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Multigrid method with eighth-order compact finite difference scheme for Helmholtz equation

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Abstract

Higher-order compact finite difference scheme with multigrid algorithm is applied in this paper for solving one-dimensional and two-dimensional inhomogeneous Helmholtz equations. In two-dimensional case, the suggested scheme has the stencil of twenty one points. An efficient solver multigrid method yields eighth-order accurate approximation on both fine and coarse grids. For the Neumann boundary condition, an eighth-order accurate representation is also developed. The accuracy and efficiency of eighth-order compact difference scheme are exhibited through graphical illustrations and computed results are drafted in tabular form to validate the numerical experiments.

Keywords: Helmhlotz equation, compact iterative scheme, multigrid method, uniform grids

(Some figures may appear in colour only in the online journal)

1. Introduction

In recent years, there has been a growing interest in the development of higher-order schemes for solving partial differential equations (PDEs). To achieve high order numerical solutions for partial differential equations, one has either to use high order accurate methods that require an increase in the stencil of grid points or increase in the number of nodes by taking smaller grid sizes. But taking smaller grid sizes need much more CPU time and storage space. Therefore, higher-order compact (HOC) methods are desirable to solve PDEs numerically. There are two approaches to compute higher-order compact schemes, that are Padé approximation by Baker *et al* [1] and the Taylor's series method by Strikwerda [2].

Compact schemes are high order explicit/implicit methods which boast high order accuracy by using smaller grid sizes. Higher-order compact schemes are frequently using almost in every area of computational problems such as the convection–diffusion problems by Zhang *et al* [3–5], the Poisson equation by Wang and Zhang *et al* [6–9] and the Helmholtz equation by Hirtum [10], Handlovičová and Riečanová [11], and Ghaffar *et al* [12–16]. Helmholtz equation is an elliptic partial differential equation having many applications in real life such as in propagation of electromagnetic waves, radar scattering, elasticity, weather prediction, acoustic wave scattering, electromagnetic fields, medical imaging, seismology, acoustics and electromagnetic radiation, noise reduction in silencers, water wave propagation, electrostatics, theoretical physics, science of optics, mechanical engineering and fluid mechanics.

In this article, we derive compact eighth-order scheme for solving one-dimensional and two-dimensional Helmholtz equations. Let Ω be an open bounded square domain that is

$$u_{xx} + u_{yy} + k^2 u = f(x, y), \qquad (x, y) \in \Omega, \tag{1}$$

where $u = u(x, y)$ is the solution and $f(x, y)$ is the forcing function, both are taken to be enough smooth and have the required continuous derivatives. Equation (1) appears from solution of the wave equation. In the above equation $k = \frac{\mathrm{i}\,\omega}{c}$ is the wave number in a dispersive medium, where ω is the wave frequency and c is the velocity of light. Helmholtz equation can also be obtained from the solution of linearized Poisson equation. In this case, $k = q\sqrt{\frac{8\,\pi\,\beta\,c}{\epsilon}}$, in which q represents the charge of an ion, β is the inverse thermal energy, c is the ionic concentration, and ϵ is the dielectric constant. To achieve the desired computational efficiency and performance of HOC schemes; a complete characterization of the truncation error must be formulated and minimized. HOC schemes for elliptic problems have been studied by Spotz and Carey [17], Ge and Zhang [18], and Sun *et al* [19].

As the second order solution can be computed by standard central difference operators to u_{xx} and u_{yy} in equation (1). Higher order discretization schemes require more complex process and computation in order to compute the coefficient matrix of small size [7, 12, 16, 20]. The interest in developing higher-order compact methods has been growing to solve PDEs [3, 4, 7, 17, 21, 22]. These schemes use the stencil of minimum three points in discretization formulas, so we call them compact. The compact finite difference (CFD) method is one of the methods, which are used to increase accuracy and efficiency of the numerical solutions of PDEs. The basic procedure for generating higher-order difference schemes is to expand the stencil of grid points. The efforts to calculate more accurate solution by using limited grid sizes have drawn the attention of researchers to develop higher-order compact (HOC) schemes for Helmholtz equation. In context of fourth-order and sixth-order discretization, many researchers have been focused on equal and unequal grid sizes [23–27]. A noticeable work for inhomogeneous Helmholtz equation has been done by Singer and Turkel [28].

Previously, some implicit HOC schemes were used for two-dimensional Stommel Ocean model and convection–diffusion equation by Chu and Fan [29, 30]. Also explicit HOC schemes with accelerating iterative methods like multigrid methods were used to solve the resulting sparse system arisen from the discretization of HOC schemes by Zhang [5, 7, 9]. As we know, it is assumed that there is no explicit compact scheme higher than sixth-order accuracy. By using the idea of Nabavi's method [25], we aim to develop a new eighth-order compact scheme for one-dimensional and two-dimensional Helmholtz equations with Dirichlet and Neumann boundary conditions. We develop a multigrid method to solve the required sparse linear system to get higher-order accurate solution on both the coarse and fine grids. The organization of the paper is as follows; in section 2, we present the main idea of the proposed eighth-order compact difference discretization strategy for one-dimensional and two-dimensional Helmholtz equations. Section 3 focuses on the development of eighth-order compact difference schemes in the case of Neumann boundary conditions. Multigrid method is discussed in section 4. Section 5 contains the series of numerical calculations that demonstrates the accuracy of our proposed eighth-order scheme. Concluding remarks are presented in section 6.

2. Materials and methods

In this section, we present the main idea of eighth-order compact difference discretization strategy for one-dimensional and two-dimensional Helmholtz equations.

2.1. Higher-order (eighth-order) compact scheme

The numerical study has been conducted to develop HOC scheme based on eighth-order approximation computed from the Helmholtz equation. This scheme is formulated for one- and two-dimensions. In this study, a uniform grid of the interval $[a, b]$ is used with N uniform segments, so that the grid spacing $\Delta\,x = \Delta\,y = h$, is along $x-$ and $y-$ directions. The first-order central difference scheme with respect to x with $u_{i,j} = u(x_{i,j})$ is defined as

$$\delta_x u_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} + \mathcal{O}(h^2), \tag{2}$$

where $(x_{i,j})$ is the grid point at ith, jth steps. Similarly the second-order two-dimensional central difference scheme with respect to x is denoted by δ_x^2 and defined as

$$\delta_x^2 u_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \mathcal{O}(h^2). \tag{3}$$

Likewise, difference operators δ_y and δ_y^2 are defined accordingly.

2.2. One-dimensional case

The one-dimensional Helmholtz equation is

$$u''(x) + k^2 u(x) = f(x) \quad \text{for} \quad x \in [a, b]. \tag{4}$$

Using Taylor's series expansion to obtain the required description as

$$u_{i+1} = u_i + h\,u_i' + \frac{h^2}{2\,!}\,u_i'' + \frac{h^3}{3\,!}\,u_i^{(3)} + \frac{h^4}{4\,!}\,u_i^{(4)} + \frac{h^5}{5\,!}\,u_i^{(5)} + \frac{h^6}{6\,!}u_i^{(6)} + \frac{h^7}{7\,!}u_i^{(7)} + \frac{h^8}{8\,!}u_i^{(8)} + \frac{h^9}{9\,!}u_i^{(9)} + \mathcal{O}(h^{10}), \tag{5}$$

$$u_{i-1} = u_i - h\,u_i' + \frac{h^2}{2\,!}\,u_i'' - \frac{h^3}{3\,!}\,u_i^{(3)} + \frac{h^4}{4\,!}\,u_i^{(4)} - \frac{h^5}{5\,!}\,u_i^{(5)} + \frac{h^6}{6\,!}u_i^{(6)} - \frac{h^7}{7\,!}u_i^{(7)} + \frac{h^8}{8\,!}u_i^{(8)} - \frac{h^9}{9\,!}u_i^{(9)} + \mathcal{O}(h^{10}). \tag{6}$$

Using the definitions of δ_x and δ_x^2 , we have

$$\delta_x u_i = \frac{u_{i+1} - u_{i-1}}{2h} = u_i' + \frac{h^2}{6}\,u_i^{(3)} + \frac{h^4}{120}\,u_i^{(5)} + \frac{h^6}{5040}u_i^{(7)} + \mathcal{O}(h^8), \tag{7}$$

$$\delta_x^2 u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = u_i'' + \frac{h^2}{12}\,u_i^{(4)} + \frac{h^4}{360}\,u_i^{(6)} + \frac{h^6}{20160}\,u_i^{(8)} + \mathcal{O}(h^8). \tag{8}$$

In order to get the eighth-order compact difference scheme for equation (4), applying δ_x^2 to $u_i^{(6)}$ using equation (3), we have

$$u_i^{(8)} = \delta_x^2\, u_i^{(6)} + \mathcal{O}(h^2). \tag{9}$$

Substituting equation (9) in equation (8), we have

$$\delta_x^2 u_i = u_i'' + \frac{h^2}{12}\, u_i^{(4)} + \frac{h^4}{360}\, u_i^{(6)} + \frac{h^6}{20160}\, \delta_x^2\, u_i^{(6)} + \mathcal{O}(h^8). \tag{10}$$

The descritized form of equation (4) is

$$u_i'' = f_i - k^2\, u_i. \tag{11}$$

From equation (11), we have $u_i^{(4)} = f_i'' - k^2 u_i''$ and $u_i^{(6)} = f_i^{(4)} - k^2 u_i^{(4)}$. Substituting in equation (10), we get

$$\delta_x^2 u_i = \left(1 - \frac{k^2h^2}{12} + \frac{k^4h^4}{360} - \frac{k^4h^6}{20160}\, \delta_x^2\right) u_i'' + \frac{h^2}{12}\left(1 - \frac{k^2h^2}{30} - \frac{k^2h^4}{1680}\, \delta_x^2\right) f_i'' + \frac{h^4}{360}\left(1 + \frac{h^2}{56}\, \delta_x^2\right) f_i^{(4)} + \mathcal{O}(h^8). \tag{12}$$

After some simplification, we get

$$\delta_x^2 u_i = \left(1 - \frac{k^2h^2}{12} + \frac{k^4h^4}{360} - \frac{k^6h^6}{20160}\right) u_i'' + \frac{h^2}{12}\left(1 - \frac{k^2h^2}{30} + \frac{k^4h^4}{1680} - \frac{k^2h^4}{1680}\, \delta_x^2\right) f_i'' + \frac{h^4}{360}\left(1 + \frac{h^2}{56}\, \delta_x^2\right) f_i^{(4)} + \mathcal{O}(h^8). \tag{13}$$

$$\Rightarrow\ u_i'' = \frac{\delta_x^2 u_i - \frac{h^2}{12}\left(1 - \frac{k^2h^2}{30} + \frac{k^4h^4}{1680} - \frac{k^2h^4}{1680}\, \delta_x^2\right) f_i'' - \frac{h^4}{360}\left(1 + \frac{h^2}{56}\, \delta_x^2\right) f_i^{(4)}}{1 - \frac{k^2h^2}{12} + \frac{k^4h^4}{360} - \frac{k^6h^6}{20160}}. \tag{14}$$

