IOP Publishing Classical and Quantum Gravity

Class. Quantum Grav. **37** (2020) 055001 (55pp) https://doi.org/10.1088/1361-6382/ab60b6

The transverse-traceless spin-2 gravitational wave cannot be a standalone observable because it is acausal

Yi-Zen Chu[1,2] **and Yen-Wei Liu**[1,3]

[1] Department of Physics, National Central University, Chungli 32001, Taiwan, Republic of China

[2] Center for High Energy and High Field Physics (CHiP), National Central University, Chungli 32001, Taiwan, Republic of China

E-mail: yenwei113@gmail.com

Received 22 July 2019, revised 6 December 2019
Accepted for publication 11 December 2019
Published 6 February 2020

Abstract

We show, through an explicit calculation of the relevant Green's functions, that the transverse-traceless (TT) portion of the gravitational perturbations of Minkowski spacetime and of spatially flat cosmologies with a constant equation-of-state w receive contributions from their isolated matter source(s) outside the past null cone of the observer. This implies the TT gravitational wave (GW) cannot be a standalone observable—despite widespread (apparent) claims in the gravitational wave literature to the contrary. About a Minkowski background, all 4 of the gauge-invariant variables—the two scalars, one vector and tensor—play crucial roles to ensure the spatial tidal forces encoded within the gauge-invariant linearized Riemann tensor are causal. These gravitational tidal forces do not depend solely on the TT graviton but rather on the causal portion of its acceleration. However, in the far zone radiative limit, the flat spacetime 'TT' graviton Green's function does reduce to the causal 'tt' ones, which are the ones commonly used to compute gravitational waveforms. Similar remarks apply to the spin-1 photon; for instance, the electric field does not depend solely on the photon, but is the causal part of its velocity. As is known within the quantum theory of photons and linearized gravitons, there are obstacles to the construction of simultaneously gauge-invariant and Lorentz-covariant descriptions of these massless spin-1 and spin-2 states. Our results transparently demonstrate that the quantum operators associated with the helicity-1 photon and helicity-2 linear graviton both violate micro-causality: namely, they do not commute outside the light cone in flat and cosmological spacetimes.

Keywords: gravitational waves, cosmology, causality

(Some figures may appear in colour only in the online journal)

[3] Author to whom any correspondence should be addressed.

Contents

1. Motivation

Students of gravitational wave (GW) physics are taught that the key observable—the fractional distortion of the arms of laser interferometers employed by detectors such as LIGO, Virgo, Kagra, etc—induced by the passage of a GW train generated by a distant astrophysical source, is directly proportional to the 'transverse-traceless' portion of the metric perturbation. Specifically, in a weakly curved spacetime[4]

$$g_{\mu\nu}[t,\vec{x}] = \eta_{\mu\nu} + h_{\mu\nu}[t,\vec{x}], \qquad |h_{\mu\nu}| \ll 1; \tag{1}$$

if X^i denotes the Cartesian coordinate vector joining one end of an interferometer arm to another, its change δX^i due to a GW signal impinging upon the detector is often claimed to be[5]

$$\delta X^i = \frac{1}{2} h_{ij}^{\mathrm{TT}} X^j, \tag{2}$$

where h_{ij}^{TT} is the 'transverse-traceless' portion of the space-space components of $h_{\mu\nu}$ in equation (1). What does 'transverse-traceless' really mean in this context? Rácz [1] and—more recently—Ashtekar and Bonga [2, 3] have pointed out, the GW literature erroneously uses two distinct notions of 'transverse-traceless' interchangeably[6]. (We shall adopt Ashtekar and Bonga's notation of 'TT' and 'tt'.) On the one hand, there is one involving the divergence-free condition,

$$\partial_i h_{ij}^{\mathrm{TT}} = \partial_i h_{ji}^{\mathrm{TT}} = 0 = \delta^{ij} h_{ij}^{\mathrm{TT}}; \tag{3}$$

while on the other hand there is one involving a transverse-projection in position space,

$$h_{ij}^{\mathrm{tt}} \equiv P_{ijab} h_{ab}. \tag{4}$$

[4] The Greek indices $\mu, \nu, \ldots,$ run from 0 to $d-1$, while the Latin ones $i, j, \ldots,$ run over spatial coordinates from 1 to $d-1$, and the 'mostly plus' sign convention for the metric is used, namely $\eta_{\mu\nu} = \mathrm{diag}[-1, +1, \ldots, +1]$. Throughout this paper, the symmetrization and anti-symmetrization of indices are denoted by the symbols $(\ldots)$ and $[\ldots]$, respectively, e.g. $T_{(\mu\nu)} \equiv \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$ and $T_{[\mu\nu]} \equiv \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$.

[5] See, for example, equation (27.26) of Thorne and Blandford [6].

[6] Frenkel and Rácz [4] have also pointed out a similar error within the electromagnetic context.

The definition of transverse-projection in equation (4) is based on the unit radial vector $\widehat{r} \equiv \vec{x}/|\vec{x}|$ pointing from the isolated astrophysical source centered at $\vec{0}$ to the observer at $\vec{x}$, namely

$$P_{ijab} \equiv P_{a(i}P_{j)b} - \frac{1}{d-2} P_{ij}P_{ab}, \tag{5}$$

$$P_{ij} \equiv \delta_{ij} - \widehat{r}_i \widehat{r}_j. \tag{6}$$

Because the rank-2 object is a projector, in the sense that

$$P_{ia}P_{aj} = P_{ij}, \tag{7}$$

and is also transverse to the radial direction,

$$\widehat{r}^i P_{ij} = 0 = P_{ij}\widehat{r}^j, \tag{8}$$

we see that the 'tt' GW in equation (4) enjoys the same traceless condition as its 'TT' counterpart in equation (3) (i.e. $\delta^{ij} h^{\mathrm{tt}}_{ij} = 0$) but is transverse to the unit radial vector

$$\widehat{r}^i h^{\mathrm{tt}}_{ij} = 0 = h^{\mathrm{tt}}_{ij}\widehat{r}^j \tag{9}$$

instead of being divergence-free.

We believe the intent of much of the contemporary gravitational literature is to claim the TT GW, obeying equation (3), to be the observable; while the tt one in equation (4) to be only an approximate expression of the same gravitational signal when the observer is very far from the source[7]. To our knowledge, the clearest enunciation of this stance may be found in the review by Flanagan and Hughes [7]. After describing how the TT piece of the gravitational perturbation of flat spacetime is the only gauge invariant portion that obeys a wave equation in section 2.2—the remaining 2 scalars and one vector obey Poisson equations—and after attempting to justify how the TT GW is the one appearing in equation (2) (see equation (3.12) of [7]) they went on in section 4.2 to assert, albeit without justification, that the far zone version of this TT GW in fact reduces to the tt one. In equation (4.23), they then expressed the final GW quadrupole formula in the latter tt form.

Other pedagogical discussions of GWs usually begin with the homogeneous TT wave solutions in perturbed Minkowski spacetime completely devoid of matter[8]:

$$h^{\mathrm{TT}}_{ij}[t,\vec{x}] = \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} \left(\widehat{P}_{ijab}[\vec{k}]\epsilon_{ab}[\vec{k}] \mathrm{e}^{\mathrm{i}k_\mu x^\mu} + \text{c.c.} \right), \tag{10}$$

[7] The exception appears to be Thorne and Blandford [6], where they went straight to the tt form of the GW (see Box 27.2) without any discussion of gauge invariance whatsoever.

[8] See, for instance, sections 35.2–35.4 of Misner, Thorne, Wheeler [8]; section 10.2 of Weinberg [9]; or section 9.1 of Schutz [10]. Briefly, one may start with the de Donder gauge condition $\partial^\mu h_{\mu\nu} = (1/2)\partial_\nu h$, where $h \equiv \eta^{\mu\nu} h_{\mu\nu}$, and solve the linearized vacuum Einstein's equations $\partial^2 (h_{\mu\nu} - (1/2)\eta_{\mu\nu} h) = 0$. By performing a gauge transformation in Fourier spacetime to set to zero the $h_{\mu 0} = h_{0\mu}$ components, one would arrive at equation (10). Often, equation (3) is called the 'TT gauge' but texts often do not caution the reader that this gauge condition can no longer be imposed once matter is introduced into the setup, i.e. once the very source of GWs is present. (Maggiore [11] does note that 'TT gauge' does not exist inside the source, but goes on to impose it anyway in the far zone.) We wish to reiterate the remarks already made in section C of [12]: if the 'TT gauge' were to exist, that would mean the initially non-trivial gauge invariant scalars and vector variables may be coordinate-transformed to zero. Furthermore, the misleading use of the phrase 'TT gauge' suggests one may even choose a different gauge to compute GW patterns—after all, one ought to be able to use any desired coordinate system—but this cannot be the case, as the GW pattern is an observable and must therefore yield a unique result.

$$k_\mu \equiv \left(-|\vec{k}|, k_i\right), \qquad k^2 \equiv k_\sigma k^\sigma = 0; \tag{11}$$

where 'c.c.' denotes the complex conjugate of the preceding term; ϵ_{ab} can be viewed as the purely spatial gravitational wave amplitude tensor; and the projector is now one in momentum/Fourier space,

$$\widehat{P}_{ijab}[\vec{k}] \equiv \widehat{P}_{a(i}\widehat{P}_{j)b} - \frac{1}{d-2}\widehat{P}_{ij}\widehat{P}_{ab}, \tag{12}$$

$$\widehat{P}_{ij} \equiv \delta_{ij} - \widehat{k}_i\widehat{k}_j, \qquad \widehat{k}_i \equiv \frac{k_i}{|\vec{k}|}. \tag{13}$$

Because in Fourier space a spatial derivative is replaced with a momentum vector, $\partial_j \to ik_j$, and because of the transverse-traceless properties

$$k^i\widehat{P}_{ijab} = k^i\widehat{P}_{jiab} = 0 = \delta^{ij}\widehat{P}_{ijab}; \tag{14}$$

the perturbations in equation (10) do indeed satisfy the 'TT' conditions in equation (3).

These GW discussions typically go on to justify equation (2) in vacuum before, as opposed to after, solving the perturbations engendered by a non-trivial source. The excuse is that, one expects these perturbations from an isolated system to approach TT plane GWs in the asymptotic far zone limit. As we shall see below, the TT and tt GWs do indeed coincide in this $r \equiv |\vec{x}| \to \infty$ limit. Hence, one might reasonably question: why bother with the distinction at all? To this end, Ashtekar and Bonga point out that the tt GWs miss the 'Coulombic aspects' that are contained in the TT ones. Moreover, in Quantum Field Theory, each mode of the superposition of TT GWs in equation (10) and not those of h_{ij}^{tt}—because it is the former that is fully gauge-invariant—would be regarded as an irreducible spin-2 graviton. Therefore, one may be led to a principled stance and insist that it is h_{ij}^{TT} that is physical.

But is the TT GW really a standalone observable? One of us (YZC) has been confused by this issue since several years ago, when he began developing a program to explore novel ways to understand the causal structure of gravitational signals in curved spacetimes—i.e. how they propagate both on and within the null cone. As highlighted in [7], the gauge invariant TT GW is a nonlocal function of the metric perturbation $h_{\mu\nu}$ in equation (1), because the TT projection process takes place in Fourier space. Since, at least about a Minkowski background, the de Donder gauge gravitational perturbation depends on its matter source in a causal manner, this suggests the TT GW may thus depend on the same matter source in an acausal manner due to this nonlocal character. This in turn would render it unphysical, as no classical physical observable should arise from outside the past light cone of the observer.

In this paper, we wish to clarify how the gauge-invariant forms of the vector potential and metric perturbations of, respectively, electromagnetism and linearized gravitation contribute to the observables of these theories. This will include understanding how all their gauge-invariant field variables, not just the dynamical massless spin-1 and spin-2 ones, play crucial roles in ensuring that their physical observables depend on their progenitors—namely, the electric current and matter stress tensor—in a causal manner. Through a concrete evaluation of the massless spin-1 photon and spin-2 graviton Green's functions, in Minkowski and spatially flat cosmological spacetimes, we will show that they are indeed acausally dependent on these sources and therefore cannot be standalone observables. However, by ensuring the rest of the gauge-invariant variables are included in the computation of the electromagnetic Faraday tensor $F_{\mu\nu}$ as well as the $\delta_1 R_{0i0j}$ components of the linearized Riemann tensor in Minkowski and the $\delta_1 C^i{}_{0j0}$ components of the linearized Weyl tensor in spatially flat cosmologies, the electromagnetic and gravitational tidal forces become strictly causal ones. In particular, we gain the following insight into the gauge-invariant content of electromagnetism and linearized General Relativity. While the magnetic field F_{ij} does depend only on the massless spin-1 photon, the electric field F_{0i} depends on the causal portion of the velocity of the photon, with its acausal portion canceled by the gauge-invariant scalar of the vector potential, in all spacetime dimensions $d \geqslant 3$. For the gravitational case, tidal forces in a flat spacetime background are encoded within the causal part of the acceleration of the massless spin-2 graviton, with the acausal portion eliminated by the two gauge-invariant scalars and one vector potential for all $d \geqslant 4$. Additionally, about a cosmological background, if the Weyl tensor describes the dominant contributions to tidal forces, then the latter appear to depend on the causal portions of both the massless spin-2 and the two gauge-invariant Bardeen scalar potentials. We view this latter analysis as a first step towards an understanding of whether the two Bardeen gauge-invariant scalar potentials ought to be considered an integral portion of gravitational waves and their associated memories in cosmological settings—even though the dynamics of General Relativity (in $3+1$ dimensions) is usually attributed exclusively to its two spin-2 degrees of freedom.

In section 2 we will define the electromagnetic and gravitational gauge invariant variables; and proceed to clarify what the relevant (classical) electromagnetic and gravitational observables are. In section 3 we will use the non-local character of the transverse projection in momentum space to argue why these gauge-invariant variables are expected to be acausal. Following which, we begin in section 4 to compute the explicit forms of the transverse-photon and TT graviton Minkowski Green's functions, confirming their acausal nature. We also compute the solutions to the gauge-invariant scalars and vectors; and combine the results to study how the electromagnetic Faraday tensor and gravitational linearized Riemann are causally dependent on their respective sources. The far zone and stationary limit are examined; and micro-causality violated is pointed out. In section 5, we move on to study similar issues but in a cosmology dominated by a cosmological constant or driven by a relativistic fluid with equation of state $0 < w \leqslant 1$. Finally, we summarize our findings and outline future directions in section 6.

2. Gauge-invariance and observables

Setup Throughout the rest of this paper, we will be studying the d-dimensional perturbed Friedmann–Lemaître–Robertson–Walker (FLRW)-like metric

$$g_{\mu\nu}[x] = a[\eta]^2 \left(\eta_{\mu\nu} + \chi_{\mu\nu}[x]\right), \qquad x^\mu \equiv (\eta, \vec{x}); \tag{15}$$

where $a = 1$ for a flat background or

$$a[\eta] = \left(\frac{\eta}{\eta_0}\right)^{\frac{2}{q_w}}, \qquad q_w \equiv (d-3) + (d-1)w. \tag{16}$$

In equation (16), if the perturbations $\chi_{\mu\nu}$ were not present, setting $w = -1$ with $\eta < 0$ yields de Sitter spacetime and $0 \leqslant w \leqslant 1$ with $\eta > 0$ a spatially flat cosmology driven by a single perfect fluid with equation-of-state w. The non-trivial perturbations $\chi_{\mu\nu}$ satisfy Einstein's equations coupled to the fluid plus a compact astrophysical system, linearized about the corresponding backgrounds. The detailed analysis can be found in sections III and IV in [12], and we will cite the relevant results below.

Let us consider an infinitesimal coordinate transformation

$$x^\mu = x'^\mu + \xi^\mu[x'], \tag{17}$$

where $x' \equiv (\eta', \vec{x}')$ and ξ^μ is small in the same sense that $h_{\mu\nu}$ is small. Then up to first order in perturbations, the metric tensor in the primed coordinate system may be written as

$$g_{\mu'\nu'}[\eta', \vec{x}'] = a[\eta']^2 \left(\eta_{\mu\nu} + \chi_{\mu'\nu'}[\eta', \vec{x}']\right), \tag{18}$$

with all the coordinate transformation induced by equation (17) attributed to that of the metric perturbation in the following way:

$$\chi_{\mu'\nu'}[x'] = \chi_{\mu\nu}[x'] + 2\eta_{\sigma(\mu}\partial_{\nu')}\xi^\sigma[x'] + 2\frac{\dot{a}[\eta']}{a[\eta']}\xi^0[x']\eta_{\mu\nu}. \tag{19}$$

(The $\chi_{\mu\nu}[\eta', \vec{x}']$ and $\xi^\mu[x']$ on the right hand side of equation (19) are the perturbation and gauge-transformation vector components in the 'old' $x^\mu \equiv (\eta, \vec{x})$ coordinate basis, but with the replacement $x \to x'$.) Next, we perform a scalar–vector–tensor decomposition of both the metric perturbations[9]

$$\chi_{00} \equiv E, \qquad \chi_{0i} \equiv \partial_i F + F_i,$$
$$\chi_{ij} \equiv D_{ij} + \partial_{(i}D_{j)} + \frac{D}{d-1}\delta_{ij} + \left(\partial_i\partial_j - \frac{\delta_{ij}}{d-1}\vec{\nabla}^2\right)K, \tag{20}$$

as well as the astrophysical stress tensor

$${}^{(a)}T_{00} \equiv \rho, \qquad {}^{(a)}T_{0i} \equiv \Sigma_i + \partial_i\Sigma, \tag{21}$$

$${}^{(a)}T_{ij} \equiv \sigma_{ij} + \partial_{(i}\sigma_{j)} + \frac{\sigma}{d-1}\delta_{ij} + \left(\partial_i\partial_j - \frac{\delta_{ij}}{d-1}\vec{\nabla}^2\right)\Upsilon; \tag{22}$$

where these variables subject to the following constraints

$$\partial_i F_i = \partial_i D_i = 0 = \delta^{ij}D_{ij} = \partial_i D_{ij}, \tag{23}$$

$$\partial_i\Sigma_i = \partial_i\sigma_i = 0 = \delta^{ij}\sigma_{ij} = \partial_i\sigma_{ij}. \tag{24}$$

We may then gather the following are gauge-invariant at first order in perturbations[10]:

$$\Phi \equiv -\frac{E}{2} + \frac{1}{a}\partial_0\left\{a\left(F - \frac{\dot{K}}{2}\right)\right\}, \tag{25}$$

$$\Psi \equiv -\frac{D - \vec{\nabla}^2 K}{2(d-1)} - \frac{\dot{a}}{a}\left(F - \frac{\dot{K}}{2}\right), \tag{26}$$

$$V_i \equiv F_i - \frac{\dot{D}_i}{2} \qquad \text{and} \qquad D_{ij} \equiv \chi_{ij}^{\mathrm{TT}}. \tag{27}$$

Within the cosmological case, the solution of D_{ij} can be found in equation (111), that of V_i in equation (119) and those of Ψ in equations (123), (125) and (130) of [12]. Φ and Ψ are related through

$$(d-3)\Psi - \Phi = 8\pi G_{\mathrm{N}}\Upsilon. \tag{28}$$

[9] See, for example, section IV in [12] for a discussion of the scalar–vector–tensor decomposition and the gauge-invariant formalism of linearized gravitation.

[10] Notice that the sign convention for the metric as well as certain gauge-invariant variables are defined differently in [12]. To change the gauge-invariant notations into those employed in [12], we follow the conversions: Φ[here] $\to \Psi$[[12]]/2, Ψ[here] $\to \Phi$[[12]]/2, V_i[here] $\to -V_i$[[12]], and D_{ij}[here] $\to -D_{ij}$[[12]].

Within the Minkowski case, on the other hand, equation (28) still holds but the solution of Ψ can be found in equation (A27); that of V_i in equation (A28); and that of D_{ij} in equation (A29) of [12]. As already alluded to, of all the gauge invariant variables in a flat background, only the tensor D_{ij} admits wave solutions.

Because of the TT constraints in equation (23), note that the tensor mode D_{ij} exists only for $d \geqslant 4$. The apparent physical importance of these field variables in equations (25)–(27) lies in the fact that, if some observable can be expressed in terms of them, then the same observable cannot be rendered trivial merely by a small change in coordinates since Φ, Ψ, V_i and D_{ij} will remain invariant.

When dealing with electromagnetism, we will set to zero the perturbations $\chi_{\mu\nu}$ in equation (15) and proceed to solve Maxwell's equations

$$\nabla_\nu F^{\mu\nu} = J^\mu, \qquad F_{\mu\nu} \equiv 2\partial_{[\mu} A_{\nu]}. \tag{29}$$

Under the gauge transformation

$$A_\mu \to A_\mu - \partial_\mu \mathcal{C}, \tag{30}$$

the vector potential A_μ leaves the Faraday tensor $F_{\mu\nu}$ invariant for an arbitrary function $\mathcal{C}$. If we perform a scalar–vector decomposition of the vector potential

$$A_i \equiv \alpha_i + \partial_i \alpha, \qquad \partial_i \alpha_i = 0, \tag{31}$$

and that of the electric current

$$J_0 \equiv -\rho, \tag{32}$$

$$J_i \equiv \Gamma_i + \partial_i \Gamma, \qquad \partial_i \Gamma_i = 0; \tag{33}$$

we may proceed to identify the following gauge-invariant scalar Φ and transverse (helicity $-$ 1) photon α_i[11]:

$$\Phi \equiv \dot{\alpha} - A_0 \qquad \text{and} \qquad \alpha_i \equiv A_i^{\mathrm{T}}. \tag{34}$$

In terms of these variables, the Faraday tensor reads

$$F_{0i} = \dot{\alpha}_i + \partial_i \Phi, \qquad F_{ij} = 2\partial_{[i}\alpha_{j]}. \tag{35}$$

We refer the reader to section V of [12] for further details; again, we will cite the relevant results below.

Electromagnetic observables In classical electromagnetism, it is the electric $F_{0i} = -F_{i0}$ and magnetic F_{ij} fields in equation (35) that are regarded as observables, because they provide the forces on electrically charged systems. We believe the situation for gravity is more subtle, however.

Gravitational observables: (simplified) Weber bar Let us begin with a small lump of freely falling material acting as a Weber-bar detector of GWs. In what follows, the assumption of freely falling detectors makes it technically advantageous to describe their trajectories using the synchronous gauge metric—where the perturbations are purely spatial:

[11] We highlight that, in the electromagnetic case here, some conventions are defined differently relative to [12]. For instance, when switching to the gauge-invariant variables used in [12], we may implement the following conversions: α_i[here] $\to -\alpha_i$[[12]], Φ[here] $\to \Phi$[[12]], ρ[here] $\to a^2\rho$[[12]], and Γ_i[here] $\to -a^2\Gamma_i$[[12]].

$$g_{\mu\nu} = a^2 \left(\eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \chi_{ij}^{(\mathrm{Synch})} \mathrm{d}x^i \mathrm{d}x^j \right). \tag{36}$$

For, if the particles comprising the detectors experience negligible inter-particle forces, then the synchronous gauge in equation (36) can be chosen such that each of them would in fact have constant spatial trajectories; specifically, for the ith particle its timelike geodesic reads

$$Z^\mu_{(i)} = \left(\eta, \vec{Z}_{(i)}\right), \qquad \vec{Z}_{(i)} \text{ constant.} \tag{37}$$

The tidal forces due to a passing GW acting between an infinitesimally nearby pair of particles, whose worldlines are joined by ℓ^μ, is given by the geodesic deviation equation

$$U^\alpha U^\beta \nabla_\alpha \nabla_\beta \ell^\mu = -R^\mu{}_{\nu\alpha\beta} U^\nu \ell^\alpha U^\beta. \tag{38}$$

The $U^\mu = a^{-1}\delta^\mu_0$ in equation (38) is the timelike geodesic vector tangent to either one of the two worldlines. For the freely falling scenario at hand, it is in fact consistent to choose ℓ^μ to be purely spatial, i.e. $\ell^0 = 0$, so that equation (38) becomes

$$\nabla_0 \nabla_0 \ell^i = -R^i{}_{0j0} \ell^j. \tag{39}$$

Minkowski analysis In a flat background, $a = 1$, we witness from equation (39) that the $\delta_1 R_{0i0j}$ components of the linearized Riemann tensor must therefore describe the first-order tidal forces between infinitesimally nearby particles within our idealized Weber bar. Any inter-particle (electromagnetic) forces that are responsible for holding the lump of material together must therefore counter the $\delta_1 R_{0i0j}$ on the right hand side of equation (39). Moreover, as long as our Weber bar's proper size is very small compared to that of the GW wavelength, the physical pattern of rarefaction and compression of the material asserted by the GW's passage must also be encoded entirely within $\delta_1 R_{0i0j}$. Now, not only do these components carry physical meaning, the entire linearized Riemann tensor is in fact gauge invariant because the full Riemann tensor is zero when evaluated on the background $g_{\mu\nu} = \eta_{\mu\nu}$. These reasons explain why we will, in the coming sections, compute $\delta_1 R_{0i0j}$ within the gauge invariant formalism:

$$\delta_1 R_{0i0j} = \delta_{ij}\ddot{\Psi} + \partial_i \partial_j \Phi + \partial_{(i}\dot{V}_{j)} - \frac{1}{2}\ddot{D}_{ij}, \qquad (a = 1). \tag{40}$$

Cosmological background If our freely falling particles were in a cosmological background, the geodesic deviation equation (39) continues to be applicable. However, the linearized Riemann tensor is no longer gauge invariant because its background value, when $g_{\mu\nu} = a^2\eta_{\mu\nu}$, is no longer zero. This renders its physical interpretation more subtle. On the other hand, the traceless portion of the Riemann tensor, i.e. the Weyl tensor $C^\mu{}_{\nu\alpha\beta}$, is conformally invariant. This means $C^\mu{}_{\nu\alpha\beta}[g_{\mu\nu} = a^2\eta_{\mu\nu}] = 0$ and the $\delta_1 C^i{}_{0j0}$ components of the linearized Weyl tensor is gauge invariant. It may be possible to argue that $\delta_1 C^i{}_{0j0}$ provides the dominant contribution to tidal forces in cosmology—for, in flat spacetime, it is exactly equivalent to the Riemann tensor whenever the zero cosmological constant form of Einstein's equations holds and the Weber bar is in a vacuum region[12]—but we shall leave the detailed analysis of this cosmological case to future work [13].

Gravitational observable: (simplified) laser interferometer We move on to consider a toy model of a freely falling laser interferometer. If we assume the proper size of the interferometer is small compared to the GW wavelength, it is reasonable to then state the observed interference pattern will be proportional to the differences in its arm-lengths. As argued in

[12] See, for e.g. equation (14) of [12] or equation (24).

section C of [12], we may again employ the synchronous gauge in equation (36) to compute the time dependent proper distance between two ends $(\eta, \vec{Y}_0)$ and $(\eta, \vec{Z}_0)$ of a single arm. (Remember the $\vec{Y}_0$ and $\vec{Z}_0$ here are constants.) Using Synge's world function, and assuming the interferometer is turned on at η' but does not operate over cosmological timescales, the fractional distortion of this $\vec{Y} \leftrightarrow \vec{Z}$ arm is (see equation (6) of [12])

$$\left(\frac{\delta L}{L_0}\right)[\eta > \eta'] = \frac{\widehat{n}^i \widehat{n}^j}{2} \int_0^1 \Delta\chi_{ij}^{(\text{Synch})} \mathrm{d}\lambda + \mathcal{O}\left[\left(\chi_{mn}^{(\text{Synch})}\right)^2, (\eta - \eta')\dot{a}[\eta']/a[\eta']\right], \tag{41}$$

where $\widehat{n} \equiv (\vec{Z}_0 - \vec{Y}_0)/|\vec{Z}_0 - \vec{Y}_0|$ is the unit radial vector pointing from one mass to the other; and the λ-integral involves a Euclidean straight line between the two end points:

$$\Delta\chi_{ij}^{(\text{Synch})} \equiv \chi_{ij}^{(\text{Synch})}\left[\eta, \vec{Y}_0 + \lambda\left(\vec{Z}_0 - \vec{Y}_0\right)\right] - \chi_{ij}^{(\text{Synch})}\left[\eta', \vec{Y}_0 + \lambda\left(\vec{Z}_0 - \vec{Y}_0\right)\right]. \tag{42}$$

Minkowski analysis Now, in the synchronous gauge of equation (36), the linearized Riemann tensor reads

$$\delta_1 R_{0i0j} = -\frac{1}{2}\ddot{\chi}_{ij}^{(\text{Synch})}. \tag{43}$$

We may thus solve for the synchronous gauge perturbation needed in equation (41) by first connecting it to the gauge-invariant variables. This, in turn, is achieved by exploiting the gauge-invariance of the linearized Riemann tensor in a Minkowski background. In other words, since equations (40) and (43) refer to the same object, we have

$$\ddot{\chi}_{ij}^{(\text{Synch})} = \ddot{D}_{ij} - 2\partial_{(i}\dot{V}_{j)} - 2\delta_{ij}\ddot{\Psi} - 2\partial_i\partial_j\Phi, \qquad (a = 1). \tag{44}$$

Equations (40), (43) and (44) inform us that the fractional distortion formula in equation (41) is therefore—at least in principle—related to the double time integral of the linearized Riemann tensor itself[13]. In any event, as long as the GW detector is sufficiently far enough from the astrophysical source, we may take the far zone limit of the right hand side of equation (44). Below, we will use methods different from those in [2, 3] to argue that, this far zone limit yields

$$\left(\ddot{D}_{ij} - 2\partial_{(i}\dot{V}_{j)} - 2\delta_{ij}\ddot{\Psi} - 2\partial_i\partial_j\Phi\right)_{\text{far zone}} = \ddot{D}_{ij}[\text{far zone}] = \ddot{\chi}_{ij}^{\text{tt}}[\text{far zone}], \qquad (a = 1); \tag{45}$$

where χ_{ij}^{tt} is the tt projection of the de Donder gauge solution as $r \equiv |\vec{x}| \to \infty$. Comparing equations (44) and (45) allows us to deduce:

$$\chi_{ij}^{(\text{Synch})}[\eta, \vec{x}] = \chi_{ij}^{\text{tt}}[\eta, \vec{x}] + (\eta - \eta')\mathcal{V}_{ij}[\eta', \vec{x}] + \mathcal{W}_{ij}[\eta', \vec{x}], \tag{46}$$

where $\mathcal{V}_{ij}$ and $\mathcal{W}_{ij}$ are the two undetermined initial conditions at η', and, on both sides, the far zone limit has been taken. One issue that is often not addressed is, why the initial conditions—the last two terms on the right hand side of equation (46)—are usually neglected. We will take the perspective that realistic GW detectors are sensitive to waves within a limited bandwidth, and since the second and third terms of equation (46) are zero-frequency 'waves' one may ignore their contributions. More explicitly, the (ω-)frequency transform of equation (46) becomes

[13] See, for example, equations (27.22) and (27.24) of [6].

$$\widetilde{\chi}_{ij}^{(\mathrm{Synch})}[\omega,\vec{x}] = \widetilde{\chi}_{ij}^{\mathrm{tt}}[\omega,\vec{x}] - (2\pi i)\delta'[\omega]\mathrm{e}^{\mathrm{i}\omega\eta'}\mathcal{V}_{ij}[\eta',\vec{x}] + (2\pi)\delta[\omega]\mathcal{W}_{ij}[\eta',\vec{x}], \quad (47)$$

where $\delta[\omega]$ and $\delta'[\omega]$ are the Dirac delta function and its derivative.

