IOP Publishing Physica Scripta
Phys. Scr. 95 (2020) 055219 (13pp) https://doi.org/10.1088/1402-4896/ab6ba2

Approximate analytical solution of the linear and nonlinear multi-pantograph delay differential equations

Mohamed S M Bahgat

Mathematics Department, Faculty of Science, Minia University, El-Minia, Egypt

E-mail: msmbahgat66@hotmail.com

Received 17 November 2019, revised 3 January 2020
Accepted for publication 14 January 2020
Published 5 March 2020

Abstract

In this paper, a combination of Laplace transform and variational iteration method are applied to get an approximate analytic solution for the multi-pantograph delay with higher order differential equations. Lagrange multiplier technique is constructed a correction functional which obtained by using Laplace transform with the variational theory. Numerical studies for the application of the present method for the considered problems are given and graphically illustrated, our proposed method is compared favorably with other methods. The simplicity and efficiency of the method.

Keywords: Pantograph delay differential equation, Laplace transform, Adomian decomposition method

(Some figures may appear in colour only in the online journal)

1. Introduction

The pantograph equation is a special type of functional differential equations with proportional delay. It arises in rather different fields of pure and applied mathematics, such as electrodynamics, control systems, number theory, probability, and quantum mechanics. Many researchers have studied the pantograph-type delay differential equation using analytical and numerical techniques.

1.1. Delay differential equations (DDEs)

Delay differential equations (DDEs) are a large and important class of dynamical systems. They often arise in either natural or technological control problems. In these systems, a controller monitors the state of the system, and makes adjustments to the system based on its observations. Since these adjustments can never be made instantaneously, a delay arises between the observation and the control action. On the other hand, many complicated physical problems described in terms of partial differential equations can be approximated by much simpler problems described in terms of DDEs. Mathematical modeling with DDEs is widely used for analysis and predictions in various areas of life sciences, for example, population dynamics, epidemiology, immunology, physiology, and neural networks. There is no doubt that some of the recent developments in the theory of DDEs have enhanced our understanding of the qualitative behavior of their solutions and have many applications in mathematical biology and other related fields. Both theory and applications of DDEs require a bit more mathematical maturity than their ODEs counterparts. The mathematical description of delay dynamical systems will naturally involve the delay parameter in some specified way. Nonlinearity and sensitivity analysis of DDEs have been studied intensely in recent years in diverse areas of science and technology, particularly in the context of chaotic dynamics, where the first investigators of modern times to study the DDEs:

$$y'(t) = f(t, y(t), y(t-\tau))$$

and its effect on simple feed-back control systems in which the communication time cannot be neglected. DDE is defined as an unknown function $y(t)$ and some of its derivatives, evaluated at arguments that differ by any of fixed number of values $\tau_1, \tau_2, \ldots, \tau_k$. The general form of the nth order DDE is given by

$$\begin{aligned} F = F(t, y(t), y(t-\tau_1), \ldots, y(t-\tau_k),\\ y'(t), y'(t-\tau_1), \ldots y'(t-\tau_k), \ldots,\\ y^{(n)}(t), y^{(n)}(t-\tau_1), \ldots, y^{(n)}(t-\tau_k)) = 0, \end{aligned}$$

where F is a given functional and $\tau_i, \forall\, i = 1, 2, \ldots, k$; are given fixed positive number called the 'time delay'. The emphasis will be, in general, on the linear equations with constant coefficients of the first order and with one delay (because as in ordinary differential equation (ODE) any differential equation with higher order than one may be transformed into a linear system of differential equations of the first order)

$$a_0 y'(t) + a_1 y'(t-\tau) + b_0 y(t) + b_1 y(t-\tau) = f(t),$$

where $f(t)$ is a given continuous function and τ is a positive constant and a_0, a_1, b_0, and b_1 are constants (if $f(t) = 0$, then this equation is said to be homogenous; otherwise it is non-homogenous). The kind of initial conditions that should be used in DDE's differ from ODE's so that one should specify in DDE's an initial function on some interval of length τ, say $[t_0 - \tau, t_0]$ and then try to find the solution of the previous equation for all $t \geqslant t_0$. Thus, we set $y(t) = \phi_0(t)$, for $t_0 - \tau \leqslant t \leqslant t_0$ where $\phi_0(t)$ is some given continuous function. Therefore the solution of DDE consist of finding a continuous extension of $\phi_0(t)$ into a function $y(t)$ which satisfies the equation for all $t \geqslant t_0$. DDE given by the previous equation can be classified into three types which are retarded, neutral and mixed. The first type means an equation where the rate of change of state variable y is determined by the present and past states of the equation where the coefficient of $y'(t-\tau)$ is zero, i.e. ($a_0 \neq 0$, $a_1 = 0$). If the rate of change of state depends on its own past values as well on its derivatives, the equation is of neutral type, the equation where the coefficient of $y(t-\tau)$ is zero, i.e. ($a_0 \neq 0$, $a_1 \neq 0$ and $b_1 = 0$), while the third type is a combination of the previous two types, i.e. ($a_0 \neq 0$, $a_1 \neq 0$, $b_0 \neq 0$ and $b_1 \neq 0$).

1.2. The pantograph equation

A pantograph is a device for collecting an electrical current to power an electric locomotive or electric multiple unit. The system is employed to make contact with an electrified overhead wire (cable). Pantographs come in all shapes and sizes depending on the speed of the locomotive or train set, power requirements, power supply systems etc. The basic parts of a pantograph is a lower arm(s) that pivot against the roof, of a carriage/loco, and is attached to upper arm(s) that is in-turn attached to a collector 'head' or 'pan'. The head is the only part of the pantograph to touch the wire pick-up. The current is collected via metalized carbon strips on the head. In the 1960s, the British Railways wanted to make the electric locomotive faster. An important construct was the pantograph, which collects current from an overhead wire. Many researchers are studied the motion of the pantograph head on an electric locomotive. In the solution procedure of this problem, they came across a special DDE of the form

$$y'(t) = ay(t) + by(\theta t), \quad t > 0.$$

where a, b are real constants and $0 < \theta < 1$, this kind of DDE was called pantograph equation. In the following years, the pantograph equation became a prime example for a DDE. The continuous and discrete cases of the pantograph equation have been well studied over the last several decades, where the continuous and discrete cases denote different examples of time scales T that are arbitrary nonempty closed subsets of real numbers. In the continuous case, which means the time scale of the real numbers $T = \mathbb{R}$, the pantograph equation is presented as a differential equation. In the discrete case, which means the time scale of the integers $T = \mathbb{Z}$, here especially the nonnegative integers $T = \mathbb{N}_0$, the pantograph equation is presented as a difference equation. The present study focuses on the pantograph equation in the quantum case, which is the time scale $T = q\mathbb{N}_0$ for $q > 1$. The general theory of calculus on the time scale $T = q\mathbb{N}_0$ is also called quantum calculus, in fact, the mathematical model of this system includes modeling the motion of the wire connected with the dynamics of the supports and modeling the dynamics of the pantograph, the system where the pantograph is collecting current from the overhead wire is modeled to determine the motion of the pantograph head. First, the overhead trolley wire is modeled. Second, the model of the pantograph is considered. Lastly, a simplified formulation of the whole problem is derived. The pantograph equations is a kind of DDEs and arise in many applications such as, cell growth, probability theory of algebraic structures, astrophysics, nonlinear dynamical systems, etc. In recent years, the multi-pantograph equations were studied by many authors numerically and analytically. For examples Li and Liu [1] applied the Runge–Kutta methods to multi-pantograph equation. Moitsheki and Makinde [2] used an application of Lie point symmetry and Adomain decomposition methods to thermal storage diffusion equations. A numerical method based on the Adomian decomposition method (ADM) which has been used from the 1970s to 1990s by George Adomian [3, 4]. The decomposition procedure of Adomian is based on the search for a solution in the form of a series with easily computed components was proposed by [5]. Since non-perturbative techniques for thermal radiation effect on natural convection past a vertical plate embedded in a saturated porous medium was proposed and developed [6] and the homotopy perturbation method was studied by [7, 8], this method has been successfully applied to solve many types of nonlinear problem [9–11]. Mirzaee and Hoseini [12] used collocation technique and matrices of Fibonacci polynomials to explain differential difference equation with positive and negative shifts. Stochastic and deterministic numerical solver has been implemented broadly in varied fields, for example nanotechnology [13], doubly singular nonlinear systems [14] and multi-point boundary value problem [15]. A new spectral Jacobi rational-Gauss collocation (JRC) method is proposed for solving the multipantograph DDEs on the half-line. The method is based on Jacobi rational functions and Gauss quadrature integration formula [16]. The Lagrange multiplier technique [17] was widely used to solve a number of nonlinear problems and it was developed into a powerful analytical method. The variational iteration method (VIM) was developed by [18–20] for solving a wide range of nonliear problems, it has been successfully applied on initial and boundary value problem. The application of VIM to differential equations usually follows the following three steps: (a) Obtaining the correction functional. (b) Identifying the Lagrange multiplier, which is determined by a simplification not reasonably explained in the literature, various authors such as, [21, 22] have identified this Lagrange multiplier via different approaches to accelerate the convergence rate of solutions. (c) Determining the initial iteration.

