = f(\phi_\Sigma) = M = 0$, or
(IV) $2(n-1)f'(\phi_\Sigma)^2 + (n-2)f(\phi_\Sigma) = 0$, $(n-1)[K_{\mu\nu}] = [K]h_{\mu\nu}$, and $M = 2\varepsilon f'(\phi_\Sigma)[K]$.

Proof. If $2(n-1)f'(\phi_\Sigma)^2 + (n-2)f(\phi_\Sigma) \neq 0$ holds, equations (3.46) and (3.47) with $t_{\mu\nu} \equiv 0$ show that there are two possibilities $[K_{\mu\nu}] = M = 0$ or $f(\phi_\Sigma) = [K] = M = 0$. If $2(n-1)f'(\phi_\Sigma)^2 + (n-2)f(\phi_\Sigma) = 0$ holds, equations (3.42) and (3.43) with $t_{\mu\nu} \equiv 0$ reduce

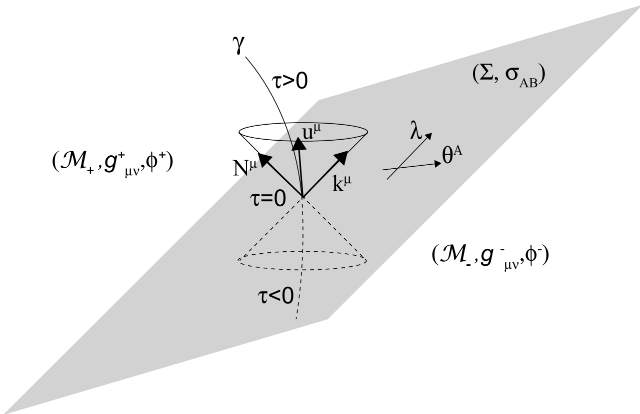

Figure 4. A null hypersurface Σ divides the spacetime into two regions $\mathcal{M}_+$ and $\mathcal{M}_-$.

to

$$f'(\phi_\Sigma)^2\left\{(n-1)[K_{\mu\nu}]-[K]h_{\mu\nu}\right\}=0, \tag{3.62}$$

$$M-2\varepsilon f'(\phi_\Sigma)[K]=0, \tag{3.63}$$

and hence there are two possibilities $f'(\phi_\Sigma)=f(\phi_\Sigma)=M=0$ or $(n-1)[K_{\mu\nu}]=[K]h_{\mu\nu}$ with $M=2\varepsilon f'(\phi_\Sigma)[K]$. ■

While the case (I) in proposition 2 is the same as that in the Einstein frame, the cases (II)–(IV) are characteristic in the Jordan frame, which suggest the possibility of a vacuum thin-shell, where the spacetime is vacuum but C^0 at Σ. (See [2] for such a vacuum thin-shell in Einstein–Gauss–Bonnet gravity.) While the constraint on the jump of the extrinsic curvature at Σ is different from $[K_{\mu\nu}]=0$ in the cases (II) and (IV), there is no constraint $[K_{\mu\nu}]$ in the case (III). We note that the first condition in the case (IV) is always satisfied in the theory with the non-minimal coupling (2.25), which does not admit the Einstein frame.

3.6.2. Einstein frame. While the total action in the Jordan frame is given by equation (3.48), it is described in the Einstein frame as

$$\begin{aligned}
I_{\rm E}=&\int_{\mathcal{M}_+}{\rm d}^nx_+\sqrt{-\bar{g}^+}\left(\frac{1}{2\kappa_n}\bar{R}^+-\frac{1}{2}(\nabla\psi^+)^2-\bar{V}(\psi^+)+\bar{\mathcal{L}}^{(m)}_{\mathcal{M}_+}\right)\\
&+\int_{\mathcal{M}_-}{\rm d}^nx_-\sqrt{-\bar{g}^-}\left(\frac{1}{2\kappa_n}\bar{R}^--\frac{1}{2}(\nabla\psi^-)^2-\bar{V}(\psi^-)+\bar{\mathcal{L}}^{(m)}_{\mathcal{M}_-}\right)\\
&+\frac{\epsilon_+}{\kappa_n}\int_{\partial\mathcal{M}_+-\Sigma_+}{\rm d}^{n-1}z_+\sqrt{|\bar{\zeta}^+|}\bar{K}^++\frac{\epsilon_-}{\kappa_n}\int_{\partial\mathcal{M}_--\Sigma_-}{\rm d}^{n-1}z_-\sqrt{|\bar{\zeta}^-|}\bar{K}^-\\
&+\frac{\varepsilon}{\kappa_n}\int_{\Sigma_+}{\rm d}^{n-1}y\sqrt{|\bar{h}|}\bar{K}^++\frac{\varepsilon}{\kappa_n}\int_{\Sigma_-}{\rm d}^{n-1}y\sqrt{|\bar{h}|}\bar{K}^-+\int_{\Sigma}{\rm d}^{n-1}y\sqrt{|\bar{h}|}\bar{\mathcal{L}}^{(m)}_{\Sigma},
\end{aligned} \tag{3.64}$$

where ϵ_+, ϵ_-, and ε independently take their values ±1 and $\psi^\pm|_\Sigma=\psi$.

Since the bulk matter fields do not contribute to the energy-momentum tensor on Σ under the assumptions (a) and (b) in section 2.2, we can write $\bar{T}_{\mu\nu}$ as

$$\bar{T}_{\mu\nu} = \Theta(l)\bar{T}^{+}_{\mu\nu} + \Theta(-l)\bar{T}^{-}_{\mu\nu}. \tag{3.65}$$

Under the assumptions that $\bar{\mathcal{L}}^{(m)}_{\mathcal{M}_\pm}$ and $\bar{\mathcal{L}}^{(m)}_{\Sigma}$ do not depend on $\psi^\pm$ and ψ, respectively, variation of the above action, with the boundary condition $\delta\bar{g}^{\pm}_{\mu\nu}|_{\partial\mathcal{M}_\pm - \Sigma_\pm} = 0 = \delta\psi^\pm|_{\partial\mathcal{M}_\pm - \Sigma_\pm}$, provides the Einstein equation (2.18) and the equation of motion for the scalar field (2.19) in the bulk spacetimes $\mathcal{M}_+$ and $\mathcal{M}_-$, as well as, the following junction conditions on Σ:

$$-\varepsilon\left([\bar{K}_{\mu\nu}] - \bar{h}_{\mu\nu}[\bar{K}]\right) = \kappa_n \bar{t}_{\mu\nu}, \tag{3.66}$$

$$\bar{M} = 0, \tag{3.67}$$

where $\bar{M} := \varepsilon\bar{n}^\mu[\partial_\mu\psi]$. $\bar{t}_{\mu\nu}$ in equation (3.66) is given by $\bar{t}_{\mu\nu} = \bar{t}_{ab}e^a_\mu e^b_\mu$, where

$$\bar{t}_{ab} := -2\frac{\partial\bar{\mathcal{L}}^{(m)}_{\Sigma}}{\partial\bar{h}^{ab}} + \bar{h}_{ab}\bar{\mathcal{L}}^{(m)}_{\Sigma}. \tag{3.68}$$

From the junction conditions (3.66) and (3.67), one can easily show the following proposition.

Proposition 3 (E-regularity and E-vacuum at non-null Σ). *Suppose in the Einstein frame that*

(i) $[\bar{g}_{\mu\nu}] = [\psi] = 0$ holds at a non-null hypersurface Σ, and
(ii) the second junction conditions at Σ are given by equations (3.66) and (3.67).

Then, the C^1 regularity at Σ is equivalent to $\bar{t}_{ab} \equiv 0$.

Proof. By lemma 2, the C^1 regularity at Σ is equivalent to $[\bar{K}_{ab}] = 0$. Then, the proposition follows from equation (3.66) and its trace. ∎

4. Junction conditions for null hypersurfaces

4.1. Setup

In the case where Σ is null, our setup follows [11]. (See also sections 3.1 and 3.11 in [3] and the book [28].) Our convention is such that $\mathcal{M}_-$ is in the past of Σ, and $\mathcal{M}_+$ is in the future. As in the case where Σ is non-null, a null hypersurface Σ can be described as $\Phi(x) = 0$. However, unlike the non-null case, the form of a unit normal vector (2.4) cannot be used in the case of null Σ since $(\nabla_\mu\Phi)(\nabla^\mu\Phi) = 0$ is then satisfied. In the null case, we introduce a normal vector k^μ to Σ as

$$k^\mu = -\nabla^\mu\Phi, \tag{4.1}$$

where the sign is chosen so that k^μ is future-directed when Φ increases toward the future. Since $(\nabla_\mu\Phi)(\nabla^\mu\Phi) = 0$ holds everywhere on Σ, its derivative is directed along k_μ so that one can introduce a scalar κ such that $\kappa k_\mu = \nabla_\mu((\nabla_\nu\Phi)(\nabla^\nu\Phi))/2$. Then, equation (4.1) shows that k^μ satisfies a geodesic equation;

$$k^\nu\nabla_\nu k_\mu = \kappa k_\mu, \tag{4.2}$$

which implies that Σ is generated by null geodesics and k^μ is tangent to these generators[8].

[8] Hereafter we use the term ‘generator’ (or null generator) as defined on page 49 of [3] below figure 2.7 therein. Note that this definition is different from the one given for *generator* within the Lie algebra/group context.

We parametrize the geodesics by λ and then a displacement along each generators is given by $\mathrm{d}x^\mu = k^\mu \mathrm{d}\lambda$. λ is an affine parameter if $\nabla_\mu((\nabla_\nu\Phi)(\nabla^\nu\Phi)) = 0$ is satisfied, namely $(\nabla_\nu\Phi)(\nabla^\nu\Phi)$ is zero not only on but also in a neighbourhood around Σ. We let the parameter λ on the null generators be one of the coordinates on Σ and the other $n-2$ coordinates θ^A are introduced to label the generators. (See figure 4.) The coordinates $y^a = (\lambda, \theta^A)$ on Σ are assumed to be the same on both sides of Σ. As for non-null Σ in the previous section, we assume that continuous canonical coordinates x^μ, distinct from $x^\mu_\pm$, can be introduced in an open region containing both sides of Σ. Hereafter we will describe geometrical and physical quantities in terms of the canonical coordinates x^μ.

The tangent vectors $e^\mu_a := \partial x^\mu/\partial y^a$ on each side of Σ are naturally separated into a null vector k^μ that is tangent to the generators, and $n-2$ spacelike vectors e^μ_A that point in the directions transverse to the generators. k^μ and e^μ_A are written as

$$k^\mu := e^\mu_\lambda = \left(\frac{\partial x^\mu}{\partial \lambda}\right)_{\theta^A}, \quad e^\mu_A = \left(\frac{\partial x^\mu}{\partial \theta^A}\right)_\lambda, \tag{4.3}$$

which satisfy

$$k^\mu k_\mu = 0 = k_\mu e^\mu_A. \tag{4.4}$$

In the canonical coordinates x^μ, both k^μ and e^μ_A are continuous across Σ and hence we have $[k^\mu] = [e^\mu_A] = 0$. The remaining inner products

$$\sigma_{\pm AB}(\lambda, \theta^C) := g^\pm_{\mu\nu} e^\mu_A e^\mu_B \tag{4.5}$$

are non-vanishing, and we assume that they are also continuous across Σ:

$$[\sigma_{AB}] := \sigma_{+AB} - \sigma_{-AB} = 0. \tag{4.6}$$

The $(n-2)$-tensor $\sigma_{AB} := \sigma_{+AB} (\equiv \sigma_{-AB})$ is the induced metric on Σ:

$$\mathrm{d}s^2_\Sigma = \sigma_{AB}\mathrm{d}\theta^A \mathrm{d}\theta^B. \tag{4.7}$$

The condition (4.6) ensures that the intrinsic geometry on Σ is well-defined. The basis is completed by adding an auxiliary null vector N^μ which satisfies

$$N^\mu N_\mu = 0, \;\; N^\mu k_\mu = -1, \;\; N_\mu e^\mu_A = 0, \tag{4.8}$$

and hence N^μ is continuous across Σ. The completeness relations of the basis are given as

$$g^{\mu\nu} = -k^\mu N^\nu - N^\mu k^\nu + \sigma^{AB} e^\mu_A e^\nu_B, \tag{4.9}$$

where the inverse metric σ^{AB} on Σ is the inverse of σ_{AB}.

