Analytical Results to Fuel-optimal Spacecraft Formation Maneuver

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Abstract: In this study, analytical results are obtained for fuel-optimal spacecraft formation maneuver, including initialization and reconfiguration. The method consists of two stages. The first is parameterization of the problem and a new form of the problem is developed for simplicity. Second, the out-of-plane and in-plane maneuver are studied with the help of a useful inequality introduced and proved, the lower bound of the fuel consumption and the corresponding constraints of the control forces are derived and obtained. The result shows that the minimum total fuel consumption depends on the relative size parameters, and that the existence of optimal control algorithms depends upon the initial conditions. Numerical simulation of application proved the validity and efficiency of the proposed method, which can also be utilized in the maneuver of PCF.

1. Introduction

Design of fuel saving control algorithms for spacecraft formation flying maneuver, including initialization and reconfiguration[1],[2],[3] has attracted worldwide attention recently. This research falls into two categories according to the different models used for the design of the algorithms: linear model or orbit element differences.

The design of the fuel saving control algorithms using linear model has appeared in many literatures, such as Hill's equations[4] (also known as Clohessy-Wiltshire equations[5]) in circular orbits, Lawden equations[6] and Tschauner-Hempel equations[7] in elliptic Keplerian orbits. Approaches used for optimization can be summarized as linear programming[8], minimum sliding mode error feedback controller[9], hybrid multi-agent optimization architecture and genetic algorithm[10], particle swarm optimization[11] niched evolutionary algorithm[12], Hamilton-Jacobian-Bellman optimality[13],[14], genetic algorithm and primer vector theory[15], [16], hybrid linear/nonlinear controller[17], analytical solutions[18], pseudo spectral homotopy algorithm[19], quadratic homotopy method[20], indirect method and successive convex programming[21] and the use of nonlinear trajectory generation software package[22]. Other research aspects include reachability and optimal phasing analysis for formation reconfiguration[23], a general method for optimal guidance of formations by optimizing the orbit design, open-time minimum-fuel problem with impulsive control[24].

Other noteworthy approaches for modeling spacecraft formations and designing control algorithms are orbit element differences[25],[26] and Theona theory[27]. Achievements in this field include analytical, two-impulse solution to achieve the expected orbital-elemental differences[28],[29], maneuver guidance with analytical performance by mapping relative orbital elements into a fuel equivalent space[30], feedback control law with guaranteed neighboring fuel-optimality[31], fuel-optimal using Gauss pseudospectral method[32],[33], and fuel-optimal maneuver using low-thrust propulsion[34].

However, the methods mentioned above have lots of problems, strong constrains and obvious shortcomings. Due to the lack of reliable analytical methods, some researchers have to rely on numerical methods. The high numerical sensitivity and nonlinearity corresponding to the bang-bang control are existed in numerical methods. Almost all methods use merely impulse or continuous thrust, which limit flexibility in formation maneuvering. Some methods even cannot guarantee the optimality of solutions.

This work obtains analytical solutions to the optimal spacecraft formation maneuver problem whose cost function expressed by characteristic velocity, with the help of the use of configuration parameters. The in-plane and out-of-plane maneuver are studied. The minimum fuel consumption and the corresponding constraints of control forces are derived. Application of orbital maneuvers is discussed and numerical simulation is illustrated.

2. Problem Formulation

The first-order approximation of relative motion for formation spacecraft expressed in Local-Vertical-Local-Horizontal (LVLH) frame is the well-known Hill's equations

$$X(t)=\underbrace{\begin{bmatrix}0 & I\\ A_1 & A_2\end{bmatrix}}_{A}X(t)+\underbrace{\begin{bmatrix}0\\ I\end{bmatrix}}_{B}U(t) \tag{1}$$