We close this section by providing an expedited method to obtain the synchronous gauge metric perturbation from the gauge-invariance of the linearized Riemann tensor, at least in $(3+1)$ dimensional Minkowski spacetime. The key is that the de Donder gauge $\partial^\mu\chi_{\mu\nu} = (1/2)\partial_\nu\chi$, with $\chi \equiv \eta^{\rho\sigma}\chi_{\rho\sigma}$, allows us to solve $\chi_{\mu\nu}$ rather easily with the massless scalar Green's function. In terms of the 'trace-reversed' variable

$$\overline{\chi}_{\mu\nu} \equiv \chi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\chi, \qquad \overline{\chi} = \eta^{\rho\sigma}\overline{\chi}_{\rho\sigma}; \quad (48)$$

the linearized Einstein's equations lead us to the following far zone solution:

$$\overline{\chi}_{\mu\nu}[\eta,\vec{x}] = \frac{4G_{\mathrm{N}}}{r}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{x}' T_{\mu\nu}[\eta - r + \vec{x}'\cdot\widehat{r},\vec{x}'], \qquad \widehat{r} \equiv \frac{\vec{x}}{|\vec{x}|}. \quad (49)$$

(Note: we have chosen $\vec{0}$ to lie within the astrophysical system.) We also have, in de Donder gauge, the following components of the linearized Riemann tensor

$$\delta_1 R_{0i0j} = \frac{1}{2}\left(2\partial_l\partial_{(i}\overline{\chi}_{j)l} - \partial_0^2\left(\overline{\chi}_{ij} - \frac{1}{2}\delta_{ij}\overline{\chi}_{ll}\right) - \frac{1}{2}\delta_{ij}\partial_l\partial_m\overline{\chi}_{lm} - \frac{1}{2}\partial_i\partial_j\overline{\chi}_{ll} - \frac{1}{2}\partial_i\partial_j\overline{\chi}_{00}\right). \quad (50)$$

Upon inserting equation (49) into equation (50), one may recognize that every spatial derivative acting on $\overline{\chi}_{\mu\nu}$ may be replaced with a time derivative, via

$$\partial_i \to -\widehat{r}^i\partial_0. \quad (51)$$

In the far zone limit we are working in, the error incurred by this replacement scales as (timescale of source)/(observer-source distance) or (characteristic size of source)/(observer-source distance), both of which are small by assumption. At this point, one would find that equation (50) has been massaged into

$$\delta_1 R_{0i0j} = -\frac{1}{2}\ddot{\chi}_{ij}^{\mathrm{tt}}, \quad (52)$$

where the tt perturbation is the following projection (see equation (5)) of the de Donder gauge solution

$$\chi_{ij}^{\mathrm{tt}} = P_{ijab}\overline{\chi}_{ab}[\text{de Donder}]. \quad (53)$$

Comparing equations (43) and (52) now hands us, for finite frequency ω,

$$\widetilde{\chi}_{ij}^{(\mathrm{Synch})}[\omega,\vec{x}] = \widetilde{\chi}_{ij}^{\mathrm{tt}}[\omega,\vec{x}] \qquad (\text{Far zone}). \quad (54)$$

Finally, we once again leave to future work [13] the connection between the cosmological synchronous gauge metric perturbation to its gauge-invariant counterparts.

3. Why are the massless spin-1 photon and spin-2 graviton acausal?

Before we proceed to tackle the computations of the massless spin-1 and spin-2 Green's functions, let us first explain why acausality is to be expected. Since this section is meant to be heuristic, we shall be content to work strictly in a Minkowski background.

Spin-1 photons We begin by recalling the fact that the Lorenz gauge vector potential, which obeys $\partial^\mu A_\mu = 0$ and $\partial^2 A_\mu = -J_\mu$, is causally dependent on the electric current J_μ. If G_d^+ denotes the retarded Green's function of the massless scalar, we have

$$A_\mu[x] = -\int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x' G_d^+[x-x'] J_\mu[x']. \tag{55}$$

In even dimensions, where G_d^+ propagates signals strictly on the null cone, A_μ is the field due to the electric current lying on the past light cone of the observer. In odd dimensions, where G_d^+ propagates signals strictly inside the null cone (at least for timelike sources [12]), A_μ is the field due to the electric current lying within the past light cone of the observer. On the other hand, the transverse spin-1 photon can be constructed from the Lorenz gauge photon via the following Fourier-space projection involving the spatial Fourier transform of the spatial components of the vector potential $\widetilde{A}_j[\eta,\vec{k}]$:

$$\alpha_i[\eta,\vec{x}] = \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} \widehat{P}_{ij}[\vec{k}] \widetilde{A}_j[\eta,\vec{k}] \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}, \tag{56}$$

where the transverse projector $\widehat{P}_{ij}[\vec{k}]$ is defined in equation (13). Now, in Fourier space, $-1/\vec{k}^2$ is simply the Euclidean Green's function

$$G_d^{(\mathrm{E})}[\vec{x}-\vec{x}'] = \frac{1}{\partial_i\partial_i}[\vec{x}-\vec{x}'] = \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{\mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{x}')}}{-\vec{k}^2}, \tag{57}$$

with the concrete expressions

$$G_{3+2\epsilon}^{(\mathrm{E,reg})}[\vec{x}-\vec{x}'] = -\frac{1}{4\pi}\left(\frac{1}{\epsilon} - \gamma - \ln[\pi] - 2\ln[\mu R]\right), \tag{58}$$

$$G_{d\geqslant 4}^{(\mathrm{E})}[\vec{x}-\vec{x}'] = -\frac{\Gamma\left[\frac{d-3}{2}\right]}{4\pi^{\frac{d-1}{2}}|\vec{x}-\vec{x}'|^{d-3}}; \tag{59}$$

where $G_3^{(\mathrm{E})}$ has been dimensional-regularized in equation (58) with an arbitrary mass scale μ introduced and γ being the Euler-Mascheroni constant, and $-\partial_i\partial_j$ is replaced with $k_i k_j$. Utilizing eqs. (57), (58) and (59) in equation (56) informs us that the transverse photon itself must therefore be related to its Lorenz gauge counterpart through the subtraction of the latter's longitudinal piece:

$$\alpha_i[\eta,\vec{x}] = A_i[\eta,\vec{x}] + A_i^{\parallel}[\eta,\vec{x}], \tag{60}$$

$$A_i^{\parallel}[\eta,\vec{x}] \equiv -\partial_i\partial_j \int_{\mathbb{R}^2} \mathrm{d}^2\vec{x}' \frac{\ln\left[|\vec{x}-\vec{x}'|\right]}{2\pi} A_j[\eta,\vec{x}'] \qquad (d=3) \tag{61}$$

$$\equiv \partial_i\partial_j \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}' \frac{\Gamma\left[\frac{d-3}{2}\right]}{4\pi^{\frac{d-1}{2}}|\vec{x}-\vec{x}'|^{d-3}} A_j[\eta,\vec{x}'] \qquad (d\geqslant 4). \tag{62}$$

We will now explain how the second term $A_i^{\parallel}[\eta,\vec{x}]$ in equation (60) is most likely acausal, because it is essentially the causal Lorenz gauge vector potential but smeared over all space, weighted by the Euclidean Green's function in equations (58) and (59). Referring to figure 1, we see that equations (61) and (62) are the weighted superposition of $A_i[\eta,\vec{x}']$ over all $\vec{x}'$, which—for a fixed $\vec{x}'$—receives signals from the electric current from the past light cone of

$(\eta, \vec{x}')$ (for even dimensions) or within it (for odd dimensions). But from the perspective of the observer at $(\eta, \vec{x})$, this means $A_i^{\parallel}$ is getting a signal from the portion of the source residing within the shaded (blue) region, which lies outside its past null cone.

Spin-2 gravitons The de Donder gauge $\partial^\mu \overline{\chi}_{\mu\nu} = (1/2)\partial_\nu \chi$, with $\chi \equiv \eta^{\rho\sigma}\chi_{\rho\sigma}$, like the Lorenz gauge for photons, yields gravitational perturbations that are causally sourced by their matter $T_{\mu\nu}$. Equation (A39) of [12] tells us

$$\overline{\chi}_{\mu\nu}[x] = -16\pi G_{\mathrm{N}} \int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x' G_d^+[x-x'] T_{\mu\nu}[x']. \tag{63}$$

The transverse-traceless spin-2 graviton is gotten from its de Donder gauge counterpart via the Fourier space projection

$$\chi_{ij}^{\mathrm{TT}}[\eta, \vec{x}] = \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} \widehat{P}_{ijmn}[\vec{k}]\, \widetilde{\overline{\chi}}_{mn}[\eta, \vec{k}] \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}, \tag{64}$$

where the projection tensor $\widehat{P}_{ijmn}[\vec{k}]$ is given in equation (12), which can also be expressed explicitly in terms of $\vec{k}$,

$$\begin{aligned}
\widehat{P}_{ijmn}[\vec{k}] &= \widehat{P}_{m(i}\widehat{P}_{j)n} - \frac{1}{d-2}\widehat{P}_{ij}\widehat{P}_{mn}, \\
&= \delta_{m(i}\delta_{j)n} - \frac{\delta_{ij}\delta_{mn}}{d-2} - \frac{\left(\delta_{m(i}k_{j)}k_n + \delta_{n(i}k_{j)}k_m\right)}{\vec{k}^2} \\
&\quad + \frac{\delta_{ij}k_m k_n + \delta_{mn}k_i k_j}{(d-2)\vec{k}^2} + \left(\frac{d-3}{d-2}\right)\frac{k_i k_j k_m k_n}{\vec{k}^4}.
\end{aligned} \tag{65}$$

The same sort of arguments made for the spin-1 photon would apply here to tell us the spin-2 graviton receives signals from $T_{\mu\nu}$ from outside the observer's past light cone. For instance, the third and fourth group of terms in the second equality in equation (65) involves two spatial derivatives acting on the weighted superposition of the de Donder GW over all space but at the same observer time η, namely

$$\sim \partial_a \partial_b \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}' \frac{\overline{\chi}_{ij}[\eta, \vec{x}']}{|\vec{x}-\vec{x}'|^{d-3}}; \tag{66}$$

whereas the last group of terms in equation (65) involves four spatial derivatives acting on a different weighted superposition of the same:

$$\sim \partial_a \partial_b \partial_c \partial_e \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}' \frac{\overline{\chi}_{ij}[\eta, \vec{x}']}{|\vec{x}-\vec{x}'|^{d-5}}. \tag{67}$$

4. Minkowski spacetime

As demonstrated in the previous section, both massless spin-1 and spin-2 fields are expected to contain the acausal information from their isolated sources. To quantify this acausality, we will in this section perform a detailed analysis of the effective Minkowski spacetime Green's functions of the electromagnetic and gravitational gauge-invariant variables.

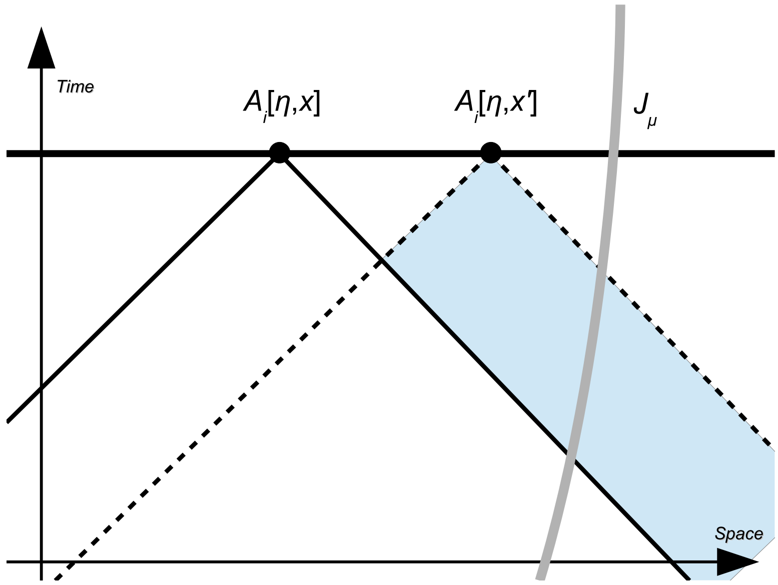

Figure 1. The $A_i[\eta, \vec{x}]$ is the Lorenz gauge photon vector potential at the observer's location $(\eta, \vec{x})$. The $A_i[\eta, \vec{x}']$ is the Lorenz gauge potential at some other spatial location but at the same time η. The solid cone is the past null cone of the observer at $(\eta, \vec{x})$; while the dotted one is that of $(\eta, \vec{x}')$. The gray thick line denotes the worldtube of the electric current. (See Brill and Goodman [5] for a closely related discussion, but from the Coulomb gauge perspective.)

4.1. Electromagnetism

We will begin with electromagnetism in all spacetime dimensions equal to or higher than three, $d \geqslant 3$; for $d = 3$ corresponds to the lowest physical dimension for spin-1 photons to exist. In addition, the discussion for the electromagnetic field here will provide a useful guide for us to tackle the more complicated and subtle case of linearized gravitation.

Field equations In terms of the gauge-invariant variables (34) and provided the electric current J^μ is conserved, the non-redundant portions of Maxwell's equations (29)—see [12] for a discussion—are the dynamical wave equation for the transverse spin-1 photon α_i,

$$\partial^2 \alpha_i = -\Gamma_i, \tag{68}$$

with $\partial^2 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$, and a Poisson's equation obeyed by the gauge-invariant scalar Φ,

$$\vec{\nabla}^2 \Phi = -\rho. \tag{69}$$

Notice both equations involve non-locality. The source Γ_i on the right hand side of equation (68) is the transverse component J_i, which (recalling arguments from the previous section) is thus a non-local functional of J_i. Whereas the Poisson equation obeyed by $\Phi[\eta, \vec{x}]$ means it is sensitive to the charge density $\rho[\eta, \vec{x}]$ on the right hand side of equation (69) at the same instant η. As we will show later in this section, only when α_i and Φ are both involved, do the physical observables—i.e. the field strength $F_{\mu\nu}$—become causally dependent on the electromagnetic current J_μ.

Spin-1 photons To solve for α_i in equation (68) through its effective Green's function convolved against its localized sources, it is convenient to first go to the Fourier space, where the transverse property is implemented through a projection of the Fourier transform of the current $\widetilde{J}_i$. For all $\vec{k} \neq \vec{0}$, the α_i can be written as the superposition

$$\alpha_i[\eta, \vec{x}] = -\int_{\mathbb{R}} \mathrm{d}\eta' \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} \widetilde{G}_d^+[\eta, \eta'; \vec{k}] \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right) \widetilde{J}_j[\eta', \vec{k}]\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}, \tag{70}$$

where $\widetilde{G}_d^+$ denotes the Fourier transform of the retarded Green's function of the massless scalar. In flat spacetime, it enjoys time-translation symmetry and reads

$$\widetilde{G}_d^+[\eta, \eta'; \vec{k}] = -\Theta[T] \frac{\sin |\vec{k}|T}{|\vec{k}|}, \tag{71}$$

with $T \equiv \eta - \eta'$. As we have seen before, $k_i k_j$ in momentum space can be pulled out with the replacement $k_i k_j \to -\partial_i \partial_j$ acting on the Fourier integral. Therefore, the expression (70) can be re-cast into the convolution of the spin-1 effective Green's function G_{ij}^+ against the local electromagnetic current,

$$\alpha_i[x] = -\int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x'\, G_{ij}^+[T, \vec{R}]\, J_j[x'], \tag{72}$$

where $\vec{R} \equiv \vec{x} - \vec{x}'$ and the G_{ij}^+ takes the form

$$\begin{aligned} G_{ij}^+[T, \vec{R}] &\equiv -\Theta[T] C_{ij}[T, \vec{R}], \\ C_{ij}[T, \vec{R}] &= \delta_{ij} C_{1,d}[T, R] + \partial_i \partial_j C_{2,d}[T, R]. \end{aligned} \tag{73}$$

The $C_{1,d}$ and $C_{2,d}$ are respectively defined to be two scalar Fourier integrals

$$C_{1,d}[T, R] \equiv \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{\sin |\vec{k}|T}{|\vec{k}|} \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{R}}, \tag{74}$$

$$C_{2,d}[T, R] \equiv \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{\sin |\vec{k}|T}{|\vec{k}|^3} \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{R}}, \tag{75}$$

with the observer-source spatial distance denoted as $R \equiv |\vec{x} - \vec{x}'|$. Hence, the effective Green's function G_{ij}^+ can be gotten explicitly through equation (73) once $C_{1,d}$ and $C_{2,d}$ are known. One of the advantages of using $C_{1,d}$ and $C_{2,d}$ to compute G_{ij}^+ is that our calculations can be simplified by exploiting the fact that they both obey the homogeneous wave equation $\partial^2 C_{1,d} = 0$ and $\partial^2 C_{2,d} = 0$ with the initial conditions

$$C_{1,d}[T = 0, R] = 0 \qquad \text{and} \qquad C_{2,d}[T = 0, R] = 0; \tag{76}$$

as well as the initial velocities

$$\dot{C}_{1,d}[T = 0, R] = \delta^{(d-1)}[\vec{x} - \vec{x}'] \qquad \text{and} \qquad \dot{C}_{2,d}[T = 0, R] = -G_d^{(\mathrm{E})}[R], \tag{77}$$

where the overdot denotes the time derivative with respect to η. In addition, $C_{1,d}$ and $C_{2,d}$ are connected via the spatial Laplacian operator or double-time derivatives:

$$\ddot{C}_{2,d}[T, R] = -C_{1,d}[T, R], \tag{78}$$

$$\vec{\nabla}^2 C_{2,d}[T, R] = -C_{1,d}[T, R]. \tag{79}$$

(Equations (76) through (79) conditions follow readily from the Fourier representations in equations (57), (74) and (75).) Moreover, note that

$$G_d^{\pm}[x-x'] = \mp\Theta[\pm T]C_{1,d}[T,R] \tag{80}$$

are the retarded $(+)$/advanced $(-)$ massless scalar Green's functions; obeying

$$\partial^2 G_d^{\pm} = \delta^{(d)}[x-x']. \tag{81}$$

In other words, $-C_{1,d}$ itself is the retarded minus advanced Green's function:

$$-C_{1,d} = G_d^{+}[x-x'] - G_d^{-}[x-x']. \tag{82}$$

Likewise, for the spin-1 case, the $-C_{ij}$ in equation (73) is the difference between the retarded Green's function in equation (73) and that of its advanced counterpart:

$$-C_{ij} = G_{ij}^{+}[x-x'] - G_{ij}^{-}[x-x']. \tag{83}$$

In Quantum Field Theory, the $C_{1,d}$ in equation (82) is proportional to the commutator of massless scalar fields. In turn, C_{ij} is proportional to the commutator of (spin-1) photon fields. Therefore, the elucidation of the (classical) causal structure of C_{ij} will also lead to insights regarding the quantization of the associated spin-1 photons.

Before moving on to the analytic solutions, let us show that the source of G_{ij}^{+} in equation (73) is an extended one, as opposed to the usual spacetime point source of, say, the massless scalar Green's function. Applying the wave operator to the expression (73) for G_{ij}^{+} hands us

$$\partial^2 G_{ij}^{+}[T,\vec{R}] = \delta_{ij}\delta^{(d)}[x-x'] - \delta[T]\partial_i\partial_j G_d^{(\mathrm{E})}[R] \tag{84}$$

$$= \delta[T]\left(\delta_{ij}\delta^{(d-1)}[\vec{x}-\vec{x}'] - \partial_i\partial_j G_d^{(\mathrm{E})}[R]\right); \tag{85}$$

where the $G_d^{(\mathrm{E})}$ is the Euclidean Green's function of equation (57) and the relations in equation (77) were employed. We may view equation (84) as a $(d-1)\times(d-1)$ matrix of massless scalar wave equations. That $-\partial_i\partial_j G_d^{(\mathrm{E})}[|\vec{x}-\vec{x}'|]$ is non-zero everywhere in space at $T=0$ tells us, for a fixed pair of indices ij, the (retarded) signal it generates likely fills all of spacetime to the future of η'. This is to be contrasted against the massless scalar Green's function equation itself in equation (81); where, because the source at x' is point-like, the signal it generates propagates only on and/or within its future light cone. If the observer at x lies outside the light cone of x', the signal $G_d^{+}[x-x']$ will be zero and causality respected. Returning to equation (84), if one continues to insist on viewing $G_{ij}^{+}[x-x']$ as the signal at x generated at x', since it is non-zero throughout all $\vec{x}$ whenever $\eta > \eta'$, once x lies outside the light cone of x' the observer at x would be led to conclude the signal is acausal.

Recursion relations In both Minkowski and spatially flat cosmologies, we are aided by the spatial-translation and spatial-parity invariance of the underlying spacetimes. In particular, these symmetries allow us to solve for $C_{1,d}$ and $C_{2,d}$ for all dimensions once we know their 3- and 4-dimensional solutions. This is because the higher-dimensional ones can be generated through the 'dimension-raising operator'

$$\mathcal{D}_R \equiv -\frac{1}{2\pi R}\frac{\partial}{\partial R}. \tag{86}$$

(See appendix (E) of [12] for a detailed discussion.) In brief, any bi-scalar function f_d that depends on space solely through $R \equiv |\vec{x}-\vec{x}'|$ and takes the same Fourier integral form

$$f_d\,[R] = \int \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} \widetilde{f}_d\left[|\vec{k}|\right] \mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{x}')} \tag{87}$$

for all relevant spacetime dimensions d, obeys the recursion relation

$$f_{d+2}[R] = -\frac{1}{2\pi R}\frac{\partial f_d[R]}{\partial R}. \tag{88}$$

This remark applies to both $C_{1,d}$ and $C_{2,d}$; specifically, we only need $C_{1,4}$ and $C_{2,4}$ to determine their counterparts in all even $d \geqslant 4$:

$$C_{1,\text{even } d\geqslant 4} = \mathcal{D}_R^{\frac{d-4}{2}} C_{1,4}, \tag{89}$$

$$C_{2,\text{even } d\geqslant 4} = \mathcal{D}_R^{\frac{d-4}{2}} C_{2,4}. \tag{90}$$

Likewise, we only need $C_{1,3}$ and $C_{2,3}$ to obtain their counterparts in all odd $d \geqslant 3$:

$$C_{1,\text{odd } d\geqslant 3} = \mathcal{D}_R^{\frac{d-3}{2}} C_{1,3}, \tag{91}$$

$$C_{2,\text{odd } d\geqslant 3} = \mathcal{D}_R^{\frac{d-3}{2}} C_{2,3}. \tag{92}$$

Also notice that, by counting the powers $|\vec{k}|$ in the integrals (74) and (75) as $|\vec{k}| \to 0$, $C_{1,d}$ is finite for all d, while $C_{2,d}$ is expected to diverge when $d \leqslant 3$. However, on physical grounds, the full effective Green's function G^{+}_{ij} should converge for all spacetime dimensions $d \geqslant 3$. This suggests that, for $d = 3$, the two spatial derivatives acting on $C_{2,3}$ in equation (73) will eliminate the divergence completely.

Time integral method According to equation (82), $-\Theta[T]C_{1,d}$ is the retarded Green's function G^{+}_{d} of the massless scalar. Because equation (82) will continue to hold even in cosmology and because the analytic position spacetime solutions to $C_{1,d}$ and G^{+}_{d} are known in all Minkowski and constant equation-of-state universes [12], we shall introduce a 'time-integral' method here that will allow us to solve the (retarded part of) $C_{2,d}$ in terms of time integrals of $C_{1,d}$. We first recall that equation (78) provides us a ordinary differential equation (ODE) relating $C_{2,d}$ to $C_{1,d}$. Integrating it twice with respect to time, and taking into account the initial conditions in equations (76) and (77),

$$C_{2,d}[T,R] = -\int_0^T \mathrm{d}\tau_2 \int_0^{\tau_2} \mathrm{d}\tau_1 C_{1,d}[\tau_1,R] + T\dot{C}_{2,d}[T=0,R] + C_{2,d}[T=0,R] \tag{93}$$

$$= -\int_0^T \mathrm{d}\tau_2 \int_0^{\tau_2} \mathrm{d}\tau_1 C_{1,d}[\tau_1,R] - TG^{(\mathrm{E})}_{d}[R]. \tag{94}$$

Now, any casual quantity $\mathfrak{Q}^{+}[\eta,\vec{x};\eta',\vec{x}']$—which we define as one that is non-zero only when $T \geqslant R \geqslant 0$—may be multiplied by $\Theta[T-R^{-}]$. While any anti-causal expression $\mathfrak{Q}^{-}[\eta,\vec{x};\eta',\vec{x}']$—which we define as one that is non-zero only when $T \leqslant -R \leqslant 0$—may be multiplied by $\Theta[-T-R^{-}]$[14]. If we then consider

$$\int_{\eta'}^{\eta'+\tau} \mathrm{d}\eta\, \mathfrak{Q}^{+}[\eta,\vec{x};\eta',\vec{x}'] = \int_{\eta'}^{\eta'+\tau} \mathrm{d}\eta\, \Theta[\eta-\eta'-R^{-}]\mathfrak{Q}^{+} = \int_0^{\tau} \mathrm{d}\tau'\Theta[\tau'-R^{-}]\mathfrak{Q}^{+}. \tag{95}$$

[14] The R^{-} guarantees that signals on the null cone proportional to $\delta[T-R]$ and its derivatives are included.

We see that, whenever $\tau > R^-$, we may set $\Theta[\tau' - R^-]$ to one for $\tau' \in (R^-, \tau)$; whereas whenever $\tau < R^-$, the latter is to be set to zero.

$$\int_0^\tau \mathrm{d}\tau' \Theta[\tau' - R^-] \mathfrak{Q}^+ = \Theta[\tau - R^-] \int_{R^-}^\tau \mathrm{d}\tau' \mathfrak{Q}^+ \bigg|_{\tau'=\eta-\eta'}. \tag{96}$$

Iterating this reasoning, we may deduce that one or multiple nested integrals of a causal quantity would return another causal one:

$$\int_0^T \mathrm{d}\tau_n \cdots \int_0^{\tau_3} \mathrm{d}\tau_2 \int_0^{\tau_2} \mathrm{d}\tau_1 \Theta[\tau_1 - R^-] \mathfrak{Q}^+ = \Theta[T - R^-] \int_{R^-}^T \mathrm{d}\tau_n \cdots \int_{R^-}^{\tau_3} \mathrm{d}\tau_2 \int_{R^-}^{\tau_2} \mathrm{d}\tau_1 \mathfrak{Q}^+ \bigg|_{\tau_1=\eta-\eta'}. \tag{97}$$

Similarly, if we instead consider

$$\int_{\eta'+\tau}^{\eta'} \mathrm{d}\eta\, \mathfrak{Q}^-[\eta, \vec{x}; \eta', \vec{x}'] = \int_{\eta'+\tau}^{\eta'} \mathrm{d}\eta\, \Theta[-(\eta - \eta') - R^-] \mathfrak{Q}^- = \int_\tau^0 \mathrm{d}\tau' \Theta[-\tau' - R^-] \mathfrak{Q}^-. \tag{98}$$

We see that, whenever $\tau < -R^-$, we may set $\Theta[-\tau' - R^-]$ to one for $\tau' \in (\tau, -R^-)$; whereas whenever $\tau \geqslant -R^-$, the latter is to be set to zero.

$$\int_0^\tau \mathrm{d}\tau' \Theta[-\tau' - R^-] \mathfrak{Q}^- = \Theta[-\tau - R^-] \int_{-R^-}^\tau \mathrm{d}\tau' \mathfrak{Q}^- \bigg|_{\tau'=\eta-\eta'}. \tag{99}$$

Iterating this reasoning, we may deduce that one or multiple nested integrals of an anti-causal quantity would return another anti-causal one:

$$\int_0^T \mathrm{d}\tau_n \cdots \int_0^{\tau_3} \mathrm{d}\tau_2 \int_0^{\tau_2} \mathrm{d}\tau_1 \Theta[-\tau_1 - R^-] \mathfrak{Q}^- = \Theta[-T - R^-] \int_{-R^-}^T \mathrm{d}\tau_n \cdots \int_{-R^-}^{\tau_3} \mathrm{d}\tau_2 \int_{-R^-}^{\tau_2} \mathrm{d}\tau_1 \mathfrak{Q}^- \bigg|_{\tau_1=\eta-\eta'}. \tag{100}$$

This discussion implies the integral of the difference between the causal Green's function and its anti-causal counterpart—recall equation (82)—namely $-C_{1,d}$, returns a causal minus anti-causal object:

$$\begin{aligned} &\int_0^T \mathrm{d}\tau' \left(G_d^+[\tau', R] - G_d^-[\tau', R]\right) \\ &\qquad = \Theta[T - R^-] \int_{R^-}^T \mathrm{d}\tau'\, G_d^+[\tau', R] - \Theta[-T - R^-] \int_{-R^-}^T \mathrm{d}\tau'\, G_d^-[\tau', R]. \end{aligned} \tag{101}$$

Furthermore, referring to equations (82) and (94), we see that the retarded portion of $C_{2,d}$—which is what we need—is gotten by integrating the retarded portion of $C_{1,d}$:

$$C_{2,d}^+[T, R] = \Theta[T - R^-] \int_{R^-}^T \mathrm{d}\tau_2 \int_{R^-}^{\tau_2} \mathrm{d}\tau_1\, G_d^+[\tau_1, R] - \Theta[T] \cdot T G_d^{(\mathrm{E})}[R], \tag{102}$$

where $C_{2,d}^+ \equiv \Theta[T] C_{2,d}$. Observe that the first term on the right hand side is strictly causal, whereas the second term arising from the initial condition is retarded but acausal because it contributes a non-zero signal outside the past null cone. As additional Minkowski and cosmological examples below will further corroborate, the 'time-integral' method not only allows us to compute (up to quadrature) the retarded part of $C_{2,d}$ from the known solutions of the massless scalar causal Green's functions, it provides a clean separation between the strictly causal versus the retarded-but-acausal terms arising from the initial conditions— even if the time-integrals themselves cannot be performed analytically.