We introduce a congruence of timelike geodesics γ that arbitrarily intersect Σ, of which tangent vector is u^μ. Geodesics are parametrized by proper time τ, which is adjusted so that $\tau = 0$ at Σ, $\tau < 0$ in $\mathcal{M}_-$, and $\tau > 0$ in $\mathcal{M}_+$. Then, the metric $g_{\mu\nu}$ and the scalar field ϕ are expressed as distribution-valued tensors in the canonical coordinates x^μ as

$$g_{\mu\nu} = \Theta(\tau) g^+_{\mu\nu} + \Theta(-\tau) g^-_{\mu\nu}, \tag{4.10}$$

$$\phi = \Theta(\tau)\phi^+ + \Theta(-\tau)\phi^-. \tag{4.11}$$

A displacement along a member of the congruence is described by

$$\mathrm{d}x^\mu = u^\mu \mathrm{d}\tau, \tag{4.12}$$

which is continuous across Σ, namely $[u^\mu] = 0$. The hypersurface Σ is described by $\tau(x^\mu) = 0$ and its normal vector k^μ is proportional to the gradient of $\tau(x^\mu)$ evaluated at Σ. Hence, the expression of k_μ compatible with equation (4.12) is

$$k_\mu = -(-k_\nu u^\nu)\frac{\partial \tau}{\partial x^\mu}. \tag{4.13}$$

4.2. Continuity of $g_{\mu\nu}$ and ϕ: first junction conditions

Differentiating equations (4.10) and (4.11) and using equation (4.13), we obtain

$$\partial_\rho g_{\mu\nu} = \Theta(\tau)\partial_\rho g^+_{\mu\nu} + \Theta(-\tau)\partial_\rho g^-_{\mu\nu} - (-k_\eta u^\eta)^{-1}[g_{\mu\nu}]k_\rho \delta(\tau), \tag{4.14}$$

$$\partial_\mu \phi = \Theta(\tau)\partial_\mu \phi^+ + \Theta(-\tau)\partial_\mu \phi^- - (-k_\eta u^\eta)^{-1}[\phi]k_\rho \delta(\tau). \tag{4.15}$$

As in the case where Σ is non-null, to removed the δ pieces appearing in the right-hand sides, which generate terms proportional to $\Theta(\tau)\delta(\tau)$, we impose $[g_{\mu\nu}] = [\phi] = 0$, namely continuity of the metric and ϕ across Σ, and then we have

$$\partial_\rho g_{\mu\nu} = \Theta(\tau)\partial_\rho g^+_{\mu\nu} + \Theta(-\tau)\partial_\rho g^-_{\mu\nu}, \tag{4.16}$$

$$\partial_\mu \phi = \Theta(\tau)\partial_\mu \phi^+ + \Theta(-\tau)\partial_\mu \phi^-. \tag{4.17}$$

Now we characterize the discontinuous behaviors of $\partial_\rho g_{\mu\nu}$ and $\partial_\mu \phi$. The continuity conditions on the fields guarantee that the tangential derivatives of the metric and scalar field are also continuous, namely

$$[\partial_\rho g_{\mu\nu}]k^\rho = 0 = [\partial_\rho g_{\mu\nu}]e^\rho_A, \tag{4.18}$$

$$[\partial_\mu \phi]k^\mu = 0 = [\partial_\mu \phi]e^\mu_A. \tag{4.19}$$

The only possible discontinuity is therefore in $N^\rho \partial_\rho g_{\mu\nu}$ and $N^\mu \partial_\mu \phi$, namely the transverse derivatives. In view of equation (4.8), there exist a tensor field $\gamma_{\mu\nu}$ and a scalar field W such that

$$[\partial_\rho g_{\mu\nu}] = -\gamma_{\mu\nu}k_\rho, \qquad [\partial_\mu \phi] = Wk_\mu. \tag{4.20}$$

Namely, $\gamma_{\mu\nu}$ and W are defined by

$$\gamma_{\mu\nu} := N^\rho[\partial_\rho g_{\mu\nu}], \qquad W := -N^\mu[\partial_\mu \phi], \tag{4.21}$$

respectively.

4.3. Discontinuity of geometric and physical quantities

We have seen $[k^\mu] = [e^\mu_A] = [N^\mu] = [u^\mu] = 0$ in the canonical coordinates x^μ. Differentiation of the metric proceeds as in the non-null case, except that we now write τ instead of l, and we use equation (4.13) to relate the gradient of τ to the null vector k^μ. Thus, we obtain a Riemann tensor that contains a singular part given by

$$R^\rho{}_{\sigma\mu\nu} = \Theta(\tau)R^{+\rho}{}_{\sigma\mu\nu} + \Theta(-\tau)R^{-\rho}{}_{\sigma\mu\nu} + \delta(\tau)\tilde{R}^\rho{}_{\sigma\mu\nu}, \tag{4.22}$$

$$\tilde{R}^\rho{}_{\sigma\mu\nu} := -(-k_\eta u^\eta)^{-1}([\Gamma^\rho_{\sigma\nu}]k_\mu - [\Gamma^\rho_{\sigma\mu}]k_\nu) \tag{4.23}$$

where $[\Gamma^{\rho}_{\sigma\nu}]$ is the jump in the Christoffel symbols across Σ.

Equation (4.20) implies

$$[\Gamma^{\rho}_{\sigma\mu}] = -\frac{1}{2}(\gamma^{\rho}{}_{\sigma}k_{\mu} + \gamma^{\rho}{}_{\mu}k_{\sigma} - \gamma_{\sigma\mu}k^{\rho}), \tag{4.24}$$

so that the δ-function part of the Riemann tensor can be written as

$$\tilde{R}^{\rho}{}_{\sigma\mu\nu} = \frac{1}{2}(-k_{\eta}u^{\eta})^{-1}(\gamma^{\rho}{}_{\nu}k_{\sigma}k_{\mu} - \gamma^{\rho}{}_{\mu}k_{\sigma}k_{\nu} - \gamma_{\sigma\nu}k^{\rho}k_{\mu} + \gamma_{\sigma\mu}k^{\rho}k_{\nu}). \tag{4.25}$$

We see that k^{μ} and $\gamma_{\mu\nu}$ give a complete characterization of the singular part of the Riemann tensor, and the δ-function terms of the Ricci tensor and the Ricci scalar are easily determined as

$$\tilde{R}_{\sigma\nu} = \frac{1}{2}(-k_{\eta}u^{\eta})^{-1}(\gamma_{\mu\nu}k^{\mu}k_{\sigma} + \gamma_{\mu\sigma}k^{\mu}k_{\nu} - \gamma k_{\sigma}k_{\nu}), \tag{4.26}$$

$$\tilde{R} = (-k_{\eta}u^{\eta})^{-1}\gamma_{\mu\nu}k^{\mu}k^{\nu}, \tag{4.27}$$

respectively, where $\gamma := \gamma^{\mu}{}_{\mu}$. Finally, the singular part of the Einstein tensor is given by

$$\tilde{G}^{\mu\nu} = \frac{1}{2}(-k_{\eta}u^{\eta})^{-1}(\gamma^{\nu}{}_{\rho}k^{\rho}k^{\mu} + \gamma^{\mu}{}_{\rho}k^{\rho}k^{\nu} - \gamma k^{\mu}k^{\nu} - g^{\mu\nu}\gamma_{\rho\sigma}k^{\rho}k^{\sigma}). \tag{4.28}$$

The factor $-k_{\eta}u^{\eta}$ depends on the choice of observers correponding to u^{μ} who makes measurements on the shell.

On the other hand, differentiating equation (4.17), we obtain

$$\partial_{\mu}\partial_{\nu}\phi = \Theta(\tau)\partial_{\mu}\partial_{\nu}\phi^{+} + \Theta(-\tau)\partial_{\mu}\partial_{\nu}\phi^{-} - (-k_{\eta}u^{\eta})^{-1}Wk_{\mu}k_{\nu}\delta(\tau), \tag{4.29}$$

where we used equations (4.13) and (4.20). From the above expression, we get

$$\nabla_{\mu}\nabla_{\nu}\phi = \Theta(\tau)\nabla_{\mu}\nabla_{\nu}\phi^{+} + \Theta(-\tau)\nabla_{\mu}\nabla_{\nu}\phi^{-} - (-k_{\eta}u^{\eta})^{-1}Wk_{\mu}k_{\nu}\delta(\tau), \tag{4.30}$$

$$\Box\phi = \Theta(\tau)\Box\phi^{+} + \Theta(-\tau)\Box\phi^{-}. \tag{4.31}$$

Finally, using the following expression

$$\nabla_{\mu}\nabla_{\nu}f(\phi) = f'(\phi)\nabla_{\mu}\nabla_{\nu}\phi + f''(\phi)(\nabla_{\mu}\phi)(\nabla_{\nu}\phi), \tag{4.32}$$

we find

$$\nabla_{\mu}\nabla_{\nu}f(\phi) = \Theta(\tau)\nabla_{\mu}\nabla_{\nu}f(\phi^{+}) + \Theta(-\tau)\nabla_{\mu}\nabla_{\nu}f(\phi^{-}) - (-k_{\eta}u^{\eta})^{-1}f'(\phi)Wk_{\mu}k_{\nu}\delta(\tau), \tag{4.33}$$

$$\Box f(\phi) = \Theta(\tau)\Box f(\phi^{+}) + \Theta(-\tau)\Box f(\phi^{-}). \tag{4.34}$$

For later use, we introduce the projections

$$\gamma_{A} := \gamma_{\mu\nu}e^{\mu}_{A}k^{\nu}, \quad \gamma_{AB} := \gamma_{\mu\nu}e^{\mu}_{A}e^{\nu}_{B}. \tag{4.35}$$

By the completeness relation (4.9), the vector $\gamma_{\mu\nu}k^{\nu}$ admits the following decomposition:

$$\gamma_{\mu\nu}k^{\nu} = \frac{1}{2}(\gamma - \sigma^{AB}\gamma_{AB})k_{\mu} + (\sigma_{AB}\gamma^{B})e^{A}_{\mu} - (\gamma_{\rho\sigma}k^{\rho}k^{\sigma})N_{\mu}. \tag{4.36}$$

For a consistency check, from the above expression, we obtain $\gamma_{\mu\nu}k^{\nu}N_{\mu} = (1/2)\gamma_{\mu\nu}(k^{\mu}N^{\nu} + k^{\nu}N^{\mu}) = \gamma_{\mu\nu}k^{\mu}N^{\nu}$, where the last equality holds because of the symmetric nature of $\gamma_{\mu\nu}$.

Since k^μ is not normal but tangent to Σ, we introduce a transverse curvature C_{ab} that properly represents the transverse derivative of the metric:

$$C_{ab} := \frac{1}{2}(\mathcal{L}_N g_{\mu\nu})e^\mu_a e^\nu_b = (\nabla_\mu N_\nu)e^\mu_a e^\nu_b, \tag{4.37}$$

where we have used that $N_\mu e^\mu_a = 0$ and an identity $(\nabla_\nu e^\mu_a)e^\nu_b \equiv (\nabla_\nu e^\mu_b)e^\nu_a$. In the canonical coordinates x^μ, the jump of the transverse curvature at Σ is given by

$$[C_{ab}] = [\nabla_\mu N_\nu]e^\mu_a e^\nu_b = \frac{1}{2}\gamma_{\mu\nu}e^\mu_a e^\nu_b. \tag{4.38}$$

We therefore have

$$[C_{\lambda\lambda}] = \frac{1}{2}\gamma_{\mu\nu}k^\mu k^\nu, \quad [C_{A\lambda}] = \frac{1}{2}\gamma_A, \quad [C_{AB}] = \frac{1}{2}\gamma_{AB}. \tag{4.39}$$

As in lemma 2 for non-null Σ, the following lemma provides several different expressions of a C^1 regular matching condition, $[g_{\alpha\beta}] = [\partial_\mu g_{\alpha\beta}] = 0$, in the case where Σ is null.

Lemma 4. *If* $[g_{\alpha\beta}] = 0$ *holds, the following five conditions are equivalent: (i)* $\tilde{R}^\rho{}_{\sigma\mu\nu} = 0$, *(ii)* $[C_{ab}] = 0$, *(iii)* $\gamma_{\mu\nu} = 0$, *(iv)* $[\partial_\mu g_{\alpha\beta}] = 0$, *and (v)* $[\Gamma^\rho_{\sigma\mu}] = 0$.

Proof. The conditions (ii) and (iii) are equivalent by equation (4.38). The conditions (iii) and (iv) are equivalent by equations (4.20) and (4.21). Now we show that the conditions (i) and (ii) are equivalent. If $[C_{ab}] = 0$ holds, we have $\gamma_{\mu\nu} = 0$ and hence $\tilde{R}^\rho{}_{\sigma\mu\nu} = 0$ is satisfied by equation (4.25). On the other hand, if $\tilde{R}^\rho{}_{\sigma\mu\nu} = 0$ holds, equation (4.25) gives

$$\gamma_{\rho\nu}k_\sigma k_\mu - \gamma_{\rho\mu}k_\sigma k_\nu - \gamma_{\sigma\nu}k_\rho k_\mu + \gamma_{\sigma\mu}k_\rho k_\nu = 0. \tag{4.40}$$

Acting $k^\rho k^\nu$, $k^\rho e^\nu_A$, and $e^\rho_A e^\nu_B$ on the above equation, we respectively obtain $\gamma_{\rho\nu}k^\rho k^\nu = 0$, $\gamma_{\rho\nu}k^\rho e^\nu_A = 0$, and $\gamma_{\rho\nu}e^\rho_A e^\nu_B = 0$, and hence $[C_{ab}] = 0$ is concluded. Since we have shown that the conditions (i)–(iv) are equivalent, we complete the proof by showing that the conditions (iii) and (v) are equivalent. The condition (iii) implies the condition (v) by equation (4.24). The condition (v) implies the condition (i) by equation (4.23), which is equivalent to the condition (iii). ■

4.4. Second junction conditions

4.4.1. Equation of motion for a scalar field. Here we derive the junction condition from the equation of motion (2.12), namely $\Pi = 0$, where Π is defined by equation (2.16). Using equations (4.27) and (4.31), we write down Π as

$$\Pi = \Theta(\tau)\Pi^+ + \Theta(-\tau)\Pi^- + \delta(\tau)\tilde{\Pi}, \tag{4.41}$$

where the δ-function part $\tilde{\Pi}$ is given by

$$\tilde{\Pi} := (-k_\eta u^\eta)^{-1} f'(\phi)\gamma_{\mu\nu}k^\mu k^\nu. \tag{4.42}$$

Thus, the equation of motion $\Pi = 0$ on Σ gives $\tilde{\Pi} = 0$, namely

$$f'(\phi)\gamma_{\mu\nu}k^\mu k^\nu = 0. \tag{4.43}$$

We shall refer to this condition as the junction condition from the equation of motion for a scalar field. For a minimally coupled scalar field, namely for $f(\phi) = 1/(2\kappa_n)$, this condition is trivially satisfied.