In this work, we propose a method based on Laplace variational iteration method (LVIM) to numerically solve the generalized pantograph equations

$$y^{(n)}(t) = \alpha(t)y(t) + \sum_{i=0}^{n-1}\beta_i(t)y^{(i)}(\phi_i(t)) + g(t), \quad 0 < t \leqslant T \tag{1}$$

with initial conditions

$$y^{(i)}(0) = \lambda_i, \quad \lambda_i \in \mathbb{R}, \quad (i = 0, 1, \cdots, n-1), \tag{2}$$

where $\alpha(t)$, $\beta_i(t)$ and $g(t)$ are analytical functions, $\phi_i \in (0, 1)$, $i = 1, 2, \cdots, n-1$. The basic motivation of this work is to apply the LVIM for solving multi-pantograph DDEs which are otherwise difficult to analyze because of their complex nature and infinite dimensionality. This study is presented as follows: In second section, we start by presenting LVIM to solve multi-pantograph DDEs. In third section, this method is shown and compared by ten examples by taking various values of t for each value of $y_n(t)$, $n = 0, 1, 2, 3\cdots$. Also , we have plotted the graphs for numerical solutions of LVIM and exact solution. Finally, we give some concluding remarks in fourth section.

2. Construction of iterative formula by LVIM

First, let us take the general nonlinear differential equation to illustrate the main idea of VIM for DDEs [22],

$$y^{(k)} + R[y] + N[y] = g(t), \tag{3}$$

with the following initial conditions

$$y^{(l)}(0) = y_0^{(l)} = \lambda_l, \quad \lambda_l \in \mathbb{R}, \quad l = 0(1)(k-1), \tag{4}$$

in which $y^{(k)} = \frac{d^k y}{dt^k}$, R is a linear operator, N is a nonlinear operator, $g(t)$ is a known continuous function, and $y^{(k)}$ is kth order derivative. The basic idea of the method VIM is construct the following correction functional for equation (3)

$$y_{n+1} = y_n + \int_0^t \mu(t, \rho)(y^{(k)} + R[y_n] + N[y_n] - g(\rho))\mathrm{d}\rho, \tag{5}$$

where $\mu(t, \rho)$ is called the general Lagrange multiplier which can be identified optimally via variation theorem and y_n, $n \geqslant 0$ is the nth order approximate solution of the exact solution $y(t)$ which will readily obtained upon using a good initial approximation $y_0(t)$, which obtained from the initial conditions (4). So when $n \to \infty$ approximate solution y_n converges to the exact solution $y(t)$. Let us give the original opinion of the Lagrange multiplier which play an important role in this work, so, the entire procedure of Lagrange multipliers is expressed as a case of algebraic equation where its solution $f(x) = 0$ could be found by

$$x_{n+1} = x_n + \mu f(x_n). \tag{6}$$

Optimality condition for the extreme $\frac{\delta x_{n+1}}{\delta x_n} = 0$ leads to

$$\mu = -\frac{1}{f'(x_n)}, \tag{7}$$

where δ is the classical variational operator. Implementing the initial point x_0 provided, the approximate solution x_{n+1} could be determined via next iterative scheme, with (6) and (7)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \quad n = 0, 1, 2, \cdots \tag{8}$$

The algorithm (8) is well known as the famous Newton–Raphson formula that possesses a quadratic convergence. In this paper, we extend this the idea to finding the unknown Lagrange multiplier. Key step is the application of Laplace transform into (3) as follows

$$s^k Y(s) - y^{(k-1)}(0) - s^{k-1}y(0) + \mathscr{L}\{R[y]\} + \mathscr{L}\{N[y]\} - \mathscr{L}\{[g(t)]\} = 0, \tag{9}$$

where $Y(s) = \mathscr{L}\{y(t)\} = \int_0^\infty \mathrm{e}^{-st}y(t)\mathrm{d}t$, with the notation used $\mathscr{L}$ to indicate Laplace transform. Hence, the algorithm of LVIM is given below:

(1) Applying the Laplace transform to (3) gives the correction functional as

$$Y_{n+1}(s) = Y_n(s) + \mu(s)(s^k Y_n(s) - y^{k-1}(0) - \cdots - s^{k-1}y(0) + \mathscr{L}\{R[y_n] + N[y_n] - g(t)\}). \tag{10}$$

(2) Considering the terms $\mathscr{L}\{R[y_n] + N[y_n]\}$ as restricted variations, we let (10) be stationary with respect to Y_n

$$\delta Y_{n+1}(s) = \delta Y_n(s) + \mu(s)(s^k \delta Y_n(s)). \tag{11}$$

From (11), we derive Lagrange multiplier as

$$\mu(s) = -\frac{1}{s^k}. \tag{12}$$

(3) Succeeding approximations can then be attained with the application of inverse Laplace transform $\mathscr{L}^{-1}$ into

(10), which gives

$$
\begin{aligned}
y_{n+1}(t) = y_n(t) - \mathscr{L}^{-1}\bigg(&\frac{1}{s^k}(s^k Y_n(s) - y^{(k-1)}(0) \\
&- \cdots - s^{k-1}y(0) + \mathscr{L}\{R[y_n] + N[y_n] - g(t)\})\bigg) \\
= \mathscr{L}^{-1}\bigg(&\frac{y(0)}{s} + \cdots + \frac{y^{(k-1)}(0)}{s^k} \\
&- \frac{1}{s^k}\mathscr{L}\{R[y_n] + N[y_n] - g(t)\}\bigg),
\end{aligned} \tag{13}
$$

with initial approximation $y_0(t)$ can be determined by

$$
\begin{aligned}
y_0(t) &= \mathscr{L}^{-1}\left(\frac{y(0)}{s} + \cdots + \frac{y^{(k-1)}(0)}{s^k}\right) \\
&= y(0) + y'(0)t + \cdots + \frac{y^{(k-1)}(0)t^{k-1}}{(k-1)!},
\end{aligned} \tag{14}
$$

Equation (14) shows that the first iteration in the classical VIM is made up by the Taylor series.

3. Illustrative examples

Present section represents the numerical examples which given to show the convergence and accuracy of LVIM, which applied to ten pantograph type DDEs. For comparison purposes, the solution intervals of problems are chosen generally the same as those in the references. The examples are computed using Maple 18. Results obtained by the presented method are compared with the exact solution of each example and found to be good agreement with each other.

3.1. Linear and nonlinear multi-pantograph delay with first order order differential equations

Example 1. Consider the following linear multi-pantograph delay equation of the first order [23]

$$
\begin{aligned}
&y'(t) = -\frac{5}{6}y(t) + 4y\left(\frac{t}{2}\right) + 9y\left(\frac{t}{3}\right) + t^2 - 1, \\
&y(0) = 1, \quad 0 < t \leqslant 1,
\end{aligned} \tag{15}
$$

for which the exact solution is

$$
y(t) = 1 + \frac{67}{6}t + \frac{1675}{72}t^2 + \frac{12157}{1296}t^3,
$$

Equation (15) can be written in the following form:

$$
y'(t) + \frac{5}{6}y(t) - 4y\left(\frac{t}{2}\right) - 9y\left(\frac{t}{3}\right) - t^2 + 1 = 0.
$$

Taking the $\mathscr{L}$, we obtain

$$
\begin{aligned}
sY(s) - y(0) - \mathscr{L}\bigg(&-\frac{5}{6}y(t) + 4y\left(\frac{t}{2}\right) \\
&+9y\left(\frac{t}{3}\right) + t^2 - 1\bigg) = 0.
\end{aligned}
$$

The iteration formula thus is

$$
\begin{aligned}
Y_{n+1}(s) = Y_n(s) + \mu(s)\bigg[&sY_n(s) - y(0) \\
&- \mathscr{L}\left(-\frac{5}{6}y_n(t) + 4y_n\left(\frac{t}{2}\right) + 9y_n\left(\frac{t}{3}\right) + t^2 - 1\right)\bigg].
\end{aligned}
$$

With equation (12), and applying inverse Laplace transform, the above iteration formula can be explicitly given as

$$
\begin{aligned}
y_{n+1}(t) = y_n(t) - \mathscr{L}^{-1}\bigg\{&\frac{1}{s}(sY_n(s) - y(0) \\
&- \mathscr{L}\left(-\frac{5}{6}y_n(t) + 4y_n\left(\frac{t}{2}\right) + 9y_n\left(\frac{t}{3}\right) + t^2 - 1\right)\bigg)\bigg\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&y_{n+1}(t) = 1 + \mathscr{L}^{-1} \\
&\left\{\frac{1}{s}\left(\mathscr{L}\left(-\frac{5}{6}y_n(t) + 4y_n\left(\frac{t}{2}\right) + 9y_n\left(\frac{t}{3}\right) + t^2 - 1\right)\right)\right\},
\end{aligned} \tag{16}
$$

with the initial iteration $y_0(t) = 1$ and applying the iteration formula (16), we attain

$$
y_1(t) = 1 + 1/6\, t(2\,t^2 + 67)
$$

$$
y_2(t) = 1 + \frac{t(24\,t^2 + 1675\,t + 804)}{72}
$$

the closed form of above solution is

$$
y_3(t) = 1 + \frac{67}{6}t + \frac{1675}{72}t^2 + \frac{12157}{1296}t^3.
$$

$$
\vdots
$$

which is exactly the same as the exact solution. It is clear from figure 1 that the first three results not only give rapidly convergent series but also an accurate calculation of the solutions. For $n \to \infty$, then $y_n(t)$ tends to the exact solution $y(t)$.