where $X(t)=[x(t),\ y(t),\ z(t),\ \dot{x}(t),\ \dot{y}(t),\ \dot{z}(t)]^{\mathrm{T}}$ are the states of the relative motion, including position and velocity, and $U(t)=[u_x(t),\ u_y(t),\ u_z(t)]^{\mathrm{T}}$ are the control forces. Sub-matrices A_1 and A_2 are given by

$$A_1=\begin{bmatrix}3n^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -n^2\end{bmatrix},\ A_2=\begin{bmatrix}0 & 2n & 0\\ -2n & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \tag{2}$$

where n denotes the mean motion of the chief orbit. The state transfer matrix of (1) is obtained as

$$\Phi(t)=e^{At}=\begin{bmatrix}4-3\cos(nt) & 0 & 0 & \dfrac{\sin(nt)}{n} & \dfrac{2}{n}(1-\cos(nt)) & 0\\ 6\sin(nt)-6nt & 1 & 0 & \dfrac{2}{n}(\cos(nt)-1) & \dfrac{4\sin(nt)-3nt}{n} & 0\\ 0 & 0 & \cos(nt) & 0 & 0 & \dfrac{\sin nt}{n}\\ 3n\sin(nt) & 0 & 0 & \cos(nt) & 2\sin(nt) & 0\\ 6n(\cos(nt)-1) & 0 & 0 & -2\sin(nt) & 4\cos(nt)-3 & 0\\ 0 & 0 & -n\sin(nt) & 0 & 0 & \cos nt\end{bmatrix} \tag{3}$$

Generally, the formation maneuver can be regarded as a controlled orbit transfer from the initial states X_0 to terminal free states $X_d(t_f)$, as follows:

$$X_d(t_f)=\Phi(t_f)X_0+\int_0^{t_f}\Phi(t_f-t)BU(t)\mathrm{d}t \tag{4}$$

where t_f denotes the terminal time. Furthermore, the target configuration should be stable, i.e., $\dot{y}_{d0}=-2nx_{d0}$. A simple form from Eq. (4) can be expressed as

$$\int_0^{t_f}\Phi(-t)BU(t)\mathrm{d}t=\Delta X_0 \tag{5}$$

where $\Delta X_0=X_{d0}-X_0$ denotes the initial error states.

It is assumed that there are three different thrusters, one for each direction. Fuel consumption due to a single impulse is proportional to the 1-norm of the control force. The optimal maneuver can be formulated as follows:

$$\begin{aligned}&\min\quad J=\int_0^{t_f}[|u_x(t)|+|u_y(t)|+|u_z(t)|]\mathrm{d}t\\ &\text{s.t.}\qquad \int_0^{t_f}\Phi(-t)BU(t)\mathrm{d}t=\Delta X_0\end{aligned} \tag{6}$$

where J is the fuel cost function. By substituting Eq. (3) into Eq. (6), one can obtain

$$\begin{aligned}
&\min \quad J=\int_0^{t_f}[|u_x(t)|+|u_y(t)|+|u_z(t)|]\mathrm{d}t \\
&\text{s.t.} \quad \int_0^{t_f}\begin{bmatrix} -\sin(nt) & 2-2\cos(nt) \\ \cos(nt) & -2\sin(nt) \\ 2\cos(nt)-2 & -4\sin(nt)+3nt \\ 2\sin(nt) & 4\cos(nt)-3 \end{bmatrix}\begin{bmatrix} u_x(t) \\ u_y(t) \end{bmatrix}\mathrm{d}t=\begin{bmatrix} n\Delta x_0 \\ \Delta\dot{x}_0 \\ n\Delta y_0 \\ \Delta\dot{y}_0 \end{bmatrix} \\
&\qquad \int_0^{t_f}\begin{bmatrix} \sin(nt) \\ \cos(nt) \end{bmatrix}u_z(t)\mathrm{d}t=\begin{bmatrix} -n\Delta z_0 \\ \Delta\dot{z}_0 \end{bmatrix}
\end{aligned} \tag{7}$$

It is noted that control forces $u_x(t)$, $u_y(t)$, and $u_z(t)$ become delta functions for impulsive formation maneuver.

