

# Finite time synchronization of fractional chaotic systems with several slaves in an optimal manner

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## Abstract

Optimal synchronization of chaotic fractional differential equations with one master and several slaves in finite time is the main aim of this paper. To achieve this goal, we convert the finite time synchronization problem to a fractional optimal control problem; then by solving it, we achieve the active control. In this way, we use Bernstein polynomials and prove that the corresponding minimization problem is a quadratic convex problem. Some examples using the famous Lorenz, Chen, Lu, and Liu systems are given to show the efficiency of the method.

Keywords: optimal synchronization, optimal control, chaotic systems, fractional calculus

(Some figures may appear in colour only in the online journal)

## 1. Introduction

The applications of fractional calculus in the last few decades have attracted the attention of many authors. Many real-world physical systems are displayed fractional order dynamics, such as electromagnetic waves [1], viscoelastic systems [2], dielectric polarization [3], and so on.

Chaotic dynamical systems are deterministic and non-linear systems, which are exponentially sensitive to initial conditions; see [4]. Chaotic dynamical systems with two close initial conditions have completely different trajectories in finite time, even though trajectories remain in a finite region. In this case, small errors in measurement of initial conditions make it impossible to predict the trajectories of system even with a complete and exact model of the system. Chaos has been seen in a many type of systems such as chemical systems, electrical systems, fluid systems, and analog computers; see [5].

Synchronization is one of the main topics in control theory, that means to enforce two or more chaotic systems have the same behavior. Synchronization is an interesting topic for its applications in secure communication [6], power electronic

systems [7], laser [8], physical systems [9], neural networks [10], stochastic complex networks [11] and so on. For example, in physics, synchronization is observed to occur between oscillators, where a collection of oscillators are observed to synchronize in a diverse variety of systems in spite of the presence of unavoidable difference between the oscillators. For more applications of synchronization see [12–16]. Synchronization of chaotic systems, for the first time, was proposed by Pecorra and Carroll in 1990 [17]. Different methods are available for synchronization of chaotic systems, depending on the structure of the systems. Recently, many types of synchronization methods have been developed by many authors, such as complete synchronization [17], generalized synchronization [18], phase synchronization [19], antiphase synchronization [20], and so forth. Several approaches have been considered for synchronization of chaotic systems such as adaptive control synchronization, synchronization using active control, synchronization using sliding mode control, and so on; see, for example, [21–24]. These studies have guaranteed that when times goes to infinity, the trajectories of the error system converge to zero, whereas, in practice, it is more precious to synchronize fractional chaotic systems in a finite time. In [25], by using finite time control techniques, continuous state feedback control approaches to solve the synchronization of chaotic systems was considered. In

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[26], a synchronization of Lorenz systems by the control theory in finite time was proposed by Patrick Louodop *et al.* They proposed a set of feedback controls that perform the synchronization of Lorenz chaotic systems according to the Lyapunov stability theory. Xi *et al* [27] proposed an approach according to a sliding mode control for the synchronization of chaotic systems. Pang *et al* [28] investigated synchronization of chaotic systems with different dimensions in a finite-time.

One of the most important approaches for the synchronization of chaotic systems is optimal control; see [29, 30]. The optimal control problem is to minimize a functional subject to a dynamic based on a set of control and state functions. The reason that we use optimal control method, is that it is possible to find an equivalent optimal control problem in finite time, then by using Bernstein polynomials (BPs) one can find approximate state and control functions. One of the advantages of this procedure is that we can find the control function straightforward, then by using this function it is possible to find the state function. While in other methods, usually finding control is not directly. In [31], an optimal control approach was used to control and synchronize chaotic systems. Optimal synchronization of chaotic systems by means of linear matrix inequality methods was proposed by Tong *et al* [32]. Karimi *et al* [33] proposed a parallel synchronization method to synchronize chaotic systems.

Many synchronization studies have been limited to one master and one slave models. Recently, novel synchronization techniques have been developed such as combination [34], combination–combination [35], and double compound synchronizations [36]. These schemes enhance the security of information transmitted via chaotic signals; see [37].

In this article, to overcome the disadvantages of combination synchronization, which involves one slave, we consider the optimal synchronization of dynamical chaotic systems with one master and several slaves and we use BPs because of the efficiency and simple applications of these polynomials. For illustrating the effectiveness of the method, some numerical examples, using the famous Lorenz, Chen, Liu, and Lu fractional chaotic systems, are given.

The organization of this paper is as follows. Section 2 includes some requirements in fractional calculus and BPs. In section 3, the problem statement and converting the finite time synchronization to a fractional optimal control problem (FOCP) are discussed. In this section, an algorithm that utilizes the BPs, is given for solving the mentioned FOCP and we prove that the minimization problem is a quadratic convex problem. In section 4, two examples are provided to illustrate the performance of the proposed method. Finally, we conclude the paper in section 5.

## 2. Some preliminaries in fractional calculus and BPs

In this section, some basic definitions of fractional derivative and integral, BPs, and their properties are given.

For fractional order derivative and integral, the following definitions are used in literature (see [38, 39] for more details).

**Definition 2.1.** The Riemann–Liouville fractional integral operator of order  $q > 0$  of a continuous function  $f$  is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^t (t - \tau)^{q-1} f(\tau) d\tau. \quad (1)$$

**Definition 2.2.** Suppose that a real function  $f(t)$  is in the space  $C^n[0, \infty)$ ; then the Riemann–Liouville derivative of order  $q$ ,  $n - 1 < q < n$ , is defined as follows:

$${}^{RL}_t D_t^q f(t) = \frac{1}{\Gamma(n - q)} \cdot \frac{d^n}{dt^n} \left[ \int_{t_0}^t \frac{f(\tau)}{(t - \tau)^{q-n+1}} d\tau \right]. \quad (2)$$

**Definition 2.3.** The fractional derivative of  $f(t)$  of order  $q$ ,  $n - 1 < q < n$ , in Caputo sense, is defined by using Riemann–Liouville integral operator as follows:

$${}_t^c D_t^q f(t) = I_t^q ({}^{RL}_t D_t^n f(t)). \quad (3)$$

So, substitution implies

$${}_t^c D_t^q f(t) = \frac{1}{\Gamma(n - q)} \int_{t_0}^t \frac{D^n f(\tau)}{(t - \tau)^{q-n+1}} d\tau. \quad (4)$$

In this paper we use fractional derivative in Caputo sense.

Fractional derivatives and integrals have several properties that we list some of them, which will be used in this paper.

- For  $q = 1$ , fractional derivatives give the same results as classical ordinary derivative  $df(t)/dt$ .
- Every fractional derivative is a linear operator. It means that if  $f$  and  $g$  are two real functions, then, for all  $c_1, c_2 \in \mathbb{R}$ , we have  $D^q(c_1 f + c_2 g) = c_1 D^q f + c_2 D^q g$ .
- The additive law exponents (semigroup property) holds. It means that  $D^{q_1}(D^{q_2} f(t)) = D^{q_2}(D^{q_1} f(t)) = D^{q_1+q_2} f(t)$ .
- For  $q > 0$ , we have  $D^q(D^{-q} f(t)) = D^0 f(t) = f(t)$ , which means that the fractional differentiation operator is a left inverse of the fractional integration of the same order  $q$ .

**Definition 2.4.** The definition of the BP of degree  $n$  in the interval  $[c, d]$  is as follows:

$$B_{i,n} \left( \frac{t - c}{d - c} \right) = \binom{n}{i} \left( \frac{d - t}{d - c} \right)^{n-i} \left( \frac{t - c}{d - c} \right)^i, \quad 0 \leq i \leq n. \quad (5)$$

Therefore, within the interval  $[0, 1]$ , the BP of degree  $n$  has the following form:

$$B_{i,n}(t) = \binom{n}{i} (1 - t)^{n-i} t^i. \quad (6)$$

The following two lemmas give some properties of BPs.