Example 2. Consider the following linear multi-pantograph delay equation of the first order [23]

$$
\begin{aligned}
&y'(t) = -3y(t) + 0.4y(0.4t) + y(0.1t), \\
&y(0) = 2.127, \quad 0 < t \leqslant 1.
\end{aligned}
$$

By applying the procedures of the example 2, according to equations (10)–(14) we obtain

$$
\begin{aligned}
&y_{n+1}(t) = 2.127 + \mathscr{L}^{-1} \\
&\times \left\{\frac{1}{s}(\mathscr{L}(-3y_n(t) + 0.4y_n(0.4t) + y_n(0.1t))\right\},
\end{aligned} \tag{17}
$$

with the intial iteration $y_0(t) = y(0) = 2.127$ and applying the iteration formula (17), we attain

$$
y_1(t) = 2.127 - 3.403\,200\,000\,t,
$$

$$
y_2(t) = 2.127 - 3.403\,200\,000\,t + 4.662\,384\,000\,t^2,
$$

$$
\begin{aligned}
y_3(t) = 2.127 &- 4.547\,378\,527\,t^3 \\
&+ 4.662\,384\,000\,t^2 - 3.403\,200\,000\,t,
\end{aligned}
$$

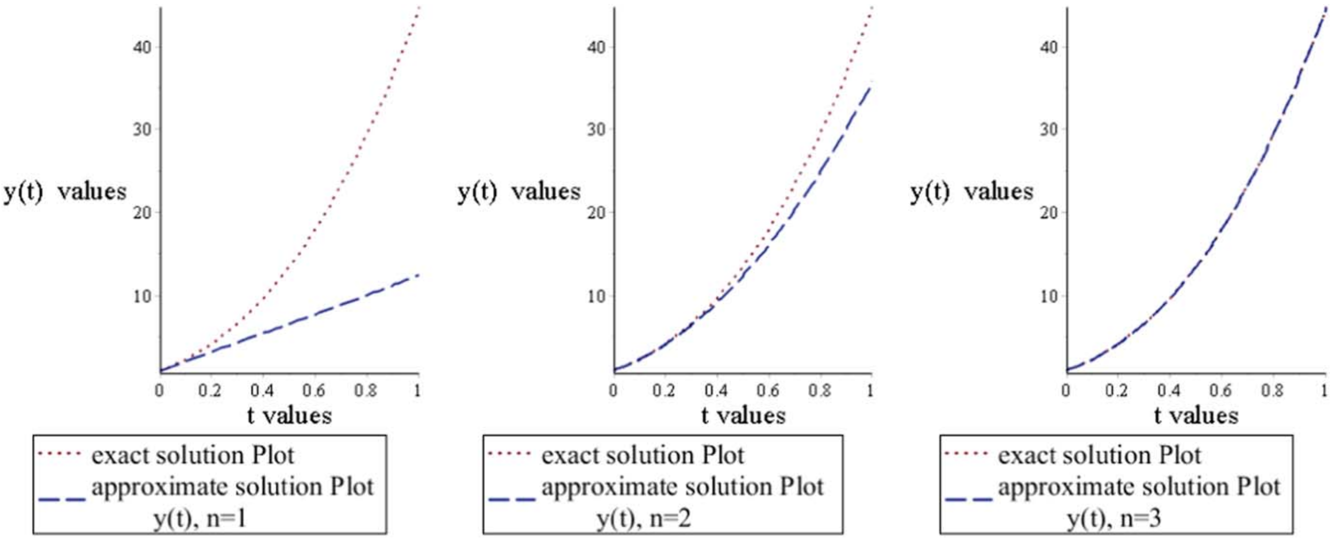

Figure 1. Comparison of exact solution and approximate solutions of example 1 for $n = 1, 2, 3$.

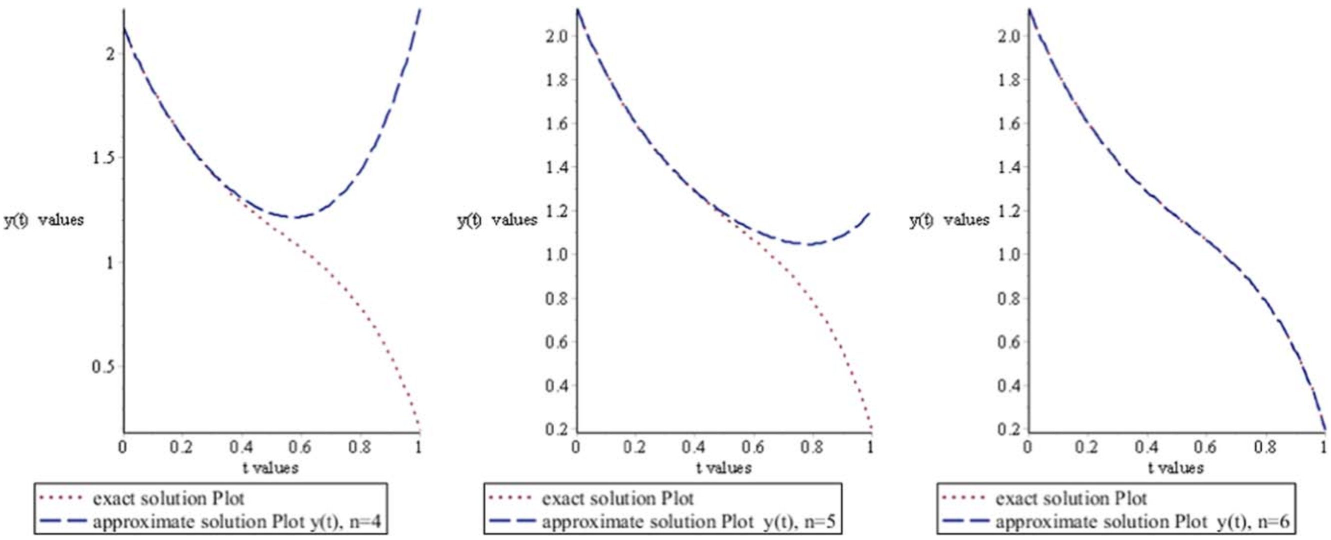

Figure 2. Comparison of exact solution and approximate solutions of example 2 for $n = 4, 5, 6$.

$$y_4(t) = 2.127 + 3.380\,293\,827\,t^4 - 4.547\,378\,526\,t^3 + 4.662\,384\,000\,t^2 - 3.403\,200\,000\,t,$$

$$y_5(t) = 2.127 - 2.021\,185\,848\,t^5 + 3.380\,293\,827\,t^4 - 4.547\,378\,526\,t^3 + 4.662\,384\,000\,t^2 - 3.403\,200\,000\,t$$

$$\vdots$$

the obtained our results for example 2 is compared with the results obtained by ADM and differetial transform method (DTM) in [23] which take the form

$$y(t) = 2.127 - 3.403\,2t + 4.662\,384t^2 - 4.547\,378\,528t^3 + 3.380\,293\,829t^4 - 2.021\,185\,85t^5 + \cdots$$

In figure 2 our results are exactly the same with above series solution in [23] at $n = 6$

Example 3. Consider the following linear multi-pantograph delay equation of the first order [24]

$$y'(t) = -y(t) + b_1(t)y\left(\frac{t}{2}\right) + b_2(t)y\left(\frac{t}{4}\right),$$
$$y(0) = 1, \quad 0 < t \leqslant 1,$$

where $b_1(t) = -\mathrm{e}^{-\frac{1}{2}t}\sin\left(\frac{1}{2}t\right)$, $\quad b_2(t) = -2\mathrm{e}^{-\frac{3}{4}t}\sin\left(\frac{1}{4}t\right)\cos\left(\frac{1}{2}t\right)$.

Following the procedures of example 1, we obtain the iteration formula

$$y_{n+1}(t) = 1 + \mathscr{L}^{-1}\left\{\frac{1}{s}\left(\mathscr{L}\left(-y_n(t) + b_1(t)y_n\left(\frac{t}{2}\right) + b_2(t)y_n\left(\frac{t}{4}\right)\right)\right)\right\}, \tag{18}$$

when applying the iteration formula (18), with the initial iteration $y_0(t) = 1$, we obtain in figure 3 the iteration solutions

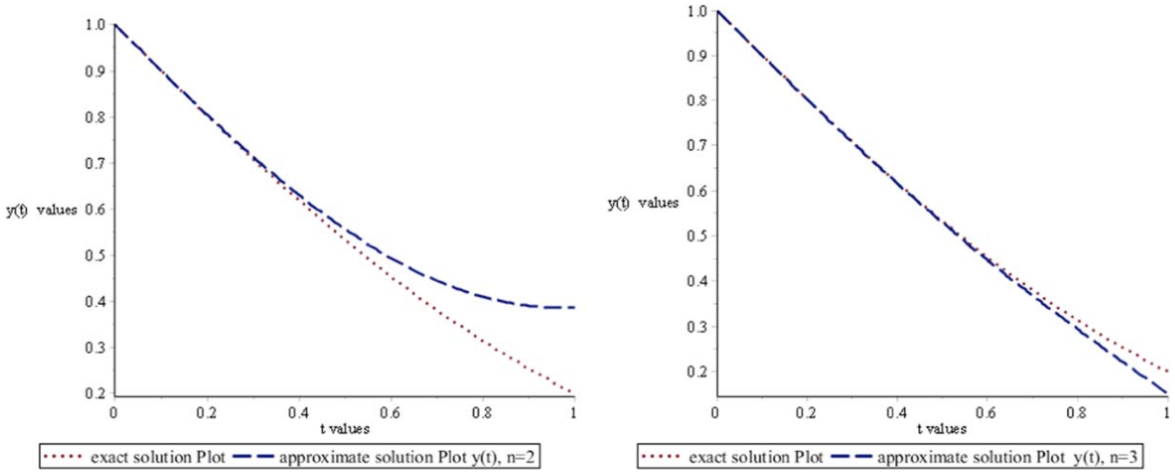

Figure 3. Comparison of exact solution and approximate solutions of example 3 for $n = 2, 3$.

at $n = 2, 3$ of the LVIM converged with the exact solution. Numerical results at $n = 3$ approximate as the exact solution.