3. Parameterization of the Problem

At first, a configuration parameters vector, denoted as $P=[p, \phi, s, l, q, \theta]^{\mathrm{T}}$, are defined

$$\begin{aligned}
&p=\sqrt{(3n\Delta x_0+2\Delta\dot{y}_0)^2+\Delta\dot{x}_0^2}\,/\,n && \phi=\arctan\tfrac{\Delta\dot{x}_0}{3n\Delta x_0+2\Delta\dot{y}_0} \\
&s=4\Delta x_0+2\Delta\dot{y}_0\,/\,n && l=\Delta y_0-2\Delta\dot{x}_0\,/\,n \\
&q=\sqrt{(n\Delta z_0)^2+\Delta\dot{z}_0^2}\,/\,n && \theta=\arctan\tfrac{n\Delta z_0}{\Delta\dot{z}_0}
\end{aligned} \tag{8}$$

where $\phi, \theta\in[0, 2\pi)$. Let us assume that $\phi=0$ when $p=0$, and $\theta=0$ when $q=0$.The error states $\Delta X(t)$ can be formulated with P as

$$\Delta X(t)=\Phi(t)\Delta X_0=X(P, t) \tag{9}$$

The detail expression is

$$\begin{aligned}
&\Delta x(t)=-p\cos(nt+\phi)+s \\
&\Delta y(t)=2p\sin(nt+\phi)+l-1.5nts \\
&\Delta z(t)=q\sin(nt+\theta) \\
&\Delta\dot{x}(t)=np\sin(nt+\phi) \\
&\Delta\dot{y}(t)=2np\cos(nt+\phi)-1.5ns \\
&\Delta\dot{z}(t)=nq\cos(nt+\theta)
\end{aligned} \tag{10}$$

Obviously, parameters p and ϕ represent the size and orientation of the in-plane motion, s and l denote the center offsets, and q and θ describe the size and orientation of the out-of-plane motion, respectively.

With the help of the relationship between ΔX_0 and P, the optimization problem Eq. (7) can be developed as

$$\begin{aligned}
&\min \quad J=\int_0^{t_f}[|u_x(t)|+|u_y(t)|+|u_z(t)|]\mathrm{d}t \\
&\text{s.t.} \quad \int_0^{t_f}[\cos(nt)u_x(t)-2\sin(nt)u_y(t)]\mathrm{d}t=np\sin\phi \\
&\qquad \int_0^{t_f}[\sin(nt)u_x(t)+2\cos(nt)u_y(t)]\mathrm{d}t=np\cos\phi \\
&\qquad \int_0^{t_f}u_y(t)\mathrm{d}t=0.5ns \\
&\qquad \int_0^{t_f}[3ntu_y(t)-2u_x(t)]\mathrm{d}t=nl \\
&\qquad \int_0^{t_f}[\sin(nt)u_z(t)]\mathrm{d}t=-nq\sin\theta \\
&\qquad \int_0^{t_f}[\cos(nt)u_z(t)]\mathrm{d}t=nq\cos\theta
\end{aligned} \tag{11}$$

Here, Eq. (11) is called the parameterized problem of the fuel-optimal formation maneuver. When compared with Eq. (7), the integral terms of constraints in Eq. (11) seem much easier to deal with.

4.Problem Solving

To discuss and solve the parameterized problem Eq. (11) directly, a useful inequality is introduced first as follows:

$$\left[\int_0^{t_f}\sqrt{f^2(t)+g^2(t)}\mathrm{d}t\right]^2 \ge \left[\int_0^{t_f} f(t)\mathrm{d}t\right]^2+\left[\int_0^{t_f} g(t)\mathrm{d}t\right]^2 \tag{12}$$

where both $f(t)$ and $g(t)$ are real integrable functions defined on interval $[0,\ t_f]$.

