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Multi-pole solutions and their asymptotic analysis of the focusing Ablowitz–Ladik equation

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Abstract

For the focusing Ablowitz–Ladik equation, the double- and triple-pole solutions are derived from its multi-soliton solutions via some limit technique. Also, the asymptotic analysis is performed for such two multi-pole solutions (MPSs) by considering the balance between exponential and algebraic terms. Like the continuous nonlinear Schrödinger equation, the discrete MPSs describe the elastic interactions of multiple solitons with the same amplitudes. But in contrast to the common multi-soliton solutions, most asymptotic solitons in the MPSs are localized in the curves of the *nt* plane, and thus they have the time-dependent velocities. In addition, the solitons' relative distances grow logarithmically with $|t|$, while the separation acceleration magnitudes decrease exponentially with their distance.

Keywords: Ablowitz–Ladik equation, multi-pole solutions, soliton interactions, asymptotic analysis

(Some figures may appear in colour only in the online journal)

1. Introduction

The existence of multi-soliton solutions is one of the most distinctive features for integrable nonlinear evolution equations (NLEEs), like the nonlinear Schrödinger equation (NLSE) [1]. In the terminology of inverse scattering transform (IST), an N-soliton solution corresponds to that the reflection coefficient admits N simple poles [1–3]. Usually, such a solution describes the elastic N-soliton collisions, where the relative distance between two interacting solitons is linear in $|t|$ [2, 4]. Recently, there is a growing interest in the degenerate cases of N-soliton solutions and their described soliton interactions (e.g. see [5–13] and references therein). Taking the focusing NLSE as an example, its bright N-soliton solutions admit two important degenerate cases. One is the soliton bound state (i.e. soliton train) when the discrete eigenvalues of the scattering problem have the same real parts [9, 14]. The bounded solitons travel with the same velocities and keep finite relative distance which varies periodically in time. The other one is the multi-pole solution (MPS) when the multiplicity of discrete eigenvalues is higher than one [2, 3, 11, 13, 15]. In the latter degenerate case, two or more solitons with equal amplitudes form a weak bound state, in which there is a strong interaction at $t\approx 0$ and the solitons' relative distance grows logarithmically with $|t|$ [2, 3, 11, 13, 15].

In the celebrated work of Zakharov and Shabat [2], they first reported the double-pole solution of the focusing NLSE as the limit of the two-soliton solution when two distinct poles coalesce into one. Subsequently, Olmedilla [3] derived the formula for an arbitrary Lth-order MPS by solving the Gel′fand–Levitan–Marchenko equations, and studied the asymptotic behavior of the double- and triple-pole solutions, which shows that the interacting solitons diverge from each other logarithmically as $|t|\to+\infty$. Recently, Schiebold [13] gave a rigorous and complete asymptotic description of the

MPSs with arbitrary order based on some operator-theoretic approach to integrable systems. It should be mentioned that the MPSs also exist in the focusing NLSE with nonzero background [11, 12], but there are no such kinds of solutions for the defocusing case [16]. In the context of optical fibers, the MPSs can describe the interactions of multiple chirped pulses of equal amplitudes, where the interaction force between two pulses decreases exponentially with their initial distance [15, 17]. Besides, the MPSs, mostly in cases of double-pole solution, have also been revealed in many other integrable NLEEs [5, 6, 13, 18–29], and the nonintegrable subcritical and supercritical NLSEs [30].

As the discrete analog of NLSE, the Ablowitz–Ladik (AL) equation [31, 32]:

$$\mathrm{i}q_{n,t} = (1 + \sigma|q_n|^2)(q_{n+1} + q_{n-1}), \tag{1}$$

where $\sigma = 1$ and -1, respectively, represent the focusing and defocusing nonlinearity, is relevant to modeling nonlinear localized waves in certain electrical and optical lattice systems [33]. Both the focusing and defocusing versions of equation (1) are integrable in the sense that their initial value problems are exactly soluble via the IST scheme [32, 34–36]. Like the continuous NLSE, equation (1) possesses the bright soliton solutions for $\sigma = 1$ [37] and the dark soliton solutions for $\sigma = -1$ [38]. Also, the focusing AL equation with the nonzero background admits the breather and rogue-wave solutions which are related to the modulation instability [39–41]. Up to now, many integrable properties of equation (1) have been detailed, like the Hamiltonian structure [42], Bäcklund transformation [43], and quasi-periodic solutions [44].

We note that there is very little attention which has been paid to the discrete MPSs of equation (1) in the existing literature. In this paper, we derive the explicit formulas of the double- and triple-pole solutions based on the two- and three-soliton solutions for the focusing AL equation. Our calculation relies on some limit technique which has been frequently used in constructing the rational localized-wave solutions via the Darboux transformation (see, for example, [6, 7, 10, 45, 46]). In principle, it is feasible to obtain he discrete MPSs with arbitrary order in such a way. On the other hand, we perform an asymptotic analysis of both the double- and triple-pole solutions based on the balance between the exponential and algebraic terms, which was recently proposed in [6]. As a result, we reveal that the double-pole solution describes the elastic interactions between two solitons having the same amplitudes. The asymptotic solitons are found to be localized in some curves of the nt plane, so that their velocities are time-dependent. Moreover, the relative distance between two solitons grows logarithmically with $|t|$, while the separation acceleration magnitude decreases exponentially with the distance. The triple-pole solution also describes the elastic interactions among three solitons with the same amplitudes, in which two pairs of asymptotic solitons lie in curves and possess the same properties as those in the double-pole case, but the third pair has a constant velocity and experiences no phase shift upon an interaction.

The structure of this paper is organized as follows: in section 2, the bright soliton solutions of equation (1) with $\sigma = 1$ are constructed by the Hirota method. In section 3, the double- and triple-pole solutions are, respectively, degenerated from the two- and three-soliton solutions by some limit technique. Also, the asymptotic behavior of such two MPSs is studied, and thus the soliton interaction properties are discussed. Finally, we address the conclusions and discussions of this paper in section 4.

2. Soliton solutions via the Hirota method

In this section, we use the Hirota method [47] to construct the bright soliton solutions of equation (1) with $\sigma = 1$. Taking the variable transformation in the form

$$q_n(t) = i^n \frac{g_n(t)}{f_n(t)}, \tag{2}$$

where $f_n(t)$ is a real-valued function and $g_n(t)$ is a complex-valued function, we obtain the bilinear form for equation (1) as follows:

$$D_t g_n \cdot f_n = \lambda(g_{n+1}f_{n-1} - g_{n-1}f_{n+1}), \tag{3}$$

$$f_n^2 + |g_n|^2 = \lambda f_{n+1}f_{n-1}, \tag{4}$$

where D_t is defined by $D_t^j g_n(t)\cdot f_n(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t'}\right)^j g_n(t)f_n(t')|_{t=t'}$ [47]. By introducing a formal parameter ε, we expand the functions g_n and f_n in the following form

$$\begin{aligned} g_n = {} & \varepsilon^1 g_n^{(1)} + \varepsilon^3 g_n^{(3)} + \varepsilon^5 g_n^{(5)} + \cdots \\ & + \varepsilon^{2M-1} g_n^{(2M-1)} + \cdots, \end{aligned} \tag{5}$$

$$\begin{aligned} f_n = {} & 1 + \varepsilon^2 f_n^{(2)} + \varepsilon^4 f_n^{(4)} + \varepsilon^6 f_n^{(6)} + \cdots \\ & + \varepsilon^{2M} f_n^{(2M)} + \cdots, \end{aligned} \tag{6}$$

where M is an arbitrary positive integer. Then, substituting g_n and f_n in equations (5) and (6) into (3) and (4) and truncating the resulting equations at different M, one can derive a chain of soliton solutions for the focusing AL equation.

For the truncation at $M = 1$, the bright one-soliton solution is obtained as

$$q_n = i^n \frac{\varepsilon g_n^{(1)}}{1 + \varepsilon^2 f_n^{(2)}}, \tag{7}$$

with

$$g_n^{(1)} = \alpha_1 \mathrm{e}^{\theta_1},\ f_n^{(2)} = \beta_1 \mathrm{e}^{\theta_1 + \bar{\theta}_1}\ (\theta_1 = \kappa_1 n + \omega_1 t),$$
$$\beta_1 = \frac{|\alpha_1|^2 \mathrm{e}^{\kappa_1 + \bar{\kappa}_1}}{(\mathrm{e}^{\kappa_1 + \bar{\kappa}_1} - 1)^2},\ \omega_1 = \mathrm{e}^{\kappa_1} - \mathrm{e}^{-\kappa_1},$$

where the bar denotes complex conjugate, κ_1 is a constant in $\mathbb{C}$ and satisfies $\kappa_1 + \bar{\kappa}_1 \neq 0$ to avoid the singularity.