Example 4 ([**16**]). Consider the following pantograph delay equation

$$y'(t) = a(t)y(t) + b_1(t)y(\gamma t) + f(t)),$$
$$y(0) = 1, \quad 0 \leqslant t \leqslant 1,$$

where $a(t) = -1, \quad b_1(t) = \frac{\gamma}{2}, \quad f(t) = -\frac{\gamma}{2}e^{\gamma t}$. The exact solution is $y(t) = e^{-t}$. Following the procedures of example 3, the resulting graph of example 4 for the presented method for $\gamma = 0.2, 0.5, 0.8, 0.9$ and the exact solution are shown in figures 4 (a) and (b). Figure 4(c) gives the comparison of the absolute error functions of different values γ obtained by LVIM and the exact solution for example 4. LVIM compares to Jacobi rational-Gauss collocation method [16] with two choices of α and β in cases of $\gamma = 0.2, 0.5, 0.8, 0.9$, we show from the figures 4(a), (b), the LVIM solution is exactly the same as the exact solution for some γ and $n = 5$. Table 1 shows that a comparison of the absolute error obtained by the JRC method [16] and the presented method LVIM. The numerical results are consistent.

Example 5. Nonlinear time-delay model in biology [25]. Our next test problem is

$$y'(t) = 2y(t)\left(1 - \frac{y(t - 0.1)}{0.5}\right), \quad y(0) = 1, \quad 0 < t \leqslant 1.$$

Following the procedures of example 3, we obtain the iteration formula

$$y_{n+1}(t) = 1 + \mathscr{L}^{-1}\left\{\frac{1}{s}\left(\mathscr{L}\left(2y_n(t)(1 - \frac{y_n(t - 0.1)}{0.5)}\right)\right)\right\}, \tag{19}$$

with the intial iteration $y_0(t) = 1$ and applying the iteration formula (19), we attain

$$y_1(t) = 1.0 - 2.0\, t$$

$$y_2(t) = 1.0 - 5.333\,333\,332\, t^3 + 6.799\,999\,998\, t^2 - 2.799\,999\,999\, t$$

$$y_3(t) = 1.0 - 16.253\,968\,24\, t^7 + 54.044\,444\,42\, t^6 - 76.074\,666\,63\, t^5 + 62.780\,444\,42\, t^4 - 35.064\,888\,88\, t^3 + 13.418\,666\,67\, t^2 - 3.413\,333\,332\, t$$

$$\begin{aligned} y_4(t) = 1.0 &- 70.451\,062\,28\, t^{15} + 554.802\,115\,7\, t^{14} \\ &- 2\,028.086\,248\, t^{13} + 4\,608.490\,744\, t^{12} \\ &- 7\,355.933\,871\, t^{11} \\ &+ 8\,822.709\,083\, t^{10} - 8\,283.933\,277\, t^9 \\ &+ 6\,249.156\,433\, t^8 - 3\,855.392\,660\, t^7 \\ &+ 1\,970.805\,774\, t^6 - 842.144\,271\,0\, t^5 \\ &+ 301.913\,823\,8\, t^4 - 90.142\,006\,96\, t^3 \\ &+ 21.830\,415\,46\, t^2 - 4.070\,717\,395\, t \\ &\vdots \end{aligned}$$

The exact solution of example 5 is not known. In [25], Dehghan and Salehi used VIM and ADM to compute approximate solutions of example 5 so, we compared our result with the result polynomial least squares method (PLSM) in [26], which takes the following polynomial,

$$y(t) = -2.578\,41\, t^7 + 15.818\,6\, t^6 - 39.594\,6\, t^5 + 52.059\, t^4 - 38.666\,5\, t^3 + 16.375\,2\, t^2 - 3.898\,39\, t + 1.$$

Figure 5 presents the comparison between our approximate solutions and the numerical solution presented in [26]. LVIM solution is exactly the same as in [26] at $n >= 5$.

3.2. Linear and nonlinear multi-pantograph delay with second order order differential equations

Example 6. Consider the following linear multi-pantograph delay equation of the second order

$$y''(t) = \frac{3}{4}y(t) + y\left(\frac{1}{2}t\right) - \frac{\frac{3}{4}}{t^2 + t + 1} - \frac{4}{t^2 + 2t + 4} + \frac{2(2t + 1)^2}{(t^2 + t + 1)^3} - \frac{2}{(t^2 + t + 1)^2}$$

with initial conditions $y(0) = 1, \quad y'(0) = -1$. The exact solution of the problem is $y(t) = \frac{1}{t^2 + t + 1}$.

Table 1. Comparison of the absolute errors for example 4.

t	JRC method [16] $\alpha=\beta=-\frac{1}{2}$	JRC method [16] $\alpha=\beta=0$ (γ=0.2)	Proposed method	JRC method [16] $\alpha=\beta=-\frac{1}{2}$	JRC method [16] $\alpha=\beta=0$ ($\gamma=0.5$)	Proposed method
0	0.0	0.0	0.0	0.0	0.0	0.0
0.2	1.94×10^{-8}	1.69×10^{-9}	8.08×10^{-6}	2.04×10^{-8}	1.55×10^{-9}	1.03×10^{-7}
0.4	2.63×10^{-8}	9.61×10^{-8}	8.08×10^{-6}	2.76×10^{-8}	9.94×10^{-8}	4.18×10^{-6}
0.6	9.18×10^{-9}	2.04×10^{-8}	2.61×10^{-4}	9.83×10^{-9}	2.22×10^{-8}	4.58×10^{-5}
0.8	2.11×10^{-8}	9.37×10^{-8}	4.29×10^{-4}	2.33×10^{-8}	9.93×10^{-8}	2.49×10^{-4}
1.0	2.20×10^{-8}	2.08×10^{-7}	1.12×10^{-3}	1.67×10^{-8}	2.18×10^{-7}	9.23×10^{-4}
		($\gamma=0.8$)			($\gamma=0.9$)	
0.1	1.33×10^{-8}	6.97×10^{-8}	8.01×10^{-9}	2.81×10^{-9}	6.38×10^{-8}	8.00×10^{-9}
0.3	1.11×10^{-9}	1.22×10^{-9}	6.3×10^{-9}	8.38×10^{-9}	2.65×10^{-9}	9.30×10^{-9}
0.5	1.08×10^{-8}	1.24×10^{-7}	3.07×10^{-7}	5.14×10^{-8}	1.08×10^{-7}	9.70×10^{-9}
0.7	5.51×10^{-8}	1.39×10^{-7}	2.99×10^{-7}	1.10×10^{-7}	1.16×10^{-7}	6.20×10^{-9}
0.9	9.36×10^{-8}	8.85×10^{-8}	2.22×10^{-6}	1.43×10^{-7}	6.37×10^{-8}	1.98×10^{-8}

Taking $\mathscr{L}$, we obtain the iteration formula

$$Y_{n+1}(s) = Y_n(s) + \mu(s)(s^2Y_n(s) - sy(0) - y'(0) - \mathscr{L}\left\{\frac{3}{4}y_n(t) + y_n\left(\frac{1}{2}t\right) - \frac{\frac{3}{4}}{t^2+t+1} - \frac{4}{t^2+2t+4} + \frac{2(2t+1)^2}{(t^2+t+1)^3} - \frac{2}{(t^2+t+1)^2}\right\}) \tag{20}$$

with the Lagrange multiplier

$$\mu = -\frac{1}{s^2}$$

Taking $\mathscr{L}^{-1}$, we obtain

$$y_{n+1}(t) = 1 - t + \mathscr{L}^{-1}\left\{\frac{1}{s^2}\left(\mathscr{L}\left(\frac{3}{4}y_n(t) + y_n\left(\frac{1}{2}t\right) - \frac{\frac{3}{4}}{t^2+t+1} - \frac{4}{t^2+2t+4} + \frac{2(2t+1)^2}{(t^2+t+1)^3} - \frac{2}{(t^2+t+1)^2}\right)\right)\right\}, \tag{21}$$

with the initial iteration $y_0(t) = y(0) + y'(0)t = 1 - t$ and applying the iteration formula (21) to obtain $y_n(t)$, $n = 1,2, \cdots$. Figure 6 shows the graphic of the approximations $y_n(t)$ for $n = 0,1,2$ and a comparison is made between values of the approximate solutions and the exact solution. In table 2, the maximum error is made between values of the approximate solutions $y_2(t)$, $y_3(t)$ and the exact solution. LVIM solution is exactly the same as the exact solution at $n = 2$.

