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A geometric measure of non-classicality

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Abstract

This paper aims to stress the role of the Cahill–Glauber quasi-probability densities in defining, detecting, and quantifying the non-classicality of field states in quantum optics. The distance between a given pure state and the set of all pure classical states is called here a geometric degree of non-classicality. As such, we investigate non-classicality of a pure single-mode state of the radiation field by using the coherent states as a reference set of pure classical states. It turns out that any such distance is expressed in terms of the maximal value of the Husimi Q function. As an insightful application we consider the de-Gaussification process produced when preparing a quantum state by adding p photons to a pure Gaussian one. For a coherent-state input, we get an analytic degree of non-classicality which compares interestingly with the previously evaluated entanglement potential. Then we show that addition of a single photon to a squeezed vacuum state causes a considerable enhancement of non-classicality, especially at weak and moderate squeezing of the original state. By contrast, addition of further photons is less effective.

Keywords: non-classicality, quasi-probability distributions, photon-added Gaussian states, distance-type measures of non-classicality

(Some figures may appear in colour only in the online journal)

1. Introduction

As first pointed out by Glauber [1], there are states of the quantum radiation field for which all the normally-ordered quantities are described by classical distributions. These states, which are now termed *classical*, have been singled out by Titulaer and Glauber [2] as possessing a well-behaved P representation of the density operator [3, 4], which is either a non-negative regular function or a distribution no more singular than Dirac's δ. The opposite situation, namely, the non-existence of the Glauber–Sudarshan P representation as a genuine probability density, is a largely accepted definition of non-classicality which generated a large amount of research in quantum optics as can be seen in the survey [5].

The main concept we exploit in the present paper dedicated to an evaluation of non-classicality for pure states emerges from quantum information. Accordingly, the distance from a given state having a specific property to a reference set of states not having it has been accepted as a measure of that property. It is notable that non-classicality of continuous-variable states was the first property proposed to be quantified by a distance-type degree [6]:

$$\mathcal{D}(\hat{\rho}) := \min_{\hat{\rho}' \in \mathcal{C}} d(\hat{\rho}, \hat{\rho}'), \qquad (1)$$

where $\mathcal{C}$ is the convex set of all the classical states and d is the distance between the density operators $\hat{\rho}$ and $\hat{\rho}'$. However, the trace metric employed by Hillery in [6] and termed as *non-classical distance* turned out to be difficult to deal with analytically. Moreover, there is no parametrization of the whole set $\mathcal{C}$ of classical states to allow the extremization procedure required by Hillery's definition (1). Subsequently, by restricting the set of all classical states to a tractable subset identified by a classicality criterion, and by using more convenient metrics, the non-classical distance was successfully applied to one-mode Gaussian states. Specifically, we mention the Hilbert–Schmidt metric used in [7, 8], the Bures metric in [9], the relative-entropy measure in [9, 10], as well as the quantum Chernoff bound in [11, 12]. In our papers [9, 10], the Bures degree of non-classicality for one-mode Gaussian states was found to fully agree with the earlier result of Lee's non-classical depth [13].

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Interest in non-classicality aspects has recently renewed being stimulated by the ongoing resource theories of various quantum properties [14]. Attempts to produce a resource theory of non-classicality [15–17] are in fact based on the identification of the set of classical states and classical operations. We point out that in [9, 10] co-authored with Scutaru, the present authors formulated a set of three requirements to make the definition (1) acceptable as a measure of non-classicality:

(C1) The degree of non-classicality vanishes if and only if the state is classical;
(C2) Classical transformations preserve the degree of non-classicality;
(C3) Non-classicality does not increase under any positive operator-valued measure.

We have there considered as being *classical* those unitary transformations in Hilbert space which map coherent states into coherent states. As such, the only one-mode classical transformations are the translations described by the displacement operators $\hat{D}(\lambda) = \exp(\lambda\hat{a}^{\dagger} - \lambda^{*}\hat{a})$ and the rotations $\hat{R}(\theta) = \exp(-\mathrm{i}\theta\hat{a}^{\dagger}\hat{a})$, written in terms of the amplitude operators of the mode $\hat{a}$ and $\hat{a}^{\dagger}$. Therefore, the requirements (C1)–(C3) are quite similar to what is considered now to be necessary for defining a resource theory of non-classicality.