Let v_i denotes the eighth-order approximation of u_i, i.e., $u_i = v_i + \mathcal{O}(h^8)$. Using equation (4) and equation (14), together with $\delta_x^2 f_i^{(4)} = \frac{f_{i+1}^{(4)} - 2f_i^{(4)} + f_{i-1}^{(4)}}{h^2} + \mathcal{O}(h^2)$, substituting for u_i'' in equation (11), thus for one-dimensional Helmholtz equation, a three-point eighth-order compact scheme is

$$a_1 v_{i+1} + a_0 v_i + a_1 v_{i-1} = b_1 f_{i-1} + b_0 f_i + b_1 f_{i+1} + c_1 f_{i-1}'' + c_0 f_i'' + c_1 f_{i+1}'' + d_1 f_{i-1}^{(4)} + d_0 f_i^{(4)} + d_1 f_{i+1}^{(4)}, \tag{15}$$

where $a_0 = \left(-2 + k^2h^2 - \frac{k^4h^4}{12} + \frac{3k^6h^6}{1120}\right)$, $a_1 = \left(1 + \frac{h^6k^6}{20160}\right)$, $b_0 = h^2\left(1 - \frac{k^2h^2}{12} + \frac{3h^4k^4}{1120}\right)$, $b_1 = \frac{h^6k^4}{20160}$, $c_0 = \frac{h^2}{12}\left(1 - \frac{9h^2k^2}{280}\right)$, $c_1 = \left(-\frac{h^6k^2}{20160}\right)$, $d_0 = \frac{3h^6}{1120}$, $d_1 = \frac{h^6}{20160}$. Also f and f'' are to be known analytically at every grid point and the R.H.S of equation (15) is known for all nodes. In case, where f is not known explicitly, then only sixth-order accurate approximation for f'' is required and fourth-order for $f^{(4)}$.

2.3. Two-dimensional case

The Helmholtz equation in two-dimensional form is

$$u_{xx} + u_{yy} + k^2 u \;=\; f(x, y), \qquad (x, y) \in \Omega. \tag{16}$$

Using Taylor's series expansion to get the required description as

$$u(i+1, j) = u_{i,j} + h\,\frac{\partial\, u_{i,j}}{\partial\, x} + \frac{h^2}{2\,!}\,\frac{\partial^2 u_{i,j}}{\partial\, x^2} + \frac{h^3}{3\,!}\,\frac{\partial^3 u_{i,j}}{\partial\, x^3} + \frac{h^4}{4\,!}\,\frac{\partial^4 u_{i,j}}{\partial\, x^4} + \frac{h^5}{5\,!}\,\frac{\partial^5 u_{i,j}}{\partial\, x^5} + \frac{h^6}{6\,!}\,\frac{\partial^6 u_{i,j}}{\partial\, x^6} + \frac{h^7}{7\,!}\,\frac{\partial^7 u_{i,j}}{\partial\, x^7} + \frac{h^8}{8\,!}\,\frac{\partial^8 u_{i,j}}{\partial\, x^8} + \frac{h^9}{9\,!}\,\frac{\partial^9 u_{i,j}}{\partial\, x^9} + \mathcal{O}(h^{10}), \tag{17}$$

$$u(i-1, j) = u_{i,j} - h\,\frac{\partial\, u_{i,j}}{\partial\, x} + \frac{h^2}{2\,!}\,\frac{\partial^2 u_{i,j}}{\partial\, x^2} - \frac{h^3}{3\,!}\,\frac{\partial^3 u_{i,j}}{\partial\, x^3} + \frac{h^4}{4\,!}\,\frac{\partial^4 u_{i,j}}{\partial\, x^4} - \frac{h^5}{5\,!}\,\frac{\partial^5 u_{i,j}}{\partial\, x^5} + \frac{h^6}{6\,!}\,\frac{\partial^6 u_{i,j}}{\partial\, x^6} - \frac{h^7}{7\,!}\,\frac{\partial^7 u_{i,j}}{\partial\, x^7} + \frac{h^8}{8\,!}\,\frac{\partial^8 u_{i,j}}{\partial\, x^8} - \frac{h^9}{9\,!}\,\frac{\partial^9 u_{i,j}}{\partial\, x^9} + \mathcal{O}(h^{10}). \tag{18}$$

Adding equations (17) and (18) which gives

$$\frac{\partial^2 u_{i,j}}{\partial\, x^2} = \frac{u_{i+1,j} - 2\, u_{i,j} + u_{i-1,j}}{h^2} - \frac{h^2}{12}\frac{\partial^4 u_{i,j}}{\partial\, x^4} - \frac{h^4}{360}\frac{\partial^6 u_{i,j}}{\partial\, x^6} - \frac{h^6}{20160}\frac{\partial^8 u_{i,j}}{\partial\, x^8} + \mathcal{O}(h^8). \tag{19}$$

Using the definition of central difference scheme in equation (19), we have

$$\delta_x^2 u_{i,j} = \frac{\partial^2 u_{i,j}}{\partial\, x^2} + \frac{h^2}{12}\frac{\partial^4 u_{i,j}}{\partial\, x^4} + \frac{h^4}{360}\frac{\partial^6 u_{i,j}}{\partial\, x^6} + \frac{h^6}{20160}\frac{\partial^8 u_{i,j}}{\partial\, x^8} + \mathcal{O}(h^8). \tag{20}$$

Similarly, we can find expression like equation (20) for y-direction. Hence, equation (16) can be written as:

$$\delta_x^2 u_{i,j} + \delta_y^2 u_{i,j} + k^2 u_{i,j} + \alpha_{i,j} = f_{i,j} + \mathcal{O}(h^8), \tag{21}$$

where

$$\alpha_{i,j} = -\frac{h^2}{12}\left[\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right]_{i,j} - \frac{h^4}{360}\left[\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right]_{i,j} - \frac{h^6}{20160}\left[\frac{\partial^8 u}{\partial x^8} + \frac{\partial^8 u}{\partial y^8}\right]_{i,j} + \mathcal{O}(h^8). \tag{22}$$

The higher-order derivatives can be obtained by differentiating equation (16). The forcing function $f(x, y)$ is also included in this process of differentiation. Applying the higher derivatives of equation (16), we can write

$$\left(\frac{\partial^4 u}{\partial x^4}\right)_{i,j} = \left[\frac{\partial^2 f}{\partial x^2} - k^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^2\,\partial y^2}\right]_{i,j},\quad \left(\frac{\partial^4 u}{\partial y^4}\right)_{i,j} = \left[\frac{\partial^2 f}{\partial y^2} - k^2\frac{\partial^2 u}{\partial y^2} - \frac{\partial^4 u}{\partial y^2\,\partial x^2}\right]_{i,j}. \tag{23}$$

Making use of equation (23) in equation (22), we get

$$\alpha_{i,j} = -\frac{h^2}{12}\left[\left\{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right\}_{i,j} - k^2\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right\}_{i,j} - 2\left\{\frac{\partial^4 u}{\partial x^2\partial y^2}\right\}_{i,j}\right] - \frac{h^4}{360}\left[\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right]_{i,j} - \frac{h^6}{20160}\left[\frac{\partial^8 u}{\partial x^8} + \frac{\partial^8 u}{\partial y^8}\right]_{i,j} + \mathcal{O}(h^8). \tag{24}$$

To obtain the higher-order approximation of $\left[\frac{\partial^4 u}{\partial x^2\,\partial y^2}\right]_{i,j}$ in equation (24), applying Taylor's series expansion such that

$$\left[\frac{\partial^4 u}{\partial^2 x\,\partial^2 y}\right]_{i,j} = \delta_x^2\,\delta_y^2\,u_{i,j} - \frac{h^2}{12}\left[\frac{\partial^6 u}{\partial x^4\,\partial y^2} + \frac{\partial^6 u}{\partial y^4\,\partial x^2}\right]_{i,j} - \frac{h^4}{360}\left[\frac{\partial^8 u}{\partial x^6\,\partial y^2} + \frac{\partial^8 u}{\partial y^6\,\partial x^2}\right]_{i,j} + \mathcal{O}(h^6). \tag{25}$$

Now making use of equation (25) in equation (24), which becomes

$$\alpha_{i,j} = \frac{h^2}{12}[-\nabla^2 f_{i,j} + k^2 f_{i,j} - k^4 u_{i,j} + 2\,\delta_x^2\,\delta_y^2\,u_{i,j}] - \frac{h^4}{360}\left[\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6} + 5\left\{\frac{\partial^6 u}{\partial x^4\partial y^2} + \frac{\partial^6 u}{\partial y^4\partial x^2}\right\}\right]_{i,j} - \frac{h^6}{2160}\left[\frac{\partial^8 u}{\partial x^6\partial y^2} + \frac{\partial^8 u}{\partial y^6\partial x^2}\right]_{i,j} - \frac{h^6}{20160}\left[\frac{\partial^8 u}{\partial x^8} + \frac{\partial^8 u}{\partial y^8}\right]_{i,j} + \mathcal{O}(h^8), \tag{26}$$

where the operator ∇^2 is Laplacian. For eighth-order compact approximation, again differentiating equation (16), we have

$$\frac{\partial^6 u_{i,j}}{\partial x^4\,\partial y^2} + \frac{\partial^6 u_{i,j}}{\partial y^4\,\partial x^2} = \frac{\partial^4 f_{i,j}}{\partial x^2\,\partial y^2} - k^2\frac{\partial^4 u_{i,j}}{\partial x^2\,\partial y^2}, \tag{27}$$

$$\left[\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right]_{i,j} = \nabla^4 f_{i,j} - k^2\nabla^4 u_{i,j} - \left[\frac{\partial^6 u}{\partial x^4\,\partial y^2} + \frac{\partial^6 u}{\partial y^4\,\partial x^2}\right]_{i,j}, \tag{28}$$

where ∇^4 is bi-harmonic operator. Making use of equation (27) in equation (28), we get

$$\left[\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right]_{i,j} = \nabla^4 f_{i,j} - k^2\nabla^2 f_{i,j} + k^4 f_{i,j} - k^6 u_{i,j} - \left[\frac{\partial^4 f}{\partial x^2\partial y^2}\right]_{i,j} + 3k^2\left[\frac{\partial^4 u}{\partial x^2\partial y^2}\right]_{i,j}. \tag{29}$$