Exact solutions in even dimensions $d \geqslant 4$ The retarded and advanced Green's functions of a massless scalar in 4D Minkowski are

$$G_4^{\pm}[x - x'] = -\frac{\delta[T \mp R]}{4\pi R}. \tag{103}$$

The δ-function teaches us that G_4^+ propagates signals strictly on the forward null cone; and the G_4^- strictly on the backward null cone. From equation (82), $C_{1,4}$ thus reads:

$$C_{1,4}[T, R] = \frac{1}{4\pi R}\left(\delta[T - R] - \delta[T + R]\right). \tag{104}$$

Of course, $C_{1,4}$ can be worked out straightforwardly from equation (74) by setting $d = 4$.

To compute $C_{2,4}$, on the other hand, we insert equation (104) into equation (94) and obtain

$$C_{2,4}[T, R] = \frac{1}{4\pi}\left(\Theta[T - R] - \Theta[-T - R]\right) + \frac{T}{4\pi R}\left(\Theta[T]\Theta[-T + R] + \Theta[-T]\Theta[T + R]\right). \tag{105}$$

We may check this result by tackling equation (75) directly. After integrating over the angular coordinates in $\vec{k}$-space,

$$C_{2,4}[T, R] = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{(2\pi)^2} \frac{\sin kT}{k} \frac{\sin kR}{kR}. \tag{106}$$

The sines can be converted into exponentials; and because there are no singularities the contour on the real line may be displaced slightly toward the positive or negative imaginary k-axis near $k = 0$. The resulting expression would consist of 4 terms, each of which would now be amendable to the residue theorem by closing the contour appropriately in the lower or upper half complex k plane.

From the retarded portion of equation (105), we find that the contribution of $C_{2,4}$ comes from both inside and outside the past light cone of the observer; however, the signal that resides within the light cone—its 'tail'—is a spacetime constant and will therefore be removed by the spatial derivatives in equation (73). In contrast, the acausal one with $T < R$ still remains and does contribute to the 4D effective Green's function G_{ij}^+, along with some additional light-cone contributions from differentiating the step functions in $C_{2,4}$.

$$G_{ij}^{(+,4\mathrm{D})}[T, \vec{R}] = -\left(\delta_{ij} - \widehat{R}_i \widehat{R}_j\right) \frac{\delta[T - R]}{4\pi R} - \Theta[T]\Theta[-T + R]\partial_i\partial_j\left(\frac{T}{4\pi R}\right). \tag{107}$$

To sum: the 4D effective Green's function $G_{ij}^{(+,4\mathrm{D})}[x - x']$ propagates signals on and outside the forward light cone of the source at x'—namely, it is acausal.

With the 4D solutions in equations (104) and (105) at hand, we may employ eqs. (89) and (90) to state:

$$C^+_{1,\mathrm{even}\ d\geqslant 4}[T, R] = \mathcal{D}_R^{\frac{d-4}{2}}\left(\frac{\delta[T - R]}{4\pi R}\right), \tag{108}$$

where $C^+_{1,d} \equiv \Theta[T]C_{1,d}$, and

$$C^+_{2,\mathrm{even}\ d\geqslant 4}[T, R] = \mathcal{D}_R^{\frac{d-4}{2}}\left(\Theta[T - R]\frac{1}{4\pi} + \Theta[T]\Theta[-T + R]\frac{T}{4\pi R}\right); \tag{109}$$

where only the retarded contributions are shown, and we highlight that the tail portion of equation (105) in higher dimensions $d \geqslant 6$ partially cancels the acausal part of it upon differentiation. Like the 4D case, the $T\mathcal{D}_R^{\frac{d-4}{2}}(4\pi R)^{-1}$ part of $C^{+}_{2,\text{even }d\geqslant 4}$ indicates the latter continues to receive acausal contribution from outside the light cone.

Exact solutions in odd dimensions $d \geqslant 3$ Odd-dimensional solutions differ from even ones due to the presence of inside-the-null-cone propagation—'tail' signals. In $(2+1)$-dimensions, the retarded Green's function of the massless scalar is

$$C^{+}_{1,3}[T,R] = -G^{+}_{3}[T,R] = \frac{\Theta[T-R]}{2\pi\sqrt{T^2-R^2}}, \tag{110}$$

which, unlike the 4D case, is pure tail. We now turn to solving the integral $C_{2,3}$, which, as we have reasoned earlier, is expected to blow up when considered alone, but its divergent piece does not really enter the physical spin-1 Green's function G^{+}_{ij}, as it will be eliminated by the two spatial derivatives $\partial_i\partial_j$ in equation (73). Despite being divergent, $C_{2,3}$ can nonetheless be regularized to a finite expression in the time-integral method, where the divergence only takes place on the initial condition. Within dimensional regularization, the resulting regularized form of it, $C^{(+,\text{reg})}_{2,3+2\epsilon}$, is given by

$$\begin{aligned} C^{(+,\text{reg})}_{2,3+2\epsilon}[T,R] = \Theta[T-R]\,&\frac{-T\ln\left[\mu\left(T+\sqrt{T^2-R^2}\right)\right]+\sqrt{T^2-R^2}}{2\pi} \\ &+\Theta[T]\Theta[-T+R]\,\frac{-T\ln[\mu R]}{2\pi}+\Theta[T]\,\frac{T}{4\pi}\left(\frac{1}{\epsilon}-\gamma-\ln[\pi]\right). \end{aligned} \tag{111}$$

By referring to equation (73), we see that both tail and the acausal parts of $C^{(\text{reg})}_{2,3+2\epsilon}$ contribute to the three-dimensional G^{+}_{ij}, with no pure light-cone signals involved.

To further justify the validity of equation (111), we independently computed finite $C_{2,5}$ using its Fourier representation in equation (75):

$$C^{+}_{2,5}[T,R] = \Theta[T-R]\,\frac{T-\sqrt{T^2-R^2}}{4\pi^2R^2}+\Theta[T]\Theta[-T+R]\,\frac{T}{4\pi^2R^2}. \tag{112}$$

This then allows us to verify $C^{+}_{2,5} = \mathcal{D}_R C^{(+,\text{reg})}_{2,3+2\epsilon}$ in equation (92). Higher odd-dimensional results follow from equations (91), (92), (110), and (111), where we can simply drop the last term of equation (111) and set the mass scale μ to one, since they will be removed in G^{+}_{ij} by the spatial derivatives in equation (73),

$$C^{+}_{1,\text{odd }d\geqslant 3}[T,R] = \mathcal{D}_R^{\frac{d-3}{2}}\left(\frac{\Theta[T-R]}{2\pi\sqrt{T^2-R^2}}\right), \tag{113}$$

$$C^{+}_{2,\text{odd }d\geqslant 3}[T,R] = \mathcal{D}_R^{\frac{d-3}{2}}\left(\Theta[T-R]\,\frac{-T\ln\left[\left(T+\sqrt{T^2-R^2}\right)\right]+\sqrt{T^2-R^2}}{2\pi}+\Theta[T]\Theta[-T+R]\,\frac{-T\ln R}{2\pi}\right). \tag{114}$$

With equations (113) and (114) plugged into equation (73), we now have the explicit spin-1 Green's function G^{+}_{ij} for all odd dimensions $d \geqslant 3$. These analytic solutions reveal that, in odd dimensions, the spin-1 photon receives not only the causal tail signals from both $C_{1,d}$ and $C_{2,d}$, with no strictly δ-function light-cone counterpart, but also the acausal contribution from $C_{2,d}[0 \leqslant T < R]$. As a result, we have explicitly shown that, in the presence of the local electromagnetic source, the spin-1 photon being acausal turns out to be a generic feature in any spacetime dimensions $d \geqslant 3$.

Scalar The scalar solution for the gauge-invariant Φ, which obeys the Poisson's equation (69), is given by a Coulomb-type form,

$$\Phi[\eta,\vec{x}] = \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}'\, G_d^{(\mathrm{E})}[R]\, J_0[\eta,\vec{x}'], \tag{115}$$

where we recall that $G_d^{(\mathrm{E})}$ takes the form of equation (58) for $d=3$ and equation (59) for $d \geqslant 4$. Clearly, this solution manifestly violates causality, in the sense that the scalar Φ is instantaneously sourced by the local charge density ρ. Therefore, neither the spin-1 photon α_i nor the scalar Φ can be a standalone observable in classical electromagnetism, which then leads us to pose the question: how do these gauge-invariant variables enter the key observable—the Faraday tensor—such that the result is causally dependent on their corresponding sources?

Faraday tensor The causal nature of $F_{\mu\nu}$ can be seen from its own wave equation, derived by taking the divergence of the identity $\partial_{[\mu}F_{\nu\sigma]} = 0$ and imposing the Maxwell eqautions (29),

$$\Box F_{\mu\nu} + R_{\rho\sigma\mu\nu}F^{\rho\sigma} + 2R^{\sigma}{}_{[\mu}F_{\nu]\sigma} = -2\nabla_{[\mu}J_{\nu]}, \qquad \Box \equiv \nabla_\sigma\nabla^\sigma. \tag{116}$$

In Minkowski spacetime, the geometric tensors vanish and the electromagnetic fields encoded within $F_{\mu\nu}$ are thus given by the massless scalar Green's function convoluted against the first derivatives of the electromagnetic sources,

$$F_{\mu\nu}[x] = -2\int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x'\, \partial_{[\mu}G_d^{+} J_{\nu]}[x']. \tag{117}$$

Here, we have dropped the surface terms at infinity when integrating by parts, which can be justified by the causal structures of equations (108) and (113) as well as the fact that those at past infinity ($T \to \infty$) in odd dimensions are negligible (see equation (113)).

Let us now recover equation (117) within the gauge-invariant formalism. We first make use of the conservation law for the electromagnetic current, $\partial_j J_j = \dot{J}_0$, to re-write the spin-1 expression (72),

$$\alpha_i[x] = -\int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x' \left(G_d^{+} J_i[x'] - \Theta[T]\,\partial_i \dot{C}_{2,d} J_0[x']\right), \tag{118}$$

where the second term is now the convolution with the charge density J_0. The surface terms from integration by parts—namely, $\int \mathrm{d}\eta'\, \mathrm{d}^{d-2}\vec{x}'\, \Theta[T]\partial_i C_{2,d} J_j$ evaluated at spatial infinity and $\int \mathrm{d}^{d-1}\vec{x}'\, \Theta[T]\partial_i C_{2,d} J_0$ at past infinity—have been neglected, as the former falls off as $R \to \infty$ in both even and odd dimensions, and the electric current is assumed to be isolated; whereas the latter, when $T \to \infty$, has zero contribution in even dimensions and becomes negligible in odd dimensions (see equations (109) and (114)). The magnetic field, according to equation (35), is therefore consistent with equation (117):

$$F_{ij}[x] = 2\partial_{[i}\alpha_{j]} = -2\int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x'\, \partial_{[i}G_d^{+} J_{j]}[x']. \tag{119}$$

This calculation shows that, despite α_i being acausal, taking the curl of the spin-1 field ends up removing its acausal information encoded in the second term of equation (118).

According to equation (35), the electric field F_{0i} is the sum of $\dot{\alpha}_i$ and $\partial_i\Phi$. Employing $\dot{C}_{2,d}\big|_{T=0} = -G_d^{(\mathrm{E})}$ and $\ddot{C}_{2,d} = -C_{1,d}$ in equations (77) and (78), the time derivative of equation (118) is

$$\dot{\alpha}_i[x] = -\int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x' \left(\dot{G}_d^+ J_i[x'] - \partial_i G_d^+ J_0[x']\right) - \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}'\, \partial_i G_d^{(\mathrm{E})} J_0[\eta, \vec{x}']. \tag{120}$$

We see that the acausal term containing $G_d^{(\mathrm{E})}$ in equation (120) is canceled by adding to $\dot{\alpha}_i$ the spatial gradient of equation (115). The result is

$$F_{0i}[x] = \dot{\alpha}_i[x] + \partial_i \Phi[x] = -\int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x' \left(\dot{G}_d^+ J_i[x'] - \partial_i G_d^+ J_0[x']\right) \tag{121}$$

which, again, agrees exactly with equation (117). To sum: the spin-1 photon α_i contains all relevant electromagnetic information, but is acausal. On the other hand, the primary role of Φ is to cancel the acausal part of $\dot{\alpha}_i$, rendering the electric field F_{0i} strictly causal. In other words, the electric field turns out to be determined by the causal portion of the velocity of the transverse spin-1 field,

$$F_{0i} = \left(\dot{\alpha}_i\right)_{\text{causal}} = \left(\dot{A}_i^{\mathrm{T}}\right)_{\text{causal}}. \tag{122}$$

Next, we move on to investigate how the spin-1 field and the Faraday tensor behave under certain physically interesting limits.

Stationary limit and Φ That the electric field in equation (122) is the causal piece of $\dot{\alpha}_i$ reveals a subtlety in the stationary limit, where the electric current is time independent. For, the first term containing $\dot{G}_d^+$ in equation (120) integrates to zero, which then informs us that

$$\dot{\alpha}_i[\vec{x}] = \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}' \partial_i G_d^{(\mathrm{E})}[\vec{x}-\vec{x}'] J_0[\vec{x}'] - \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}'\, \partial_i G_d^{(\mathrm{E})}[\vec{x}-\vec{x}'] J_0[\vec{x}'] = 0; \tag{123}$$

because in the second term of equation (120),

$$\int_{\mathbb{R}} \mathrm{d}\eta' G_d^+[\eta-\eta', \vec{x}-\vec{x}'] = G_d^{(\mathrm{E})}[\vec{x}-\vec{x}']. \tag{124}$$

In words: within the stationary limit, the causal structure of $\dot{\alpha}_i$ itself becomes degenerate—the otherwise causal and acausal terms in equation (120) cancel one another.

At first sight, equation (69) appears to tell us Φ is the Coulomb potential of a static charge distribution. This seems to be further reinforced by the fact that $\partial_i \Phi$ from equation (115) is the sole contribution to the electric field F_{0i} in equation (121), since $\dot{\alpha}_i = 0$. But the interpretation that $\partial_i \Phi$ is (the dominant piece of) the electric force becomes erroneous once there is the mildest non-trivial time dependence in the electric current—as already pointed out—because Φ is purely acausal and hence cannot be a standalone physical observable. Instead, the gradient of equation (115) cancels the (normally acausal) second term in the first equality of equation (123) and thus equation (122) continues to hold:

$$F_{0i}[\vec{x}] = \left(\dot{\alpha}_i\right)_{\text{causal}} = \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}' \partial_i G_d^{(\mathrm{E})}[\vec{x}-\vec{x}'] J_0[\vec{x}']. \tag{125}$$

Far-zone limit Provided that the observer is very far from the isolated sources, the leading-order term of the field, which scales as $1/r^{\frac{d}{2}-1}$, corresponds to the radiative piece that is capable of carrying energy-momentum to infinity. To extract the leading contribution of the spin-1 field in such a regime, firstly, we re-express the spin-1 commutator C_{ij}, by explicitly carrying out the spatial derivatives assuming $R \neq 0$, in the following form[15]

$$C_{ij}[T,\vec{R}] = P_{ij}[\vec{R}]\, C_{1,d}[T,R] + \Pi_{ij}[\vec{R}]\, 2\pi C_{2,d+2}[T,R], \tag{126}$$

where the projection tensor P_{ij} and Π_{ij}, respectively, are defined as

$$P_{ij}[\vec{R}] \equiv \delta_{ij} - \widehat{R}_i\widehat{R}_j, \tag{127}$$

$$\Pi_{ij}[\vec{R}] \equiv -\delta_{ij} + (d-1)\widehat{R}_i\widehat{R}_j; \tag{128}$$

the $\widehat{R} \equiv (\vec{x}-\vec{x}')/|\vec{x}-\vec{x}'|$ is pointing from the source location $\vec{x}'$ to the observer $\vec{x}$; and $C_{2,d+2}$ in the second term of equations (126) is the $(d+2)$-dimensional form of equations (90) and (92) but its R is the one in $d-1$ spatial dimensions. Note also that, to reach equation (126), we have employed the homogeneous wave equation, $\partial^2 C_{2,d} = 0$, with $R \neq 0$, as well as the conversion $\ddot{C}_{2,d} = -C_{1,d}$, to relate its second spatial derivatives to $C_{1,d}$. Also, it can be checked directly that the expression (126) for $R \neq 0$ is indeed divergenceless. Altogether, as long as the observer is away from the source, we have an alternative expression for the spin-1 effective Green's function by inserting equation (126) into $G^+_{ij} = -\Theta[T]C_{ij}$. The purpose of putting C_{ij} in this form is that, the dominant far-zone contribution of the field can be extracted simply by comparing $C_{1,d}$ and $C_{2,d+2}$. Furthermore, each term in equation (126) is manifestly finite for all spacetime dimensions in which photons exist, since there is no divergence incurred in $C_{1,d}$ and $C_{2,d+2}$ for $d \geqslant 3$.

If τ_c and r_c are respectively the characteristic time scale and proper size of the source, and r is the observer-source distance, the far zone is defined as the limits $\tau_c/r \ll 1$ and $r_c/r \ll 1$. To perform this limit on the Green's function, we will work in frequency ω-space. Specifically, the far zone then translates into $|\omega| r \gg 1$. We shall be content in extracting the leading expressions in the limit $|\omega| R \gg 1$.

In terms of the superposition of individual frequencies, the spin-1 field α_i can be written as

$$\alpha_i[\eta,\vec{x}] = -\int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}' \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\pi}\, \widetilde{G}^+_{ij}[\omega,\vec{R}]\mathrm{e}^{-\mathrm{i}\omega\eta}\widetilde{J}_j[\omega,\vec{x}'], \tag{129}$$

with $\widetilde{G}^+_{ij}[\omega,R]$ being the frequency transform of the spin-1 effective Green's function,

$$\begin{aligned}\widetilde{G}^+_{ij}[\omega,\vec{R}] &= \int_{\mathbb{R}} \mathrm{d}T\, G^+_{ij}[T,\vec{R}]\, \mathrm{e}^{\mathrm{i}\omega T} \\ &= P_{ij}[\vec{R}]\, \widetilde{G}^+_d[\omega,R] - \Pi_{ij}[\vec{R}]\, 2\pi \widetilde{C}^+_{2,d+2}[\omega,R],\end{aligned} \tag{130}$$

where we have assumed $R \neq 0$ and used the expression (126). The $\widetilde{G}^+_d[\omega,R]$ and $\widetilde{C}^+_{2,d+2}[\omega,R]$, respectively, denote the frequency transforms of the massless scalar Green's function G^+_d and $C^+_{2,d+2}$[16].

Spin-1 photons in even dimensions $d \geqslant 4$ A direct calculation starting from equations (108) and (109) tells us, in all even $d = 4 + 2n \geqslant 4$,

[15] At $R = 0$, when two spatial derivatives $\partial_i\partial_j$ act on $1/R^{d-3}$, terms involving $\delta^{(d-1)}[\vec{x}-\vec{x}']$ could arise. However, if the observer is away from the source, then those local terms will not contribute to the effective Green's function, and therefore, we can simply ignore them in the calculation.

[16] The $\widetilde{G}^+_d[\omega,R]$ in equation (130) is the frequency transform of $G^+_d[\eta-\eta';R]$; this is to be distinguished from $\widetilde{G}^+_d[\eta,\eta';\vec{k}]$ in equation (70), which is the spatial-Fourier transform of the same $G^+_d[\eta-\eta';R]$.

$$\widetilde{G}^{+}_{4+2n}[\omega,R]=-\mathcal{D}^{n}_{R}\left(\frac{\mathrm{e}^{\mathrm{i}\omega R}}{4\pi R}\right)=-\frac{\mathrm{i}\omega^{2n+1}}{2(2\pi)^{n+1}(\omega R)^{n}}h^{(1)}_{n}[\omega R], \tag{131}$$

$$\widetilde{C}^{+}_{2,6+2n}[\omega,R]=\mathcal{D}^{n}_{R}\left(\frac{\mathrm{e}^{\mathrm{i}\omega R}}{8\pi^{2}R^{2}(\mathrm{i}\omega)}-\frac{\mathrm{e}^{\mathrm{i}\omega R}}{8\pi^{2}R^{3}(\mathrm{i}\omega)^{2}}\right)-\frac{(2n+1)!!\,\omega^{2n+1}}{2(2\pi)^{n+2}(\omega R)^{2n+3}}, \tag{132}$$

where $h^{(1)}_{n}$ is the spherical Hankel function of the first kind. Notice that the last term in equation (132) does not contain the factor $\mathrm{e}^{\mathrm{i}\omega R}$; it describes the non-propagating portion of the signal in frequency space, which in turn arises from the acausal effect found in position space. In the limit $|\omega|R\gg 1$, equations (131) and (132) behave asymptotically as

$$\widetilde{G}^{+}_{4+2n}[\omega,R]=\frac{(-1)^{n+1}\mathrm{i}^{n}\omega^{2n+1}}{2(2\pi\omega R)^{n+1}}\,\mathrm{e}^{\mathrm{i}\omega R}\left(1+\mathcal{O}\left[\frac{1}{\omega R}\right]\right), \tag{133}$$

$$\widetilde{C}^{+}_{2,6+2n}[\omega,R]=\frac{(-1)^{n+1}\mathrm{i}^{n}\omega^{2n+1}}{2(2\pi\omega R)^{n+1}}\,\mathrm{e}^{\mathrm{i}\omega R}\cdot\frac{\mathrm{i}}{2\pi\omega R}\left(1+\mathcal{O}\left[\frac{1}{\omega R}\right]\right), \tag{134}$$

which reveals that, for any fixed dimension $d=4+2n$ and at leading order, the acausal $\widetilde{C}^{+}_{2,6+2n}[\omega,R]$ term is suppressed as $1/(\omega R)$ relative to $\widetilde{G}^{+}_{4+2n}$[17]. Therefore, at leading $1/(\omega R)$ order, the effective Green's function $\widetilde{G}^{+}_{ij}[\omega,\vec{R}]$, in frequency space, is exclusively dependent on $\widetilde{G}^{+}_{d}[\omega,R]$; moreover, with the assumption $r_c/r\ll 1$, its far-zone leading contribution can be extracted from the first term of equation (130), which is given by

$$\widetilde{G}^{+}_{ij}[\omega,\vec{R}]=P_{ij}\widetilde{G}^{(+,\,\mathrm{fz})}_{4+2n}[\omega;\vec{x},\vec{x}']\left(1+\mathcal{O}\left[\frac{1}{\omega r},\frac{r_c}{r}\right]\right), \tag{135}$$

where P_{ij} is the far-zone spatial projector defined in equation (6) and

$$\widetilde{G}^{(+,\,\mathrm{fz})}_{4+2n}[\omega;\vec{x},\vec{x}']\equiv\frac{(-1)^{n+1}(\mathrm{i}\omega)^{n}}{2(2\pi)^{n+1}}\frac{\mathrm{e}^{\mathrm{i}\omega(r-\vec{x}'\cdot\widehat{r})}}{r^{n+1}}. \tag{136}$$

By performing the inverse frequency transform, in the far-zone radiative limit, the transverse spin-1 photon $\alpha_i=A^{\mathrm{T}}_{i}$ reduces to a transverse projection in space:

$$\lim_{r\to\infty}\alpha_i\to A^{\mathrm{t}}_{i},\qquad A^{\mathrm{t}}_{i}[x]\equiv P_{ij}\left(-\int_{\mathbb{R}^{d-1,1}}\mathrm{d}^{d}x'\,G^{(+,\,\mathrm{fz})}_{d}[T;\vec{x},\vec{x}']J_{j}[x']\right), \tag{137}$$

where $G^{(+,\,\mathrm{fz})}_{d}[T;\vec{x},\vec{x}']$ is the far-zone contribution of the massless scalar Green's function,

$$\begin{aligned}G^{(+,\,\mathrm{fz})}_{d}[T;\vec{x},\vec{x}']&=\int_{\mathbb{R}}\frac{\mathrm{d}\omega}{2\pi}\widetilde{G}^{(+,\,\mathrm{fz})}_{d}[\omega;\vec{x},\vec{x}']\mathrm{e}^{-\mathrm{i}\omega T}\\&=-\frac{1}{2(2\pi r)^{\frac{d-2}{2}}}\left(\frac{\partial}{\partial\eta}\right)^{\frac{d-4}{2}}\delta\left[T-r+\vec{x}'\cdot\widehat{r}\right].\quad(\text{even }d)\end{aligned} \tag{138}$$

Spin-1 photons in odd dimensions $d\geqslant 3$ For $d=3+2n$, we can frequency transform the retarded position-space solutions (113) and (114) to obtain

$$\widetilde{G}^{+}_{3+2n}[\omega>0,R]=-\frac{\mathrm{i}\omega^{2n}}{4(2\pi\omega R)^{n}}H^{(1)}_{n}[\omega R], \tag{139}$$

[17] Strictly speaking, equation (133) applies only for $n>0$. There are no $1/(\omega R)$ corrections in $(3+1)$-dimensions, because equation (131) informs us that $\widetilde{G}^{+}_{4}[\omega,R]=-\mathrm{e}^{\mathrm{i}\omega R}/(4\pi R)$.

$$\widetilde{C}^{+}_{2,5+2n}[\omega > 0, R] = \frac{\mathrm{i}\omega^{2n}}{4(2\pi\omega R)^{n+1}} H^{(1)}_{n+1}[\omega R] - \frac{2^n n!\, \omega^{2n}}{(2\pi)^{n+2}(\omega R)^{2n+2}}, \tag{140}$$

where $H^{(1)}_n$ is the Hankel function of the first kind and its differential recursion relation has been employed, and these expressions are only valid for positive frequencies $\omega > 0$; however, since G^+_d and $C^+_{2,d+2}$ are real, the negative-frequency modes can be expressed in terms of the complex conjugates of equations (139) and (140), $\widetilde{G}^+_d[-\omega, R] = \widetilde{G}^{+*}_d[\omega, R]$ and $\widetilde{C}^+_{2,d+2}[-\omega, R] = \widetilde{C}^{+*}_{2,d+2}[\omega, R]$, where the asterisk '$*$' denotes complex conjugation. As in the even-dimensional case, the non-propagating piece of signals also shows up in the second term of equation (140). At leading $1/(\omega R)$ order, the Hankel function goes asymptotically to $H^{(1)}_n[\omega R] = \sqrt{2/\pi\omega R}\, \mathrm{e}^{\mathrm{i}(\omega R - n\pi/2 - \pi/4)} + \mathcal{O}\left[1/(\omega R)^{3/2}\right]$, so we can read off the leading-order pieces of equations (139) and (140) accordingly,

$$\widetilde{G}^{+}_{3+2n}[\omega > 0, R] = -\frac{\mathrm{i}\omega^{2n}}{2(2\pi\omega R)^{n+\frac{1}{2}}} \mathrm{e}^{\mathrm{i}\left(\omega R - \frac{n\pi}{2} - \frac{\pi}{4}\right)} \left(1 + \mathcal{O}\left[\frac{1}{\omega R}\right]\right), \tag{141}$$

$$\widetilde{C}^{+}_{2,5+2n}[\omega > 0, R] = -\frac{\mathrm{i}\omega^{2n}}{2(2\pi\omega R)^{n+\frac{1}{2}}} \mathrm{e}^{\mathrm{i}\left(\omega R - \frac{n\pi}{2} - \frac{\pi}{4}\right)} \cdot \frac{\mathrm{i}}{2\pi\omega R} \left(1 + \mathcal{O}\left[\frac{1}{\omega R}\right] + \mathcal{O}\left[\frac{1}{(\omega R)^{n+\frac{1}{2}}}\right]\right), \tag{142}$$

from which we infer that, unlike the even-dimensional results, amplitudes of these tail signals contain fractional powers of frequencies. And, these asymptotic behaviors tell us that, in the far-zone regime $|\omega| R \gg 1$, the massless scalar Green's function $\widetilde{G}^+_d[\omega, R]$ still dominates over $\widetilde{C}^+_{2,d+2}[\omega, R]$ here. Hence, we can extract the far-zone leading order in $1/r$ piece of $\widetilde{G}^+_{ij}[\omega, \vec{R}]$ in the same way,

$$\widetilde{G}^{+}_{ij}[\omega > 0, \vec{R}] = P_{ij} \widetilde{G}^{(+,\mathrm{fz})}_{3+2n}[\omega; \vec{x}, \vec{x}'] \left(1 + \mathcal{O}\left[\frac{1}{\omega r}, \frac{r_c}{r}\right]\right), \tag{143}$$

with $\widetilde{G}^{(+,\mathrm{fz})}_{3+2n}$ defined by

$$\widetilde{G}^{(+,\mathrm{fz})}_{3+2n}[\omega; \vec{x}, \vec{x}'] \equiv -\frac{\mathrm{i}\omega^{n-\frac{1}{2}}}{2(2\pi)^{n+\frac{1}{2}}} \frac{\mathrm{e}^{\mathrm{i}\left(\omega(r - \vec{x}'\cdot\widehat{r}) - \frac{n\pi}{2} - \frac{\pi}{4}\right)}}{r^{n+\frac{1}{2}}}. \tag{144}$$

Consequently, in the radiative limit $r \to \infty$, we reach the same conclusion stated in equation (137) for odd dimensions $d = 3 + 2n$ as well, where $G^{(+,\mathrm{fz})}_d$ is given instead by

$$G^{(+,\mathrm{fz})}_d[T; \vec{x}, \vec{x}'] = -\frac{1}{(2\pi r)^{\frac{d-2}{2}}} \mathrm{Re}\left[\int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \mathrm{i}\omega^{\frac{d-4}{2}} \mathrm{e}^{\mathrm{i}\left(\omega(r - \vec{x}'\cdot\widehat{r} - T) - \frac{(d-2)\pi}{4}\right)}\right]. \quad (\text{odd } d). \tag{145}$$

Thus, based on the frequency-space analysis, the fact that $\alpha_i \equiv A^{\mathrm{T}}_i \to A^{\mathrm{t}}_i$ as $r \to \infty$ holds generically in any spacetime dimensions $d \geqslant 3$, clearly demonstrating that the acausal portion of the spin-1 field actually contributes negligibly to the far-zone signals.