**Lemma 2.1** (See [40]). If  $\Phi_m(t) = [B_{0,m}(t), B_{1,m}(t), \dots, B_{m,m}(t)]^T$  and  $T_m = [1, t, t^2, \dots, t^m]^T$ , then

$$\Phi_m(t) = \Lambda T_m(t), \quad (7)$$

where  $\Lambda = (\gamma_{ij})$ ,  $1 \leq i, j \leq m+1$ , is an  $(m+1) \times (m+1)$  matrix and

$$\gamma_{i+1,j+1} = \begin{cases} (-1)^{j-i} \binom{n}{i} \binom{m-i}{j-i} & ; i \leq j, \\ 0 & ; i > j, \end{cases} \quad 0 \leq i, j \leq m. \quad (8)$$

**Lemma 2.2** (See [41]). Let  $L^2[0, 1]$  be a Hilbert space with inner product  $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$ . Assume that  $B = \text{spn}\{B_{0,m}, B_{1,m}, \dots, B_{m,m}\}$ . If  $y \in L^2[0, 1]$ , then  $y$  has a unique best approximation of order  $m$  out of  $B$  as follows:

$$\sum_{i=0}^m c_i B_{i,m}(t) = C^T \Phi_m(t), \quad t \in [0, 1], \quad (9)$$

where the unique vector  $C$  is defined as  $C = W^{-1} \langle y, \Phi_m \rangle$  in which

$$\langle y, \Phi_m \rangle = \int_0^1 y \Phi_m dt = [\langle y, B_{0,m} \rangle, \langle y, B_{1,m} \rangle, \dots, \langle y, B_{m,m} \rangle]^T \quad (10)$$

and the entries of matrix  $W = (W_{i+1,j+1})$ ,  $0 \leq i, j \leq m$ , are defined as:

$$W_{i+1,j+1} = \int_0^1 B_{i,m}(t) B_{j,m}(t) dt = \frac{\binom{m}{i} \binom{m}{j}}{(2m+1) \binom{2m}{j+i}}. \quad (11)$$

### 3. Optimal synchronization of fractional dynamical systems using BPs

#### 3.1. Problem statement

In this section, optimal synchronization of fractional chaotic system with one master and several slaves by using BPs is considered.

**Remark 3.1.** The main reason of using BPs is the efficiency and simple applications of these polynomials. In fact, by BPs we can easily convert our FOCP to optimization of a convex performance index subject to linear constraints where the new problem is tractable.

On the other hand the accuracy of solutions of the FOCPs using BPs is better than the accuracy of solutions whose achieved by using other methods, for example Legendre polynomials method (see [38]). Also as we see in section 4.2, the synchronization errors whose achieved using BPs method

are less than the synchronization errors where achieved when Lagrange multipliers method is employed.

Consider the following system:

$$\text{(master)} \quad D^q x = f(x), \quad (12)$$

and

$$\text{(slave)} \quad D^q y_i = f_i(y_i), \quad i = 1, 2, \dots, s. \quad (13)$$

Here  $x = (x_1, x_2, \dots, x_n)^T$  and  $y_i = (y_{i1}, y_{i2}, \dots, y_{in})^T$  are real vectors in  $\mathbb{R}^n$  and  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $f_i: \mathbb{R}^n \rightarrow \mathbb{R}^n$  are continuous vector functions for  $i = 1, 2, \dots, s$ .

Suppose that, the term  $\sum_{i=1}^s c_i y_i$ , where  $c_i$ 's,  $i = 1, 2, \dots, s$ , are constant diagonal matrices, is as output of slave systems that must synchronize with output of master system (12). Now from the slave systems (13), one can easily find:

$$D^q \left( \sum_{i=1}^s c_i y_i \right) = \sum_{i=1}^s c_i f_i(y_i). \quad (14)$$

Adding the control vector function  $U = U(x, y_1, y_2, \dots, y_s)$  to (14) and subtracting from the master system (12) yield:

$$D^q \left( \sum_{i=1}^s c_i y_i - x \right) = \sum_{i=1}^s c_i f_i(y_i) - f(x) + U. \quad (15)$$

Hence

$$D^q e = \sum_{i=1}^s c_i f_i(y_i) - f(x) + U, \quad (16)$$

where  $e = \sum_{i=1}^s c_i y_i - x$  is the error of synchronization. For convenience, we use the notation  $U = [U_1, U_2, \dots, U_n]^T$ , where  $U_1 = U_1(x, y_1, y_2, \dots, y_s)$ ,  $U_2 = U_2(x, y_1, y_2, \dots, y_s)$ , ...,  $U_n = U_n(x, y_1, y_2, \dots, y_s)$ .

Now we have the following lemma.

**Lemma 3.1.** The error system (16) can be written as follows:

$$D^q e = Ae + F(x, y_1, y_2, \dots, y_s) + U, \quad (17)$$

where  $A \in \mathbb{R}^{n \times n}$  and

$$F(x, y_1, y_2, \dots, y_s) = \begin{bmatrix} F_1(x, y_1, y_2, \dots, y_s) \\ F_2(x, y_1, y_2, \dots, y_s) \\ \vdots \\ F_n(x, y_1, y_2, \dots, y_s) \end{bmatrix} \quad (18)$$

is nonlinear term.

**Proof.** The master system (12) and slave systems (13) can be decomposed to linear and nonlinear parts as follows:

$$\text{master:} \quad D^q x = Mx + g(x) \quad (19)$$

$$\text{slaves:} \quad D^q y_i = A_i y_i + g_i(y_i), \quad i = 1, 2, \dots, s, \quad (20)$$

where  $M$  and  $A_i$  are real  $n \times n$  matrices and  $g(x)$  and  $g_i(y_i)$  are nonlinear parts of master and slave systems, respectively. Since the matrices  $c_i$ ,  $i = 1, 2, \dots, s$ , are diagonal, the error system (16) can be written as:

$$D^q e = \sum_{i=1}^s A_i c_i y_i + \sum_{i=1}^s c_i g_i(y_i) - Mx - g(x) + U. \quad (21)$$

Therefore

$$\begin{aligned} D^q e &= A_1 c_1 y_1 + A_2 c_2 y_2 + \cdots + A_s c_s y_s - A_1 x - A_2 x \\ &\quad - \cdots - A_s x + \sum_{i=1}^s c_i g_i(y_i) \\ &\quad + \sum_{i=1}^s A_i x - Mx - g(x) + U. \end{aligned} \quad (22)$$

So we have:

$$\begin{aligned} D^q e &= A_1 \left( \sum_{i=1}^s c_i y_i \right) + A_2 \left( \sum_{i=1}^s c_i y_i \right) \\ &\quad + \cdots + A_s \left( \sum_{i=1}^s c_i y_i \right) - A_1 x - A_2 x - \cdots - A_s x \\ &\quad - \sum_{i,j=1, i \neq j}^s A_i c_j y_j + \sum_{i=1}^s c_i g_i(y_i) \\ &\quad + \sum_{i=1}^s A_i x - Mx - g(x) + U. \end{aligned} \quad (23)$$

$$\begin{aligned} \Rightarrow D^q e &= A_1 e + A_2 e + \cdots + A_s e - \sum_{i,j=1, i \neq j}^s A_i c_j y_j \\ &\quad + \sum_{i=1}^s c_i g_i(y_i) + \sum_{i=1}^s A_i x - Mx - g(x) + U. \end{aligned} \quad (24)$$

Hence we have:

$$D^q e = Ae + F(x, y_1, y_2, \dots, y_s) + U, \quad (25)$$

where  $A = A_1 + A_2 + \cdots + A_s$  and

$$\begin{aligned} F(x, y_1, y_2, \dots, y_s) &= - \sum_{i,j=1, i \neq j}^s A_i c_j y_j \\ &\quad + \sum_{i=1}^s c_i g_i(y_i) + \sum_{i=1}^s A_i x - Mx - g(x), \end{aligned} \quad (26)$$

therefore proof is complete.  $\square$

By considering  $U = -F(x, y_1, y_2, \dots, y_m) + u$ , where  $u = (u_1, u_2, \dots, u_n)^T$ , we have the following dynamical error system:

$$D^q e(t) = Ae(t) + u. \quad (27)$$

Motivated by the definition of the finite time synchronization in [26, 42, 43], we say that optimal finite time synchronization on the interval  $[0, t_f]$  occurs if a performance index subject to dynamical system (27) is minimized and

$$\lim_{t \rightarrow t_f} e(t) = 0. \quad (28)$$

Now we consider the following finite time optimal control problem:

$$\min J[e(\cdot), u(\cdot)] = \frac{1}{2} \int_0^{t_f} [e^T(t) Q e(t) + u^T(t) R u(t)] dt \quad (29)$$

s.t.

$$D^q e(t) = Ae(t) + u, \quad (30)$$

$$e(t_f) = 0, \quad (31)$$

$$e(0) = \sum_{i=1}^s c_i y_i(0) - x(0). \quad (32)$$

In the above FOCP, since our dynamical systems are continuous, condition (31) is equivalent to  $\lim_{t \rightarrow t_f} e(t) = 0$ . Our aim is to obtain the optimal control vector  $u$  such that the quadratic cost functional (29) is minimized. In this case optimal finite time synchronization occurs.