Example 7. Consider the initial value problem of second-order nonlinear differential equation with pantograph delay [27]

$$y''(t) = -y\left(\frac{t}{2}\right) - y^2(t) + \sin^4(t) + \sin^2(t) + 8,$$
$$y(0) = 2, \quad y'(0) = 0, \quad 0 < t \leqslant 1$$

which has the exact solution $y(t) = \frac{5-\cos 2t}{2}$.

Following the procedures of example 6, we obtain the iteration formula

$$y_{n+1}(t) = 2 + \mathscr{L}^{-1}\left\{\frac{1}{s^2}\left(\mathscr{L}\left(y_n\left(\frac{t}{2}\right) - y_n^2(t) + \sin^4(t) + \sin^2(t) + 8\right)\right)\right\}, \tag{22}$$

with the intial iteration $y_0(t) = 2$ and applying the iteration formula (19), we attain

$$y_1(t) = 3/2 - 1/16(\cos(t))^4 + \frac{9(\cos(t))^2}{16} + \frac{23\,t^2}{16}$$

$$y_2(t) = \frac{29431735}{18874368} + 1/4\cos(t) + \frac{\cos(8\,t)}{2097152} - \frac{529\,t^6}{7680} + \frac{130941\,t^2}{65536} - \frac{\cos(6\,t)}{18432} - \frac{5543\,t^4}{12288} + \frac{\cos(2\,t)(368\,t^2+405)}{2048} - \frac{\cos(4\,t)(46\,t^2+231)}{32768} + \frac{(-5888\sin(2\,t) + 23\sin(4\,t))t}{16384}$$

$$\vdots$$

From Figure 7, we also observe the convergence of the approximations to the exact solution.

Example 8. Consider the initial value problem of of the second order nonlinear variable coefficient differential equation with pantograph delay [27]:

$$y''(t) = y(t) - \frac{8}{t^2}y^2\left(\frac{t}{2}\right),$$
$$y(0) = 1, \quad y'(0) = 1, \quad 0 < t \leqslant 3$$

which has the exact solution $y(t) = te^{-t}$.

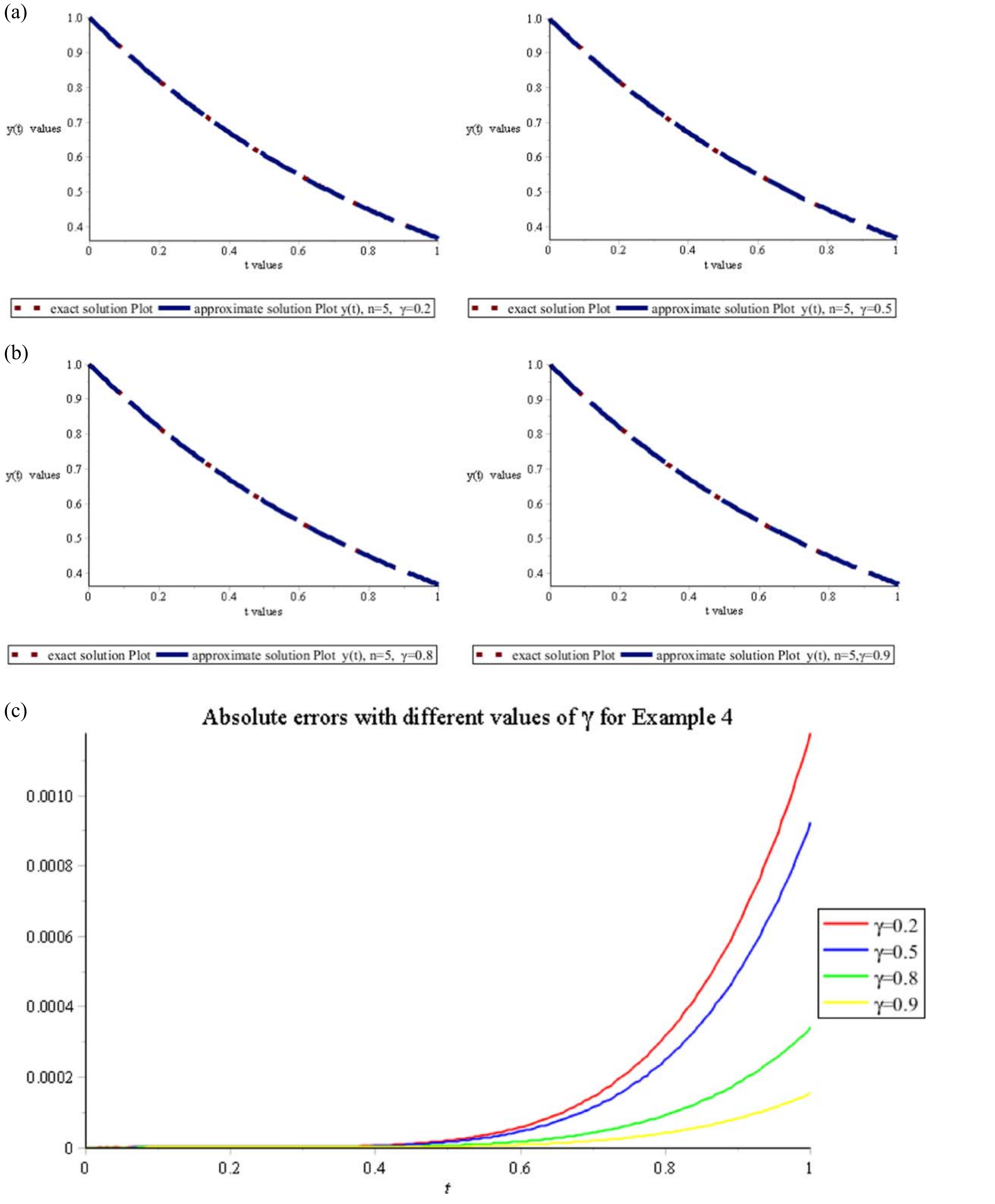

Figure 4. (a) Comparison of exact solution and approximate solutions of example 4 for $n = 5$, $\gamma = 0.2, 0.5$. (b) Comparison of exact solution and approximate solutions of example 4 for $n = 5$, $\gamma = 0.0.8, 0.9$. (c) The Abolute errors with different values of γ and $n = 5$ for example 4.

Following the procedures of example 6, we attain with the intial iteration $y_0(t) = t$

$$y_1(t) = t + 1/6\, t^2(t - 6),$$

$$y_2(t) = t - \frac{t^2(t^4 - 108\, t^3 + 1200\, t^2 - 4320\, t + 8640)}{8640},$$

$$\begin{aligned} y_3(t) = &-\frac{t^{12}}{5045118566400} + \frac{t^{11}}{9732096000} - \frac{293\, t^{10}}{17915904000} \\ &+ \frac{31t^9}{39813120} - \frac{463\, t^8}{23224320} + \frac{1573\, t^7}{2903040} \\ &- \frac{599\, t^6}{86400} + \frac{59\, t^5}{1440} - 1/6\, t^4 + 1/2\, t^3 - t^2 + t \end{aligned}.$$

$$\vdots$$

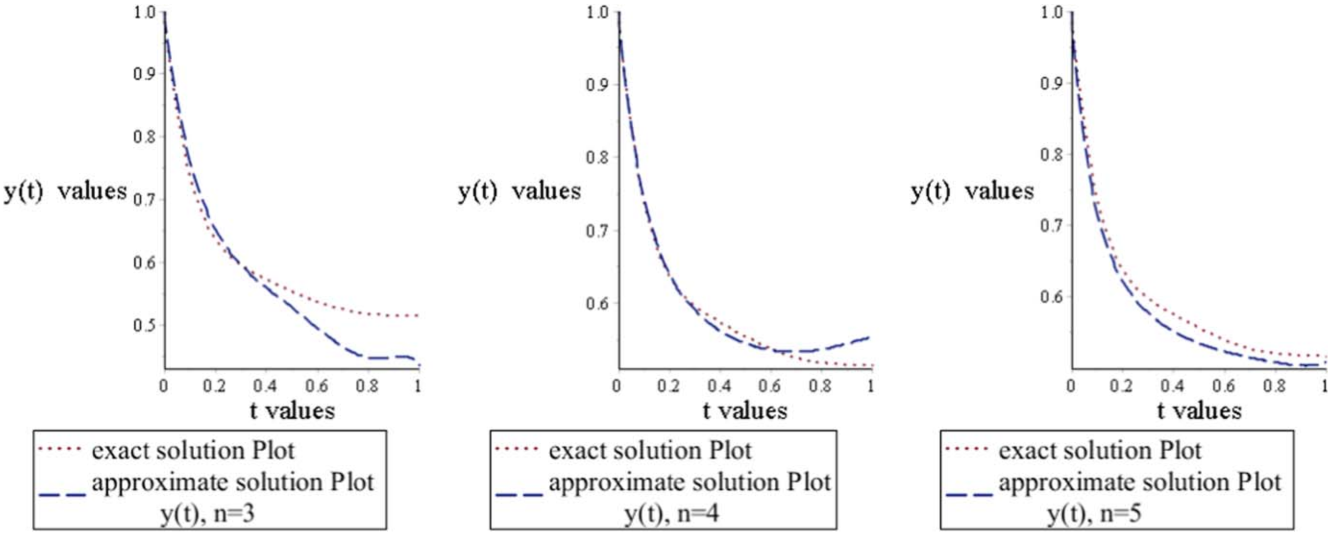

Figure 5. Comparison of our approximate solutions and the computed results in [26] for example 5 at $n = 3, 4, 5$.