Summary: far zone transverse Green's functions To sum, the massless spin-1 transverse photon α_i in the radiative regime will coincide with another notion of the 'transverse' vector potential A^{t}_i. While α_i in equation (56) involves a transverse projection in Fourier space, the A^{t}_i in equation (137) is a local-in-space transverse projection of the far-zone Lorenz-gauge causal solution for the vector potential, i.e. $\widehat{r}^i A^{\mathrm{t}}_i = 0$, which consists solely of the light-cone signals in even dimensions. In 4D Minkowski spacetime, that the two different notions of the transverse vector potential overlap in the far zone has already been pointed out in [2, 3]. The method used here allows us to generalize the conclusion to all dimensions.

Faraday tensor Since we have already shown the spin-1 photons α_i, for all $d \geqslant 3$, reduce asymptotically to causal A_i^{t} in the far zone, we then expect in this regime the magnetic and electric fields, equations (119) and (122), to become

$$F_{ij} \approx -2\,\widehat{r}_{[i}\dot{A}^{\mathrm{t}}_{j]}, \tag{146}$$

$$F_{0i} \approx \dot{A}^{\mathrm{t}}_i. \tag{147}$$

Here, the far-zone limit has been taken on both sides of the equations, and we have also used the results in eqs. (133), (134), (141) and (142) to deduce—as far as the leading-order contribution is concerned—the replacement rule $\partial_i = -\partial_{i'} = -\widehat{R}_i\partial_0$ holds at the leading $1/(\omega R)$ level, and after which the dominant far-zone contribution in terms of r can be extracted readily. In addition, the far-zone expressions (146) and (147) can also be checked for consistency through equation (117), by using the replacement rule as well as the conservation law for the charge current.

Commutator of spin-1 photons As already alluded to, the results for the retarded Green's function of the massless spin-1 α_i are intimately related to the commutator of these photon operators in Quantum Field Theory. Let us first consider a free scalar field ϕ as a simple example. Its commutator is related to $C_{1,d}$ in the following manner:

$$\left[\phi[x], \phi[x']\right] = -\mathrm{i}C_{1,d}[T, R]. \tag{148}$$

According to equation (82), since the retarded/advanced Green's functions on the right hand side are strictly zero outside the null cone, this $C_{1,d}$ consists of only causal information—i.e. it too is zero whenever the two spacetime points are spacelike: $(x-x')^2 > 0$. In contrast, because the spin-1 Green's functions are non-zero outside the light cone, according to equation (83), the non-interacting spin-1 commutator is therefore acausal:

$$\left[\alpha_i[x], \alpha_j[x']\right] = -\mathrm{i}C_{ij}[T, \vec{R}]. \tag{149}$$

In quantum field theory, operators that commute outside the light cone are said to obey micro-causality. Free spin-1 photons are therefore seen to violate micro-causality. It is likely that this acausal character of their commutator is a manifestation of the known tension between Lorentz covariance and gauge invariance when constructing massless helicity-1 theories in flat spacetime.

4.2. Linearized gravitation

We now turn to the linearized theory of General Relativity in a Minkowski background, as described in section 2. The relevant Green's functions will be computed analytically for all spacetime dimensions $d \geqslant 4$; i.e. excluding those without spin-2 degrees of freedom.

Field equations The gauge-invariant form of the linearized Einstein's equations can be expressed in terms of the variables defined in equations (25)–(27), where, as a constrained system, the full set of gauge-invariant field equations can be reduced to four fundamental ones, i.e. equation (28) and the following three [12]:

$$(d-2)\vec{\nabla}^2\Psi = 8\pi G_{\mathrm{N}}\rho, \tag{150}$$

$$\vec{\nabla}^2 V_i = -16\pi G_{\mathrm{N}}\Sigma_i, \tag{151}$$

$$\partial^2 D_{ij} = -16\pi G_{\mathrm{N}}\sigma_{ij}; \tag{152}$$

where the source terms ρ, Σ_i, and σ_{ij} refer to different parts of the scalar–vector–tensor decomposition of the astrophysical stress-energy tensor ${}^{(\mathrm{a})}T_{\mu\nu}$ (see equations (21) and (22)). These four independent equations, along with the law of energy-momentum conservation, $\partial^{\mu}\,{}^{(\mathrm{a})}T_{\mu\nu} = 0$, already imply the other three remaining ones in the linearized Einstein's equations. We see that only the spin-2 graviton field, $D_{ij} \equiv \chi_{ij}^{\mathrm{TT}}$, admits dynamical wave solutions, sourced by the TT portion of ${}^{(\mathrm{a})}T_{\mu\nu}$, while the Bardeen scalar potential Ψ, as well as the vector mode V_i, obey Poisson-type ones. This set of equations appear to be similar to their electromagnetic counterparts, and thus, by the same arguments used earlier, we already expect these gauge-invariant variables to be acausal in nature once the GW sources are taken into account. In this sense, none of these gauge-invariant variables—including the spin-2 D_{ij}—may be regarded as a standalone observable. Indeed, as we will see in the subsequent discussion, the linearized Riemann tensor $\delta_1 R_{0i0j}$, discussed in section 2, in close analogy to the field strength $F_{\mu\nu}$ for electromagnetism, does require all their contributions to become a causal object.

Spin-2 gravitons The analytic solutions for the effective Green's functions are crucially important for capturing the propagation of wave signals in this linearized system. Here, we start with the massless spin-2 field D_{ij}, obeying the wave equation (152). Since the TT projection of the source takes place locally in Fourier space, as long as $\vec{k} \neq \vec{0}$, we can firstly express D_{ij} as

$$D_{ij}[\eta,\vec{x}] = -16\pi G_{\mathrm{N}}\int_{\mathbb{R}} \mathrm{d}\eta' \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}}\, \widetilde{G}_d^{+}[\eta,\eta';\vec{k}]\, \widehat{P}_{ijmn}[\vec{k}]\, {}^{(\mathrm{a})}\widetilde{T}_{mn}[\eta',\vec{k}]\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}, \tag{153}$$

where $\widetilde{G}_d^{+}[\eta,\eta';\vec{k}]$ is given in equation (71), and the spin-2 TT projector $\widehat{P}_{ijmn}[\vec{k}]$ is defined in equation (12). Then, the expression (153), with each k_j in equation (65) replaced by a spatial derivative via $\partial_j \to ik_j$, can be re-written as the spin-2 effective Green's function G^{+}_{ijmn} convolved against the local stress-energy tensor of the source,

$$D_{ij}[x] = -16\pi G_{\mathrm{N}}\int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x'\, G^{+}_{ijmn}[T,\vec{R}]\, {}^{(\mathrm{a})}T_{mn}[x'], \tag{154}$$

where the spin-2 effective Green's function G^{+}_{ijmn} is given by the following tensor structure,

$$\begin{aligned} G^{+}_{ijmn}[T,\vec{R}] &= -\,\Theta[T]C_{ijmn}[T,\vec{R}], \\ C_{ijmn}[T,\vec{R}] &= \left(\delta_{m(i}\delta_{j)n} - \frac{\delta_{ij}\delta_{mn}}{d-2}\right)C_{1,d}[T,R] + \Bigg(\delta_{m(i}\partial_{j)}\partial_n + \delta_{n(i}\partial_{j)}\partial_m \\ &\quad - \frac{\delta_{ij}\partial_m\partial_n - \delta_{mn}\partial_i\partial_j}{d-2}\Bigg)C_{2,d}[T,R] + \left(\frac{d-3}{d-2}\right)\partial_i\partial_j\partial_m\partial_n C_{3,d}[T,R], \end{aligned} \tag{155}$$

with $C_{1,d}$ and $C_{2,d}$ defined previously in equations (74) and (75), and $C_{3,d}$ defined by

$$C_{3,d}[T,R] \equiv \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{\sin|\vec{k}|T}{|\vec{k}|^5}\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{R}}. \tag{156}$$

Compared with the spin-1 photon case, even though the whole tensor structure of C_{ijmn} here is very different than that of C_{ij} (see equation (73)), the first two terms have structural similarity to C_{ij}, and the scalar Fourier integrals $C_{1,d}$ and $C_{2,d}$ have already been dealt with analytically;

the only new term that remains to be computed is $C_{3,d}$. Moreover, it can be checked readily that, by employing the relations $\vec{\nabla}^2 C_{2,d} = -C_{1,d}$ and $\vec{\nabla}^2 C_{3,d} = -C_{2,d}$ (see equations (74), (75), and (156)), the expression (155) indeed satisfies the TT conditions $\delta^{ij} G^{+}_{ijmn} = 0$ and $\partial_i G^{+}_{ijmn} = \partial_j G^{+}_{ijmn} = 0$. We shall soon deploy the time-integral method, which amounts to solving

$$\ddot{C}_{3,d} = -C_{2,d}. \tag{157}$$

By integrating equation (157) twice, followed by recalling equation (94), we have

$$C_{3,d}[T,R] = -\int_0^T \mathrm{d}\tau_2 \int_0^{\tau_2} \mathrm{d}\tau_1\, C_{2,d}[\tau_1,R] + T\dot{C}_{3,d}[T=0,R] + C_{3,d}[T=0,R], \tag{158}$$

$$= \int_0^T \mathrm{d}\tau_4 \int_0^{\tau_4} \mathrm{d}\tau_3 \int_0^{\tau_3} \mathrm{d}\tau_2 \int_0^{\tau_2} \mathrm{d}\tau_1\, C_{1,d}[\tau_1,R] + \frac{T^3}{6} G^{(\mathrm{E})}_d[R] + T D_d[R], \tag{159}$$

where the initial conditions $C_{3,d}[T=0,R] = 0$ and $\dot{C}_{3,d}[T=0,R] = D_d[R]$ have been employed (see equation (156)), with D_d defined by

$$D_d[R] \equiv \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{e^{i\vec{k}\cdot\vec{R}}}{\vec{k}^4}, \tag{160}$$

whose concrete position-space expressions read

$$D^{(\mathrm{reg})}_{3+2\epsilon}[R] = -\frac{R^2}{16\pi}\left(\frac{1}{\epsilon} - \big(\gamma + \ln[\pi] - 1\big) - 2\ln[\mu R]\right), \tag{161}$$

$$D^{(\mathrm{reg})}_{4}[R] = -\frac{R}{8\pi}, \tag{162}$$

$$D^{(\mathrm{reg})}_{5+2\epsilon}[R] = \frac{1}{16\pi^2}\left(\frac{1}{\epsilon} - \gamma - \ln[\pi] - 2\ln[\mu R]\right), \tag{163}$$

$$D_{d\geqslant 6}[R] = \frac{\Gamma\left[\frac{d-5}{2}\right]}{16\pi^{\frac{d-1}{2}} R^{d-5}}. \tag{164}$$

Note that D_3, D_4, and D_5 are in the dimensional-regularized forms. Finally, parallel to equation (84) in the photon case, acausality encoded in G^{+}_{ijmn} can be seen at the level of its wave equation,

$$\begin{aligned}\partial^2 G^{+}_{ijmn}[T,\vec{R}] =& \delta[T]\Bigg\{\left(\delta_{m(i}\delta_{j)n} - \frac{\delta_{ij}\delta_{mn}}{d-2}\right)\delta^{(d-1)}[\vec{x}-\vec{x}'] - \Bigg(\delta_{m(i}\partial_{j)}\partial_n + \delta_{n(i}\partial_{j)}\partial_m \\ &- \frac{\delta_{ij}\partial_m\partial_n - \delta_{mn}\partial_i\partial_j}{d-2}\Bigg) G^{(\mathrm{E})}_d[R] + \left(\frac{d-3}{d-2}\right)\partial_i\partial_j\partial_m\partial_n D_d[R]\Bigg\}.\end{aligned} \tag{165}$$

The last two terms in equation (165) correspond to the acausal contributions to the signal attributed to $x' \equiv (\eta', \vec{x}')$, but arising from a non-zero source smeared over the rest of the equal-time spatial hypersurface at $\eta = \eta'$.

To compute $C_{3,d}$ in equation (156), we first notice that the integral itself diverges when $d \leqslant 5$, inferred from the power of $|\vec{k}|$ in the limit $|\vec{k}| \to 0$. However, in the physical spacetime dimensions where spin-2 gravitons exist, namely $d \geqslant 4$, those divergences in $C_{3,4}$ and $C_{3,5}$ are expected to be removed by the multiple spatial derivatives in the spin-2 effective Green's function (155). Furthermore, since the dimension-raising operator still applies to $C_{3,d}$, in either even or odd spacetime dimensions, the lowest-dimensional form is already adequate for us to generate all the higher-dimensional ones.

Exact solutions in even dimensions $d \geqslant 4$ Since the analytic forms of $C_{1,d}$ and $C_{2,d}$, for all even $d \geqslant 4$, have been obtained in equations (108) and (109), we now focus on the $C_{3,d}$ term needed in equation (155). Similar methods used to compute $C_{2,d}$ will be performed to tackle this new integral. Here, we start with $C_{3,4}$, the lowest even dimension for the spin-2 graviton to exist. Even though $C_{3,4}$ itself is a divergent integral, we can still extract its regularized finite contribution through the time-integral method, as we did for $C_{2,3}$ in the spin-1 calculation. Utilizing dimensional-regularization, the result turns out to be finite:

$$C_{3,4}^{(+,\mathrm{reg})}[T,R] = \Theta[T-R]\,\frac{-R^2-3T^2}{24\pi} + \Theta[T]\Theta[-T+R]\,\frac{-3R^2T-T^3}{24\pi R}. \tag{166}$$

Similar to $C_{2,4}$, the tail function in equation (166), when plugged into equation (155), will be eliminated by the spatial derivatives in G^{+}_{ijmn}. Whereas the acausal portion of the signals, from the second term of equation (166), will still remain. In addition, this regularized form can be justified by checking whether this expression, with one dimension-rasing operator acting on it, coincides with $C^{+}_{3,6}$ obtained by a direct contour-integral calculation:

$$C^{+}_{3,6}[T,R] = \Theta[T-R]\,\frac{1}{24\pi^2} + \Theta[T]\Theta[-T+R]\,\frac{3R^2T-T^3}{48\pi^2R^3}. \tag{167}$$

(One may check, indeed, that $C_{3,6} = \mathcal{D}_R C_{3,4}^{(+,\mathrm{reg})}$.) With this solution at hand, applying dimension-raising operators to it then produces all the higher even-dimensional results,

$$C^{+}_{3,\mathrm{even}\ d\geqslant 4}[T,R] = \mathcal{D}_R^{\frac{d-4}{2}}\left(\Theta[T-R]\,\frac{-R^2-3T^2}{24\pi} + \Theta[T]\Theta[-T+R]\,\frac{-3R^2T-T^3}{24\pi R}\right). \tag{168}$$

Now, plugging equation (168) for $C_{3,d}$, along with known $C_{1,d}$ and $C_{2,d}$, into the spin-2 effective Green's function (155), we find that the spin-2 causal structure is analogous to that of the spin-1 field. More precisely, for all even dimensions $d \geqslant 4$, no tail signals exist in G^{+}_{ijmn}, but the spin-2 graviton receives not only the light-cone signals but also the acausal ones from both $C_{2,d}$ and $C_{3,d}$.

Exact solutions in odd dimensions $d \geqslant 5$ For odd spacetime dimensions, we begin with $d = 5$, since TT gravitons are non-existent in lower odd dimensions. To calculate $C_{3,5}$, we can make use of the time-integral method to first extract the regularized $C_{3,3}$, namely inserting equation (110) into equation (159) along with the dimensional-regularized initial conditions:

$$\begin{aligned}C_{3,3+2\epsilon}^{(+,\mathrm{reg})}[T,R] =& \Theta[T-R]\,\frac{1}{8\pi}\left(-\left(T^2-R^2\right)^{\frac{3}{2}} - \frac{1}{9}\left(13R^2+2T^2\right)\sqrt{T^2-R^2} + T\left(R^2+\frac{2}{3}T^2\right)\right.\\ &\left.\times \ln\left[\mu\left(T+\sqrt{T^2-R^2}\right)\right]\right) + \Theta[T]\Theta[-T+R]\,\frac{T\ln[\mu R]}{8\pi}\left(R^2+\frac{2}{3}T^2\right)\\ &-\Theta[T]\frac{T}{16\pi}\left(\left(R^2+\frac{2}{3}T^2\right)\left(\frac{1}{\epsilon}-\gamma-\ln[\pi]\right)+R^2\right).\end{aligned} \tag{169}$$

By applying the raising operator $\mathcal{D}_R$ once,

$$C_{3,5+2\epsilon}^{(+,\mathrm{reg})}[T,R] = \Theta[T-R]\,\frac{1}{24\pi^2}\left(\left(2+\frac{T^2}{R^2}\right)\sqrt{T^2-R^2}-3T\ln\left[\mu\left(T+\sqrt{T^2-R^2}\right)\right]-\frac{T^3}{R^2}\right) + \Theta[T]\Theta[-T+R]\,\frac{1}{24\pi^2}\left(-\frac{T^3}{R^2}-3T\ln[\mu R]\right)+\Theta[T]\,\frac{T}{16\pi^2}\left(\frac{1}{\epsilon}-\gamma-\ln[\pi]\right), \tag{170}$$

which is still a regularized expression of the divergent integral $C_{3,5}$, and again, whose validity can be justified in the same way. Through a direct computation of the finite integral $C_{3,7}$, as we did for $C_{2,5}$, we can show that the resulting expression of $C_{3,7}$, namely

$$C_{3,7}^{+}[T,R] = \Theta[T-R]\,\frac{2\left(T^2-R^2\right)^{\frac{3}{2}}+3R^2T-2T^3}{48\pi^3R^4}+\Theta[T]\Theta[-T+R]\,\frac{3R^2T-2T^3}{48\pi^3R^4}, \tag{171}$$

is simply $\mathcal{D}_R C_{3,5+2\epsilon}^{(+,\mathrm{reg})}$. Following through the same procedures, we can extend the analytic solution (170) to all odd dimensions $d \geqslant 5$ via dimension-rasing operators, where, similar to the odd-dimensional photon case, we may discard the last term of equation (170) and set $\mu = 1$ as they will be eliminated in the physical G_{ijmn}^{+} (see equation (155)),

$$C_{3,\mathrm{odd}\ d\geqslant 5}^{+}[T,R] = \mathcal{D}_R^{\frac{d-5}{2}}\left(\Theta[T-R]\,\frac{1}{24\pi^2}\left(\left(2+\frac{T^2}{R^2}\right)\sqrt{T^2-R^2}-3T\ln\left[\left(T+\sqrt{T^2-R^2}\right)\right]-\frac{T^3}{R^2}\right) + \Theta[T]\Theta[-T+R]\,\frac{1}{24\pi^2}\left(-\frac{T^3}{R^2}-3T\ln R\right)\right). \tag{172}$$

As with spin-1 photons in odd dimensions, the spin-2 effective Green's function (155) in this case explicitly reveals that, besides pure tails from $C_{1,d}$, the spin-2 graviton receives extra tail and acausal contributions from both $C_{2,d}$ and $C_{3,d}$ for all odd $d \geqslant 5$; and moreover, no signals traveling strictly on its past light cone—namely, no δ-function light-cone contributions.

Therefore, the acausal nature of the TT spin-2 field in all relevant spacetime dimensions is explicitly confirmed by the analytic solutions of G_{ijmn}^{+} obtained in this section. Now, we proceed to solve for other gauge-invariant variables involved in this system.

Bardeen scalars In a flat background, one of the Bardeen scalar potentials Ψ obeys the Poisson's equation (150), which then leads to a Coulomb-type solution,

$$\Psi[\eta,\vec{x}] = \frac{8\pi G_{\mathrm{N}}}{d-2}\int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}'\, G_d^{(\mathrm{E})}[R]\,{}^{(\mathrm{a})}T_{00}[\eta,\vec{x}']. \tag{173}$$

This transparently shows the acausal character of Ψ, since it is instantaneously sourced by the local matter energy density. The other Bardeen potential Φ is related to Ψ via equation (28); recall that Υ is the nonlocal scalar portion of decomposed ${}^{(\mathrm{a})}T_{ij}$ (see equation (22)), which, in Fourier space with $\vec{k} \neq \vec{0}$, is given by the following local projection [12],

$$\widetilde{\Upsilon}[\eta,\vec{k}] = -\left(\frac{d-1}{d-2}\right)\frac{1}{\vec{k}^4}\left(k_ik_j-\frac{\vec{k}^2}{d-1}\delta_{ij}\right){}^{(\mathrm{a})}\widetilde{T}_{ij}[\eta,\vec{k}]. \tag{174}$$

By virtue of this local decomposition, we can inverse Fourier transform equation (174) to yield

$$\begin{aligned}\Phi[\eta,\vec{x}] =&(d-3)\Psi[\eta,\vec{x}]\\ &+\frac{8\pi G_{\mathrm{N}}}{d-2}\int_{\mathbb{R}^{d-1}}\mathrm{d}^{d-1}\vec{x}'\left(G_d^{(\mathrm{E})}[R]^{(\mathrm{a})}T_{ll}[\eta,\vec{x}']-(d-1)\partial_i\partial_j D_d[R]^{(\mathrm{a})}T_{ij}[\eta,\vec{x}']\right),\end{aligned} \tag{175}$$

where ${}^{(\mathrm{a})}T_{ll}$ denotes the spatial trace of the (ij) components of the matter stress-energy tensor, ${}^{(\mathrm{a})}T_{ll}\equiv\delta^{ij}\,{}^{(\mathrm{a})}T_{ij}$, and $D_d[R]$ is defined in equation (160). Inserting equation (173) into equation (175), we can express Φ itself in terms of the following convolution,

$$\begin{aligned}\Phi[\eta,\vec{x}] =&\frac{8\pi G_{\mathrm{N}}}{d-2}\int_{\mathbb{R}^{d-1}}\mathrm{d}^{d-1}\vec{x}'\left((d-3)G_d^{(\mathrm{E})}[R]^{(\mathrm{a})}T_{00}[\eta,\vec{x}']+G_d^{(\mathrm{E})}[R]^{(\mathrm{a})}T_{ll}[\eta,\vec{x}']\right.\\ &\left.-(d-1)\partial_i\partial_j D_d[R]^{(\mathrm{a})}T_{ij}[\eta,\vec{x}']\right),\end{aligned} \tag{176}$$

where we see that Φ is effectively dependent on different components of the matter stress-energy tensor, weighted either by $G_d^{(\mathrm{E})}$ or D_d on the instantaneous $\eta=\eta'$ surface.

Vector potential The gauge-invariant vector mode V_i, in linearized gravity, also obeys the Poisson's equation (151), but is instead sourced by the nonlocal transverse part of ${}^{(\mathrm{a})}T_{0i}$ (see eq.(21)). As before, since the decomposition is local in momentum space,

$$\widetilde{\Sigma}_i[\eta,\vec{k}]=\left(\delta_{ij}-\frac{k_i k_j}{\vec{k}^2}\right){}^{(\mathrm{a})}\widetilde{T}_{0j}[\eta,\vec{k}], \tag{177}$$

the solution of equation (151) can first be cast into a Fourier form using equation (177), and then translated back to the position space,

$$V_i[\eta,\vec{x}]=16\pi G_{\mathrm{N}}\int_{\mathbb{R}^{d-1}}\mathrm{d}^{d-1}\vec{x}'\left(\partial_i\partial_j D_d[R]\,{}^{(\mathrm{a})}T_{0j}[\eta,\vec{x}']-G_d^{(\mathrm{E})}[R]^{(\mathrm{a})}T_{0i}[\eta,\vec{x}']\right), \tag{178}$$

which is yet again an instantaneous acausal signal. Therefore, in Minkowski background, other than the spin-2 graviton field D_{ij}, the rest of the gauge-invariant variables depend exclusively on the weighted superposition of the matter sources evaluated at the instantaneous observer time η.

Linearized Riemann tensor In our discussion of gravitational observables in section 2, we have argued that, in a free-falling synchronous-gauge setup, the $\delta_1 R_{0i0j}$ components of the linearized Riemann tensor encode the gravitational tidal forces exerted upon the neighboring test particles in the geodesic deviation equation (see equation (39)). And, being also gauge-invariant in Minkowski spacetime, it would reasonably be regarded as a classical physical observable and expected to be strictly causal as well.

As with the Faraday tensor in electromagnetism, it can be directly shown via its second-order wave equation that the linearized Riemann tensor is causal with respect to the flat background. Firstly, by taking the divergence of the Bianchi identity obeyed by the exact Riemann tensor, followed by imposing Einstein's equations, one may obtain

$$\Box R^{\rho\sigma}{}_{\mu\nu}+\left[\nabla^\lambda,\nabla^{[\rho}\right]R^{\sigma]}{}_{\lambda\mu\nu}=32\pi G_{\mathrm{N}}\nabla^{[\rho}\nabla_{[\mu}\left(T_{\nu]}{}^{\sigma]}-\delta^{\sigma]}_{\nu]}\frac{T}{(d-2)}\right). \tag{179}$$

The linearized version in Minkowski background is thus

$$\partial^2 \delta_1 R^{\rho\sigma}{}_{\mu\nu} = 32\pi G_{\mathrm{N}}\, \partial^{[\rho}\partial_{[\mu}\left({}^{(\mathrm{a})}T_{\nu]}{}^{\sigma]} - \delta^{\sigma]}_{\nu]}\frac{{}^{(\mathrm{a})}T}{(d-2)}\right), \tag{180}$$

where ${}^{(\mathrm{a})}T$ is now the trace of the matter stress tensor in flat spacetime, namely ${}^{(\mathrm{a})}T \equiv \eta^{\mu\nu}\,{}^{(\mathrm{a})}T_{\mu\nu}$[18]. The wave equation (180) then leads immediately to the fact that the linearized Riemann tensor is causally sourced by the second derivatives of the astrophysical stress-energy tensor, from which its $\delta_1 R_{0i0j}$ components can be expressed explicitly as

$$\begin{aligned}\delta_1 R_{0i0j}[x] =& 8\pi G_{\mathrm{N}} \int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x' \Bigg\{ \ddot{G}^+_d \left({}^{(\mathrm{a})}T_{ij}[x'] + \frac{\delta_{ij}}{(d-2)}\left({}^{(\mathrm{a})}T_{00}[x'] - {}^{(\mathrm{a})}T_{ll}[x']\right)\right) - 2\partial_{(i}\dot{G}^+_d\,{}^{(\mathrm{a})}T_{j)0}[x'] \\ &+ \frac{1}{(d-2)}\partial_i\partial_j G^+_d \left((d-3)\,{}^{(\mathrm{a})}T_{00}[x'] + {}^{(\mathrm{a})}T_{ll}[x']\right)\Bigg\}. \end{aligned}\tag{181}$$

To arrive at equation (181), we have integrated by parts and dropped the boundary contributions evaluated at the spatial and past infinity. Not only does equation (181) show $\delta_1 R_{0i0j}$ is completely causal for all $d \geqslant 4$, it also provides a check for our calculations in the gauge-invariant approach.

As we have shown earlier, all the gauge-invariant variables—the two scalars Ψ and Φ, one vector V_i, and one tensor D_{ij}—are acausal. From the similar issue encountered in the electromagnetic case, we would expect that, in describing the physical GW observables, a mutual cancellation of the acausal contributions must occur among these variables. Let us now check this statement more carefully using their analytic solutions. Given in equation (40) is the linearized Riemann tensor $\delta_1 R_{0i0j}$ expressed in terms of four gauge-invariant variables in Minkowski spacetime, all of which are non-trivial whenever matter sources are present. Notice that the spin-2 graviton field D_{ij} enters $\delta_1 R_{0i0j}$ through its acceleration; before taking time derivatives, the spin-2 field given in equation (154) can firstly be re-cast into another convolution,

$$\begin{aligned} D_{ij}[\eta,\vec{x}] =& -16\pi G_{\mathrm{N}} \int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x' \Bigg\{ G^+_d \left({}^{(\mathrm{a})}T_{ij}[x'] + \frac{\delta_{ij}}{d-2}\left({}^{(\mathrm{a})}T_{00}[x'] - {}^{(\mathrm{a})}T_{ll}[x']\right)\right) - 2\,\Theta[T]\partial_{(i}\dot{C}_{2,d}\,{}^{(\mathrm{a})}T_{j)0}[x'] \\ &+ \frac{1}{d-2}\Theta[T]\partial_i\partial_j C_{2,d}\left((d-3)\,{}^{(\mathrm{a})}T_{00}[x'] + {}^{(\mathrm{a})}T_{ll}[x']\right)\Bigg\} \\ &+ \frac{16\pi G_{\mathrm{N}}}{d-2}\int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}'\left(\delta_{ij} G^{(\mathrm{E})}_d\,{}^{(\mathrm{a})}T_{00}[\eta,\vec{x}'] + (d-3)\partial_i\partial_j D_d\,{}^{(\mathrm{a})}T_{00}[\eta,\vec{x}']\right), \end{aligned}\tag{182}$$

where we have utilized the conservation of the matter stress-energy tensor, ${}^{(\mathrm{a})}\dot{T}_{0j} = \partial_i{}^{(\mathrm{a})}T_{ij}$ and ${}^{(\mathrm{a})}\ddot{T}_{00} = \partial_i\partial_j{}^{(\mathrm{a})}T_{ij}$, the conversion properties $\ddot{C}_{2,d} = -C_{1,d}$ and $\ddot{C}_{3,d} = -C_{2,d}$, and the initial conditions $\dot{C}_{2,d}\big|_{T=0} = -G^{(\mathrm{E})}_d$ and $\dot{C}_{3,d}\big|_{T=0} = D_d$. Also, as in the spin-1 case, the boundary terms from integrations by parts all vanish, which can be justified using the analytic expressions of $C_{2,d}$ and $C_{3,d}$ obtained above. Note also that, in equation (182), the second integral is performed on the equal-time surface, which is clearly acausal, but the whole expression for D_{ij} is not yet a clean separation based on causality, since $C_{2,d}$ in the first integral still contains both causal and acausal pieces. Furthermore, we highlight that the energy-momentum

[18] Strictly speaking, the matter stress tensor in equation (179), when perturbed around a background Minkowski spacetime $g_{\mu\nu} = \eta_{\mu\nu} + \chi_{\mu\nu}$, would typically admit an infinite series in $\chi_{\mu\nu}$. Whereas the matter stress tensor in equation (180) does not contain $\chi_{\mu\nu}$. The matter stress tensor appearing everywhere else in this paper denotes the latter.

conservation law is always assumed throughout this paper. However, this is no longer true in a self-gravitating system, such as the in-spiraling pairs of black holes/neutron stars whose GWs LIGO have detected to date. Conceptually speaking, to make the theory self-consistent, non-linear corrections of gravity must be incorporated into the right-hand side of the linearized wave equation, so that the conservation of total stress tensor remains valid at linear level. We hope to address this subtlety more systematically in future work.