For the truncation at $M = 2$, the bright two-soliton solution is derived as

$$q_n = i^n \frac{\varepsilon g_n^{(1)} + \varepsilon^3 g_n^{(3)}}{1 + \varepsilon^2 f_n^{(2)} + \varepsilon^4 f_n^{(4)}}, \tag{8}$$

with

$$\begin{aligned}
g_n^{(1)} &= \alpha_1 e^{\theta_1} + \alpha_2 e^{\theta_2},\\
f_n^{(2)} &= \beta_{11} e^{\theta_1+\bar\theta_1} + \beta_{12} e^{\theta_1+\bar\theta_2} + \beta_{21} e^{\theta_2+\bar\theta_1} + \beta_{22} e^{\theta_2+\bar\theta_2},\\
g_n^{(3)} &= \gamma_1 e^{\theta_1+\theta_2+\bar\theta_1} + \gamma_2 e^{\theta_1+\theta_2+\bar\theta_2},\ \ f_n^{(4)} = \delta e^{\theta_1+\theta_2+\bar\theta_1+\bar\theta_2},\\
\theta_i &= \kappa_i n + \omega_i t,\ \ \omega_i = e^{\kappa_i} - e^{-\kappa_i}\ (i = 1, 2),\\
\delta &= \frac{|\alpha_1|^2|\alpha_2|^2 (e^{\kappa_1} - e^{\kappa_2})^2 (e^{\bar\kappa_1} - e^{\bar\kappa_2})^2 e^{\kappa_1+\bar\kappa_1+\kappa_2+\bar\kappa_2}}{(e^{\kappa_1+\bar\kappa_1} - 1)^2 (e^{\kappa_2+\bar\kappa_1} - 1)^2 (e^{\kappa_1+\bar\kappa_2} - 1)^2 (e^{\kappa_2+\bar\kappa_2} - 1)^2},\\
\beta_{ij} &= \frac{\alpha_i \bar\alpha_j e^{\kappa_i+\bar\kappa_j}}{(e^{\kappa_i+\bar\kappa_j} - 1)^2} \quad (i, j = 1, 2),\\
\gamma_i &= \frac{(e^{\kappa_i} - e^{\kappa_{3-i}})^2 |\alpha_i|^2 \alpha_{3-i} e^{2\bar\kappa_i}}{(e^{\kappa_i+\bar\kappa_i} - 1)^2 (e^{\kappa_{3-i}+\bar\kappa_i} - 1)^2} \quad (i, j = 1, 2),
\end{aligned}$$

where $\kappa_{1,2}$ are two constants in $\mathbb{C}$ and satisfy $\kappa_i + \bar\kappa_j \neq 0\ (i, j = 1, 2)$ in order to avoid the singularity.

For the truncation at $M = 3$, the bright three-soliton solution is presented as follows:

$$q_n = i^n \frac{\varepsilon g_n^{(1)} + \varepsilon^3 g_n^{(3)} + \varepsilon^5 g_n^{(5)}}{1 + \varepsilon^2 f_n^{(2)} + \varepsilon^4 f_n^{(4)} + \varepsilon^6 f_n^{(6)}}, \tag{9}$$

with

$$\begin{aligned}
g_n^{(1)} &= \alpha_1 e^{\theta_1} + \alpha_2 e^{\theta_2} + \alpha_3 e^{\theta_3},\\
f_n^{(2)} &= \sum_{1\leqslant i,j\leqslant 3} \beta_{ij} e^{\theta_i+\bar\theta_j},\\
g_n^{(3)} &= \sum_{\substack{i<j,\\ 1\leqslant i,\ j,\ k\leqslant 3}} \gamma_{ijk} e^{\theta_i+\theta_j+\bar\theta_k},\\
f_n^{(4)} &= \sum_{\substack{i<j,\ k<l,\\ 1\leqslant i,\ j,\ k,\ l\leqslant 3}} \delta_{ijkl} e^{\theta_i+\theta_j+\bar\theta_k+\bar\theta_l},\\
g_n^{(5)} &= \sum_{\substack{i<j,\\ 1<i,\ j\leqslant 3}} \zeta_{ij} e^{\theta_1+\theta_2+\theta_3+\bar\theta_i+\bar\theta_j},\\
f_n^{(6)} &= \eta e^{\theta_1+\theta_2+\theta_3+\bar\theta_1+\bar\theta_2+\bar\theta_3},\\
\beta_{ij} &= \frac{\alpha_i \bar\alpha_j e^{\kappa_i+\bar\kappa_j}}{(e^{\kappa_i+\bar\kappa_j} - 1)^2}\ (i, j = 1, 2, 3),\\
\gamma_{ijk} &= \frac{\alpha_i \alpha_j \bar\alpha_k (e^{\kappa_i} - e^{\kappa_j})^2 e^{2\bar\kappa_k}}{(e^{\kappa_i+\bar\kappa_k} - 1)^2 (e^{\kappa_j+\bar\kappa_k} - 1)^2}\ (i < j;\ i, j, k = 1, 2, 3),\\
\delta_{ijkl} &= \frac{\alpha_i \alpha_j \bar\alpha_k \bar\alpha_l (e^{\kappa_i} - e^{\kappa_j})^2 (e^{\bar\kappa_k} - e^{\bar\kappa_l})^2 e^{\kappa_i+\kappa_j+\bar\kappa_k+\bar\kappa_l}}{(e^{\kappa_i+\bar\kappa_k} - 1)^2 (e^{\kappa_j+\bar\kappa_k} - 1)^2 (e^{\kappa_i+\bar\kappa_l} - 1)^2 (e^{\kappa_j+\bar\kappa_l} - 1)^2}\\
&\quad \times (i < j;\ k < l;\ i, j, k, l = 1, 2, 3),
\end{aligned}$$

$$\begin{aligned}
\zeta_{ij} &= \frac{\alpha_1 \alpha_2 \alpha_3 \bar\alpha_i \bar\alpha_j (e^{\kappa_1} - e^{\kappa_2})^2 (e^{\kappa_1} - e^{\kappa_3})^2 (e^{\kappa_2} - e^{\kappa_3})^2 (e^{\bar\kappa_i} - e^{\bar\kappa_j})^2 e^{2(\bar\kappa_i+\bar\kappa_j)}}{(e^{\kappa_1+\bar\kappa_i} - 1)^2 (e^{\kappa_2+\bar\kappa_i} - 1)^2 (e^{\kappa_3+\bar\kappa_i} - 1)^2 (e^{\kappa_1+\bar\kappa_j} - 1)^2 (e^{\kappa_2+\bar\kappa_j} - 1)^2 (e^{\kappa_3+\bar\kappa_j} - 1)^2}\\
&\qquad (i < j;\ i, j = 1, 2, 3),\\
\eta &= \frac{|\ \alpha_1 \alpha_2 \alpha_3 (e^{\kappa_1} - e^{\kappa_2})^2 (e^{\kappa_1} - e^{\kappa_3})^2 (e^{\kappa_2} - e^{\kappa_3})^2|^2 e^{\kappa_1+\kappa_2+\kappa_3+\bar\kappa_1+\bar\kappa_2+\bar\kappa_3}}{(e^{\kappa_1+\bar\kappa_1} - 1)^2 (e^{\kappa_2+\bar\kappa_2} - 1)^2 (e^{\kappa_3+\bar\kappa_3} - 1)^2 |\ (e^{\kappa_1+\bar\kappa_2} - 1)^2 (e^{\kappa_3+\bar\kappa_1} - 1)^2 (e^{\kappa_3+\bar\kappa_2} - 1)^2|^2},\\
\theta_i &= \kappa_i n + \omega_i t,\ \ \omega_i = e^{\kappa_i} - e^{-\kappa_i}\quad (i = 1, 2, 3),
\end{aligned}$$

where $\kappa_{1,2,3}$ are three constants in $\mathbb{C}$ and satisfy $\kappa_i + \bar\kappa_j \neq 0$ $(1 \leqslant i, j \leqslant 3)$ in order to avoid the singularity. Continually in a similar manner, one can obtain more higher-order multi-soliton solutions when the truncation is taken at $M > 3$.