Using equations (28) and (29) in equation (26), we have

$$\alpha_{i,j} = \frac{h^2}{12}[-\nabla^2 f + 2\delta_x^2\delta_y^2 u + k^2 f - k^4 u]_{i,j} - \frac{h^4}{360}\left[\nabla^4 f - k^2\nabla^2 f + k^4 f - k^6 u + 4\left\{\frac{\partial^4 f}{\partial x^2\,\partial y^2}\right\} - 2k^2\left\{\frac{\partial^4 u}{\partial x^2\partial y^2}\right\}\right]_{i,j} - \frac{h^6}{181440}\left[9\left\{\frac{\partial^8 u}{\partial x^8} + \frac{\partial^8 u}{\partial y^8}\right\} + 84\left\{\frac{\partial^8 u}{\partial x^6\partial y^2} + \frac{\partial^8 u}{\partial y^6\partial x^2}\right\}\right]_{i,j} + \mathcal{O}(h^8). \tag{30}$$

Now

$$\left(\frac{\partial^8 u}{\partial x^8}\right)_{i,j} = \left[\frac{\partial^6 f}{\partial x^6} - k^2\frac{\partial^6 u}{\partial x^6} - \frac{\partial^8 u}{\partial x^6\partial y^2}\right]_{i,j} \quad \text{and}$$
$$\left(\frac{\partial^8 u}{\partial y^8}\right)_{i,j} = \left[\frac{\partial^6 f}{\partial y^6} - k^2\frac{\partial^6 u}{\partial y^6} - \frac{\partial^8 u}{\partial y^6\partial x^2}\right]_{i,j}. \tag{31}$$

Also

$$\left[\frac{\partial^8 u}{\partial x^6\partial y^2}\right]_{i,j} = \left[\frac{\partial^6 f}{\partial x^4\partial y^2} - k^2\frac{\partial^6 u}{\partial x^4\partial y^2} - \frac{\partial^8 u}{\partial x^4\partial y^4}\right]_{i,j},$$
$$\left[\frac{\partial^8 u}{\partial y^6\partial x^2}\right]_{i,j} = \left[\frac{\partial^6 f}{\partial y^4\partial x^2} - k^2\frac{\partial^6 u}{\partial y^4\partial x^2} - \frac{\partial^8 u}{\partial y^4\partial x^4}\right]_{i,j}.$$

Therefore

$$\left[\frac{\partial^8 u}{\partial x^6\partial y^2} + \frac{\partial^8 u}{\partial y^6\partial x^2}\right]_{i,j} = \left[\frac{\partial^6 f}{\partial x^4\partial y^2} + \frac{\partial^6 f}{\partial y^4\partial x^2} - k^2\left\{\frac{\partial^6 u}{\partial x^4\partial y^2} + \frac{\partial^6 u}{\partial y^4\partial x^2}\right\} - 2\left\{\frac{\partial^8 u}{\partial x^4\partial y^4}\right\}\right]_{i,j}. \tag{32}$$

Making use of equations (31) and (32) in equation (30), we have

$$\begin{aligned}\alpha_{i,j} = {}& \frac{h^2}{12}[-\nabla^2 f + 2\delta_x^2\delta_y^2 u + k^2 f - k^4 u]_{i,j} \\ & - \frac{h^4}{360}\left[\nabla^4 f - k^2\nabla^2 f + k^4 f - k^6 u + 4\left\{\frac{\partial^4 f}{\partial x^2\partial y^2}\right\} - 2k^2\left\{\frac{\partial^4 u}{\partial x^2\partial y^2}\right\}\right]_{i,j} \\ & - \frac{h^6}{181440}\left[9\left\{\frac{\partial^6 f}{\partial x^6} + \frac{\partial^6 f}{\partial y^6} - k^2\left(\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right)\right\} + 75\left\{\frac{\partial^8 u}{\partial x^6\partial y^2} + \frac{\partial^8 u}{\partial y^6\partial x^2}\right\}\right]_{i,j} + \mathcal{O}(h^8).\end{aligned} \tag{33}$$

Making use of equations (29) and (32), equation(33) yields

$$\begin{aligned}\alpha_{i,j} = {}& \frac{h^2}{12}[-\nabla^2 f + 2\delta_x^2\delta_y^2 u + k^2 f - k^4 u]_{i,j} \\ & - \frac{h^4}{360}\left[\nabla^4 f - k^2\nabla^2 f + k^4 f - k^6 u + 4\left\{\frac{\partial^4 f}{\partial x^2\partial y^2}\right\} - 2k^2\left\{\frac{\partial^4 u}{\partial x^2\partial y^2}\right\}\right]_{i,j} - \frac{h^6}{181440}\left[9\nabla^6 f - 9k^2\nabla^4 f \right. \\ & \left. + 9k^4\nabla^2 f - 9k^6 f + 9k^8 u - 66k^2\left\{\frac{\partial^4 f}{\partial x^2\partial y^2}\right\}\right]_{i,j} \\ & + \frac{h^6}{181440}\left[44k^4\left\{\frac{\partial^4 u}{\partial x^2\partial y^2}\right\} + 150\left\{\frac{\partial^8 u}{\partial x^4\partial y^4}\right\} - 75\left\{\frac{\partial^6 f}{\partial x^4\partial y^2} + \frac{\partial^6 f}{\partial y^4\partial x^2}\right\}\right]_{i,j} + \mathcal{O}(h^8).\end{aligned} \tag{34}$$

Finally, the eighth-order compact approximation for two-dimensional Helmholtz equation leads to

$$\begin{aligned}&(\delta_x^2 + \delta_y^2)v_{i,j} + \frac{h^2k^4}{12}v_{i,j} + \frac{h^4k^4}{360}v_{i,j} - \frac{h^6k^2}{20160}v_{i,j} \\ &+ \frac{h^2}{6}(\delta_x^2\,\delta_y^2)v_{i,j} + \frac{k^2h^4}{180}(\delta_x^2\,\delta_y^2)v_{i,j} + \frac{44h^6k^4}{181440}(\delta_x^2\,\delta_y^2)v_{i,j} \\ &+ \frac{5h^6}{6048}(\delta_x^4\,\delta_y^4)v_{i,j} \\ &= \left[\frac{181440 + h^2k^2(-15120 + 50h^2k^2 - 9h^4k^4)}{181440}\right]f_{i,j} \\ &+ h^2\left[\frac{15120 - 504h^2k^2 + 9h^4k^4}{181440}\right]\nabla^2 f_{i,j} \\ &+ \left[\frac{h^4(504 - 9h^2k^2)}{181440}\right]\nabla^4 f_{i,j} + \frac{h^6}{20160}(\nabla^6 f_{i,j}) \\ &+ \left[\frac{h^4(2016 - 66h^2k^2)}{181440}\right]\frac{\partial^4 f_{i,j}}{\partial x^2\partial y^2} \\ &+ \frac{5h^6}{12096}\left[\frac{\partial^6 f}{\partial x^4\partial y^2} + \frac{\partial^6 f}{\partial y^4\partial x^2}\right]_{i,j},\end{aligned} \tag{35}$$

where $v_{i,j}$ is the approximated value of $u_{i,j}$ satisfying the formulation of equation (16), that is $u_{i,j} = v_{i,j} + \mathcal{O}(h^8)$. Equation (35) can be expressed in more general form as

$$\begin{aligned}&a_0\, v_{i,j} + a_1\, L_{i,j}^1 + a_2\, L_{i,j}^2 + a_3\, L_{i,j}^3 + a_4\, L_{i,j}^4 \\ &\quad = b_0\, f_{i,j} + b_1\, R_{i,j}^1 + b_2\, R_{i,j}^2 + b_3\, R_{i,j}^3 + b_4\, R_{i,j}^4,\end{aligned} \tag{36}$$

where

$$\begin{aligned}a_0 &= -\left[\frac{708264 - 4032h^2k^2 - 176h^4k^2 - 15120h^4k^4 - 504h^6k^4 + 9h^8k^2}{181440}\right], \\ a_1 &= \left[\frac{117360 - 2016h^2k^2 - 88h^4k^2}{181440}\right], \\ a_2 &= \left[\frac{32640 + 1008h^2k^2 + 44h^4k^2}{181440}\right], \\ a_3 &= \frac{-5}{1512}, \\ a_4 &= \frac{5}{6048},\end{aligned}$$

$$
\begin{aligned}
L^1_{i,j} &= v_{i,j-1} + v_{i,j+1} + v_{i-1,j} + v_{i+1,j},\\
L^2_{i,j} &= v_{i-1,j-1} + v_{i+1,j-1} + v_{i+1,j+1} + v_{i-1,j+1},\\
L^3_{i,j} &= v_{i+2,j+1} + v_{i+2,j-1} + v_{i-2,j+1} + v_{i-2,j-1}\\
&\quad + v_{i+1,j+2} + v_{i+1,j-2} + v_{i-1,j+2} + v_{i-1,j-2},\\
L^4_{i,j} &= v_{i+2,j+2} + v_{i+2,j-2} + v_{i-2,j+2} + v_{i-2,j-2},
\end{aligned} \tag{37}
$$

and the R.H.S of equation (36) has the following coefficients

$$
\begin{aligned}
b_0 &= -\left[\frac{133212h^2 + 1654h^4k^2 - 36h^6k^4}{181440}\right],\\
b_1 &= \left[\frac{10248h^2 - 336h^4k^2 + 9h^6k^4}{181440}\right],\\
b_2 &= \left[\frac{1416h^2 - 66h^4k^2}{181440}\right],\\
b_3 &= \frac{109h^2}{60480},\quad b_4 = \frac{5h^2}{1206},
\end{aligned}
$$

$$
\begin{aligned}
R^1_{i,j} &= f_{i,j-1} + f_{i,j+1} + f_{i+1,j} + f_{i-1,j},\\
R^2_{i,j} &= f_{i-1,j-1} + f_{i+1,j-1} + f_{i+1,j+1} + f_{i-1,j+1},\\
R^3_{i,j} &= f_{i+2,j} + f_{i-2,j} + f_{i,j+2} + v_{i,j-2},\\
R^4_{i,j} &= f_{i+2,j+1} + f_{i-2,j+1} + f_{i+2,j-1} + f_{i-2,j-1}\\
&\quad + f_{i+1,j+2} + f_{i+1,j-2} + f_{i-1,j+2} + f_{i-1,j-2}.
\end{aligned} \tag{38}
$$

Thus, equation (36) is the compact eighth-order approximation for Helmholtz equation in two-dimensions. Therefore, this equation leads to the form of $Av = Bf$, in which A and B are symmetric and block pentadiagonal sparse matrices. The final linear system can be formulated for every node. Also, in this equation, derivative of the forcing function f can be determined numerically.