In the present paper we restrict ourselves to investigate non-classicality of pure continuous-variable states. What is really encouraging in defining a distance-type measure for pure states was discovered in the early days of quantum optics: a clear identification of the set of classical pure states. Indeed, Cahill [18] and later on Hillery [19] proved that the only pure states that are classical are the coherent ones: all other pure (Gaussian and non-Gaussian) states are non-classical. Therefore, a geometric measure of non-classicality for a pure state $|\psi\rangle$ could be its distance to the set of all classical pure states, namely, the coherent ones. It appears to us that this measure was first used in [20] to quantify non-classicality of some popular pure states such as Fock states, squeezed vacuum, as well as even and odd coherent states. Soon after, in [21] a comparison between this distance-type measure and Lee's non-classical depth for pure states has revealed an advantage of the geometric degree which turned out to be more sensitive.

The plan of our paper is as follows. In section 2 we discuss the role played by the quasi-probability distributions introduced by Cahill and Glauber [22] in describing and measuring non-classicality. Section 3 investigates first the non-classicality of an interesting pure state which is important for experiments: a coherent state with p added photons [23]. We give here an analytic form of the defined geometric degree of non-classicality and compare it with some previously used indicators of non-classicality. Another relevant example we are interested in is provided by the squeezed vacuum states (SVSs) whose non-classicality enhancement by addition of photons is here analysed. The concluding section 4 stresses the usefulness of the geometric measure and its consistency with other evaluations of the amount of non-classicality.

2. Non-classicality and quasi-probability distributions

In continuous-variable settings, phase-space formalism allows us to conveniently describe quantum states using the generalised quasi-probability distributions [22]:

$$W(\beta, s) = \frac{1}{\pi}\int \mathrm{d}^2\lambda \exp(\beta\lambda^{*} - \beta^{*}\lambda)\,\chi(\lambda, s), \tag{2}$$

which are in fact the Fourier transforms of the s-ordered characteristic functions (CFs) of the density operator $\hat{\rho}$,

$$\chi(\lambda, s) := \exp\left(\frac{s}{2}|\lambda|^2\right)\mathrm{Tr}[\hat{\rho}\hat{D}(\lambda)]. \tag{3}$$

The most significant quasi-probability distributions arise from the normally ordered CF ($s = 1$, $\frac{1}{\pi}W(\beta, 1) =: P(\beta)$ is the Glauber–Sudarshan P function), symmetrically ordered CF ($s = 0$, $W(\beta, 0) =: W(\beta)$ is the Wigner function), and anti-normally ordered CF ($s = -1$, $\frac{1}{\pi}W(\beta, -1) =: Q(\beta)$ is the Husimi Q function).

It is quite interesting that all the three above-defined quasi-probability distributions play a role in describing non-classicality, especially in the pure-state case. First, the behaviour of the P representation *defines* the concept of non-classicality in quantum optics: non-classical states are those having a P representation which is negative or more singular than Dirac's δ. Second, according to the Hudson's theorem [24], only the pure Gaussian states possess a non-negative Wigner function, all the other non-Gaussian pure states display negativities of their Wigner distributions. Signatures of non-classicality could be thus identified through the negativity of the Wigner function, for pure and even some mixed non-Gaussian states [25]. Therefore, the Wigner function *detects* non-classicality of all pure states except for the Gaussian ones. Finally, to see the role played by the Q function in describing non-classicality, let us define the geometric degree of non-classicality of a pure one-mode state $|\Psi\rangle$:

$$\mathcal{D}_Q(|\Psi\rangle\langle\Psi|) := 1 - \max_{\{\beta\}} |\langle\beta|\Psi\rangle|^2, \tag{4}$$

where $\{|\beta\rangle\}$ is the set of the coherent states. The following remarks are at hand when looking at the definition (4):