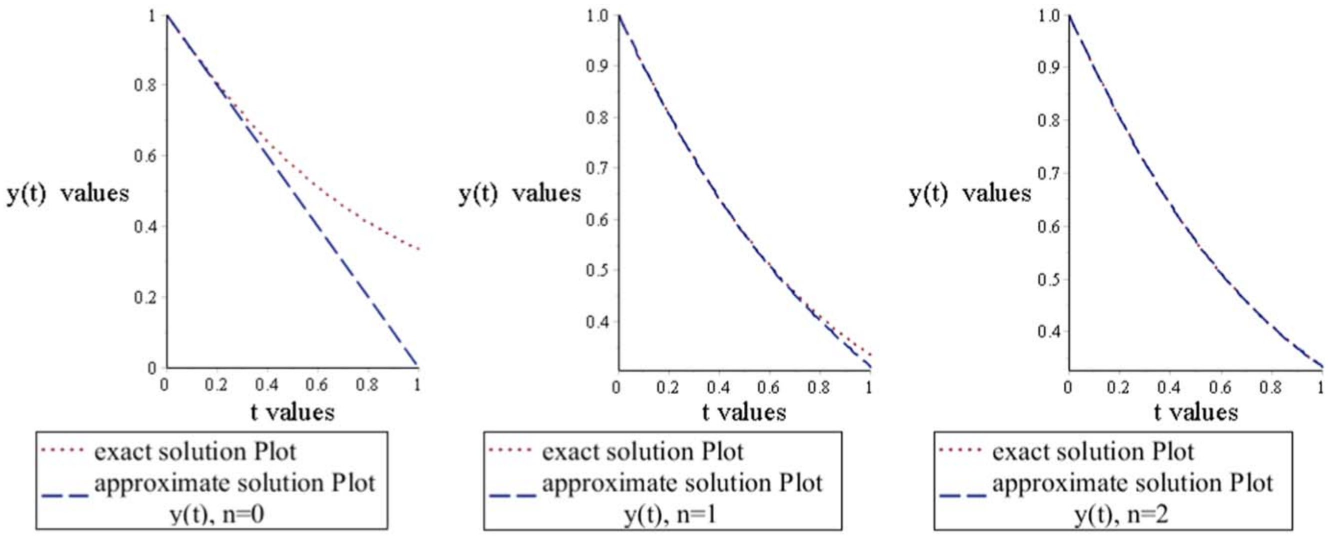

Figure 6. Comparison of exact solution and approximate solutions of example 6 for $n = 0, 1, 2$.

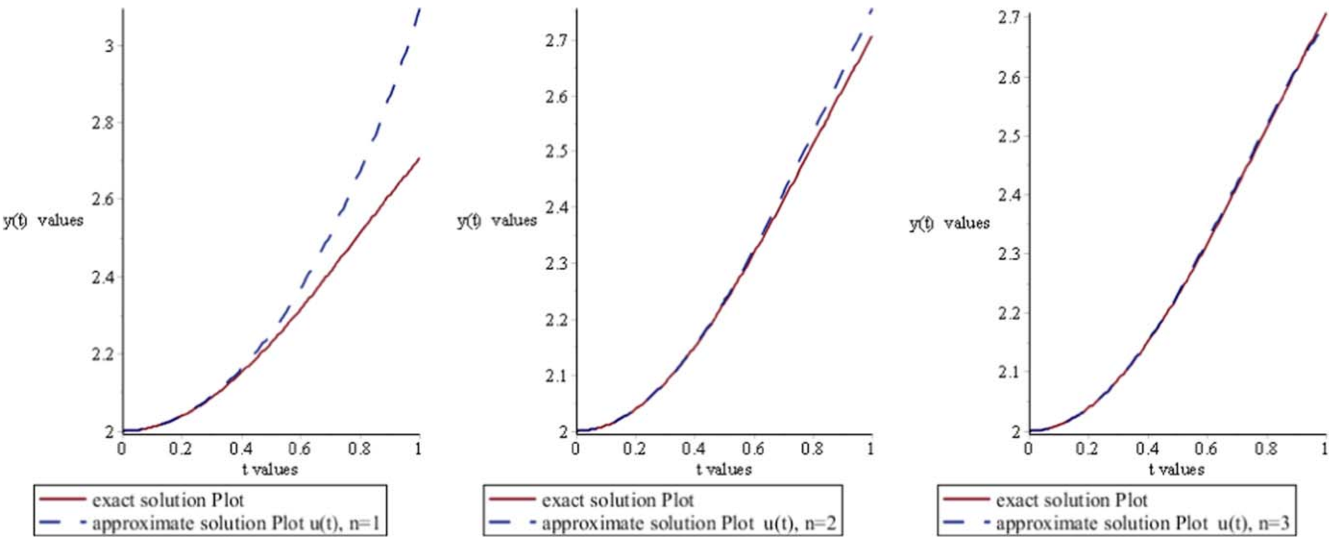

Figure 7. Comparison of exact solution and approximate solutions of example 7 for $n = 1, 2, 3$.

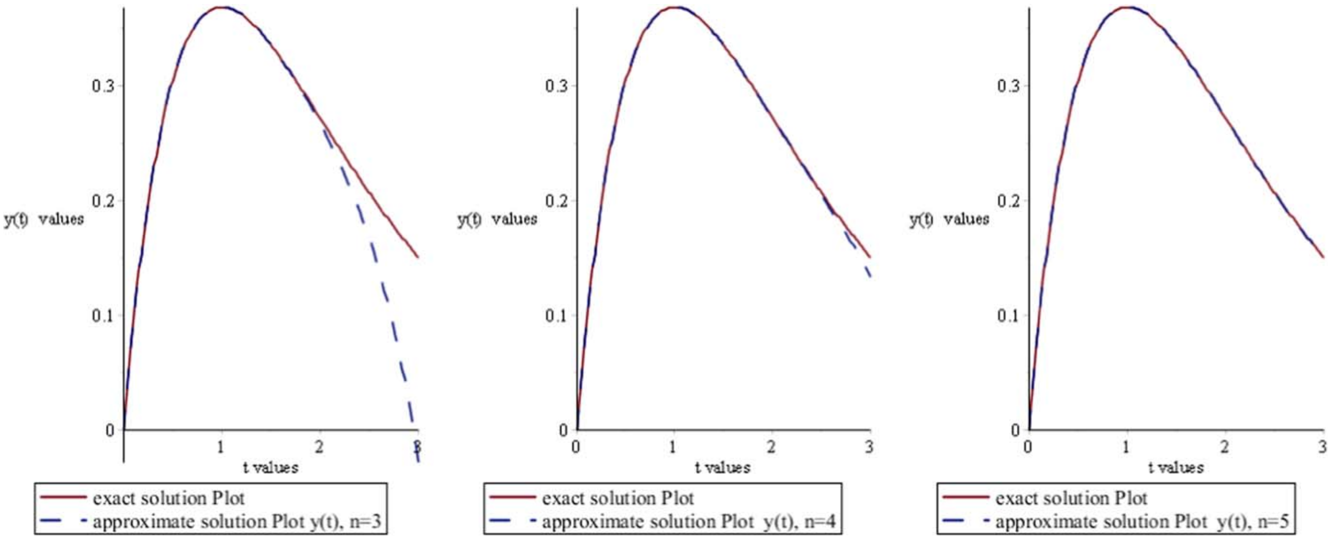

Figure 8. Comparison of exact solution and approximate solutions of example 8 for $n = 3, 4, 5$.

Table 2. Comparison of the LVIM and exact solutions for example 6.

t	$y_2(t)$	$y_3(t)$	$y_{\text{exact}}(t)$	Max. Error
0.0	1.00	1.000 000 01	1.00	1.3×10^{-9}
0.2	0.806 451 594	0.806 451 63	0.806 451 612 9	2.1×10^{-8}
0.4	0.641 024 53	0.641 025 735	0.641 025 641 0	1.1×10^{-5}
0.6	0.510 187 06	0.510 203 989	0.510 204 081 6	1.7×10^{-4}
0.8	0.409 720 46	0.409 835 207 2	0.409 836 065 6	1.2×10^{-3}
1.0	0.332 832 27	0.333 327 511	0.333 333 333 3	5.0×10^{-3}

Table 3. Comparison of the absolute maximum errors for example 8.

n	1	2	3	4	5	6	7
LVIM method	2.0×10^{-1}	5.61×10^{-3}	1.6×10^{-5}	2.76×10^{-7}	3.6×10^{-9}	1.1×10^{-10}	3.1×10^{-12}
PIA(1,1) method [27]	2.01×10^{-1}	5.61×10^{-3}	1.59×10^{-5}	2.76×10^{-7}	3.68×10^{-9}	6.76×10^{-11}	4.94×10^{-13}

Comparison of the approximate solutions $y_n(t)$, $n = 1, 2, 3, 4, 5$ with the exact solution $y(t) = te^{-t}$ is illustrated in figure 8. It clear from figure 8 the numerical results at $n = 5$ approximate as the exact solution.