3. Eighth-order accurate approximation of the boundary

The formula derived in equation (36) can be used for all points using Dirichlet boundary condition, while in the case of Neumann boundary condition, eighth-order approximation is developed for one-dimension as well as for two-dimensions. We introduce a ghost point $i = -1$ and consider the coordinate line $i = 0$ in equation (36). We specify both the Helmholtz equation and the Neumann boundary condition at the boundary $i = 0$.

3.1. Neumann boundary condition (one-dimensional case)

For such boundary condition, it is consider that

$$u_i'(x) = \alpha, \tag{39}$$

where α is any constant. To get the eighth-order approximation of equation (39), we have

$$\delta_x u_i = u_i' + \frac{h^2}{6}u_i^{(3)} + \frac{h^4}{120}u_i^{(5)} + \frac{h^6}{5040}u_i^{(7)} + \mathcal{O}(h^8). \tag{40}$$

Applying δ_x^2 to $u_i^{(5)}$ using equation (3), we have $u_i^{(7)} = \delta_x^2 u_i^{(5)} + \mathcal{O}(h^2)$, therefore

$$
\begin{aligned}
\delta_x u_i &= u_i' + \frac{h^2}{6}u_i^{(3)} + \frac{h^4}{120}u_i^{(5)}\\
&\quad + \frac{h^6}{5040}[\delta_x^2 u_i^{(5)} + \mathcal{O}(h^2)] + \mathcal{O}(h^8).
\end{aligned} \tag{41}
$$

Differentiating equation (11), we get $u_i^{(3)} = f_i' - k^2 u_i'$ and $u_i^{(5)} = f_i^{(3)} - k^2 u_i^{(3)}$. Substituting these equations together with equation (2) in equation (41), we get

$$
\begin{aligned}
\delta_x u_i &= \left[1 - \frac{h^2k^2}{6} + \frac{h^4k^4}{120} + \frac{h^6k^4}{5040}\delta_x^2\right]u_i'\\
&\quad + \frac{h^2}{6}\left[1 - \frac{h^2k^2}{20} - \frac{h^4k^2}{840}\delta_x^2\right]f_i'\\
&\quad + \frac{h^4}{120}\left[1 + \frac{h^2}{42}\delta_x^2\right]f_i^{(3)} + \mathcal{O}(h^8).
\end{aligned} \tag{42}
$$

Now using $\delta_x^2 f_i' = \frac{f'_{i+1} - 2f'_i + f'_{i-1}}{h^2} + \mathcal{O}(h^2)$, $\delta_x^2 f_i^{(3)} = \frac{f^{(3)}_{i+1} - 2f^{(3)}_i + f^{(3)}_{i-1}}{h^2} + \mathcal{O}(h^2)$ and equation (41), we get equation of the form

$$
\begin{aligned}
\frac{u_{i+1} - u_{i-1}}{2h} &= \left[1 - \frac{h^2k^2}{6} + \frac{h^4k^4}{120} + \frac{h^6k^4}{5040}\delta_x^2\right]u_i'\\
&\quad + \frac{h^2}{6}\left[1 - \frac{k^2h^2}{21}\right]f_i' - \frac{k^2h^4}{5040}[f'_{i+1} + f'_{i+1}]\\
&\quad + \frac{h^4}{126}f_i^{(3)} + \frac{h^4}{5040}[f^{(3)}_{i+1} + f^{(3)}_{i-1}] + \mathcal{O}(h^8).
\end{aligned} \tag{43}
$$

After some simplification, we get

$$
\begin{aligned}
\frac{u_{i+1} - u_{i-1}}{2h} &= \left[1 - \frac{h^2k^2}{6} + \frac{h^4k^4}{120} - \frac{h^6k^6}{5040}\right]u_i'\\
&\quad + \frac{h^2}{6}\left[1 - \frac{k^2h^2}{21} + \frac{k^4h^4}{840}\right]f_i' - \frac{k^2h^4}{5040}[f'_{i+1} + f'_{i+1}]\\
&\quad + \frac{h^4}{126}f_i^{(3)} + \frac{h^4}{5040}[f^{(3)}_{i+1} + f^{(3)}_{i-1}] + \mathcal{O}(h^8).
\end{aligned} \tag{44}
$$

Consider the formula for discrete form using equation (39), we have

$$
\begin{aligned}
u_{i+1} - u_{i-1} &= 2\,\alpha\,h + \frac{h^3}{3}f_i' + \frac{h^5}{60}f_i^{(3)}\\
&\quad + \frac{h^5}{2520}[f^{(3)}_{i+1} + f^{(3)}_{i-1}] + \mathcal{O}(h^8).
\end{aligned} \tag{45}
$$

Replacing u_i by v_i for $i = 0$, we get

$$v_1 - v_{-1} = 2\,\alpha\, h + \frac{h^3}{3} f_0^{'} + \frac{h^5}{60} f_0^{(3)} + \frac{h^5}{2520}[f_1^{(3)} + f_{-1}^{(3)}] + \mathcal{O}(h^8). \quad (46)$$

We want to get rid of v_{-1} , as we have no value at the point x_{-1}. Making use of equation (15) for $i = 0$, we get

$$a_1 v_1 + a_0 v_0 + a_1 v_{-1} = b_1 f_{-1} + b_0 f_0 + b_1 f_1 + c_1 f_{-1}'' + c_0 f_0'' + c_1 f_1'' + d_1 f_{-1}^{(4)} + d_0 f_0^{(4)} + d_1 f_1^{(4)}. \quad (47)$$

Making use of equations (46), (47) to get rid of v_{-1} and obtain the pursuit approximation at point x on the boundary, we have

$$\begin{aligned} 2\,v_1 - 2\,v_0 = {} & 2\,\alpha\, a_1 h\left[1 - \frac{h^2k^2}{6} + \frac{h^4k^4}{120} - \frac{h^6k^6}{5040}\right] \\ & + \frac{a_1 h^2}{6}\left[1 - \frac{k^2h^2}{21} + \frac{k^4h^4}{840}\right] f_0^{'} - \frac{a_1 k^2 h^4}{5040}[f_1^{'} + f_{-1}^{'}] \\ & + \frac{a_1 h^4}{60}\left[\frac{20}{21}\right] f_0^{(3)} + \frac{a_1 h^4}{5040}[f_1^{(3)} + f_{-1}^{(3)}] \\ & + h^2 f_0 + b_1 f_{-1} + b_0 f_0 + b_1 f_1 \\ & + c_1 f_{-1}^{''} + c_0 f_0^{''} + c_1 f_1^{''} + d_1 f_{-1}^{(4)} + d_0 f_0^{(4)} + d_1 f_1^{(4)}. \end{aligned} \quad (48)$$

It is assumed that all the parameters used in the R.H.S of equation (48) are known and the forcing function is also known explicitly.

3.2. Neumann boundary condition (two-dimensional case)

Assume that

$$\left.\frac{\partial u}{\partial x}\right|_{x\,=\,0} = g(y). \quad (49)$$

Using equation (17) and (18) in equation (16), we have

$$\delta_x\, u_{i,j} = \frac{\partial\, u_{i,j}}{\partial\, x} + \frac{h^2}{6}\frac{\partial^3 u_{i,j}}{\partial\, x^3} + \frac{h^4}{120}\frac{\partial^5 u_{i,j}}{\partial\, x^5} + \frac{h^6}{720}\frac{\partial^7 u_{i,j}}{\partial\, x^7} + \mathcal{O}(h^8). \quad (50)$$

From equation (16), we have

$$\left(\frac{\partial^3 u}{\partial\, x^3}\right)_{i,j} = \left[\frac{\partial f}{\partial x} - k^2\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x \partial y^2}\right]_{i,j}. \quad (51)$$

Higher-order approximation of $\left(\frac{\partial^3 u}{\partial\, x\,\partial\, y^2}\right)$ is required in the above equation (51), that is

$$\begin{aligned} \left(\frac{\partial^3 u}{\partial\, x\,\partial\, y^2}\right)_{i,j} = {} & \delta_x\,\delta_y^2\, u_{i,j} \\ & - \frac{h^2}{12}\left[2\frac{\partial^5 u}{\partial\, x^3\,\partial\, y^2} + \frac{\partial^5 u}{\partial\, x\,\partial\, y^4}\right]_{i,j} \\ & - \frac{h^4}{360}\left[\frac{\partial^7 u}{\partial\, x\,\partial\, y^6} + 3\frac{\partial^7 u}{\partial\, x^5\,\partial\, y^2}\right]_{i,j} \\ & + \mathcal{O}(h^6). \end{aligned} \quad (52)$$

Putting equation (52) in equation (51) gives us

$$\begin{aligned} \left(\frac{\partial^3 u}{\partial\, x^3}\right)_{i,j} = {} & \left[\frac{\partial f}{\partial\, x} - k^2\frac{\partial\, u}{\partial\, x}\right]_{i,j} \\ & + \frac{h^2}{12}\left[2\,\frac{\partial^5 u}{\partial\, x^3\,\partial\, y^2} + \frac{\partial^5 u}{\partial\, x\,\partial\, y^4}\right]_{i,j} \\ & - \delta_x\,\delta_y^2\, u_{i,j} + \frac{h^4}{360}\left[\frac{\partial^7 u}{\partial\, x\,\partial\, y^6} + 3\frac{\partial^7 u}{\partial\, x^5\,\partial\, y^2}\right]_{i,j} \\ & + \mathcal{O}(h^6). \end{aligned} \quad (53)$$

From equation (16), we have

$$\left(\frac{\partial^5 u}{\partial\, x^5}\right)_{i,j} = \left[\frac{\partial^3 f}{\partial\, x^3}\right]_{i,j} - k^2\left[\frac{\partial^3 u}{\partial\, x^3}\right]_{i,j} - \left[\frac{\partial^5 u}{\partial\, x^3\,\partial\, y^2}\right]_{i,j} + \mathcal{O}(h^2). \quad (54)$$

In equation (54), $\left(\frac{\partial^5 u}{\partial\, x^3\,\partial\, y^2}\right)$ can be expressed up to fourth-order as

$$\left(\frac{\partial^5 u}{\partial\, x^3\,\partial\, y^2}\right)_{i,j} = \delta_x^3\,\delta_y^2\, u_{i,j} - \frac{h^2}{12}\left[\frac{\partial^7 u}{\partial\, x^3\,\partial\, y^4} + 2\,\frac{\partial^7 u}{\partial\, x^5\,\partial\, y^2}\right]_{i,j} + \mathcal{O}(h^4). \quad (55)$$