### 3.2. Optimal control of fractional order systems by BPs

After converting the finite time synchronization problem to the FOCP (29)–(32), we use the BPs to solve the mentioned FOCP. In this way, we review the main result of the approximate method for numerically solving FOCPs by BPs [38]. We can map the interval  $[0, t_f]$  onto  $[0, 1]$ , using the change of variable  $\theta = \frac{t}{t_f}$ , whenever we need. So without use of generality, we solve the FOCP (29)–(32) in  $[0, 1]$ . Consider the following optimal control problem:

$$\min J[e(\cdot), u(\cdot)] = \frac{1}{2} \int_0^1 [e^T(t) Q e(t) + u^T(t) R u(t)] dt \quad (33)$$

s.t.

$$D^q e_i(t) = \sum_{j=1}^n a_{ij}(t) e_j(t) + u_i(t), \quad i = 1, \dots, n, \quad (34)$$

$$e_i(1) = 0, \quad i = 1, \dots, n, \quad (35)$$

$$e_i(0) = e_{i,0}, \quad i = 1, \dots, n, \quad (36)$$

where  $e(t) = [e_1(t), \dots, e_n(t)]^T$  and  $u(t) = [u_1(t), \dots, u_n(t)]^T$ . Moreover  $Q$  and  $R$  are positive semidefinite and positive definite,  $n \times n$  symmetric matrices in the performance index, respectively. Also,  $a_{ij}(t)$  are continuous functions which are the coefficients of  $e_j(t)$  for  $(1 \leq j \leq n)$ .

By using the method that we discussed in [38] to approximate fractional derivative of BPs in the Caputo sense, we use an operational matrix as follows:

$$D^q \Phi_m(t) \approx D_q \Phi_m(t), \quad (37)$$

where  $D_q = \Lambda \tilde{\Sigma} P^T$ ,

$$\begin{aligned} \tilde{\Sigma} &= (\tilde{\Sigma}_{i+1, j+1}), \quad \tilde{\Sigma}_{i+1, j+1} \\ &= \begin{cases} \frac{\Gamma(1+j)}{\Gamma(1+j-q)}, & i, j = [q], \dots, m \text{ and } i = j, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \quad (38)$$

and  $P$  is an  $(m+1) \times (m+1)$  matrix with zero vector in  $[q]$ th column and the vector  $P_i$  in  $(i+1)$ th column's for  $i = [q], \dots, m$ . Here  $P_i = W^{-1}\bar{P}_i$  and  $\bar{P}_i = [\bar{P}_{i,0}, \bar{P}_{i,1}, \dots, \bar{P}_{i,m}]$ , in which the  $\bar{P}_i$ s,  $i = [q], \dots, m$ , have the following form:

$$\begin{aligned}\bar{P}_{i,j} &= \int_0^1 t^{i-q} B_{j,m}(t) dt \\ &= \frac{m! \Gamma(j+i-q+1)}{j! \Gamma(m+i-q+2)}, \quad 0 \leq j \leq m.\end{aligned}\quad (39)$$

Now we assume that

$$\begin{aligned}e_i(t) &\approx E_i^T \Phi_m(t), \quad i = 1, \dots, n, \\ u_i(t) &\approx U_i^T \Phi_m(t), \quad i = 1, \dots, n,\end{aligned}\quad (40)$$

where the entries  $E_i = [E_i(0), \dots, E_i(m)]^T$  and  $U_i = [U_i(0), \dots, U_i(m)]^T$  are the Bernstein coefficients of the approximation of  $e_i(t)$  and  $u_i(t)$ , respectively. So

$$D^q e_i(t) \approx E_i(t) D^q \Phi_m(t) \approx E_i(t) D_q \Phi_m(t). \quad (41)$$

Therefore, the system (34) is approximated as:

$$E_i^T D_q \Phi_m(t) = \sum_{j=1}^r (A^{i,j})^T \Phi_m(t) \Phi_m^T(t) E_j + U_i^T \Phi_m(t), \quad (42)$$

or

$$E_i^T D_q \Phi_m(t) = \sum_{j=1}^r \Phi_m^T(t) \tilde{A}^{i,j} E_j + U_i^T \Phi_m(t), \quad (43)$$

for  $1 \leq i \leq n$ , where  $(A^{i,j}) \in \mathbb{R}^{m+1}$  are the known BP coefficients of  $a_{i,j}(t)$ , where can be obtained by equation (9). Also the known matrix  $\tilde{A}^{i,j} \in \mathbb{R}^{(m+1) \times (m+1)}$  in (43) can be achieved such that

$$(A^{i,j})^T \Phi_m(t) \Phi_m^T(t) = \Phi_m^T(t) \tilde{A}^{i,j}, \quad i, j = 1, 2, \dots, n. \quad (44)$$

We need to write the initial condition  $e_i(0) = e_{i,0}$  in (36) in terms of BPs as follows:

$$e_{i,0} = [E_i(0), \dots, E_i(m)] \Phi_m(0) = [E_i(0), \dots, E_i(m)] \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (45)$$

or

$$E_i(0) = e_{i,0}, \quad i = 1, \dots, n. \quad (46)$$

Also the conditions (35) can be written as follows:

$$E_i^T \Lambda \mathbf{1}_{m+1} = 0, \quad i = 1, \dots, n, \quad (47)$$

where  $\mathbf{1}_{m+1} = [1, 1, \dots, 1]^T \in \mathbb{R}^{m+1}$ . On the other hand, from (40) and (11), we can approximate the performance index

(33) by BPs as follows:

$$\begin{aligned}J[e(\cdot), u(\cdot)] &= \frac{1}{2} \int_0^1 [e^T(t) Q e(t) + u^T(t) R u(t)] dt \\ &= \frac{1}{2} \int_0^1 \left[ \sum_{i,j=1}^n Q_{ij} e_i(t) e_j(t) + \sum_{i,j=1}^n R_{ij} u_i(t) u_j(t) \right] dt \\ &\approx \frac{1}{2} \int_0^1 \left[ \sum_{i,j=1}^n Q_{ij} \sum_{k,l=0}^m E_i(k) E_j(l) B_{k,m}(t) B_{l,m}(t) \right. \\ &\quad \left. + \sum_{i,j=1}^n R_{ij} \sum_{k,l=0}^m U_i(k) U_j(l) B_{k,m}(t) B_{l,m}(t) \right] dt \\ &= \frac{1}{2} \left[ \sum_{i,j=1}^n \sum_{k,l=0}^m Q_{ij} E_i(k) E_j(l) \int_0^1 B_{k,m}(t) B_{l,m}(t) dt \right. \\ &\quad \left. + \sum_{i,j=1}^n \sum_{k,l=0}^m R_{ij} U_i(k) U_j(l) \int_0^1 B_{k,m}(t) B_{l,m}(t) dt \right] \\ &= \frac{1}{2} \left[ \sum_{i,j=1}^n \sum_{k,l=0}^m E_i(k) Q_{ij} W_{kl} E_j(l) \right. \\ &\quad \left. + \sum_{i,j=1}^n \sum_{k,l=0}^m U_i(k) Q_{ij} W_{kl} U_j(l) \right] \\ &= \frac{1}{2} [E^T (Q \otimes W) E + U^T (R \otimes W) U],\end{aligned}\quad (48)$$

where

$$\begin{aligned}E &= [E_1(0), \dots, E_1(m), E_2(0), \dots, E_2(m), \\ &\quad \dots, E_n(0), \dots, E_n(m)] \in \mathbb{R}^{n(m+1)}, \\ U &= [U_1(0), \dots, U_1(m), U_2(0), \dots, U_2(m), \\ &\quad \dots, U_n(0), \dots, U_n(m)] \in \mathbb{R}^{n(m+1)},\end{aligned}\quad (49)$$

and the notation  $\otimes$  is the Kronecker product [44]. In other words, we have

$$J[e(\cdot), u(\cdot)] \approx J[E(\cdot), U(\cdot)] = \frac{1}{2} [E^T U^T] H \begin{bmatrix} E \\ U \end{bmatrix}, \quad (50)$$

where the matrix  $H \in \mathbb{R}^{2n(m+1) \times 2n(m+1)}$  equals

$$H = \begin{bmatrix} Q \otimes W & 0 \\ 0 & R \otimes W \end{bmatrix}. \quad (51)$$

From the above discussion, we can approximate FOCP (33)–(36) by the following optimization problem:

$$\min J[E(\cdot), U(\cdot)] = \frac{1}{2} [E^T U^T] H \begin{bmatrix} E \\ U \end{bmatrix}, \quad (52)$$

s.t.

$$E_i^T D_q - \sum_{j=1}^n E_j^T (\tilde{A}^{i,j})^T - U_i^T = 0, \quad i = 1, \dots, n, \quad (53)$$

$$E_i^T \Lambda \mathbf{1}_{m+1} = 0, \quad i = 1, \dots, n, \quad (54)$$

$$E_i(0) = e_{i,0}, \quad i = 1, \dots, n. \quad (55)$$

Now we show that the quadratic optimization problem (52)–(55) is a convex optimization problem.