The absolute maximum errors for different values of n are given in table 3, and it is revealed that the error decreases continually as n increase

3.3. Linear and nonlinear multi-pantograph delay with third order order differential equations

Example 9. Consider the linear pantograph equation of third order

$$y'''(t) - y(t) - 2y(t-1) = 2e^{1-t}, \quad y(0) = 1,$$
$$y'(0) = -1, \quad y''(0) = 1 \quad 0 < t \leqslant 1$$

which has the exact solution $y(t) = e^{-t}$

Taking the $\mathscr{L}$, we obtain

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) + \mathscr{L}(y(t) + 2y(t-1) - 2e^{1-t}) = 0.$$

The iteration formula thus is

$$Y_{n+1}(s) = Y_n(s) + \mu(s)[s^3Y(s) - s^2y(0) - sy'(0) - y''(0) - \mathscr{L}(-y(t) - 2y(t-1) + 2e^{1-t})].$$

With equation (12), and applying inverse Laplace transform, the above iteration formula can be explicitly given as

$$y_{n+1}(t) = y_n(t) - \mathscr{L}^{-1}\left\{\frac{1}{s^3}(s^3Y(s) - s^2y(0) - sy'(0) - y''(0) - \mathscr{L}(-y(t) - 2y(t-1) + 2e^{1-t}))\right\}$$

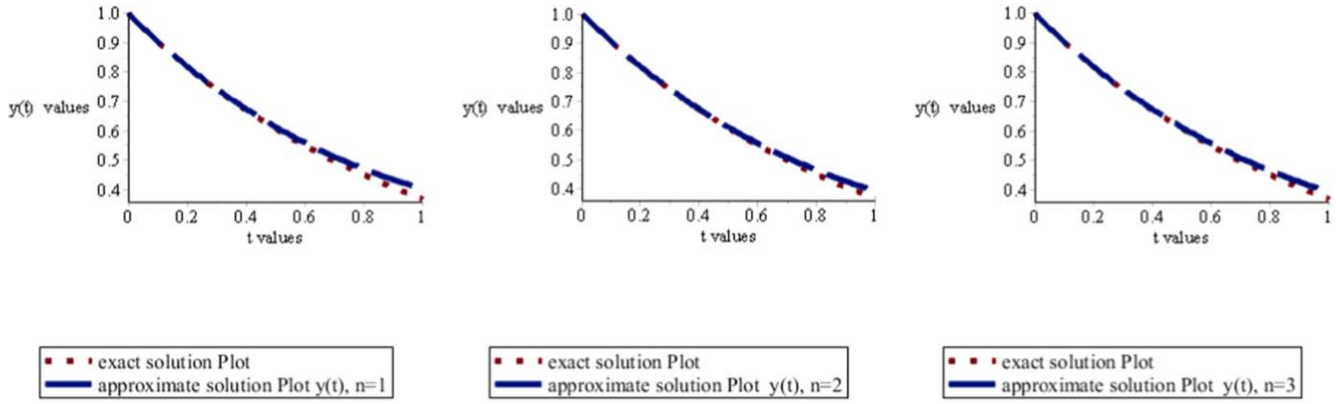

Figure 9. Comparison of exact solution and approximate solutions of example 9 for $n = 1, 2, 3$.

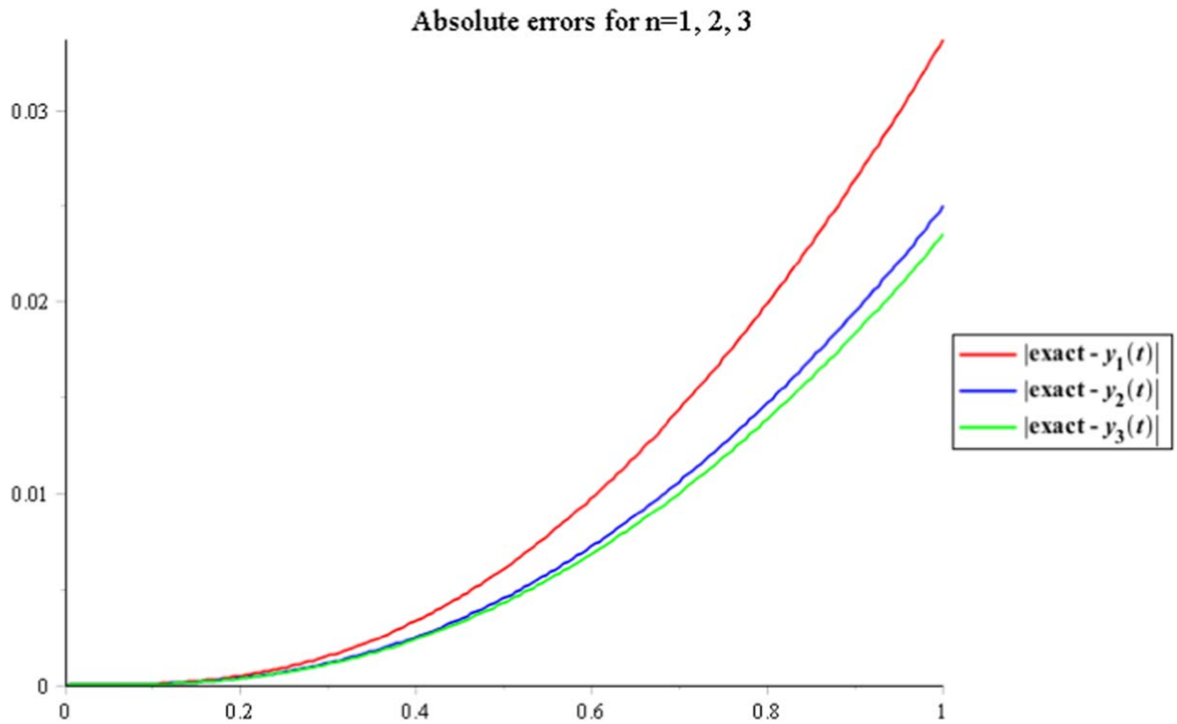

Figure 10. Graph of absolute errors for three iteration solutions for the multi-pantograph delay using LVIM at $n = 1, 2, 3$ for example 9.

Therefore,

$$y_{n+1}(t) = 1 - t + \frac{t^2}{2} + \mathscr{L}^{-1} \times \left\{ \frac{1}{s^3} (\mathscr{L}(-y(t) - 2y(t-1) + 2\mathrm{e}^{1-t})) \right\}. \tag{23}$$

with the initial iteration $u_0(t) = 1 - t - \frac{t^2}{2}$ and applying the iteration formula (16), we attain

$$y_1(t) = \frac{56}{15} - \frac{143\,t}{24} + 5\,t^2 - 1/40\,t^5 + 1/3\,t^4 - \frac{25\,t^3}{12} + \mathrm{e}t^2 - 4\,\mathrm{e}t + 5\,\mathrm{e} - 2\,\mathrm{e}^{2-t},$$

$$y_2(t) = 1 - t + 1/2\,t^2 - \frac{t^7}{240} + \frac{t^8}{4480} + \frac{31\,t^6}{720} + 4\,\mathrm{e} + 4\,\mathrm{e}^2 + 2\,\mathrm{e}^2t^2 + 2\,\mathrm{e}t^2 - 2\,t^3\mathrm{e} - \frac{127\,t^3}{90} + \frac{5\,t^4\mathrm{e}}{12} + \frac{155\,t^4}{288} - 1/20\,t^5\mathrm{e} - \frac{7\,t^5}{40} - 4\,\mathrm{e}^{2-t} - 4\,\mathrm{e}^{1-t} - 4\,\mathrm{e}^2t - 4\,\mathrm{e}t,$$

$$y_3 = (t)1 - t + 1/2\,t^2 - \frac{t^9}{2520} - \frac{t^{11}}{1478400} + \frac{t^{10}}{44800} + 8\,\mathrm{e}^3 - 8\,\mathrm{e}^{3-t} - \frac{239\,t^7}{10080} + \frac{149\,t^8}{40320} + \frac{1267\,t^6}{10800} - 12\,\mathrm{e}^2t + 6\,\mathrm{e}^2t^2 - \frac{217\,t^3\mathrm{e}}{45} + \frac{215\,t^4\mathrm{e}}{144} - 2/5\,t^5\mathrm{e}$$