Using equation (55) in equation (54), we have

$$\begin{aligned} \left(\frac{\partial^5 u}{\partial\, x^5}\right)_{i,j} = {} & \left[\frac{\partial^3 f}{\partial\, x^3}\right]_{i,j} - k^2\left[\frac{\partial f}{\partial\, x} + k^2\frac{\partial\, u}{\partial\, x}\right]_{i,j} + k^2\,\delta_x\,\delta_y^2\, u_{i,j} \\ & - \delta_x^3\,\delta_y^2\, u_{i,j} + \frac{h^2}{12}\left[\frac{\partial^7 u}{\partial\, x^3\,\partial\, y^4} + 2\,\frac{\partial^7 u}{\partial\, x^5\,\partial\, y^2}\right]_{i,j} + \mathcal{O}(h^4). \end{aligned} \quad (56)$$

Using equations (53) and (54) in equation (50), we have

$$\delta_x u_{i,j} - \frac{k^4h^4}{120}\left[1 - \frac{k^2h^2}{6}\right]\left(\frac{\partial u}{\partial x}\right)_{i,j} + \frac{h^2}{6}\left[1 + \frac{k^2h^2}{20}\right]\delta_x \delta_y^2 u_{i,j} + \frac{h^4}{120}\left[1 + \frac{k^2h^2}{8}\right]\delta_x^3 \delta_y^2 u_{i,j} + \frac{h^6k^4}{720}\delta_x \delta_y^4 u_{i,j}$$
$$= \left[1 - \frac{k^2h^2}{6}\right]\left(\frac{\partial u}{\partial x}\right)_{i,j} + \frac{h^2}{6}\left[1 + \frac{k^4h^4}{120}\right]\left(\frac{\partial f}{\partial x}\right)_{i,j} + \frac{h^2}{120}\left[1 - \frac{k^2h^2}{10}\right]\left(\frac{\partial^3 f}{\partial x^3}\right)_{i,j} + \frac{h^4}{72}\left[1 - \frac{k^2h^2}{20}\right]\left(\frac{\partial^3 f}{\partial x\,\partial y^2}\right)_{i,j} + \frac{h^4}{720}\left(\frac{\partial^5 f}{\partial x^5}\right)_{i,j} + \frac{h^6}{432}\left[1 + \frac{k^2h^2}{8}\right]\left(\frac{\partial^5 f}{\partial x^3\,\partial y^2}\right)_{i,j} + \mathcal{O}(h^8). \quad (57)$$

In equation (57), $\left(\frac{\partial u}{\partial x}\right)_{i,j}$ is approximated as

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \delta_x u_{i,j} + \beta h^2 \delta_x \delta_y^2 u_{i,j}, \quad (58)$$

where β is any constant. Making use of equation (58) in equation (57) and putting $i = 0$, we have

$$L_1^*(v_{1,j} - v_{-1,j}) + L_2^*(v_{2,j} - v_{-2,j}) + L_3^*(v_{1,j+1} + v_{1,j-1} - v_{-1,j+1} - v_{-1,j-1}) + L_4^*(v_{2,j+1} + v_{2,j-1} - v_{-2,j+1} - v_{-2,j-1})$$
$$= \left[1 - \frac{k^2h^2}{6}\right]g + \frac{h^2}{6}\left[1 + \frac{k^4h^4}{120}\right]\left(\frac{\partial f}{\partial x}\right)_{0,j} + \frac{h^2}{120}\left[1 - \frac{k^2h^2}{10}\right]\left(\frac{\partial^3 f}{\partial x^3}\right)_{0,j} + \frac{h^4}{720}\left(\frac{\partial^5 f}{\partial x^5}\right)_{0,j} + \frac{h^4}{72}\left[1 - \frac{k^2h^2}{20}\right]\left(\frac{\partial^3 f}{\partial x\,\partial y^2}\right)_{0,j} + \frac{h^6}{432}\left[1 + \frac{k^2h^2}{8}\right]\left(\frac{\partial^5 f}{\partial x^3\partial y^2}\right)_{0,j}, \quad (59)$$

where $g = g(y)$ and

$$L_1^* = 1 - \frac{k^4h^4}{120}\left[1 - \frac{k^2h^2}{120}\right](1 - 2\beta) - \frac{1}{6h}\left[1 + \frac{k^2h^2}{20}\right],$$
$$L_3^* = \frac{120 + \beta}{240h}\left[1 + \frac{k^2h^2}{120}\right],$$
$$L_2^* = -\frac{k^4h^4}{120}\beta + \frac{1}{12h}\left[1 + \frac{k^2h^2}{20}\right] - \frac{k^4h}{180} - \frac{1}{40h}\left[1 + \frac{k^2h^2}{8}\right],$$
$$L_4^* = \frac{1}{240}\left[1 - \frac{k^2h^2}{20}\right].$$

Putting $i = 0$ in equation (36), we have

$$a_1 v_{0,j} + a_2(v_{1,j} + v_{-1,j} + v_{0,j+1} + v_{0,j-1}) + a_3(v_{1,j+1} + v_{1,j-1} + v_{-1,j+1} + v_{-1,j-1}) + a_4(v_{2,j+1} + v_{2,j-1} + v_{-2,j+1} + v_{-2,j-1} + v_{1,j+2} + v_{1,j-2} + v_{-1,j+2} + v_{-1,j-2}) + a_5(v_{2,j+2} + v_{2,j-2} + v_{-2,j+2} + v_{-2,j-2})$$
$$= b_0 f_{0,j} + b_1 R_{0,j}^1 + b_2 R_{0,j}^2 + b_3 R_{0,j}^3 + b_4 R_{0,j}^4. \quad (60)$$

Adding equation (60) with μ times of equation (59), we get formula for the boundary nodes as

$$a_1 v_{0,j} + a_2(2v_{1,j} + v_{0,j+1} + v_{0,j-1}) + a_3(v_{1,j+1} + v_{1,j-1}) + a_4(v_{2,j+1} + v_{2,j-1} + v_{-2,j+1} + v_{-2,j-1} + v_{1,j+2} + v_{1,j-2} + v_{-1,j+2} + v_{-1,j-2}) + a_5(v_{2,j+2} + v_{2,j-2} + v_{-2,j+2} + v_{-2,j-2})$$
$$= \left[\frac{181440 + h^2k^2(-15120 + 50h^2k^2 - 9h^4k^4)}{181440}\right]f_{i,j} + h^2\left[\frac{15120 - 504h^2k^2 + 9h^4k^4}{181440}\right]\nabla^2 f_{0,j} + \left[\frac{h^4(504 - 9h^2k^2)}{181440}\right]\nabla^4 f_{0,j} + \frac{h^6}{20160}(\nabla^6 f_{0,j}) + \left[\frac{h^4(2016 - 66h^2k^2)}{181440}\right]\frac{\partial^4 f_{0,j}}{\partial x^2\,\partial y^2}$$
$$+ \frac{5h^6}{12096}\left[\frac{\partial^6 f}{\partial x^4\partial y^2} + \frac{\partial^6 f}{\partial y^4\partial x^2}\right]_{0,j} + \left[1 - \frac{k^2h^2}{6}\right]g + \frac{h^2}{6}\left[1 + \frac{k^4h^4}{120}\right]\left(\frac{\partial f}{\partial x}\right)_{0,j} + \frac{h^2}{120}\left[1 - \frac{k^2h^2}{10}\right]\left(\frac{\partial^3 f}{\partial x^3}\right)_{0,j} + \frac{h^4}{720}\left(\frac{\partial^5 f}{\partial x^5}\right)_{0,j} + \frac{h^4}{72}\left[1 - \frac{k^2h^2}{20}\right]\left(\frac{\partial^3 f}{\partial x\partial y^2}\right)_{0,j} + \frac{h^6}{432}\left[1 + \frac{k^2h^2}{8}\right]\left(\frac{\partial^5 f}{\partial x^3\partial y^2}\right)_{0,j}, \quad (61)$$

where

$$\mu = \frac{a_2}{L_2^*} = \frac{a_3}{L_3^*} = \frac{a_4}{L_4^*} = \frac{1}{h}.$$

4. Multigrid method

The results obtained from discretization through compact higher-order schemes are in linear systems of sparse nature. Multigrid methods are used to solve these systems efficiently. Multigrid method utilizes some relaxation methods to remove high frequency error. In order to smooth the errors, this method makes the use of coarse grid correction. Multigrid method has been extensively used for several elliptic problems such as Poisson and Helmholtz equations [8, 31]. We are using V-cycle multigrid method to solve linear system arisen from the discretization of higher-order compact difference scheme. To show the performance and to match the results of HOC scheme, we use the full weighting projection operator on uniform grids. In multigrid method, we are using a point Guass–Seidel as a smoother, because in general, mesh coarsening strategy work very well with equal mesh sizes discretized some elliptic equations [32].

In this strategy, instead of point relaxation, the line relaxation is used because in removing high frequency errors, the line Guass–Seidel relaxation method is very efficient. For our particular problem in this article, only one of the x- or y-directions be the dominant direction. Let the dominant direction be the x-direction, and on each successive grid only line relaxation is applied along the x-direction. The coefficient matrix obtained from the eighth-order compact scheme under this ordering is in the form of block pentadiagonal matrix such that each block is of order $(N-1)$. It is observed that the coefficient matrix U is of the order $(N-1)\times(N-1)$, such that $U=\text{pentadiag}[U_2, U_1, U_0, U_1, U_2]$ and $U_0=\text{pentadiag}[0, a_3, a_4, a_3, 0]$, $U_1=\text{pentadiag}[a_1, a_2, a_3, a_2, a_1]$, $U_2=\text{pentadiag}[a_0, a_1, 0, a_1, a_0]$ are of $(N-1)$ pentadiagonal symmetric sub-matrices. Let on each jth line, the part of the solution vector u_j representing the grid points and f_j be the corresponding R.H.S part of the forcing function. Therefore U_0 needs only one factorization on each grid level. In multigrid method with any smoother such as Gauss-Seidel, Jacobi or LU-decomposition relaxations, bilinear interpolation is used to transfer corrected value from a coarse grid to a fine grid. In order to update the residual on a coarse grid, full-weighting scheme is also used.

4.1. Principles of multigrid method

The Multigrid method has mainly two components, namely, error smoothing and correction on coarse grid level.