4.4.2. Einstein equations. Next let us derive the junction conditions from the Einstein equation (2.11), namely $E_{\mu\nu} = T_{\mu\nu}$, where $E_{\mu\nu}$ is defined by equation (2.15). Using equations (4.28), (4.33) and (4.34), we write down $E_{\mu\nu}$ as

$$E_{\mu\nu} = \Theta(\tau)E^{+}_{\mu\nu} + \Theta(-\tau)E^{-}_{\mu\nu} + \delta(\tau)\tilde{E}_{\mu\nu}, \tag{4.44}$$

where, the δ-function part $\tilde{E}_{\mu\nu}$ is given by

$$\begin{aligned}\tilde{E}_{\mu\nu} :&= (-k_\eta u^\eta)^{-1} \\ &\times \left\{ f(\phi)(\gamma_{\nu\rho}k^\rho k_\mu + \gamma_{\mu\rho}k^\rho k_\nu - g_{\mu\nu}\gamma_{\rho\sigma}k^\rho k^\sigma) + (2f'(\phi)W - f(\phi)\gamma)\, k_\mu k_\nu \right\}.\end{aligned} \tag{4.45}$$

Under the assumptions (a) and (b) in section 2.2, we can write the bulk energy-momentum tensor $T_{\mu\nu}$ as

$$T_{\mu\nu} = \Theta(\tau)T^{+}_{\mu\nu} + \Theta(-\tau)T^{-}_{\mu\nu}, \tag{4.46}$$

which means that the bulk matter fields do not contribute to the energy-momentum tensor on Σ. By equations (4.44) and (4.46), the Einstein equations $E_{\mu\nu} = T_{\mu\nu}$ on Σ give $\tilde{E}_{\mu\nu} = 0$, which we shall refer as the junction conditions from the Einstein equations, which are the conditions for vacuum Σ.

For embedding configurations of Σ with $\tilde{E}_{\mu\nu} \neq 0$, the Einstein equations require an additional matter field on Σ for consistency, so that Σ is no more vacuum. The Einstein equations on Σ then become

$$\tilde{E}_{\mu\nu} = t_{\mu\nu}, \tag{4.47}$$

where $t_{\mu\nu}$ is the thin-shell energy-momentum tensor on Σ, which is written as

$$\begin{aligned}t_{\mu\nu} &= (-k_\eta u^\eta)^{-1} \\ &\times \left\{ f(\phi)(\gamma_{\nu\rho}k^\rho k_\mu + \gamma_{\mu\rho}k^\rho k_\nu - g_{\mu\nu}\gamma_{\rho\sigma}k^\rho k^\sigma) + (2f'(\phi)W - f(\phi)\gamma)\, k_\mu k_\nu \right\}.\end{aligned} \tag{4.48}$$

The expression of $t_{\mu\nu}$ can be simplified if we decompose it in the basis $\{k^\mu, e^\mu_A, N^\mu\}$. Using equation (4.36) and involving once more the completeness relation (4.9), $t_{\mu\nu}$ is written as

$$t_{\mu\nu} = (-k_\eta u^\eta)^{-1} \left\{ \mu k_\mu k_\nu + j_A(k_\mu e^A_\nu + e^A_\mu k_\nu) + p\sigma_{AB}e^A_\mu e^B_\nu \right\} \tag{4.49}$$

with

$$\mu := 2f'(\phi)W - f(\phi)\sigma^{AB}\gamma_{AB}, \tag{4.50}$$

$$j_A := f(\phi)\sigma_{AB}\gamma^B, \tag{4.51}$$

$$p := -f(\phi)\gamma_{\mu\nu}k^\mu k^\nu \tag{4.52}$$

$t_{\mu\nu}$ is the surface energy-momentum tensor of the null shell, where the factor $(-k_\eta u^\eta)^{-1}$ represents the dependence of the observers which make measurements on the shell. The quantities μ, j^A, and p are respectively interpreted as the shell's surface density, a surface current, and an isotropic surface pressure [11]. In the special case where j^A and p are vanishing, the surface energy-momentum tensor (4.49) is equivalent to a null dust fluid.

Equations (4.2) and (4.8) show

$$\kappa = -N_\mu k^\nu \nabla_\nu k^\mu = -N_\mu e^\nu_\lambda \nabla_\nu e^\mu_\lambda = (\nabla_\nu N_\mu) e^\nu_\lambda e^\mu_\lambda = C_{\lambda\lambda} \tag{4.53}$$

where we used equation (4.37) at the last equality. Combined with equation (4.39), we obtain

$$[\kappa] = [C_{\lambda\lambda}] = \frac{1}{2}\gamma_{\mu\nu} k^\mu k^\nu. \tag{4.54}$$

In the general case $f'(\phi) \neq 0$, the junction condition (4.43) gives $\gamma_{\mu\nu}k^\mu k^\nu = 0$. Then, equations (4.52) and (4.54) show $p = 0$ and $[\kappa] = 0$, respectively. Thus, in scalar–tensor theories, the surface energy-momentum tensor of a null shell must be pressureless in contrast to general relativity. In additon, by equation (4.2), $[\kappa] = 0$ implies that we can use the same affine parameter for the null generators in both sides of the null hypersurface Σ [4].

By equation (4.39), the surface quantities (4.50)–(4.52) can be expressed in terms of the transverse curvature such that

$$\mu = 2f'(\phi)W - 2f(\phi)\sigma^{AB}[C_{AB}], \tag{4.55}$$

$$j^A = 2f(\phi)\sigma^{AB}[C_{\lambda B}], \tag{4.56}$$

$$p = -2f(\phi)[C_{\lambda\lambda}]. \tag{4.57}$$

In summary, we have obtained the junction conditions (4.43) and (4.49) in the Jordan frame at a null hypersurface Σ as

$$t_{\mu\nu} = (-k_\eta u^\eta)^{-1}\left\{\mu k_\mu k_\nu + j_A(k_\mu e^A_\nu + e^A_\mu k_\nu) + p\sigma_{AB}e^A_\mu e^B_\nu\right\}, \tag{4.58}$$

$$f'(\phi)[C_{\lambda\lambda}] = 0, \tag{4.59}$$

where μ, j^A, and p are defined by equations (4.50)–(4.52) (or equivalently equations (4.55)–(4.57)) and $\gamma_{\mu\nu}$ and W are defined by equation (4.21).

4.5. Conditions for C^1 matching and vacuum Σ

4.5.1. Jordan frame. Now let us study the conditions for a C^1 matching at Σ and also for vacuum Σ.

Proposition 4 (J-regularity at null Σ). *Let ϕ_Σ be the value of ϕ at a null hypersurface Σ. Suppose in the Jordan frame that*

(i) $f(\phi)$ is a C^1 function,
(ii) $[\sigma_{AB}] = [\phi] = 0$ holds at Σ, and
(iii) the second junction conditions at Σ are given by equations (4.58) and (4.59).

Then, the C^1 regularity at Σ implies $j_A = p = 0$ and $\mu = 2f'(\phi_\Sigma)W$. If $f(\phi_\Sigma) \neq 0$ holds, $j_A = p = 0$ and $\mu = 2f'(\phi_\Sigma)W$ at Σ imply $[C_{\lambda\lambda}] = [C_{\lambda A}] = \sigma^{AB}[C_{AB}] = 0$.

Proof. By lemma 4, the C^1 regularity at Σ is equivalent to $[C_{ab}] = 0$. Then, the proposition follows from equations (4.55)–(4.57). ■

The above proposition suggests a possibility of a C^1 matching at Σ with non-vanishing μ if $f'(\phi_\Sigma)W \neq 0$ holds. This *non-vacuum C^1 matching* is characteristic in the Jordan frame and clearly shows that $[C_{ab}] = 0$ and $t_{\mu\nu} = 0$ are not equivalent in this frame. We also note that $\sigma^{AB}[C_{AB}] = 0$ is a weaker condition than $[C_{AB}] = 0$.

Now let us obtain conditions for vacuum Σ ($t_{\mu\nu} \equiv 0$) in the case where Σ is null.

Proposition 5 (J-vacuum at null Σ). *Let ϕ_Σ be the value of ϕ at a null hypersurface Σ. Then, under the assumptions (i)–(iii) in proposition 4, $t_{\mu\nu} \equiv 0$ is realized at Σ only in one of the following three cases:*

(I) $f(\phi_\Sigma) = 0$ *and* $f'(\phi_\Sigma) = 0$,
(II) $f(\phi_\Sigma) = 0$ *and* $[C_{\lambda\lambda}] = W = 0$, *or*
(III) $[C_{\lambda\lambda}] = 0$, $f'(\phi_\Sigma)W = f(\phi_\Sigma)\sigma^{AB}[C_{AB}]$, *and* $[C_{\lambda A}] = 0$.

Proof. With $t_{\mu\nu} = 0$, equations (4.58) and (4.59) reduce to

$$f'(\phi)W = f(\phi)\sigma^{AB}[C_{AB}], \tag{4.60}$$

$$f(\phi)\sigma^{AB}[C_{\lambda B}] = 0, \tag{4.61}$$

$$f(\phi)[C_{\lambda\lambda}] = 0, \tag{4.62}$$

$$f'(\phi)[C_{\lambda\lambda}] = 0. \tag{4.63}$$

The proposition follows from equations (4.60)–(4.63). ■

4.5.2. Einstein frame. Under the assumptions (a) and (b) in section 2.3, we can write the bulk energy-momentum tensor $\bar{T}_{\mu\nu}$ as

$$\bar{T}_{\mu\nu} = \Theta(\tau)\bar{T}^+_{\mu\nu} + \Theta(-\tau)\bar{T}^-_{\mu\nu}, \tag{4.64}$$

which means that the bulk matter fields do not contribute to the energy-momentum tensor $\bar{t}_{\mu\nu}$ on Σ. Since the junction condition (4.43) is trivially satisfied, the junction conditions in the Einstein frame are

$$\bar{t}_{\mu\nu} = (-\bar{k}_\eta \bar{u}^\eta)^{-1}\left\{\bar{\mu}\bar{k}_\mu\bar{k}_\nu + \bar{j}_A(\bar{k}_\mu e^A_\nu + e^A_\mu \bar{k}_\nu) + \bar{p}\bar{\sigma}_{AB}e^A_\mu e^B_\nu\right\}, \tag{4.65}$$

where

$$\bar{\mu} := -\kappa_n^{-1}\bar{\sigma}^{AB}[\bar{C}_{AB}], \tag{4.66}$$

$$\bar{j}^A := \kappa_n^{-1}\bar{\sigma}^{AB}[\bar{C}_{\lambda B}], \tag{4.67}$$

$$\bar{p} := -\kappa_n^{-1}[\bar{C}_{\lambda\lambda}]. \tag{4.68}$$

Therefore, $\bar{t}_{\mu\nu} \equiv 0$ at Σ is equivalent to

$$\bar{\sigma}^{AB}[\bar{C}_{AB}] = [\bar{C}_{\lambda B}] = [\bar{C}_{\lambda\lambda}] = 0. \tag{4.69}$$

The following proposition clarifies the relation between vacuum Σ ($\bar{t}_{\mu\nu} \equiv 0$) and a C^1 matching at Σ in the Einstein frame.

Proposition 6 (E-regularity and E-vacuum at null Σ). *Suppose in the Einstein frame that*

(i) $[\bar{\sigma}_{AB}] = [\psi] = 0$ *holds at a null hypersurface* Σ*, and*
(ii) the second junction condition at Σ *is given by equation (4.65).*

Then, the C^1 *regularity at* Σ *implies* $\bar{t}_{\mu\nu} = 0$. $\bar{t}_{\mu\nu} = 0$ *at* Σ *implies* $[\bar{C}_{\lambda\lambda}] = [\bar{C}_{\lambda A}] = \bar{\sigma}^{AB}[\bar{C}_{AB}] = 0$.

Proof. By lemma 4, the C^1 regularity at Σ is equivalent to $[\bar{C}_{ab}] = 0$. Then, the proposition follows from equations (4.66)–(4.68). ■

While proposition 3 shows that $\bar{t}_{\mu\nu} \equiv 0$ and $[\bar{K}_{\mu\nu}] = 0$ are equivalent in the case where Σ is non-null, proposition 6 shows that $\bar{t}_{\mu\nu} = 0$ is just a necessary condition for $[\bar{C}_{ab}] = 0$ in the case where Σ is null because $\bar{\sigma}^{AB}[\bar{C}_{AB}] = 0$ is weaker than $[\bar{C}_{AB}] = 0$.