Proof: Let us define a plane curve, denoted by L, with a parametric equation as follows:

$$L(t):\begin{cases} x(t)=\int_0^t f(t)\mathrm{d}t \\ y(t)=\int_0^t g(t)\mathrm{d}t \end{cases};\ t\in[0,\ t_f] \tag{13}$$

Obviously, the initial point $L(0)=(0,\ 0)$ is with origin O, as illustrated in Fig.1. The length of L can be derived as

$$h=\int_0^{t_f}\sqrt{x'^2(t)+y'^2(t)}\mathrm{d}t=\int_0^{t_f}\sqrt{f^2(t)+g^2(t)}\mathrm{d}t \tag{14}$$

where $(\cdot)'$ denotes the derivation with respect to t. In plane Oxy, the line segment OL_f has the minimum distance than that of any curve connected to points O and L_f, hence

$$h^2 \ge |OL_f|^2 = x^2(t_f)+y^2(t_f) \tag{15}$$

where $L_f=L(t_f)$ is the terminal point of L. By substituting Eqs. (13) and (14) into Eq. (15), we can obtain

$$\left[\int_0^{t_f}\sqrt{f^2(t)+g^2(t)}\mathrm{d}t\right]^2 \ge \left[\int_0^{t_f} f(t)\mathrm{d}t\right]^2+\left[\int_0^{t_f} g(t)\mathrm{d}t\right]^2 \tag{16}$$

When the following equation

$$\left[\int_0^{t_f}\sqrt{f^2(t)+g^2(t)}\mathrm{d}t\right]^2 = \left[\int_0^{t_f} f(t)\mathrm{d}t\right]^2+\left[\int_0^{t_f} g(t)\mathrm{d}t\right]^2 \tag{17}$$

holds, $L(t)$ becomes a beeline OL_f, and its property can be obtained as

$$\begin{gathered} k(t)=\frac{y'(t)}{x'(t)}=\frac{f(t)}{g(t)}=\frac{\int_0^{t_f} f(t)\mathrm{d}t}{\int_0^{t_f} g(t)\mathrm{d}t}=\text{const} \\ f(t)\int_0^{t_f} f(t)\mathrm{d}t\ge 0;\quad g(t)\int_0^{t_f} g(t)\mathrm{d}t\ge 0 \end{gathered} \tag{18}$$

where $k(t)$ denotes the slope of L. Equation (18) implies that the terminal point L_f determinates the nature of curve L, which satisfies Eq. (17).

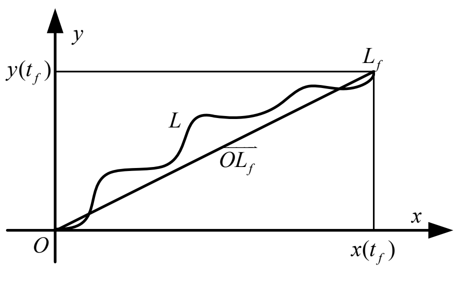


Fig.1 Curve *L*

4.1 Out-of-plane

The optimization problem of out-of-plane motion can be obtained from Eq. (11)

$$\begin{aligned} \min \quad & J_z = \int_0^{t_f} |u_z(t)|\,\mathrm{d}t \\ \text{s.t.} \quad & \int_0^{t_f} [\sin(nt)u_z(t)]\mathrm{d}t = -nq\sin\theta \\ & \int_0^{t_f} [\cos(nt)u_z(t)]\mathrm{d}t = nq\cos\theta \end{aligned} \tag{19}$$

Considering the inequality Eq. (12) and constraints in Eq. (19), the cost function J_z becomes

$$\begin{aligned} J_z &= \int_0^{t_f} |u_z(t)|\,\mathrm{d}t \\ &= \int_0^{t_f} \sqrt{[\sin(nt)u_z(t)]^2 + [\cos(nt)u_z(t)]^2}\,\mathrm{d}t \\ &\ge \sqrt{\left\{\int_0^{t_f} [\sin(nt)u_z(t)]\mathrm{d}t\right\}^2 + \left\{\int_0^{t_f} [\cos(nt)u_z(t)]\mathrm{d}t\right\}^2} \\ &= nq \end{aligned} \tag{20}$$