3. Multi-pole solutions

Based on the multi-soliton solutions in section 2, we will construct the discrete MPSs of equation (1) with $\sigma = 1$ by letting $\kappa_2, \ldots, \kappa_M \to \kappa_1$. To calculate the limits by the Taylor expansion [6, 7, 10, 45, 46], we take

$$\kappa_m = \kappa_1(1 + \mu_m \tilde\varepsilon)\ (2 \leqslant m \leqslant M), \tag{10}$$

where $\tilde\varepsilon$ is a small parameter, and $\mu_m = 1$ or -1. Also, we introduce $\tilde\varepsilon$ into θ_m $(1 \leqslant m \leqslant M)$ in the form

$$\begin{aligned}
\theta_1 &= \kappa_1 n + \omega_1 t + \sum_{i=1}^{\infty} s_i \tilde\varepsilon^i,\\
\theta_m &= \kappa_m n + \omega_m t + \sum_{i=1}^{\infty} x_i^{(m)} s_i \tilde\varepsilon^i\ (2 \leqslant m \leqslant M),
\end{aligned} \tag{11}$$

where s_i's are arbitrary parameters (which will appear in the MPSs) in $\mathbb{C}$, $x_i^{(m)}$'s $(2 \leqslant m \leqslant M)$ are real constants to be determined. Then, we expand $f_n^{(2m)}$ and $g_n^{(2m-1)}$ $(1 \leqslant m \leqslant M)$ in the Taylor series of $\tilde\varepsilon$ as follows:

$$\begin{aligned}
f_n^{(2m)}(t) &= \sum_{j=0}^{\infty} f_n^{(2m,j)}(t) \tilde\varepsilon^j,\\
g_n^{(2m-1)}(t) &= \sum_{j=0}^{\infty} g_n^{(2m-1,j)}(t) \tilde\varepsilon^j,
\end{aligned} \tag{12}$$

where $f_n^{(2m,j)}$ and $g_n^{(2m-1,j)}$ are defined by

$$\begin{aligned}
f_n^{(2m,j)} &= \frac{1}{j!} \frac{d^j f_n^{(2m)}}{d\tilde\varepsilon^j}\bigg|_{\tilde\varepsilon=0},\\
g_n^{(2m-1,j)} &= \frac{1}{j!} \frac{d^j g_n^{(2m-1)}}{d\tilde\varepsilon^j}\bigg|_{\tilde\varepsilon=0}.
\end{aligned} \tag{13}$$

Note that the coefficients of the first several terms in the expansions of $f_n^{(2m)}$ and $g_n^{(2m-1)}$ $(2 \leqslant m \leqslant M)$ are zeroes. Thus, with substitution of (12) into (2), the limits at $\tilde\varepsilon \to 0$ just yield the one-soliton solution when ε is regarded as a parameter independent of $\tilde\varepsilon$. To overcome this difficulty, one must consider the contribution from all the terms $f_n^{(2m)}$ and $g_n^{(2m-1)}$ $(1 \leqslant m \leqslant M)$ in deriving the MPSs. Observing that the constant '1' always appears in the denominators of multi-soliton solutions, we assume that

$$\varepsilon = \tilde{\varepsilon}^{-r_M}\ (r_M \in \mathbb{R}^+), \tag{14}$$

where r_M is to be determined, so that the constant terms are dominant in all the expansions of $\varepsilon^{2m} f_n^{(2m)}$ and $\varepsilon^{2m-1} g_n^{(2m-1)}$ $(1 \leqslant m \leqslant M)$. Meanwhile, one need to make all the negative power terms of $\tilde{\varepsilon}$ vanish by imposing some constraints on μ_m's and $x_i^{(m)}$'s. After those manipulations, the limits of multi-soliton solutions at $\tilde{\varepsilon} \to 0$ will give rise to the MPSs correctly.

3.1. Double-pole solution

For the two-soliton solution (8), we let $\kappa_2 = \kappa_1(1 + \tilde{\varepsilon})$ (i.e. $\mu_2 = 1$) and calculate the Taylor expansions of $g_n^{(1)}$, $g_n^{(3)}$, $f_n^{(2)}$ and $f_n^{(4)}$ in $\tilde{\varepsilon}$. It turns out that $g_n^{(3,0)} = g_n^{(3,1)} = f_n^{(4,0)} = f_n^{(4,1)} = f_n^{(4,2)} = f_n^{(4,3)} = 0$ are identically satisfied, and that $g_n^{(1,0)}$, $g_n^{(3,2)}$, $f_n^{(2,0)}$, $f_n^{(2,1)}$ all become 0 if and only if $\alpha_1 + \alpha_2 = 0$. Thus, we take $\varepsilon = 1/\tilde{\varepsilon}$ (i.e. $r_2 = 1$) and $\alpha_2 = -\alpha_1$, which suffices to ensure that all the negative power terms of $\tilde{\varepsilon}$ vanish and the constant terms are dominant both in the numerator and denominator. Still, it requires that $x_1^{(2)} \neq 1$ in order to make sure the free parameter s_1 appear in the solution. Without loss of generality, we assume $x_1^{(2)} = -1$. Then, the limit of solution (8) as $\tilde{\varepsilon} \to 0$ yields the double-pole solution of equation (1) with $\sigma = 1$ as follows:

$$q_n = i^n \frac{g_n^{(1,1)} + g_n^{(3,3)}}{1 + f_n^{(2,2)} + f_n^{(4,4)}}, \tag{15}$$

with

$$g_n^{(1,1)} = -\alpha_1 e^{\xi_1}(T_1 + \xi_1),$$

$$f_n^{(2,2)} = \beta_{11} e^{\xi_1 + \bar{\xi}_1}\Bigg\{|T_1 + \xi_1|^2 - 2\coth(\kappa_{1R})\mathrm{Re}\,[\bar{\kappa}_1(T_1 + \xi_1)] + |\kappa_1|^2\left[\frac{1}{2}\mathrm{csch}^2(\kappa_{1R}) + \coth^2(\kappa_{1R})\right]\Bigg\},$$

$$g_n^{(3,3)} = \frac{\alpha_1 \beta_{11} \kappa_1^2 e^{2\xi_1 + \bar{\xi}_1}[\bar{T}_1 + \bar{\xi}_1 - 2\bar{\kappa}_1 \coth(\kappa_{1R})]}{4\sinh^2(\kappa_{1R})},$$

$$f_n^{(4,4)} = \frac{\beta_{11}^2 |\kappa_1|^4 e^{2(\xi_1 + \bar{\xi}_1)}}{16\sinh^4(\kappa_{1R})},$$

where $T_1 = (w_1 - \omega_1)t - 2s_1$, $\xi_1 = \kappa_1 n + \omega_1 t$, $\beta_{11} = \frac{|\alpha_1|^2 \mathrm{csch}^2(\kappa_{1R})}{4}$, $\omega_1 = 2\sinh(\kappa_1)$, $w_1 = 2\kappa_1\cosh(\kappa_1)$, the subscripts R and I represent the real and imaginary parts of κ_1, respectively. In the following, we use the improved asymptotic analysis method in [6] to study the asymptotic behavior of solution (15), and then reveal its described dynamics of soliton interactions.

To begin with, we explain that the asymptotic solitons in solution (15) cannot be located in any straight line $\mathcal{L}$: $\kappa_{1R}\, n - ct = \text{const}$. In fact, one can regard that solution (15) is explicitly dependent only on ξ_{1R} and t because of the relation

$$\xi_{1I} = \frac{\kappa_{1I}}{\kappa_{1R}}\xi_{1R} + 2\left[\cosh(\kappa_{1R})\sin(\kappa_{1I})t - \frac{\kappa_{1I}}{\kappa_{1R}}\sinh(\kappa_{1R})\cos(\kappa_{1I})t\right], \tag{16}$$

where $\xi_{1R} = \kappa_{1R}\, n + 2\sinh(\kappa_{1R})\cos(\kappa_{1I})t$. Thus, one needs to consider the asymptotic behavior of ξ_{1R} along the line $\mathcal{L}$ as $|t| \to \infty$. Looking at the difference $\xi_{1R} - (\kappa_{1R}\, n - ct) = [c + 2\sinh(\kappa_{1R})\cos(\kappa_{1I})]t$, we have

$$\xi_{1R} \to \begin{cases} \pm\infty, & c \neq -2\sinh(\kappa_{1R})\cos(\kappa_{1I}), \\ O(1), & c = -2\sinh(\kappa_{1R})\cos(\kappa_{1I}). \end{cases} \tag{17}$$

Associated with the two cases in equation (17), the dominant behavior of solution (15) can be given by

$$q_n \sim \begin{cases} i^n \dfrac{4\alpha_1 \sinh^2(\kappa_{1R})(\bar{T}_1 + \bar{\xi}_1 + 2\bar{s}_1)}{\beta_{11}\bar{\kappa}_1^2 e^{\bar{\xi}_1}}, & \xi_{1R} \to +\infty, \\ -i^n \dfrac{\alpha_1 (T_1 + \xi_1 + 2s_1)}{e^{-\xi_1}}, & \xi_{1R} \to -\infty, \\ i^n\left[\dfrac{\alpha_1 \kappa_1^2 e^{\xi_1}}{4\sinh^2(\kappa_{1R})(T_1 + \xi_1 + 2s_1)} - \dfrac{\alpha_1 e^{-\bar{\xi}_1}}{\beta_{11}(\bar{T}_1 + \bar{\xi}_1 + 2\bar{s}_1)}\right], & \xi_{1R} = O(1). \end{cases} \tag{18}$$

Noticing that $e^{\xi_{1R}} \gg |T_1 + \xi_1 + 2s_1|$ $(\xi_{1R} \to +\infty)$, $e^{-\xi_{1R}} \gg |T_1 + \xi_1 + 2s_1|$ $(\xi_{1R} \to -\infty)$ and $|T_1 + \xi_1 + 2s_1| \gg e^{\pm\xi_{1R}}$ $(\xi_{1R} = O(1))$, the asymptotic limits in equation (18) are all 0 when $|t| \to \infty$. That is, solution (15) has no asymptotic soliton lying in a straight line.