5. Relation between the conditions in Jordan and Einstein frames

In this section, we study the relation between C^1 matchings in Jordan and Einstein frames and also the relation between the conditions for vacuum Σ. As seen in section 2.4, the matter Lagrangian densities may introduce anomalies which violate the correspondence between the Jordan and Einstein frames. Indeed, as shown in appendix A in the case where Σ is non-null, there is a proper mapping between the Jordan frame (3.48) and the Einstein frame (3.64) only when the non-minimal coupling $f(\phi)$ does not satisfy equation (2.25) and the extra matter fields in the bulk and on Σ are conformal invariant, namely $T_{\mu\nu} = \bar{T}_{\mu\nu}$ and $t_{ab} = \bar{t}_{ab}$, including vacuum cases. Only in such cases, there exists a proper correspondence between the field equations and junction conditions in two frames.

Even if there is no proper correspondence between them, one can study the relation of the C^1 regular matchings in the Jordan and Einstein frames because it is a purely geometrical concept. Naively thinking, the C^1 regular matchings in two frames seem to be equivalent; however, we will see that there are some exceptional cases. We first show the following lemma for later use.

Lemma 5. *Let* ϕ_Σ *be the value of* ϕ *at a null or non-null hypersurface* Σ*. If* $f(\phi)$ *is a* C^1 *function and not in the exceptional form (2.25), then* $[\phi] = 0$ *and* $[\psi] = 0$ *are equivalent. If* $f(\phi_\Sigma) \neq 0$ *holds in addition, then* $[g_{\mu\nu}] = 0$ *and* $[\bar{g}_{\mu\nu}] = 0$ *are equivalent.*

Proof. Since $f(\phi)$ is in the C^1-class and not in the exceptional form (2.25), equation (2.22) with a fixed sign in the right-hand side shows that $\psi(\phi)$ is a continuous and monotonic function. Thus, there exists a continuous inverse function $\phi(\psi)$ and hence $[\phi] = 0$ and $[\psi] = 0$ are equivalent. Since $f(\phi_\Sigma)$ is assumed to be non-zero and finite at Σ, the relation $\bar{g}_{\mu\nu} = (2\kappa_n f(\phi))^{2/(n-2)} g_{\mu\nu}$ shows that $[g_{\mu\nu}] = 0$ is equivalent to $[\bar{g}_{\mu\nu}] = 0$. ■

In the case of $f(\phi_\Sigma) = 0$, the geometric information in the other frame cannot be obtained so that one has to study the other frame individually. Actually, $f(\phi_\Sigma) = 0$ is a part of the J-vacuum conditions (II) and (III) for non-null Σ in proposition 2 as well as the J-vacuum conditions (I) and (II) for null Σ in proposition 5. In the following subsections, we will see that $2(n-1)f'(\phi_\Sigma)^2 + (n-2)f(\phi_\Sigma) = 0$ is also such an exceptional case.

5.1. Non-null hypersurfaces

Using equations (A.4) and (2.22), we obtain

$$\varepsilon\bar{n}^{\mu}\partial_{\mu}\psi = \pm\varepsilon(2\kappa_n f(\phi))^{-1/(n-2)}\sqrt{\frac{2(n-1)f'(\phi)^2 + (n-2)f(\phi)}{2(n-2)\kappa_n f(\phi)^2}}n^{\mu}\partial_{\mu}\phi, \tag{5.1}$$

while equation (A.6) gives

$$\bar{K}_{\mu\nu} = (2\kappa_n f(\phi))^{1/(n-2)}\left(K_{\mu\nu} + \frac{f'(\phi)}{(n-2)f(\phi)}n^{\sigma}(\partial_{\sigma}\phi)h_{\mu\nu}\right). \tag{5.2}$$

From the above equations, we first clarify the relations of the C^1 regularity at Σ in the Jordan and Einstein frames.

Proposition 7 (Relation of C^1-regularities at non-null Σ). *Let ϕ_Σ be the value of ϕ at a non-null hypersurface Σ. Suppose that $f(\phi)$ is a C^1 function, not in the exceptional form (2.25), and satisfies $f(\phi_\Sigma) \neq 0$. Then,*

(i) under the assumptions in proposition 1 in the Jordan frame, $[K_{\mu\nu}] = 0$ implies $[\bar{K}_{\mu\nu}] = 0$, and
(ii) under the assumptions in proposition 3 in the Einstein frame, $[\bar{K}_{\mu\nu}] = 0$ implies $[K_{\mu\nu}] = 0$ if $2(n-1)f'(\phi_\Sigma)^2 + (n-2)f(\phi_\Sigma) \neq 0$.

Proof. By lemma 5, $[\phi] = [g_{\mu\nu}] = 0$ and $[\psi] = [\bar{g}_{\mu\nu}] = 0$ are equivalent. Then, equations (5.1) and (5.2) give

$$\bar{M} = \pm(2\kappa_n f(\phi))^{-1/(n-2)}\sqrt{\frac{2(n-1)f'(\phi)^2 + (n-2)f(\phi)}{2(n-2)\kappa_n f(\phi)^2}}M, \tag{5.3}$$

$$[\bar{K}_{\mu\nu}] = (2\kappa_n f(\phi))^{1/(n-2)}\left([K_{\mu\nu}] + \frac{\varepsilon f'(\phi)}{(n-2)f(\phi)}Mh_{\mu\nu}\right). \tag{5.4}$$

$[K_{\mu\nu}] = 0$ in the Jordan frame implies $M = t_{\mu\nu} = 0$ by the junction conditions (3.42) and (3.43), which shows $[\bar{K}_{\mu\nu}] = \bar{M} = 0$ by equations (5.3) and (5.4). On the other hand, $[\bar{K}_{\mu\nu}] = 0$ in the Einstein frame implies $\bar{M} = \bar{t}_{\mu\nu} = 0$ by the junction conditions (3.66) and (3.67), which shows $[K_{\mu\nu}] = M = 0$ by equations (5.3) and (5.4). ■

The above proposition does not assume a proper correspondence between the Jordan and Einstein frames. If there is, the statement (i) leads $\bar{t}_{\mu\nu} = 0$ by proposition 3, while the statement (ii) leads $t_{\mu\nu} = 0$ by proposition 2.

Now we clarify the relation of the vacuum Σ conditions in the case where there is a proper correspondence between two frames.

Proposition 8 (Relation of vacuum non-null Σ). *Let ϕ_Σ be the value of ϕ at a non-null hypersurface Σ. Suppose that*

(i) $f(\phi)$ is a C^1 function, not in the exceptional form (2.25), and satisfies $f(\phi_\Sigma) \neq 0$,
(ii) there is a proper correspondence between the Jordan and Einstein frames, and
(iii) the assumptions in proposition 1 in the Jordan frame and in proposition 3 in the Einstein frame hold.

Then, J-vacuum condition (I) or (IV) in proposition 2 implies E-vacuum. E-vacuum implies the J-vacuum condition (I) if $2(n-1)f'(\phi_\Sigma)^2 + (n-2)f(\phi_\Sigma) \neq 0$ holds.

Proof. By equations (5.3) and (5.4), both J-vacuum conditions (I) and (IV) in proposition 2 imply $\bar{M} = [\bar{K}_{\mu\nu}] = 0$, which shows $\bar{t}_{\mu\nu} = 0$ by proposition 3. By proposition 3, E-vacuum $\bar{t}_{\mu\nu} = 0$ is equivalent to $[\bar{K}_{\mu\nu}] = 0$, which shows $t_{\mu\nu} = 0$ by proposition 7. ∎

5.2. Null hypersurfaces

While $\bar{u}^\mu = u^\mu$ holds, $\bar{g}_{\mu\nu} = (2\kappa_n f(\phi))^{2/(n-2)} g_{\mu\nu}$ shows that the relations between the pseudo-orthonormal basis and the induced metric in the Jordan and Einstein frames are

$$\bar{N}^\mu = (2\kappa_n f(\phi))^{-1/(n-2)} N^\mu, \quad \bar{k}^\mu = (2\kappa_n f(\phi))^{-1/(n-2)} k^\mu, \tag{5.5}$$

$$\bar{\sigma}^{AB} = (2\kappa_n f(\phi))^{-2/(n-2)} \sigma^{AB}, \quad \bar{e}^\mu_A = e^\mu_A, \tag{5.6}$$

which satisfy the following completeness condition in the Einstein frame:

$$\bar{g}^{\mu\nu} = -\bar{k}^\mu \bar{N}^\nu - \bar{N}^\mu \bar{k}^\nu + \bar{\sigma}^{AB} \bar{e}^\mu_A \bar{e}^\nu_B. \tag{5.7}$$

Now let us clarify the relations of the C^1 regularity at Σ in the Jordan and Einstein frames.

Proposition 9 (Relation of C^1-regularities at null Σ). *Let ϕ_Σ be the value of ϕ at a null hypersurface Σ. Suppose that $f(\phi)$ is a C^1 function, not in the exceptional form (2.25), and satisfies $f(\phi_\Sigma) \neq 0$. Then, the following two statements hold:*

(i) $[C_{ab}] = f'(\phi_\Sigma) W = 0$ *in the Jordan frame implies* $[\bar{C}_{ab}] = 0$*, and*
(ii) $[\bar{C}_{ab}] = \bar{W} = 0$ *in the Einstein frame implies* $[C_{ab}] = 0$ *if* $2(n-1) f'(\phi_\Sigma)^2 + (n-2) f(\phi_\Sigma) \neq 0$.

Proof. Since $f(\phi_\Sigma)$ is non-zero and finite, equations (5.5) and (5.6) show that $[N^\mu] = [k^\mu] = [\sigma_{AB}] = [e^\mu_A] = 0$ are equivalent to $[\bar{N}^\mu] = [\bar{k}^\mu] = [\bar{\sigma}_{AB}] = [\bar{e}^\mu_A] = 0$. Also, by lemma 5, $[\phi] = 0$ and $[\psi] = 0$ are equivalent. Then, the following relations

$$-\bar{N}^\mu \partial_\mu \psi = -\sqrt{\frac{2(n-1) f'(\phi)^2 + (n-2) f(\phi)}{2(n-2)\kappa_n f(\phi)^2}} (2\kappa_n f(\phi))^{-1/(n-2)} N^\mu \partial_\mu \phi, \tag{5.8}$$

$$\begin{aligned} \bar{N}^\rho \partial_\rho \bar{g}_{\mu\nu} &= (2\kappa_n f(\phi))^{-1/(n-2)} N^\rho \partial_\rho ((2\kappa_n f(\phi))^{2/(n-2)} g_{\mu\nu}) \\ &= (2\kappa_n f(\phi))^{1/(n-2)} \left(\frac{2}{n-2} f(\phi)^{-1} f'(\phi) N^\rho (\partial_\rho \phi) g_{\mu\nu} + N^\rho \partial_\rho g_{\mu\nu} \right), \end{aligned} \tag{5.9}$$

give

$$\bar{W} = \sqrt{\frac{2(n-1) f'(\phi)^2 + (n-2) f(\phi)}{2(n-2)\kappa_n f(\phi)^2}} (2\kappa_n f(\phi))^{-1/(n-2)} W \tag{5.10}$$

$$[\bar{C}_{ab}] = (2\kappa_n f(\phi))^{1/(n-2)} \left(-\frac{1}{n-2} \frac{f'(\phi)}{f(\phi)} W g_{\mu\nu} e^\mu_a e^\nu_b + [C_{ab}] \right), \tag{5.11}$$

where we used $[C_{ab}] = [\nabla_\mu N_\nu] e^\mu_a e^\nu_b = (1/2) N^\rho [\partial_\rho g_{\mu\nu}] e^\mu_a e^\nu_b$. The proposition follows from equations (5.10) and (5.11). ∎

The above proposition is purely geometrical and does not assume the second junction conditions. In fact, even with the second junction conditions, the C^1 matching conditions at null Σ in the Jordan and Einstein frames are not equivalent.

Next we clarify the relations of vacuum Σ conditions in the case where there is a proper correspondence between two frames.

Proposition 10 (Relation of vacuum null Σ). *Let ϕ_Σ be the value of ϕ at a null hypersurface Σ. Suppose that*

(i) $f(\phi)$ is a C^1 function, not in the exceptional form (2.25), and satisfies $f(\phi_\Sigma) \neq 0$,
(ii) there is a proper correspondence between the Jordan and Einstein frames, and
(iii) the assumptions in proposition 4 in the Jordan frame and in proposition 6 in the Einstein frame hold.

Then, the J-vacuum condition (III) in proposition 5 is equivalent to E-vacuum.

Proof. Equation (5.11) gives

$$[\bar{C}_{\lambda\lambda}] = (2\kappa_n f(\phi))^{1/(n-2)}[C_{\lambda\lambda}], \tag{5.12}$$

$$[\bar{C}_{\lambda B}] = (2\kappa_n f(\phi))^{1/(n-2)}[C_{\lambda B}], \tag{5.13}$$

$$[\bar{C}_{AB}] = (2\kappa_n f(\phi))^{1/(n-2)}\left(-\frac{1}{n-2}\frac{f'(\phi)}{f(\phi)}W\sigma_{AB} + [C_{AB}]\right), \tag{5.14}$$

$$\bar{\sigma}^{AB}[\bar{C}_{AB}] = (2\kappa_n f(\phi))^{-1/(n-2)}\left(-\frac{f'(\phi)}{f(\phi)}W + \sigma^{AB}[C_{AB}]\right). \tag{5.15}$$

E-vacuum ($\bar{t}_{\mu\nu} = 0$) is equivalent to equation (4.69). By equations (5.12), (5.13) and (5.15), equation (4.69) is equivalent to the J-vacuum condition (III) in proposition 5. ■

5.3. Examples of vacuum C^1 matching at null hypersurface

Here we present two examples of the vacuum C^1 matching at a null hypersurface. Since extra matter fields do not exist in the bulk spacetime, there is a proper correspondence in the Jordan and Einstein frames in both cases.