Consequently, nq is a lower bound of J_z. When following equations:

$$\begin{aligned} & \frac{u_z^*(t)\sin(nt)}{u_z^*(t)\cos(nt)} = \frac{-q\sin\theta}{q\cos\theta} = \frac{q\sin(-\theta)}{q\cos(-\theta)} \\ & u_z^*(t)q\sin(nt)\sin(-\theta) \ge 0 \\ & u_z^*(t)q\cos(nt)\cos(-\theta) \ge 0 \end{aligned} \tag{21}$$

hold, J_z reaches its global minimum value $J_z^* = nq$, and the corresponding optimal control force $u_z^*(t)$ is

$$u_z^*(t)\begin{cases} \ge 0; & t = kT - T_\theta \\ \le 0; & t = (k+0.5)T - T_\theta \\ = 0; & otherwise \end{cases} \tag{22}$$

where $T = 2\pi / n$ is the chief orbital period, $k \in Z^+$, $T_\theta = \theta / n$. It must be noted that $u_z^*(t)$ has nonzero value only in a series of separate instants with interval $0.5T$, while its absolute integral value nq is always nonzero in most applications. Then, $u_z^*(t)$ becomes the sum of a series of corresponding delta functions as follows:

$$u_z^*(t) = \sum_{j=1}^{m_z^+} u_z^{*+}(j)\delta(t - k_j^+T + T_\theta) + \sum_{j=1}^{m_z^-} u_z^{*-}(j)\delta(t - k_j^-T - 0.5T + T_\theta) \tag{23}$$

where subscript "+" denotes the positive impulses, while "–" denotes the negative ones. m_z^+ and m_z^- denote the counts of the positive and negative impulses, respectively, and k_j^+ and $k_j^- \in Z^+$.

Equation (23) indicates that the optimal maneuver of the out-of-plane motion must be impulsive control process, which satisfies

$$\begin{aligned} & \sum_{j=1}^{m_z^+} u_z^{*+}(j) - \sum_{j=1}^{m_z^-} u_z^{*-}(j) = nq \\ & t_z^+(j) = k_j^+T - T_\theta,\ t_z^-(j) = (k_j^- + 0.5)T - T_\theta \end{aligned} \tag{24}$$

Equation (24) is the analytical solution to optimization problem Eq. (19). Clearly, any set of impulses, which is agreeable with Eq. (24), is a proper, optimal, and open control algorithm. The simplest set only contains a positive impulse nq exerted at time $t = T - T_\theta$, or a negative impulse $-nq$ exerted at time $t = 1.5T - T_\theta$.

4.2 In-plane

According to Eq. (11), the optimization problem of in-plane motion can be written as

$$\begin{aligned}
&\min && J_{xy}=\int_0^{t_f}[|u_x(t)|+|u_y(t)|]\mathrm{d}t \\
&\text{s.t.} && \begin{cases}\int_0^{t_f}[\cos(nt)u_x(t)-2\sin(nt)u_y(t)]\mathrm{d}t=np\sin\phi \\ \int_0^{t_f}[\sin(nt)u_x(t)+2\cos(nt)u_y(t)]\mathrm{d}t=np\cos\phi \\ \int_0^{t_f}u_y(t)\mathrm{d}t=0.5ns \\ \int_0^{t_f}[3ntu_y(t)-2u_x(t)]\mathrm{d}t=nl\end{cases}
\end{aligned}\tag{25}$$