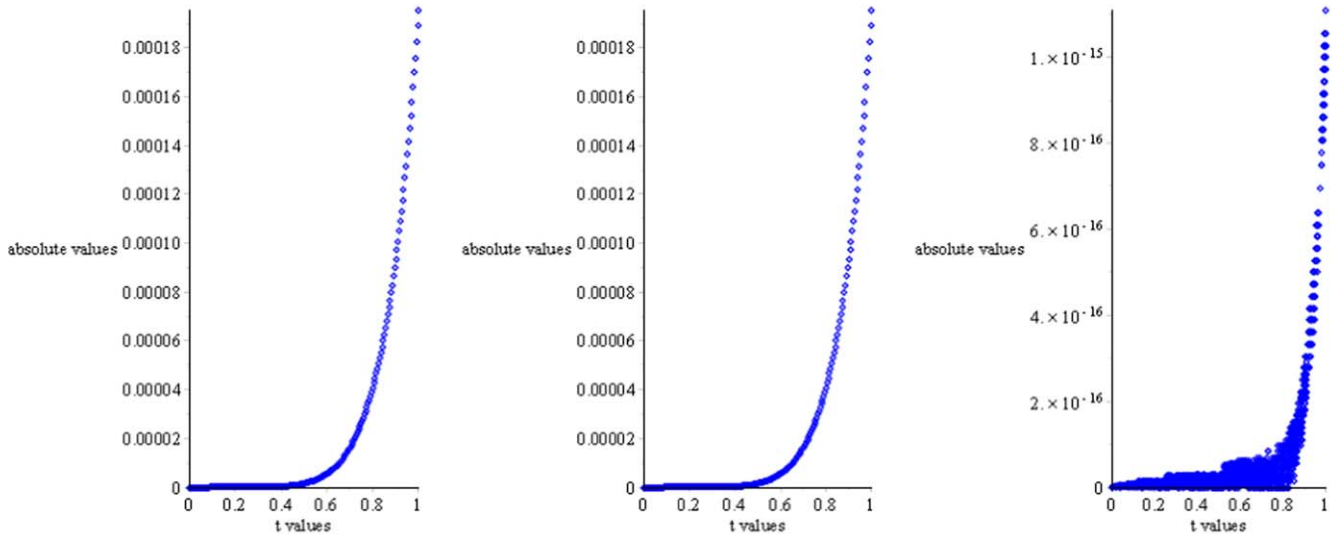

Figure 11. Graph of absolute errors for the first three iteration solutions for the multi-pantograph delay by using LVIM solution at $n = 1, 2, 3$ for example 10.

$$\begin{aligned}&- 8\,\mathrm{e}^3 t\\ &+ 4\,\mathrm{e}^3 t^2 - 4\,\mathrm{e}^2 t^3 + 5/6\,\mathrm{e}^2 t^4 - 1/10\,\mathrm{e}^2 t^5\\ &+ \frac{31\,t^6 \mathrm{e}}{360}\\ &- \frac{t^7 \mathrm{e}}{120} + \frac{t^8 \mathrm{e}}{2240} + 12\,\mathrm{e}^2 - 12\,\mathrm{e}^{2-t} - \frac{5137\,t^5}{14400}\\ &+ \frac{25327\,t^4}{30240} - \frac{41705\,t^3}{24192} + 3\,\mathrm{e}t^2 - 6\,\mathrm{e}t + 6\,\mathrm{e}\\ &- 6\,\mathrm{e}^{1-t}\\ &\vdots\end{aligned}$$

It is clear from figure 9 that the first three results not only give rapidly convergent series but also an accurate calculation of the solutions. For $n \to \infty$, then $y_n(t)$ tends to the exact solution $y(t)$. In figure 10 shows the absolute error for $n = 3$ is less than the absolute errors at $n = 1, 2$.

Example 10. Consider the nonlinear pantograph equation of third order [28]:

$$y'''(t) = -1 + 2y^2\left(\frac{t}{2}\right), \quad y(0) = 0,$$
$$y'(0) = 1, \quad y''(0) = 0 \quad 0 < t \leqslant 1$$

which has the exact solution $y(t) = \sin(t)$

Figure 11 display comparison of the absolute error functions of the first three iteration solutions of LVIM with the exact solution for example 10

4. Conclusion

It has been the aim of this paper to show that the Lagrange multiplier is proposed from the Laplace transform and incorporated with the method VIM to approximate the solution of multi-pantograph linear and nonlinear DDEs with different order. We attain the high approximate solutions or the exact solutions within a few iterations. It is concluded from figures that the successive approximations method is an accurate and converge very rapidly in physical problems. Some numerical examples have been provided to illustrate that the techniques is reliable and powerful method for the delay type of equations. Comparative results showed that the method is efficient in both linear and nonlinear problems, while for the linear problems studied, LVIM method gave almost the same results as JRC [16] and PIA [27] methods, it gave better results for the nonlinear problems. On the other hand, the results obtained in the first iteration y_1 of the LVIM is more convergent than those obtained in other Methods; this validates the performance of the present method. Faster convergence is achieved with LVIM when compared to another.

Acknowledgments

The author is very grateful to the editor and referees for their helpful, useful and detailed comments for improving the paper.

References

[1] Li D and Liu M Z 2005 Runge–Kutta methods for the multi-pantograph delay equation *Applied Mathematics and Computation* **163** 383–95
[2] Moitsheki R J and Makinde O D 2008 Application of Lie point symmetry and Adomain decomposition methods to thermal storage diffusion equations *J. Phys.: Conf. Ser.* **128** 012058
[3] Adomian G 1994 *Solving Frontier Problems of Physics: The Decomposition Method* (Boston: Kluwer Academic Publishers)
[4] Adomian G and R Rach R 1993 Analytic solution of nonlinear boundary-value problems in several dimensions by decomposition *J. Math. Anal. Appl.* **174** 118–37

[5] Mehdi D and Fatemeh S 2008 The use of the decomposition procedure of Adomian for solving a delay differential equation arising in electrodynamics *Phys. Scr.* **78** 5065004
[6] Makinde O D and Moitsheki R J 2008 On non-perturbative techniques for thermal radiation effect on natural convection past a vertical plate embedded in a saturated porous medium *Math. Probl. Eng.* **2008** 689074
[7] He J 1999 Homotopy perturbation technique *Comput. Methods Appl. Mech. Eng.* **178** 257–62
[8] He J 2006 Some asymptotic methods for strongly nonlinear equations *Int. J. Mod. Phys.* B **20** 1141–99
[9] Feng X, Mei L and He G 2007 An efficient algorithm for solving Troesch's problem *Appl. Math. Comput.* **189** 500–07
[10] Feng X, He Y and Meng J 2008 Application of the homotopy perturbation method to Bratu-type equations *Topol. Methods Nonlinear Anal.* **31** 243–52
[11] Feng X and He Y 2011 Modified homotopy perturbation method for solving the Stokes equations *Comput. Math. Appl.* **61** 2262–6
[12] Mirzaee F and Hoseini S F 2016 Application of Fibonacci collocation method for solving Volterra-Fredholm integral equations *Appl. Math. Comput.* **273** 637–44
[13] Mehmood A, Zameer A and Raja M A Z 2018 Intelligent computing to analyze the dynamics of Magnetohydrodynamic flow over stretchable rotating disk model *Appl. Soft Comput.* **67** 8–28
[14] Raja M A Z, Mehmood J, Sabir Z, Nasab A K and Manzar M A 2017 Numerical solution of doubly singular nonlinear systems using neural networks-based integrated intelligent computing *Neural Computing and Applications* **31** 793–812
[15] Sabir Z and Raja M 2014 Numeric treatment of nonlinear second order multi-point boundary value problems using ANN, GAs and sequential quadratic programming technique *Int. J. Ind. Eng. Comput.* **5** 431–42
[16] Doha E H, Bhrawy A H and Hafez R M 2017 Numerical algorithm for solving multi-pantograph delay equations on the half-line using jacobi rational functions with convergence analysis *Acta Math. Appl. Sin.* **33** 297–310
[17] He J H 1998 Approximate analytical solution for seepage flow with fractional derivatives in porous media *Comput. Methods Appl. Mech. Eng.* **167** 57–68
[18] Batiha B, Noorani M S M and Hashim I 2007 Approximate analytical solution of the coupled sine-Gordon equation using the variational iteration method *Phys. Scr.* **76** 445–8
[19] He J H 2007 Variational iteration method- some recent results and new interpretations *J. Comput. Appl. Math.* **207** 3–17
[20] He J H and Wu X H 2007 Variational iteration method- new development and applications *Comp. Appl. Math.* **54** 881–94
[21] Wu G C 2013 Challenge in the variational iteration method—A new approach to identification of the Lagrange multipliers *J. King Saudi Univ.* **25** 175–8
[22] Wu G C and Baleanu D 2013 Variational iteration method for fractional calculus- a universal approach by Laplace transform *Adv. Difference Equ.* **18** 2013
[23] Cakir M and Arslan D 2015 The adomian decomposition method and the differential transform method for numerical solution of multi-pantograph delay *Differ. Equ. Appl. Math.* **6** 1332–43
[24] Feng X 2013 An analytic study on the multi-pantograph delay equations with variable coefficients *Appl. Bull. Math. Soc. Sci. Math. Roumanie Tome* **56** 205–15 (https://www.jstor.org/stable/43679311)
[25] Dehghan M and Salehi R 2010 Solution of a nonlinear time-delay model in biology via semi-analytical approaches *Computer Physics Communications* **181** 1255–65
[26] Căruntu B and Bota C 2014 Analytical Approximate Solutions for a General Class of Nonlinear Delay Differential Equations *The Scientific World Journal* **2014** 631416
[27] Bahşi M M and Çevik M 2015 Numerical Solution of Pantograph-Type Delay Differential Equations Using Perturbation-Iteration Algorithms *Journal of Applied Mathematics* **2015** 139821
[28] Öztürk Y 2016 Approximate solution of generalized pantograph equations with variable coefficients by operational method *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)* **7** 66