5.3.1. Roberts-(A)dS solution in the Einstein frame ($n = 4$). Let us consider the Einstein-Λ system with a massless scalar field ϕ in four dimensions, of which action is given by

$$I_{\rm E} = \int_{\mathcal{M}} {\rm d}^4x\sqrt{-\bar{g}}\left(\frac{1}{2\kappa}(\bar{R} - 2\Lambda) - \frac{1}{2}(\bar{\nabla}\psi)^2\right) + \frac{\varepsilon}{\kappa}\int_{\partial\mathcal{M}} {\rm d}^3x\sqrt{|\bar{h}|}\bar{K}, \tag{5.16}$$

which corresponds to the Einstein frame with $\bar{V}(\psi) = \Lambda/\kappa$.

In this system, we consider the following topological generalization of Roberts-(A)dS solution [29, 30]:

$$\begin{aligned} {\rm d}s^2 &= \bar{g}_{\mu\nu}{\rm d}x^\mu{\rm d}x^\nu \\ &= \left(1 - \frac{\Lambda}{6}uv\right)^{-2}\left(-2{\rm d}u{\rm d}v + (-kuv + D_1v^2 + D_2u^2)\bar{\eta}_{AB}(z){\rm d}\theta^A{\rm d}\theta^B\right), \end{aligned} \tag{5.17}$$

in the coordinates $x^\mu = (u, v, \theta^A)$, where $A, B = 2, 3$. In the above solution, D_1 and D_2 are constants and $\bar{\eta}_{AB}$ is the metric on a two-dimensional space of constant curvature with its Gauss curvature $k = 1, 0, -1$. For $k^2 - 4D_1D_2 > 0$, the scalar field ψ is real and given by

$$\psi = \begin{cases} \pm\frac{1}{\sqrt{2\kappa}}\ln\left|\frac{u\sqrt{k^2-4D_1D_2}+(ku-2D_1v)}{u\sqrt{k^2-4D_1D_2}-(ku-2D_1v)}\right| + \psi_0 & \text{for } D_1 \neq 0, \\ \pm\frac{1}{\sqrt{2\kappa}}\ln\left|D_2 - k\frac{v}{u}\right| + \psi_1 & \text{for } D_1 = 0, \end{cases} \tag{5.18}$$

where ψ_0 and ψ_1 are constants. For $k^2 - 4D_1D_2 < 0$, ψ is ghost and given by

$$\psi = \pm \mathrm{i}\sqrt{\frac{2}{\kappa}}\left[\arctan\left(\frac{ku - 2D_1v}{u\sqrt{4D_1D_2 - k^2}}\right) + \mathrm{sign}(D_1v)\frac{\pi}{2}\right] + \psi_2, \tag{5.19}$$

where ψ_2 is a pure imaginary constant. If $k^2 - 4D_1D_2 = 0$, the field equations give ψ =constant and $\bar{R}^{\mu\nu}{}_{\rho\sigma} = (\Lambda/3)(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho)$, namely the spacetime is maximally symmetric. With $\Lambda = 0$, the solution (5.17) reduces to the Roberts solution [31].

In [32], it has been presented that a vacuum C^1 matching is possible between two Roberts-(A)dS spacetimes with different values of D_2 at a null hypersurface Σ given by $u = 0$. The induced metric $\bar{h}_{ab}$ on Σ is

$$\mathrm{d}s^2_\Sigma = \bar{h}_{ab}\mathrm{d}y^a\mathrm{d}y^b = D_1v^2\bar{\eta}_{AB}\mathrm{d}\theta^A\mathrm{d}\theta^B (= \bar{\sigma}_{AB}\mathrm{d}x^A\mathrm{d}x^B) \tag{5.20}$$

and therefore $D_1 \neq 0$ is required, where $y^a = (v, \theta^A)$ is a set of coordinates on Σ. For $D_1 \neq 0$, the value of ψ on Σ is constant containing ψ_0 or ψ_2, so we can always set ψ be continuous at Σ by choosing the value of ψ_0 or ψ_2 in the spacetime attached.

The basis vectors of Σ defined by $\bar{e}^\mu_a := \partial x^\mu/\partial y^a$ are

$$\bar{e}^\mu_v\frac{\partial}{\partial x^\mu} = \bar{k}^\mu = \frac{\partial}{\partial v}, \qquad \bar{e}^\mu_A\frac{\partial}{\partial x^\mu} = \delta^\mu{}_A\frac{\partial}{\partial\theta^A}, \tag{5.21}$$

and the bases are completed by $\bar{N}_\mu\mathrm{d}x^\mu = -\mathrm{d}v$. They satisfy $\bar{N}_\mu\bar{e}^\mu_v(\equiv \bar{N}_\mu\bar{k}^\mu) = -1$ and $\bar{N}_\mu\bar{e}^\mu_A = 0$ on Σ. Using the following expression

$$\begin{aligned}\bar{\nabla}_\nu\bar{N}_\mu &= \partial_\nu\bar{N}_\mu - \bar{\Gamma}^\alpha_{\nu\mu}\bar{N}_\alpha = -\bar{\Gamma}^v_{\nu\mu}\bar{N}_v \\ &= \frac{1}{2}\bar{g}^{vu}(\partial_\nu\bar{g}_{\mu u} + \partial_\mu\bar{g}_{\nu u} - \partial_u\bar{g}_{\mu\nu}),\end{aligned} \tag{5.22}$$

and $\bar{g}^{uv} = -(1 - \Lambda uv/6)^2$, we compute the non-zero components of $\bar{C}_{ab}$ as

$$\bar{C}_{vv} = (\bar{\nabla}_\nu\bar{N}_\mu)\bar{e}^\mu_v\bar{e}^\nu_v = \bar{g}^{vu}\partial_v\bar{g}_{vu} = \frac{\Lambda}{3}u\left(1 - \frac{\Lambda}{6}uv\right)^{-1}, \tag{5.23}$$

$$\begin{aligned}\bar{C}_{AB} &= (\bar{\nabla}_\nu\bar{N}_\mu)\bar{e}^\mu_A\bar{e}^\nu_B = -\frac{1}{2}\bar{g}^{vu}\partial_u\bar{g}_{AB} \\ &= \frac{1}{2}\left\{\frac{\Lambda}{3}v\left(1 - \frac{\Lambda}{6}uv\right)^{-1}(-kuv + D_1v^2 + D_2u^2) + (-kv + 2D_2u)\right\}\bar{\eta}_{AB},\end{aligned} \tag{5.24}$$

and hence

$$\bar{C}_{vv}|_\Sigma = 0, \qquad \bar{C}_{AB}|_\Sigma = \frac{1}{2}v\left(\frac{1}{3}\Lambda D_1v^2 - k\right)\bar{\eta}_{AB}. \tag{5.25}$$

Since $\bar{h}_{ab}$ and $\bar{C}_{ab}|_\Sigma$ do not contain D_2, two Roberts-(A)dS spacetimes (5.17) with the same nonzero D_1 but different D_2 can be attached at $u=0$ in a C^1 regular manner, where $[\bar{h}_{ab}]=[\bar{C}_{ab}]=0$ are realized. Then, by proposition 6, $\bar{t}_{\mu\nu}=0$ holds and hence there is no massive thin-shell at Σ. As a special case, a Roberts-(A)dS spacetime can be attached to the past (A)dS spacetime at $u=0$ and the resulting spacetime may represent black-hole or naked-singularity formation from a regular initial datum.

Lastly, let us see whether $\bar{W}:=-\bar{N}^\mu[\partial_\mu\psi]$ is vanishing or not at $u=0$. With the following expression;

$$\bar{N}^\mu\frac{\partial}{\partial x^\mu}=\left(1-\frac{1}{6}\Lambda uv\right)^2\frac{\partial}{\partial u}, \tag{5.26}$$

we obtain

$$(\bar{N}^\rho\partial_\rho\psi)|_\Sigma=\pm\frac{\sqrt{k^2-4D_1D_2}}{\sqrt{2\kappa}D_1 v} \tag{5.27}$$

for $k^2-4D_1D_2\neq 0$ with $D_1\neq 0$. Since the above expression contains both D_1 and D_2, $\bar{W}\neq 0$ holds when two Roberts-(A)dS spacetimes (5.17) with the same nonzero D_1 but different D_2 are attached at $u=0$. Therefore, proposition 9 does not work and the C^1 regularity at Σ in the Jordan frame is not clear in this case. However, since $\bar{t}_{\mu\nu}=0$ holds at Σ, $t_{\mu\nu}=0$ also holds in the Jordan frame under the assumptions in proposition 10.

5.3.2. Generalized Xu solution in the Jordan frame ($n=3$). Another example is presented in the three-dimensional gravity coupled to a non-minimally self-interacting scalar field ϕ in the presence of a negative cosmological constant Λ:

$$\begin{aligned} I_{\rm J}=&\int {\rm d}^3x\sqrt{-g}\left(\frac{1}{2\kappa}(R-2\Lambda)-\frac{1}{2}(\nabla\phi)^2-\frac{1}{16}R\phi^2-\alpha\phi^6\right)\\ &+\frac{\varepsilon}{\kappa}\int_{\partial\mathcal{M}}{\rm d}^2x\sqrt{|h|}\left(1-\frac{\kappa}{8}\phi^2\right)K, \end{aligned} \tag{5.28}$$

which is the Jordan frame with

$$f(\phi)=\frac{1}{2\kappa}\left(1-\frac{\kappa}{8}\phi^2\right), \tag{5.29}$$

$$V(\phi)=-\frac{1}{\kappa l^2}+\alpha\phi^6, \tag{5.30}$$

where l is the AdS radius defined by $l^{-2}:=-\Lambda$. To simplify the expressions in the following argument, we introduce a constant β defined by

$$\beta:=\frac{512\alpha l^2-\kappa^2}{8\kappa l^2}. \tag{5.31}$$

In this system, there is the following generalized Xu solution [33, 34]:

$${\rm d}s^2=-f(v,r){\rm d}v^2+2{\rm d}v{\rm d}r+r^2{\rm d}\theta^2, \tag{5.32}$$

$$\phi(v,r)=\frac{a(v)}{\sqrt{r+\kappa a(v)^2/8}}, \tag{5.33}$$

Table 1. Summary of the main results in the present paper. In all the cases, continuity of the metric and scalar field at Σ is required as the first junction conditions.

Σ	Frame	2nd junction conditions	C^1 matching	Vacuum Σ	J-E relation
Non-null	Jordan	Equations (3.42) & (3.43)	Proposition 1	Proposition 2	Propositions 7 & 8
	Einstein	Equations (3.66) & (3.67)	Proposition 3	Proposition 3	
Null	Jordan	Equations (4.58) & (4.59)	Proposition 4	Proposition 5	Propositions 9 & 10
	Einstein	Equation (4.65)	Proposition 6	Proposition 6	

where the metric function $f(v,r)$ is given by

$$f(v,r) = \frac{r^2}{l^2} - B_0 a(v) - \frac{B_0 \kappa a(v)^3}{12r}. \tag{5.34}$$

Here B_0 is a parameter in the solution and the function $a(v)$ is given by

$$a(v) = \frac{2B_0}{3\kappa} v \tag{5.35}$$

for $\beta = 0$ and

$$\frac{1}{2}\ln\left(\frac{(a-a_0)^2}{a^2+a_0 a+a_0^2}\right) - \sqrt{3}\arctan\left(\frac{2a+a_0}{\sqrt{3}a_0}\right) = \frac{3a_0^2\beta}{4}\left(v - \frac{2\sqrt{3}\pi}{9\beta a_0^2}\right) \tag{5.36}$$

for $\beta \neq 0$. Here B_0 is an integration constant and a_0 is defined by

$$a_0 := -\epsilon \left|\frac{8B_0}{3\kappa\beta}\right|^{1/3}, \tag{5.37}$$

where ϵ is the sign of $8B_0/(3\kappa\beta)$, and hence we have $B_0 = -(3/8)\kappa\beta a_0^3$. In the solutions (5.35) and (5.36), we have set another integration constant such that $a(0) = 0$ without loss of generality. $B_0 = 0$ gives the massless BTZ spacetime and the behavior of a near $v = 0$ for $\beta \neq 0$ is given by

$$a(v) \simeq \frac{2B_0}{3\kappa} v, \tag{5.38}$$

which is the same as equation (5.35) for $\beta = 0$.

Actually, the generalized Xu solution (5.32) for $v \geqslant 0$ can be attached to the massless BTZ spacetime for $v \leqslant 0$ with $\phi \equiv 0$ (and hence $[\phi] = 0$ is realized). On the null hypersurface Σ defined by $v = 0$, we install coordinates $y^a = (\lambda, \theta^A)$ which are the same on both past and future sides of Σ. Here λ is an arbitrary parameter on the null generators of Σ and θ^A label the generators, where the index A is always $A = 1$ in the three-dimensional case. We identify $-r$ with λ and set $\theta^A = \theta$ on Σ in the spacetime (5.32).