4.1.1. Error smoothing. The simplest iterative techniques such as Jacobi and Guass–Seidel methods are very slow in convergence for large system of equations but they are very fast in error smoothing. Due to this reason, these methods are very effective at smoothing the high frequency error while leaving relatively unchanged the low frequency error. These methods smooth the error but not necessary to reduce its size. This property can be well estimated on a coarse grid.

4.1.2. Correction on coarse grid. The coarse grid correction is a good instructional tool to understand how more complicated multigrid methods work. It is a two-grid iterative method. Let us consider a linear system in forward form as

$$\mathscr{L}u=f, \tag{62}$$

where $\mathscr{L}$ is the operator (linear). The goal is to find the vector u, in simplest form as

$$u=\mathscr{L}^{-1}f. \tag{63}$$

Let v be the approximation to the solution u in above equation. Then the error e is estimated as $e=u-v$. The

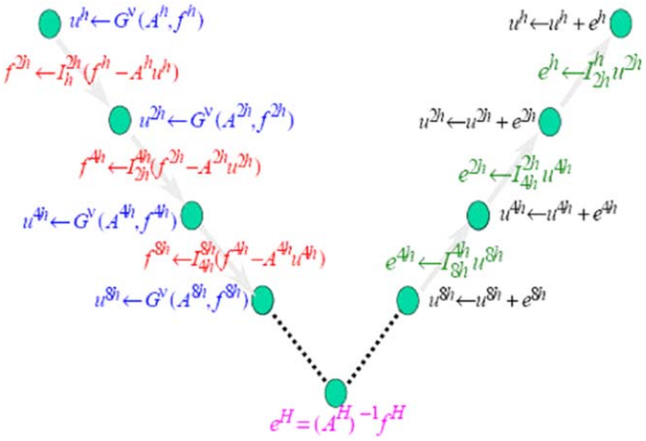

Figure 1. One cycle of two-grid multigrid method.

defect in equation (63) gives the residual equation, i.e.

$$\mathscr{L}e=f-\mathscr{L}v \;=\; r, \tag{64}$$

where r is the residual. This residual equation is used to relax on the error e. One of these functions and operators has a corresponding function on the coarse grid. The coarse grid correction starts with an initial guess on the finer grid. The above equation (64) is expansive as the original equation (62). If we replace $\mathscr{L}$ by $\widetilde{\mathscr{L}}$ (in Jacobi method, diagonal D of $\mathscr{L}$ is used for approximation), an approximation of the error can be found, and then use to correct v and repeat the process until its convergence.

4.1.3. Multigrid algorithm.

Algorithm 1. These multigrid parameters are arranged on each level as:

Let v_1 represents pre-smoothing steps.
Let v_2 represents post-smoothing steps.
Let r represents the number of multigrid cycles, here V-cycle is used with $r=1$.
Multigrid Cycle (FAS)

$$\phi^h \longleftarrow \text{FASCYC}^h(\phi^h, f^h, v_1, v_2, r).$$

1. If Ω^h represents the coarsest grid, then using a time marching scheme to solve equation (36) and then finish, else continue the pre-smoothing steps:

$$\phi^h \;\longleftarrow\; \text{SMOOTHER}(\phi^h, f^h, v_1, \text{tol}), \quad (\text{Pre}-\text{smoothing}).$$

2. Restriction:

$$\phi^{2h}=I_h^{2h}\,\phi^h, \quad \overline{\phi}^{2h}=\phi^{2h},$$
$$f^{2h}=I_h^{2h}(f^h-N^h\phi^h)+N^{2h}\phi^{2h},$$
$$\phi^{2h}\longleftarrow \text{FASCYC}^{2h}(\phi^{2h}, f^{2h}, v_1, v_2).$$

3. Interpolation:

$$\phi^h \longleftarrow \phi^h + I_{2h}^h(\phi^{2h}-\overline{\phi}^{2h}).$$

4.

$$\phi^h \;\longleftarrow\; \text{SMOOTHER}^{v_2}(\phi^h, f^h, v_2), \quad (\text{Post}-\text{smoothing})$$

here I_h^{2h} is the restriction operator and I_{2h}^h is the interpolation.

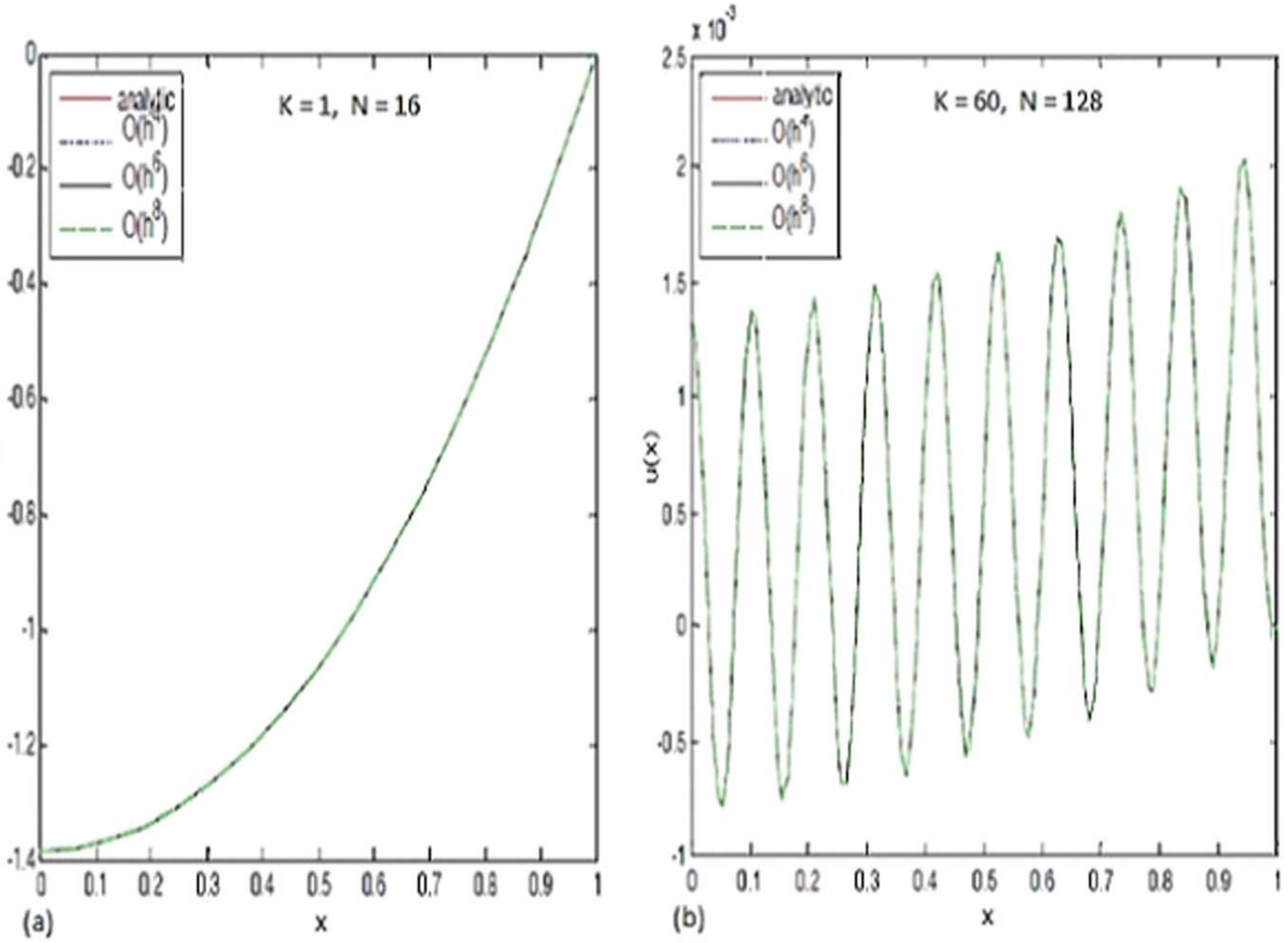

Figure 2. Exact solution (a) M_4, M_6, M_8, $N = 16$, $k = 1$ (b) M_4, M_6, M_8, $N = 128$, $k = 60$ for problem-1.

Table 1. Error norm, CPU (seconds) and order of convergence for different schemes for problem-1, $\|e\|_2$, $N = 8, 16, 32, 64, 128$, $k = 10$.

N	$M_4(\|e\|_2)$	CPU	V	$M_6(\|e\|_2)$	CPU	V	$M_8(\|e\|_2)$	Order	CPU	V
8	$7.3343\ e^{-4}$	0.013	2	$8.4231\ e^{-5}$	0.054	2	$9.2388\ e^{-6}$	—	0.058	2
16	$2.5403\ e^{-4}$	0.025	2	$6.3223\ e^{-6}$	0.060	2	$4.9343\ e^{-8}$	7.82	0.166	2
32	$2.1323\ e^{-5}$	0.046	2	$1.3321\ e^{-7}$	0.087	2	$2.5985\ e^{-10}$	7.96	1.130	2
64	$1.8559\ e^{-6}$	0.154	2	$2.9027\ e^{-9}$	0.313	2	$5.3108\ e^{-12}$	8.04	1.430	2
128	$1.6322\ e^{-7}$	1.840	2	$6.3872\ e^{-11}$	2.224	2	$0.5245\ e^{-14}$	8.10	2.262	2

The iterations due to multigrid method are typically converge independent of the problem size. Hence, in a few number of iterations, it will provide the optimal solution with desired accuracy, independent of the mesh size. Although best accuracy can be expected in order to fine the mesh size, hence more and more iterations are needed to be applied. This effect contributes a logarithmic factor to estimate the complexity.

In this technique, a multigrid algorithm is applied on the coarser (lower) level Ω_{2h} to provide a good initial guess, and then interpolate its results to next finer (higher) level. This is the so-called full multigrid method. This method starts discretization on the coarsest level with an exact solver. The obtained results are then interpolated to the next finer level, where some cycles of the multigrid method (V or W) are applied and the results are then refined using multigrid correction. Again these results are interpolated to the finer grid, where further a few cycles of multigrid be sufficient to produce the desire accuracy for final answer. Typically this algorithm requires one or two V-cycles on each level. This method produces solutions at a toll that is proportional to the number of unknowns which is the optimal complexity of full multigrid method [32]. Every cycle of multigrid algorithm consists of pre-smoothing, correction on coarse grid level, and post-smoothing steps.

Table 2. Error norm, CPU (seconds) and order of convergence for different schemes for problem-2, $\|e\|_2$, $N = 8, 16, 32, 64, 128$, $k = 10$.