Now, we proceed to take double-time derivative of the spin-2 field in equation (182),

$$\begin{aligned}\ddot{D}_{ij}[\eta,\vec{x}] = &-16\pi G_{\rm N}\int_{\mathbb{R}^{d-1,1}}{\rm d}^d x'\left\{\ddot{G}_d^+\left({}^{(\rm a)}T_{ij}[x'] + \frac{\delta_{ij}}{d-2}\left({}^{(\rm a)}T_{00}[x'] - {}^{(\rm a)}T_{ll}[x']\right)\right) - 2\partial_{(i}\dot{G}_d^{+\,(\rm a)}T_{j)0}[x']\right.\\ &\left.+\frac{1}{d-2}\partial_i\partial_j G_d^+\left((d-3)\,{}^{(\rm a)}T_{00}[x'] + {}^{(\rm a)}T_{ll}[x']\right)\right\}\\ &+16\pi G_{\rm N}\int_{\mathbb{R}^{d-1}}{\rm d}^{d-1}\vec{x}'\left\{-2\partial_m\partial_{(i}G_d^{(\rm E)\,(\rm a)}T_{j)m}[\eta,\vec{x}'] + \frac{1}{d-2}\left(\delta_{ij}\partial_m\partial_n G_d^{(\rm E)\,(\rm a)}T_{mn}[\eta,\vec{x}']\right.\right.\\ &\left.\left.+\partial_i\partial_j G_d^{(\rm E)}\left((d-3)^{(\rm a)}T_{00}[\eta,\vec{x}'] + {}^{(\rm a)}T_{ll}[\eta,\vec{x}']\right)\right) + \left(\frac{d-3}{d-2}\right)\partial_i\partial_j\partial_m\partial_n D_d{}^{(\rm a)}T_{mn}[\eta,\vec{x}']\right\},\end{aligned} \tag{183}$$

where the same properties used in the previous calculation have been employed to carry out the differentiation[19], and we observe that the first integral in equation (183) is completely causal and is exactly -2 times the expression in equation (181), whereas the second one is acausally performed over the equal-time hypersurface, which is therefore expected to connect to the other gauge-invariant variables. According to equation (40), the scalar and vector contributions to $\delta_1 R_{0i0j}$ come from $\ddot{\Psi}$, $\partial_i\partial_j\Phi$, and $\partial_{(i}\dot{V}_{j)}$, the explicit forms of which can be readily deduced from equations (173), (176) and (178),

$$\ddot{\Psi}[\eta,\vec{x}] = \frac{8\pi G_{\rm N}}{d-2}\int_{\mathbb{R}^{d-1}}{\rm d}^{d-1}\vec{x}'\,\partial_m\partial_n G_d^{(\rm E)\,(\rm a)}T_{mn}[\eta,\vec{x}'], \tag{184}$$

$$\begin{aligned}\partial_i\partial_j\Phi[\eta,\vec{x}] =&\frac{8\pi G_{\rm N}}{d-2}\int_{\mathbb{R}^{d-1}}{\rm d}^{d-1}\vec{x}'\left((d-3)\partial_i\partial_j G_d^{(\rm E)\,(\rm a)}T_{00}[\eta,\vec{x}'] + \partial_i\partial_j G_d^{(\rm E)\,(\rm a)}T_{ll}[\eta,\vec{x}']\right.\\ &\left.-(d-1)\partial_i\partial_j\partial_m\partial_n D_d{}^{(\rm a)}T_{mn}[\eta,\vec{x}']\right),\end{aligned} \tag{185}$$

$$\partial_{(i}\dot{V}_{j)}[\eta,\vec{x}] = -16\pi G_{\rm N}\int_{\mathbb{R}^{d-1}}{\rm d}^{d-1}\vec{x}'\left(\partial_m\partial_{(i}G_d^{(\rm E)\,(\rm a)}T_{j)m}[\eta,\vec{x}'] - \partial_i\partial_j\partial_m\partial_n D_d{}^{(\rm a)}T_{mn}[\eta,\vec{x}']\right), \tag{186}$$

where the conservation law of ${}^{(\rm a)}T_{\mu\nu}$ allows us to switch between different components of the stress-energy tensor. As it turns out, the scalar and vector contributions, added together in accord with equation (40), do conspire to cancel the acausal portion of the acceleration of the spin-2 TT graviton completely, i.e. the second integral of equation (183). As a

[19] Whenever a second time derivative acts on the expression involving a step function $\Theta[T]$, we make use of the following simplification for any function $F[\eta,\eta']$,

$$\partial_\eta^2\left(\Theta[T]F[\eta,\eta']\right) = \Theta[T]\ddot{F}[\eta,\eta'] + \dot{\delta}[T]F[\eta',\eta'] + \delta[T]\dot{F}[\eta,\eta']\big|_{\eta=\eta'},$$

where we have made the replacement $T\dot{\delta}[T] \to -\delta[T]$, which results from differentiating the identity $T\delta[T] = 0$ with respect to η. Notice that the last two terms only contribute at $\eta = \eta'$, and this property will also be utilized later on in the cosmological case.

result, the remaining part of equation (40) is then strictly causal and exactly consistent with equation (181),

$$\delta_1 R_{0i0j} = \delta_{ij}\ddot{\Psi} + \partial_i\partial_j\Phi + \partial_{(i}\dot{V}_{j)} - \frac{1}{2}\ddot{D}_{ij} = -\frac{1}{2}\left(\ddot{D}_{ij}\right)_{\text{causal}} = -\frac{1}{2}\left(\ddot{\chi}^{\text{TT}}_{ij}\right)_{\text{causal}}, \tag{187}$$

which is valid in general weak-field situations. The physical insight gained from the gauge-invariant formalism is that the information regarding gravitational tidal forces is exclusively encoded within the causal part of the acceleration of the spin-2 field; whereas the acausal portion of D_{ij} is completely canceled by the gauge-invariant Bardeen scalars and vector mode. This situation is very similar to that of the spin-1 field describing the electric field in electromagnetism.

Stationary limit and Φ Like the photon case, the limit where the stress tensor ${}^{(\mathrm{a})}T_{\mu\nu}$ becomes time-independent leads to a degenerate causal structure for the acceleration of the spin-2 graviton; namely, its otherwise causal and acausal pieces cancel. In such a situation, one may further verify from eqs. (173), (178), and (183) that $\ddot{\Psi} = \dot{V}_i = \ddot{D}_{ij} = 0$, leaving the tidal forces to depend only on Φ:

$$\delta_1 R_{0i0j} = \partial_i\partial_j\Phi. \tag{188}$$

Since equation (187) holds in general, we may maintain that, despite appearances, it is really the acausal pieces of $\ddot{D}_{ij}$—which are equal in magnitude but opposite in sign to the causal ones—that are canceling the $\partial_i\partial_j\Phi$. This interpretation ensures that causality is respected once there is the slightest time-dependence in the ${}^{(\mathrm{a})}T_{\mu\nu}$.

$$\delta_1 R_{0i0j}[\vec{x}] = -\frac{1}{2}\left(\ddot{D}_{ij}\right)_{\text{causal}} = \frac{8\pi G_{\mathrm{N}}}{d-2}\int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}'\,\partial_i\partial_j G^{(\mathrm{E})}_d\left((d-3)\,{}^{(\mathrm{a})}T_{00}[\vec{x}'] + {}^{(\mathrm{a})}T_{ll}[\vec{x}']\right). \tag{189}$$

Far-zone limit To extract the far-zone GW signals generated by the isolated astrophysical systems, we perform the same frequency space analysis for the spin-2 effective Green's function here as for its spin-1 counterpart. Before taking the far-zone limit, we first re-cast the spin-2 effective Green's function (155) into the one analogous to equation (126) for spin-1 photons, by carrying out all the spatial derivatives involved in C_{ijmn} while avoiding the point $R = 0$,

$$C_{ijmn}[T,\vec{R}] = P_{ijmn}[\vec{R}]\,C_{1,d}[T,R] + \Pi_{ijmn}[\vec{R}]\,2\pi C_{2,d+2}[T,R] + \Xi_{ijmn}[\vec{R}]\,4\pi^2 C_{3,d+4}[T,R], \tag{190}$$

where $P_{ijmn}[\vec{R}]$ denotes the TT spatial projector based on the unit vector $\widehat{R}$,

$$P_{ijmn}[\vec{R}] \equiv P_{m(i}[\vec{R}]\,P_{j)n}[\vec{R}] - \frac{1}{d-2}P_{ij}[\vec{R}]\,P_{mn}[\vec{R}], \tag{191}$$

with $P_{ij}[\vec{R}]$ given in equation (127), and the other symmetric tensor structures $\Pi_{ijmn}[\vec{R}]$ and $\Xi_{ijmn}[\vec{R}]$, respectively, are defined as

$$\Pi_{ijmn}[\vec{R}] = -2P_{ijmn}[\vec{R}] + \frac{d(d-3)}{d-2}\left(\delta_{m(i}\widehat{R}_{j)}\widehat{R}_n + \delta_{n(i}\widehat{R}_{j)}\widehat{R}_m - 2\widehat{R}_i\widehat{R}_j\widehat{R}_m\widehat{R}_n\right) \tag{192}$$

$$\Xi_{ijmn}[\vec{R}] = \left(\frac{d-3}{d-2}\right)\Big(\delta_{ij}\delta_{mn} + 2\delta_{m(i}\delta_{j)n} - (d+1)\Big(\delta_{ij}\widehat{R}_m\widehat{R}_n + \delta_{mn}\widehat{R}_i\widehat{R}_j + 2\delta_{m(i}\widehat{R}_{j)}\widehat{R}_n + 2\delta_{n(i}\widehat{R}_{j)}\widehat{R}_m\Big) + (d+1)(d+3)\widehat{R}_i\widehat{R}_j\widehat{R}_m\widehat{R}_n\Big). \tag{193}$$

We have taken advantage of the homogeneous wave equation obeyed by both $C_{2,d}$ and $C_{3,d}$, along with the properties $\ddot{C}_{2,d} = -C_{1,d}$ and $\ddot{C}_{3,d} = -C_{2,d}$, to relate different scalar functions. Here, we highlight again that, for a fixed d, the notations $C_{2,d+2}$ and $C_{3,d+4}$ used in equation (190) represent their corresponding $(d+2)$ and $(d+4)$-dimensional functional forms, but the R is the one in $d-1$ spatial dimensions. Essentially, as long as $R \neq 0$, equation (190) is equivalent to its original expression (155), and, as explained in the similar spin-1 situation, this form is useful for the far-zone analysis and is manifestly finite in all relevant spacetime dimensions, because $C_{1,d}$, $C_{2,d+2}$, and $C_{3,d+4}$ all converge for $d \geqslant 4$. As a consistency check of equation (190), its TT properties can be shown explicitly by a direct calculation.

The leading contribution of GWs, responsible for the far-zone tidal forces, can be extracted from the spin-2 effective Green's function using equation (190) in frequency space. The relative amplitudes of the three scalar functions can in turn be directly compared in the limit $|\omega|R \gg 1$. To begin, we express the spin-2 field D_{ij} in terms of the superposition of monochromatic modes,

$$D_{ij}[\eta, \vec{x}] = -16\pi G_{\rm N} \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}' \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\pi} \, \widetilde{G}^{+}_{ijmn}[\omega, \vec{R}] \, \mathrm{e}^{-\mathrm{i}\omega\eta} \, {}^{(\mathrm{a})}\widetilde{T}_{mn}[\omega, \vec{x}'], \tag{194}$$

where $\widetilde{G}^{+}_{ijmn}[\omega, \vec{R}]$ is the frequency transform of the spin-2 effective Green's function assuming $R \neq 0$,

$$\begin{aligned} \widetilde{G}^{+}_{ijmn}[\omega, \vec{R}] &= \int_{\mathbb{R}} \mathrm{d}T \, G^{+}_{ijmn}[T, \vec{R}] \, \mathrm{e}^{\mathrm{i}\omega T} \\ &= P_{ijmn}[\vec{R}] \, \widetilde{G}^{+}_{d}[\omega, R] - \Pi_{ijmn}[\vec{R}] \, 2\pi \widetilde{C}^{+}_{2,d+2}[\omega, R] - \Xi_{ijmn}[\vec{R}] \, 4\pi^2 \widetilde{C}^{+}_{3,d+4}[\omega, R], \end{aligned} \tag{195}$$

with $\widetilde{G}^{+}_{d}[\omega, R]$, $\widetilde{C}^{+}_{2,d+2}[\omega, R]$, and $\widetilde{C}^{+}_{3,d+4}[\omega, R]$, respectively, defined to be the frequency transforms of their real-space counterparts, G^{+}_{d}, $C^{+}_{2,d+2}$, and $C^{+}_{3,d+4}$. As we did for the far-zone spin-1 waves, we now take the limit of equation (195) as $|\omega|R \to \infty$, from which to extract the dominant spin-2 GWs in the radiative regime. Since we have calculated $\widetilde{G}^{+}_{d}[\omega, R]$ and $\widetilde{C}^{+}_{2,d+2}[\omega, R]$ earlier in the photon case, $\widetilde{C}^{+}_{3,d+4}[\omega, R]$ is the only term left to evaluate here.

Spin-2 gravitons in even dimensions $d \geqslant 4$ For even-dimensional spacetimes, $\widetilde{G}^{+}_{d}[\omega, R]$ and $\widetilde{C}^{+}_{2,d+2}[\omega, R]$, for $d = 4 + 2n$, have been obtained in equations (131) and (132). In the same way, with the analytic expression of $C_{3,d}$ given in equation (168), its frequency transform $\widetilde{C}^{+}_{3,d+4}[\omega, R]$ can be computed straightforwardly,

$$\begin{aligned} \widetilde{C}^{+}_{3,8+2n}[\omega, R] =& \mathcal{D}^{n}_{R} \left(\frac{\mathrm{e}^{\mathrm{i}\omega R}}{16\pi^3 R^3 (\mathrm{i}\omega)^2} - \frac{3\mathrm{e}^{\mathrm{i}\omega R}}{16\pi^3 R^4 (\mathrm{i}\omega)^3} + \frac{3\mathrm{e}^{\mathrm{i}\omega R}}{16\pi^3 R^5 (\mathrm{i}\omega)^4} \right) - \frac{(2n+1)!! \, \omega^{2n+1}}{4(2\pi)^{n+3} (\omega R)^{2n+3}} \\ &- \frac{(2n+3)!! \, \omega^{2n+1}}{2(2\pi)^{n+3} (\omega R)^{2n+5}}, \end{aligned} \tag{196}$$

which, analogous to $\widetilde{C}^{+}_{2,d+2}[\omega, R]$, comprises the non-propagating modes associated with the acausal effect, as well as the propagating ones with the $\mathrm{e}^{\mathrm{i}\omega R}$ factor. Moreover, $\widetilde{C}^{+}_{3,8+2n}[\omega, R]$ turns out to be suppressed relative to $\widetilde{G}^{+}_{d}[\omega, R]$ and $\widetilde{C}^{+}_{2,d+2}[\omega, R]$ when $|\omega|R \to \infty$ (see equations (131) and (132)). More explicitly, at leading $1/(\omega R)$ order, it behaves like

$$\widetilde{C}^{+}_{3,8+2n}[\omega, R] = \frac{(-1)^{n+1} \mathrm{i}^{n} \omega^{2n+1}}{2(2\pi\omega R)^{n+1}} \, \mathrm{e}^{\mathrm{i}\omega R} \cdot \frac{1}{(2\pi\omega R)^2} \left(1 + \mathcal{O}\left[\frac{1}{\omega R} \right] \right). \tag{197}$$

Hence, as inferred from the asymptotic behaviors of three scalar functions (133), (134) and (197), the pure causal one, $\widetilde{G}_d^+[\omega, R]$, is still the dominant contribution to the spin-2 effective Green's function in the limit $|\omega|R \gg 1$. In close analogy with the spin-1 case, the spin-2 GWs in the radiative zone is dominated by the first term of equation (195). That is, under the far-zone assumptions $|\omega|r \gg 1$ and $r_c/r \ll 1$, the leading $1/r$ piece of $\widetilde{G}^+_{ijmn}$ is given by

$$\widetilde{G}^+_{ijmn}[\omega, \vec{R}] = P_{ijmn}\widetilde{G}^{(+,\,\mathrm{fz})}_{4+2n}[\omega; \vec{x}, \vec{x}'] \left(1 + \mathcal{O}\left[\frac{1}{\omega r}, \frac{r_c}{r}\right]\right), \tag{198}$$

where P_{ijmn}, the far-zone 'tt' projector, is defined in equation (5) and $\widetilde{G}^{(+,\,\mathrm{fz})}_{4+2n}$ given in equation (136). Accordingly, as already alluded to in section 2, the spin-2 TT graviton D_{ij}, in the far-zone radiative regime ($r \to \infty$), reduces to the causal 'tt' GWs,

$$\lim_{r\to\infty} D_{ij} \to \chi^{\mathrm{tt}}_{ij}, \qquad \chi^{\mathrm{tt}}_{ij}[x] \equiv P_{ijmn}\left(-16\pi G_{\mathrm{N}} \int_{\mathbb{R}^{d-1,1}} \mathrm{d}^d x'\, G_d^{(+,\,\mathrm{fz})}[T; \vec{x}, \vec{x}']^{(\mathrm{a})} T_{mn}[x']\right), \tag{199}$$

where $G_d^{(+,\,\mathrm{fz})}[T; \vec{x}, \vec{x}']$, as before, denotes the far-zone version of the massless scalar Green's function, which, for even $d \geqslant 4$, is given in equation (138) consisting of pure light-cone signals. This χ^{tt}_{ij} is thus the tt projection of the far-zone de Donder-gauge solution of the metric perturbations, $\chi^{\mathrm{tt}}_{ij} = P_{ijmn}\overline{\chi}_{mn}[\text{de Donder}]$. In other words, like the consequence of the far-zone spin-1 field (137), the two distinct notions of 'transverse-traceless' metric perturbations, $D_{ij} = \chi^{\mathrm{TT}}_{ij}$ and χ^{tt}_{ij}, are shown to coincide as $r \to \infty$, where the acausal effect in D_{ij} becomes sufficiently insignificant.

Spin-2 gravitons in odd dimensions $d \geqslant 5$ Following the similar procedures, we are able to extract the far-zone portion of the spin-2 GWs for odd dimensions as well. Odd-dimensional $\widetilde{G}_d^+[\omega, R]$ and $\widetilde{C}^+_{2,d+2}[\omega, R]$ for $d = 5 + 2n$ and positive frequencies can be obtained simply by replacing $n \to n + 1$ in equations (139) and (140). And, given equation (172), $\widetilde{C}^+_{3,d+4}[\omega, R]$ can be tackled similarly to $\widetilde{C}^+_{2,d+2}[\omega, R]$,

$$\widetilde{C}^+_{3,9+2n}[\omega > 0, R] = \frac{\mathrm{i}\omega^{2n+2}}{4(2\pi\omega R)^{n+3}} H^{(1)}_{n+3}[\omega R] - \frac{2^n (n+1)!\,\omega^{2n+2}}{(2\pi)^{n+4}(\omega R)^{2n+4}} - \frac{2^{n+2}(n+2)!\,\omega^{2n+2}}{(2\pi)^{n+4}(\omega R)^{2n+6}}, \tag{200}$$

which, as in the even-dimensional case, resembles the structure of $\widetilde{C}^+_{2,d+2}[\omega, R]$ in equation (140), and tends to be more suppressed than both $\widetilde{G}_d^+[\omega, R]$ and $\widetilde{C}^+_{2,d+2}[\omega, R]$ at leading $1/(\omega R)$ order. That is, as $|\omega|R \gg 1$, the asymptotic behavior of $\widetilde{C}^+_{3,d+4}[\omega, R]$ reads

$$\widetilde{C}^+_{3,9+2n}[\omega > 0, R] = \frac{-\mathrm{i}\omega^{2n+2}}{2(2\pi\omega R)^{n+\frac{3}{2}}} \mathrm{e}^{\mathrm{i}\left(\omega R - \frac{(n+1)\pi}{2} - \frac{\pi}{4}\right)} \cdot \frac{1}{(2\pi\omega R)^2}\left(1 + \mathcal{O}\left[\frac{1}{\omega R}\right] + \mathcal{O}\left[\frac{1}{(\omega R)^{n+\frac{1}{2}}}\right]\right), \tag{201}$$

where the expression has been factorized into the leading $1/(\omega R)$ piece of $\widetilde{G}^+_{5+2n}[\omega, R]$ times the suppression factor. Likewise, among the three scalar functions in equation (195), $\widetilde{G}_d^+[\omega, R]$ continues to be the dominant contribution in the limit $|\omega|R \to \infty$. As a result, the far-zone behavior of $\widetilde{G}^+_{ijmn}[\omega, R]$ here admits the same 'tt' structure as equation (198) for even d,

$$\widetilde{G}^+_{ijmn}[\omega > 0, \vec{R}] = P_{ijmn}\widetilde{G}^{(+,\,\mathrm{fz})}_{5+2n}[\omega; \vec{x}, \vec{x}'] \left(1 + \mathcal{O}\left[\frac{1}{\omega r}, \frac{r_c}{r}\right]\right), \tag{202}$$

where $\widetilde{G}^{(+,\,\mathrm{fz})}_{5+2n}$ is given in equation (144) with n replaced by $n + 1$; and P_{ijmn} in equation (5). A similar line of arguments then reveals that the spin-2 TT graviton D_{ij}, in odd dimensions

$d \geqslant 5$, also reduces to $\chi^{\rm tt}_{ij}$ as $r \to \infty$, where the acausal nature of D_{ij} becomes trivial; namely, the feature (199) still holds here, with odd-dimensional $G^{(+,\,\rm fz)}_d$ given in equation (145).

Linearized Riemann tensor Through the analysis of the spin-2 effective Green's function, we have just shown that, in the radiative limit, the spin-2 TT GWs in fact coincide with the tt ones, $D_{ij} \to \chi^{\rm tt}_{ij}$, for all spacetime dimensions $d \geqslant 4$. For this reason, the far-zone version of the tidal forces (52) for all $d \geqslant 4$, as well as the statement (45), follows immediately from equation (187) and the fact that $\chi^{\rm tt}_{ij}$ is completely causal. This result can alternatively be derived from the expression (181) for $\delta_1 R_{0i0j}$, by repeatedly employing the replacement rule $\partial_i = -\partial_{i'} = -\widehat{R}_i\partial_0$ and the conservation of the matter stress tensor in the intermediate steps before reaching the final far-zone expression. Furthermore, the far-zone connection between $\chi^{\rm tt}_{ij}$ and the synchronous-gauge metric $\chi^{\rm (Synch)}_{ij}$ can be made via equation (46), where as explained in section 2 the initial conditions could be dropped for GW detectors sensitive to only finite frequencies. In particular, the fractional distortion spin-2 pattern of the laser interferometer, described in equation (41) are exclusively attributed to the causal $\chi^{\rm tt}_{ij}$. Such a characterization of the GW observables in terms of $\chi^{\rm tt}_{ij}$, however, is legitimate only when the GW detector is sufficiently far away from the matter sources.

Commutator of spin-2 gravitons Earlier in this section, we have shown micro-causality is violated for the massless spin-1 photons (see equation (149)). A similar line of reasoning then reveals that the massless spin-2 graviton field violates micro-causality too. For, the tensor structure C_{ijmn} in equation (155) is related to the commutator of the corresponding quantum operators via the relationship

$$\left[D_{ij}[x], D_{mn}[x']\right] = -\mathrm{i}C_{ijmn}[T, \vec{R}]. \tag{203}$$

The acausal nature of C_{ijmn} immediately tells us that the massless spin-2 gravitons do not commute at spacelike separations. Once again, it is likely that this violation of micro-causality is linked to the tension between gauge invariance and Lorentz covariance when constructing massless helicity-2 quantum Fock states.

5. Spatially flat cosmologies with constant *w*

We now move on to the spatially-flat cosmological background, driven by a perfect fluid with a constant equation-of-state w. Again, we will consider both the electromagnetism and linearized gravitation cases, where the dynamics of the linearized gravitational system, unlike that of electromagnetism, has non-trivial dependence on w. In cosmology, there is no longer a time-translation symmetry, and the full analytic expressions for spin-1 and spin-2 effective Green's functions may generally be difficult to attain. On the other hand, the background space-translation symmetry is still preserved, so the similar Fourier-space analysis exploited in Minkowski spacetime continues to apply in the cosmological setup. At a more technical level, the translation symmetry in space would still allow us to utilize the time-integral method to express the spin-1 and spin-2 effective Green's functions in terms of the analytic solutions found in [12], so that the corresponding causal structures can be analyzed.

5.1. Electromagnetism

We start with the electromagnetic field in the cosmological background spacetime, described by the metric (15) with $\chi_{\mu\nu}$ set to zero, and our focus here will be the causal structure of the theory in the gauge-invariant content for all $d \geqslant 3$.

Field equations In spatially-flat cosmologies, the Maxwell's equations (29), in terms of the gauge-invariant variables (34), are translated into a set of two independent field equations,

$$-\frac{\partial_0\left(a^{d-4}\,\dot{\alpha}_i\right)}{a^{d-4}}+\vec{\nabla}^2\alpha_i=-a^2\Gamma_i, \tag{204}$$

$$\vec{\nabla}^2\Phi=-a^2\rho. \tag{205}$$

Notice that the spatial components of equation (29) encode not only equation (204) but also the equation $\partial_0\left(a^{d-4}\,\Phi\right)/a^{d-4}=a^2\Gamma$, which is in fact redundant, as is already implied by the Poisson's equation (205) inserted into the conservation law of the charge current $\nabla_\mu J^\mu=0$,

$$\frac{\partial_0\left(a^{d-2}J_0\right)}{a^{d-2}}=\partial_i J_i. \tag{206}$$

Moreover, equations (204) and (205) together show that, except the re-scaling $J_\mu\to a^2 J_\mu$, the theory is conformally invariant when $d=4$, and, like its Minkowski counterpart, only the spin-1 photon field α_i admits dynamical wave solutions, with the scalar Φ still obeying a Poisson-type equation.