Next, we consider that the asymptotic solitons are localized in some curve $\mathcal{C}$ in the nt-plane. Along the curve $\mathcal{C}$, ξ_{1R} goes to ∞ or $-\infty$ when $|t| \to \infty$ because the slope of $\mathcal{C}$ is not a constant. Moreover, one should note that with replacement of ξ_{1I} by ξ_{1R} via (16), solution (15) contains only the terms t, ξ_{1R} and $e^{\xi_{1R}}$, and that there are the inequality relations $e^{\xi_{1R}} \gg \xi_{1R}(\xi_{1R} \to +\infty)$ and $e^{\xi_{1R}} \ll \xi_{1R}(\xi_{1R} \to -\infty)$. Therefore, in order to obtain the expressions of asymptotic solitons, we must consider the balance between t and $e^{\xi_{1R}}$, that is

$$t \sim w(n, t) e^{p\,\xi_{1R}}\ (|t| \to +\infty), \tag{19}$$

where $w(n, t) = O(1)$, p is a constant to be determined. For different values of p, solution (15) has the following dominant behavior:

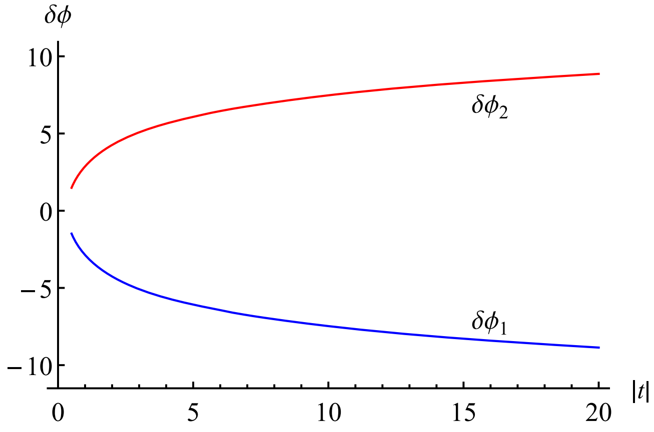


Figure 1. The phase shifts between $q_{n,i}^{+}$ and $q_{n,i}^{-}$ $(i=1,\,2)$ versus $|t|$, where the parameters are selected as $\alpha_1=\frac{1}{2}+\frac{1}{2}\mathrm{i}$, $\kappa_1=\frac{4}{3}+\mathrm{i}$ and $s_1=\frac{1}{2}$.

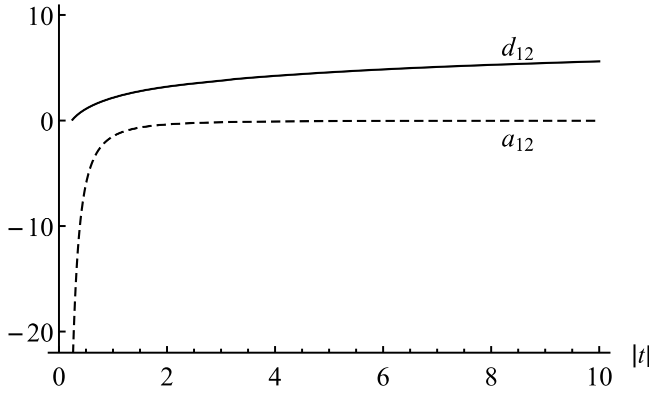


Figure 2. The two-soliton relative distance (solid) and separation acceleration (dashed) versus $|t|$, where the parameters are the same as those in figure 1.

$$q_n\sim\begin{cases}\dfrac{i^n\alpha_1\kappa_{1R}\kappa_1^2\mathrm{e}^{\mathrm{i}\xi_{1I}}}{4\varrho\sinh^2(\kappa_{1R})t\mathrm{e}^{-\xi_{1R}}}, & p>1\ \ (\xi_{1R}\to+\infty),\\[2ex] \dfrac{4i^n\alpha_1\kappa_{1R}\mathrm{e}^{\mathrm{i}\xi_{1I}}}{\sinh^2(\kappa_{1R})\left[\frac{16\varrho}{\kappa_1^2}\frac{t}{\mathrm{e}^{\xi_{1R}}}+\frac{\beta_{11}\kappa_{1R}^2\bar{\kappa}_1^2}{\bar{\varrho}\sinh^4(\kappa_{1R})}\frac{\mathrm{e}^{\xi_{1R}}}{t}\right]}, & p=1\ \ (\xi_{1R}\to+\infty),\\[2ex] \dfrac{4i^n\alpha_1\sinh^2(\kappa_{1R})\bar{\varrho}t\mathrm{e}^{\mathrm{i}\xi_{1I}}}{\beta_{11}\kappa_{1R}\bar{\kappa}_1^2\mathrm{e}^{\xi_{1R}}}, & 0<p<1\ \ (\xi_{1R}\to+\infty),\end{cases} \tag{20}$$

$$q_n\sim\begin{cases}\dfrac{-i^n\alpha_1\kappa_{1R}\mathrm{e}^{\mathrm{i}\xi_{1I}}}{\beta_{11}\bar{\varrho}t\mathrm{e}^{\xi_{1R}}}, & -1<p<0\ \ (\xi_{1R}\to-\infty),\\[2ex] \dfrac{-i^n\alpha_1\kappa_{1R}\varrho\mathrm{e}^{\mathrm{i}\xi_{1I}}}{\beta_{11}|\varrho|^2t\mathrm{e}^{\xi_{1R}}+\frac{\kappa_{1R}^2}{t\mathrm{e}^{\xi_{1R}}}}, & p=-1\ \ (\xi_{1R}\to-\infty),\\[2ex] \dfrac{-i^n\alpha_1\kappa_{1R}\mathrm{e}^{\mathrm{i}\xi_{1I}}}{\beta_{11}\bar{\varrho}t\mathrm{e}^{\xi_{1R}}}, & p<-1\ \ (\xi_{1R}\to-\infty),\end{cases} \tag{21}$$

with $\varrho=2\mathrm{i}\kappa_{1R}\cosh(\kappa_{1R})\sin(\kappa_{1I})-2\mathrm{i}\,\kappa_{1I}\sinh(\kappa_{1R})\cos(\kappa_{1I})+\kappa_{1R}(w_1-\omega_1)$, where we have substituted equation (16) for ξ_{1I} before determining the dominant balance in solution (15).

As suggested by equations (20) and (21), the asymptotic solitons of solution (15) are formed only when the balance $t\mathrm{e}^{\xi_{1R}}=O(1)$ or $t\mathrm{e}^{-\xi_{1R}}=O(1)$ is met. In view that such a balance might occur as $t\to+\infty$ or $t\to-\infty$, we can obtain four asymptotic solitons with their expressions given as follows:

$$\begin{aligned}q_n\to q_{n,1}^{+}&=\frac{i^n\alpha_1\kappa_{1R}\kappa_1^2\bar{\varrho}|\sinh(\kappa_{1R})|\,\mathrm{e}^{\mathrm{i}\xi_{1I}}}{|\alpha_1\kappa_{1R}\kappa_1^2\varrho|}\\&\quad\times\mathrm{sech}\left[\xi_{1\mathrm{R}}+\ln\left(\frac{|\alpha_1\kappa_{1\mathrm{R}}\kappa_1^2|}{8|\varrho\sinh^3(\kappa_{1\mathrm{R}})|\,\mathrm{t}}\right)\right]\\&\quad(t\mathrm{e}^{-\xi_{1R}}=O(1),\ \ t\to+\infty),\end{aligned} \tag{22}$$

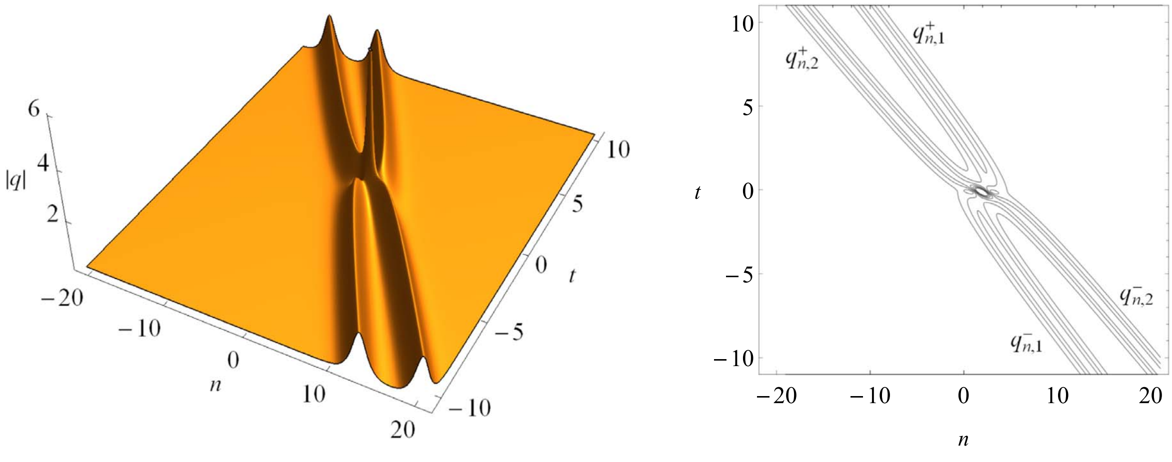
$$\begin{aligned}q_n\to q_{n,1}^{-}&=\frac{i^n\alpha_1\kappa_{1R}\varrho|\sinh(\kappa_{1R})|\,\mathrm{e}^{\mathrm{i}\xi_{1I}}}{|\alpha_1\kappa_{1R}\varrho|}\\&\quad\times\mathrm{sech}\left[\xi_{1\mathrm{R}}+\ln\left(\frac{-|\alpha_1\varrho|\,\mathrm{t}}{2|\kappa_{1\mathrm{R}}\sinh(\kappa_{1\mathrm{R}})|}\right)\right]\\&\quad(t\mathrm{e}^{\xi_{1R}}=O(1),\ \ t\to-\infty),\end{aligned} \tag{23}$$