(i) $\mathcal{D}_Q$ is an exact distance-type measure for pure states because the reference set of classical pure states is exhaustive.
(ii) $\mathcal{D}_Q$ is defined in terms of the maximum of the Q function,

$$Q(\beta) := \frac{1}{\pi}\langle\beta|\Psi\rangle\langle\Psi|\beta\rangle. \tag{5}$$

(iii) $\mathcal{D}_Q$ obviously meets the requirements (C1)–(C2): $0 \leqq \mathcal{D}_Q \leqq 1$, $\mathcal{D}_Q = 0$ for coherent states; $\mathcal{D}_Q$ is invariant under displacements and rotations.

To conclude, the maximum of the Q function is a reliable *measure* of non-classicality. The advantages of a geometric degree of non-classicality for pure states were recognized long ago. $\mathcal{D}_Q$ was first evaluated in some simple cases

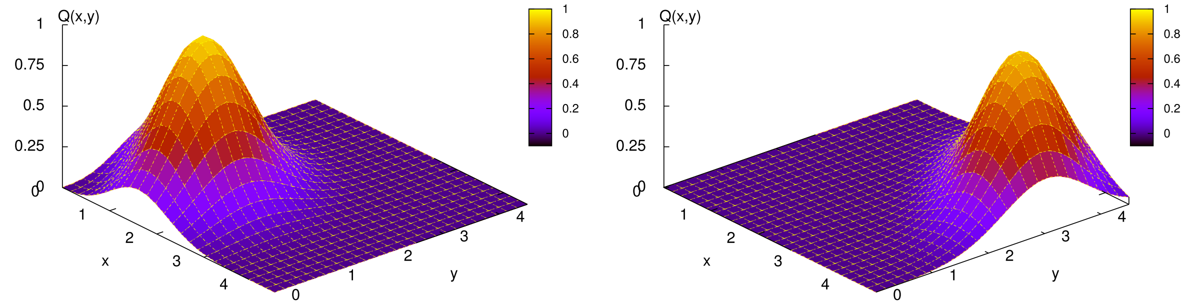

Figure 1. The Q function of a p-photon-added coherent state. The parameters of the state are: $p = 1$, $\mathfrak{Re}(\alpha) = \mathfrak{Im}(\alpha) = 2$ (left) and $p = 10$, $\mathfrak{Re}(\alpha) = \mathfrak{Im}(\alpha) = 2$ (right). For the sake of simplicity, we have denoted $x := \mathfrak{Re}(\beta)$, $y := \mathfrak{Im}(\beta)$.

including Gaussian pure states [20]. For non-Gaussian pure states, some examples were examined [21] showing that the distance-type measures are more appropriate to quantify non-classicality than the non-classical depth. Recently, in [26], improved upper and lower bounds on Hillery's non-classical distance were written in terms of the Q function of the given state.

3. Exactly solvable examples: photon-added pure Gaussian states

Let us consider the output state generated by the addition of p photons to an arbitrary pure state $|\Psi_0\rangle$:

$$|\Psi_p\rangle := \mathcal{N}_p(\hat{a}^\dagger)^p|\Psi_0\rangle, \qquad (p = 1, 2, 3, \ldots), \tag{6}$$

where $\mathcal{N}_p$ is a normalization factor whose squared modulus reads:

$$|\mathcal{N}_p|^2 = [\langle\Psi_0|\ \hat{a}^p(\hat{a}^\dagger)^p|\Psi_0\rangle]^{-1}. \tag{7}$$

Accordingly, the normalization constant is determined by the pth-order anti-normally ordered correlation function of the input state $|\Psi_0\rangle$. Further, by employing in equation (5) the eigenvalue equation of the annihilation operator, $\hat{a}|\beta\rangle = \beta|\beta\rangle$, we find that the Q function of the output state (6) is proportional to that of the input state $|\Psi_0\rangle$:

$$Q_p(\beta) = |\mathcal{N}_p|^2|\beta|^{2p}Q_0(\beta), \qquad (p = 1, 2, 3, \ldots). \tag{8}$$