Spin-1 photons To solve for the transverse spin-1 field α_i in the cosmological system, we first re-write its wave equation (204) as

$$\left\{\partial^2+\frac{(d-4)}{4}\left((d-4)\mathcal{H}^2+2\dot{\mathcal{H}}\right)\right\}\left(a^{\frac{d-4}{2}}\alpha_i\right)=-a^{\frac{d}{2}}\Gamma_i, \tag{207}$$

where $\mathcal{H}\equiv\dot{a}/a$, denoting the conformal Hubble parameter. Then, following the same manipulations in Fourier space performed in Minkowski spacetime, we are able to express the spin-1 field α_i in terms of the following convolution based on equation (207),

$$a[\eta]^{\frac{d-4}{2}}\alpha_i[\eta,\vec{x}]=-\int_{\mathbb{R}^{d-1}}\mathrm{d}^{d-1}\vec{x}'\int_{\eta_\mathrm{p}}^{\eta_\mathrm{f}}\mathrm{d}\eta'\,a[\eta']^{\frac{d}{2}}G_{ij}^{(\gamma,+)}[\eta,\eta';R]\,J_j[\eta',\vec{x}'], \tag{208}$$

where the time interval of integration $(\eta_\mathrm{p},\eta_\mathrm{f})$ covers all the possible values of η for an expanding universe, and the spin-1 effective Green's function $G_{ij}^{(\gamma,+)}$ is given by

$$G_{ij}^{(\gamma,+)}[\eta,\eta';\vec{R}]=-\Theta[T]C_{ij}^{(\gamma)}[\eta,\eta';\vec{R}],$$
$$C_{ij}^{(\gamma)}[\eta,\eta';\vec{R}]=\delta_{ij}C_{1,d}^{(\gamma)}[\eta,\eta';R]+\partial_i\partial_j C_{2,d}^{(\gamma)}[\eta,\eta';R], \tag{209}$$

with $C_{1,d}^{(\gamma)}$ and $C_{2,d}^{(\gamma)}$, respectively, defined by

$$C_{1,d}^{(\gamma)}[\eta,\eta';R]\equiv\int_{\mathbb{R}^{d-1}}\frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}}\,\widetilde{C}_{1,d}^{(\gamma)}\left[\eta,\eta';|\vec{k}|\right]\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{R}}, \tag{210}$$

$$C_{2,d}^{(\gamma)}[\eta,\eta';R]\equiv\int_{\mathbb{R}^{d-1}}\frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}}\,\widetilde{C}_{1,d}^{(\gamma)}\left[\eta,\eta';|\vec{k}|\right]\,\frac{\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{R}}}{\vec{k}^2}; \tag{211}$$

the Fourier transform of $C_{1,d}^{(\gamma)}$ can be equivalently expressed in terms of the following decomposition,

$$\widetilde{C}_{1,d}^{(\gamma)}\left[\eta,\eta';|\vec{k}|\right]=\mathrm{i}\left(v_{|\vec{k}|}^{(\gamma)}[\eta]\,v_{|\vec{k}|}^{(\gamma)*}[\eta']-v_{|\vec{k}|}^{(\gamma)*}[\eta]\,v_{|\vec{k}|}^{(\gamma)}[\eta']\right), \tag{212}$$

where $v_{|\vec{k}|}^{(\gamma)}$ are in fact the mode functions of the massless scalar field that satisfies the homogeneous (i.e. $\Gamma_i=0$) form of the wave equation (207), the Fourier-space version of which is therefore obeyed by $v^{(\gamma)}_{|\vec{k}|}$ itself and $\widetilde{C}^{(\gamma)}_{1,d}$; moreover, $v^{(\gamma)}_{|\vec{k}|}$ have been normalized so that the initial condition imposed on the time derivative of $\widetilde{C}^{(\gamma)}_{1,d}$, namely $\dot{\widetilde{C}}^{(\gamma)}_{1,d}\big|_{\eta=\eta'} = 1$, coincides with the Wronskian condition for the mode functions, $v^{(\gamma)}_{|\vec{k}|}\dot{v}^{(\gamma)*}_{|\vec{k}|} - v^{(\gamma)*}_{|\vec{k}|}\dot{v}^{(\gamma)}_{|\vec{k}|} = i$. In this language, the properties of $C^{(\gamma)}_{1,d}$ and $C^{(\gamma)}_{2,d}$ become more transparent. Specifically, both of the scalar functions obey the homogeneous wave equation associated with the wave operator in equation (207), and the equal-time initial conditions for $C^{(\gamma)}_{1,d}$, $C^{(\gamma)}_{2,d}$, and their velocities, can be immediately read off,

$$C^{(\gamma)}_{1,d}\big|_{\eta=\eta'} = C^{(\gamma)}_{2,d}\big|_{\eta=\eta'} = 0, \tag{213}$$

$$\dot{C}^{(\gamma)}_{1,d}\big|_{\eta=\eta'} = -\partial_{\eta'} C^{(\gamma)}_{1,d}\big|_{\eta=\eta'} = \delta^{(d-1)}[\vec{x}-\vec{x}'], \tag{214}$$

$$\dot{C}^{(\gamma)}_{2,d}\big|_{\eta=\eta'} = -\partial_{\eta'} C^{(\gamma)}_{2,d}\big|_{\eta=\eta'} = -G^{(\mathrm{E})}_d[R]. \tag{215}$$

It turns out that $C^{(\gamma)}_{1,d}$ and $C^{(\gamma)}_{2,d}$ are the cosmological generalization of their Minkowski counterparts (74) and (75), and $C^{(\gamma)}_{ij}$ is connected to the commutator of the massless spin-1 photons in the cosmological background,

$$\left[\alpha_i[x], \alpha_j[x']\right] = -\mathrm{i}C^{(\gamma)}_{ij}[\eta,\eta';\vec{R}]. \tag{216}$$

When specializing to the constant-w cosmologies considered in this section, where $\mathcal{H} = 2/(q_w\eta)$ with $(\eta_\mathrm{p},\eta_\mathrm{f}) = (-\infty, 0)$ for $w < -(d-3)/(d-1)$ and $(\eta_\mathrm{p},\eta_\mathrm{f}) = (0,\infty)$ for $w > -(d-3)/(d-1)$, the analytic solution of the massless scalar Green's function $G^{(\gamma,+)}_d = -\Theta[T]C^{(\gamma)}_{1,d}$ for $d \geqslant 3$ has been derived in [12] via Nariai's ansatz (see equations (205) and (206) of [12]), instead of computing equation (210); the result shows that $G^{(\gamma,+)}_d$ contains pure causal signals propagating either on or inside the light cone. Since $C^{(\gamma)}_{1,d}$ is known, we may employ the time integral method to compute $C^{(\gamma)}_{2,d}$ without resorting to tackling the integral in equation (211) directly. The homogeneous counterpart of equations (207) together with the relation $-\vec{\nabla}^2 C^{(\gamma)}_{2,d} = C^{(\gamma)}_{1,d}$ leads us to

$$\frac{-\partial_\eta\left(a^{2\alpha}\,\partial_\eta\left(a^{-\alpha}\,C^{(\gamma)}_{2,d}\right)\right)}{a^\alpha} = C^{(\gamma)}_{1,d}, \qquad \alpha = \frac{d-4}{2}. \tag{217}$$

With the initial conditions (213) and (215), we are able to write down

$$\begin{aligned} C^{(\gamma)}_{2,d}[\eta,\eta';R] = &- a[\eta]^{\frac{d-4}{2}} \int_{\eta'}^{\eta} \mathrm{d}\eta_2\, a[\eta_2]^{-(d-4)} \int_{\eta'}^{\eta_2} \mathrm{d}\eta_1\, a[\eta_1]^{\frac{d-4}{2}}\, C^{(\gamma)}_{1,d}[\eta_1,\eta';R] \\ &- G^{(\mathrm{E})}_d[R]\left(\frac{a[\eta]}{a[\eta']}\right)^{\frac{d-4}{2}} \int_{\eta'}^{\eta} \mathrm{d}\eta_1\, \left(\frac{a[\eta']}{a[\eta_1]}\right)^{d-4}. \end{aligned} \tag{218}$$

If we only consider the retarded piece ($\eta > \eta'$) of $C^{(\gamma)}_{2,d}$, then as explained in section 4 since $C^{(\gamma)}_{1,d}[\eta,\eta';R]$ is strictly causal (i.e. its retarded part is proportional to $\Theta[T-R^-]$), the first term of equation (218) would in turn yield a strictly causal contribution. Whereas the second term is a smooth function of spacetime, consisting of the signals that pervade all physical spacetime points with $\eta > \eta'$, including the region outside the light cone. Hence, the time-integral method has cleanly elucidated the causal structure of $C^{(\gamma)}_{2,d}$, even if the integrals cannot be performed in closed form: acausality is present for all $d \geqslant 3$ and is encoded only in the second term of equation (218). Moreover, it also implies the spin-1 quantum operator violates micro-causality in constant-w cosmologies (see equation (216)).

Scalar The Poisson's equation (205) for scalar Φ can be solved immediately by utilizing the Euclidean Green's function,

$$\Phi[\eta,\vec{x}] = \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}'\, a[\eta]^2 G^{(\mathrm{E})}_d[R]\, J_0[\eta,\vec{x}'], \tag{219}$$

which, except for the factor $a[\eta]^2$, is precisely the same as its Minkowski counterpart (115). This in turn implies that, despite the distinct waveforms of the spin-1 field in cosmology, the acausal portion of its velocity must take such a simple form to ensure a causal electric field, as we will demonstrate below.

Faraday tensor Let us now turn to the causal structure of the Faraday tensor $F_{\mu\nu}$ within the cosmological context, where $F_{\mu\nu}$ in the gauge-invariant formalism still takes the form of equation (35). To make the causality analysis more transparent, we first re-cast the spin-1 photon field (208) in the following convolution that involves both time and spatial components of the electric current J_μ,

$$\alpha_i[\eta,\vec{x}] = -\int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}' \int_{\eta_\mathrm{p}}^{\eta_\mathrm{f}} \mathrm{d}\eta'\, a[\eta]^{-\frac{d-4}{2}} a[\eta']^{\frac{d}{2}} \Bigg(G^{(\gamma,+)}_d J_i[\eta',\vec{x}'] + \Theta[T]\, a[\eta']^{\frac{d-4}{2}} \partial_{\eta'} \left(a[\eta']^{-\frac{d-4}{2}} \partial_i C^{(\gamma)}_{2,d} \right) J_0[\eta',\vec{x}'] \Bigg), \tag{220}$$

where we have employed the conservation law (206) and removed the boundary terms that arise from integration by parts[20]. The resulting expression is in fact the cosmological generalization of equation (118), and notice that, due to the lack of time-translation symmetry, the time derivative of $C^{(\gamma)}_{1,d}$ or $C^{(\gamma)}_{2,d}$ with respect to η' cannot simply be replaced with the negative of that with respect to η.

Start with the (ij) components of $F_{\mu\nu}$. Taking the spatial curl of either equation (208) or equation (220) gives rise to the magnetic field in the cosmological background (see equation (35)),

$$F_{ij}[\eta,\vec{x}] = 2\partial_{[i}\alpha_{j]} = -2\int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}' \int_{\eta_\mathrm{p}}^{\eta_\mathrm{f}} \mathrm{d}\eta'\, a[\eta]^{-\frac{d-4}{2}} a[\eta']^{\frac{d}{2}} \partial_{[i} G^{(\gamma,+)}_d J_{j]}[\eta',\vec{x}'], \tag{221}$$

which is completely causal for $d \geqslant 3$, since the acausal effect of the spin-1 photons, encapsulated in the second term of equation (208) or equation (220), is eliminated by the curl operation here—as was the case in the Minkowski background. Next, to obtain the electric field F_{0i}, we first compute the velocity of the spin-1 field by taking the time derivative of equation (220), with the expression (218) substituted for $C^{(\gamma)}_{2,d}$,

[20] With the assumption that the electric current is sufficiently localized, the boundary contributions evaluated at spatial infinity are zero, whereas the ones at past infinity, namely at $\eta' = -\infty$ for $w < -(d-3)/(d-1)$ and $\eta' = 0$ for $w > -(d-3)/(d-1)$, still require further justification. However, those boundary terms at past infinity are in fact the surface integrals of $C^{(\gamma)}_{2,d}$ at $\eta' = -\infty$ or $\eta' = 0$, indicating the fact that they satisfy the homogeneous wave equation and do not alter the exact inhomogeneous solution to the spin-1 wave equation. We hope to clarify this issue further in our later work.

$$\dot{\alpha}_i[\eta,\vec{x}] = -\int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}' \int_{\eta_\mathrm{p}}^{\eta_\mathrm{f}} \mathrm{d}\eta'\, a[\eta]^{-\frac{d-4}{2}} a[\eta']^{\frac{d}{2}} \Bigg\{ \left(\dot{G}_d^{(\gamma,+)}[\eta,\eta';R] - \frac{(d-4)}{q_w \eta} G_d^{(\gamma,+)}[\eta,\eta';R] \right) J_i[\eta',\vec{x}'] \\ + \left(\frac{a[\eta']}{a[\eta]}\right)^{\frac{d-4}{2}} \partial_{\eta'} \left(a[\eta']^{-\frac{d-4}{2}} \int_{\eta'}^{\eta} \mathrm{d}\eta_1\, a[\eta_1]^{\frac{d-4}{2}} \partial_i G_d^{(\gamma,+)}[\eta_1,\eta';R] \right) J_0[\eta',\vec{x}'] \Bigg\} \\ - \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}'\, a[\eta]^2 \partial_i G_d^{(\mathrm{E})}[R] J_0[\eta,\vec{x}'], \tag{222}$$

where the spacetime integral involving $G_d^{(\gamma,+)}$ is purely causal, whereas the last term is an acausal instant-time-surface integral. Then, summing the expression (222) for $\dot{\alpha}_i$ and the gradient of Φ in equation (219) amounts to canceling the acausal term in the last line of equation (222), and yields the purely causal electric field F_{0i},

$$F_{0i}[\eta,\vec{x}] = \dot{\alpha}_i[\eta,\vec{x}] + \partial_i \Phi[\eta,\vec{x}] \\ = -\int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}' \int_{\eta_\mathrm{p}}^{\eta_\mathrm{f}} \mathrm{d}\eta'\, a[\eta]^{-\frac{d-4}{2}} a[\eta']^{\frac{d}{2}} \Bigg\{ \left(\dot{G}_d^{(\gamma,+)}[\eta,\eta';R] - \frac{(d-4)}{q_w \eta} G_d^{(\gamma,+)}[\eta,\eta';R] \right) J_i[\eta',\vec{x}'] \\ + \left(\frac{a[\eta']}{a[\eta]}\right)^{\frac{d-4}{2}} \partial_{\eta'} \left(a[\eta']^{-\frac{d-4}{2}} \int_{\eta'}^{\eta} \mathrm{d}\eta_1\, a[\eta_1]^{\frac{d-4}{2}} \partial_i G_d^{(\gamma,+)}[\eta_1,\eta';R] \right) J_0[\eta',\vec{x}'] \Bigg\}. \tag{223}$$

This result extends equation (122) to the cosmological context; i.e. the electric field is still the causal piece of the velocity of the spin-1 photon α_i. Furthermore, as a simple check of consistency, one can show equation (223) does reduce to its Minkowski counterpart (121), by setting $a \to 1$ and assuming $G_d^{(\gamma,+)}$ takes its Minkowski form G_d^+ with time-translation symmetry.

Although we have shown explicitly that causality is preserved for the electromagnetic observables in constant-w spatially-flat cosmologies, the second line of equation (223) still involves a time integral from some initial time η' to the present η. However, by introducing a new Green's function $G_d^{(\gamma|\mathrm{time},+)}$ obeying

$$\left\{ \partial^2 + \frac{(d-4)\,(q_w + d - 4)}{q_w^2 \eta^2} \right\} G_d^{(\gamma|\mathrm{time},+)}[\eta,\eta';R] = \delta^{(d)}[x - x'], \tag{224}$$

the effective Green's function of F_{0i} in equation (223) can be further reduced to a localized form,

$$F_{0i}[\eta,\vec{x}] = -\int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}' \int_{\eta_\mathrm{p}}^{\eta_\mathrm{f}} \mathrm{d}\eta'\, a[\eta]^{-\frac{d-4}{2}} a[\eta']^{\frac{d}{2}} \Bigg\{ \Bigg(\dot{G}_d^{(\gamma,+)}[\eta,\eta';R] \\ - \frac{(d-4)}{q_w \eta} G_d^{(\gamma,+)}[\eta,\eta';R] \Bigg) J_i[\eta',\vec{x}'] - \partial_i G_d^{(\gamma|\mathrm{time},+)}[\eta,\eta';R] J_0[\eta',\vec{x}'] \Bigg\}. \tag{225}$$

We will leave the detailed derivation of equation (225) to our future work [13]. It turns out that the result (225) is exactly consistent with F_{0i} computed directly using the solution of the generalized Lorenz-gauge vector potential A_μ in [12]; the analytic solutions of $G_d^{(\gamma,+)} = -\Theta[T]\mathcal{G}^{(\gamma|\mathrm{space})}$ and $G_d^{(\gamma|\mathrm{time},+)} = -\Theta[T]\mathcal{G}^{(\gamma|\mathrm{time})}$ can be found in equations (B38), (B39), (B40), and (B41) of [12].

5.2. Linearized gravitation

Linearized gravity coupled to the isolated astrophysical sources in cosmology is, in fact, a more sophisticated system, due to the additional first-order perturbations of the very fluid driving cosmic expansion. A detailed analysis of this linearized gravitational system, described in terms of the perturbed metric (15), has been performed in [12] within a constant equation-of-state universe, where the background perfect fluid is modeled through an effective field theory description. As explicitly demonstrated in [12], the field equations for the gauge-invariant metric perturbations, deduced from the full linearized Einstein's equations, can be put in a decoupled form with no perturbed fluid quantities involved. The set of resulting equations then reveals that the dynamics of Bardeen scalar potentials varies in different ranges of the equation-of-state w. Here, we will focus on the causal structure of the system in the de Sitter ($w = -1$) and relativistic-fluid ($0 < w \leqslant 1$) cases for all $d \geqslant 4$.

Field equations In terms of the gauge-invariant variables formed in equations (25), (26), and (27), the relevant equations are derived from Einstein's equations of this cosmological system, linearized about the spatially-flat background with a constant equation-of-state w. Based on the results obtained in [12], the character of the decoupled field equation of the Bardeen scalar Ψ for $w = -1$ is distinct from the $0 < w \leqslant 1$ case. Hence, in what follows, we will consider these two cases separately.

Field equations for $w = -1$ When $w = -1$, there is no fluid and the background geometry is de Sitter spacetime. The corresponding gauge-invariant equations are given by equations (28) and (151), both of which remain unchanged, and

$$(d-2)\vec{\nabla}^2\Psi = 8\pi G_{\mathrm{N}}\left(\rho + (d-1)\mathcal{H}\Sigma\right), \tag{226}$$

$$\begin{aligned} -\ddot{D}_{ij} - (d-2)\mathcal{H}\dot{D}_{ij} + \vec{\nabla}^2 D_{ij} &= a^2\,\overline{\Box}^{(\mathrm{S})} D_{ij} \\ &\equiv \frac{a^2\,\partial_\mu\left(\sqrt{|\overline{g}|}\,\overline{g}^{\mu\nu}\partial_\nu D_{ij}\right)}{\sqrt{|\overline{g}|}} = -16\pi G_{\mathrm{N}}\sigma_{ij}, \end{aligned} \tag{227}$$

where $\overline{g}$ denotes the determinant of the background metric $\overline{g}_{\mu\nu} = a^2\eta_{\mu\nu}$, the scale factor is $a[\eta] = -1/(H\eta)$, with H denoting the constant Hubble parameter, and the conformal Hubble parameter reads $\mathcal{H} = -1/\eta$. Compared with their Minkowski counterparts (150) and (152), both Ψ and D_{ij} retain similar dynamics in the de Sitter case here; the Bardeen scalar Ψ still obeys a Poisson-type equation, sourced not only by the local energy density ρ, but also by the non-local longitudinal part Σ of ${}^{(\mathrm{a})}T_{0i}$, while the spin-2 field D_{ij} obeys a tensor wave equation in de Sitter background.

Field equations for $0 < w \leqslant 1$ For a physical relativistic equation-of-state within the range $0 < w \leqslant 1$, the field equations (28), (151), and (227) still hold with $\mathcal{H} = 2/(q_w\eta)$—recall equation (16)—but the Bardeen scalar Ψ now obeys a dynamical wave-like equation [12], instead of being governed by a Poisson-type one[21],

[21] The left-hand side of equation (228) can also be re-expressed in terms of the d'Alembertian ${}^{(\Psi)}\Box$ associated with

$${}^{(\Psi)}g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \left(\frac{\eta}{\eta_0}\right)^{\frac{4(q_w+d-2)}{(d-2)q_w}}\left(-\mathrm{d}\eta^2 + w^{-1}\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}\right).$$

See equation (128) of [12].

$$-\ddot{\Psi} - \left(q_w + d - 2\right)\mathcal{H}\dot{\Psi} + w\vec{\nabla}^2\Psi = -8\pi G_{\mathrm{N}}\left(\frac{\partial_0\left(a^{d-2}\Sigma\right)}{(d-2)a^{d-2}} - \frac{w\rho}{(d-2)} + \mathcal{H}\dot{\Upsilon}\right), \tag{228}$$

which implies the existence of the acoustic cone, $\sqrt{w}\,T = R$, on which these scalar gravitational signals propagate at speed $\sqrt{w}$. It is worth noting that there exists no counterpart of this phenomenon in Minkowski and de Sitter spacetimes; this change in the character of the scalar equation for Ψ is presumably tied to the dynamics of the background fluid.

In addition, as we have already noticed from our previous calculations, the energy-momentum conservation of the astrophysical sources, $\overline{\nabla}^{\mu}\,{}^{(\mathrm{a})}T_{\mu\nu} = 0$, where $\overline{\nabla}_{\mu}$ is the covariant derivative associated with the background metric, will be crucial in extracting the relevant effective Green's functions and their causal structures. For later convenience, in spatially-flat cosmologies, the conservation law can be re-expressed as

$$\partial_i {}^{(\mathrm{a})}T_{ij} = \frac{\partial_0\left(a^{d-2}\,{}^{(\mathrm{a})}T_{0j}\right)}{a^{d-2}}, \tag{229}$$

$$\partial_j {}^{(\mathrm{a})}T_{0j} = \frac{\partial_0\left(a^{d-2}\,{}^{(\mathrm{a})}T_{00}\right)}{a^{d-2}} - \mathcal{H}\left({}^{(\mathrm{a})}T_{00} - {}^{(\mathrm{a})}T_{ll}\right). \tag{230}$$

Spin-2 gravitons The spin-2 wave equation for either $w = -1$ or $0 < w \leqslant 1$ takes the same form as equation (227) with $\mathcal{H} = 2/(q_w\eta)$; therefore, for both cases, the method used in the previous photon computation can be applied directly to solving equation (227) via the spin-2 effective Green's function convolved against the local matter sources.

The first step is to re-cast the tensor wave equation (227) into a conformal re-scaled form,

$$\left\{\partial^2 + \frac{(d-2)\left(d-2-q_w\right)}{q_w^2\eta^2}\right\}\left(a^{\frac{d-2}{2}}D_{ij}\right) = -16\pi G_{\mathrm{N}} a^{\frac{d-2}{2}}\sigma_{ij}, \tag{231}$$

and then, a similar procedure of implementing the local Fourier-space projection of equation (231) leads us to the following convolution for the spin-2 gravitons,

$$a[\eta]^{\frac{d-2}{2}}D_{ij}[\eta,\vec{x}] = -16\pi G_{\mathrm{N}}\int_{\mathbb{R}^{d-1}}\mathrm{d}^{d-1}\vec{x}'\int_{\eta_{\mathrm{p}}}^{\eta_{\mathrm{f}}}\mathrm{d}\eta'\,a[\eta']^{\frac{d-2}{2}}G^{(g,+)}_{ijmn}[\eta,\eta';R]\,{}^{(\mathrm{a})}T_{mn}[\eta',\vec{x}'], \tag{232}$$

where the time interval $(\eta_{\mathrm{p}},\eta_{\mathrm{f}})$ corresponds to $(-\infty,0)$ for $w=-1$ and $(0,\infty)$ for $0 < w \leqslant 1$, and $G^{(g,+)}_{ijmn}$ refers to the spin-2 effective Green's function,

$$\begin{aligned}
G^{(g,+)}_{ijmn}[\eta,\eta';\vec{R}] &= -\Theta[T]C^{(g)}_{ijmn}[\eta,\eta';\vec{R}],\\
C^{(g)}_{ijmn}[\eta,\eta';\vec{R}] &= \left(\delta_{m(i}\delta_{j)n} - \frac{\delta_{ij}\delta_{mn}}{d-2}\right)C^{(g)}_{1,d}[\eta,\eta';R] + \left(\delta_{m(i}\partial_{j)}\partial_n + \delta_{n(i}\partial_{j)}\partial_m\right.\\
&\quad \left. - \frac{\delta_{ij}\partial_m\partial_n - \delta_{mn}\partial_i\partial_j}{d-2}\right)C^{(g)}_{2,d}[\eta,\eta';R] + \left(\frac{d-3}{d-2}\right)\partial_i\partial_j\partial_m\partial_n C^{(g)}_{3,d}[\eta,\eta';R],
\end{aligned} \tag{233}$$

which has the same tensor structure as equation (155), with the scalar functions $C^{(g)}_{1,d}$, $C^{(g)}_{2,d}$, and $C^{(g)}_{3,d}$ generalized to their cosmological versions,

$$C^{(g)}_{1,d}[\eta,\eta';R] \equiv \int_{\mathbb{R}^{d-1}}\frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}}\,\widetilde{C}^{(g)}_{1,d}\left[\eta,\eta';|\vec{k}|\right]\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{R}}, \tag{234}$$

$$C^{(g)}_{2,d}[\eta,\eta';R] \equiv \int_{\mathbb{R}^{d-1}}\frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}}\,\widetilde{C}^{(g)}_{1,d}\left[\eta,\eta';|\vec{k}|\right]\,\frac{\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{R}}}{\vec{k}^2}, \tag{235}$$

$$C^{(g)}_{3,d}[\eta,\eta';R] \equiv \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}} \widetilde{C}^{(g)}_{1,d}\left[\eta,\eta';|\vec{k}|\right] \frac{\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{R}}}{\vec{k}^4}; \tag{236}$$

the Fourier transform of $C^{(g)}_{1,d}$ is denoted by $\widetilde{C}^{(g)}_{1,d}$. For $G^{(g,+)}_{ijmn}$ to be a Green's function, the $C^{(g)}$s in equations (233) must obey the homogeneous version of equation (231). This implies, for instance,

$$\widetilde{C}^{(g)}_{1,d}\left[\eta,\eta';|\vec{k}|\right] = \mathrm{i}\left(v^{(g)}_{|\vec{k}|}[\eta]\, v^{(g)*}_{|\vec{k}|}[\eta'] - v^{(g)*}_{|\vec{k}|}[\eta]\, v^{(g)}_{|\vec{k}|}[\eta']\right), \tag{237}$$

where $v^{(g)}_{|\vec{k}|}$ are the mode functions obeying the same homogeneous wave equation as $\widetilde{C}^{(g)}_{1,d}$, and are normalized to match the Wronskian condition $v^{(g)}_{|\vec{k}|}\dot{v}^{(g)*}_{|\vec{k}|} - v^{(g)*}_{|\vec{k}|}\dot{v}^{(g)}_{|\vec{k}|} = i$, or the initial condition $\dot{\widetilde{C}}^{(g)}_{1,d}\big|_{\eta=\eta'} = 1$. Furthermore, because their Fourier transforms indicate $-\vec{\nabla}^2 C^{(g)}_{3,d} = C^{(g)}_{2,d}$ and $-\vec{\nabla}^2 C^{(g)}_{2,d} = C^{(g)}_{1,d}$, the homogeneous equations for $C^{(g)}_{1,d}$, $C^{(g)}_{2,d}$, and $C^{(g)}_{3,d}$ translate to the relations:

$$\frac{-\partial_\eta\left(a^{2\alpha}\,\partial_\eta\left(a^{-\alpha}\,C^{(g)}_{2,d}\right)\right)}{a^\alpha} = C^{(g)}_{1,d}, \tag{238}$$

$$\frac{-\partial_\eta\left(a^{2\alpha}\,\partial_\eta\left(a^{-\alpha}\,C^{(g)}_{3,d}\right)\right)}{a^\alpha} = C^{(g)}_{2,d}, \qquad \alpha = \frac{d-2}{2}. \tag{239}$$

We may now apply the time-integral method here to relate $C^{(g)}_{2,d}$ and $C^{(g)}_{3,d}$ to $C^{(g)}_{1,d}$, without evaluating their Fourier transform integrals, since $C^{(g)}_{1,d}$ itself has already been derived in [12] (see equations (112) and (113) therein). As we shall witness, this will also yield a clean elucidation of their causal structure. To this end, note that the initial conditions for the $C^{(g)}$s at $\eta=\eta'$ may be identified from their Fourier transforms and the anti-symmetric nature of the mode decomposition in equation (237):

$$C^{(g)}_{1,d}\big|_{\eta=\eta'} = C^{(g)}_{2,d}\big|_{\eta=\eta'} = C^{(g)}_{3,d}\big|_{\eta=\eta'} = 0, \tag{240}$$

$$\dot{C}^{(g)}_{1,d}\big|_{\eta=\eta'} = -\partial_{\eta'}C^{(g)}_{1,d}\big|_{\eta=\eta'} = \delta^{(d-1)}[\vec{x}-\vec{x}'], \tag{241}$$

$$\dot{C}^{(g)}_{2,d}\big|_{\eta=\eta'} = -\partial_{\eta'}C^{(g)}_{2,d}\big|_{\eta=\eta'} = -G^{(\mathrm{E})}_d[R], \tag{242}$$

$$\dot{C}^{(g)}_{3,d}\big|_{\eta=\eta'} = -\partial_{\eta'}C^{(g)}_{3,d}\big|_{\eta=\eta'} = D_d[R]. \tag{243}$$

With equations (240), (242) and (243) imposed, $C^{(g)}_{2,d}$ and $C^{(g)}_{3,d}$ can both be expressed in terms of (the known) $C^{(g)}_{1,d}$ by integrating equations (238) and (239),

$$\begin{aligned} C^{(g)}_{2,d}[\eta,\eta';R] = &- a[\eta]^{\frac{d-2}{2}} \int_{\eta'}^{\eta} \mathrm{d}\eta_2\, a[\eta_2]^{-(d-2)} \int_{\eta'}^{\eta_2} \mathrm{d}\eta_1\, a[\eta_1]^{\frac{d-2}{2}}\, C^{(g)}_{1,d}[\eta_1,\eta';R] \\ &- G^{(\mathrm{E})}_d[R]\left(\frac{a[\eta]}{a[\eta']}\right)^{\frac{d-2}{2}} \int_{\eta'}^{\eta} \mathrm{d}\eta_1 \left(\frac{a[\eta']}{a[\eta_1]}\right)^{d-2}, \end{aligned} \tag{244}$$

$$C_{3,d}^{(g)}[\eta,\eta';R] = a[\eta]^{\frac{d-2}{2}} \int_{\eta'}^{\eta} \mathrm{d}\eta_4\, a[\eta_4]^{-(d-2)} \int_{\eta'}^{\eta_4} \mathrm{d}\eta_3\, a[\eta_3]^{d-2} \int_{\eta'}^{\eta_3} \mathrm{d}\eta_2\, a[\eta_2]^{-(d-2)} \int_{\eta'}^{\eta_2} \mathrm{d}\eta_1\, a[\eta_1]^{\frac{d-2}{2}} C_{1,d}^{(g)}[\eta_1,\eta';R]$$
$$+ G_d^{(\mathrm{E})}[R] \left(\frac{a[\eta]}{a[\eta']}\right)^{\frac{d-2}{2}} \int_{\eta'}^{\eta} \mathrm{d}\eta_3\, a[\eta_3]^{-(d-2)} \int_{\eta'}^{\eta_3} \mathrm{d}\eta_2\, a[\eta_2]^{d-2} \int_{\eta'}^{\eta_2} \mathrm{d}\eta_1 \left(\frac{a[\eta']}{a[\eta_1]}\right)^{d-2}$$
$$+ D_d[R] \left(\frac{a[\eta]}{a[\eta']}\right)^{\frac{d-2}{2}} \int_{\eta'}^{\eta} \mathrm{d}\eta_1 \left(\frac{a[\eta']}{a[\eta_1]}\right)^{d-2}. \tag{245}$$

Although the integrals in the first lines of equations (244) and (245) can be difficult to carry out, just like equation (218) for $C_{2,d}^{(\gamma)}$, the causal structures of these expressions can still be readily identified. Firstly, the retarded portion of $C_{1,d}^{(g)}$, or equivalently $G_d^{(g,+)} = -\Theta[T]C_{1,d}^{(g)}$, is composed only of the causal signals [12]. Then, as already discussed in section 4, the first terms of both equations (244) and (245) are causal as well when $\eta > \eta'$; whereas the remaining terms of $C_{2,d}^{(g)}$ and $C_{3,d}^{(g)}$, associated with their initial conditions, being non-zero for all $\eta > \eta'$, admit contributions from outside the light cone. For this reason, we see that, after plugging equations (244) and (245) into equation (233), the spin-2 effective Green's function $G_{ijmn}^{(g,+)}$ is acausal for all $d \geqslant 4$.