$$\begin{aligned}q_n\to q_{n,2}^{+}&=-\frac{i^n\alpha_1\kappa_{1R}\varrho|\sinh(\kappa_{1R})|\,\mathrm{e}^{\mathrm{i}\xi_{1I}}}{|\alpha_1\kappa_{1R}\varrho|}\\&\quad\times\mathrm{sech}\left[\xi_{1\mathrm{R}}+\ln\left(\frac{|\alpha_1\varrho|\,\mathrm{t}}{2|\kappa_{1\mathrm{R}}\sinh(\kappa_{1\mathrm{R}})|}\right)\right]\\&\quad(t\mathrm{e}^{\xi_{1R}}=O(1),\ \ t\to+\infty),\end{aligned} \tag{24}$$

$$\begin{aligned}q_n\to q_{n,2}^{-}&=-\frac{i^n\alpha_1\kappa_{1R}\kappa_1^2\bar{\varrho}|\sinh(\kappa_{1R})|\,\mathrm{e}^{\mathrm{i}\xi_{1I}}}{|\alpha_1\kappa_{1R}\kappa_1^2\varrho|}\\&\quad\times\mathrm{sech}\left[\xi_{1\mathrm{R}}+\ln\left(\frac{-|\alpha_1\kappa_{1\mathrm{R}}\kappa_1^2|}{8|\varrho\sinh^3(\kappa_{1\mathrm{R}})|\,\mathrm{t}}\right)\right]\\&\quad(t\mathrm{e}^{-\xi_{1R}}=O(1),\ t\to-\infty),\end{aligned} \tag{25}$$

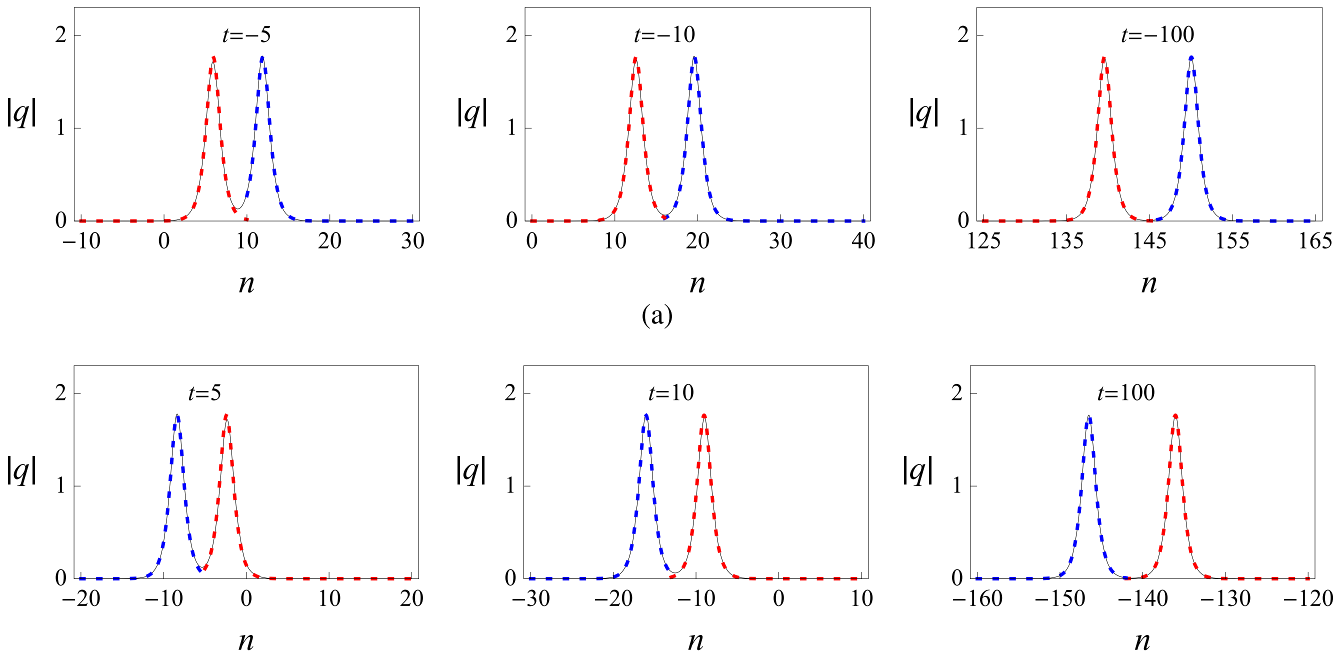
where we have labeled the above four asymptotic solitons based on that the velocity of $q_{n,i}^{+}$ at $t>0$ is the same as that of $q_{n,i}^{-}$ at $t<0$ $(i=1,\,2)$, as given in equation (27).

It can be found that the asymptotic solitons $q_{n,1}^{\pm}$ and $q_{n,2}^{\pm}$ in equations (22)–(25) are all of the bright type and have the



(a) (b)

Figure 3. (a) 3D plot of the double-pole solution (15). (b) Contour plot of the double-pole solution. The parameters are the same as those in figure 1.



(a) (b)

Figure 4. (a) Comparison of the asymptotic expressions $q^-_{n,1}$ (red dashed) and $q^-_{n,2}$ (blue dashed) with the exact solution (15) (black solid) when $t<0$; (b) comparison of the asymptotic soliton $q^+_{n,1}$ (red dashed) and $q^+_{n,2}$ (blue dashed) with the exact solution (15) (black solid) when $t>0$. The selection of parameters follows that in figure 1.

same amplitudes

$$A_1^{\pm} = A_2^{\pm} = |\sinh(\kappa_{1R})|. \tag{26}$$

Note that the center trajectories of $q^{\pm}_{n,1}$ and $q^{\pm}_{n,2}$ are along four curves in the nt plane. Accordingly, the soliton velocities are time-dependent and they can be explicitly given by

$$v_1^{\pm} = \frac{1/|t| - 2\sinh(\kappa_{1R})\cos(\kappa_{1I})}{\kappa_{1R}},$$
$$v_2^{\pm} = -\frac{1/|t| + 2\sinh(\kappa_{1R})\cos(\kappa_{1I})}{\kappa_{1R}}. \tag{27}$$

Also, the phase shifts between $q^+_{n,i}$ and $q^-_{n,i}$ $(i=1,2)$ vary with $|t|$ in a logarithmical law (see figure 1):

$$\delta\phi_1 = -\delta\phi_2 = 2\ln\left(\frac{|\kappa_{1R}\kappa_1|}{2|\varrho\sinh(\kappa_{1R})||t|}\right). \tag{28}$$

Since v_i^+ has the same value at t as that of v_i^- at $-t$ and $\delta\phi_1$ is always opposite to $\delta\phi_2$ at the same t, thus the double-pole solution (15) describes the elastic interactions between two solitons having equal amplitudes.

Moreover, we obtain the relative distance between $q^{+}_{n,1}$ and $q^{+}_{n,2}$ (or between $q^{-}_{n,1}$ and $q^{-}_{n,2}$) as follows:

$$d_{12} = \frac{2}{|\kappa_{1R}|}\ln\left(\frac{2|\varrho\sinh(\kappa_{1R})||t|}{|\kappa_{1R}\kappa_1|}\right) \left(|t| > \frac{|\kappa_{1R}\kappa_1|}{2|\varrho\sinh(\kappa_{1R})|}\right). \tag{29}$$

Meanwhile, we calculate the second derivative of d_{12} with respect to $|t|$:

$$a_{12} = -\frac{8|\varrho|^2\sinh^2(\kappa_{1R})}{|\kappa_{1R}^3\kappa_1^2|}e^{-|\kappa_{1R}|d_{12}}, \tag{30}$$

which represents the acceleration that two solitons separate from each other. Here, $a_{12} < 0$ says that the two-soliton separation speed is decreasing as $|t|$ increases. Thus, we obtain that the relative distance between two solitons grows logarithmically with $|t|$, whereas their separation acceleration magnitude decreases exponentially with the distance and finally tends to 0 (see figure 2). This is in contrast with the $\kappa_1 \neq \kappa_2$ case in the two-soliton solution (8), where the relative distance between two solitons is linear in $|t|$.