**Lemma 3.2.** The matrix  $W \in \mathbb{R}^{(m+1) \times (m+1)}$  in (11) is a positive definite matrix.

**Proof.** By the definition of the matrix  $W$  in (11), for all nonzero  $z \in \mathbb{R}^{(m+1)}$ , we have

$$\begin{aligned} z^T W z &= \sum_{i=0}^m \sum_{j=0}^m z_i z_j W_{ij} \\ &= \sum_{i=0}^m \sum_{j=0}^m z_i z_j \int_0^1 B_{i,m}(t) B_{j,m}(t) dt \\ &= \int_0^1 \left[ \sum_{i=0}^m \sum_{j=0}^m z_i z_j B_{i,m}(t) B_{j,m}(t) \right] dt \\ &= \int_0^1 \left[ \sum_{i=0}^m z_i B_{i,m}(t) \sum_{j=0}^m z_j B_{j,m}(t) \right] dt \\ &= \int_0^1 \left[ \sum_{i=0}^m z_i B_{i,m}(t) \right]^2 dt. \end{aligned} \quad (56)$$

Since the set of BPs  $\{B_{0,m}, B_{1,m}, \dots, B_{n,m}\}$  is linear independent and  $z \neq 0$ , we know that  $[\sum_{i=0}^m z_i B_{i,m}(t)]^2$  is a nonzero and non-negative polynomial. Therefore (56) shows that  $z^T W z > 0$ . Hence  $W$  is a positive definite matrix.  $\square$

**Theorem 3.1.** The quadratic optimization problem (52)–(55) is a convex optimization problem.

**Proof.** We know that  $Q$  and  $R$  are, respectively, positive semidefinite and positive definite matrices. Also lemma 3.2 shows that the matrix  $W$  is positive definite. On the other hand [44, corollary 4.2.13], shows that the Kronecker product of symmetric positive semidefinite matrices is a positive semidefinite matrix. Therefore matrices  $Q \otimes W$  and  $R \otimes W$  are positive semidefinite. So we conclude that the matrix

$$H = \begin{bmatrix} Q \otimes W & 0 \\ 0 & R \otimes W \end{bmatrix} \quad (57)$$

is positive semidefinite. Thus the objective function  $J[E(\cdot), U(\cdot)]$  in (52) is convex.  $\square$

**Lemma 3.3 ([38]).** Suppose that  $E^T \Phi_m(t) = \sum_{j=0}^m E_j B_{j,m}(t)$  be the BP of degree  $m$  that approximate the function  $e \in L^2[0, 1]$ . Then  $D^q \{E^T \Phi_m(t)\}$  tends to  $D^q e$ , as  $m$  tends to infinity.

**Theorem 3.2.** If  $m$  (the degree of BPs), tends to infinity, the approximate solutions  $\bar{e}(\cdot) = \bar{E}^T \Phi_m(\cdot)$  and  $\bar{u}(\cdot) = \bar{U}^T \Phi_m(\cdot)$  where  $(\bar{E}, \bar{U})$  is the optimal solution of (52)–(55), converges to exact solutions  $e^*(\cdot)$  and  $u^*(\cdot)$ , respectively.

**Proof.** Consider the problem (52)–(55) and let  $\Omega_m$  is the set of all  $(E^T, U^T) \Phi_m(t)$  where  $(E^T, U^T)$  satisfies constraints (53) and (54). By the convergence property of BPs, for each  $(E_1, U_1) \Phi_m(t) \in \Omega_m$ , there exists a unique pair of functions  $(e_1(t), u_1(t))$  that  $(E_1, U_1) \Phi_m(t) \rightarrow (e_1(t), u_1(t))$  as  $m \rightarrow \infty$  for  $t \in [0, 1]$ . By lemma 3.3, if  $m \rightarrow \infty$ , then  $E_1^T D_q \Phi_m(t) \rightarrow D^q e_1(t)$ , therefore if we let  $\Omega$  be the set of all  $(e(t), u(t))$  that satisfies conditions (34)–(36), obviously we

**Table 1.** The amounts of  $\Delta_i$ ,  $i = 1, 2, 3$ , for optimal control problem (75) for  $m = 6, 8, 10, 12$ .

	$\Delta_1$	$\Delta_2$	$\Delta_3$
$m = 6$	0.489 867 4373	2.118 433 695	9.507 202 864
$m = 8$	0.348 315 3800	2.039 159 612	9.507 180 182
$m = 10$	0.313 605 4828	1.987 205 664	9.507 172 230
$m = 12$	0.310 728 4229	1.986 039 407	9.507 168 683

see that  $(e_1(t), u_1(t)) \in \Omega$ . Hence when  $m$  tends to infinity, each element in  $\Omega_m$  converges to one in  $\Omega$ .

On the other hand, when  $m \rightarrow \infty$ , we have:

$$J_1^m = (E_1^T \Phi_m(t), U_1^T \Phi_m(t)) \rightarrow J_1, \quad (58)$$

where  $J_1^m$  is the value of performance index (33) corresponding to the pair  $(E_1^T, U_1^T) \Phi_m(t)$ , and  $J_1$  is the value of performance index (33) corresponding to the feasible pair  $(e_1(t), u_1(t))$ . Therefore we have:

$$\Omega_1 \subseteq \dots \subseteq \Omega_m \subseteq \Omega_{m+1} \subseteq \dots \subseteq \Omega, \quad (59)$$

hence

$$\inf_{\Omega_1} J^1 \geq \dots \geq \inf_{\Omega_m} J^m \geq \inf_{\Omega_{m+1}} J^{m+1} \geq \dots \geq \inf_{\Omega} J. \quad (60)$$

Now let  $\xi_m = \inf_{\Omega_m} J^m$ , then the sequence  $\{\xi_m\}$  is bounded and monotone, thus it converges to a number  $\xi \geq \inf_{\Omega} J$ . We need to show that  $\xi = \inf_{\Omega} J$ . By the definition of  $\inf$ , for any given  $\epsilon > 0$ , there exists a pair  $(e(t), u(t))$  in  $\Omega$ , such that:

$$J(e(t), u(t)) < \inf_{\Omega} J + \epsilon. \quad (61)$$

Since  $J(e(t), u(t))$  is continuous, there exists a  $N_\epsilon$ , where if  $N_\epsilon < m$ ,

$$|J(e(t), u(t)) - J(E^T \Phi_m(t), U^T \Phi_m(t))| < \epsilon. \quad (62)$$

Therefore if  $N_\epsilon < m$ , from (61) and (62) we have:

$$J(E^T \Phi_m(t), U^T \Phi_m(t)) < J(e(t), u(t)) + \epsilon < \inf_{\Omega} J + 2\epsilon, \quad (63)$$

furthermore we know:

$$\inf_{\Omega} J \leq \xi_m \leq J(E^T \Phi_m(t), U^T \Phi_m(t)), \quad (64)$$

so

$$\inf_{\Omega} J \leq \xi_m < \inf_{\Omega} J + 2\epsilon. \quad (65)$$

Therefore for any  $\epsilon > 0$  we have:

$$0 \leq \xi_m - \inf_{\Omega} J \leq \epsilon. \quad (66)$$

So

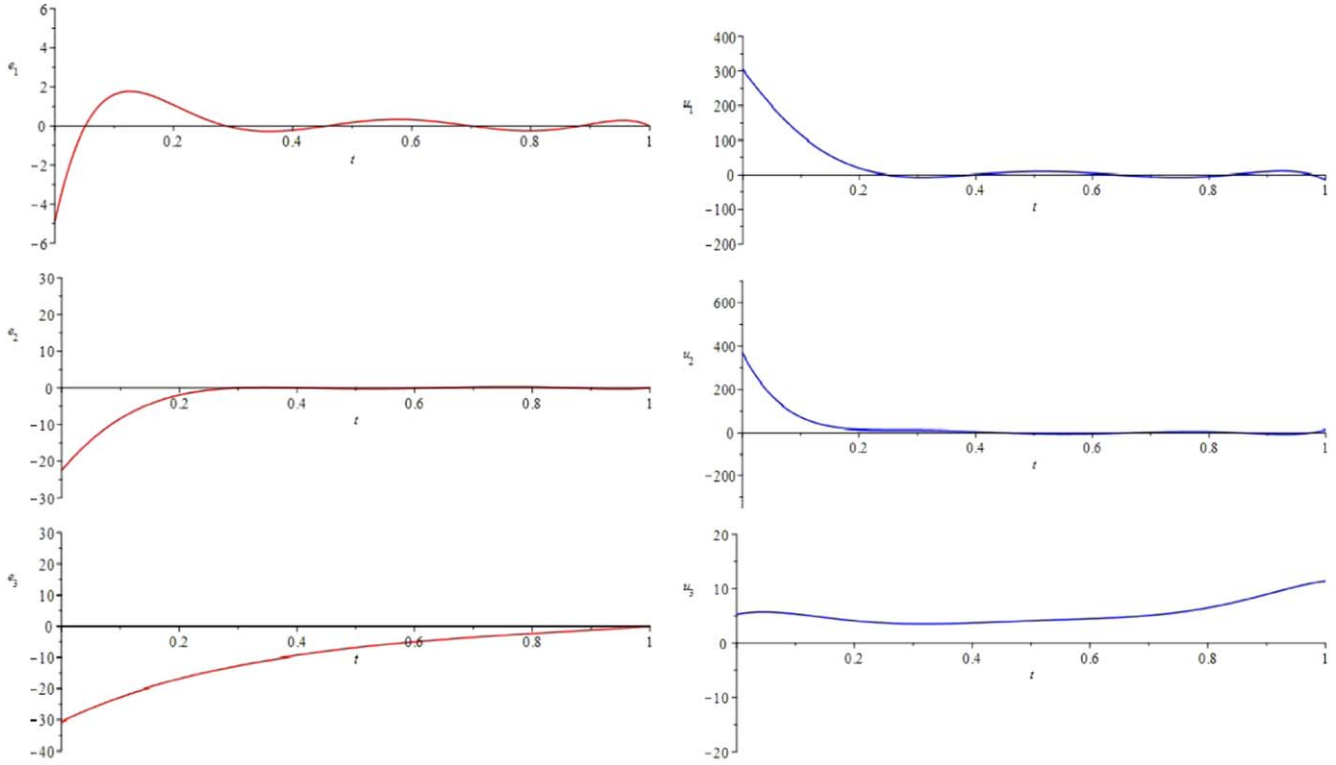
$$\xi = \lim_{m \rightarrow \infty} \xi_m = \inf_{\Omega} J, \quad (67)$$

hence proof is complete.  $\square$

## 4. Numerical examples

Two illustrative test problems for optimal finite time synchronization, are given in this section. For solving the corresponding





**Figure 1.** Error trajectories  $e_i$  and controls  $u_i$ ,  $i = 1, 2, 3$ , for optimal control problem (75) for  $m = 6$  and  $q = 0.98$ .

quadratic minimization problem, we use ‘QPSolve’ in Maple (2018) environment.

#### 4.1. Optimal synchronization between fractional systems of Chen, Lorenz, and Liu

In the first example, we consider the finite time optimal synchronization of the Chen system as master and the combination of Lorenz and Liu systems as slave.

Consider the following Chen system as master:

$$\begin{cases} D^q x_1 = \alpha(x_2 - x_1), \\ D^q x_2 = (\gamma - \alpha)x_1 - x_1x_3 + \gamma x_2, \\ D^q x_3 = x_1x_2 - \beta x_3. \end{cases} \quad (68)$$

Here  $x = [x_1, x_2, x_3]^T$  and initial point is  $x(0) = [10, 25, 36]^T$ . This system has chaotic behavior for  $\alpha = 35$ ,  $\beta = 3$ ,  $\gamma = 28$ , and  $q > 0.83$  (see [45]).

The following two systems are Lorenz and Liu systems as slave 1 and 2, respectively.

$$\begin{cases} D^q y_{11} = \sigma(y_{12} - y_{11}), \\ D^q y_{12} = ry_{11} - y_{12} - y_{11}y_{13}, \\ D^q y_{13} = y_{11}y_{12} - \mu y_{13}. \end{cases} \quad (69)$$

Here  $y_1 = [y_{11}, y_{12}, y_{13}]^T$  and the initial point is  $y_1(0) = [10, 5, 10]^T$ . This system has chaotic behavior for  $\sigma = 10$ ,  $r = 28$ ,  $\mu = \frac{8}{3}$  and  $q > 0.92$  (see [46, 47]).

$$\begin{cases} D^q y_{21} = -ay_{21} - ly_{22}^2, \\ D^q y_{22} = by_{22} - ky_{21}y_{23}, \\ D^q y_{23} = -cy_{23} + my_{21}y_{22}. \end{cases} \quad (70)$$

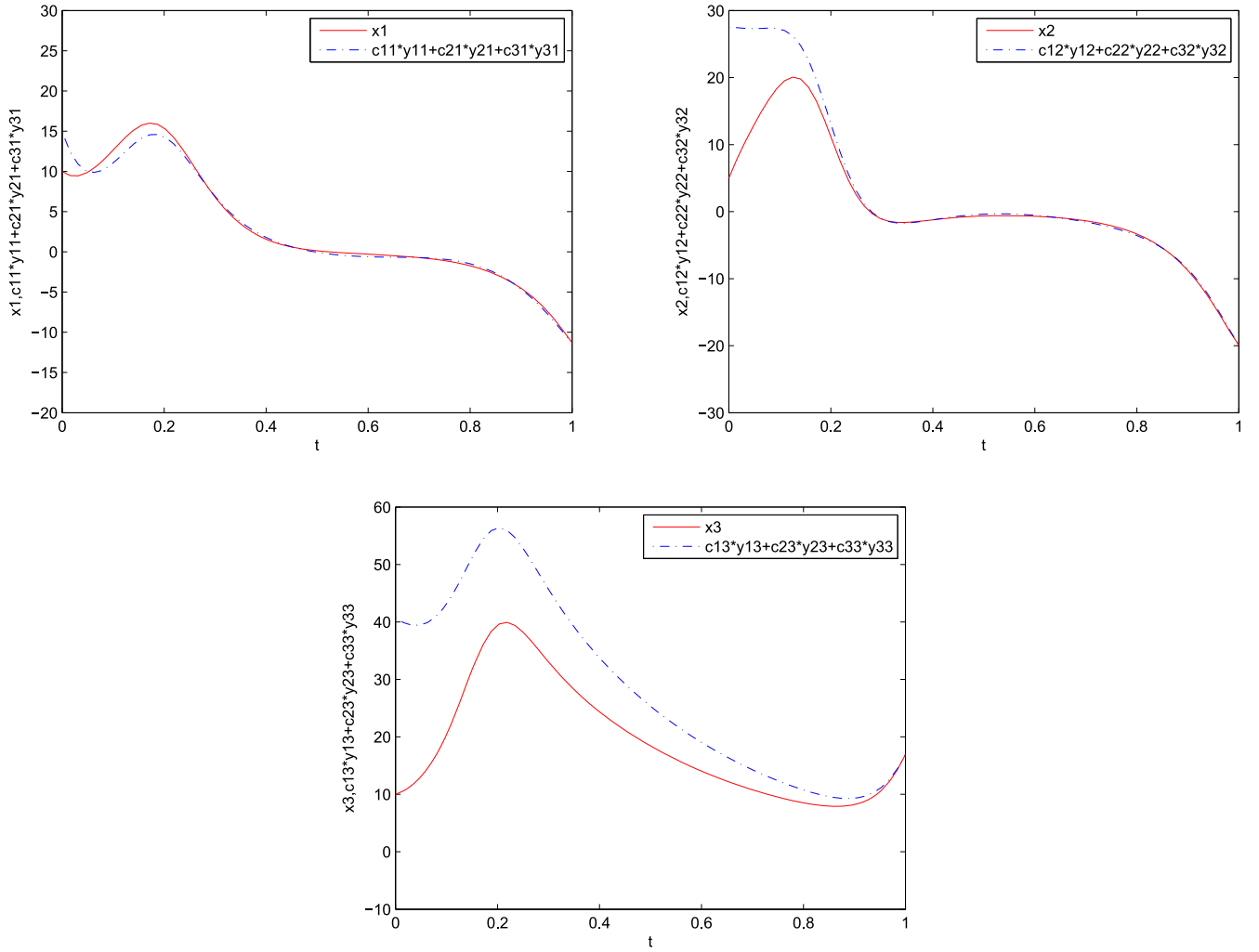
Here  $y_2 = [y_{21}, y_{22}, y_{23}]^T$  and the initial point is  $y_2(0) = [0.2, 0, 0.5]^T$ . This system has chaotic behavior for  $a = 1$ ,  $b = 2.5$ ,  $c = 5$ ,  $k = 4$ ,  $l = 1$ ,  $m = 4$ , and  $q > 0.92$  (see [48]).