The parametric equations $x^\mu = x^\mu(\lambda, \theta^A)$ describing Σ are $v = 0$, $r = -\lambda$, and $\theta = \theta$. The line element on Σ is one-dimensional and given by

$$\mathrm{d}s_\Sigma^2 = h_{ab}\mathrm{d}y^a\mathrm{d}y^b = \lambda^2\mathrm{d}\theta^2 (= \sigma_{AB}\mathrm{d}\theta^A\mathrm{d}\theta^B), \tag{5.39}$$

where $y^a = (\lambda, \theta)$ is a set of coordinates on Σ. Using them, we obtain the tangent vectors of Σ defined by $e^\mu_a := \partial x^\mu/\partial y^a$ as

$$e^\mu_\lambda \frac{\partial}{\partial x^\mu} = -\frac{\partial}{\partial r}, \qquad e^\mu_\theta \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \theta}. \tag{5.40}$$

An auxiliary null vector N^μ given by

$$N^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial v} + \frac{1}{2} f(0, r) \frac{\partial}{\partial r} \tag{5.41}$$

completes the basis. The expression $N_\mu dx^\mu = -(f(0, r)/2)dv + dr$ shows $N_\mu N^\mu = 0$, $N_\mu e^\mu_\lambda = -1$, and $N_\mu e^\mu_\theta = 0$. Then, the only nonvanishing component of the transverse curvature $C_{ab} := (\nabla_\nu N_\mu) e^\mu_a e^\nu_b$ of Σ is

$$C_{\theta\theta} = \frac{1}{2} r f(0, r) = \frac{r}{2l^2}. \tag{5.42}$$

Since equations (5.39) and (5.42) do not contain B_0, $[\sigma_{AB}] = [C_{ab}] = [\phi] = 0$ is realized and therefore a C^1 regular matching is achieved at Σ.

On the other hand, $W \neq 0$ holds at $v = 0$ because equations (5.41) and (5.33) show

$$(N^\mu \nabla_\mu \phi)|_\Sigma = \frac{2B_0}{3\kappa r^{1/2}}, \tag{5.43}$$

which contains B_0. Nevertheless, since equation (5.29) shows $f'(\phi_\Sigma) = 0$ with $\phi_\Sigma = 0$, the J-vacuum condition (III) in proposition 5 is satisfied at $v = 0$. In this case, since $f(\phi_\Sigma) = 1/2\kappa \neq 0$ holds, both C^1 regularity and vacuum Σ are realized in the Einstein frame by propositions 9 and 10.

6. Summary

In the present paper, we have studied junction conditions in a large class of scalar–tensor theories in arbitrary $n(\geqslant 3)$ dimensions. We have treated both null and non-null junction hypersurfaces Σ under the three assumptions: (a) the Lagrangian density for bulk matter fields does not depend on the scalar field, (b) the matter fields are minimally coupled to gravity, and (c) the energy-momentum tensor on Σ does not contain the same scalar field in the bulk spacetime. As a consequence of the assumptions (a) and (b), the bulk energy-momentum tensor does not contribute to the energy-momentum tensor on Σ.

While the metric and scalar field must be continuous on Σ as the first junction conditions, the jumps of their first derivatives and the matter field on Σ are related as the second junction conditions given from the Einstein equations and the equation of motion for the bulk scalar field treated as distributions [11]. We have confirmed that the resulting junction conditions are compatible with the ones derived in the variational method pioneered in [13] in the case of non-null Σ.

To the best of the authors' knowledge, the junction conditions in the Jordan frame have been derived for the first time in the present paper in the cases of spacelike Σ with $n \neq 4$ and null Σ with arbitrary $n(\geqslant 3)$. In the Jordan frame, the junction conditions at timelike Σ were previously derived in four dimensions [18] and also in arbitrary dimensions [20]. In [17], the authors derived the junction conditions at non-null Σ in a general scalar–tensor theory which contains our action (2.10) in the absence of additional matter fields. However, a signature ε (which is s in [17]) is missing in the surface term B_4 in equation (3.3) in [17], which is required to be consistent with the result in general relativity.

Subsequently, we have clarified the C^1 regular matching conditions and the vacuum conditions at Σ both in the Jordan and Einstein frames. At non-null Σ in the Einstein frame, the C^1 regularity (E-regularity) is equivalent to the vacuum Σ condition (E-vacuum). In the Jordan frame, in contrast, while the C^1 regularity (J-regularity) implies vacuum Σ (J-vacuum), J-vacuum does not necessarily imply J-regularity. In other words, J-regularity is a sufficient condition for J-vacuum which suggests a possibility of *vacuum* thin-shell at non-null Σ in the Jordan frame.

The situations are different in the case where Σ is null. In this case, E-regularity and E-vacuum are even not equivalent. Actually, E-regularity is a sufficient condition for E-vacuum so that there is a possibility of vacuum thin-shell at null Σ. To compound matters, J-regularity and J-vacuum do not necessarily imply each other, which suggests that both non-vacuum C^1 regular matching and vacuum thin-shell may be possible at null Σ in the Jordan frame. The main results obtained in the present paper are summarized in table 1.

Lastly, we have clarified the relations between the sufficient conditions for the C^1 regularity in the Jordan and Einstein frames and also between the vacuum Σ conditions, which allow us to identify the properties of the junction hypersurface Σ in the other frame. We have adopted these results to two concrete exact solutions; The Roberts-(A)dS solution in the Einstein frame in four dimensions and the generalized Xu solution in the Jordan frame in three dimensions.

As demonstrated in these two examples, all the results in the present paper may provide a firm basis for applications in a variety of contexts, which would clarify the effects of the non-minimal coupling of the scalar field to gravity. Additionally, to construct concrete examples of the following configurations is an interesting task: (i) a vacuum thin-shell at null Σ in the Einstein frame, (ii) a vacuum thin-shell at null and non-null Σ in the Jordan frame, and (iii) a non-vacuum at null Σ in the Jordan frame. We leave these problems for future investigations.

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Appendix A. Transformation from Jordan to Einstein frame

In this appendix, we consider a conformal transformation from the following action in the Jordan frame:

$$\begin{aligned} I_{\rm J} =& \int_{\mathcal{M}} \mathrm{d}^n x\sqrt{-g}\biggl(f(\phi)R - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\biggr) + 2\varepsilon\int_{\Sigma} \mathrm{d}^{n-1}y\sqrt{|h|}f(\phi)K \\ &+ \int_{\mathcal{M}} \mathrm{d}^n x\sqrt{-g}\mathcal{L}^{(m)}_{\mathcal{M}} + \int_{\Sigma} \mathrm{d}^{n-1}x\sqrt{|h|}\mathcal{L}^{(m)}_{\Sigma}. \end{aligned} \tag{A.1}$$

By a conformal transformation $g_{\mu\nu}(x) = \Omega(x)^2\bar{g}_{\mu\nu}$ in n dimensions, the Ricci scalar is transformed as

$$R = \Omega^{-2}\left\{\bar{R} - 2(n-1)\bar{\Box}\ln\Omega - (n-1)(n-2)(\bar{\nabla}_\rho\ln\Omega)(\bar{\nabla}^\rho\ln\Omega)\right\}, \tag{A.2}$$

which is shown by the following transformation of the Christoffel symbol:

$$\begin{aligned}
\Gamma^{\mu}_{\rho\sigma} &= \frac{1}{2} g^{\mu\alpha}(\partial_\sigma g_{\alpha\rho} + \partial_\rho g_{\alpha\sigma} - \partial_\alpha g_{\rho\sigma}) \\
&= \frac{1}{2}\Omega^{-2}\bar{g}^{\mu\alpha}\left\{\partial_\sigma(\Omega^2\bar{g}_{\alpha\rho}) + \partial_\rho(\Omega^2\bar{g}_{\alpha\sigma}) - \partial_\alpha(\Omega^2\bar{g}_{\rho\sigma})\right\} \\
&= \bar{\Gamma}^{\mu}_{\rho\sigma} + (\partial_\sigma \ln\Omega)\bar{g}^{\mu}{}_{\rho} + (\partial_\rho \ln\Omega)\bar{g}^{\mu}{}_{\sigma} - (\partial_\alpha \ln\Omega)\bar{g}^{\mu\alpha}\bar{g}_{\rho\sigma}.
\end{aligned} \tag{A.3}$$

We consider the case where the matching non-null hypersurface Σ is described by $\Phi(x) = 0$ in both frames. In this case, the unit orthogonal vector of Σ is transformed as

$$\bar{n}_\mu := (\varepsilon\bar{g}^{\rho\sigma}\nabla_\rho\Phi\nabla_\sigma\Phi)^{-1/2}\nabla_\mu\Phi = \Omega^{-1}(\varepsilon g^{\rho\sigma}\nabla_\rho\Phi\nabla_\sigma\Phi)^{-1/2}\nabla_\mu\Phi = \Omega^{-1}n_\mu \tag{A.4}$$

and hence the projection tensor is transformed as

$$\bar{h}_{\mu\nu} := \bar{g}_{\mu\nu} - \varepsilon\bar{n}_\mu\bar{n}_\nu = \Omega^{-2}(g_{\mu\nu} - \varepsilon n_\mu n_\nu) = \Omega^{-2}h_{\mu\nu}. \tag{A.5}$$

Using these results, one can show that the extrinsic curvature and its trace are transformed as

$$\begin{aligned}
K_{\mu\nu} &= h^{\rho}_{(\mu}h^{\sigma}_{\nu)}\nabla_\rho n_\sigma = \bar{h}^{\rho}_{(\mu}\bar{h}^{\sigma}_{\nu)}(\partial_\rho n_\sigma - \Gamma^{\alpha}_{\rho\sigma}n_\alpha) \\
&= \bar{h}^{\rho}_{(\mu}\bar{h}^{\sigma}_{\nu)}\left\{\partial_\rho(\Omega\bar{n}_\sigma) - \left(\bar{\Gamma}^{\alpha}_{\rho\sigma} + (\partial_\sigma\ln\Omega)\bar{g}^{\alpha}{}_{\rho} + (\partial_\rho\ln\Omega)\bar{g}^{\alpha}{}_{\sigma} - (\partial_\beta\ln\Omega)\bar{g}^{\alpha\beta}\bar{g}_{\rho\sigma}\right)\Omega\bar{n}_\alpha\right\} \\
&= \Omega\bar{h}^{\rho}_{(\mu}\bar{h}^{\sigma}_{\nu)}\left\{\bar{\nabla}_\rho\bar{n}_\sigma + (\partial_\beta\ln\Omega)\bar{g}^{\alpha\beta}\bar{g}_{\rho\sigma}\bar{n}_\alpha\right\} = \Omega\left\{\bar{K}_{\mu\nu} + \bar{h}_{\mu\nu}(\partial_\beta\ln\Omega)\bar{n}^\beta\right\}
\end{aligned} \tag{A.6}$$

and

$$\begin{aligned}
K &= g^{\mu\nu}K_{\mu\nu} = \Omega^{-1}\bar{g}^{\mu\nu}\left\{\bar{K}_{\mu\nu} + \bar{h}_{\mu\nu}(\partial_\sigma\ln\Omega)\bar{n}^\sigma\right\} \\
&= \Omega^{-1}\bar{K} + (n-1)\Omega^{-1}(\partial_\sigma\ln\Omega)\bar{n}^\sigma.
\end{aligned} \tag{A.7}$$

Putting the above expressions into the action (A.1) in the Jordan frame, we obtain

$$\begin{aligned}
I_{\mathrm{J}} &= \int_{\mathcal{M}} \mathrm{d}^n x\sqrt{-\bar{g}}\left\{\Omega^{n-2}f(\phi)\left(\bar{R} - 2(n-1)\bar{\Box}\ln\Omega - (n-1)(n-2)(\bar{\nabla}_\rho\ln\Omega)(\bar{\nabla}^\rho\ln\Omega)\right)\right. \\
&\quad \left. - \frac{1}{2}\Omega^{n-2}(\bar{\nabla}\phi)^2 - \Omega^n V(\phi)\right\} + 2\varepsilon\int_\Sigma \mathrm{d}^{n-1}x\sqrt{|\bar{h}|}\Omega^{n-2}f(\phi)\left(\bar{K} + (n-1)(\bar{\nabla}_\sigma\ln\Omega)\bar{n}^\sigma\right) \\
&\quad + \int_{\mathcal{M}} \mathrm{d}^n x\sqrt{-\bar{g}}\Omega^n\mathcal{L}^{(m)}_{\mathcal{M}} + \int_\Sigma \mathrm{d}^{n-1}x\sqrt{|\bar{h}|}\Omega^{n-1}\mathcal{L}^{(m)}_\Sigma \\
&= \int_{\mathcal{M}} \mathrm{d}^n x\sqrt{-\bar{g}}\left\{\Omega^{n-2}f(\phi)\left(\bar{R} - (n-1)(n-2)(\bar{\nabla}_\rho\ln\Omega)(\bar{\nabla}^\rho\ln\Omega)\right)\right. \\
&\quad \left. + 2(n-1)\bar{\nabla}^\rho(\Omega^{n-2}f(\phi))\bar{\nabla}_\rho\ln\Omega - \frac{1}{2}\Omega^{n-2}(\bar{\nabla}\phi)^2 - \Omega^n V(\phi)\right\} \\
&\quad + 2\varepsilon\int_\Sigma \mathrm{d}^{n-1}x\sqrt{|\bar{h}|}\Omega^{n-2}f(\phi)\bar{K} + \int_{\mathcal{M}} \mathrm{d}^n x\sqrt{-\bar{g}}\Omega^n\mathcal{L}^{(m)}_{\mathcal{M}} + \int_\Sigma \mathrm{d}^{n-1}x\sqrt{|\bar{h}|}\Omega^{n-1}\mathcal{L}^{(m)}_\Sigma,
\end{aligned} \tag{A.8}$$

where we used the Stokes' theorem (2.5) at the second equality.