In figures 3(a) and (b), we illustrate a two-soliton interaction described by solution (15) and mark all the asymptotic solitons. To show the validity of our asymptotic analysis, we compare the asymptotic solitons $q^{\pm}_{n,i}$ $(i = 1, 2)$ with the exact solution (15) in figure 4. It can be seen that there is a good agreement for the asymptotic expressions to approximate solution (15) at large values of $|t|$.

3.2. Triple-pole solution

Based on the three-soliton solution (9), we let $\varepsilon = 1/\tilde{\varepsilon}^2$, $\kappa_2 = \kappa_1(1+\tilde{\varepsilon})$ and $\kappa_3 = \kappa_1(1-\tilde{\varepsilon})$ (i.e. $r_3 = 2$, $\mu_2 = 1$ and $\mu_3 = -1$). Similarly, for making all the terms like $\tilde{\varepsilon}^{-i}$ $(1 \leqslant i \leqslant 4)$ vanish, we must take $\alpha_2 = -\frac{\alpha_1}{2}$, $\alpha_3 = -\alpha_1 - \alpha_2$ and $x_1^{(2)} + x_1^{(3)} = 2$. Still, $x_2^{(2)} + x_2^{(3)} \neq 2$ and $x_1^{(3)} \neq 1$ are required so that the free parameters s_1 and s_2 both appear in the solution. Without loss of generality, we choose $x_1^{(2)} = 0$, $x_2^{(2)} = 1$, $x_1^{(3)} = 2$ and $x_2^{(3)} = -1$. Thus, taking the limit of solution (9) as $\tilde{\varepsilon} \to 0$, we have the triple-pole solution of equation (1) with $\sigma = 1$ as follows:

$$q_n = i^n\frac{g_n^{(1,1)} + g_n^{(3,3)} + g_n^{(5,5)}}{1 + f_n^{(2,2)} + f_n^{(4,4)} + f_n^{(6,6)}}, \tag{31}$$

where

$$g_n^{(1,1)} = -\frac{1}{2}\alpha_1 e^{\xi_1}[T_2^2 + 2\kappa_1^2\sinh(\kappa_1)t - 2s_2],$$

$$\begin{aligned} f_n^{(2,2)} = \beta_{11}e^{\xi_1+\bar{\xi}_1}\Bigg\{&\frac{|\kappa_1|^4[33 + 26\cosh(2\kappa_{1R}) + \cosh(4\kappa_{1R})]}{32\sinh^4(\kappa_{1R})} \\ &+ \frac{1}{4}|T_2^2 - 2s_2|^2 + \frac{|\kappa_1|^2[11\cosh(\kappa_{1R}) + \cosh(3\kappa_{1R})]}{4\sinh^3(\kappa_{1R})} \\ &\times \mathrm{Re}(\bar{\kappa}_1 T_2) + \frac{|\kappa_1|^2[2 + \cosh(2\kappa_{1R})]}{2\sinh^2(\kappa_{1R})}|T_2|^2 \\ &+ \coth(\kappa_{1R})\mathrm{Re}[\kappa_1 T_2(\bar{T}_2^2 - 2\bar{s}_2)] \\ &+ \frac{2 + \cosh(2\kappa_{1R})}{4\sinh^2(\kappa_{1R})}\mathrm{Re}[\bar{\kappa}_1^2(T_2^2 - 2s_2)]\Bigg\}, \end{aligned}$$

$$\begin{aligned} g_n^{(3,3)} = \frac{\beta_{11}\kappa_1^2\alpha_1 e^{2\xi_1+\bar{\xi}_1}}{2^7\sinh^6(\kappa_{1R})}\{&|\kappa_1|^4[20 + 20\cosh(2\kappa_{1R}) \\ &+ 2\cosh(4\kappa_{1R})] - 8\sinh^4(\kappa_{1R})|T_2^2 - 2s_2|^2 \\ &- \bar{\kappa}_1^2[4\cosh(2\kappa_{1R}) + 4\cosh(4\kappa_{1R}) - 8] \\ &\times (T_2^2 - 2s_2) + \kappa_1^2\sinh^2(2\kappa_{1R})(\bar{T}_2^2 - 2\bar{s}_2) \\ &+ \bar{\kappa}_1|\kappa_1|^2[44\sinh(2\kappa_{1R}) + 8\sinh(4\kappa_{1R})]T_2 \\ &+ \bar{\kappa}_1[8\sinh(4\kappa_{1R}) - 16\sinh(2\kappa_{1R})]T_2|T_2|^2 \\ &- 16\bar{\kappa}_1\sinh^2(\kappa_{1R})\sinh(2\kappa_{1R})\bar{T}_2(T_2^2 - 2s_2) \\ &+ |\kappa_1|^2[8\cosh(2\kappa_{1R}) + 8\cosh(4\kappa_{1R}) - 16]|T_2|^2 \\ &+ \kappa_1|\kappa_1|^2[8\sinh^2(\kappa_{1R}) + 2\sinh(4\kappa_{1R})]\bar{T}_2 \\ &+ \bar{\kappa}_1^2[8\cosh(2\kappa_{1R}) + 8\cosh(4\kappa_{1R}) - 16]T_2^2 \\ &+ 16\sinh^4(\kappa_{1R})T_2^2(\bar{T}_2^2 - 2\bar{s}_2) \\ &+ 8\kappa_1\sinh^2(\kappa_{1R})\sinh(2\kappa_{1R})T_2(\bar{T}_2^2 - 2\bar{s}_2)\}, \end{aligned}$$

$$\begin{aligned} f_n^{(4,4)} &= \frac{\beta_{11}^2|\kappa_1|^4 e^{2\xi_1+2\bar{\xi}_1}}{2^8\sinh^4(\kappa_{1R})} \\ &\times \Bigg\{\frac{|\kappa_1|^4[195 + 212\cosh(2\kappa_{1R}) + 49\cosh(4\kappa_{1R})]}{8\sinh^4(\kappa_{1R})} \\ &+ 4|T_2^2 - 2s_2|^2 - \frac{[10 + 14\cosh(2\kappa_{1R})]\mathrm{Re}[\bar{\kappa}_1^2(T_2^2 - 2s_2)]}{\sinh^2(\kappa_{1R})} \\ &+ \frac{[20 + 28\cosh(2\kappa_{1R})]\mathrm{Re}[\bar{\kappa}_1^2 T_2^2]}{\sinh^2(\kappa_{1R})} + 16|T_2|^4 \\ &+ \frac{|\kappa_1|^2[32\sinh(2\kappa_{1R}) + 14\sinh(4\kappa_{1R})]\mathrm{Re}(\bar{\kappa}_1 T_2)}{\sinh^4(\kappa_{1R})} \\ &- 32\coth(\kappa_{1R})\mathrm{Re}[\bar{\kappa}_1\bar{T}_2(T_2^2 - 2s_2)] \\ &- 16\,\mathrm{Re}[\bar{T}_2^2(T_2^2 - 2s_2)] + \frac{|\kappa_1|^2[40 + 32\cosh(2\kappa_{1R})]}{\sinh^2(\kappa_{1R})}|T_2|^2 \\ &+ 64\coth(\kappa_{1R})\mathrm{Re}(\bar{\kappa}_1 T_2)|T_2|^2\Bigg\}, \end{aligned}$$

$$\begin{aligned} g_n^{(5,5)} = \frac{\beta_{11}^2\kappa_1^4|\kappa_1|^4 e^{3\xi_1+2\bar{\xi}_1}}{2^{11}\sinh^{10}(\kappa_{1R})}\{&4\sinh^2(\kappa_{1R})(\bar{T}_2^2 - 2\bar{s}_2) \\ &- 8\sinh^2(\kappa_{1R})\bar{T}_2^2 - 13\bar{\kappa}_1^2 \\ &- 17\bar{\kappa}_1^2\cosh(2\kappa_{1R}) - 12\bar{\kappa}_1\sinh(2\kappa_{1R})\bar{T}_2\}, \end{aligned}$$

$$f_n^{(6,6)} = \frac{\beta_{11}^3|\kappa_1|^{12}e^{3\xi_1+3\bar{\xi}_1}}{[2\sinh(\kappa_{1R})]^{12}},$$

$$T_2 = 2[\sinh(\kappa_1) - \kappa_1\cosh(\kappa_1)]t - \xi_1 + s_1.$$

Next, in order to understand the soliton interactions in the triple-pole solution, we also make an asymptotic analysis of solution (31) by the same procedure in section 3.1.