As we know that the Q function for a Gaussian state [27] is an exponential of the variables β and β^*, equation (8) shows us that addition of photons is a non-Gaussian operation. Moreover, according to a theorem of Lee, by removing the vacuum from the Fock-state expansion of a classical state, we get a non-classical output [28]. This can be done by adding photons to a classical state such as a coherent one [23] or a thermal one [29]. We can say that photon-added states are both non-classical and non-Gaussian. This is important because the non-Gaussianity of states and operations turned out to be an indispensable property for some protocols in quantum information with continuous variables such as: entanglement distillation [30], as well as distillation of any other quantum resource [31], quantum error correction [32], and universal quantum computation [33, 34]. For the sake of simplicity and in view of physical relevance, we chose to discuss here non-classicality for two classes of states obtained by addition of photons to pure Gaussian ones: photon-added coherent states and photon-added SVSs.

The p-photon-added coherent state,

$$|\Psi_p(\alpha)\rangle := \frac{1}{[p!L_p(-|\alpha|^2)]^{1/2}}(\hat{a}^\dagger)^p|\alpha\rangle, \tag{9}$$

was formally defined and characterized from the quantum optical perspective in [23]. It was found that its P distribution is highly singular and its Wigner function displays negative values. Other non-classical properties such as sub-Poissonian statistics and amplitude squeezing were also analysed. Note that in equation (9), $L_p(u)$ is a Laguerre polynomial, which is always positive for negative values of its argument, as shown by equations (7), p. 188 and/or (14), p.189 in [35].

A significant piece of progress was the experimental preparation and investigation of a single-photon-added coherent state by Zavatta *et al* [36, 37]. Remark that single-photon-added coherent states are interesting from a fundamental point of view as they represent the result of the simplest excitation of a classical light field. It can be generated by injecting a coherent state $|\alpha\rangle$ into the signal mode of an optical parametric amplifier and a conditioning of the state based on measurements on the idler mode to select the one-photon excitation. Moreover, the experiment allowed the tomographic reconstruction of the Wigner function. Its negativity was detected and analysed while a significant degree of squeezing was also found.

Let us now proceed to find $\mathcal{D}_Q$ defined in equation (4) for the state (9). The Q function for $|\ \Psi_p(\alpha)\rangle$ is [23]:

$$\begin{aligned} Q_p(\alpha, \beta) &= \frac{1}{\pi}|\langle\beta|\Psi_p(\alpha)\rangle|^2 \\ &= \frac{1}{\pi}\frac{|\beta|^{2p}}{p!L_p(-|\alpha|^2)}\exp(-|\beta - \alpha|^2), \end{aligned} \tag{10}$$

The single-peaked aspect of Q_p can be seen in figure 1 for different numbers of added photons to the same coherent state.

Maximization of $Q_p(\beta)$ with respect to the variables $\mathfrak{Re}(\beta)$ and $\mathfrak{Im}(\beta)$ is routinely performed to nicely give a