Setting $\Omega = (2\kappa_n f(\phi))^{-1/(n-2)}$, we obtain the action in the Einstein frame:

$$\begin{aligned}
I_{\mathrm{E}} &= \int_{\mathcal{M}} \mathrm{d}^n x\sqrt{-\bar{g}}\left\{\frac{1}{2\kappa_n}\bar{R} - \frac{2(n-1)f'^2 + (n-2)f}{4(n-2)\kappa_n f^2}(\bar{\nabla}\phi)^2 - \Omega^n V(\phi)\right\} \\
&\quad + \frac{\varepsilon}{\kappa_n}\int_\Sigma \mathrm{d}^{n-1}x\sqrt{|\bar{h}|}\bar{K} + \int_{\mathcal{M}} \mathrm{d}^n x\sqrt{-\bar{g}}\Omega^n\mathcal{L}^{(m)}_{\mathcal{M}} + \int_\Sigma \mathrm{d}^{n-1}x\sqrt{|\bar{h}|}\Omega^{n-1}\mathcal{L}^{(m)}_\Sigma.
\end{aligned} \tag{A.9}$$

By a redefinition of the scalar field (2.22), we finally write down the action in the Einstein frame in the following canonical form:

$$I_{\rm E} = \int_{\mathcal{M}} \mathrm{d}^n x\sqrt{-\bar{g}}\left\{\frac{1}{2\kappa_n}\bar{R} - \frac{1}{2}(\bar{\nabla}\psi)^2 - \bar{V}(\psi)\right\}$$
$$+ \frac{\varepsilon}{\kappa_n}\int_{\Sigma}\mathrm{d}^{n-1}x\sqrt{|\bar{h}|}\bar{K} + \int_{\mathcal{M}}\mathrm{d}^n x\sqrt{-\bar{g}}\bar{\mathcal{L}}^{(m)}_{\mathcal{M}} + \int_{\Sigma}\mathrm{d}^{n-1}x\sqrt{|\bar{h}|}\bar{\mathcal{L}}^{(m)}_{\Sigma}, \tag{A.10}$$

where

$$\bar{V}(\psi) := (2\kappa_n f(\phi(\psi)))^{-n/(n-2)}V(\phi(\psi)), \tag{A.11}$$

$$\bar{\mathcal{L}}^{(m)}_{\mathcal{M}} := (2\kappa_n f(\phi(\psi)))^{-n/(n-2)}\mathcal{L}^{(m)}_{\mathcal{M}}, \tag{A.12}$$

$$\bar{\mathcal{L}}^{(m)}_{\Sigma} := (2\kappa_n f(\phi(\psi)))^{-(n-1)/(n-2)}\mathcal{L}^{(m)}_{\Sigma}. \tag{A.13}$$

As explained in section 2.4, for a proper mapping between the bulk equations in the Jordan and Einstein frames, assumptions in lemma 1 are required. For a proper mapping between the junction conditions in two frames, one needs $\sqrt{-g}\mathcal{L}^{(m)}_{\mathcal{M}} = \sqrt{-\bar{g}}\bar{\mathcal{L}}^{(m)}_{\mathcal{M}}$ in addition, which includes the vacuum case $\mathcal{L}^{(m)}_{\Sigma} = \bar{\mathcal{L}}^{(m)}_{\Sigma} \equiv 0$.

Appendix B. Junction conditions from variational principle for non-null Σ

In this appendix, we derive the junction conditions in the Jordan frame by the variational principle in the case where the matching hypersurface Σ is non-null. For this purpose it is convenient to start with the following action:

$$I_0 = I_{\mathcal{M}} + I_{\partial\mathcal{M}}, \tag{B.1}$$

where the bulk ($I_{\mathcal{M}}$) and boundary ($I_{\partial\mathcal{M}}$) actions are given by

$$I_{\mathcal{M}} := \int_{\mathcal{M}}\mathrm{d}^n x\sqrt{-g}\left(f(\phi)R - \frac{1}{2}g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi) - V(\phi) + \mathcal{L}^{(m)}\right), \tag{B.2}$$

$$I_{\partial\mathcal{M}} := 2\varepsilon\int_{\partial\mathcal{M}}\mathrm{d}^{n-1}y\sqrt{|h|}f(\phi)K. \tag{B.3}$$

In general relativity ($f(\phi) = 1/2\kappa_n$), equation (B.3) reduces to the Gibbons–Hawking term [35]. Such a boundary term has been constructed also in Einstein–Gauss–Bonnet gravity [36, 37] and further generalized in Lovelock gravity [38], which is the most general quasi-linear second-order theory of gravity in arbitrary dimensions [39].

In the following, we assume that $g_{\mu\nu}$ and ϕ are continuous at $\partial\mathcal{M}$ and matter Lagrangian density $\sqrt{-g}\mathcal{L}^{(m)}_{\mathcal{M}}$ does not depend on ϕ.

B.1. Useful formulae

For variation, we will use

$$\delta g^{\mu\alpha} = -g^{\nu\alpha}g^{\mu\rho}\delta g_{\nu\rho}, \qquad \delta g_{\nu\alpha} = -g_{\mu\alpha}g_{\nu\rho}\delta g^{\mu\rho}. \tag{B.4}$$

Jacobi's formula shows

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{B.5}$$

and

$$\delta\sqrt{|h|} = -\frac{1}{2}\sqrt{|h|}h_{ab}\delta h^{ab} = -\frac{1}{2}\sqrt{|h|}h_{\mu\nu}\delta h^{\mu\nu} = -\frac{1}{2}\sqrt{|h|}h_{\mu\nu}\delta g^{\mu\nu}. \tag{B.6}$$

While $\Gamma^\rho_{\mu\nu}$ is not a tensor, its variation $\delta\Gamma^\rho_{\mu\nu}$ is a tensor given by

$$\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}(\nabla_\nu\delta g_{\alpha\mu} + \nabla_\mu\delta g_{\alpha\nu} - \nabla_\alpha\delta g_{\mu\nu}), \tag{B.7}$$

which gives

$$\delta R^\rho{}_{\sigma\mu\nu} = \nabla_\mu\delta\Gamma^\rho_{\nu\sigma} - \nabla_\nu\delta\Gamma^\rho_{\mu\sigma}, \tag{B.8}$$

$$\delta R_{\sigma\nu} = \nabla_\rho\delta\Gamma^\rho_{\nu\sigma} - \nabla_\nu\delta\Gamma^\rho_{\rho\sigma}. \tag{B.9}$$

We can rewrite the term $f(\phi)g^{\mu\nu}\delta R_{\mu\nu}$ such that

$$f(\phi)g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\rho J^\rho - (\nabla_\mu\nabla_\rho f(\phi))\delta g^{\rho\mu} + (\Box f(\phi))g_{\mu\nu}\delta g^{\mu\nu}, \tag{B.10}$$

where

$$J^\rho := f(\phi)\Big(-\nabla_\mu\delta g^{\rho\mu} + \nabla_\alpha(g_{\mu\nu}g^{\rho\alpha}\delta g^{\mu\nu})\Big) + (\nabla_\mu f(\phi))\delta g^{\rho\mu} - (\nabla_\alpha f(\phi))g_{\mu\nu}g^{\rho\alpha}\delta g^{\mu\nu}. \tag{B.11}$$

From equation (2.4), we obtain

$$\delta n_\mu = \frac{1}{2}\varepsilon n_\mu n^\alpha n^\beta\delta g_{\alpha\beta}, \tag{B.12}$$

$$\delta n^\mu = -g^{\mu\alpha}n^\beta\delta g_{\alpha\beta} + \frac{1}{2}\varepsilon n^\mu n^\alpha n^\beta\delta g_{\alpha\beta}. \tag{B.13}$$

Using this, after lengthy but straightforward calculations, we obtain

$$\begin{aligned}\delta K = &-\frac{1}{2}(\nabla^\alpha n^\beta)\delta g_{\alpha\beta} + \frac{1}{2}\varepsilon n^\alpha n^\mu g^{\beta\nu}(\nabla_\mu n_\nu)\delta g_{\alpha\beta} \\ &- \frac{1}{2}n^\beta h^{\alpha\mu}(\nabla_\mu\delta g_{\alpha\beta} - \nabla_\beta\delta g_{\alpha\mu}) - \frac{1}{2}h^\rho_\mu\nabla_\rho(h^{\mu\alpha}n^\beta\delta g_{\alpha\beta}).\end{aligned} \tag{B.14}$$

B.2. Variation with respect to ϕ

First let us consider variation with respect to ϕ. Using integration by parts and used the Stokes' theorem (2.5), variation of the bulk action (B.2) leads

$$\delta_\phi I_{\mathcal{M}} = \int_{\mathcal{M}} \mathrm{d}^n x\sqrt{-g}\Big(\Box\phi + f'(\phi)R - V'(\phi)\Big)\delta\phi - \int_{\partial\mathcal{M}} \mathrm{d}^{n-1}y\sqrt{|h|}(\varepsilon n^\mu\nabla_\mu\phi)\delta\phi. \tag{B.15}$$

On the other hand, variation of the boundary term (B.3) simply leads

$$\delta_\phi I_{\partial\mathcal{M}} = 2\varepsilon\int_{\partial\mathcal{M}} \mathrm{d}^{n-1}y\sqrt{|h|}f'(\phi)K\delta\phi. \tag{B.16}$$

Since matter Lagrangian density $\sqrt{-g}\mathcal{L}_{\mathcal{M}}^{(m)}$ does not depend on ϕ, variation of the total action (B.1) with respect to ϕ reduces to the following form:

$$\delta_\phi I_0 = \int_{\mathcal{M}} \mathrm{d}^n x \sqrt{-g}\mathcal{E}_{(\phi)}\delta\phi + \int_{\partial\mathcal{M}} \mathrm{d}^{n-1} y \sqrt{|h|}\mathcal{F}_{(\phi)}\delta\phi, \tag{B.17}$$

where

$$\mathcal{E}_{(\phi)} := \Box\phi + f'(\phi)R - V'(\phi), \tag{B.18}$$

$$\mathcal{F}_{(\phi)} := \varepsilon\left(2f'(\phi)K - n^\mu\nabla_\mu\phi\right). \tag{B.19}$$

B.3. Variation with respect to $g^{\mu\nu}$

Next let us consider variation with respect to $g^{\mu\nu}$. Using integration by parts and the Stokes' theorem (2.5), variation of the bulk action (B.2) leads

$$\begin{aligned}\delta_g I_{\mathcal{M}} &= \int_{\mathcal{M}} \mathrm{d}^n x\sqrt{-g}\left(\mathcal{E}_{\mu\nu} - \frac{1}{2}T_{\mu\nu}\right)\delta g^{\mu\nu} + \varepsilon\int_{\partial\mathcal{M}} \mathrm{d}^{n-1}y\sqrt{|h|}n_\rho J^\rho \\ &= \int_{\mathcal{M}} \mathrm{d}^n x\sqrt{-g}\left(\mathcal{E}_{\mu\nu} - \frac{1}{2}T_{\mu\nu}\right)\delta g^{\mu\nu} \\ &\quad + \varepsilon\int_{\partial\mathcal{M}} \mathrm{d}^{n-1}y\sqrt{|h|}\bigg\{f(\phi)n^\sigma h^{\mu\nu}\Big(\nabla_\mu(\delta g_{\sigma\nu}) - \nabla_\sigma(\delta g_{\mu\nu})\Big) \\ &\quad - n^\sigma(\nabla_\mu f(\phi))g^{\mu\nu}\delta g_{\sigma\nu} + n^\sigma(\nabla_\sigma f(\phi))g^{\mu\nu}\delta g_{\mu\nu}\bigg\},\end{aligned} \tag{B.20}$$

where J^ρ is defined by equation (B.11) and

$$\begin{aligned}\mathcal{E}_{\mu\nu} :&= f(\phi)G_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\left(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right) \\ &\quad - \frac{1}{2}(\nabla_\mu\phi)(\nabla_\nu\phi) - \nabla_\mu\nabla_\nu f(\phi) + g_{\mu\nu}\Box f(\phi),\end{aligned} \tag{B.21}$$

with

$$T_{\mu\nu} := -2\frac{\partial\mathcal{L}_{\mathcal{M}}^{(m)}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\mathcal{M}}^{(m)}. \tag{B.22}$$