On one side, along the straight line $\kappa_{1R}\,n - ct = \mathrm{const}$ with $c = -2\sinh(\kappa_{1R})\cos(\kappa_{1I})$, we derive the following

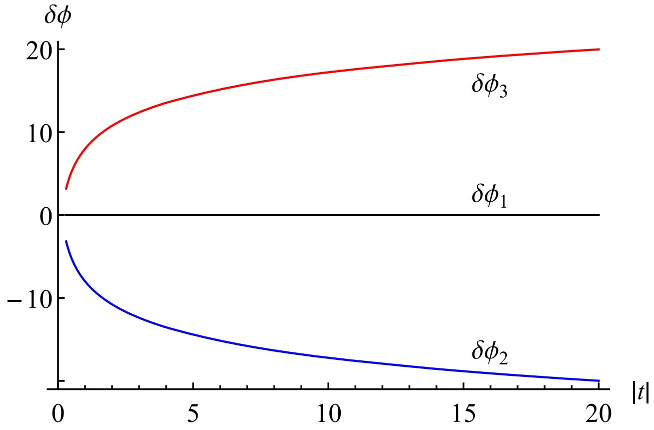


Figure 5. The phase shifts between $q_{n,i}^{+}$ and $q_{n,i}^{-}$ $(1 \leqslant i \leqslant 3)$ versus $|t|$, where the parameters are selected as $\alpha_1 = 2 + 2\mathrm{i}$, $\kappa_1 = \frac{5}{3} + \frac{4}{3}\mathrm{i}$ and $s_1 = s_2 = 1$.

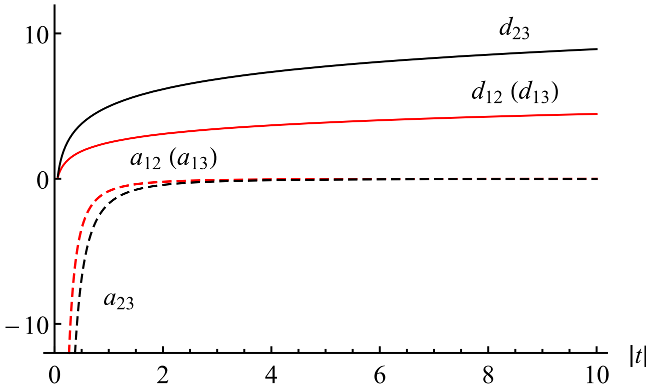


Figure 6. The two-soliton relative distances d_{12} (red solid), d_{13} (red solid), d_{23} (black solid) and separation accelerations a_{12} (red dashed), a_{13} (red dashed), a_{23} (black dashed) versus $|t|$, where the parameters are the same as those in figure 5.

asymptotic expression:

$$q_n \to q_{n,1}^{\pm} = \frac{i^n \alpha_1 \kappa_1^2 \sinh(\kappa_{1R}) \mathrm{e}^{\mathrm{i}\xi_{1I}}}{|\alpha_1||\kappa_1|^2} \times \operatorname{sech}\left[\xi_{1R} + \ln\left(\frac{|\alpha_1||\kappa_1|^2}{8|\sinh(\kappa_{1R})|^3}\right)\right] (t \to \pm\infty), \tag{32}$$

which represents a pair of bright solitons lying in the same straight line $\xi_{1R} + \ln\left(\frac{|\alpha_1||\kappa_1|^2}{8|\sinh(\kappa_{1R})|^3}\right) = 0$ as $t \to \pm\infty$. On the other side, via equation (16) we substitute ξ_{1I} for ξ_{1R} in solution (31), and then find that the asymptotic solitons are formed when the dominant balance $t \sim \mathrm{e}^{\frac{1}{2}\xi_{1R}}$ or $t \sim \mathrm{e}^{-\frac{1}{2}\xi_{1R}}$ is reached as $|t| \to \infty$. Hence, we can obtain four asymptotic bright solitons with the curved center trajectories, and their expressions are given by

$$\begin{aligned} q_n \to q_{n,2}^{+} &= -\frac{i^n \alpha_1 \kappa_1^4 \bar{\varrho}^2 |\sinh(\kappa_{1R})|\, \mathrm{e}^{\mathrm{i}\xi_{1I}}}{|\alpha_1 \kappa_1^4 \varrho^2|} \\ &\quad \times \operatorname{sech}\left[\xi_{1R} + 2\ln\left(\frac{|\alpha_1^{\frac{1}{2}} \kappa_{1R} \kappa_1^2|}{4|\varrho \sinh^{\frac{5}{2}}(\kappa_{1R})|\, t}\right)\right] \\ &\quad (t\mathrm{e}^{-\frac{1}{2}\xi_{1R}} = O(1),\ t \to +\infty), \end{aligned} \tag{33}$$

$$\begin{aligned} q_n \to q_{n,2}^{-} &= -\frac{i^n \alpha_1 \varrho^2 |\sinh(\kappa_{1R})|\, \mathrm{e}^{\mathrm{i}\xi_{1I}}}{|\alpha_1 \varrho^2|} \\ &\quad \times \operatorname{sech}\left[\xi_{1R} + 2\ln\left(\frac{-|\alpha_1^{\frac{1}{2}} \varrho|\, t}{2|\kappa_{1R} \sinh^{\frac{1}{2}}(\kappa_{1R})|}\right)\right] \\ &\quad \times (t\mathrm{e}^{\frac{1}{2}\xi_{1R}} = O(1),\ t \to -\infty), \end{aligned} \tag{34}$$

$$\begin{aligned} q_n \to q_{n,3}^{+} &= -\frac{i^n \alpha_1 \varrho^2 |\sinh(\kappa_{1R})|\, \mathrm{e}^{\mathrm{i}\xi_{1I}}}{|\alpha_1 \varrho^2|} \\ &\quad \times \operatorname{sech}\left[\xi_{1R} + 2\ln\left(\frac{|\alpha_1^{\frac{1}{2}} \varrho|\, t}{2|\kappa_{1R} \sinh^{\frac{1}{2}}(\kappa_{1R})|}\right)\right] \\ &\quad \times (t\mathrm{e}^{\frac{1}{2}\xi_{1R}} = O(1),\ t \to +\infty), \end{aligned} \tag{35}$$

$$\begin{aligned} q_n \to q_{n,3}^{-} &= -\frac{i^n \alpha_1 \kappa_1^4 \bar{\varrho}^2 |\sinh(\kappa_{1R})| \mathrm{e}^{\mathrm{i}\xi_{1I}}}{|\alpha_1 \kappa_1^4 \varrho^2|} \\ &\quad \times \operatorname{sech}\left[\xi_{1R} + 2\ln\left(\frac{-|\alpha_1^{\frac{1}{2}} \kappa_{1R} \kappa_1^2|}{4|\varrho \sinh^{\frac{5}{2}}(\kappa_{1R})| t}\right)\right] \\ &\quad \times (t\mathrm{e}^{-\frac{1}{2}\xi_{1R}} = O(1),\ t \to -\infty), \end{aligned} \tag{36}$$

where we have also labeled the above four asymptotic solitons in view that the velocity of $q_{n,i}^{+}$ at $t > 0$ is the same as that of $q_{n,i}^{-}$ at $t < 0$ $(i = 2, 3)$, as given in equation (37).

Like the double-pole case, $q_{n,i}^{\pm}$ in equations (32)–(36) have the same amplitudes $A_i^{\pm} = |\sinh(\kappa_{1R})|$ $(1 \leqslant i \leqslant 3)$. Also, the velocities of asymptotic solitons $q_{n,2}^{\pm}$ and $q_{n,3}^{\pm}$ are time-dependent:

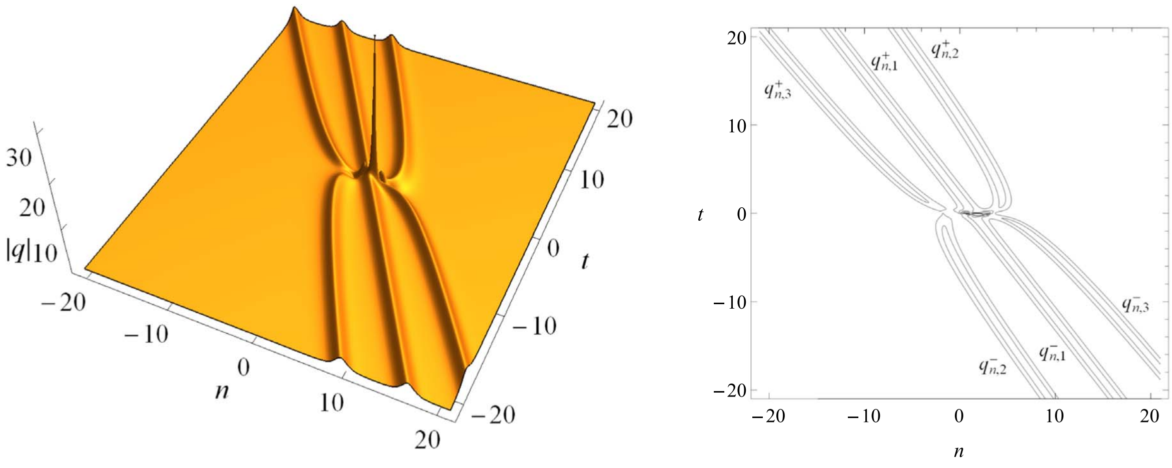
$$\begin{aligned} v_2^{\pm} &= \frac{2/|t| - 2\sinh(\kappa_{1R})\cos(\kappa_{1I})}{\kappa_{1R}}, \\ v_3^{\pm} &= -\frac{2/|t| + 2\sinh(\kappa_{1R})\cos(\kappa_{1I})}{\kappa_{1R}}, \end{aligned} \tag{37}$$

and their phase shifts change with $|t|$ in the following logarithmical manner (see figure 5):

$$\delta\phi_2 = -\delta\phi_3 = 4\ln\left(\frac{|\kappa_{1R}\kappa_1|}{\sqrt{2}|\varrho \sinh(\kappa_{1R})||t|}\right). \tag{38}$$

However, $q_{n,1}^{\pm}$ have the constant velocities

$$v_1^{\pm} = -\frac{2\sinh(\kappa_{1R})\cos(\kappa_{1I})}{\kappa_{1R}}, \tag{39}$$



(a) (b)

Figure 7. (a) 3D plot of the triple-pole solution (31); (b) Contour plot of the triple-pole solution. The parameters are the same as those in figure 5.