At the quantum level, therefore, the free massless spin-2 operator D_{ij} necessarily violates micro-causality in spatially flat cosmologies:

$$[D_{ij}[x], D_{mn}[x']] = -\mathrm{i}C_{ijmn}^{(g)}[\eta,\eta';\vec{R}]. \tag{246}$$

For later convenience, the expression (232) for spin-2 gravitons can be re-cast into another form analogous to their Minkowski counterpart (182),

$$D_{ij}[\eta,\vec{x}] = -16\pi G_{\mathrm{N}} \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}' \int_{\eta_{\mathrm{p}}}^{\eta_{\mathrm{f}}} \mathrm{d}\eta' \left(\frac{a[\eta']}{a[\eta]}\right)^{\frac{d-2}{2}} \Bigg\{ G_d^{(g,+)} \left({}^{(\mathrm{a})}T_{ij}[\eta',\vec{x}'] - \frac{\delta_{ij}}{d-2} {}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}'] \right)$$
$$+ 2\Theta[T]a[\eta']^{\frac{d-2}{2}} \partial_{\eta'} \left(a[\eta']^{-\frac{d-2}{2}} \partial_{(i} C_{2,d}^{(g)} \right) {}^{(\mathrm{a})}T_{j)0}[\eta',\vec{x}'] + \frac{\delta_{ij}}{d-2} \Theta[T] a[\eta']^{\frac{d-2}{2}}$$
$$\times \left(\partial_{\eta'}^2 \left(a[\eta']^{-\frac{d-2}{2}} C_{2,d}^{(g)} \right) {}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] + \mathcal{H}[\eta'] \partial_{\eta'} \left(a[\eta']^{-\frac{d-2}{2}} C_{2,d}^{(g)} \right) \left({}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] - {}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}'] \right) \right)$$
$$+ \frac{1}{d-2} \Theta[T] \partial_i \partial_j C_{2,d}^{(g)} {}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}'] - \left(\frac{d-3}{d-2}\right) \Theta[T] a[\eta']^{\frac{d-2}{2}} \Bigg(\partial_{\eta'}^2 \left(a[\eta']^{-\frac{d-2}{2}} \partial_i \partial_j C_{3,d}^{(g)} \right) {}^{(\mathrm{a})}T_{00}[\eta',\vec{x}']$$
$$+ \mathcal{H}[\eta'] \partial_{\eta'} \left(a[\eta']^{-\frac{d-2}{2}} \partial_i \partial_j C_{3,d}^{(g)} \right) \left({}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] - {}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}'] \right) \Bigg) \Bigg\}$$
$$+ \frac{16\pi G_{\mathrm{N}}}{d-2} \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}' \left(\delta_{ij} G_d^{(\mathrm{E})} {}^{(\mathrm{a})}T_{00}[\eta,\vec{x}'] + (d-3)\partial_i \partial_j D_d {}^{(\mathrm{a})}T_{00}[\eta,\vec{x}'] \right), \tag{247}$$

where ${}^{(\mathrm{a})}T_{ll} \equiv \delta^{ij}\, {}^{(\mathrm{a})}T_{ij}$, and we have used the conservation laws given in equations (229) and (230) as well as the initial conditions (240), (242) and (243), and removed all the boundary contributions that result from integrations by parts[22]. The convolution in equation (247) now involves different components of ${}^{(\mathrm{a})}T_{\mu\nu}$ from that in equation (232). Below, the former would

[22] Similar to the spin-1 case, the surface integrals upon integration by parts always involve $C_{2,d}^{(g)}$ and $C_{3,d}^{(g)}$; therefore, when evaluated at past infinity, i.e. $\eta' = -\infty$ for $w = -1$ or $\eta' = 0$ for $0 < w \leqslant 1$, those terms obey the spin-2 homogeneous wave equation and will not change the inhomogeneous solution obtained here.

help us identify how the acausal portions of the spin-2 contribution to the Weyl tensor are canceled by those from other gauge-invariant variables.

Bardeen scalars Unlike the spin-2 gravitons, the field equation for the Bardeen scalar Ψ no longer takes an universal form for both $w = -1$ and $0 < w \leqslant 1$, which will be solved separately for each case in the following. Once the solution of Ψ is obtained, the other Bardeen scalar potential Φ, related to Ψ via the formula in equation (28), is given immediately by equation (175).

Solutions for $w = -1$ In de Sitter background ($w = -1$), the Poisson-type equation (226) obeyed by Ψ involves a non-local function Σ of the matter source (see equation (21)), whose momentum-space counterpart for $\vec{k} \neq \vec{0}$ reads

$$\widetilde{\Sigma}[\eta, \vec{k}] = \frac{k_j}{i\vec{k}^2}\,{}^{(\mathrm{a})}\widetilde{T}_{0j}[\eta, \vec{k}]. \tag{248}$$

Once again, the solution to equation (226) can be readily derived by implementing the Fourier transform of equation (226) with equation (248),

$$\Psi[\eta, \vec{x}] = \frac{8\pi G_{\mathrm{N}}}{d-2} \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}' \left(G_d^{(\mathrm{E})}[R]\,{}^{(\mathrm{a})}T_{00}[\eta, \vec{x}'] + (d-1)\mathcal{H}[\eta]\, \partial_j D_d[R]\,{}^{(\mathrm{a})}T_{0j}[\eta, \vec{x}'] \right), \tag{249}$$

which is again the weighted superposition of local source terms over the equal-time hypersurface.

Solutions for $0 < w \leqslant 1$ When $0 < w \leqslant 1$, the Bardeen scalar Ψ becomes dynamical in the sense of being governed by the wave equation (228), from which we see that the propagation of scalar GWs is in general different than that of the spin-2 ones. However, the strategy of solving the spin-2 wave equation in light of causality still applies here for Ψ-waves.

In the same vein, the scalar wave equation (228) can firstly be re-written as a re-scaling form,

$$\begin{aligned} &\left\{ \partial^2_{(w)} + \frac{(d-2)\,(q_w + d - 2)}{q_w^2 \eta^2} \right\} \left(a^{\frac{1}{2}(q_w+d-2)} \Psi \right) \\ &\qquad = -8\pi G_{\mathrm{N}}\, a^{\frac{1}{2}(q_w+d-2)} \left(\frac{\partial_0\left(a^{d-2}\Sigma\right)}{(d-2)a^{d-2}} - \frac{w\rho}{(d-2)} + \mathcal{H}\dot{\Upsilon} \right), \end{aligned} \tag{250}$$

where $\partial^2_{(w)} \equiv -\partial_\eta^2 + w\vec{\nabla}^2$. Then, using equations (174) and (248) through the same procedure employed for the spin-2 wave equation, we have

$$\begin{aligned} a[\eta]^{\frac{1}{2}(q_w+d-2)}\Psi[\eta, \vec{x}] &= \frac{8\pi G_{\mathrm{N}}}{d-2} \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}' \int_0^\infty \mathrm{d}\eta'\, \Theta[T] a[\eta']^{\frac{1}{2}(q_w+d-2)} \\ &\times w^{-\frac{d-3}{2}} \Bigg\{ -\partial_j C^{(w)}_{2,d}\left[\eta, \eta'; \tfrac{R}{\sqrt{w}}\right] a[\eta']^{-(d-2)} \partial_{\eta'} \left(a[\eta']^{d-2}\,{}^{(\mathrm{a})}T_{0j}[\eta', \vec{x}'] \right) - C^{(w)}_{1,d}\left[\eta, \eta'; \tfrac{R}{\sqrt{w}}\right]{}^{(\mathrm{a})}T_{00}[\eta', \vec{x}'] \\ &+ \mathcal{H}[\eta'] \left((d-1)w\, \partial_i \partial_j C^{(w)}_{3,d}\left[\eta, \eta'; \tfrac{R}{\sqrt{w}}\right]{}^{(\mathrm{a})}\dot{T}_{ij}[\eta', \vec{x}'] + C^{(w)}_{2,d}\left[\eta, \eta'; \tfrac{R}{\sqrt{w}}\right]{}^{(\mathrm{a})}\dot{T}_{ll}[\eta', \vec{x}'] \right) \Bigg\}, \end{aligned} \tag{251}$$

where the scalar functions $C^{(w)}_{1,d}$, $C^{(w)}_{2,d}$, and $C^{(w)}_{3,d}$, respectively, are defined in a manner similar to equations (234), (235), and (236),

$$C^{(w)}_{1,d}\left[\eta, \eta'; \tfrac{R}{\sqrt{w}}\right] \equiv w^{\frac{d-1}{2}} \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}}\, \widetilde{C}^{(w)}_{1,d}\left[\eta, \eta'; \sqrt{w}|\vec{k}|\right] \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{R}}, \tag{252}$$

$$C_{2,d}^{(w)}\left[\eta,\eta';\frac{R}{\sqrt{w}}\right]\equiv w^{\frac{d-3}{2}}\int_{\mathbb{R}^{d-1}}\frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}}\,\widetilde{C}_{1,d}^{(w)}\left[\eta,\eta';\sqrt{w}|\vec{k}|\right]\,\frac{\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{R}}}{\vec{k}^2},\tag{253}$$

$$C_{3,d}^{(w)}\left[\eta,\eta';\frac{R}{\sqrt{w}}\right]\equiv w^{\frac{d-5}{2}}\int_{\mathbb{R}^{d-1}}\frac{\mathrm{d}^{d-1}\vec{k}}{(2\pi)^{d-1}}\,\widetilde{C}_{1,d}^{(w)}\left[\eta,\eta';\sqrt{w}|\vec{k}|\right]\,\frac{\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{R}}}{\vec{k}^4},\tag{254}$$

in which $\widetilde{C}_{1,d}^{(w)}$ represents the Fourier transform of $C_{1,d}^{(w)}$ with respect to $\vec{R}/\sqrt{w}$, and it obeys the Fourier-transformed homogeneous wave equation of equation (250) with initial conditions specified by $\widetilde{C}_{1,d}^{(w)}\big|_{\eta=\eta'}=0$ and $\dot{\widetilde{C}}_{1,d}^{(w)}\big|_{\eta=\eta'}=-\partial_{\eta'}\widetilde{C}_{1,d}^{(w)}\big|_{\eta=\eta'}=1$[23]. Analogous to equations (212) and (237), $\widetilde{C}_{1,d}^{(w)}$ admits the following decomposition in terms of the mode functions $v_{\sqrt{w}|\vec{k}|}^{(w)}$ that obey the same homogeneous wave equation in momentum space,

$$\widetilde{C}_{1,d}^{(w)}\left[\eta,\eta';\sqrt{w}|\vec{k}|\right]=\mathrm{i}\left(v_{\sqrt{w}|\vec{k}|}^{(w)}[\eta]\,v_{\sqrt{w}|\vec{k}|}^{(w)*}[\eta']-v_{\sqrt{w}|\vec{k}|}^{(w)*}[\eta]\,v_{\sqrt{w}|\vec{k}|}^{(w)}[\eta']\right),\tag{255}$$

where the Wronskian condition, $v_{\sqrt{w}|\vec{k}|}^{(w)}\dot{v}_{\sqrt{w}|\vec{k}|}^{(w)*}-v_{\sqrt{w}|\vec{k}|}^{(w)*}\dot{v}_{\sqrt{w}|\vec{k}|}^{(w)}=i$, is fulfilled to be consistent with the initial condition $\dot{\widetilde{C}}_{1,d}^{(w)}\big|_{\eta=\eta'}=1$. By construction, these $C^{(w)}$s are solutions to the homogeneous version of equation (250). Moreover, their equal-time initial conditions may be readily identified,

$$C_{1,d}^{(w)}\big|_{\eta=\eta'}=C_{2,d}^{(w)}\big|_{\eta=\eta'}=C_{3,d}^{(w)}\big|_{\eta=\eta'}=0,\tag{256}$$

$$\dot{C}_{1,d}^{(w)}\big|_{\eta=\eta'}=-\partial_{\eta'}C_{1,d}^{(w)}\big|_{\eta=\eta'}=w^{\frac{d-1}{2}}\delta^{(d-1)}[\vec{x}-\vec{x}'],\tag{257}$$

$$\dot{C}_{2,d}^{(w)}\big|_{\eta=\eta'}=-\partial_{\eta'}C_{2,d}^{(w)}\big|_{\eta=\eta'}=-w^{\frac{d-3}{2}}G_d^{(\mathrm{E})}[R],\tag{258}$$

$$\dot{C}_{3,d}^{(w)}\big|_{\eta=\eta'}=-\partial_{\eta'}C_{3,d}^{(w)}\big|_{\eta=\eta'}=w^{\frac{d-5}{2}}D_d[R].\tag{259}$$

Exploiting their Fourier representations to observe that $-w\vec{\nabla}^2C_{2,d}^{(w)}=C_{1,d}^{(w)}$ and $-w\vec{\nabla}^2C_{3,d}^{(w)}=C_{2,d}^{(w)}$, we see that the homogeneous cousins of equation (250) are

$$\frac{-\partial_\eta\left(a^{2\alpha}\,\partial_\eta\left(a^{-\alpha}\,C_{2,d}^{(w)}\right)\right)}{a^\alpha}=C_{1,d}^{(w)},\tag{260}$$

$$\frac{-\partial_\eta\left(a^{2\alpha}\,\partial_\eta\left(a^{-\alpha}\,C_{3,d}^{(w)}\right)\right)}{a^\alpha}=C_{2,d}^{(w)},\qquad\alpha=-\frac{d-2}{2}.\tag{261}$$

[23] The factor of w appearing in equation (252) has been arranged such that the corresponding massless scalar Green's function, $G_d^{(w,+)}=-\Theta[T]C_{1,d}^{(w)}$ with re-scaled coordinates $(\eta,\vec{y})\equiv(\eta,\vec{x}/\sqrt{w})$, obeys the wave equation

$$\left\{-\partial_\eta^2+\frac{(d-2)\,(q_w+d-2)}{q_w^2\eta^2}+\vec{\nabla}_{\vec{y}}^2\right\}G_d^{(w,+)}[\eta,\eta';|\vec{y}-\vec{y}'|]=\delta[\eta-\eta']\delta^{(d-1)}[\vec{y}-\vec{y}'],$$

where $\vec{\nabla}_{\vec{y}}^2$ is the spatial Laplacian with respect to $\vec{y}$. The analytic solution of $G_d^{(w,+)}$ for $0<w\leqslant 1$ has been obtained in [12].

At this point we may integrate these equations to express $C^{(w)}_{2,d}$ and $C^{(w)}_{3,d}$ in terms of $C^{(w)}_{1,d}$, which had been derived analytically in [12] (see equations (131) and (132) of [12]). The resulting expressions are

$$\begin{aligned} C^{(w)}_{2,d}\left[\eta,\eta';\tfrac{R}{\sqrt{w}}\right] =& - a[\eta]^{-\frac{d-2}{2}}\int_{\eta'}^{\eta}\mathrm{d}\eta_2\, a[\eta_2]^{d-2}\int_{\eta'}^{\eta_2}\mathrm{d}\eta_1\, a[\eta_1]^{-\frac{d-2}{2}}C^{(w)}_{1,d}\left[\eta_1,\eta';\tfrac{R}{\sqrt{w}}\right] \\ &- w^{\frac{d-3}{2}}G^{(\mathrm{E})}_d[R]\left(\frac{a[\eta']}{a[\eta]}\right)^{\frac{d-2}{2}}\int_{\eta'}^{\eta}\mathrm{d}\eta_1\left(\frac{a[\eta_1]}{a[\eta']}\right)^{d-2}, \end{aligned} \tag{262}$$

$$\begin{aligned} C^{(w)}_{3,d}\left[\eta,\eta';\tfrac{R}{\sqrt{w}}\right] =& a[\eta]^{-\frac{d-2}{2}}\int_{\eta'}^{\eta}\mathrm{d}\eta_4\, a[\eta_4]^{(d-2)}\int_{\eta'}^{\eta_4}\mathrm{d}\eta_3\, a[\eta_3]^{-(d-2)} \\ &\times\int_{\eta'}^{\eta_3}\mathrm{d}\eta_2\, a[\eta_2]^{(d-2)}\int_{\eta'}^{\eta_2}\mathrm{d}\eta_1\, a[\eta_1]^{-\frac{d-2}{2}}C^{(w)}_{1,d}\left[\eta_1,\eta';\tfrac{R}{\sqrt{w}}\right] \\ &+ w^{\frac{d-3}{2}}G^{(\mathrm{E})}_d[R]\left(\frac{a[\eta']}{a[\eta]}\right)^{\frac{d-2}{2}}\int_{\eta'}^{\eta}\mathrm{d}\eta_3\, a[\eta_3]^{d-2}\int_{\eta'}^{\eta_3}\mathrm{d}\eta_2\, a[\eta_2]^{-(d-2)}\int_{\eta'}^{\eta_2}\mathrm{d}\eta_1\left(\frac{a[\eta_1]}{a[\eta']}\right)^{d-2} \\ &+ w^{\frac{d-5}{2}}D_d[R]\left(\frac{a[\eta']}{a[\eta]}\right)^{\frac{d-2}{2}}\int_{\eta'}^{\eta}\mathrm{d}\eta_1\left(\frac{a[\eta_1]}{a[\eta']}\right)^{d-2}, \end{aligned} \tag{263}$$

which bear close resemblance to equations (244) and (245). The retarded part of $C^{(w)}_{1,d}$, or the massless scalar Green's function $G^{(w,+)}_d = -\Theta[T]C^{(w)}_{1,d}$, has been shown to contain only the causal scalar GW signals propagating either on or within the acoustic cone [12]. Applying the arguments in section 4—the first terms of equations (262) and (263) are causal when $\eta > \eta'$ because $C^{(w)}_{1,d}$ is; while the rest of the terms are non-zero both inside and outside the light cone of $(\eta',\vec{x}')$. The Bardeen scalar Ψ is therefore acausal for all relevant spacetime dimensions (see equation (251)).

Alternatively, we can perform integration-by-parts and employ the energy-momentum conservation laws (229) and (230), as well as the properties of $C^{(w)}_{2,d}$ and $C^{(w)}_{3,d}$, to re-express the effective Green's function of Ψ in equation (251) as

$$\begin{aligned} \Psi[\eta,\vec{x}] =& \frac{8\pi G_{\mathrm{N}}}{d-2}\int_{\mathbb{R}^{d-1}}\mathrm{d}^{d-1}\vec{x}'\int_0^{\infty}\mathrm{d}\eta'\left(\frac{a[\eta']}{a[\eta]}\right)^{\frac{1}{2}(q_w+d-2)} w^{-\frac{d-3}{2}}\Bigg\{G^{(w,+)}_d\,{}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] \\ &- \Theta[T]a[\eta']^{-\frac{1}{2}(q_w+d-2)}\partial_{\eta'}\left(a[\eta']^{\frac{1}{2}(q_w+d-2)}\mathcal{H}[\eta']\,C^{(w)}_{2,d}\right){}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}'] + (d-2)\Theta[T]a[\eta']^{-\frac{1}{2}(q_w-d+2)} \\ &\times\partial_{\eta'}\left(a[\eta']^{\frac{1}{2}(q_w-d+2)}\mathcal{H}[\eta']\,C^{(w)}_{2,d}\right){}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] + (d-2)\Theta[T]\mathcal{H}[\eta']^2\,C^{(w)}_{2,d}\left({}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] - {}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}']\right) \\ &- \Theta[T]a[\eta']^{-\frac{1}{2}(q_w-d+2)}\partial_{\eta'}\left(a[\eta']^{-(d-2)}\partial_{\eta'}\left(a[\eta']^{\frac{1}{2}(q_w+d-2)}\,C^{(w)}_{2,d}\right)\right){}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] \\ &- \Theta[T]a[\eta']^{-\frac{1}{2}(q_w+d-2)}\mathcal{H}[\eta']\partial_{\eta'}\left(a[\eta']^{\frac{1}{2}(q_w+d-2)}\,C^{(w)}_{2,d}\right)\left({}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] - {}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}']\right) \\ &- w(d-1)\Theta[T]a[\eta']^{-\frac{1}{2}(q_w-d+2)}\partial^2_{\eta'}\left(a[\eta']^{-(d-2)}\partial_{\eta'}\left(a[\eta']^{\frac{1}{2}(q_w+d-2)}\mathcal{H}[\eta']\,C^{(w)}_{3,d}\right)\right){}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] \\ &- w(d-1)\Theta[T]a[\eta']^{-\frac{1}{2}(q_w-d+2)}\mathcal{H}[\eta']\partial_{\eta'}\left(a[\eta']^{-(d-2)}\partial_{\eta'}\left(a[\eta']^{\frac{1}{2}(q_w+d-2)}\mathcal{H}[\eta']\,C^{(w)}_{3,d}\right)\right)\left({}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] \right. \\ &\left. - {}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}']\right)\Bigg\} + \frac{8\pi G_{\mathrm{N}}}{d-2}\int_{\mathbb{R}^{d-1}}\mathrm{d}^{d-1}\vec{x}'\left(G^{(\mathrm{E})}_d\,{}^{(\mathrm{a})}T_{00}[\eta,\vec{x}'] + (d-1)\mathcal{H}[\eta]\partial_j D_d\,{}^{(\mathrm{a})}T_{0j}[\eta,\vec{x}']\right), \end{aligned} \tag{264}$$

where all the boundary terms that arise from the integrations by parts have been discarded[24]. This form shows more transparently the convolution with the local matter stress-energy tensor, and will be used for our later analysis of the physical observables.

Vector potential According to [12], the vector mode V_i, in de Sitter space ($w = -1$), obeys the Poisson-type equation (151), while for $0 < w \leqslant 1$, if perturbations are assumed to be negligible in the far past, then the same vector equation, i.e. equation (151), is satisfied as well. Therefore, in both cases, the solution of V_i is that in equation (178).

Linearized Weyl tensor As we have discussed in section 2, the linearized Riemann tensor in the cosmological background is no longer gauge invariant due to its non-zero background value. In cosmological spacetimes, which are conformally flat, the causal and gauge-invariant counterpart to the linearized Riemann in flat spacetimes is the linearized Weyl tensor $\delta_1 C^{\mu}{}_{\nu\alpha\beta}$. More specifically, since the Weyl tensor $C^{\mu}{}_{\nu\alpha\beta}$ is conformally invariant, it is zero when evaluated on the unperturbed cosmological geometry $g_{\mu\nu} = a^2\eta_{\mu\nu}$ and must therefore be gauge-invariant at first order in $\chi_{\mu\nu}$. Furthermore, its exact wave equation is simply the traceless part of equation (179); but since it is zero at zeroth order, the first order Weyl tensor $\delta_1 C^{\mu}{}_{\nu\alpha\beta}$ must therefore obey an equation involving the wave operator with respect to the background FLRW metric.

Motivated by these considerations, we shall proceed to calculate

$$\delta_1 C^i{}_{0j0} = \left(\frac{d-3}{d-2}\right)\left\{\left(\partial_i\partial_j - \frac{\delta_{ij}}{d-1}\vec{\nabla}^2\right)(\Phi+\Psi) + \partial_{(i}\dot{V}_{j)} - \frac{1}{2}\left(\ddot{D}_{ij} + \frac{1}{d-3}\vec{\nabla}^2 D_{ij}\right)\right\}. \tag{265}$$

It is likely that $\delta_1 C^i{}_{0j0}$ encodes the dominant contributions to the first-order tidal forces described in equation (39); but we shall leave this analysis to future work [13]. Here, we will instead focus on the causal structure of this quantity with respect to the background spacetime.

Linearized Weyl tensor for $w = -1$ Within the de Sitter case, plugging into equation (265) the solution of D_{ij} in equation (247), those of Ψ and Φ in equations (249) and (175), and that of V_i in equation (178), with $\mathcal{H}[\eta] = -1/\eta$ and $(\eta_{\rm p}, \eta_{\rm f}) = (-\infty, 0)$, we find that, after employing the conservation conditions (229) and (230), the scalars and vector act to cancel the acausal signals from the tensor contributions to Weyl. In more detail,

$$\begin{aligned}\delta_1 C^i{}_{0j0} = &-\frac{1}{2}\left(\frac{d-3}{d-2}\right)\left(\ddot{D}_{ij} + \frac{1}{d-3}\vec{\nabla}^2 D_{ij}\right)_{\rm causal} \\ &+ \frac{8\pi G_{\rm N}}{d-2}\left({}^{(\rm a)}T_{ij} - \frac{\delta_{ij}}{d-1}\left((d-3)\,{}^{(\rm a)}T_{00} + 2\,{}^{(\rm a)}T_{ll}\right)\right);\end{aligned} \tag{266}$$

where the first line of equation (266) denotes the causal part of the spin-2 contributions that depend exclusively on the retarded Green's function $G_d^{(g,+)}[\eta, \eta'; R]$,

[24] As previously reasoned in the spin-1 and spin-2 cases, discarding the boundary contributions at past infinity does not affect the inhomogeneous solutions of the Bardeen scalars, since the corresponding homogeneous wave equation is obeyed by those surface terms, which correspond to evaluating the surface integrals of $C^{(w)}_{2,d}$ and $C^{(w)}_{3,d}$ at $\eta' = -\infty$ (for $w = -1$) or $\eta' = 0$ (for $0 < w \leqslant 1$).

$$\begin{aligned}
&\left(\ddot{D}_{ij}[\eta,\vec{x}] + \frac{1}{d-3}\vec{\nabla}^2 D_{ij}[\eta,\vec{x}]\right)_{\text{causal}} \\
&= -16\pi G_{\text{N}}\left(\frac{d-2}{d-3}\right)\int_{\mathbb{R}^{d-1}}\mathrm{d}^{d-1}\vec{x}'\int_{\eta_{\text{p}}}^{\eta_{\text{f}}}\mathrm{d}\eta'\left(\frac{a[\eta']}{a[\eta]}\right)^{\frac{d-2}{2}}\left\{\left(\ddot{G}_d^{(g,+)} - \frac{2(d-3)}{q_w\eta}\dot{G}_d^{(g,+)}\right.\right. \\
&\quad\left.+\frac{(d-2)\,(q_w+d-4)}{q_w^2\eta^2}G_d^{(g,+)}\right)\left({}^{(\mathrm{a})}T_{ij}[\eta',\vec{x}'] - \frac{\delta_{ij}}{d-2}{}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}']\right) \\
&\quad+2a[\eta']^{\frac{d-2}{2}}\partial_{\eta'}\left(a[\eta']^{-\frac{d-2}{2}}\left(\partial_{(i}G_d^{(g,+)} - \frac{2(d-3)}{q_w\eta}a[\eta]^{-\frac{d-2}{2}}\int_{\eta'}^{\eta}\mathrm{d}\eta_1\,a[\eta_1]^{\frac{d-2}{2}}\partial_{(i}G_d^{(g,+)}\right)\right){}^{(\mathrm{a})}T_{j)0}[\eta',\vec{x}'] \\
&\quad+\frac{\delta_{ij}}{d-2}a[\eta']^{\frac{d-2}{2}}\left(\partial_{\eta'}^2\left(a[\eta']^{-\frac{d-2}{2}}\left(G_d^{(g,+)} - \frac{2(d-3)}{q_w\eta}a[\eta]^{-\frac{d-2}{2}}\int_{\eta'}^{\eta}\mathrm{d}\eta_1\,a[\eta_1]^{\frac{d-2}{2}}G_d^{(g,+)}\right)\right){}^{(\mathrm{a})}T_{00}[\eta',\vec{x}']\right. \\
&\quad+\left.\frac{2}{q_w\eta'}\partial_{\eta'}\left(a[\eta']^{-\frac{d-2}{2}}\left(G_d^{(g,+)} - \frac{2(d-3)}{q_w\eta}a[\eta]^{-\frac{d-2}{2}}\int_{\eta'}^{\eta}\mathrm{d}\eta_1\,a[\eta_1]^{\frac{d-2}{2}}G_d^{(g,+)}\right)\right)\left({}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] - {}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}']\right)\right) \\
&\quad+\frac{1}{d-2}\left(\partial_i\partial_j G_d^{(g,+)} - \frac{2(d-3)}{q_w\eta}a[\eta]^{-\frac{d-2}{2}}\int_{\eta'}^{\eta}\mathrm{d}\eta_1\,a[\eta_1]^{\frac{d-2}{2}}\partial_i\partial_j G_d^{(g,+)}\right){}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}'] \\
&\quad+\left(\frac{d-3}{d-2}\right)a[\eta']^{\frac{d-2}{2}}a[\eta]^{\frac{d-2}{2}}\left(\partial_{\eta'}^2\left(a[\eta']^{-\frac{d-2}{2}}\left(\int_{\eta'}^{\eta}\mathrm{d}\eta_2\,a[\eta_2]^{-(d-2)}\int_{\eta'}^{\eta_2}\mathrm{d}\eta_1\,a[\eta_1]^{\frac{d-2}{2}}\partial_i\partial_j G_d^{(g,+)}\right.\right.\right. \\
&\quad-\left.\left.\frac{2(d-3)}{q_w\eta}a[\eta]^{-(d-2)}\int_{\eta'}^{\eta}\mathrm{d}\eta_3\,a[\eta_3]^{d-2}\int_{\eta'}^{\eta_3}\mathrm{d}\eta_2\,a[\eta_2]^{-(d-2)}\int_{\eta'}^{\eta_2}\mathrm{d}\eta_1\,a[\eta_1]^{\frac{d-2}{2}}\partial_i\partial_j G_d^{(g,+)}\right)\right){}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] \\
&\quad+\frac{2}{q_w\eta'}\partial_{\eta'}\left(a[\eta']^{-\frac{d-2}{2}}\left(\int_{\eta'}^{\eta}\mathrm{d}\eta_2\,a[\eta_2]^{-(d-2)}\int_{\eta'}^{\eta_2}\mathrm{d}\eta_1\,a[\eta_1]^{\frac{d-2}{2}}\partial_i\partial_j G_d^{(g,+)}\right.\right. \\
&\quad-\left.\left.\frac{2(d-3)}{q_w\eta}a[\eta]^{-(d-2)}\int_{\eta'}^{\eta}\mathrm{d}\eta_3\,a[\eta_3]^{d-2}\int_{\eta'}^{\eta_3}\mathrm{d}\eta_2\,a[\eta_2]^{-(d-2)}\int_{\eta'}^{\eta_2}\mathrm{d}\eta_1\,a[\eta_1]^{\frac{d-2}{2}}\partial_i\partial_j G_d^{(g,+)}\right)\right) \\
&\quad\left.\left.\times\left({}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] - {}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}']\right)\right)\right\},
\end{aligned} \tag{267}$$

while the second line of equation (266) consists solely of the stress-energy tensor of the GW source evaluated at the observer location. (Recall that $a[\eta] = -1/(H\eta)$ and $q_w = -2$ in de Sitter spacetime.) As long as the observer at $(\eta, \vec{x})$ is not located at the source, these ${}^{(\mathrm{a})}T_{\mu\nu}[\eta,\vec{x}]$ terms in equation (266) are zero[25].