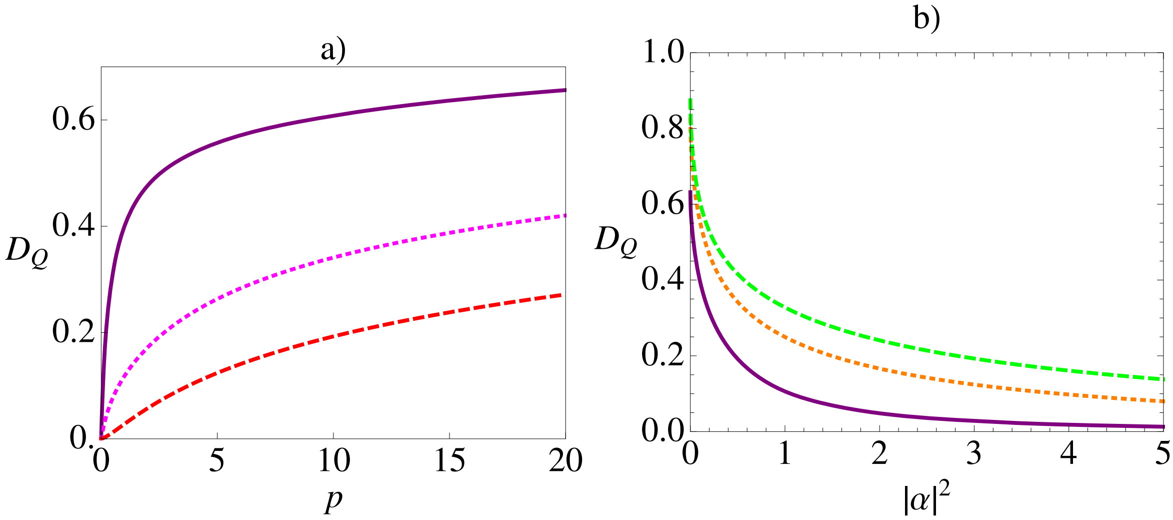

Figure 2. Photon-added coherent states: (a) $\mathcal{D}_Q$ increases with the number p of added photons. The coherent mean occupancies are $|\alpha|^2 = 3$ (red dashed plot); 0.9 (magenta dotted curve) and 0.1 (purple line). (b) Non-classicality degree is much larger for smaller coherent mean occupancies. We have used $p = 1$ (purple plot), $p = 5$ (red dotted plot), and $p = 10$ (green dashed curve).

general formula:

$$Q_p^{\max}(\alpha) = \frac{1}{\pi}\frac{1}{p!L_p(-|\alpha|^2)}\left[\frac{|\alpha|}{2}(\sqrt{1 + 4p/|\alpha|^2} + 1)\right]^{2p} \times \exp\left[-\frac{|\alpha|^2}{4}(\sqrt{1 + 4p/|\alpha|^2} - 1)^2\right]. \tag{11}$$

The geometric degree of non-classicality $\mathcal{D}_Q(|\Psi_p(\alpha)\rangle\langle\Psi_p(\alpha)|) \coloneqq 1 - \pi Q_p^{\max}(\alpha)$ depends therefore only on the number p of added photons and the mean occupancy of the coherent state $\langle\hat{a}^\dagger\hat{a}\rangle = |\alpha|^2$. Plots of $\mathcal{D}_Q$ with respect to the number p of added photons and the coherent mean occupancy $|\alpha|^2$ in figure 2 show that non-classicality increases with the number of added photons and decreases drastically with the intensity of the coherent beam. The incipient values on the plots of figure 2(b) are the degrees of non-classicality of the Fock states with $p = 1, 5, 10$, respectively. Indeed, when taking the limit $\alpha = 0$ in equation (11) we get the maximal Q function for the number state $|p\rangle$:

$$Q_p^{\max}(0) = \frac{1}{\pi}\frac{1}{p!}\left(\frac{p}{e}\right)^p, \qquad (p = 1, 2, 3, \ldots). \tag{12}$$

Equation (12) coincides with a directly obtained formula for the maximal Q function of a Fock state. Let us write the corresponding degree of non-classicality of a Fock state:

$$\mathcal{D}_Q(|p\rangle\langle p|) = 1 - \frac{1}{p!}\left(\frac{p}{e}\right)^p, \qquad (p = 1, 2, 3, \ldots). \tag{13}$$

In view of the inequality

$$\frac{Q_{p+1}^{\max}(0)}{Q_p^{\max}(0)} = \frac{1}{e}\left(1 + \frac{1}{p}\right)^p < 1, \qquad (p = 1, 2, 3, \ldots), \tag{14}$$

the sequence (12) is strictly decreasing and therefore the geometric degree of non-classicality (13) increases with the number of added photons. For large photon numbers, we write equation (13) using Stirling's approximation, equation (3), p. 47 in [38]:

$$\mathcal{D}_Q(|p\rangle\langle p|) \approx 1 - \frac{1}{\sqrt{2\pi p}}, \qquad (p \gg 1). \tag{15}$$

According to figure 2(b), non-classicality degrees $\mathcal{D}_Q$ converge to 0 for higher coherent intensities $|\alpha|^2$, but are well separated for low beam intensity, being larger for higher numbers of photons added to a weak coherent state.