Considering inequality Eq. (12) and the first two constraints in Eq. (25), the cost function J_{xy} becomes

$$\begin{aligned}
J_{xy}&=\int_0^{t_f}[|u_x(t)|+|u_y(t)|]\mathrm{d}t \\
&\geq 0.5\int_0^{t_f}\sqrt{u_x^2(t)+4u_y^2(t)}\mathrm{d}t \\
&\geq 0.5\int_0^{t_f}\sqrt{\begin{matrix}[\cos(nt)u_x(t)-2\sin(nt)u_y(t)]^2\\+[\sin(nt)u_x(t)-2\cos(nt)u_y(t)]^2\end{matrix}}\mathrm{d}t \\
&=0.5np
\end{aligned}\tag{26}$$

It has been shown that $0.5np$ is a lower bound of J_{xy}. When $J_{xy}=0.5np$, the cost function reaches its minimum value, and the corresponding optimal control forces satisfy

$$\begin{cases}u_x^*(t)\equiv 0 \\ \dfrac{u_y^*(t)\sin(nt)}{u_y^*(t)\cos(nt)}=\dfrac{-p\sin\phi}{p\cos\phi}=\dfrac{p\sin(-\phi)}{p\cos(-\phi)} \\ u_y^*(t)p\sin(nt)\sin(-\phi)\geq 0 \\ u_y^*(t)p\cos(nt)\cos(-\phi)\geq 0\end{cases}\tag{27}$$

With a similar analysis in above-mentioned discussions on m_y^+, $u_y^*(t)$ is also observed to be the sum of a series of the corresponding delta functions as follows:

$$u_y^*(t)=\sum\nolimits_{j=1}^{m_y^+}u_y^{*+}(j)\delta(t-k_j^+T+T_\phi)+\sum\nolimits_{j=1}^{m_y^-}u_y^{*-}(j)\delta(t-k_j^-T-0.5T+T_\phi)\tag{28}$$

where subscript "+" denotes the positive impulses, while "–" denotes the negative impulses. m_y^+ and m_y^- denote the counts of the positive and negative impulses, respectively, and k_j^+ and $k_j^-\in Z^+$. It must be noted that there is no force or impulse exerted along the *x*-axis. The optimal maneuver of in-plane motion in this case is impulsive control strategy that satisfies

$$\begin{aligned}&\sum\nolimits_{j=1}^{m_z^+}u_y^{*+}(j)-\sum\nolimits_{j=1}^{m_z^-}u_y^{*-}(j)=0.5np \\ &t_y^+(j)=k_j^+T-T_\phi,\ t_y^-(j)=(k_j^-+0.5)T-T_\phi\end{aligned}\tag{29}$$

Furthermore, according to the third constraint of problem Eq. (25), the cost function also satisfies

$$J_{xy}=\int_0^{t_f}[|u_x(t)|+|u_y(t)|]\mathrm{d}t\geq\left|\int_0^{t_f}u_y(t)\mathrm{d}t\right|=0.5n|s|\tag{30}$$

Equation (30) shows that J_{xy} has another lower bound $0.5n|s|$. When $J_{xy}=0.5n|s|$ holds, the optimal control forces become

$$u_x^*(t)\equiv 0,\ u_y^*(t)s\geq 0;\ \forall t\in[0,\ t_f]\tag{31}$$

It has been shown that the optimal force along the *x*-axis is zero, while it is either positive or negative along the *y*-axis during formation maneuver.

From the previous arguments, the lower bound of cost function J_{xy} is

$$J_{xy}^*=\inf J_{xy}\geq\max\{0.5n|s|,\ 0.5np\}\tag{32}$$

If $p>|s|$, from previous discussions, in this case, the optimal control strategy must be impulsive control with fuel consumption $0.5np$. By substituting Eq. (29) into Eq. (25), the constraints of the optimal impulses can be obtained as

$$\begin{aligned}&\sum_{j=1}^{m_y^+} u_y^{*+}(j) - \sum_{j=1}^{m_y^-} u_y^{*-}(j) = 0.5np\\ &\sum_{j=1}^{m_y^+} u_y^{*+}(j) + \sum_{j=1}^{m_y^-} u_y^{*-}(j) = 0.5ns\\ &\sum_{j=1}^{m_y^+} u_y^{*+}(j) t_y^+(j) + \sum_{j=1}^{m_y^-} u_y^{*-}(j) t_y^-(j) = l/3\end{aligned} \tag{33}$$