[25] This calculation is greatly simplified by first using the commutator $C_{1,d}^{(g)}$, and only re-expressing the final result in terms of the massless scalar Green's function via $G_d^{(g,+)} = -\Theta[T]C_{1,d}^{(g)}$ at the very end. In particular, we notice that a local term will show up in the conversion involving a second time derivative, namely $\ddot{G}_d^{(g,+)} = -\delta^{(d)}[x-x'] - \Theta[T]\ddot{C}_{1,d}^{(g)}$ or $\partial_{\eta'}^2 G_d^{(g,+)} = -\delta^{(d)}[x-x'] - \Theta[T]\partial_{\eta'}^2 C_{1,d}^{(g)}$.A simple check of equation (266) can be made by taking the limit of equation (267) as $a \to 1$ and assuming $G_d^{(g,+)}$ takes the form of G_d^+ in Minkowski spacetime. We may then show explicitly that the first line of equation (266) reduces to $\delta_1 R_{0i0j}$ given in equation (181), and the resulting $\delta_1 {C^i}_{0j0}$ is consistent with its Minkowski counterpart obtained from the solutions derived in the last section. Moreover, the Minkowski form of $\delta_1 {C^i}_{0j0}$ also agrees with the relationship between the Riemann and the Weyl tensor,

$$ {C^\rho}_{\sigma\mu\nu} = {R^\rho}_{\sigma\mu\nu} - \frac{16\pi G_{\text{N}}}{d-2}\left(\delta^\rho_{[\mu}T_{\nu]\sigma} - g_{\sigma[\mu}{T_{\nu]}}^\rho - \delta^\rho_{[\mu}g_{\nu]\sigma}\frac{2g^{\alpha\beta}T_{\alpha\beta}}{d-1}\right), \tag{268}$$

linearized about the Minkowski background; where $T_{\mu\nu}$ is the total energy-momentum tensor of matter and Einstein's equation has been imposed on the trace parts of the Riemann tensor.

To sum: the result in equation (266) reveals that $\delta_1 {C^i}_{0j0}$ on a de Sitter background receives only signals from the spin-2 sector, as long as the observer is away from the isolated matter source(s) of GWs.

Analogous to the localization of the effective Green's function of F_{0i} shown in equation (225), the expression of equation (267) in de Sitter spacetime can be further simplified in a localized manner by introducing two additional massless scalar Green's functions $G_d^{(\mathrm{V},+)}$ and $G_d^{(\mathrm{Tr},+)}$ that, respectively, obey the following wave equations,

$$\left\{\partial^2 + \frac{(d-4)(d-2)}{4\eta^2}\right\} G_d^{(\mathrm{V},+)}[\eta,\eta';R] = \delta^{(d)}[x-x'], \tag{269}$$

$$\left\{\partial^2 + \frac{(d-6)(d-4)}{4\eta^2}\right\} G_d^{(\mathrm{Tr},+)}[\eta,\eta';R] = \delta^{(d)}[x-x']. \tag{270}$$

The effective Green's function of $\delta_1 {C^i}_{0j0}$ in equation (266) can then be localized accordingly in terms of $G_d^{(g,+)}$, $G_d^{(\mathrm{V},+)}$, and $G_d^{(\mathrm{Tr},+)}$,

$$\begin{aligned}
\delta_1 {C^i}_{0j0}[\eta,\vec{x}] &= 8\pi G_{\mathrm{N}} \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}' \int_{-\infty}^{0} \mathrm{d}\eta' \left(\frac{\eta}{\eta'}\right)^{\frac{d-2}{2}} \left\{ \left(\ddot{G}_d^{(g,+)} + \frac{(d-3)}{\eta}\dot{G}_d^{(g,+)} + \frac{(d-2)(d-6)}{4\eta^2} G_d^{(g,+)} \right) \right. \\
&\times \left({}^{(\mathrm{a})}T_{ij}[\eta',\vec{x}'] - \frac{\delta_{ij}}{d-2} {}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}'] \right) - 2\eta^{-\frac{d-4}{2}} \partial_\eta \left(\eta^{\frac{d-4}{2}} \partial_{(i} G_d^{(\mathrm{V},+)} \right) {}^{(\mathrm{a})}T_{j)0}[\eta',\vec{x}'] \\
&- \frac{\delta_{ij}}{d-2} \left((\eta\eta')^{-\frac{d-4}{2}} \partial_\eta \partial_{\eta'} \left((\eta\eta')^{\frac{d-4}{2}} G_d^{(\mathrm{V},+)} \right) {}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] + \eta'^{-1} \eta^{-\frac{d-4}{2}} \partial_\eta \left(\eta^{\frac{d-4}{2}} G_d^{(\mathrm{V},+)} \right) {}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}'] \right) \\
&\left. + \frac{1}{d-2} \partial_i \partial_j G_d^{(\mathrm{Tr},+)} \left((d-3)^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] + {}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}'] \right) \right\} \\
&+ \frac{8\pi G_{\mathrm{N}}}{d-2} \left({}^{(\mathrm{a})}T_{ij}[\eta,\vec{x}] - \frac{\delta_{ij}}{d-1} \left((d-3)^{(\mathrm{a})}T_{00}[\eta,\vec{x}] + 2^{(\mathrm{a})}T_{ll}[\eta,\vec{x}] \right) \right),
\end{aligned} \tag{271}$$

which will be explained in more detail in [13]. Equation (271) turns out to be consistent with $\delta_1 {C^i}_{0j0}$ computed from the generalized de Donder gauge $\overline{\chi}_{\mu\nu}$ solution obtained in [14]; the analytic solutions of $G_d^{(g,+)} = -\Theta[T]\mathcal{G}^{(\mathrm{T})}$, $G_d^{(\mathrm{V},+)} = -\Theta[T]\mathcal{G}^{(\mathrm{V})}$, and $G_d^{(\mathrm{Tr},+)} = -\Theta[T]\mathcal{G}^{(\mathrm{Tr})}$ can be found in equations (28), (29), (33), (34), (38), and (39) of [14].

Linearized Weyl tensor for $0 < w \leqslant 1$ To obtain $\delta_1 {C^i}_{0j0}$ for a relativistic equation-of-state w within $0 < w \leqslant 1$, we insert into equation (265) equation (247) for D_{ij}, equations (264) and (175) for Ψ and Φ, and equation (178) for V_i; recalling that $\mathcal{H}[\eta] = 2/(q_w\eta)$ and $(\eta_{\mathrm{p}}, \eta_{\mathrm{f}}) = (0, \infty)$. A direct computation then reveals that an exact cancellation of the acausal signals takes place again in equation (265), so that

$$\begin{aligned}
\delta_1 {C^i}_{0j0} &= \left(\frac{d-3}{d-2}\right) \left\{ \left(\left(\partial_i \partial_j - \frac{\delta_{ij}}{d-1} \vec{\nabla}^2 \right) (\Phi + \Psi) \right)_{\text{causal}} - \frac{1}{2} \left(\ddot{D}_{ij} + \frac{1}{d-3} \vec{\nabla}^2 D_{ij} \right)_{\text{causal}} \right\} \\
&\quad + \frac{8\pi G_{\mathrm{N}}}{d-2} \left({}^{(\mathrm{a})}T_{ij} - \frac{\delta_{ij}}{d-1} \left((d-3)^{(\mathrm{a})}T_{00} + 2^{(\mathrm{a})}T_{ll} \right) \right).
\end{aligned} \tag{272}$$

The causal portion of the spin-2 sector takes precisely the same form as equation (267) but with scale factor given in equation (16) and $(\eta_{\mathrm{p}}, \eta_{\mathrm{f}}) = (0, \infty)$. On the other hand, the causal contributions of the Bardeen scalar potentials in the first line are given by

$$
\begin{aligned}
&\left(\left(\partial_i\partial_j - \frac{\delta_{ij}}{d-1}\vec{\nabla}^2\right)\left(\Phi[\eta,\vec{x}] + \Psi[\eta,\vec{x}]\right)\right)_{\text{causal}} \\
&= -8\pi G_{\rm N}\int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}' \int_0^\infty \mathrm{d}\eta' \left(\frac{a[\eta']}{a[\eta]}\right)^{\frac{1}{2}(q_w+d-2)} w^{-\frac{d-1}{2}} \left\{ \frac{\delta_{ij}}{d-1}\left(\left(\ddot{G}_d^{(w,+)} - \frac{(d-2)\,(q_w+d-2)}{q_w^2\eta^2}\,G_d^{(w,+)}\right.\right.\right. \\
&- a[\eta']^{-\frac{1}{2}(q_w-3d+8)}\partial_{\eta'}\left(a[\eta']^{-(2d-5)}\partial_{\eta'}\left(a[\eta']^{\frac{1}{2}(q_w+d-2)}\,G_d^{(w,+)}\right)\right) - (d-2)\frac{2q_w+4(d-3)}{q_w^2\eta'^2}\,G_d^{(w,+)}\Big)\,{}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] \\
&+ \frac{2q_w-4(d-2)}{q_w^2\eta'^2}\,G_d^{(w,+)}\,{}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}']\Big) + \delta_{ij} w\, a[\eta']^{-\frac{1}{2}(q_w-d+2)}\left(\partial_{\eta'}^2\left(a[\eta']^{-(d-2)}\partial_{\eta'}\left(\frac{2}{q_w\eta'}a[\eta']^{\frac{1}{2}(q_w+d-2)}a[\eta]^{-\frac{d-2}{2}}\right.\right.\right. \\
&\times \int_{\eta'}^{\eta}\mathrm{d}\eta_2\, a[\eta_2]^{(d-2)}\int_{\eta'}^{\eta_2}\mathrm{d}\eta_1\, a[\eta_1]^{-\frac{d-2}{2}}G_d^{(w,+)}\Big)\Big)\Big)\,{}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] + \frac{2}{q_w\eta'}\partial_{\eta'}\left(a[\eta']^{-(d-2)}\partial_{\eta'}\left(\frac{2}{q_w\eta'}a[\eta']^{\frac{1}{2}(q_w+d-2)}\right.\right. \\
&\times a[\eta]^{-\frac{d-2}{2}}\int_{\eta'}^{\eta}\mathrm{d}\eta_2\, a[\eta_2]^{(d-2)}\int_{\eta'}^{\eta_2}\mathrm{d}\eta_1\, a[\eta_1]^{-\frac{d-2}{2}}G_d^{(w,+)}\Big)\Big)\left({}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] - {}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}']\right)\Big) \\
&+ w\left(\left(a[\eta']^{-\frac{1}{2}(q_w-3d+8)}\partial_{\eta'}\left(a[\eta']^{-(2d-5)}\partial_{\eta'}\left(a[\eta']^{\frac{1}{2}(q_w+d-2)}a[\eta]^{-\frac{d-2}{2}}\int_{\eta'}^{\eta}\mathrm{d}\eta_2\, a[\eta_2]^{(d-2)}\int_{\eta'}^{\eta_2}\mathrm{d}\eta_1\, a[\eta_1]^{-\frac{d-2}{2}}\partial_i\partial_j G_d^{(w,+)}\right)\right)\right.\right. \\
&+ (d-2)\frac{2q_w+4(d-3)}{q_w^2\eta'^2}a[\eta]^{-\frac{d-2}{2}}\int_{\eta'}^{\eta}\mathrm{d}\eta_2\, a[\eta_2]^{(d-2)}\int_{\eta'}^{\eta_2}\mathrm{d}\eta_1\, a[\eta_1]^{-\frac{d-2}{2}}\partial_i\partial_j G_d^{(w,+)}\Big)\,{}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] \\
&- \frac{2q_w-4(d-2)}{q_w^2\eta'^2}a[\eta]^{-\frac{d-2}{2}}\int_{\eta'}^{\eta}\mathrm{d}\eta_2\, a[\eta_2]^{(d-2)}\int_{\eta'}^{\eta_2}\mathrm{d}\eta_1\, a[\eta_1]^{-\frac{d-2}{2}}\partial_i\partial_j G_d^{(w,+)}\,{}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}']\Big) \\
&- w\,\partial_i\partial_j G_d^{(w,+)}\,{}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] - (d-1)w^2 a[\eta']^{-\frac{1}{2}(q_w-d+2)}\left(\partial_{\eta'}^2\left(a[\eta']^{-(d-2)}\partial_{\eta'}\left(\frac{2}{q_w\eta'}\,a[\eta']^{\frac{1}{2}(q_w+d-2)}a[\eta]^{-\frac{d-2}{2}}\right.\right.\right. \\
&\times \int_{\eta'}^{\eta}\mathrm{d}\eta_4\, a[\eta_4]^{(d-2)}\int_{\eta'}^{\eta_4}\mathrm{d}\eta_3\, a[\eta_3]^{-(d-2)}\int_{\eta'}^{\eta_3}\mathrm{d}\eta_2\, a[\eta_2]^{(d-2)}\int_{\eta'}^{\eta_2}\mathrm{d}\eta_1\, a[\eta_1]^{-\frac{d-2}{2}}\partial_i\partial_j G_d^{(w,+)}\Big)\Big)\,{}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] \\
&+ \frac{2}{q_w\eta'}\partial_{\eta'}\left(a[\eta']^{-(d-2)}\partial_{\eta'}\left(\frac{2}{q_w\eta'}\,a[\eta']^{\frac{1}{2}(q_w+d-2)}a[\eta]^{-\frac{d-2}{2}}\int_{\eta'}^{\eta}\mathrm{d}\eta_4\, a[\eta_4]^{(d-2)}\int_{\eta'}^{\eta_4}\mathrm{d}\eta_3\, a[\eta_3]^{-(d-2)}\int_{\eta'}^{\eta_3}\mathrm{d}\eta_2\, a[\eta_2]^{(d-2)}\right.\right. \\
&\times \int_{\eta'}^{\eta_2}\mathrm{d}\eta_1\, a[\eta_1]^{-\frac{d-2}{2}}\partial_i\partial_j G_d^{(w,+)}\Big)\Big)\left({}^{(\mathrm{a})}T_{00}[\eta',\vec{x}'] - {}^{(\mathrm{a})}T_{ll}[\eta',\vec{x}']\right)\Big)\Big\}. \qquad (273)
\end{aligned}
$$

Observe that equation (273) is fully determined by the massless scalar Green's function $G_d^{(w,+)}\left[\eta,\eta';\frac{R}{\sqrt{w}}\right]$, and whose contributions to $\delta_1 C^i{}_{0j0}$ are therefore restricted either on or inside the acoustic cone[26].

The physically intriguing feature of the relativistic w result in equation (272) is that, not only do spin-2 gravitons contribute to $\delta_1 C^i{}_{0j0}$, it appears the Bardeen scalars do so as well. To be sure, however, it would be prudent to obtain a more explicit expression for equation (273). In an upcoming work, we hope to tackle this important step towards a more comprehensive understanding of gravitational tidal forces within a cosmological setting.

[26] Once again, notice that in equation (273), we have switched from $C_{1,d}^{(w)}$ to $G_d^{(w,+)}$, where the local terms are incurred in the conversion between their second time derivatives, $\ddot{G}_d^{(w,+)} = -w^{\frac{d-1}{2}}\delta^{(d)}[x-x'] - \Theta[T]\ddot{C}_{1,d}^{(w)}$ and $\partial_{\eta'}^2 G_d^{(w,+)} = -w^{\frac{d-1}{2}}\delta^{(d)}[x-x'] - \Theta[T]\partial_{\eta'}^2 C_{1,d}^{(w)}$. Similarly, the Minkowski counterpart of $\delta_1 C^i{}_{0j0}$ is again recovered by letting $a \to 1$, replacing $G_d^{(g,+)}$ with G_d^+ in equation (267), and assuming no scalar contributions of equation (273) to equation (272).

6. Summary, discussions, and future directions

In this paper, we have sought to clarify the physical roles played by the TT and tt gravitational perturbations; as well as the analogous issues for the spin-1 photon. Even though the TT GW is gauge-invariant—it remains un-altered under an infinitesimal change in coordinates—it is acausal. Since the bulk of the paper involves heavy mathematical analysis for arbitrary dimensions and cosmological equations-of-state, we summarize here the 4D Minkowski case for the reader's convenience.

Let us begin with the electromagnetic sector. The gauge-invariant 4D transverse photon, which obeys $\partial_i \alpha_i = 0$, cannot be a standalone observable because its solution

$$\alpha_i[\eta, \vec{x}] = -\int_{\mathbb{R}^{3,1}} \mathrm{d}^4 x' G^{+}_{ij}[T, \vec{R}] J_j[\eta', \vec{x}'] \qquad (T \equiv \eta - \eta' \text{ and } \vec{R} \equiv \vec{x} - \vec{x}') \tag{274}$$

receives contributions from portions of the electric current $J_j[\eta', \vec{x}']$ lying outside the past lightcone of the observer at $(\eta, \vec{x})$. This is because, the photon retarded Green's function

$$G^{+}_{ij}[T, \vec{R}] = -\delta_{ij} \frac{\delta[T-R]}{4\pi R} - \frac{1}{4\pi} \partial_i \partial_j \left(\Theta[T-R] + \frac{T}{R} \Theta[T] \Theta[R-T] \right) \tag{275}$$

contains an acausal portion: $G^{(+,\text{acausal})}_{ij} = -(4\pi)^{-1} \Theta[T] \Theta[R-T] T \partial_i \partial_j R^{-1}$. However, since this acausal term of the photon Green's function is part of a pure gradient, namely $-(4\pi)^{-1} \partial_i \partial_j \big(\Theta[T] \Theta[R-T] T/R \big)$, the magnetic field $F_{ij} = \partial_i \alpha_j - \partial_j \alpha_i$—which involves its curl—is therefore entirely causal. The electric field, on the other hand, is the sum of the photon velocity $\dot{\alpha}_i$ and the gradient of the gauge-invariant scalar potential,

$$F_{0i} = \dot{\alpha}_i + \partial_i \Phi. \tag{276}$$

In detail, integration-by-parts (IBPs) and the conservation of the electric current yield

$$\dot{\alpha}_i[\eta, \vec{x}] = \left(\dot{\alpha}_i \right)_{\text{causal}} + \int_{\mathbb{R}^3} \mathrm{d}^3 \vec{x}' \partial_i \frac{J_0[\eta, \vec{x}']}{4\pi R}; \tag{277}$$

where we have denoted the causal part of the photon velocity as

$$\left(\dot{\alpha}_i \right)_{\text{causal}} \equiv \int_{\mathbb{R}^{3,1}} \mathrm{d}^4 x' \left(\partial_\eta \frac{\delta[T-R]}{4\pi R} J_i[\eta', \vec{x}'] - \partial_i \frac{\delta[T-R]}{4\pi R} J_0[\eta', \vec{x}'] \right). \tag{278}$$

Whereas, the gradient of the scalar potential is

$$\partial_i \Phi = -\int_{\mathbb{R}^3} \mathrm{d}^3 \vec{x}' \partial_i \frac{J_0[\eta, \vec{x}']}{4\pi R}. \tag{279}$$

Adding equations (277) and (279) to obtain equation (276), we see that the sole purpose of Φ —as far as electromagnetic fields are concerned—is to cancel the acausal part of the photon velocity. This in turn ensures the electric field of equation (276), in a given inertial frame, is the causal part of the latter; namely, $F_{0i} = (\dot{\alpha}_i)_{\text{causal}}$.

We have also pointed out: upon quantization, these transverse massless spin-1 photon operators violate microcausality, because their Green's functions do not vanish at spacelike intervals.

The transverse-traceless graviton, which obeys $\partial_i D_{ij} = 0 = \delta^{ij} D_{ij}$, also cannot be a standalone observable—for very similar reasons as its acausal transverse photon counterpart. Its solution

$$D_{ij}[\eta,\vec{x}] = -16\pi G_{\rm N} \int_{\mathbb{R}^{3,1}} {\rm d}^4x' G^+_{ijmn}[T,\vec{R}] T_{mn}[\eta',\vec{x}'] \tag{280}$$

receives signals from regions of the stress tensor $T_{mn}[\eta',\vec{x}']$ outside the past null cone of the observer at $(\eta,\vec{x})$. For, this TT graviton Green's function reads

$$\begin{aligned} G^+_{ijmn}[T,\vec{R}] = &-\left(\delta_{m(i}\delta_{j)n} - \frac{\delta_{ij}\delta_{mn}}{2}\right)\frac{\delta[T-R]}{4\pi R} \\ &- \frac{1}{4\pi}\left(\delta_{m(i}\partial_{j)}\partial_n + \delta_{n(i}\partial_{j)}\partial_m - \frac{\delta_{ij}\partial_m\partial_n - \delta_{mn}\partial_i\partial_j}{2}\right)\left(\Theta[T-R] + \frac{T}{R}\Theta[T]\Theta[R-T]\right) \\ &+ \frac{1}{48\pi}\partial_i\partial_j\partial_m\partial_n\left(\Theta[T-R]\left(R^2+3T^2\right) + \Theta[T]\Theta[R-T]\,\frac{3R^2T+T^3}{R}\right); \end{aligned} \tag{281}$$

which contains the acausal terms $-\Theta[T]\Theta[R-T]T\big(\delta_{m(i}\partial_{j)}\partial_n + \delta_{n(i}\partial_{j)}\partial_m - (1/2)(\delta_{ij}\partial_m\partial_n - \delta_{mn}\partial_i\partial_j)\big)(4\pi R)^{-1}$ and $\Theta[T]\Theta[R-T]\partial_i\partial_j\partial_m\partial_n\big\{(3R^2T+T^3)(48\pi R)^{-1}\big\}$.

On the other hand, the dominant physical tidal forces ought to be encoded within the linearized Riemann tensor, which in turn involves all the gauge-invariant variables, not just the spin-2 graviton. In particular, its $0i0j$ components—which are usually associated with the spatial tidal forces in a given nearly-Lorentzian inertial frame—are

$$\delta_1 R_{0i0j} = \delta_{ij}\ddot{\Psi} + \partial_i\partial_j\Phi + \partial_{(i}\dot{V}_{j)} - \frac{1}{2}\ddot{D}_{ij}. \tag{282}$$

As one may expect from the preceding discussion for the spin-1 photon, the linearized Riemann $\delta_1 R_{0i0j}$ really only depends on the causal part of the spin-2 graviton acceleration:

$$\delta_1 R_{0i0j} = -\frac{1}{2}\left(\ddot{D}_{ij}\right)_{\rm causal}; \tag{283}$$

where, upon IBPs and invoking the conservation of the energy-momentum-shear-stress tensor, we have

$$\begin{aligned} \left(\ddot{D}_{ij}\right)_{\rm causal} = 4G_{\rm N}\int_{\mathbb{R}^{3,1}} {\rm d}^4x' \Bigg\{ &\partial_\eta^2 \frac{\delta[T-R]}{R}\left({}^{(\rm a)}T_{ij}[\eta',\vec{x}'] + \frac{\delta_{ij}}{2}\left({}^{(\rm a)}T_{00}[\eta',\vec{x}'] - {}^{(\rm a)}T_{ll}[\eta',\vec{x}']\right)\right) \\ &- 2\partial_\eta\partial_{(i}\frac{\delta[T-R]}{R}{}^{(\rm a)}T_{j)0}[\eta',\vec{x}'] + \frac{1}{2}\partial_i\partial_j\frac{\delta[T-R]}{R}\left({}^{(\rm a)}T_{00}[\eta',\vec{x}'] + {}^{(\rm a)}T_{ll}[\eta',\vec{x}']\right)\Bigg\}. \end{aligned} \tag{284}$$

The sole purpose of the rest of the gauge-invariant variables (Ψ,Φ,V_i), as far as the $\delta_1 R_{0i0j}$ components are concerned, is to cancel the acausal part of the graviton acceleration. Moreover, all of them are needed to ensure causality. We may verify these claims by simply comparing the following expressions.

$$\begin{aligned} \ddot{D}_{ij}[\eta,\vec{x}] = &\left(\ddot{D}_{ij}\right)_{\rm causal} \\ &- 4G_{\rm N}\int_{\mathbb{R}^{3,1}} {\rm d}^3\vec{x}' \Bigg\{ -2\partial_m\partial_{(i}\frac{1}{R}{}^{(\rm a)}T_{j)m}[\eta,\vec{x}'] + \frac{1}{2}\Bigg(\delta_{ij}\partial_m\partial_n\frac{1}{R}{}^{(\rm a)}T_{mn}[\eta,\vec{x}'] \\ &+ \partial_i\partial_j\frac{1}{R}\left({}^{(\rm a)}T_{00}[\eta,\vec{x}'] + {}^{(\rm a)}T_{ll}[\eta,\vec{x}']\right)\Bigg) + \frac{1}{4}\partial_i\partial_j\partial_m\partial_n R^{(\rm a)}T_{mn}[\eta,\vec{x}']\Bigg\} \end{aligned} \tag{285}$$

$$\begin{aligned}
\ddot{\Psi}[\eta,\vec{x}] &= -4G_{\mathrm{N}} \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}' \, \frac{1}{4}\partial_m\partial_n \frac{1}{R}^{(\mathrm{a})} T_{mn}[\eta,\vec{x}'], \\
\partial_i\partial_j\Phi[\eta,\vec{x}] &= -4G_{\mathrm{N}} \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}' \, \frac{1}{4}\left(\partial_i\partial_j \frac{1}{R}^{(\mathrm{a})} T_{00}[\eta,\vec{x}'] + \partial_i\partial_j \frac{1}{R}^{(\mathrm{a})} T_{ll}[\eta,\vec{x}'] - \frac{3}{2}\partial_i\partial_j\partial_m\partial_n R^{(\mathrm{a})} T_{mn}[\eta,\vec{x}']\right), \\
\partial_{(i}\dot{V}_{j)}[\eta,\vec{x}] &= 4G_{\mathrm{N}} \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}' \left(\partial_m\partial_{(i} \frac{1}{R}^{(\mathrm{a})} T_{j)m}[\eta,\vec{x}'] - \frac{1}{2}\partial_i\partial_j\partial_m\partial_n R^{(\mathrm{a})} T_{mn}[\eta,\vec{x}']\right). \qquad (286)
\end{aligned}$$

These massless spin-2 graviton fields, upon quantization, would violate microcausality, because their Green's functions do not vanish at spacelike intervals.

It is worth highlighting, we are not asserting that relativists are computing gravitational wave-forms wrongly. In the far zone, $|\omega|r \gg 1$, we have shown that the distortion of space due to GWs (at finite frequencies) do reduce to the tt ones gotten by performing a local-in-space projection of the (de Donder gauge) spatial perturbations χ_{ij}. These tt GWs, as opposed to their TT counterparts, are in fact the ones computed in the gravitational literature. On the other hand, within this far zone, these tt GWs in fact coincide with the TT ones, because the acausal parts of the latter begin at higher orders in $1/(\omega r)$.

In a cosmology driven by a relativistic fluid, we have uncovered tentative evidence that the Bardeen scalar potentials contribute to gravitational tidal forces, and their wave-like solutions could therefore be legitimately dubbed 'scalar gravitational waves' in this sense. More work would be required to confirm or deny this [13]. Nonetheless, if the Bardeen Ψ and Φ are indeed an integral part of cosmological GWs, we hope this work constitutes the first step towards illuminating not only their associated GW patterns but also potential scalar GW memory effects.

Let us end on a more speculative note. Even though the TT graviton is acausal and cannot be a standalone observable within classical physics, it may be produced quantum mechanically—and independently of other gauge-invariant perturbations—during a (still hypothetical) exponentially expanding phase of the early universe[27]. On the other hand, we have also pointed out that the quantum operators associated with both the free massless spin-1 and spin-2 particles violate micro-causality. Is it possible to exploit this violation to ascertain whether *B*-modes in the Cosmic Microwave Background, if we ever detect them, were truly engendered by *quantum* fluctuations of spacetime itself? Or, for the photon case, are there laboratory experiments involving quantum generation of photons that could not only serve as analogs to the inflationary scenario, but also allow the quantum nature of their production mechanism to be probed directly?

Acknowledgments

YZC is supported by the Ministry of Science and Technology of the R.O.C. under the grant 106-2112-M-008-024-MY3. He also thanks the organizers of the 2018 TGWG Conference (at Tamkang University, Taipei), where he gave a talk about this work; as well as Lior Burko and Feng-Li Lin for questions and Jan Steinhoff for constructive comments. YWL wishes to gratefully acknowledge the support of Shih-Chang Lee and the Department of Physics at National Central University during the completion of the most part of this work. He is supported by the Ministry of Science and Technology of the R.O.C. under Project No. MOST

[27] We note in passing: when constructing the quantum field theory of photons and gravitons in 4D flat spacetime, Weinberg [15] had to add non-local terms to the Hamiltonian of these massless spin-1 and spin-2 gauge theories in order to preserve the Lorentz invariance of the *S*-matrix.

108-2811-M-008-503. They would also like to acknowledge Bonga, Rácz, Flanagan and Hughes for e-mail exchanges.

ORCID iDs

Yi-Zen Chu https://orcid.org/0000-0001-9302-2214
Yen-Wei Liu https://orcid.org/0000-0002-0516-8246

References

[1] Rácz I 2009 Gravitational radiation and isotropic change of the spatial geometry (arXiv:0912.0128 [gr-qc])
[2] Ashtekar A and Bonga B 2017 On the ambiguity in the notion of transverse traceless modes of gravitational waves *Gen. Relativ. Gravit.* **49** 122
[3] Ashtekar A and Bonga B 2017 On a basic conceptual confusion in gravitational radiation theory *Class. Quantum Grav.* **34** 20LT01
[4] Frenkel A and Rácz I 2015 On the use of projection operators in electrodynamics *Eur. J. Phys.* **36** 015022
[5] Brill O L and Goodman B 1967 Causality in the Coulomb Gauge *Am. J. Phys.* **35** 832
[6] Thorne K S and Blandford R D 2017 *Modern Classical Physics Optics, Fluids, Plasmas, Elasticity, Relativity, and Statistical Physics* (Princeton, NJ: Princeton University Press)
[7] Flanagan E E and Hughes S A 2005 The Basics of gravitational wave theory *New J. Phys.* **7** 204
[8] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco, CA: W. H. Freeman)
[9] Weinberg S 1972 *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (New York: Wiley)
[10] Schutz B F 1985 *A First Course in General Relativity* (Cambridge: Cambridge University Press)
[11] Maggiore M 2007 *Gravitational Waves. Vol. 1: Theory and Experiments* (Oxford: Oxford University Press)
[12] Chu Y Z 2017 More on cosmological gravitational waves and their memories *Class. Quantum Grav.* **34** 194001
[13] Chu Y-Z and Liu Y-W in progress
[14] Chu Y Z 2017 Gravitational wave memory in dS_{4+2n} and 4D cosmology *Class. Quantum Grav.* **34** 035009
[15] Weinberg S 1965 Photons and gravitons in perturbation theory: derivation of Maxwell's and Einstein's equations *Phys. Rev.* **138** B988