Non-classicality of a p-photon-added coherent state was discussed in [39] by evaluating its entanglement potential (EP). This measure of non-classicality was defined as the entanglement generated by a non-classical state when mixed with the vacuum state in a balanced beam splitter [40]. As such, the task of measuring non-classicality is transferred to an evaluation of a two-mode entanglement which is not a simpler problem in most cases of interest. Fortunately, for some lower numbers of added photons to a coherent state the EP could be evaluated in [39]. This gives us the opportunity to compare our present distance-type results to those given by a non-classicality measure of a totally different origin, the EP. Our formula (11) depicted in figure 2 describes the behaviour of $\mathcal{D}_Q$ in close agreement with the investigation in [39]. It is sufficient to compare our figure 2(b) and the corresponding figure 3 in [39] to see their similarity. We have to remark that only for single-photon-added coherent states was possible to have an analytic EP in [39]. Even for $p = 2$ the EP-evaluation was performed only numerically. Other findings in [39] based on observing the negativity of the Wigner function enhance the idea of consistency between $\mathcal{D}_Q$ and other indicators of non-classicality.

The second example we want to address now is the modification of non-classicality by adding photons to a non-classical state, namely an SVS, $|\Psi_{\rm SV}(r, \varphi)\rangle = |\Psi_0(r, \varphi)\rangle$; its

Q function has the explicit expression [27]:

$$Q_0(r,\,\varphi,\,\beta) = \frac{1}{\pi\cosh r}\times\exp\{-|\beta|^2[1-(\tanh r)\cos(\varphi-2\arg(\beta))]\}, \quad (16)$$

where r is the squeeze parameter and φ is the squeeze angle. Notice first that the maximum of the Q function for an SVS is reached for $\beta_{\max}=0$ and has the value

$$Q_0^{\max}(r) = (\pi\cosh r)^{-1}. \quad (17)$$

Further, the maximum of $Q_p(\beta)$, equation (8), is found to be at $|\beta_{\max}|^2 = p\ e^r\cosh r$ and $\arg(\beta_{\max}) = \frac{1}{2}\varphi$. The next step to apply equation (8) is to evaluate the pth-order anti-normallyordered correlation function for an SVS. Its derivation parallels that of the normally-ordered one in [41]. We find the general formula:

$$\begin{aligned}&\langle\Psi_{\mathrm{SV}}(r,\,\varphi)|\hat{a}^p(\hat{a}^\dagger)^p|\Psi_{\mathrm{SV}}(r,\,\varphi)\rangle\\ &= p!\,(\cosh r)^{2p}\ {}_2F_1\left(-\frac{p}{2},\,-\frac{p-1}{2};\,1;\,(\tanh r)^2\right),\\ &(p=0,\,1,\,2,\,3,\,\ldots).\end{aligned} \quad (18)$$

The above Gauss hypergeometric function ${}_2F_1$ is a polynomial in the variable $\tanh r$, of degree either p if p is even or $p-1$ if p is odd, as seen from equation (2), p. 56 in [38]. We finally get the ratio of maximal values

$$\frac{Q_p^{\max}(r)}{Q_0^{\max}(r)} = \frac{1}{p!}\left(\frac{p\ e^{r-1}}{\cosh r}\right)^p\times\left[{}_2F_1\left(-\frac{p}{2},\,-\frac{p-1}{2};\,1;\,(\tanh r)^2\right)\right]^{-1}. \quad (19)$$

By specializing equation (19) to the simplest cases $p=1$ and $p=2$, we find the inequalities:

$$Q_1^{\max}(r) = \frac{2}{e}\,\frac{1}{1+\exp(-2r)}\,Q_0^{\max}(r) < Q_0^{\max}(r); \quad (20)$$

$$Q_2^{\max}(r) = \frac{16}{3}\frac{1}{e^2}\times\frac{1}{1+\frac{2}{3}\exp(-2r)+\exp(-4r)}Q_0^{\max}(r) < Q_0^{\max}(r). \quad (21)$$

Both equations (20) and (21) show that non-classicality is enhanced by adding photons to this non-classical state.