We put  $c_1 = \text{diag}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$  and  $c_2 = \text{diag}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ , so we must find the control term  $U = [U_1, U_2, U_3]^T$  such that if we add the control function  $U$  to  $c_1y_1 + c_2y_2$ , then this new system synchronizes with Chen system. We have

$$D^q(c_1y_1 + c_2y_2 - x) = \begin{bmatrix} \frac{1}{2}D^qy_{11} + \frac{1}{2}D^qy_{21} - D^qx_1 \\ \frac{1}{2}D^qy_{12} + \frac{1}{2}D^qy_{22} - D^qx_2 \\ \frac{1}{2}D^qy_{13} + \frac{1}{2}D^qy_{23} - D^qx_3 \end{bmatrix}. \quad (71)$$

Adding the control  $U = [U_1, U_2, U_3]^T$  to the system (71) and defining  $e = c_1y_1 + c_2y_2 - x$  as the error, we have

$$D^q(e) = \begin{bmatrix} \frac{1}{2}\sigma(y_{12} - y_{11}) - \frac{1}{2}ay_{21} - \frac{1}{2}ly_{22}^2 - \alpha(x_2 - x_1) + U_1 \\ \frac{1}{2}ry_{11} - \frac{1}{2}y_{12} - \frac{1}{2}y_{11}y_{13} + \frac{1}{2}by_{22} - \frac{1}{3}ky_{21}y_{23} - (\gamma - \alpha)x_1 + x_1x_3 - \gamma x_2 + U_2 \\ \frac{1}{2}y_{11}y_{12} - \frac{1}{2}\mu y_{13} - \frac{1}{2}cy_{23} + \frac{1}{2}my_{21}y_{22} - x_1x_2 + \beta x_3 + U_3 \end{bmatrix}. \quad (72)$$



**Figure 2.** Synchronization between the chaotic system (68) and the linear combination of (69) and (70) for  $m = 6$  and  $q = 0.98$ .

Here  $e(t) = [e_1(t), e_2(t), e_3(t)]^T$  is the error of the system. problem:  
Consequently, we have the following dynamical error system:

$$\begin{cases} D^q e_1 = -\sigma e_1 + \sigma e_2 + F_1 + U_1, \\ D^q e_2 = r e_1 + b e_2 + F_2 + U_2, \\ D^q e_3 = -\mu e_3 + F_3 + U_3, \end{cases} \quad (73)$$

$$\min J[e(\cdot), u(\cdot)] = \frac{1}{2} \int_0^1 [e_1^2(t) + e_2^2(t) + e_3^2(t) + u_1^2(t) + u_2^2(t) + u_3^2(t)] dt$$

where

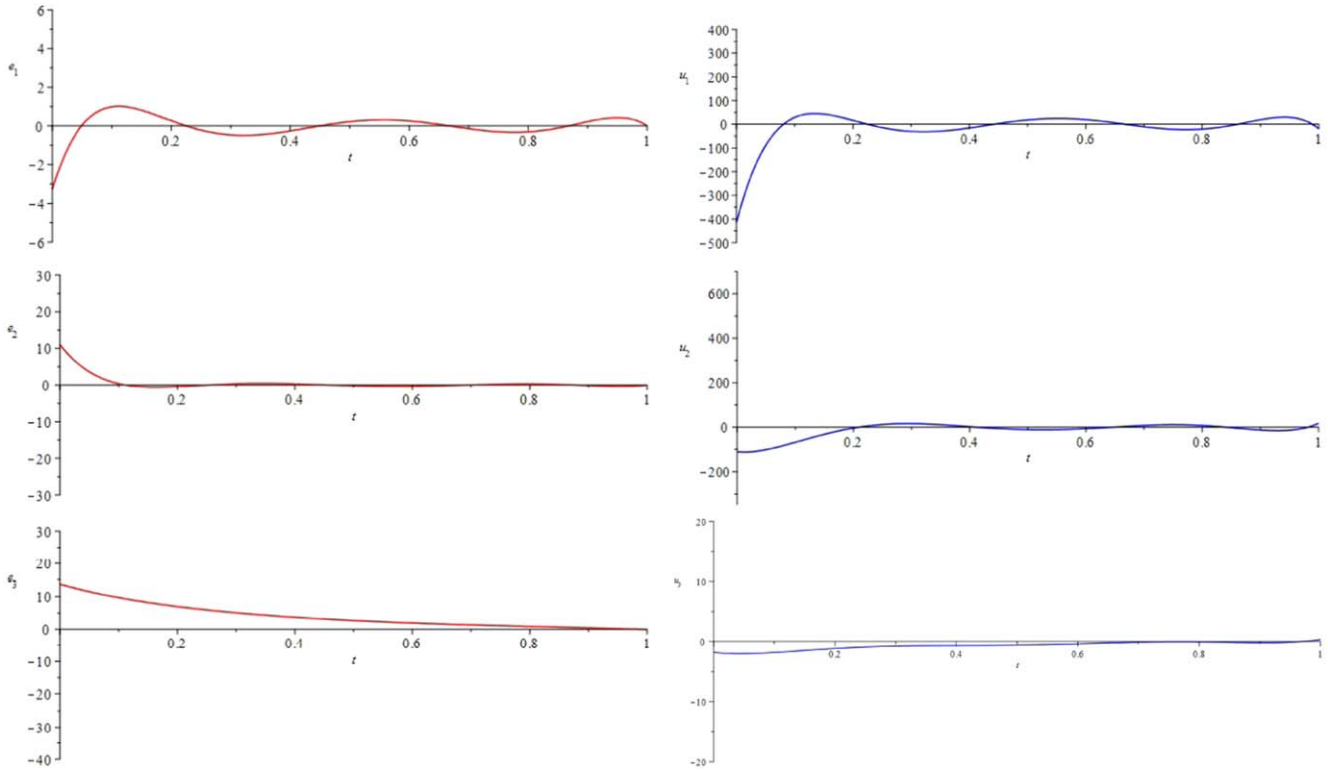
$$\begin{cases} F_1 = \frac{1}{2}(\sigma - a)y_{21} - \frac{1}{2}\sigma y_{22} - \frac{1}{2}ly_{22}^2 + (\alpha - \sigma)x_1 + (\sigma - \alpha)x_2, \\ F_2 = -\frac{1}{2}ry_{21} - \frac{1}{2}(b + 1)y_{12} - \frac{1}{2}y_{11}y_{13} - \frac{1}{2}ky_{21}y_{23} + (\alpha - \sigma)x_1 + (\alpha - \sigma)x_2 + x_1x_3, \\ F_3 = \frac{1}{2}(\mu - c)y_{23} + \frac{1}{2}y_{11}y_{12} + \frac{1}{2}my_{21}y_{22} - \frac{1}{2}x_1x_2 + \left(\frac{1}{2}\beta - \mu\right)x_3. \end{cases} \quad (74)$$

s.t.

By considering  $U = -F(x, y_1, y_2, y_3) + u$ , where  $u = [u_1, u_2, u_3]^T$ , we have the following optimal control

$$\begin{cases} D^q e_1 = -\sigma e_1 + \sigma e_2 + u_1, \\ D^q e_2 = r e_1 + b e_2 + u_2, \\ D^q e_3 = -\mu e_3 + u_3, \end{cases} \quad (75)$$





**Figure 3.** Error trajectories  $e_i$  and controls  $u_i$ ,  $i = 1, 2, 3$ , for the optimal control problem (87) for  $m = 6$  and  $q = 0.98$ .

$$\begin{aligned} e(1) &= 0, \\ e(0) &= \frac{1}{2}y_1(0) + \frac{1}{2}y_2(0) - x(0). \end{aligned}$$

We use BPs of degree  $m = 6, 8, 10, 12$  to solve the problem (75), and we define the following criteria for comparing the effect of  $m$  on the error trajectories:

$$\Delta_i = \int_0^1 |e_i(t)| dt, \quad i = 1, 2, 3. \quad (76)$$

For  $m = 6, 8, 10, 12$ , the amounts of  $\Delta_i$  are shown in table 5.