N	$M_4(\|e\|_2)$	CPU	V	$M_6(\|e\|_2)$	CPU	V	$M_8(\|e\|_2)$	Order	CPU	V
8	5.3326 e^{-3}	0.044	2	2.8252 e^{-7}	0.057	2	9.6152 e^{-7}	—	0.062	2
16	3.7689 e^{-4}	0.050	2	3.6245 e^{-9}	0.073	2	6.1581 e^{-9}	7.82	0.181	2
32	1.7264 e^{-5}	0.068	2	5.1201 e^{-11}	0.092	2	3.3108 e^{-11}	7.90	1.142	2
64	7.7715 e^{-6}	0.098	2	8.8792 e^{-13}	0.219	2	2.4610 e^{-13}	8.04	1.980	2
128	4.6821 e^{-8}	0.145	2	1.3291 e^{-14}	1.916	2	6.6321 e^{-16}	8.10	2.960	2

Table 3. Comparison of error norm of different schemes for problem-2, where $e^{-8} = 10^{-8}\ \|e\|_2$, $N = 8, 16, 32, 64, 128$ and $k = 1, 10, 20, 30, 50, 100$.

HOC	k	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
Sixth-order	1	5.2002 e^{-6}	7.8890 e^{-8}	7.2290 e^{-10}	4.1098 e^{-11}	7.9322 e^{-13}
	10	3.7586 e^{-6}	5.6585 e^{-8}	6.9975 e^{-10}	3.233 e^{-11}	6.9691 e^{-13}
	20	3.5541 e^{-6}	5.6002 e^{-8}	6.1040 e^{-10}	3.1255 e^{-11}	6.6251 e^{-13}
	30	3.1983 e^{-6}	5.2100 e^{-8}	6.1033 e^{-10}	3.1043 e^{-11}	6.4629 e^{-13}
	50	3.1287 e^{-6}	5.1520 e^{-8}	6.1530 e^{-10}	3.6054 e^{-11}	6.1690 e^{-13}
	100	3.4161 e^{-6}	4.2104 e^{-8}	2.2610 e^{-10}	2.6041 e^{-11}	6.0331 e^{-13}
Eighth-order	1	4.2122 e^{-7}	3.3211 e^{-9}	7.4412 e^{-11}	5.8234 e^{-13}	8.9240 e^{-16}
	10	9.6152 e^{-7}	6.1581 e^{-9}	3.3108 e^{-11}	3.4610 e^{-13}	6.6321 e^{-16}
	20	9.7110 e^{-7}	6.1218 e^{-9}	3.6152 e^{-11}	3.1961 e^{-13}	6.5126 e^{-16}
	30	3.9328 e^{-7}	4.1155 e^{-9}	2.6001 e^{-11}	3.3087 e^{-13}	6.3370 e^{-16}
	50	3.7822 e^{-7}	4.8676 e^{-9}	1.5226 e^{-11}	3.3350 e^{-13}	6.2164 e^{-16}
	100	3.2844 e^{-7}	2.8600 e^{-9}	1.1710 e^{-11}	3.3421 e^{-13}	5.9756 e^{-16}

4.1.4. Two-grid algorithm

Algorithm 2. Multigrid parameters are arranged on each level as:

Let v_1 represents the pre-smoothing steps.
Let v_2 represents the post-smoothing steps.

$$u_h^{n+1} = \mathrm{TWOGRID}^h(u_h^n, \mathscr{L}_h, f^h, v_1, nv_2).$$

1. Using the appropriate smoother to smooth the required solution on the fine grid level, i.e.

$$u_h^n = \mathrm{SMOOTH}^{v_1}(u_h^n, \mathscr{L}_{2h}, f^h).$$

2. Compute the residual on fine grid $\bar{r}_h{}^n = f_h - \mathscr{L}_h\, \bar{u}_h{}^n$.
3. Restrict $\bar{r}_h{}^n$ to coarse grid $\bar{r}_{2h}^n = I_h^{2h}\ \bar{r}_h{}^n$.
4. Solve on the coarse grid $\mathscr{L}_{2h}\, \tilde{e}_{2h}^n = \bar{r}_{2h}^n$.
5. Interpolate the correct value $\tilde{e}_{2h}^n$ to the fine grid $\tilde{e}_h{}^n = I_{2h}^h\ \tilde{e}_{2h}^n$.
6. Compute the next approximation $u_h^n = \bar{u}_h{}^n + \tilde{e}_h{}^n$.
7. Apply v_2 steps of smoother (post-smoothing), i.e.

$$u_h^{n+1} = \mathrm{SMOOTHER}^{v_2}(u_h^n, \mathscr{L}_{2h}, f^h).$$

Figure 1 shows the V-cycle of two-grid multigrid method.

5. Results and discussion

In this section, we give some computational output in order to test efficiency and feasibility of the newly developed schemes given in equations (15) and (36) with multigrid method. Some numerical experiments with known exact solutions are considered to solve a one-dimensional and two-dimensional Helmholtz equations on the unit square domain. The forcing function and the Dirichlet and/or Neumann boundary conditions are applied on all sides of a unit square domain.

Example 1.

$$\begin{cases} u_{xx} + k^2 u = e^x + x^2, & 0 \leqslant x \leqslant 1 \\ u(0) = u(1) = 0. \end{cases}$$

The exact solution for this problem is

$$u(x) = A\cos(k\ x) + B\sin(k\ x) + \left[\frac{1}{k^2}\right]x^2 - \frac{2}{k^4} + \left[\frac{1}{k^2+1}\right]e^x,$$

where

$$A = \frac{2}{k^4} - \frac{1}{[k^2+1]},$$

$$B = \frac{-A\cos(k)}{\sin(k)} + \frac{\left[\frac{-1}{k^2} + \frac{2}{k^4} - \frac{e}{(k^2+1)}\right]}{\sin(k)}.$$

The eigenvalues of this problem are given in [33] as $\left(n+\frac{1}{2}\right)^2\pi^2$, for $n = 0, 1, 2, 3, \ldots$.

Table 4. Error norm, CPU (seconds) and order of convergence for different schemes for problem-3, $\|e\|_2$, $N = 8, 16, 32, 64, 128$, $k = 10$.

N	$M_4(\|e\|_2)$	CPU	V	$M_6(\|e\|_2)$	CPU	V	$M_8(\|e\|_2)$	Order	CPU	V
8	$8.6724\,e^{-3}$	0.350	2	$5.9244\,e^{-6}$	0.576	2	$9.9760\,e^{-7}$	—	0.694	2
16	$4.9726\,e^{-4}$	0.472	2	$7.8334\,e^{-8}$	0.683	2	$4.3078\,e^{-9}$	7.86	1.180	2
32	$1.3286\,e^{-5}$	0.660	2	$9.6190\,e^{-10}$	0.861	2	$1.7163\,e^{-11}$	7.98	1.526	2
64	$5.9240\,e^{-6}$	0.762	2	$1.8782\,e^{-11}$	0.930	2	$6.5516\,e^{-14}$	8.08	1.986	2
128	$2.3294\,e^{-7}$	1.416	2	$1.7575\,e^{-13}$	2.552	2	$2.3850\,e^{-16}$	8.10	2.880	2

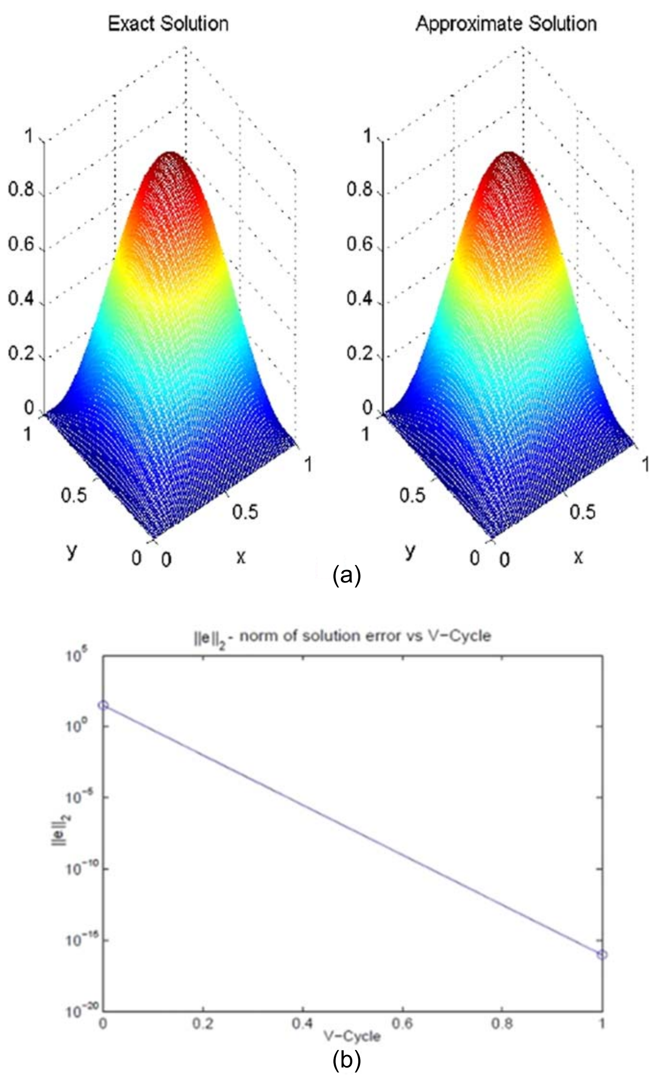

Figure 3. (a) Left: exact solution. Right: computed solution obtained through HOC scheme. (b) Error norm. The error vector $e_{ij} = u_{ij} - v_{ij}$ for different values of N and $k = 10$ for problem-2.