where

$$t_y^+(j) = k_j^+ T - T_\phi,\ t_y^-(j) = (k_j^- + 0.5)T - T_\phi \tag{34}$$

and $u_y^{*+}(j) \geq 0$, $u_y^{*-}(j) \leq 0$. Equation (37) can be developed into a new form as follows:

$$\begin{aligned}&\sum_{j=1}^{m_y^+} u_y^{*+}(j) = \frac{n(p+s)}{4} > 0\\ &\sum_{j=1}^{m_y^-} u_y^{*-}(j) = -\frac{n(p-s)}{4} < 0\\ &\sum_{j=1}^{m_y^+} u_y^{*+}(j) k_j^+ + \sum_{j=1}^{m_y^-} u_y^{*-}(j) k_j^- = \frac{4l}{3T} + \frac{nsT_\phi}{2T} + \frac{n(p-s)}{8}\end{aligned} \tag{35}$$

From Eqs. (38) and (39), there are many feasible optimal solutions to formation maneuver problem when $p > |s|$. Generally, at least three impulses are needed because of the same number of constraints in Eq. (39). Let us suppose that there are two negative impulses and one positive impulse. Thus, Eqs. (38) and (39) become

$$\begin{aligned}&t_y^+(1) = k_1^+ T - T_\phi, && u_y^{*+}(1) = 0.25n(p+s)\\ &t_y^-(1) = (k_1^- + 0.5)T - T_\phi, && u_y^{*-}(1) = -0.25\alpha n(p-s)\\ &t_y^-(2) = (k_2^- + 0.5)T - T_{\phi,} && u_y^{*-}(2) = -0.25(1-\alpha)n(p-s)\end{aligned} \tag{36}$$

where $\alpha \in [0,\ 1]$, which satisfies

$$(p+s)k_1^+ - (p-s)k_2^- + (p-s)[k_2^- - k_1^-]\alpha = \frac{4l}{3\pi} + \frac{2sT_\phi}{T} + \frac{(p-s)}{2} \tag{37}$$

Equation (41) can be called as the basic equation of the optimal three-impulse maneuver, where $p > |s|$.

One of the feasible solutions of Eq. (41) can be derived as follows:

After defining $\tilde{k}^- = k_2^- + (k_1^- - k_2^-)\alpha \geq 1$, $\sigma = \dfrac{p+s}{p-s} > 0$, and $b = \dfrac{1}{p+s}\left(\dfrac{4l}{3\pi} + \dfrac{2sT_\phi}{T} + \dfrac{p-s}{2}\right)$, Eq.(41) can be rewritten

$$k_1^+ = \sigma \tilde{k}^- + b \tag{38}$$

Considering $k_1^+ \in Z^+$ and $\tilde{k}^- \geq 1$, k_1^+ can be chosen as

$$k_1^+ = \max\{\lceil \sigma + b \rceil,\ 1\} \tag{39}$$

Then

$$\tilde{k}^- = \frac{k_1^+ + b}{\sigma},\ k_1^- = \lfloor \tilde{k}^- \rfloor + 1,\ k_2^- = \lfloor \tilde{k}^- \rfloor,\ \alpha = \tilde{k}^- - \lfloor \tilde{k}^- \rfloor \tag{40}$$

where $\lceil\ \rceil$ denotes the ceil integer of a real number and $\lfloor\ \rfloor$ denotes the floor integer of a real number. By integrating Eqs. (40), (43), and (44), an analytical solution of the optimal three-impulse maneuver can be obtained in this case.