The approximated state  $e(\cdot)$  and control  $u(\cdot)$  for  $m = 6$  are found as follows:

$$\begin{aligned} e_1(t) &\cong -1167.975t^6 + 3946.925t^5 - 5171.689t^4 \\ &\quad + 3289.847t^3 - 1032.385t^2 + 140.176t - 4.9, \\ e_2(t) &\cong 341.729t^6 - 759.042t^5 + 266.587t^4 \\ &\quad + 537.663t^3 - 555.849t^2 + 191.412t - 22.5, \\ e_3(t) &\cong -39.756t^6 + 153.243t^5 - 247.410t^4 \\ &\quad + 238.942t^3 - 169.575t^2 + 95.306t - 30.75, \end{aligned} \quad (77)$$

and

$$\begin{aligned} u_1(t) &\cong -16\,469.761t^6 + 44\,949.535t^5 - 41\,454.817t^4 \\ &\quad + 11\,465.049t^3 + 3526.089t^2 - 2335.634t \\ &\quad + 305.194, \\ u_2(t) &\cong 30\,703.739t^6 - 102\,763.284t^5 + 135\,275.944t^4 \\ &\quad - 88\,917.381t^3 + 30\,616.924t^2 \\ &\quad - 5271.738t + 368.893, \\ u_3(t) &\cong -671.188t^6 + 2069.803t^5 - 2406.108t^4 \\ &\quad + 1310.164t^3 - 318.089t^2 \\ &\quad + 21.542t + 5.280. \end{aligned} \quad (78)$$

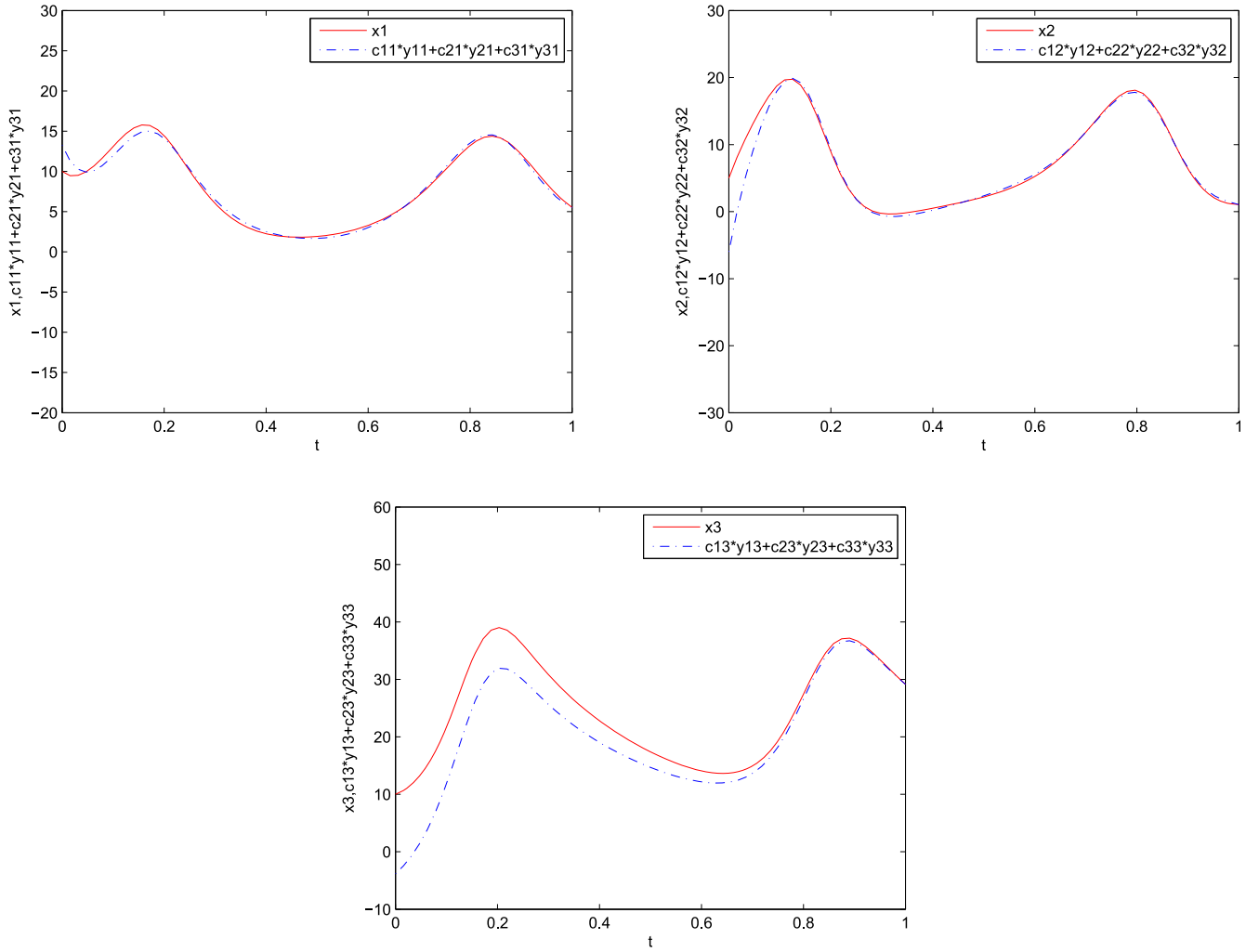
Figure 1 shows error trajectories  $e_i$  and controls  $u_i$ ,  $i = 1, 2, 3$ , of the optimal control problem (75), and figure 2 shows the synchronization for  $m = 6$  and  $q = 0.98$ .

#### 4.2. Finite time optimal synchronization between fractional systems of Lorenz, Chen, Lu, and Liu

As the second example, we consider optimal synchronization of the fractional Lorenz system as master and the combination of three fractional systems as slave.

Consider the following Lorenz system as master:

$$\begin{cases} D^q x_1 = \sigma(x_2 - x_1), \\ D^q x_2 = rx_1 - x_2 - x_1x_3, \\ D^q x_3 = x_1x_2 - \mu x_3, \end{cases} \quad x(0) = [10, 5, 10]^T, \quad \text{and} \quad [\sigma, r, \mu] = \left[10, 28, \frac{8}{3}\right]. \quad (79)$$



**Figure 4.** Synchronization between the chaotic system (79) and the linear combination of (80)–(82) for  $m = 6$  and  $q = 0.98$ .

The following three systems are Chen, Lu, and Liu systems as slave 1, 2, and 3, respectively. We put  $c_1 = \text{diag}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ ,  $c_2 = \text{diag}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ , and  $c_3 = \text{diag}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ , so one needs to find the control term

$$\begin{cases} D^q y_{11} = \alpha(y_{12} - y_{11}), \\ D^q y_{12} = (\gamma - \alpha)y_{11} - y_{11}y_{13} + \gamma y_{12}, \\ D^q y_{13} = y_{11}y_{12} - \beta y_{13}, \end{cases} \quad y_1(0) = [10, 25, 36]^T, \quad \text{and} \quad [\alpha, \beta, \gamma] = [35, 3, 28]. \quad (80)$$

$$\begin{cases} D^q y_{21} = \rho(y_{22} - y_{21}), \\ D^q y_{22} = -y_{21}y_{23} + s y_{22}, \\ D^q y_{23} = y_{21}y_{22} - d y_{23}, \end{cases} \quad y_2(0) = [10, 23, 35]^T, \quad \text{and} \quad [\rho, s, d] = [35, 3, 28]. \quad (81)$$

The system (81) has the chaotic behavior for  $q > 0.3$  (see [49])

$U = [U_1, U_2, U_3]^T$  such that if we add the control function  $U$  to  $c_1 y_1 + c_2 y_2 + c_3 y_3$ , then this new system synchronizes

$$\begin{cases} D^q y_{31} = -a y_{31} - l y_{32}^2, \\ D^q y_{32} = b y_{32} - k y_{31} y_{33}, \\ D^q y_{33} = -c y_{33} + m y_{31} y_{32}, \end{cases} \quad y_3(0) = [0.2, 0, 0.5]^T, \quad \text{and} \quad [a, b, c, k, l, m] = [1, 2.5, 5, 4, 1, 4]. \quad (82)$$

with the Lorenz system. We have

$$D^q(c_1y_1 + c_2y_2 + c_3y_3 - x) = \begin{bmatrix} \frac{1}{3}D^qy_{11} + \frac{1}{3}D^qy_{21} + \frac{1}{3}D^qy_{31} - D^qx_1 \\ \frac{1}{3}D^qy_{12} + \frac{1}{3}D^qy_{22} + \frac{1}{3}D^qy_{32} - D^qx_2 \\ \frac{1}{3}D^qy_{13} + \frac{1}{3}D^qy_{23} + \frac{1}{3}D^qy_{33} - D^qx_3 \end{bmatrix}. \quad (83)$$