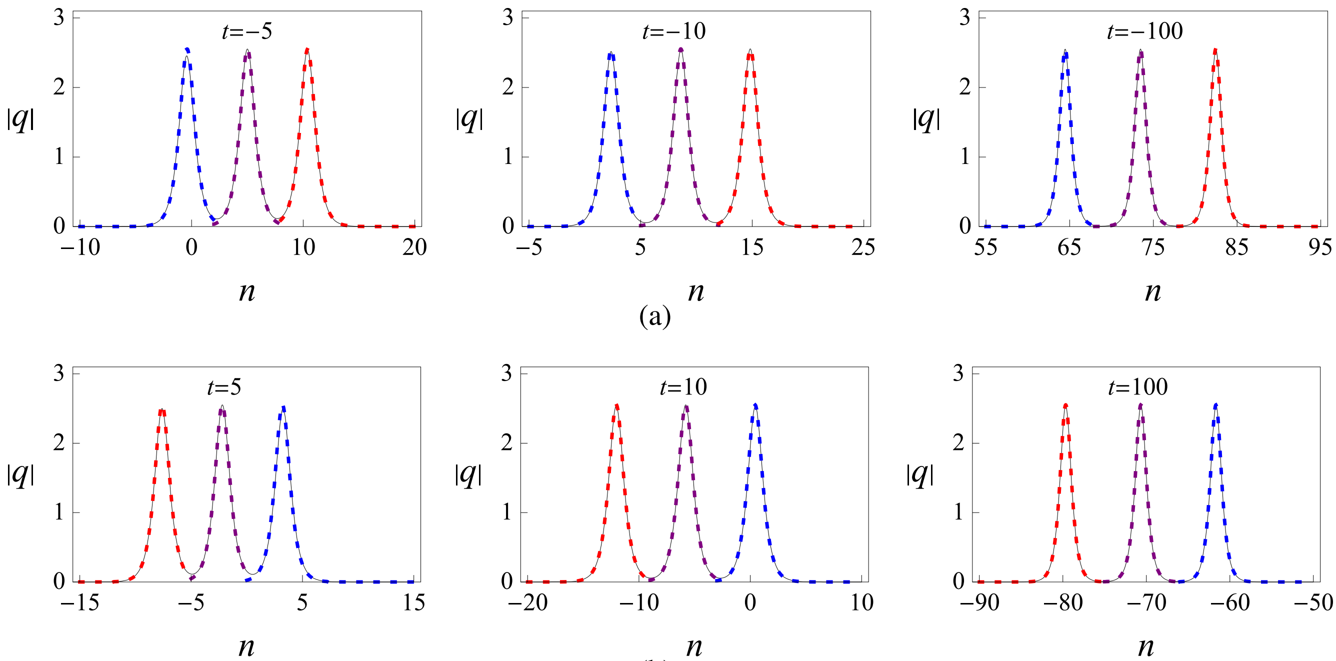


Figure 8. (a) Comparison of the asymptotic expressions $q^-_{n,1}$ (purple dashed), $q^-_{n,2}$ (blue dashed) and $q^-_{n,3}$ (red dashed) with the exact solution (31) (black solid) when $t<0$; (b) comparison of the asymptotic soliton $q^+_{n,1}$ (purple dashed), $q^+_{n,2}$ (blue dashed) and $q^+_{n,3}$ (red dashed) with the exact solution (31) (black solid) when $t>0$. The selection of parameters follows that in figure 5.

and experience no phase shift before and after interaction. Therefore, the triple-pole solution (31) describes the elastic interactions among three solitons with equal amplitudes.

Moreover, we derive the relative distances between two solitons among $q^+_{n,i}$ (or $q^-_{n,i}$) $(1\leqslant i\leqslant 3)$ as

$$d_{12} = d_{13} = \frac{1}{2}d_{23} = \frac{1}{|\kappa_{1R}|}\ln\left(\frac{2|\varrho|^2\sinh^2(\kappa_{1R})t^2}{\kappa_{1R}^2|\kappa_1|^2}\right) \times\left(|t| > \frac{|\kappa_{1R}\kappa_1|}{\sqrt{2}|\varrho||\sinh(\kappa_{1R})|}\right), \tag{40}$$

and calculate the second derivatives of d_{ij} $(1\leqslant i<j\leqslant 3)$ with respect to $|t|$:

$$a_{12} = a_{13} = -\frac{4|\varrho|^2\sinh^2(\kappa_{1R})}{|\kappa_{1R}^3\kappa_1^2|}\mathrm{e}^{-|\kappa_{1R}|d_{12}},$$
$$a_{23} = -\frac{8|\varrho|^2\sinh^2(\kappa_{1R})}{|\kappa_{1R}^3\kappa_1^2|}\mathrm{e}^{-\frac{|\kappa_{1R}|}{2}d_{23}}, \tag{41}$$

which are the accelerations that two solitons diverge from each other. Also, we obtain that the solitons' relative distances grow logarithmically with $|t|$, while the separation acceleration magnitudes decrease exponentially with the distance and finally tend to 0 (see figure 6).

In figures 7(a) and (b), we show the three-soliton interactions in solution (31) and the distribution of all asymptotic solitons. In addition, figure 8 presents the comparison of the asymptotic solitons $q_{n,i}^{\pm}$ $(1 \leqslant i \leqslant 3)$ with the exact solution (31) at different values of t. One can see that the asymptotic expressions give a good approximation to solution (31) when $|t| \gg 1$.

4. Conclusions and discussions

In this paper, for the focusing AL equation, we have derived the double- and triple-pole solutions (15) and (31) as the degenerate cases of the two- and three-soliton solutions, respectively. Via the same limit technique in section 3, one can continue to construct an arbitrary Mth ($M > 3$) order MPS. For doing so, we need to set $\varepsilon = 1/\tilde{\varepsilon}^{M-1}$ and the Mth order MPS can be obtained in the form

$$q_n = i^n \frac{g_n^{(1,1)} + g_n^{(3,3)} + \cdots + g_n^{(2M-1,2M-1)}}{1 + f_n^{(2,2)} + f_n^{(4,4)} + \cdots + f_n^{(2M,2M)}}. \tag{42}$$

We believe that this method can be used to construct the MPSs for other soliton equations [5, 20–23, 25, 29, 48, 49, 50] if their multi-soliton solutions are available. However, by the same procedure we have not obtained the MPSs from the dark multi-soliton solutions of equation (1) with $\sigma = -1$. It is worthwhile to make clear whether there is no dark MPSs in the defocusing AL equation, similar to the defocusing case of NLSE [16]. In the future, we will continue to work on this problem by some other methods, e.g. the IST or Darboux transformation.¼

On the other hand, we have performed a detailed asymptotic analysis of the double- and triple-pole solutions. It should be noted that an improved asymptotic analysis method [6], which relies on the balance between exponential and algebraic terms, has been used in obtaining the expressions of all asymptotic solitons. On this basis, we have revealed that the asymptotic solitons in solution (15) are localized in some curves of the nt plane and their velocities are time-dependent; and that two pairs of asymptotic solitons in solution (31) lie in some curves, while the third pair (which has a constant velocity and experiences no phase shift) is along a straight line. Within our knowledge, we infer that when M is odd there are $M - 1$ pairs of solitons moving in the logarithmic curves and one pair of solitons lying in a straight line, while for even M all the solitons are localized in the logarithmic curves. Similarly to the continuous MPSs of the focusing NLSE [3, 13, 15], we have also revealed the discrete MPSs describe the elastic interactions of multiple solitons with the same amplitudes, in which the solitons' relative distances grow logarithmically with $|t|$, while the separation acceleration magnitudes decrease exponentially with their relative distances.

Finally, we point out that it might be an interesting issue to study the soliton interaction behavior at $t \approx 0$ in the MPSs. As shown in figures 3(a) and 7(a), the interactions of multiple solitons with equal amplitudes may cause the occurrence of an instant large-amplitude wave. Qualitatively, the amplitude of such an instant wave is related to s_i $(1 \leqslant i \leqslant M - 1)$, but those parameters are all absent in the expressions of asymptotic solitons. Hence, one has to study the MPSs themselves so as to accurately understand the scenarios of soliton interactions in the near-field region $|t| \ll 1$. However, this is a challenging work since a large amount of calculations will be involved.

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