5. Applications: Maneuver of PCF

PCF is a formation for which deputy spacecraft is distributed on a circle, as seen from Earth. The PCF can be described with the radius R and angular position γ. The relative configuration parameters vector is

$$P=[0.5R,\ \gamma,\ 0,\ 0,\ R,\ \gamma+0.5\pi]^{\mathrm{T}}$$

with

$$R=\sqrt{R_0^2+R_d^2-2R_0R_d\cos(\gamma_d-\gamma_0)}$$

$$\gamma=\arctan\frac{R_d\sin\gamma_d-R_0\sin\gamma_0}{R_d\cos\gamma_d-R_0\cos\gamma_0}$$

where (R_0,γ_0) and (R_d,γ_d) denote the initial and target PCF, respectively. From previous discussions, the optimal PCF reconfiguration is an impulsive maneuver process. For the out-of-plane motion, the optimal impulse is

$$t_z^+=(1.75-0.5\gamma/\pi)T,\quad u_z^*=nR$$

For in-plane motion, consider $p\geq s$, and a feasible optimal solution is three impulses. According to Eqs.(40), (43), and (44), $k_1^+=2$, $k_1^-=2$, $k_2^-=1$, $\alpha=0.5$, and the three impulses are

$$t_y^+(1)=(2-0.5\gamma/\pi)T \qquad u_y^{*+}(1)=nR/8$$
$$t_y^-(1)=(1.5-0.5\gamma/\pi)T \qquad u_y^{*-}(1)=-nR/16$$
$$t_y^-(2)=(2.5-0.5\gamma/\pi)T \qquad u_y^{*-}(2)=-nR/16$$

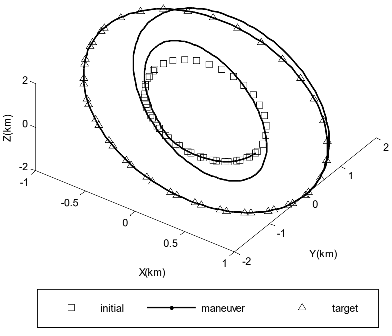
The whole fuel consumption is

$$J=1.5nR=1.25n\sqrt{R_0^2+R_d^2-2R_0R_d\cos(\gamma_d-\gamma_0)}$$

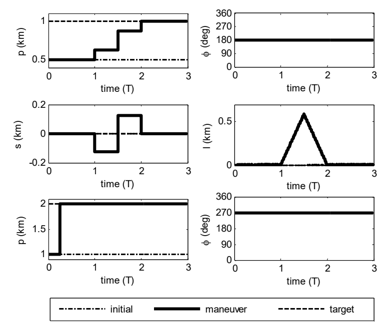
It can be observed that the minimum fuel consumption is $1.25n$ times larger than the relative size R.

Let us consider that the target spacecraft runs in a circular orbit with 800 km height, 1000 m (π) initial PCF, and 2000 m (π) target PCF. Then, the relative PCF (1000 m, 0) can be calculated.

Simulation results are shown in Fig.2. In Fig.2(a), the solid line denotes the maneuver process from initial PCF (dashed point line) to target PCF (dash line). In Fig.2(b), the solid lines represent the parameters of maneuver process. It must be noted that the phase parameters ϕ and θ are constant during reconfiguration maneuver, as shown in Fig.2(b). This indicates that the impulses are exerted at times when $\Delta x(t)=0$ or $\Delta z(t)=0$, according to Eqs. (8) and (9).



(a) Reconfiguration process



(b) Parameters

Fig.2 Fuel-optimal PCF reconfiguration

6.Conclusions

Based on the well-known Hill's equations, this paper has discussed the fuel-optimal formation maneuver in the first-order approximation. It has been shown that the low bound of the fuel consumption only depends on the relative size parameters. The initial conditions affect the existence of the optimal solution of the in-plane maneuver. In the application of formation reconfiguration, the optimal solution must be impulsive control because both the initial and target configurations are stable, and a good choice is a set of impulses. The method proposed in this paper can also be employed in PCF. Further work will focus on nonlinear optimal maneuver under various constraints. And problems will be investigated later in terms of T-H equations.

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