Adding the control  $U = [U_1, U_2, U_3]^T$  to the system (83) and defining  $e = c_1y_1 + c_2y_2 + c_3y_3 - x$  as the error, we have

$$D^q(e) = \begin{bmatrix} \frac{1}{3}\alpha(y_{12} - y_{11}) + \frac{1}{3}\rho(y_{22} - y_{21}) - \frac{1}{3}ay_{31} - \frac{1}{3}ly_{32}^2 - \sigma(x_2 - x_1) + u_1 \\ \frac{1}{3}(\gamma - \alpha)y_{11} - \frac{1}{3}y_{11}y_{13} + \frac{1}{3}\gamma y_{12} - \frac{1}{3}y_{21}y_{23} + \frac{1}{3}sy_{22} + \frac{1}{3}by_{32} - \frac{1}{3}ky_{31}y_{33} - rx_1 + x_2 + x_1x_3 + u_2 \\ \frac{1}{3}y_{11}y_{12} - \frac{1}{3}\beta y_{13} + \frac{1}{3}y_{21}y_{22} - \frac{1}{3}dy_{23} - \frac{1}{3}cy_{33} + \frac{1}{3}my_{31}y_{32} - x_1x_2 + \mu x_3 + u_3 \end{bmatrix}. \quad (84)$$

Here  $e(t) = [e_1(t), e_2(t), e_3(t)]^T$  is the error of the system. Consequently, we have the following error system:

$$\begin{cases} D^qe_1 = -\alpha e_1 + \rho e_2 + F_1 + U_1, \\ D^qe_2 = \gamma e_1 + be_2 + F_2 + U_2, \\ D^qe_3 = -\beta e_3 + F_3 + U_3, \end{cases} \quad (85)$$

where

$$\begin{cases} F_1 = -\frac{1}{3}(\alpha - \beta)y_{12} + \frac{1}{3}(\alpha - \rho)y_{21} + \frac{1}{3}(\alpha - a)y_{31} + \frac{1}{3}\rho y_{32} - \frac{1}{3}ly_{32}^2 + (\sigma - \alpha)x_1 + (\rho - \sigma)x_2, \\ F_2 = -\frac{1}{3}\alpha y_{11} + \frac{1}{3}(\gamma - b)y_{12} + \frac{1}{3}(s - b)y_{22} - \frac{1}{3}y_{11}y_{13} - \frac{1}{3}y_{21}y_{23} - \frac{1}{3}ky_{31}y_{33} - \frac{1}{3}\gamma y_{31} + (\gamma - r)x_1 \\ \quad + (b + 1)x_2 + x_1x_3, \\ F_3 = \frac{1}{3}(\beta - d)y_{23} + \frac{1}{3}(\beta - c)y_{33} + \frac{1}{3}y_{11}y_{12} + \frac{1}{3}y_{21}y_{22} + \frac{1}{3}my_{31}y_{32} + (\mu - \beta)x_3 - x_1x_2. \end{cases} \quad (86)$$

By considering  $U = -F(x, y_1, y_2, y_3) + u$ , where  $u = (u_1, u_2, u_3)^T$ , we have the following optimal control problem:

$$\min J[e(\cdot), u(\cdot)] = \frac{1}{2} \int_0^1 [e_1^2(t) + e_2^2(t) + e_3^2(t) + u_1^2(t) + u_2^2(t) + u_3^2(t)] dt$$

s.t.

$$\begin{cases} D^qe_1 = -\alpha e_1 + \rho e_2 + u_1, \\ D^qe_2 = \gamma e_1 + be_2 + u_2, \\ D^qe_3 = -\beta e_3 + u_3, \end{cases} \quad (87)$$

$$e(1) = 0,$$

$$e(0) = \frac{1}{3}y_1(0) + \frac{1}{3}y_2(0) + \frac{1}{3}y_3(0) - x(0).$$

We used BPs of degree  $m = 6$  to solve the problem (87). The approximated state  $e(\cdot)$  and control  $u(\cdot)$  are found as follows:

$$\begin{aligned} e_1(t) &\approx -1.135.862t^6 + 3702.051t^5 - 4639.111t^4 \\ &\quad + 2789.061t^3 - 813.154t^2 + 100.281t - 3.267, \\ e_2(t) &\approx 1401.223t^6 - 4756.921t^5 + 6294.860t^4 \\ &\quad - 4088.983t^3 + 1344.621t^2 - 205.801t + 11, \\ e_3(t) &\approx 42.870t^6 - 159.627t^5 + 242.563t^4 \\ &\quad - 205.680t^3 + 117.178t^2 - 51.137t + 13.833, \end{aligned} \quad (88)$$

and

$$\begin{aligned} u_1(t) &\approx -90.791.509t^6 + 296.471.020t^5 - 374.287.226t^4 \\ &\quad + 229.036.481t^3 - 69.462.960t^2 \\ &\quad + 9430.300t - 414.024, \\ u_2(t) &\approx 31.803.295t^6 - 95.814.588t^5 + 107.672.974t^4 \\ &\quad - 54.501.116t^3 + 11.213.813t^2 \\ &\quad - 247.082t - 110.396, \\ u_3(t) &\approx 714.417t^6 - 2206.093t^5 + 2577.153t^4 \\ &\quad - 1414.946t^3 + 358.641t^2 - 32.563t - 1.405. \end{aligned} \quad (89)$$

Figure 3 shows error trajectories  $e_i$  and controls  $u_i$ ,  $i = 1, 2, 3$ , of the optimal control problem (87), and figure 4 shows the synchronization for  $m = 6$  and  $q = 0.98$ .

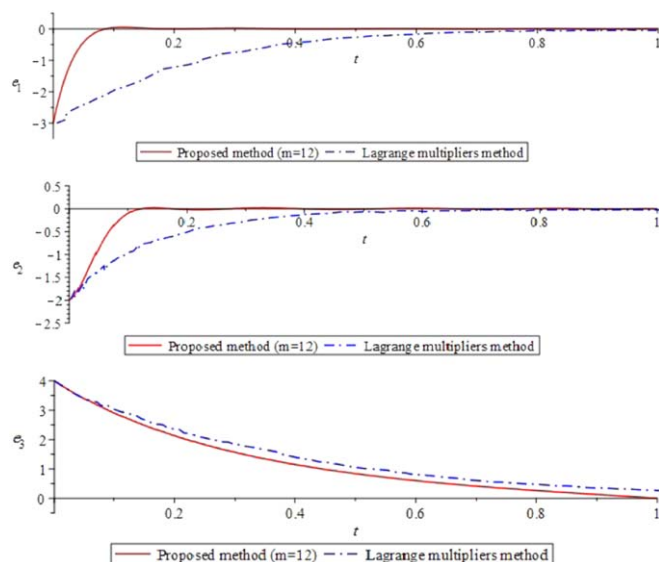
Note that for  $c_1 = I_{3 \times 3}$  and  $c_2 = c_3 = 0_{3 \times 3}$ , and the performance index

$$\int_0^1 \frac{1}{2}(e_1^2(t) + e_2^2(t) + e_3^2(t)) + 5(u_1^2(t) + u_2^2(t) + u_3^2(t)) \quad (90)$$

and initial condition  $x(0) = [2, 3, 5]^T$  and  $y_1(0) = [0, 0, 9]$ , the above optimal synchronization problem has been considered using Lagrange multipliers in [50]. As we see in table 2, the synchronization errors (76), that we achieved are less than what has been obtained in [50]. Figure 5, compares the error trajectories of the proposed method and the Lagrange multipliers method indicated in [50].

**Table 2.** The amounts of  $\Delta_i$ ,  $i = 1, 2, 3$ , using BPs and Lagrange multipliers method.

	$\Delta_1$	$\Delta_2$	$\Delta_3$
BPs method	0.086 425 9132	0.106 315 3229	1.195 901 717
Lagrange multipliers method	0.628 952 1064	0.287 716 8601	1.315 940 760

**Figure 5.** Comparing error trajectories  $e_i$ ,  $i = 1, 2, 3$ , for BPs and Lagrange multipliers method.

## 5. Conclusion

In this paper, BPs method for finite time synchronization of fractional chaotic systems (one master and combination of several slaves) have been suggested. The method approximates the difficult problem of optimal synchronization of fractional chaotic dynamical systems in finite time, by using a quadratic programming problem, where the new mathematical programming problem is intuitive and easy to solve. The proposed method is applied for the synchronization the of some well-known chaotic systems as case studies to show the high performance of the method.

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