Nevertheless, it is instructive to analyse the general formula (19) in two limit cases: zero squeezing ($r=0$) and very strong squeezing ($r\to\infty$). The first one reduces to that of the Fock states, already discussed above (equations (12)–(15)). In order to get a convenient expression valid in the strong-squeezing regime, we employ equation (19) in conjunction with two well-known identities, equations (14), p. 61, and (15), p. 5 in [38]: Gauss's summation formula,

$$\begin{aligned}&{}_2F_1(a,\,b;\,c;\,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\\ &(c\neq 0,\,-1,\,-2,\,-3,\,\ldots,\,\Re(c-a-b)>0),\end{aligned} \quad (22)$$

and, respectively, Legendre's duplication formula,

$$2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \pi^{\frac{1}{2}}\,\Gamma(2z). \quad (23)$$

We find the limit ratio

$$\lim_{r\to\infty}\frac{Q_p^{\max}(r)}{Q_0^{\max}(r)} = \left(\frac{p}{e}\right)^p\frac{\sqrt{\pi}}{\Gamma\left(p+\frac{1}{2}\right)},\qquad (p=1,\,2,\,3,\,\ldots), \quad (24)$$

as well as a stronger inequality than equation (14):

$$\lim\frac{Q_{p+1}^{\max}(r)}{Q_p^{\max}(r)} = \frac{1}{e}\left(1+\frac{1}{p}\right)^p\frac{p+1}{p+\frac{1}{2}} < 1,\quad (p=1,\,2,\,3,\,\ldots). \quad (25)$$

Accordingly, for $r\to\infty$, the sequence $\{Q_p^{\max}(r)\}$ strictly decreases to a vanishing limit. For large photon numbers, we apply again Stirling's approximation, equation (3), p. 47 in [38], to get an asymptotic expression of equation (24):

$$\lim_{r\to\infty}\frac{Q_p^{\max}(r)}{Q_0^{\max}(r)} = \frac{1}{\sqrt{2}},\qquad (p\gg 1). \quad (26)$$

In conclusion, in the strong-squeezing regime ($r\to\infty$), the geometric degree of non-classicality $\mathcal{D}_Q(|\Psi_p(r,\,\varphi)\rangle\langle\Psi_p(r,\,\varphi)|)$ strictly increases with the number of added photons reaching a limit equal to unity.

However, figure 3 displays an interesting feature of the degree of non-classicality $\mathcal{D}_Q(|\Psi_p(r,\,\varphi)\rangle\langle\Psi_p(r,\,\varphi)|)$ as a function of the mean photon number $\langle\hat{a}^\dagger\hat{a}\rangle = (\sinh r)^2$ in the input SVS. The graph for any number of added photons has a minimum for a rather small value of r which essentially separates the ranges of weak and strong squeezing. Although in both limit cases of very weak and very strong squeezing the hierarchy in the number of added photons is strictly observed, this does not happen for intermediate squeezing, i.e. in the neighbourhood of the minima, where various graphs cross each other. It is worth to point out a fact which is important from an experimental perspective. Addition of a single photon to an SVS provides a substantial enhancement of non-classicality at weak and moderate squeezing, while addition of a larger number of photons does not make any significant difference in non-classicality.

4. Conclusions

The use of the quasi-probability densities introduced by Cahill and Glauber [22] has a long and successful history in quantum optics. The present paper emphasizes their special significance for non-classicality matters. Thus the P representation is essential in *defining* the non-classicality concept, the Wigner function *detects* non-classicality by its negativity, and the Q function is a proper *quantifier* for the non-classicality of pure states. When dealing with pure states, all the distance-type measures of non-classicality defined with respect to the unique set of pure classical states, namely, the