On the other hand, variation of the boundary term (B.3) leads

$$\begin{aligned}\delta_g I_{\partial\mathcal{M}} &= \varepsilon\int_{\partial\mathcal{M}} \mathrm{d}^{n-1}y\sqrt{|h|}\bigg[f(\phi)\Big\{(Kh^{\alpha\beta} - K^{\alpha\beta})\delta g_{\alpha\beta} - n^\beta h^{\alpha\mu}(\nabla_\mu\delta g_{\alpha\beta} - \nabla_\beta\delta g_{\alpha\mu})\Big\} \\ &\quad + (\nabla_\rho f(\phi))h^{\rho\alpha}n^\beta\delta g_{\alpha\beta} - h^\rho_\mu\nabla_\rho(f(\phi)h^{\mu\alpha}n^\beta\delta g_{\alpha\beta})\bigg].\end{aligned} \tag{B.23}$$

The last term in the above integrand is a total derivative term on $\partial\mathcal{M}$ and becomes a surface integral at $\partial\partial\mathcal{M}$, namely the boundary of $\partial\mathcal{M}$ because for a given vector v^μ, we have

$$h_\mu^{\ \rho}\nabla_\rho v^\mu = h^{\mu\rho}\nabla_\rho v_\mu = e_a^\mu e_b^\rho h^{ab}\nabla_\rho v_\mu = h^{ab}D_b v_a = D_a v^a. \tag{B.24}$$

Hence, we have

$$\begin{aligned}\delta_g(I_{\mathcal{M}}+I_{\partial\mathcal{M}}) = &\int_{\mathcal{M}} \mathrm{d}^n x\sqrt{-g}\left(\mathcal{E}_{\mu\nu}-\frac{1}{2}T_{\mu\nu}\right)\delta g^{\mu\nu}\\ &-\varepsilon\int_{\partial\mathcal{M}}\mathrm{d}^{n-1}y\sqrt{|h|}\left\{f(\phi)(Kh_{\mu\nu}-K_{\mu\nu})+n^\sigma(\nabla_\sigma f(\phi))h_{\mu\nu}\right\}\delta g^{\mu\nu}\\ &+(\text{surface integral at }\partial\partial\mathcal{M}),\end{aligned} \tag{B.25}$$

where we used equation (B.4).

Now we have shown that variation of the total action (B.1) with respect to $g^{\mu\nu}$ reduces to the following form:

$$\begin{aligned}\delta_g I_0 = \int_{\mathcal{M}}\mathrm{d}^n x\sqrt{-g}\left(\mathcal{E}_{\mu\nu}-\frac{1}{2}T_{\mu\nu}\right)\delta g^{\mu\nu} - \int_{\partial\mathcal{M}}\mathrm{d}^{n-1}y\sqrt{|h|}\mathcal{F}_{\mu\nu}\delta g^{\mu\nu}\\ +(\text{Surface integral at }\partial\partial\mathcal{M}),\end{aligned} \tag{B.26}$$

where

$$\mathcal{F}_{\mu\nu} := \varepsilon f(\phi)(Kh_{\mu\nu}-K_{\mu\nu})+\varepsilon n^\sigma(\nabla_\sigma f(\phi))h_{\mu\nu}. \tag{B.27}$$

B.4. Derivation of the junction conditions

Assume that the spacetime $\mathcal{M}$ consists of two parts $\mathcal{M}_+$ and $\mathcal{M}_-$. In a situation where $\mathcal{M}_+$ and $\mathcal{M}_-$ are connected at a non-null hypersurface Σ as described in figure 3, we propose the following action

$$\begin{aligned}I_{\mathrm{J}} = &\int_{\mathcal{M}_+}\mathrm{d}^n x_+\sqrt{-g^+}\left(f(\phi^+)R^+ - \frac{1}{2}(\nabla\phi^+)^2 - V(\phi^+) + \mathcal{L}^{(m)}_{\mathcal{M}_+}\right)\\ &+\int_{\mathcal{M}_-}\mathrm{d}^n x_-\sqrt{-g^-}\left(f(\phi^-)R^- - \frac{1}{2}(\nabla\phi^-)^2 - V(\phi^-) + \mathcal{L}^{(m)}_{\mathcal{M}_-}\right)\\ &+2\epsilon_+\int_{\partial\mathcal{M}_+-\Sigma_+}\mathrm{d}^{n-1}z_+\sqrt{|\zeta^+|}f(\phi^+)K^+ + 2\epsilon_-\int_{\partial\mathcal{M}_--\Sigma_-}\mathrm{d}^{n-1}z_-\sqrt{|\zeta^-|}f(\phi^-)K^-\\ &+2\varepsilon\int_{\Sigma_+}\mathrm{d}^{n-1}y\sqrt{|h|}f(\phi)K^+ + 2\varepsilon\int_{\Sigma_-}\mathrm{d}^{n-1}y\sqrt{|h|}f(\phi)K^- + \int_{\Sigma}\mathrm{d}^{n-1}y\sqrt{|h|}\mathcal{L}^{(m)}_{\Sigma},\end{aligned} \tag{B.28}$$

where $\Sigma_\pm$ are the sides of Σ with normal vectors $n^\mu_\pm$ pointing outward, so that the boundary of each $\mathcal{M}_\pm$ is $\partial\mathcal{M}_\pm = (\partial\mathcal{M}_\pm - \Sigma_\pm)\cup\Sigma_\pm$, and ϵ_+, ϵ_-, ε independently take their values ± 1 and $\phi^\pm|_\Sigma = \phi$. Here we used $z^i_\pm$ and $\zeta^\pm_{ij}$ for the coordinates and induced metric on the boundaries $\partial\mathcal{M}_\pm - \Sigma_\pm$, respectively.

From the results obtained in the previous subsections, variation of the action (B.28) with the boundary conditions $\delta g^{\pm\mu\nu} = \delta\phi^\pm = 0$ at $\partial\mathcal{M}_\pm - \Sigma_\pm$ leads

$$\begin{aligned}\delta_g I_\mathrm{J} = &\int_{\mathcal{M}_+} \mathrm{d}^n x_+ \sqrt{-g^+}\left(\mathcal{E}^+_{\mu\nu} - \frac{1}{2}T^+_{\mu\nu}\right)\delta g^{+\mu\nu} + \int_{\mathcal{M}_-} \mathrm{d}^n x_- \sqrt{-g^-}\left(\mathcal{E}^-_{\mu\nu} - \frac{1}{2}T^-_{\mu\nu}\right)\delta g^{-\mu\nu} \\ &- \int_{\Sigma_+} \mathrm{d}^{n-1}y\sqrt{|h|}\mathcal{F}^+_{\mu\nu}\delta g^{\mu\nu} - \int_{\Sigma_-} \mathrm{d}^{n-1}y\sqrt{|h|}\mathcal{F}^-_{\mu\nu}\delta g^{\mu\nu} + \int_{\Sigma} \mathrm{d}^{n-1}y\sqrt{|h|}\left(-\frac{1}{2}t_{\mu\nu}\right)\delta g^{\mu\nu} \\ &+ (\text{Surface integral at } \partial\partial\mathcal{M}^\pm),\end{aligned} \tag{B.29}$$

where

$$t_{\mu\nu} := -2\frac{\partial \mathcal{L}^{(m)}_\Sigma}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}^{(m)}_\Sigma, \tag{B.30}$$

and

$$\begin{aligned}\delta_\phi I_\mathrm{J} = &\int_{\mathcal{M}_+} \mathrm{d}^n x_+ \sqrt{-g^+}\mathcal{E}^+_{(\phi)}\delta\phi^+ + \int_{\mathcal{M}_-} \mathrm{d}^n x_- \sqrt{-g^-}\mathcal{E}^-_{(\phi)}\delta\phi^- \\ &+ \int_{\Sigma_+} \mathrm{d}^{n-1}y\sqrt{|h|}\mathcal{F}^+_{(\phi)}\delta\phi + \int_{\Sigma_-} \mathrm{d}^{n-1}y\sqrt{|h|}\mathcal{F}^-_{(\phi)}\delta\phi.\end{aligned} \tag{B.31}$$

Choosing the normal vector n^μ to Σ such that it points from M_- to M_+, we have $n^\mu_- = n^\mu = -n^\mu_+$ and hence $K^+(n^\mu_+) = -K^+(n^\mu)$ and $K^-(n^\mu_-) = K^-(n^\mu)$. They show $\mathcal{F}^+_{\mu\nu}(n^\mu_+) = -\mathcal{F}^+_{\mu\nu}(n^\mu)$, $\mathcal{F}^-_{\mu\nu}(n^\mu_-) = \mathcal{F}^-_{\mu\nu}(n^\mu)$, $\mathcal{F}^+_{(\phi)}(n^\mu_+) = -\mathcal{F}^+_{(\phi)}(n^\mu)$, and $\mathcal{F}^-_{(\phi)}(n^\mu_+) = \mathcal{F}^-_{(\phi)}(n^\mu)$, and consequently we have

$$\int_{\Sigma_+} \mathrm{d}^{n-1}y\sqrt{|h|}\mathcal{F}^+_{\mu\nu}\delta g^{\mu\nu} = -\int_{\Sigma} \mathrm{d}^{n-1}y\sqrt{|h|}\mathcal{F}^+_{\mu\nu}\delta g^{\mu\nu}, \tag{B.32}$$

$$\int_{\Sigma_-} \mathrm{d}^{n-1}y\sqrt{|h|}\mathcal{F}^-_{\mu\nu}\delta g^{\mu\nu} = \int_{\Sigma} \mathrm{d}^{n-1}y\sqrt{|h|}\mathcal{F}^-_{\mu\nu}\delta g^{\mu\nu}, \tag{B.33}$$

$$\int_{\Sigma_+} \mathrm{d}^{n-1}y\sqrt{|h|}\mathcal{F}^+_{(\phi)}\delta\phi = -\int_{\Sigma} \mathrm{d}^{n-1}y\sqrt{|h|}\mathcal{F}^+_{(\phi)}\delta\phi, \tag{B.34}$$

$$\int_{\Sigma_-} \mathrm{d}^{n-1}y\sqrt{|h|}\mathcal{F}^-_{(\phi)}\delta\phi = \int_{\Sigma} \mathrm{d}^{n-1}y\sqrt{|h|}\mathcal{F}^-_{(\phi)}\delta\phi. \tag{B.35}$$

Therefore, equations (B.29) and (B.31) finally reduce to

$$\begin{aligned}\delta_g I_\mathrm{J} = &\int_{\mathcal{M}_+} \mathrm{d}^n x_+ \sqrt{-g^+}\left(\mathcal{E}^+_{\mu\nu} - \frac{1}{2}T^+_{\mu\nu}\right)\delta g^{+\mu\nu} + \int_{\mathcal{M}_-} \mathrm{d}^n x_- \sqrt{-g^-}\left(\mathcal{E}^-_{\mu\nu} - \frac{1}{2}T^-_{\mu\nu}\right)\delta g^{-\mu\nu} \\ &+ \int_{\Sigma} \mathrm{d}^{n-1}y\sqrt{|h|}\left(\mathcal{F}^+_{\mu\nu} - \mathcal{F}^-_{\mu\nu} - \frac{1}{2}t_{\mu\nu}\right)\delta g^{\mu\nu} + (\text{Surface integral at } \partial\partial\mathcal{M}^\pm),\end{aligned} \tag{B.36}$$

and

$$\delta_\phi I_{\mathrm{J}} = \int_{\mathcal{M}_+} \mathrm{d}^n x_+ \sqrt{-g^+}\mathcal{E}^+_{(\phi)}\delta\phi^+ + \int_{\mathcal{M}_-} \mathrm{d}^n x_- \sqrt{-g^-}\mathcal{E}^-_{(\phi)}\delta\phi^- - \int_\Sigma \mathrm{d}^{n-1}y\sqrt{|h|}(\mathcal{F}^+_{(\phi)} - \mathcal{F}^-_{(\phi)})\delta\phi. \tag{B.37}$$

Hence, by the variational principle, we obtain the Einstein equations $\mathcal{E}^\pm_{\mu\nu} = (1/2)T^\pm_{\mu\nu}$ and the equation of motion for a scalar field $\mathcal{E}^\pm_{(\phi)} = 0$ in the bulk spacetimes $\mathcal{M}_\pm$ as well as the junction conditions $[\mathcal{F}_{\mu\nu}] = (1/2)t_{\mu\nu}$ and $[\mathcal{F}_{(\phi)}] = 0$ on Σ. The bulk field equations are in the following form:

$$2f(\phi)G_{\mu\nu} + g_{\mu\nu}\left(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right) - (\nabla_\mu\phi)(\nabla_\nu\phi) - 2\nabla_\mu\nabla_\nu f(\phi) + 2g_{\mu\nu}\Box f(\phi) = T_{\mu\nu}, \tag{B.38}$$

$$\Box\phi + f'(\phi)R - V'(\phi) = 0, \tag{B.39}$$

where we have omitted $\pm$ sign for simplicity, while the junction conditions at Σ are written as

$$2\varepsilon f(\phi)\left([K]h_{\mu\nu} - [K_{\mu\nu}]\right) + 2\varepsilon f'(\phi)n^\sigma[\nabla_\sigma\phi]h_{\mu\nu} = t_{\mu\nu}, \tag{B.40}$$

$$2f'(\phi)[K] - n^\mu[\nabla_\mu\phi] = 0. \tag{B.41}$$

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