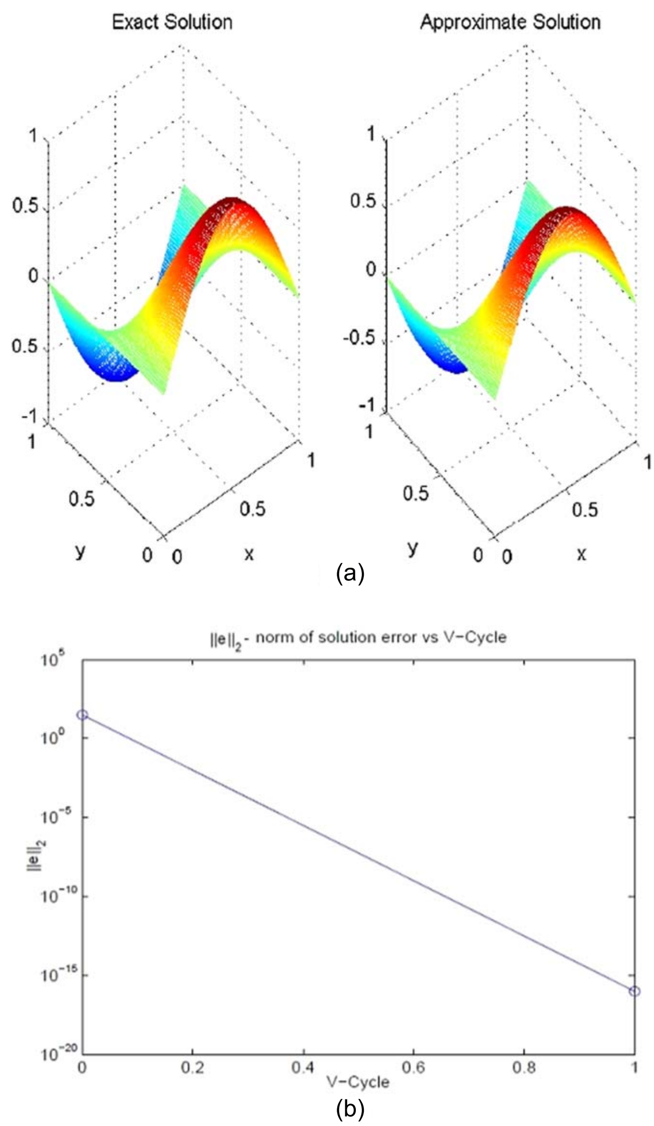

Figure 4. (a) Left: exact solution. Right: computed solution obtained through HOC scheme. (b) Error norm. The error vector $e_{ij} = u_{ij} - v_{ij}$ for different values of N and $k = 10$ for problem-3.

When k^2 equal to one of these eigenvalues, then the problem has no solution. The error norm of our numerical scheme is shown in table 1 and figure 2(a) compare these solutions to the exact solution. In order to see the behavior of approximate solution when k gets close to $\pi/2$, the scheme becomes sensitive and the accuracy is poor as shown in figure 2(b).

At each level, one pre-smoothing and one post-smoothing multigrid iteration is applied. The procedure of iteration is started initially with zero data and the process is stopped when the norm of the residual vector is reduced by 10^{-16}. The reported error is the maximum absolute error between the exact solution and the computed solution on the finest grid points. For comparison between the exact solution and the numerical solution, $\|e\|_2$-norm is used. The matrix $\|e\|_2$-norm

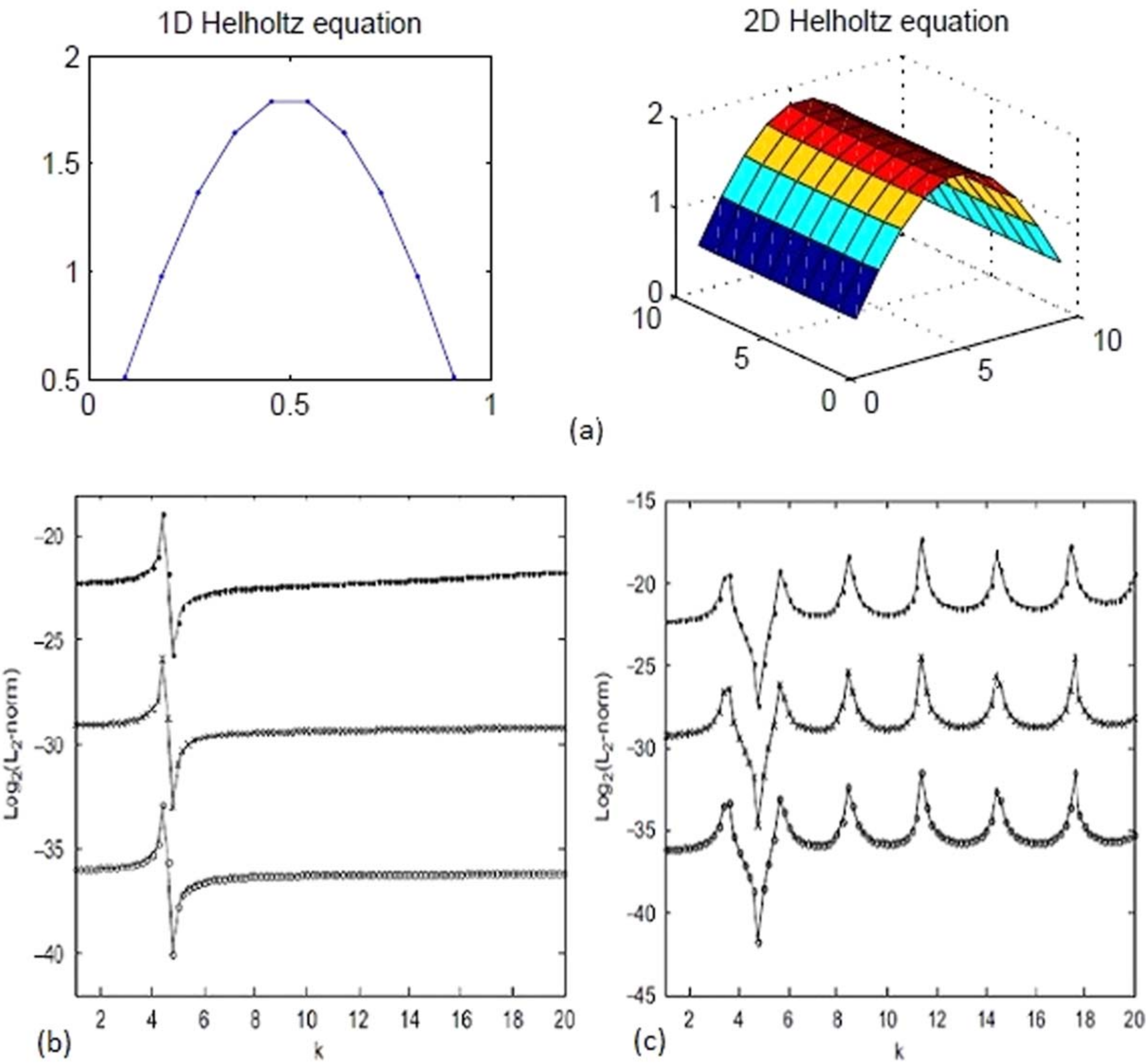

Figure 5. (a) Left: 1-D Helmholtz. Right: 2-D Helmholtz through HOC schemes. (b) $\|e\|_2$ for $\bullet$, $N = 8$; $\star$, $N = 16$; $\circ$, $N = 32$ and $k = 4.44$ for problem-2. (c) $\|e\|_2$ for $\bullet$, $N = 8$; $\star$, $N = 16$; $\circ$, $N = 32$ and $k = 5.66$ for problem-3.

of the error vector is defined as

$$\|e\|_2 = \sqrt{\sum_{i,j=0}^{N} e_{i,j}^2}, \tag{65}$$

where N is the number of nodes, $e_{ij} = u_{ij} - v_{ij}$ is the error vector, and k is the wave number. The metric order is given as

$$\text{Order} = \log_2\left(\frac{\text{Error}(N_1)}{\text{Error}(N_2)}\right),$$

where $\text{Error}(N_1)$ and $\text{Error}(N_2)$ are computed absolute errors for two consecutive grids with $N_1 + 1$ and $N_2 + 1$ points. Also the grid point N_1 is half of N_2. Furthermore, M_4 is multigrid based on the fourth-order compact scheme, M_6 is multigrid based on the sixth-order compact scheme and M_8 is multigrid based on the eighth-order compact scheme.

Example 2.

$$u_{xx} + u_{yy} = [k^2 - 2\pi^2]\sin(\pi x)\sin(\pi y);$$
$$0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1. \tag{66}$$

For this problem, the exact solution is $u(x, y) = \sin(\pi y)\sin(\pi x)$.

Example 3.

$$u_{xx} + u_{yy} = [k^2 - 2\pi^2]\sin(\pi y)\cos(\pi x),$$
$$0 \leqslant x \leqslant 1 \text{ and } 0 \leqslant y \leqslant 1. \tag{67}$$

With the pure Dirichlet boundary conditions on three sides and Neumann boundary condition only on left side of the domain as $u_x(0, y) = 0$, $u(1, y) = -\sin y$, $u(x, 0) = 0$, $u(x, 1) = 0$. The exact solution for problem 3 is $u(x, y) = \cos(\pi x)\sin(\pi y)$.

The eighth-order compact scheme is compared against the fourth-order scheme and sixth-order compact scheme in terms of accuracy, convergence order and CPU timing. The $\|e\|_2$-norm for different values of N are presented in tables 1–4. We have also examined the behavior of eighth-order compact scheme for different values of k in table 3. It is observed that the scheme is sensitive for $4 \leqslant k \leqslant 5$ as shown in figures 3–5. It is also observed that accuracy of the scheme is poor at some particular values of k and this fact can be explained through eigenvalues analysis. The estimated eigenvalues of problems 2 and 3 are given as $\lambda_{m,n} = k^2 - \pi^2(m^2 + n^2)$ and $\lambda_{m,n} = k^2 - \pi^2[(m + 1/2)^2 + n^2]$ respectively, in [25, 28]. From the results, it is observed that near $k = 4.44$, the eigenvalues tend to zero and problem 2 is unstable, and when $k = 5.66$, the eigenvalues tend to zero and problem 3 is unstable. Hence the accuracy of the scheme is poor. The error does not decrease further by increasing the value of k continuously. Tables 2 and 4 show the error-norm for different nodes taking $k = 10$ for problem 2 and problem 3, respectively. Table 3 shows the error-norm for different values of wave number k and nodes for problem 2. The scheme behaves robustly with respect to the wave number k and N.

When the value of wave number k changes from 1 up to 100, the error norm is given in table 3.

In both examples 2 and 3, the error is reduced upto 10^{-16}, therefore the norm of error in both examples is the same which can be observed from figures 3(b) and 4(b).

6. Conclusion

This work concerns with the development of higher-order compact finite difference discretization scheme. The important findings of this article are listed below.

- An efficient and highly accurate computational framework is built to solve Helmholtz equation using high order discretization scheme and multigrid method.
- A higher-order compact finite difference scheme is applied for discretizing one-dimensional and two-dimensional Helmholtz equations and its accuracy is investigated.
- Compact higher-order schemes have the advantage of higher-order accuracy.
- In two-dimensional case, the matrix is being pentadiagonal that can be efficiently solved by multigrid method.
- An eighth-order scheme for the Neumann boundary condition is also developed.
- Our numerical results show that multigrid method with HOC scheme has the required accuracy for solution of Helmholtz equation.

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