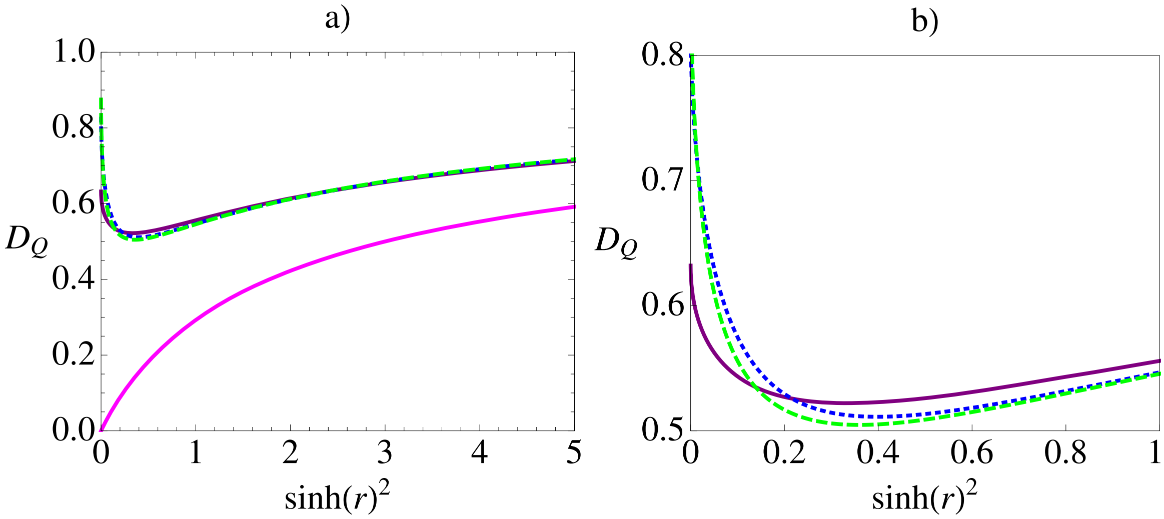

Figure 3. Photon-added squeezed vacuum states: (a) $\mathcal{D}_Q$ has a minimum at small mean occupancies $\langle \hat{a}^\dagger \hat{a} \rangle = (\sinh r)^2$ of the input SVS. At larger squeezing input, its increase is monotonic and very weakly dependent on the number p of added photons. The magenta plot starting from origin is the non-classicality degree $\mathcal{D}_Q$ of the input SVS which is much lower than all the corresponding added-photon ones. However, the degrees of non-classicality $\mathcal{D}_Q$ of the original SVS and its p-photon-added relatives tend to the maximal value 1 for strong squeezing. (b) A better view of the minimum at weak input squeezing. We have used $p = 1$ (purple plot), $p = 5$ (blue dotted plot), and $p = 10$ (green dashed curve).

coherent ones, are expressed in terms of maximal value of the Q function of the given state. This happens because, in the pure-state case, all the metrics having the suitable abilities to define a distance-type degree, such as Hilbert–Schmidt, Hellinger, and Bures metrics, depend only on the Q function of the state. We can say that $\mathcal{D}_Q$ is a properly defined measure of non-classicality because the reference set of classical coherent states is exhaustive. Interestingly, a similar geometric measure was proposed long ago by Shimony for the entanglement of pure two-mode states [42]. The chosen reference set was there composed of all pure product states. According to recent research on non-classicality [14, 17, 31] and non-Gaussianity [43], some protocols in quantum information processing perform better when using non-Gaussian resources and operations. This is a good reason for a future extension of non-classicality quantification to a larger sector of non-Gaussian pure states such as photon-added generalized coherent states [44–46].

As applications of quantifying non-classicality of pure states we have here considered states obtained by adding photons to Gaussian states. Our equation (11) for photon-added coherent states and (19) for photon-added SVSs are quite remarkable because an analytic degree of non-classicality is difficult to obtain for non-Gaussian states, which have in general a more complicated structure. The states we have dealt with in this paper are important from the experimental feasibility as well. Along these lines we have found that a coherent state of weak intensity gains more non-classicality by addition of photons than a strong-intensity one. We have also shown that even when a single photon is added to an SVS, this considerably enhances the amount of non-classicality of the input state especially at weak values of squeezing.

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